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UNIT 10

Solution of Trigonometric Equations

INTRODUCTION

- Trigonometric equations involve functions like $\sin\theta, \cos\theta$ and $\tan\theta$ are solved to find the angle(s) that satisfy a given condition.
- In this unit, we solve such equations using the periodic, even/odd, and translation properties of trigonometric functions. For instance, equations such as $\sin\theta = k, \cos\theta = k$ or $\tan\theta = k$ or are solved by considering the periodic nature of these functions.
- Next, we also explore graphical methods to solve equations like $\sin\theta = \frac{\theta}{2}, \cos\theta = \theta$ or $\tan\theta = 2\theta$ within specific intervals. The general solution of these equations incorporates the periodic behavior of trigonometric functions to identify all possible values of.
- Finally, we apply the concepts of inverse trigonometric functions to real-life problems in engineering and architecture, such as determining heights, calculating angles of elevation and depression, and designing accurate scale models.

TRIGONOMETRIC EQUATION

Definition: A trigonometric equation is an equation that involves one or more trigonometric functions such as sine, cosine, tangent etc. and their inverses.

For example,

$\sin x = \frac{1}{\sqrt{2}}$, $2\cos^2\theta - 5\cos\theta + 1 = 0$, $2\tan^{-1}x = \frac{\pi}{2}$ and $\sqrt{3}\cot x - \csc x - 1 = 0$ are trigonometric equations.

- An unknown value of the angle that satisfy the given trigonometric equation is called a **solution** of the equation.
- The solutions that lie within the interval $[0, 2\pi]$ are called the **principal solutions.**
- Due to the periodic nature of trigonometric functions, these equations have infinitely many solutions. Once a solution is found within one period, adding or subtracting multiples of the period generates all possible solutions called the general solution.

For example, Consider the equation: $\cos x = \frac{1}{2}$

Since cosine is positive, so x lies in the I and IV quadrant.

$$x = \frac{\pi}{3} \text{ and } x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \text{ where } x \in [0, 2\pi]$$

So, $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ are the solution of the equation $\cos x = \frac{1}{2}$ and are called the **principal Solutions.**

Since 2π is the period of $\cos x$. So, all possible solutions of $\cos x = \frac{1}{2}$ are $\frac{\pi}{3} + 2n\pi$ and $\frac{5\pi}{3} + 2n\pi, n \in Z$.

Hence, **general Solution** is given by $x = \left\{\frac{\pi}{3} + 2n\pi\right\} \cup \left\{\frac{5\pi}{3} + 2n\pi\right\}$.

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Example 1: Solve $\sin x = -\frac{\sqrt{3}}{2}$

Solution:

$$\sin x = -\frac{\sqrt{3}}{2}$$

Reference angle: $x_R = \frac{\pi}{3}$

As sine is –ve in III and IV quadrants, so

III-Quad.	IV-Quad.
$x = \pi + x_R$	$x = 2\pi - x_R$
$= \pi + \frac{\pi}{3} = \frac{4\pi}{3}$	$= 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

As 2π is the period of $\sin x$, so

General values of $x = \frac{4\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in Z$

General solution $= \left\{\frac{4\pi}{3} + 2n\pi\right\} \cup \left\{\frac{5\pi}{3} + 2n\pi\right\}, n \in Z$

Example 2: Solve $2\sec^2 x - \frac{8}{3} = 0$.

