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UNIT 11

Vector Valued Functions and Their Derivatives

INTRODUCTION

- Vector-valued functions return vectors with both magnitude and direction to model multidimensional quantities such as position, velocity, or force in 2D or 3D, unlike scalar functions which give a single value.
- Their derivative provides the instantaneous rate of change, yielding the velocity vector essential for analyzing motion, with applications in engineering, computer graphics, and physics, from robotic arms to planetary orbits.
- This chapter covers definitions, component forms, operations, and differentiation techniques for analyzing curves and solving real-world problems involving velocity and acceleration.

For example, a particle moving along a path in 2D might have its position at time t given by: $\vec{r}(t)=t\hat{i}+t^2\hat{j}$.

This means at time t, the particle is at the point (t, t^2) (Figure 11.1).

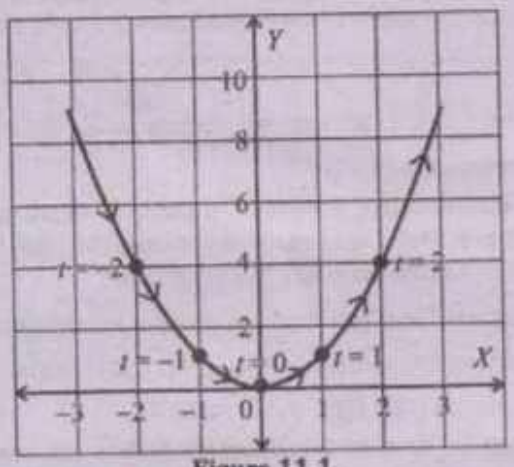

Figure 11.1

- A real-world example of a vector-valued function in three-dimensional space is the position of a satellite orbiting Earth.
- Suppose the vector-valued function describes a satellite's position at time t (in hours):

$$\vec{r}(t)=(4000\cos t)\hat{i}+(4000\sin t)\hat{j}+3000\,t\,\hat{j}$$

Here, the components represent the satellite's coordinates in space: $x = 4000\cos t$, $y = 4000\sin t$, and $z = 3000t$ (in kilometers). The x - and y-components model the satellite's circular orbit in the xy-plane with a radius of 4000 km, while the z-component indicates a linear ascent along the z-axis at a rate of 3000 km per hour.

- This function maps time to a position vector, describing the satellite's trajectory through space, which is essential for tracking and communication in aerospace engineering (Figure 11.2).

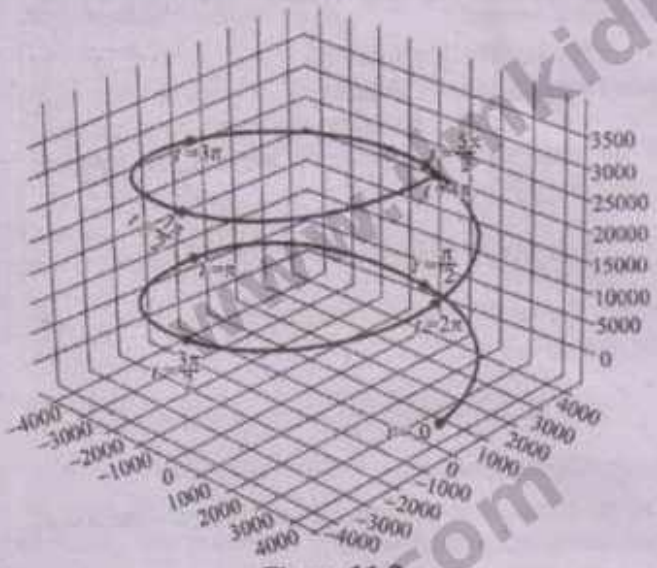

Figure 11.2

Content Outline

- Scalar and Vector-Valued Functions of a Single Variable
- Domain and Range of a Vector-Valued Function
- Identifying Scalar and Vector Valued Functions

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SCALAR AND VECTOR-VALUED FUNCTIONS OF A SINGLE VARIABLE

Scalar Function:

A scalar function is a function that assigns a single real number (a scalar) to each point in its domain. Think of it as a rule that assigns one value to a position in space, like measuring temperature at a specific point.

Mathematically, if to each value of a scalar variable t in the domain of a function $f(t)$, there corresponds a unique scalar f, then f is called a scalar function of t. A real-valued function is commonly called a scalar function.

For example, $f(t) = 3t + 2\cos t$ is a scalar function.

Vector-Valued Function (Vector Function):

A vector-valued function in a single variable is a function that takes one variable as input and returns a vector as output. Each component of the output vector is itself a scalar function of the input. Typically, a vector-valued function is written as:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along the x-, y- and z-axes, and the scalar functions $f(t), g(t)$, and $h(t)$ are the components of $\vec{r}(t)$.

Constructing a Vector-Valued Function

Step-1: Determine the Components: Identify how each coordinate changes with respect to t.

Step-2: Combine into Vector Form:

Express the components as a single vector using either component notation $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ or unit vectors $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$.

Step-3: Specify the Domain: Define the range of t where the function is valid (e.g., $t \geq 0$) for time-dependent motion.

1. Projectile Motion

When a ball is thrown at an angle, its position over time follows a parabolic path under the influence of gravity. If the initial velocity is v_0 at an angle θ, the position vector is:

$$\vec{r}(t) = (v_0\cos\theta)t\hat{i} + \left((v_0\sin\theta)t - \frac{1}{2}gt^2\right)\hat{j}$$

where, $(v_0\cos\theta)t$ = x-component (horizontal motion at constant speed) and

$(v_0\sin\theta)t - \frac{1}{2}gt^2$ = y-component (vertical motion with gravity deceleration).

This model simulates the trajectories of objects like soccer balls, arrows, or rockets in 2D.

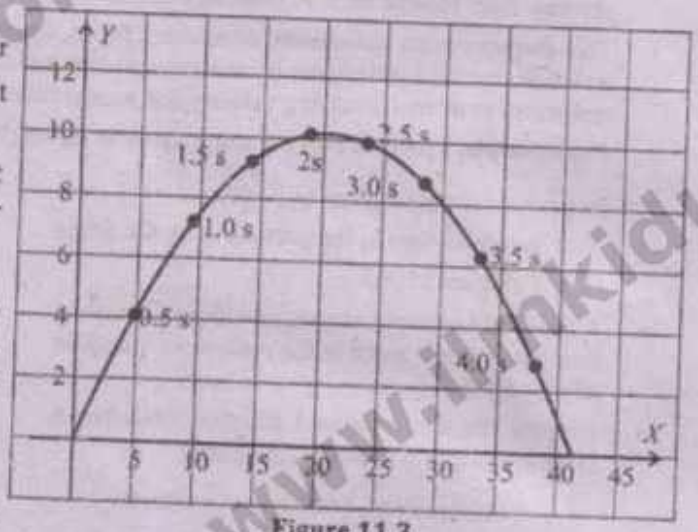

Figure 11.3

2. Flight Path of a Drone

A drone taking off and moving in a helical search pattern can be modeled as:

$$\vec{r}(t) = 5\cos t\hat{i} + 5\sin t\hat{j} + 0.5t\hat{k}$$

Where x- and y-components represent the circular motion with a 5-meter radius, and the z-component represents the steady ascent at 0.5 meters per second. This describes real-world applications like surveillance, search missions, or aerial photography.

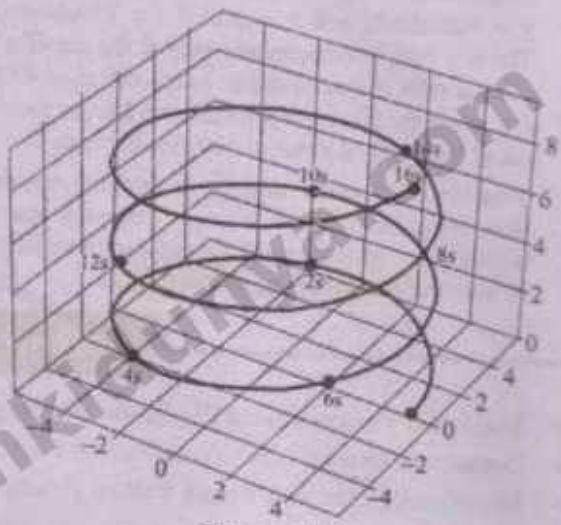

Figure 11.4

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DOMAIN AND RANGE OF A VECTOR-VALUED FUNCTION

The domain of a vector-valued function $\mathbf{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ is the set of all real numbers t for which every component function $f(t), g(t)$, and $h(t)$ is defined.

Steps to Find the Domain of a Vector-Valued Function

Step-1: Determine the domain of each component separately.

Step-2: Find the intersection of the domains of each component $f(t), g(t)$, and $h(t)$. The domain of $\mathbf{r}(t)$ is the intersection of the domains of $f(t), g(t)$, and $h(t)$. The range of a vector-valued function is the set of all possible output vectors $\mathbf{r}(t)$ as t varies over the domain.

Example 1: Find the domain and range of the vector-valued function $\vec{r}(t) = \sqrt{4-t^2}\,\hat{i} + \frac{1}{t-1}\hat{j}$.

Solution:

$$\vec{r}(t) = \sqrt{4-t^2}\,\hat{i} + \frac{1}{t-1}\hat{j}$$

For Domain:

$$x(t) = \sqrt{4-t^2}$$

As $4 - t^2 \geq 0$

$\Rightarrow \quad t^2 \leq 4$

$\sqrt{t^2} \leq \sqrt{4}$

$-2 \leq t \leq 2$

$D_x = [-2, 2]$

$$y(t) = \frac{1}{t-1}$$

As $t - 1 \neq 0$

$\Rightarrow \quad t \neq 1$

$D_y = R - \{1\}$

$D_y = (-\infty, 1) \cup (1, \infty)$

$\text{Domain} = D_x \cap D_y$

$= [-2, 2] \cap ((-\infty, 1) \cup (1, \infty))$

$= [-2, 1) \cup (1, 2]$

$\text{Range} = \{\langle x(t), y(t)\rangle \mid t \in [-2,1) \cup (1,2]\}$

$$\text{Range} = \left\{\left\langle \sqrt{4-t^2}, \frac{1}{t-1}\right\rangle \middle| \, t \in [-2,1) \cup (1,2]\right\}$$

IDENTIFYING SCALAR AND VECTOR VALUED FUNCTIONS

Understanding the distinction between scalar and vector-valued functions is crucial for advanced studies in calculus, physics, and engineering. These functions are fundamental in describing quantities that vary with respect to one or more variables, such as time or position.

Key Identification Criteria

Scalar Function

(i) Scalar-valued functions produce a single numerical value (e.g., a real number).

(ii) Scalar-valued functions are typically denoted by lowercase letters (e.g., $f(x)$).

(iii) Scalar-valued functions describe quantities like temperature or potential energy.

In a heat transfer problem, the temperature $T(x, y, z) = 100 - x^2 - y^2 - z^2$ describes the temperature at a point (x, y, z) in a three-dimensional space.

Identification: The function takes a position (x, y, z) and outputs a single number T (temperature in degrees), making it scalar-valued.

Engineers use such functions to model heat distribution in materials, such as in designing cooling systems for electronic devices.

Vector-valued function:

(i) Vector-valued functions produce a vector, often written as an ordered tuple or in terms of component functions (e.g., $\vec{f}(t) = \langle f_1(t), f_2(t)\rangle$ (2D) or $\vec{g}(t) = \langle g_1(t), g_2(t), g_3(t)\rangle$ (3D).

(ii) Vector-valued functions are often denoted by boldface letters or with an arrow (e.g., $\boldsymbol{f}(t)$ or $\vec{f}(t)$).

(iii) Vector-valued functions describe quantities such as position, velocity, and force.

Consider the function $\vec{r}(t) = (\cos(t), \sin(t), t)$. Here, the input is a single scalar variable t (often representing time), and the output is a 3-dimensional vector. For instance, if $t = \frac{\pi}{2}$ then $\vec{r}\left(\frac{\pi}{2}\right) = \left\langle \cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right), \frac{\pi}{2}\right\rangle = \left\langle 0, 1, \frac{\pi}{2}\right\rangle$. The output is a vector, indicating a point in 3D space. This function traces out a helix.

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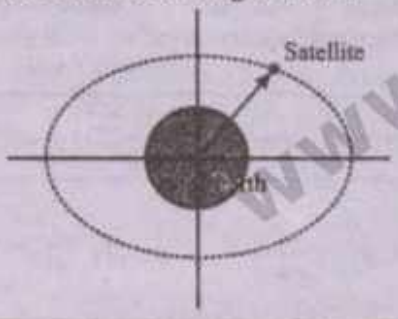
EXERCISE 11.1

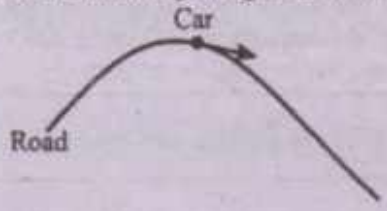
Q1. Give two real-life examples where vector-valued functions are necessary.

Solution:

Example 1:
The position of a satellite orbiting the earth.



Example 2:
The velocity of a car moving along a curved road.



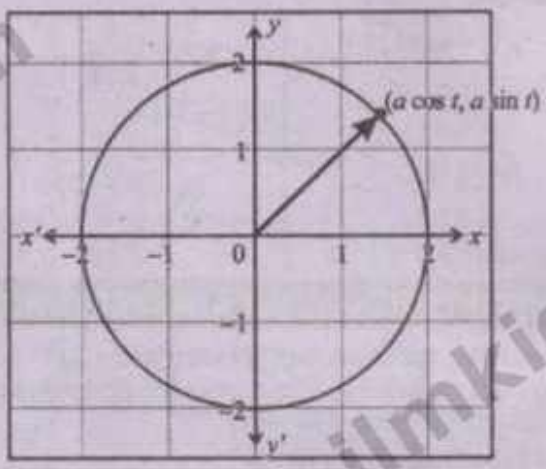
Q2. Construct a vector function for a particle moving in a circle.

Solution:
Parametric equations of a circle with centre $O(0, 0)$ and radius $a(>0)$ are

$$x = a\cos t, y = a\sin t$$

Vector function: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$$\vec{r}(t) = a\cos\hat{i} + a\sin\hat{j}$$



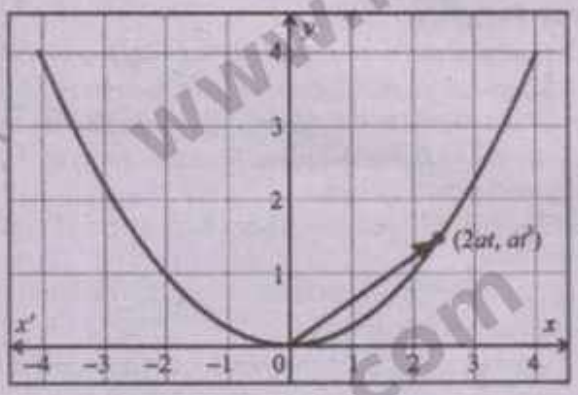
Q3. Write a vector function for a car moving along a parabolic path.

Solution:
Parametric equation of a parabola are

$$x = 2at, y = at^2 \text{ where } a \in R$$

Vector function: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$$\vec{r}(t) = 2at\hat{i} + at^2\hat{j}$$



Q4. Create a 3D vector function for a drone flying upward in a spiral.

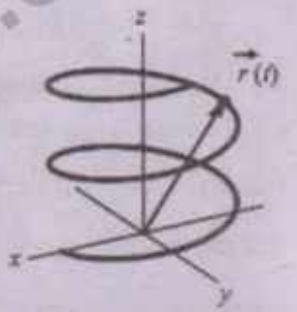
Solution:
Since a drone is flying upward in a spiral, so

$$x = a\cos t, y = a\sin t,$$
$$z = bt$$

where $a > 0$ is the radius of spiral, and $b > 0$ is the vertical speed.

Vector function: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\vec{r}(t) = a\cos t\hat{i} + a\sin t\hat{j} + bt\hat{k}$$



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Q5. Given $x(t) = t^2 + 1, y(t) = \sqrt{t}$, construct $\vec{r}(t)$.

Solution:

Given parametric equations:

$$x(t) = t^2 + 1,\ y(t) = \sqrt{t}$$

Vector function: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$$\vec{r}(t) = (t^2+1)\hat{i} + \sqrt{t}\hat{j}$$

Q6. Given $x(t) = \frac{1}{t+4}, y(t) = \ln\sqrt{t}, z(t) = \sec^{-1}(t)$, construct $\vec{r}(t)$.

Solution:

Given parametric equations:

$$x(t) = \frac{1}{t+4}, y(t) = \ln\sqrt{t}, z(t) = \sec^{-1}(t)$$

Vector function: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\vec{r}(t) = \frac{1}{t+4}\hat{i} + \ln\sqrt{t}\hat{j} + \sec^{-1}t\hat{k}$$

Q7. Find the domain and range of the following vector-valued functions.

(i) $\vec{r}(t) = \sqrt{t}\hat{i} + \ln t\hat{j}$ (ii) $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$

(iii) $\vec{r}(t) = t\hat{i} + t^2\hat{j} + e^t\hat{k}$ (iv) $\vec{r}(t) = \sin(t)\hat{j} + \cos(t)\hat{j} + t\hat{k}$

Solution:

(i) **Vector function:** $\vec{r}(t) = \sqrt{t}\hat{i} + \ln t\hat{j}$

For Domain:

$x(t) = \sqrt{t}$

As $t \geq 0$

$\Rightarrow D_x = [0, \infty)$

$y(t) = \ln t$

As $t > 0$

$\Rightarrow D_y = (0, \infty)$

Domain $= D_x \cap D_y$

$= [0, \infty) \cap (0, \infty) = (0, \infty)$

Range $= \{\langle x(t), y(t)\rangle \mid t \in (0, \infty)\}$

$= \{\langle \sqrt{t}, \ln t\rangle \mid t \in (0, \infty)\}$

(ii) $\vec{r}(t) = a\cos\hat{i} + a\sin\hat{j}$

For Domain:

Since $x(t) = \cos t$ and $y(t) = \sin t$ defined for all $t \in R$, so

Domain $= (-\infty, \infty)$

Range $= \{\langle x(t), y(t)\rangle \mid t \in (-\infty, \infty)\}$

$= \{\langle \cos t, \sin t\rangle \mid t \in (-\infty, \infty)\}$

(iii) $\vec{r}(t) = t\hat{i} + t^2\hat{j} + e^t\hat{k}$

For Domain:

Since $x(t) = t, y(t) = t^2$ and $z(t) = e^t$ are defined for all $t \in R$, so

Domain $= (-\infty, \infty)$

Range $= \{\langle x(t), y(t), z(t)\rangle \mid t \in (-\infty, \infty)\}$

$= \{\langle t, t^2, e^t\rangle \mid t \in (-\infty, \infty)\}$

(iv) $\vec{r}(t) = \sin t\hat{i} + \cos t\hat{j} + t\hat{k}$

For Domain:

Since $x(t) = \sin t, y(t) = \cos t$ and $z(t) = t$ are defined for all $t \in R$, so

Domain $= (-\infty, \infty)$

Range $= \{\langle x(t), y(t), z(t)\rangle \mid t \in (-\infty, \infty)\}$

$= \{\langle \sin t, \cos t, t\rangle \mid t \in (-\infty, \infty)\}$

Q8. Determine which of the following functions are scalar functions or vector-valued functions. Give appropriate reasoning to support the answer.