Solution:

$$2\sec^2 x - \frac{8}{3} = 0$$
$$2\sec^2 x = \frac{8}{3}$$
$$\sec^2 x = \frac{4}{3}$$
$$\sec x = \pm\frac{2}{\sqrt{3}}$$
$$\cos x = \pm\frac{\sqrt{3}}{2}$$

When $\cos x = \frac{\sqrt{3}}{2}$:

Reference angle: $x_R = \frac{\pi}{6}$

As cosine is +ve in I and IV quadrants, so

I-Quad.	IV-Quad.
$x = x_R$	$x = 2\pi - x_R$
$x = \frac{\pi}{6}$	$x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

As 2π is the period of $\cos x$, so

General values of $x = \frac{\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi, n \in Z$

When $\cos x = \frac{-\sqrt{3}}{2}$:

Reference angle: $x_R = \frac{\pi}{6}$

As cosine is –ve in II and III quadrants, so

II-Quad.	III-Quad.
$x = \pi - x_R$	$x = \pi + x_R$
$= \pi - \frac{\pi}{6}$	$= \pi + \frac{\pi}{6}$
$= \frac{5\pi}{6}$	$= \frac{7\pi}{6}$

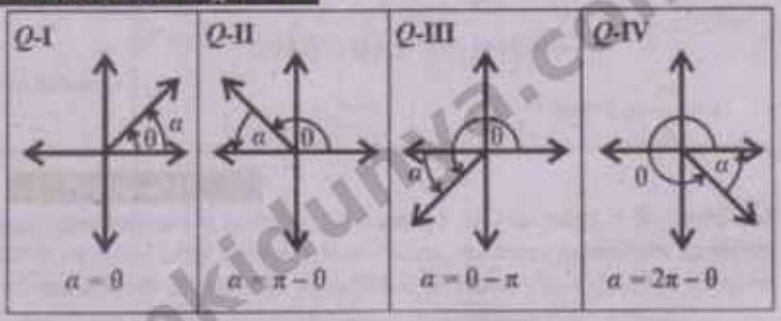
As 2π is the period of $\cos x$, so

General values of $x = \frac{5\pi}{6} + 2n\pi, \frac{7\pi}{6} + 2n\pi, n \in Z$

General solution $= \left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi\right\} \cup \left\{\frac{7\pi}{6} + 2n\pi\right\} \cup \left\{\frac{11\pi}{6} + 2n\pi\right\}, n \in Z$

Reference Angle

Let θ be a non-quadrental angle in standard position. The reference angle for θ is the acute angle α that the terminal side of θ makes with the x-axis. The absolute values of the corresponding trigonometric ratios of α and θ are same. For example, for θ = 120°, we have θ = 60° and for θ = –30°, we have α = 30°. Reference angle is used to find the principal solutions of trigonometric equations.



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Example 3: Solve $\sqrt{3}\cot x - \csc x - 1 = 0$.

Solution:

$$\sqrt{3}\cot x - \csc x - 1 = 0 \quad \ldots(1)$$

$$\sqrt{3}\frac{\cos x}{\sin x} - \frac{1}{\sin x} - 1 = 0$$

Multiply both sides by $\sin x$ $(x \neq n\pi)$

$$\sqrt{3}\cos x - 1 - \sin x = 0$$

$$\sqrt{3}\cos x - 1 = \sin x$$

Squaring both sides

$$(\sqrt{3}\cos x - 1)^2 = \sin^2 x$$

$$3\cos^2 x + 1 - 2\sqrt{3}\cos x = 1 - \cos^2 x$$

$$4\cos^2 x - 2\sqrt{3}\cos x = 0$$

$$2\cos x(2\cos x - \sqrt{3}) = 0$$

Either $2\cos x = 0$ or $2\cos x - \sqrt{3} = 0$

$\cos x = 0$ or $\cos x = \frac{\sqrt{3}}{2}$

When $\cos x = 0$:

$\Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$ (Quadrental angles)

Check:

Put $x = \frac{\pi}{2}$ in eq. (1)

$$\text{L.H.S} = \sqrt{3}\cot\frac{\pi}{2} - \csc\frac{\pi}{2} - 1$$

$$= \sqrt{3}(0) - 1 - 1 = -2 \neq \text{R.H.S}$$

So $x = \frac{\pi}{2}$ is an extraneous root.