(i) $f(t) = (3t^2 - 5)^2$ (ii) $h(t) = \langle t^2, e^t, \ln\sqrt{t}\rangle$

(iii) $g(t) = \tan(t) + \csc(t)$ (iv) $j(t) = \left\langle \frac{1}{t^2+1}, \frac{1}{t^2+1} \right\rangle$

Solution:

(i) $f(t) = (3t^2 - 5)^2$ (Scalar function)

Reason: Function has only one output

(ii) $h(t) = \langle t^2, e^t, \ln\sqrt{t}\rangle$ (vector-valued function)

Reason: Function has three outputs

(iii) $g(t) = \tan(t) + \csc(t)$ (Scalar function)

Reason: Function has only one output

(iv) $j(t) = \left\langle \frac{1}{t^2+1}, \frac{1}{t^2+1} \right\rangle$ (Vector-valued function)

Reason: Function has two outputs

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DERIVATIVE OF A VECTOR-VALUED FUNCTION

A vector-valued function of a single variable is a function that maps a scalar (usually time t) to a vector in $\mathbb{R}^n$. For a function in three-dimensional space, it is typically written as:

$$\vec{r}(t)=\langle f_1(t), f_2(t), f_3(t)\rangle = f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}$$

Where $\hat{i},\hat{j},\hat{k}$ are the unit vectors associated with the rectangular coordinates and $f_1(t), f_2(t)$, and $f_3(t)$ are scalar functions of t, are the components of the vectorvalued function $\vec{r}(t)$.

The derivative of a vector-valued function $\vec{r}(t)$ with respect to t is defined as:

$$\frac{d\vec{r}}{dt}=\vec{r}'(t)=\lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}$$

If the limit exists, then $\mathbf{r}'(t)$ is computed by differentiating each component:

$$\mathbf{r}'(t)=\langle f_1'(t), f_2'(t), f_3'(t)\rangle$$
$$= f_1'(t)\hat{i}+f_2'(t)\hat{j}+f_3'(t)\hat{k}$$

Example 4: Find the first derivative of $\vec{r}(t)=t^2\hat{i}+\sin t\hat{j}+e^t\hat{k}$ with respect to t.

Solution:

$$\vec{r}(t)=t^2\hat{i}+\sin t\hat{j}+e^t\hat{k}$$

Differentiating w.r.t. 't'

$$\frac{d}{dt}(\vec{r})=\frac{d}{dt}(t^2\hat{i}+\sin t\hat{j}+e^t\hat{k})$$

$$\frac{d\vec{r}}{dt}=\frac{d}{dt}(t^2)\hat{i}+\frac{d}{dt}(\sin t)\hat{j}+\frac{d}{dt}(e^t)\hat{k}$$

$$\frac{d\vec{r}}{dt}=2t\hat{i}+\cos t\hat{j}+e^t\hat{k}$$

Example 5: Find the first derivative of $\vec{r}(t)=\sec t\hat{i}+\tan t\hat{j}-\cos t\hat{k}$ at $t=\frac{\pi}{4}$.

Solution:

$$\vec{r}(t)=\sec t\hat{i}+\tan t\hat{j}-\cos t\hat{k}$$

Differentiating w.r.t. 't'

$$\frac{d}{dt}(\vec{r})=\frac{d}{dt}(\sec t\hat{i}+\tan t\hat{j}-\cos t\hat{k})$$

$$\frac{d\vec{r}}{dt}=\frac{d}{dt}(\sec t)\hat{i}+\frac{d}{dt}(\tan t)\hat{j}-\frac{d}{dt}(\cos t)\hat{k}$$

$$\frac{d\vec{r}}{dt}=\sec t\tan t\hat{i}+\sec^2 t\hat{j}+\sin t\hat{k}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=\frac{\pi}{4}}=\sec\frac{\pi}{4}\tan\frac{\pi}{4}\hat{i}+\left(\sec\frac{\pi}{4}\right)^2\hat{j}+\sin\frac{\pi}{4}\hat{k}$$

$$=(\sqrt{2})(1)\hat{i}+(\sqrt{2})^2\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$$

$$=\sqrt{2}\hat{i}+2\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$$

EXERCISE 11.2

Q1. Find the first-order derivatives of the following vector-valued functions with respect to t.

(i) $\vec{r}(t)=(t^2+1)\hat{i}+(t^3+t^2+3t)\hat{j}$

(ii) $\vec{r}(t)=(t\sin t-\cos t)\hat{i}+(\cot(t)+\ln t)\hat{j}+(\ln(1+t^2))\hat{k}$

(iii) $\vec{r}(t)=\frac{2}{\sqrt{1-t^2}}\hat{i}+\frac{\sqrt{3}}{1+t^2}\hat{k}$

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(i) $\vec{r}(t)=(t^2+1)\hat{i}+(t^3+t^2+3t)\hat{j}$

Solution:

Differentiating w.r.t. 't'

$$\frac{d}{dt}(\vec{r})=\frac{d}{dt}\left((t^2+1)\hat{i}+(t^3+t^2+3t)\hat{j}\right)$$

$$\frac{d\vec{r}}{dt}=\frac{d}{dt}(t^2+1)\hat{i}+\frac{d}{dt}(t^3+t^2+3t)\hat{j}=(2t+0)\hat{i}+(3t^2+2t+3(1))\hat{j}$$

$$\frac{d\vec{r}}{dt}=(2t)\hat{i}+(3t^2+2t+3)\hat{j}$$

(ii) $\vec{r}(t)=(t\sin t-\cos t)\hat{i}+(\cot(t)+\ln t)\hat{j}+(\ln(1+t^2))\hat{k}$

Solution:

Differentiating w.r.t. 't'

$$\frac{d}{dt}(\vec{r})=\frac{d}{dt}\left\{(t\sin t-\cos t)\hat{i}+(\cot(t)+\ln t)\hat{j}+\ln(1+t^2)\hat{k}\right\}$$

$$\frac{d\vec{r}}{dt}=\frac{d}{dt}(t\sin t-\cos t)\hat{i}\frac{d}{dt}+(\cot(t)+\ln t)\hat{j}+\frac{d}{dt}\ln(1+t^2)\hat{k}$$

$$\frac{d\vec{r}}{dt}=\left(t\cos t+\sin t(1)-(-\sin t)\hat{i}+\left(-\csc^2 t+\frac{1}{t}\right)\hat{j}+\frac{1}{1+t^2}(2t)\hat{k}\right)$$

$$\frac{d\vec{r}}{dt}=(t\cos t+2\sin t)\hat{i}+\left(\frac{1}{t}-\csc^2 t\right)\hat{i}+\frac{2t}{1+t^2}\hat{k}$$

(iii) $\vec{r}(t)=\frac{2}{\sqrt{1-t^2}}\hat{i}+\frac{\sqrt{3}}{1+t^2}\hat{k}$

Solution:

Differentiating w.r.t. 't'

$$\frac{d}{dt}(\vec{r})=\frac{d}{dt}\left(\frac{2}{\sqrt{1-t^2}}\hat{i}+\frac{\sqrt{3}}{1+t^2}\hat{k}\right)$$

$$\frac{d\vec{r}}{dt}=\frac{d}{dt}\left[2(1-t^2)^{-\frac{1}{2}}\right]\hat{i}+\frac{d}{dx}[\sqrt{3}(1+t^2)^{-1}]\hat{k}$$

$$=2\left(\frac{-1}{2}\right)(1-t^2)^{-\frac{1}{2}-1}(-2t)\hat{i}+(-1)\sqrt{3}(1+t^2)^{-1-1}(2t)\hat{k}$$

$$=2t(1-t^2)^{-\frac{3}{2}}\hat{i}-\sqrt{3}(2t)(1+t^2)^{-2}\hat{k}$$

$$\frac{d\vec{r}}{dt}=\frac{2t}{(1-t^2)^{\frac{3}{2}}}\hat{i}-\frac{2\sqrt{3}t}{(1+t^2)^2}\hat{k}$$

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Q2. Find the first-order derivatives of the following vector-valued functions with respect to t at the specified values of t.

(i) $\vec{r}(t)=\left(\frac{t^2+1}{\sin t}\right)\hat{i}+(\sqrt{t}e^t)\hat{j}$ at $t=\frac{\pi}{2}$ (ii) $\vec{r}(t)=(t\sqrt{1+2})\hat{i}+t\ln(t+\sqrt{t^2+1})\hat{j}$ at $t=0$

(iii) $\vec{r}(t)=\tan^{-1}t\hat{i}+\frac{\sin^2 t}{t}\hat{j}+(1+t)^{\frac{1}{3}}\hat{k}$ at $t=0$ (iv) $\vec{r}(t)=te^{-t}\hat{i}+\cos(3t)\hat{j}+t\ln(1+t^2)\hat{k}$ at $t=0$

(i) $\vec{r}(t)=\left(\frac{t^2+1}{\sin t}\right)\hat{i}+(\sqrt{t}e^t)\hat{j}$

Solution:

Differentiating w.r.t 't'

$$\frac{d}{dt}(\vec{r})=\frac{d}{dt}\left\{\left(\frac{t^2+1}{\sin t}\right)\hat{i}+\left(\sqrt{t}e^t\right)\hat{j}\right\}$$

$$\frac{d\vec{r}}{dt}=\frac{d}{dt}\left(\frac{t^2+1}{\sin t}\right)\hat{i}+\frac{d}{dt}(\sqrt{t}e^t)\hat{j}$$

$$=\frac{\sin t(2t+0)-(t^2+1)\cos t}{\sin^2 t}\hat{i}+\left(\sqrt{t}e^t+e^t\frac{1}{2\sqrt{t}}\right)\hat{j}=\frac{2t\sin t-(t^2-1)\cos t}{\sin^2 t}\hat{i}+e^t\left(\frac{2t+1}{2\sqrt{t}}\right)\hat{j}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=\frac{\pi}{2}}=\frac{2\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2}-\left(\left(\frac{\pi}{2}\right)^2+1\right)\cos\left(\frac{\pi}{2}\right)}{\left(\sin\frac{\pi}{2}\right)^2}\hat{i}+\left(\frac{2\left(\frac{\pi}{2}\right)+1}{2\sqrt{\left(\frac{\pi}{2}\right)}}\right)e^{\frac{\pi}{2}}\hat{j}$$

$$=\frac{\pi(1)-\left(\frac{\pi}{4}+1\right)(0)}{(1)^2}\hat{i}+\left(\frac{\pi+1}{\sqrt{4\frac{\pi}{2}}}\right)e^{\frac{\pi}{2}}\hat{j}=\frac{(\pi-0)}{1}\hat{i}+\left(\frac{\pi+1}{\sqrt{2\pi}}\right)e^{\frac{\pi}{2}}\hat{j}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=\frac{\pi}{2}}=\pi\hat{i}+\left(\frac{\pi+1}{\sqrt{2\pi}}\right)e^{\frac{\pi}{2}}\hat{j}$$