Put $x = \frac{3\pi}{2}$ in eq. (1)

$$\text{L.H.S} = \sqrt{3}\cot\frac{3\pi}{2} - \csc\frac{3\pi}{2} - 1$$

$$= \sqrt{3}(0) - (-1) - 1 = 0 = \text{R.H.S}$$

$\Rightarrow \quad x = \frac{3\pi}{2}$ is a root.

As 2π is the period of $\cos x$, so

General values of $x = \frac{3\pi}{2} + 2n\pi, n \in Z$

When $\cos x = \frac{\sqrt{3}}{2}$:

Reference angle: $x_R = \frac{\pi}{6}$

As cosine is +ve in I and IV quadrants, so

I-Quad.	IV-Quad.
$x = x_R$	$x = 2\pi - x_R$
$x = \frac{\pi}{6}$	$x = 2\pi - \frac{\pi}{6}$
	$x = \frac{11\pi}{6}$

Check:

Put $x = \frac{\pi}{6}$ in eq. (1)

$$\text{L.H.S} = \sqrt{3}\cot\frac{\pi}{6} - \csc\frac{\pi}{6} - 1$$

$$= \sqrt{3}\cdot\sqrt{3} - 2 - 1 = 0 = \text{R.H.S}$$

$\Rightarrow \quad x = \frac{\pi}{6}$ is a root

As 2π is the period of $\cos x$, so

General values of $x = \frac{\pi}{6} + 2n\pi, n \in Z$

Put $x = \frac{11\pi}{6}$ in eq. (1)

$$\text{L.H.S} = \sqrt{3}\cot\frac{11\pi}{6} - \csc\frac{11\pi}{6} - 1$$

$$= \sqrt{3}(-\sqrt{3}) - (-2) - 1$$

$$= -3 + 2 - 1 = -2 \neq \text{R.H.S}$$

$\Rightarrow \quad x = \frac{11\pi}{6}$ is an extraneous root.

General solution $= \left\{\frac{3\pi}{2} + 2n\pi\right\} \cup \left\{\frac{\pi}{6} + 2n\pi\right\}, n \in Z$

Remember!

Sometimes it is necessary to square both sides of a trigonometric equation to solve it. However, this operation can introduce extraneous roots i.e., solutions that appear valid but don't actually satisfy the original equation. Therefore, it is essential to verify every potential solution by substituting it back into the original equation before accepting it as valid.

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EXERCISE 10.1

Solve the following trigonometric equation:

Q1. $\sin x = \frac{1}{\sqrt{2}}$

Solution:

$$\sin x = \frac{1}{\sqrt{2}}$$

Reference angle: $x_R = \frac{\pi}{4}$

As sine is +ve in I and II quadrants, so

I-Quad	II-Quad
$x = x_R$	$x = \pi - x_R$
$x = \frac{\pi}{4}$	$x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

As 2π is the period of $\sin x$, so

General values of $x = \frac{\pi}{4} + 2n\pi, \frac{3\pi}{4} + 2n\pi, n \in Z$

General solution $= \left\{\frac{\pi}{4} + 2n\pi\right\} \cup \left\{\frac{3\pi}{4} + 2n\pi\right\}, n \in Z$

Q2. $2\cos x = -1$

Solution:

$$2\cos x = -1 \Rightarrow \cos x = \frac{-1}{2}$$

Reference angle: $x_R = \frac{\pi}{3}$

As cosine is –ve in II and III quadrants, so

II-Quad	III-Quad
$x = \pi - x_R$	$x = \pi + x_R$
$x = \pi - \frac{\pi}{3}$	$x = \pi + \frac{\pi}{3}$
$x = \frac{2\pi}{3}$	$x = \frac{4\pi}{3}$

As 2π is the period of $\cos x$, so

General values of $x = \frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi, n \in Z$

General solution $= \left\{\frac{2\pi}{3} + 2n\pi\right\} \cup \left\{\frac{4\pi}{3} + 2n\pi\right\}, n \in Z$