(ii) $\vec{r}(t)=(t\sqrt{1+2})\hat{i}+t\ln(t+\sqrt{t^2+1})\hat{j}$

Solution:

Differentiating w.r.t. 't'

$$\frac{d}{dt}(\vec{r})=\frac{d}{dt}\{(t\sqrt{t+2})\hat{i}+t\ln(t+\sqrt{t^2+1})\hat{j}\}$$

$$\frac{d\vec{r}}{dt}=\frac{d}{dt}(t\sqrt{t+2})\hat{i}+\frac{d}{dx}t\ln(t+\sqrt{t^2+1})\hat{j}$$

$$=\left(t.\frac{1}{2\sqrt{t+2}}+\sqrt{t+2}(1)\right)\hat{i}+\frac{1}{t+\sqrt{t^2+1}}\left(1+\frac{1}{2\sqrt{t^2+1}}(2t)\right)\hat{j}$$

$$=\frac{t+2(\sqrt{t+2})^2}{2\sqrt{t+2}}\hat{i}+\frac{1}{t+\sqrt{t^2+1}}\left(\frac{\sqrt{t^2+1}+t}{\sqrt{t^2+1}}\right)\hat{j}=\frac{t+2t+4}{2\sqrt{t+2}}\hat{i}+\frac{1}{\sqrt{t^2+1}}\hat{j}$$

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$$\frac{d\vec{r}}{dt} = \frac{3t+4}{2\sqrt{t+2}}\hat{i} + \frac{1}{\sqrt{t^2+1}}\hat{j}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = \frac{0+4}{2\sqrt{0+2}}\hat{i} + \frac{1}{\sqrt{0^2+1}}\hat{j}$$

$$= \frac{2}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{1}}\hat{j} = \sqrt{2}\hat{i} + \hat{j}$$

(iii) $\vec{r}(t) = \tan^{-1}t\hat{i} + \frac{\sin^2 t}{t}\hat{j} + (1+t)^{\frac{1}{3}}\hat{k}$

Solution:

Differentiating w.r.t. 't'

$$\frac{d}{dt}(\vec{r}) = \frac{d}{dt}(\tan^{-1}t)\hat{i} + \frac{d}{dt}\left(\frac{\sin^2 t}{t}\right)\hat{j} + \frac{d}{dt}(1+t)^{\frac{1}{3}}\hat{k}$$

$$\frac{d\vec{r}}{dt} = \frac{1}{1+t^2}\hat{i} + \frac{t\cos^2 t(2t) - \sin^2(1)}{t^2}\hat{j} + \frac{1}{3}(1+t)^{\frac{1}{3}-1}\hat{k}$$

$$= \frac{1}{1+t^2}\hat{i} + \frac{2t^2\cos^2 t - \sin^2 t}{t^2}\hat{j} + \frac{1}{3}(1+t)^{\frac{-2}{3}}\hat{k} = \frac{1}{1+t^2}\hat{i} + \left(2\cos^2 t - \frac{\sin t^2}{t^2}\right)\hat{j} + \frac{1}{3(1+t)^{\frac{2}{3}}}\hat{k}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = \frac{1}{1+0^2}\hat{i} + \left(2\cos(0)^2 - \lim_{t\to 0}\frac{\sin t^2}{t^2}\right)\hat{j} + \frac{1}{3(1+0)^{\frac{2}{3}}}\hat{k}$$

$$= \hat{i} + (2-1)\hat{j} + \frac{1}{3(1)^{\frac{2}{3}}}\hat{k} \qquad \because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$= \hat{i} + \hat{j} + \frac{1}{3}\hat{k}$$

(iv) $\vec{r}(t) = te^{-t}\hat{i} + \cos(3t)\hat{j} + \ln(1+t^2)\hat{k}$

Solution:

Differentiating w.r.t. 't'

$$\frac{d}{dt}(\vec{r}) = \frac{d}{dt}(t\cdot e^{-t})\hat{i} + \frac{d}{dt}(\cos(3t))\hat{j} + \frac{d}{dt}\ln(1+t^2)\hat{k}$$

$$\frac{d\vec{r}}{dt} = (te^{-t}(-1) + e^{-t}(1))\hat{i} - \sin 3t(3)\hat{j} + \frac{1}{1+t^2}(2t)\hat{k}$$

$$= e^{-t}(-t+1)\hat{i} - (3\sin 3t)\hat{j} + \frac{2t}{1+t^2}\hat{k}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = e^{-0}(0+1)\hat{i} - 2\sin 3(0)\hat{j} + \frac{2(0)}{1+0^2}\hat{k}$$

$$= 1(1)\hat{i} - 3(0)\hat{j} + \frac{0}{1}\hat{k} = \hat{i} + 0\hat{j} + 0\hat{k}$$

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VELOCITY AND ACCELERATION

If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then $\frac{d}{dt}\vec{r}(t)$ represent the velocity vector of the particle, which is tangent to the curve, i.e. $\frac{d}{dt}\vec{r}(t)=\vec{v}(t)$, and the derivative of the velocity vector with respect to time t is the acceleration of the particle, denoted by $\mathbf{a}$ i.e. $\frac{d}{dt}\vec{v}(t)=\vec{a}(t)=\frac{d^2}{dt^2}\vec{r}(t)$.

Example 6: Vibrating conveyor belt in a factory sorting system

A small metal part moves on a conveyor belt in a factory. The belt transports the part eastward (x-direction) while simultaneously vibrating sideways (north-south, y-direction) to help sort items into bins. The position of the part at time t seconds is given by $\vec{r}(t)=3t^2\hat{i}+2\sin t\hat{j}$ in meters. At a specific moment, $t=\frac{\pi}{2}$ an engineer needs to analyze the part's motion to ensure it enters the correct bin. Hence, find the following quantities at $t=\frac{\pi}{2}$.

(a) velocity (b) acceleration (c) speed (d) direction of motion.

Solution:

$$\vec{r}(t)=3t^2\hat{i}+2\sin t\,\hat{j}$$

(a) Velocity:

$$\vec{v}(t)=\frac{d\vec{r}}{dt}=\frac{d}{dt}(3t^2\hat{i}+2\sin t\hat{j})$$

$$=\frac{d}{dt}(3t^2)\hat{i}+\frac{d}{dt}(2\sin t)\hat{j}$$

$$\vec{v}(t)=6t\hat{i}+2\cos t\hat{j}$$

At $t=\frac{\pi}{2}$:

$$\vec{v}\left(\frac{\pi}{2}\right)=6\left(\frac{\pi}{2}\right)\hat{i}+2\cos\frac{\pi}{2}\hat{j}$$

$$\vec{v}\left(\frac{\pi}{2}\right)=3\pi\hat{i}+0\hat{j}$$

Interpretation: The part is moving purely eastward (x-direction) at $3\pi\,m/s$ with no north-south velocity.

(b) Acceleration:

$$\vec{a}(t)=\frac{d\vec{v}}{dt}=\frac{d}{dt}(6t\hat{i}+2\cos t\hat{j})$$

$$\vec{a}(t)=6\hat{i}-2\sin t\hat{j}$$

At $t=\frac{\pi}{2}$:

$$\vec{a}\left(\frac{\pi}{2}\right)=6\hat{i}-2\sin\frac{\pi}{2}\hat{j}$$

$$\vec{a}\left(\frac{\pi}{2}\right)=6\hat{i}-2(1)\hat{j}$$

$$=6\hat{i}-2\hat{j}$$

Interpretation: The part is accelerating eastward at $6m/s^2$ and southward (negative y-direction) at $2m/s^2$.

(c) Speed:

$$\text{Speed}=|\vec{v}(t)|=\sqrt{(6t)^2+(2\cos t)^2}$$

$$\left|\vec{v}\left(\frac{\pi}{2}\right)\right|=\sqrt{\left(6\left(\frac{\pi}{2}\right)\right)^2+\left(2\cos\frac{\pi}{2}\right)^2}$$

$$=\sqrt{(3\pi)^2+(0)^2}=3\pi$$

$$\approx 9.42\ m/s$$

(d) Direction of motion:

$$\hat{v}(t)=\frac{\vec{v}(t)}{|\vec{v}(t)|}$$

At $t=\frac{\pi}{2}$:

$$\hat{v}\left(\frac{\pi}{2}\right)=\frac{\vec{v}\left(\frac{\pi}{2}\right)}{\left|\vec{v}\left(\frac{\pi}{2}\right)\right|}$$

$$=\frac{3\pi}{3\pi}\hat{i}=1\hat{i}+0\hat{j}$$

Interpretation: The direction of motion is due east (positive x-direction)

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Example 7: Drone inspecting a cylindrical water tower

A drone is programmed to fly in a helical (spiral) path around a large cylindrical water tower to inspect its outer surface for cracks or rust. The drone maintains a constant horizontal distance of 2 meters from the tower's center while climbing upward at a steady rate. The drone's position in 3D space at time t seconds is given by:

$$r(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 3t\hat{k}$$

Here:

- $x(t) = 2\cos t$ **: east-west position (amplitude 2 m)**
- $y(t) = 2\cos t$ **: north-south position (amplitude 2 m)**
- $z(t) = 3t$ **: Vertical height (climbs 3 m/s)**

At a specific moment $t = \pi$ seconds, the drone operator needs to analyze its motion to ensure smooth inspection. Find the following quantities at $t = \pi$ (a) velocity, (b) acceleration, (c) speed, and (d) direction of motion at $t = \pi$.

Solution:

$$\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 3t\hat{k}$$

(a) Velocity:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 3t\hat{k})$$

$$\vec{v}(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 3\hat{k}$$

At $t = \pi$:

$$\vec{v}(\pi) = -2\sin\pi\,\hat{i} + 2\cos\pi\,\hat{j} + 3\hat{k}$$

$$= 0\hat{i} - 2\hat{j} + 3\hat{k}$$

Interpretation: At this moment, the drone has no east-west motion, is moving southward at $2m/s$ and climbing vertically upward at $3m/s$.

(b) Acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}(-2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 3\hat{k})$$

$$\vec{a}(t) = -2\cos t\,\hat{i} - 2\sin t\,\hat{j} + 0\hat{k}$$

At $t = \pi$:

$$\vec{a}(\pi) = -2\cos\pi\,\hat{i} - 2\sin\pi\,\hat{j} + 0\hat{k}$$

$$= -2(-1)\hat{i} - 2(0)\hat{j} + 0\hat{k}$$

$$= 2\hat{i} + 0\hat{j} + 0\hat{k}$$

Interpretation: The drone is accelerating eastward at 2 m/s^2. This centripetal acceleration points towards the centre of the circular path (the tower's axis), keeping the drone in its circular horizontal trajectory.

(c) Speed:

$$\text{Speed} = |\vec{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 9}$$

$$|\vec{v}(t)| = \sqrt{4(\sin^2 t + \cos^2 t) + 9}$$

$$= \sqrt{4(1) + 9}$$

$$= \sqrt{13}$$

At $t = \pi$:

$$|\vec{v}(\pi)| = \sqrt{13}$$

$$\approx 3.6056 m/s$$

Interpretation: The drone's speed is constant at about 3.61 m/s throughout its helical climb, ideal for capturing stable inspection footage.

(d) Direction of motion:

$$\hat{v}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

$$= \frac{-2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 3\hat{k}}{\sqrt{13}}$$

At $t = \pi$:

$$\hat{v}(\pi) = \frac{0\hat{i} - 2\hat{j} + 3\hat{k}}{\sqrt{13}}$$

$$= 0\hat{i} - \frac{2}{\sqrt{3}}\hat{j} + \frac{3}{\sqrt{13}}\hat{k}$$

Interpretation: The drone is moving in the south-up vertical plane. The direction has no east-west component.

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EXERCISE 11.3

In 1-5, each problem involves a vector-valued function representing the particle's position in the *xy*-plane or in 3D space. Find the velocity, acceleration, speed, and direction of motion at a specified value of *t*. Interpret the results by describing the particle's motion at that instant, including how fast and in what direction it is moving, whether it is speeding up or slowing down, and how its path is curving.

Q1. $\vec{r}(t)=2\sin t\hat{i}+3t\hat{j}$ **at** $t=\frac{\pi}{2}$

Solution:

$$\vec{r}(t)=2\sin t\hat{i}+3t\hat{j}$$

(a) Velocity:

$$\vec{v}(t)=\frac{d\vec{r}}{dt}=\frac{d}{dt}(2\sin t\hat{i}+3t\hat{j})$$

$$\vec{v}(t)=2\cos t\hat{i}+3\hat{j}$$

At $t=\frac{\pi}{2}$:

$$\vec{v}\left(\frac{\pi}{2}\right)=2\cos\frac{\pi}{2}\hat{i}+3\hat{j}=0\hat{i}+3\hat{j}$$

(b) Acceleration:

$$\vec{a}(t)=\frac{d\vec{v}}{dt}=\frac{d}{dt}(2\cos t\hat{i}+3\hat{j})$$

$$\vec{a}(t)=-2\sin t\hat{i}+0\hat{j}$$

At $t=\frac{\pi}{2}$:

$$\vec{a}\left(\frac{\pi}{2}\right)=-2\sin\frac{\pi}{2}\hat{i}=-2(1)\hat{i}=-2\hat{i}$$

(c) Speed:

$$\text{Speed}=|\vec{v}(t)|=\sqrt{0^2+3^2}=3$$

At $t=\frac{\pi}{2}$:

$$\left|\vec{v}\left(\frac{\pi}{2}\right)\right|=3\text{ units/sec}$$

(d) Direction of motion:

$$\hat{v}(t)=\frac{\vec{v}(t)}{|\vec{v}(t)|}=\frac{2\cos t\hat{i}+3\hat{j}}{3}$$

At $t=\frac{\pi}{2}$:

$$\hat{v}\left(\frac{\pi}{2}\right)=\frac{\vec{v}\left(\frac{\pi}{2}\right)}{\left|\vec{v}\left(\frac{\pi}{2}\right)\right|}=\frac{0\hat{i}+3\hat{j}}{3}=\frac{3\hat{j}}{3}=\hat{j}$$

Interpretation: At $t=\frac{\pi}{2}$, the particle moves vertically upward with a speed of 3 units per second. The acceleration in the negative *x*-direction indicates a force pulling the particle leftward, possibly due to sinusoidal motion in the *x*-components.

Q2. $\vec{r}(t)=e^t\hat{i}+e^{-t}\hat{j}$ **at** $t=0$

Solution:

$$\vec{r}(t)=e^t\hat{i}+e^{-t}\hat{j}$$

(a) Velocity:

$$\vec{v}(t)=\frac{d\vec{r}}{dt}=\frac{d}{dt}(e^t\hat{i}+e^{-t}\hat{j})$$

$$\vec{v}(t)=e^t\hat{i}+e^{-t}(-1)\hat{j}=e^t\hat{i}-e^{-t}\hat{j}$$

At $t=0$:

$$\vec{v}(0)=e^0\hat{i}+e^{-0}\hat{j}=\hat{i}-\hat{j}$$

(b) Acceleration:

$$\vec{a}(t)=\frac{d\vec{v}}{dt}=\frac{d}{dt}(e^t\hat{i}+e^{-t}\hat{j})$$

$$\vec{a}(t)=e^t\hat{i}-e^{-t}(-1)\hat{j}=e^t\hat{i}+e^{-t}\hat{j}$$

At $t=0$:

$$\vec{a}(0)=e^0\hat{i}+e^{-0}\hat{j}=\hat{i}+\hat{j}$$

(c) Speed:

$$\text{Speed}=|\vec{v}(t)|=\sqrt{(e^t)^2+(-e^{-t})^2}=\sqrt{e^{2t}+e^{-2t}}$$

At $t=0$:

$$|\vec{v}(0)|=\sqrt{e^0+e^0}=\sqrt{1+1}=\sqrt{2}\text{ units/sec}$$

(d) Direction of Motion:

$$\hat{v}(t)=\frac{\vec{v}(t)}{|\vec{v}(t)|}=\frac{e^t\hat{i}-e^{-t}\hat{j}}{\sqrt{e^{2t}+e^{-2t}}}$$

At $t=0$:

$$\hat{v}(0)=\frac{\vec{v}(0)}{|\vec{v}(0)|}=\frac{\hat{i}-\hat{j}}{\sqrt{2}}=\frac{1}{2}\hat{i}-\frac{1}{\sqrt{2}}\hat{j}$$

Interpretation: The particle moves at an angle of 45° downward and to the right with a speed of $\sqrt{2}$ units per second. The acceleration suggests increasing velocity in both direction, consistent with the exponential growth and decay of the components.

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Q3. $\vec{r}(t) = \cos t\hat{i} + \sin t\hat{j} + 2t\hat{k}$ at $t = \pi$

Solution:

$$\vec{r}(t) = \cos t\hat{i} + \sin t\hat{j} + 2t\hat{k}$$

(a) Velocity:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\cos t\hat{i} + \sin t\hat{j} + 2t\hat{k})$$

$$\vec{v}(t) = -\sin t\hat{i} + \cos t\hat{j} + 2\hat{k}$$

At $t = \pi$:

$$\vec{v}(\pi) = -\sin\pi\hat{i} + \cos\pi\hat{j} + 2\hat{k}$$

$$\vec{v}(\pi) = 0\hat{i} - 1\hat{j} + 2\hat{k}$$

(b) Acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}(-\sin t\hat{i} + \cos t\hat{j} + 2\hat{k})$$

$$\vec{a}(t) = -\cot t\hat{i} - \sin t\hat{j} + 0\hat{k}$$

At $t = \pi$:

$$\vec{a}(\pi) = -\cot\pi\hat{i} - \sin\pi\hat{j} + 0\hat{k}$$

$$= -(-1)\hat{i} - 0\hat{j} + 0\hat{k}$$

$$\vec{a}(\pi) = \hat{i} + 0\hat{j} + 0\hat{k}$$

(c) Speed:

$$\text{Speed} = |\vec{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 2^2}$$

$$|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{1+4} = \sqrt{5}$$

At $t = \pi$:

$$|\vec{v}(\pi)| = \sqrt{5} \text{ units/sec}$$

(d) Direction of motion:

$$\hat{v}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{-\sin t\hat{i} + \cos t\hat{j} + 2\hat{k}}{\sqrt{5}}$$

At $t = \pi$:

$$\hat{v}(\pi) = \frac{\vec{v}(\pi)}{|\vec{v}(\pi)|} = \frac{0\hat{i} - 1\hat{j} + 2\hat{k}}{\sqrt{5}}$$

$$= 0\hat{i} - \frac{1}{\sqrt{5}}\hat{j} + \frac{2}{\sqrt{5}}\hat{k}$$

Interpretation: The particle follows a helical path, moving downward and upward with a speed of $\sqrt{5}$ units per second. The acceleration in the positive x-direction reflects the circular motion in the xy-plane.

Q4. $\vec{r}(t) = t^2\hat{i} + t\hat{j} + t^3\hat{k}$ at $t = 2$

Solution:

$$\vec{r}(t) = t^2\hat{i} + t\hat{j} + t^3\hat{k}$$

(a) Velocity:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(t^2\hat{i} + t\hat{j} + t^3\hat{k})$$

$$\vec{v}(t) = 2t\hat{i} + 1\hat{j} + 3t^2\hat{k}$$

At $t = 2$:

$$\vec{v}(2) = 2(2)\hat{i} + 1\hat{j} + 3(2)^2\hat{k}$$

$$\vec{v}(2) = 4\hat{i} + 1\hat{j} + 12\hat{k}$$

(b) Acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}(2t\hat{i} + 1\hat{j} + 3t^2\hat{k})$$

$$\vec{a}(t) = 2\hat{i} + 0\hat{j} + 6t\hat{k}$$

At $t = 2$:

$$\vec{a}(2) = 2\hat{i} + 0\hat{j} + 6(2)\hat{k}$$

$$\vec{a}(2) = 2\hat{i} + 0\hat{j} + 12\hat{k}$$

(c) Speed:

$$\text{Speed} = |\vec{v}(t)| = \sqrt{(2t)^2 + 1^2 + (3t^2)^2}$$

$$|\vec{v}(t)| = \sqrt{4t^2 + 1 + 9t^4}$$

At $t = 2$:

$$|\vec{v}(2)| = \sqrt{4(2)^2 + 1 + 9(2)^4} = \sqrt{16 + 1 + 144}$$

$$= \sqrt{161} \text{ units/sec}$$

(d) Direction of Motion:

$$\hat{v}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{2t\hat{i} + 1\hat{j} + 3t^2\hat{k}}{\sqrt{161}}$$

At $t = 2$:

$$\hat{v}(2) = \frac{\vec{v}(2)}{|\vec{v}(2)|} = \frac{4\hat{i} + 1\hat{j} + 6\hat{k}}{\sqrt{161}}$$

Interpretation: At $t = 2$, in the z-direction a speed of $\sqrt{161}$ units per second. The acceleration indicates increasing velocity in the x- and z-direction, consistent with the polynomial nature of the path.

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Q5. $\vec{r}(t) = e^t \sin t\hat{i} + (t^3 - 2t)\hat{j} + \arctan(2t)\hat{k}$ **at** $t = 0$

Solution:

$$\vec{r}(t) = e^t \sin t\hat{i} + (t^3 - 2t)\hat{j} + \tan(2t)\hat{k}$$

(a) Velocity:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}[e^t \sin t\hat{i} + (t^3 - 2t)\hat{j} + \tan^{-1}(2t)\hat{k}]$$

$$= \frac{d}{dt}(e^t \sin t)\hat{i} + \frac{d}{dt}(t^3 - 2t)\hat{j} + \frac{d}{dt}(\tan^{-1}(2t))\hat{k}$$

$$= (e^t \cos t + \sin t e^t)\hat{i} + (3t^2 - 2)\hat{j} + \frac{1}{1+(2t)^2}(2)\hat{k}$$

$$\vec{v}(t) = e^t(\cos t + \sin t)\hat{i} + (3t^2 - 2)\hat{j} + \frac{2}{1+4t^2}\hat{k}$$

At $t = 0$:

$$\vec{v}(0) = e^0(\cos 0 + \sin 0)\hat{i} + (3(0)^2 - 2)\hat{j} + \frac{2}{1+4(0)^2}\hat{k}$$

$$= 1(1+0)\hat{i} + (0-2)\hat{j} + \frac{2}{1}\hat{k} = 1\hat{i} - 2\hat{j} + 2\hat{k}$$

(b) Acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}[e^t(\cos t + \sin t)\hat{i} + (3t^2 - 2)\hat{j} + \frac{2}{1+4t^2}\hat{k}]$$

$$\vec{a}(t) = \frac{d}{dt}e^t(\cos t + \sin t)\hat{i} + \frac{d}{dt}(3t^2 - 2)\hat{j} + \frac{d}{dt}2(1+4t^2)^{-1}\hat{k}$$

$$\vec{a}(t) = [e^t(-\sin t + \cos t) + (\cos t + \sin t)e^t]\hat{i} + (6t - 0)\hat{j} + 2(-1)(1+4t^2)^{-2}(8t)\hat{k}$$

$$\vec{a}(t) = e^t(-\sin t + \cos t + \cos t + \sin t)\hat{i} + 6\hat{j} - \frac{16t}{(1+4t^2)^2}\hat{k}$$

$$\vec{a}(t) = 2e^t \cos t\hat{i} + 6\hat{j} - \frac{16t}{(1+4t^2)^2}\hat{k}$$

At $t = 0$:

$$\vec{a}(0) = 2e^0 \cos 0\hat{i} + 6(0)\hat{j} - \frac{16(0)}{(1+4(0)^2)^2}\hat{k}$$

$$\vec{a}(0) = 2(1)(1)\hat{i} + 0\hat{j} - 0\hat{k}$$

$$\vec{a}(0) = 2\hat{i} + 0\hat{j} - 0\hat{k}$$

(c) Speed:

$$\text{Speed} = |\vec{v}(t)| = \sqrt{e^{2t}(\cos t + \sin t)^2 + (3t^2 - 2)^2 + \frac{4}{(1+4t^2)^2}}$$

At $t = 0$:

$$|\vec{v}(0)| = \sqrt{e^0(\cos 0 + \sin 0)^2 + (3(0)^2 - 2)^2 + \frac{4}{(1+4(0)^2)^2}}$$

$$|\vec{v}(0)| = \sqrt{1 + (1+0)^2 + (0-2)^2 + \frac{4}{(1+0)^2}} = \sqrt{1+4+4}$$

$$|\vec{v}(0)| = \sqrt{9} = 3 \text{ unit/sec}$$

(d) Direction of motion:

$$\hat{v}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{e^t(\cos t + \sin t)\hat{i} + (3t^2 - 2)\hat{j} + \frac{2}{1+4t^2}\hat{k}}{3}$$

At $t = 0$:

$$\hat{v}(0) = \frac{\vec{v}(0)}{|\vec{v}(0)|} = \frac{1\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

Interpretation: The particle moves at a speed 3 units per seconds. The acceleration in the positive y-direction suggests a force due to oscillatory motion in the y-and z-direction.

Q6. A cannonball is fired from the ground at an angle 30° with an initial velocity of 100 m/s. Assuming gravity acts downward ($g = 9.8$ m/s^2) and neglecting air resistance.

(a) Formulate the position vector $\vec{r}(t)$ of the cannonball as a function of time.

(b) Find the velocity vector at $t = 5$sec.

(c) Determine the time of the flight (when the cannonball hits the ground).

Solution:

(a) Position vector for projectile motion is

$$\vec{r}(t) = (v_0 \cos\theta)t\hat{i} + \left(h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right)\hat{j}$$

Since the cannonball is tired from the ground, so

$h = 0$, $v_0 = 100$, $\theta = 30°$ and $g = 9.8$

$$\Rightarrow \vec{r}(t) = (100\cos 30°)t\hat{i} + \left(0 + (100\sin 30°)t - \frac{1}{2}(9.8)t^2\right)\hat{j}$$

$$\vec{r}(t) = 100\left(\frac{\sqrt{3}}{2}\right)t\hat{i} + \left(\left(100.\frac{1}{2}\right)t - 4.9t^2\right)\hat{j}$$

$$\vec{r}(t) = 50\sqrt{3}t\hat{i} + (50t - 4.9t^2)\hat{j} \quad \ldots(1)$$

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(b) As we know

$$\vec{v}(t) = \frac{d}{dt}(\vec{r})$$

$$= \frac{d}{dt}[50\sqrt{3}t\hat{i} + (50t - 4.9t^2)\hat{j}]$$

$$= \frac{d}{dt}(50\sqrt{3}t)\hat{i} + \frac{d}{dt}(50 - 4.9t^2)\hat{j}$$

$$= 50\sqrt{3}(1)\hat{i} + (50 - 4.9(2)t)\hat{j}$$

$$\vec{v}(t) = 50\sqrt{3}\hat{i} + (50 - 9.8t)\hat{j}$$

At $t = 5$ sec:

$$\vec{v}(5) = 50\sqrt{3}\hat{i} + (50 - 9.8(5))\hat{j}$$

$$= 50\sqrt{3}\hat{i} + (50 - 49)\hat{j}$$

$$= 50\sqrt{3}\hat{i} + \hat{j}$$

$$\vec{v}(5) \approx 86.6\hat{i} + \hat{j}$$

(c) If the cannonball hits the ground, then

$$y(t) = 0$$

$$\Rightarrow \quad 50t - 4.9t^2 = 0 \quad \text{Using eq. (1)}$$

$$t(50 - 4.9t) = 0$$

$$\Rightarrow \quad t = 0 \quad \text{(Time of launch at ground level)}$$

or $\quad 50 - 4.9t = 0$

$$4.9t = 50$$

$$t = \frac{50}{4.9}$$

$$t \approx 10.29 \text{ seconds (Flight time)}$$

Q7. A comet orbits the Sun in an elliptical path such that its position (in AU) at time t (in years) is described by, $x = 2\cos t, y = \sin t$.

(a) Write the position vector $\vec{r}(t)$

(b) Find the velocity and speed at $t = \frac{\pi}{2}$

(c) Show that the acceleration vector always points toward the origin.

Solution:

(a) Given: $x(t) = 2\cos t, y(t) = \sin t$

Position vector: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$$\vec{r}(t) = 2\cos t\hat{i} + \sin t\hat{j} \qquad \ldots(1)$$

(b) Velocity:

As we know

$$\vec{v}(t) = \frac{d}{dt}(\vec{r})$$

$$= \frac{d}{dt}(2\cos t\hat{i} + \sin t\hat{j})$$

$$\vec{v}(t) = -2\sin t\hat{i} + \cos t\hat{j}$$

At $t = \frac{\pi}{2}$:

$$\vec{v}\left(\frac{\pi}{2}\right) = -2\sin\frac{\pi}{2}\hat{i} + \cos\frac{\pi}{2}\hat{j}$$

$$\vec{v}\left(\frac{\pi}{2}\right) = -2(1)\hat{i} + (0)\hat{j} = -2\hat{i} + 0\hat{j}$$

Speed:

$$\text{Speed} = |\vec{v}(t)| = \sqrt{(-2\sin t)^2 + (\cos t)^2}$$

At $t = \frac{\pi}{2}$:

$$\left|\vec{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{\left(-2\sin\frac{\pi}{2}\right)^2 + \left(\cos\frac{\pi}{2}\right)^2}$$

$$= \sqrt{(-2(1))^2 + (0)^2} = \sqrt{4}$$

$$\left|\vec{v}\left(\frac{\pi}{2}\right)\right| = 2 \text{ AU/year}$$

(c) Acceleration:

As we know

$$\vec{a}(t) = \frac{d}{dt}(\vec{v})$$

$$= \frac{d}{dt}(-2\sin t\hat{i} + \cos t\hat{j})$$

$$= -2\cos t\hat{i} - \sin t\hat{j} = -(2\cos t\hat{i} + \sin t\hat{j})$$

$$\Rightarrow \quad \vec{a}(t) = -\vec{r}(t) \qquad \text{From eq. (1)}$$

which shows that acceleration vector always points towards origin.

Q8. A robotic arm moves such that: Its x-position increases linearly at 2 m/s, Its y position follows $y(t) = t^2$, Its z-position is given by $z(t) = 1 - e^{-t}$.

(a) Construct the position vector $\vec{r}(t)$

(b) Find the velocity vector at $t = 1$sec

(c) Compute the acceleration at $t = 1$sec

Solution:

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(a) As x-position increases linearly at $2m/s$, so

$$x(t) = 2t \qquad \because s = vt$$
$$y(t) = t^2,\ z(t) = 1 - e^{-t}$$

Position vector: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\vec{r}(t) = 2t\hat{i} + t^2\hat{j} + (1 - e^{-t})\hat{k}$$

(b) Velocity:

As we know

$$\vec{v}(t) = \frac{d}{dt}(\vec{r})$$
$$= \frac{d}{dt}(2t\hat{i} + t^2\hat{j} + (1 - e^{-t})\hat{k}$$
$$= 2(1)\hat{i} + 2t\hat{j} + (0 - e^{-t}(-1))\hat{k}$$
$$\vec{v}(t) = 2\hat{i} + 2t\hat{j} + e^{-t}\hat{k}$$

At $t = 1$ sec:

$$\vec{v}(1) = 2\hat{i} + 2(1)\hat{j} + e^{-1}\hat{k}$$
$$\vec{v}(1) = 2\hat{i} + 2\hat{j} + \frac{1}{e}\hat{k}$$

(c) Acceleration:

As we know

$$\vec{a}(t) = \frac{d}{dt}(\vec{v})$$
$$= \frac{d}{dt}(2\hat{i} + 2t\hat{j} + e^{-t}\hat{k})$$
$$= 0\hat{i} + 2(1)\hat{j} + e^{-t}(-1)\hat{k}$$
$$\vec{a}(t) = 0\hat{i} + 2\hat{j} - e^{-t}\hat{k}$$

At $t = 1$ sec:

$$\vec{a}(1) = 0\hat{i} + 2\hat{j} - e^{-t}\hat{k} = 2\hat{j} - \frac{1}{e}\hat{k}$$

Q9. A particle moves along a circular path, described by the position vector $\vec{r}(t) = R\cos\omega t\hat{i} + R\sin\omega t\hat{j}$
Prove that its velocity is perpendicular to its position vector.

Solution:

Position vector: $\vec{r}(t) = R\cos\omega t\hat{i} + R\sin\omega t\hat{j}$

Velocity: $\vec{v}(t) = \frac{d\vec{r}}{dt}$

$$= \frac{d}{dt}(R\cos\omega t\hat{i} + R\sin\omega t\hat{j})$$
$$= R(-\sin\omega t)\omega\hat{i} + R(\cos\omega t)\omega\hat{j}$$
$$\vec{v}(t) = (-R\omega\sin\omega t)\hat{i} + (R\omega\cos\omega t)\hat{j}$$

Consider

$$\vec{v}(t)\cdot\vec{r}(t) = [(-R\omega\sin\omega t)\hat{i} + (R\omega\cos\omega t)\hat{j}].[(R\cos\omega t)\hat{i} + (R\sin\omega t)\hat{j}]$$
$$= (-R\omega\sin\omega t)(R\cos\omega t) + (R\omega\cos\omega t)(R\sin\omega t)$$
$$= -R\omega\sin\omega t.\cos\omega t + R^2\omega\sin\omega t\cos\omega t$$
$$\Rightarrow \quad \vec{v}(t)\cdot\vec{r}(t) = 0$$
$$\Rightarrow \quad \vec{v}(t) \perp \vec{r}(t) \quad \text{(Proved)}$$

Q10. A Ferris wheel of radius 20 m rotates with an angular speed of 0.2 rad/sec.

(a) Construct its position vector $\vec{r}(t)$ **(b) Find the passenger's velocity at $t = 10$sec**

(c) Show that the acceleration is centripetal (toward the center)

Solution:

(a) Position vector for circular motion in a $2D$ plane is:

$$\vec{r}(t) = (R\cos\omega t)\hat{i} + (R\sin\omega t)\hat{j}$$

Put $R = 20$, $\omega = 0.2$, we get

$$\vec{r}(t) = (20\cos(0.2)t)\hat{i} + (20\sin(0.2)t)\hat{j}$$
$$= 20\cos(0.2t)\hat{i} + 20\sin(0.2t)\hat{j} \quad \ldots(1)$$

(b) Velocity:

$$\vec{v}(t) = \frac{d}{dt}(\vec{r})$$
$$= \frac{d}{dt}\left(20\cos(0.2t)\hat{i} + (20\sin(0.2t)\hat{j}\right)$$
$$= -20\sin(0.2t)(0.2)\hat{i} + 20\cos(0.2t)(0.2)\hat{j}$$
$$\vec{v}(t) = -4\sin(0.2t)\hat{i} + 4\cos(0.2t)\hat{j}$$

At $t = 10$ sec:

$$\vec{v}(10) = -4\sin(0.2\times10)\hat{i} + 4\cos(0.2\times10)\hat{j}$$
$$\vec{v}(10) = -4\sin2\hat{i} + 4\cos2\hat{j}$$

(c) Acceleration:

$$\vec{a}(t) = \frac{d}{dx}(\vec{v})$$

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$$= \frac{d}{dx}\left[-4\sin(0.2t)\hat{i} + 4\cos(0.2t)\hat{j}\right]$$

$$= -4\cos(0.2t)(0.2)\hat{i} - 4\sin(0.2t)(0.2)\hat{j}$$

$$= -0.08\cos(0.2t)\hat{i} - 0.8\sin(0.2t)\hat{j}$$

$$= -0.04[20\cos(0.2t)\hat{i} + 20\sin(0.2t)\hat{j}]$$

$\vec{a}(t) = (-0.04)\vec{r}(t)$ From eq. (1)

which means acceleration points towards origin.

Q11. A quarterback throws a football with a position vector $\vec{r}(t) = 20t\hat{i} + (20t - 4.9t^2)\hat{j}$

(a) Find the time of flight before it hits the ground.

(b) At which time did football reach its maximum height?

(c) What is the maximum height reached?

Solution:

$$\vec{r}(t) = 20t\hat{i} + (20t - 4.9t^2)\hat{j}$$

(a) If the football hits the ground, then

$$y(t) = 0$$

$$20t - 4.9t^2 = 0$$

$$t(20 - 4.9t) = 0$$

$\Rightarrow$ $t = 0$ (Time to kicked the ball from ground)

or $20 - 4.9t = 0$

$$4.9t = 20 \Rightarrow t = \frac{20}{4.9}$$

$t \approx 4.08$ seconds (Flight time)

(b) At maximum height:

$$v_y = 0$$

$$\Rightarrow \frac{d}{dt}(y(t)) = 0$$

$$\frac{d}{dt}(20t - 4.9t^2) = 0$$

$$20 - 4.9(2t) = 0 \Rightarrow 9.8t = 20$$

$$t = \frac{20}{9.8} \approx 2.04 \text{ seconds}$$

(c) Max. height $= y(2.04)$

$$= 20(2.04) - 4.9(2.04)^2$$

$$= 40.8 - 4.9(4.1616)$$

$$= 4.8 - 20.39$$

Max. height ≈ 20.41 meters

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Multiple Choice Questions (MCQs)

1. **A scalar function assigns each input value a single:**
   A) vector B) real number C) matrix D) ordered pair

2. **A vector-valued function of a single variable takes one variable as input and returns a _____ as output.**
   A) scalar B) vector C) matrix D) constant

3. **A typical three-dimensional vector-valued function is written as:**
   A) $\vec{r}(t)=f(t)+g(t)+h(t)$ B) $\vec{r}(t)=f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$
   C) $\vec{r}(t)=f(t)g(t)h(t)$ D) $\vec{r}(t)=f(t)\hat{j}+g(t)\hat{k}+h(t)\hat{i}$

4. **In $\vec{r}(t)=f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along the:**
   A) x-, y- and z-axes B) time, speed and acceleration axes
   C) positive y-axis only D) coordinate planes

5. **The component functions $f(t)$, $g(t)$, and $h(t)$ of $\vec{r}(t)=f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$ are:**
   A) vectors B) scalar functions
   C) matrices D) constants only

6. **The vector-valued function $\vec{r}(t)=\langle f(t), g(t), h(t)\rangle$ represents a point in:**
   A) one-dimensional space only B) two-dimensional space only
   C) three-dimensional space D) no coordinate space

7. **In the projectile model, gravity affects the:**
   A) x-component only B) y-component only
   C) x- and y-components equally D) z-component only

8. **In the drone model $\vec{r}(t)=5\hat{i}+5\hat{j}+0.5\hat{k}$ the horizontal circular path has radius:**
   A) 0.5 m B) 2.5 m C) 5 m D) 10 m

9. **For $\sqrt{4-t^2}$, the corresponding interval for t is:**
   A) $[-2, 2]$ B) $(-2, 2)$ C) $(-\infty, \infty)$ D) $[0, 2]$

10. **For $\frac{1}{t-1}$, the restriction is:**
    A) $t=1$ B) $t\neq 1$ C) $t>1$ only D) $t<1$ only

11. **The domain of $\vec{r}(t)=\sqrt{4-t^2}\,\hat{i}+\frac{1}{t-1}\hat{j}$ is:**
    A) $[-2, 2]$ B) $[-2, 1)\cup(1, 2]$ C) $(-2, 1)\cup(1, 2)$ D) $(-\infty, -2]\cup[2, \infty)$

12. **For $\vec{r}(t)=\langle \cos t, \sin t, t\rangle$ at $t=\frac{\pi}{2}$, the output is:**
    A) $\left\langle 1, 0, \frac{\pi}{2}\right\rangle$ B) $\left\langle 0, 1, \frac{\pi}{2}\right\rangle$ C) $\langle 0, -1, \pi\rangle$ D) $\langle 1, 1, \pi\rangle$

13. **The derivative of $\vec{r}(t)=f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}$ is:**
    A) $\langle f_1+f_2+f_3\rangle$ B) $\langle f_1'(t), f_2'(t), f_3'(t)\rangle$ C) $\langle f_1, f_2, f_3\rangle$ D) $\langle f_1^2, f_2^2, f_3^2\rangle$

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14. For $\vec{r}(t)=t^2\hat{i}+\sin t\hat{j}+e^t\hat{k}$ is:

A) $2t\hat{i}+\cos t\hat{j}+e^t\hat{k}$ B) $t\hat{i}+\sin t\hat{j}+e^t\hat{k}$

C) $2\hat{i}+\cos t\hat{j}+e^t\hat{k}$ D) $2t\hat{i}-\cos t\hat{j}-e^t\hat{k}$

15. **Velocity of a particle with position vector $\vec{r}(t)$ is:**

A) $r(t)$ B) $r'(t)$ C) $r''(t)$ D) $|r(t)|$

16. **Acceleration of a particle with position vector $\vec{r}(t)$ is:**

A) $r(t)$ B) $r'(t)$ C) $r''(t)$ D) $|r'(t)|$

17. **Speed is the:**

A) vector sum of acceleration B) magnitude of the velocity vector

C) magnitude of position only D) derivative of speed

18. **The direction of motion is given by the:**

A) unit tangent vector B) position vector

C) acceleration scalar D) domain

19. **For $\vec{r}(t)=\cos t\hat{i}+\sin t\hat{j}$ the path in the xy-plane is:**

A) a circle of radius 1 B) a parabola

C) a straight line D) a hyperbola

20. **For $\vec{r}(t)=t^2\hat{i}+t\hat{j}+t^3\hat{k}$ at t = 2, the velocity is:**

A) $4\hat{i}+1\hat{j}+12\hat{k}$ B) $2\hat{i}+\hat{j}+8\hat{k}$

C) $4\hat{i}+2\hat{j}+6\hat{k}$ D) $2\hat{i}+2\hat{j}+12\hat{k}$

ANSWER KEY

1.	B	2.	B	3.	B	4.	A	5.	B	6.	C	7.	B	8.	C	9.	A	10.	B
11.	B	12.	B	13.	B	14.	A	15.	B	16.	C	17.	B	18.	A	19.	A	20.	A

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PAIRING SCHEME / PAPER PATTERN

Time Allowed: 3 Hours **Total Marks: 100**

The Mathematics paper for Grade 12 (2nd Year) will be worth 100 marks and will have a duration of 3 hours.

Objective (MCQs)	**Q1.** There will be 20 Multiple Choice Questions (MCQs) from the entire textbook. The distribution is as follows: One MCQ each from chapters 4, 8, 9 and 11; two MCQs each from chapters 1, 5, 6, 7 and 10; and three MCQs each from chapters 2 and 3.	**1 × 20 = 20**
Part-I: Subjective (This section contains three questions)	This section contains three questions. The details are as follows: **Q2.** Students are required to attempt any 8 out of 12 short questions. The details are as follows: (i) Four short questions will be selected from unit 1 and unit 7, two from each unit. (ii) Five short questions will be selected from unit 2. (iii) Three short questions will be selected from unit 4.	**2 × 8 = 16**
	Q3. Students are required to attempt any 8 out of 12 short questions. The details are as follows: (i) Five short questions will be selected from unit 3. (ii) Three short questions will be selected from unit 5. (iii) Four short questions will be selected from unit 6.	**2 × 8 = 16**
	Q4. Students are required to attempt any 9 out of 13 short questions. The details are as follows: (i) Nine short questions will be selected from units 8, 9 and 10, three from each unit. (ii) Four short questions will be selected from unit 11.	**2 × 9 = 18**
Part-II: Subjective	This section contains five long questions, each bifurcated into two parts **a** & **b** (carrying 5 marks each), and students have to attempt any three questions. The details are as follows: **Q5.** (a) One long question will be selected from chapter 1. (b) One long question will be selected from chapter 2. **Q6.** (a) One long question will be selected from chapter 2. (b) One long question will be selected from chapter 3. **Q7.** (a) One long question will be selected from chapter 4. (b) One long question will be selected from chapter 5. **Q8.** (a) One long question will be selected from chapter 6. (b) One long question will be selected from chapter 7. **Q9.** (a) One long question will be selected from chapter 11. (b) One long question will be selected from chapter 10 or chapter 8.	**3 × 10 = 30**

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xiv. **On a displacement-time graph, the slope represents:**
A) acceleration B) velocity C) distance D) time

xv. **On a velocity-time graph, the area under the graph represents:**
A) acceleration B) speed C) displacement D) force

xvi. **The bisection method can be applied on $[a, b]$ only if:**
A) $f(a)\cdot f(b)>0$ B) $f(a)\cdot f(b)<0$ C) $f(a)=f(b)$ D) $f(a)=0$ only

xvii. **The domain of $\sin^{-1}x$ is:**
A) R B) $[-1, 1]$ C) $(0, \infty)$ D) $[0, \pi]$

xviii. **The period of $\sin x$ is:**
A) π B) $\frac{\pi}{2}$ C) 2π D) 4π

xix. **The general solution of $\cos x = 1$ $(n \in Z)$ is:**
A) $x = 2n\pi$ B) $x = n\pi$ C) $x = n\pi + \frac{\pi}{2}$ D) $x=(2n+1)\pi$

xx. **If $\vec{r}(t)=t\hat{i}+t^2\hat{j}$, then $r'(t)=$**
A) $\hat{i}+2t\hat{j}$ B) $t\hat{i}+2t\hat{j}$ C) $\hat{i}+t^2\hat{j}$ D) $2t\hat{j}$

Subjective Type

(Section-I)

Time Allowed: 2:30 hours **Max. Marks: 80**

Q2. Write short answers to any EIGHT (8) questions: $2\times 8 = 16$

(i) Using the laws of logarithms, evaluate: $\log_{10}100+\log_{10}1000$.

(ii) Prove the hyperbolic identity: $\cosh^2 x-\sinh^2 x=1$.

(iii) A body covers 150 km in 3 hours in a straight line. Find its average velocity in km/h.

(iv) A car's velocity increases uniformly from 10 m/s to 30 m/s in 4 s. Find its acceleration.

(v) If $y=x^4$, find the second derivative $\frac{d^2y}{dx^2}$.

(vi) Find $\frac{dy}{dx}$ by implicit differentiation if $x^2+y^2=25$.

(vii) Using the chain rule, differentiate $y=(2x+1)^5$.

(viii) Find the interval in which $f(x)=x^2-4x$ is increasing.

(ix) Find the linearization of $f(x)=\sqrt{x}$ at $x=4$.

(x) Solve the differential equation $\frac{dy}{dx}=3x^2$.

(xi) Verify that $y=e^x$ is a solution of $\frac{dy}{dx}=y$.

(xii) Form the differential equation whose general solution is $y=cx$.

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Q3. Write short answers to any EIGHT (8) questions: **2 × 8 = 16**

(i) Evaluate: $\int(3x^2+2x+1)dx$.

(ii) Evaluate by substitution: $\int 2x\cos(x^2)dx$.

(iii) Evaluate by parts: $\int x\,e^x\,dx$.

(iv) Evaluate the definite integral: $\int_0^1 x\,dx$.

(v) Evaluate: $\int \sec^2 x\,dx$.

(vi) Write the line with slope 3 and y-intercept -2 (slope-intercept form).

(vii) Find the equation of the line through $(1, 2)$ and $(3, 6)$.

(viii) Find the x- and y-intercepts of the line $2x + 3y = 6$.

(ix) Find the centre and radius of the circle $x^2+y^2-4x-6y+9=0$.

(x) Find the focus and directrix of the parabola $y^2=8x$.

(xi) Write the semi-major and semi-minor axes of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$.

(xii) Find the length of the tangent from $(5, 0)$ to the circle $x^2+y^2=9$.

Q4. Write short answers to any NINE (9) questions: **2 × 9 = 16**

(i) Show that a root of $f(x)=x^3-x-1$ lies between 1 and 2 (change of sign).

(ii) Write the Newton-Raphson iteration formula.

(iii) For $f(x)=x^2-2$ on $[1, 2]$, do one bisection step and give the first approximation.

(iv) Find the principal value of $\sin^{-1}\left(\frac{1}{2}\right)$.

(v) Prove that $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$.

(vi) Evaluate $\cos^{-1}(-1)$.

(vii) Solve $2\cos x=-1$ and write its general solution.

(viii) Solve $3\tan^2 x=1$.

(ix) Solve $2\cos x\sin x-\sin x=0$.

(x) Find the domain of $\vec{r}(t)=\sqrt{t}\,\hat{i}+\left[\frac{1}{t-1}\right]\hat{j}$.

(xi) If $\vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}$, find $r'(t)$.

(xii) Find the magnitude of the vector $\vec{r}=3\hat{i}+4\hat{j}$.

(xiii) Write the formulas for velocity and acceleration for a position vector $\vec{r}(t)$.

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(Section-II)

Note: Attempt any THREE (03) questions. **3 × 10 = 30**

Q5. **(a)** Sketch the graph of $y=2(x-1)(x-3)$ by the factor method, showing x-intercepts, vertex and y-intercept. (5)

(b) Find the local maximum and minimum values of $f(x)=x^3-3x^2+4$ using the second-derivative test. (5)

Q6. **(a)** Find the equations of the tangent and normal to the curve $y=x^2-3x+2$ at $x=2$. (5)

(b) Find the area bounded by the curve $y=x^2$, the x-axis and the lines $x=0$ and $x=3$. (5)

Q7. **(a)** A population grows so that $\frac{dP}{dt}=kP$. Solve it to show $P=P_0e^{kt}$. (5)

(b) The vertices of a triangle are $A(2,-1), B(4,3), C(6,-1)$. Find the equations of its three medians. (5)

Q8. **(a)** Find the equation of the circle with centre (3, 4) passing through the origin. State its radius. (5)

(b) A cyclist travels 12 km in 30 min, rests 15 min, then 9 km in 45 min. Find the average velocity of each stage and of the whole journey (km/h). (5)

Q9. **(a)** Given $\vec{r}(t)=t^2\hat{i}+(t^3-t)\hat{j}$, find $r'(t)$ and $r''(t)$, and the velocity and acceleration at $t=1$. (5)

(b) Solve $2\cos^2 x-3\cos x+1=0$ and write its general solution. (5)

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