Q3. $3\tan^2 x = 1$

Solution:

$$3\tan^2 x = 1 \Rightarrow \tan^2 x = \frac{1}{3} \Rightarrow \tan x = \pm\frac{1}{\sqrt{3}}$$

When $\tan x = \frac{1}{\sqrt{3}}$:

Reference angle: $x_R = \frac{\pi}{6}$

As $\tan x$ is +ve in I quadrants, so

$$x = x_R = \frac{\pi}{6} \text{ where } x \in [0, \pi]$$

As period of $\tan x$ is π, so

General values of $x = \frac{\pi}{6} + n\pi, n \in Z$

When $\tan x = \frac{-1}{\sqrt{3}}$:

Reference angle: $x_R = \frac{\pi}{6}$

As $\tan x$ is –ve in II quadrant, so

$$x = \pi - x_R = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \in [0, \pi]$$

As period of $\tan x$ is π, so

General values of $x = \frac{5\pi}{6} + n\pi, n \in Z$

General solution $= \left\{\frac{\pi}{6} + n\pi\right\} \cup \left\{\frac{5\pi}{6} + n\pi\right\}, n \in Z$

Q4. $9 - 3\csc^2 x = 5$

Solution:

$$9 - 3\csc^2 x = 5$$

$$9 - 5 = 3\csc^2 x \Rightarrow \csc^2 x = \frac{4}{3}$$

$$\csc x = \pm\frac{2}{\sqrt{3}} \Rightarrow \sin x = \pm\frac{\sqrt{3}}{2}$$

When $\sin x = \frac{\sqrt{3}}{2}$:

Reference angle: $x_R = \frac{\pi}{3}$

As sine is +ve in I and II quadrants, so

I-Quad.	II-Quad.
$x = x_R$	$x = \pi - x_R$
$x = \frac{\pi}{3}$	$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

When $\sin x = \frac{-\sqrt{3}}{2}$:

Reference angle: $x_R = \frac{\pi}{3}$

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As sine is –ve in III and IV quadrants, so

III-Quad.	IV-Quad.
$x = \pi + x_R$	$x = 2\pi - x_R$
$= \pi + \frac{\pi}{3} = \frac{4\pi}{3}$	$= 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

As 2π is the period of $\sin x$, so

General values of $x = \frac{\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in Z$

General solution $= \left\{\frac{\pi}{3} + 2n\pi\right\} \cup \left\{\frac{2\pi}{3} + 2n\pi\right\} \cup \left\{\frac{4\pi}{3} + 2n\pi\right\} \cup \left\{\frac{5\pi}{3} + 2n\pi\right\}, n \in Z$

Q5. $2\cos^2 x - 3\cos x + 1 = 0$

Solution:

$$2\cos^2 x - 3\cos x + 1 = 0$$
$$2\cos^2 x - 2\cos x - \cos x + 1 = 0$$
$$2\cos x(\cos x - 1) - 1(\cos x - 1) = 0$$
$$(\cos x - 1)(2\cos x - 1) = 0$$

Either $\cos x - 1 = 0$ or $2\cos x - 1 = 0$

$\cos x = 1$ or $\cos x = \frac{1}{2}$

When $\cos x = 1$:

$\Rightarrow \quad x = 0$ (Quadrental angle)

As 2π is the period of $\cos x$, so

General value of $x = 0 + 2n\pi = 2n\pi, n \in Z$

When $\cos x = \frac{1}{2}$:

Reference angle: $x_R = \frac{\pi}{3}$

As cosine is +ve in I and IV quadrants, so

I-Quad	IV-Quad
$x = x_R$	$x = 2\pi - x_R$
$x = \frac{\pi}{3}$	$x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

As 2π is the period of $\cos x$, so

General values of $x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in Z$

General solution $= \{2n\pi\} \cup \left\{\frac{\pi}{3} + 2n\pi\right\} \cup \left\{\frac{5\pi}{3} + 2n\pi\right\}, n \in Z$

Q6. $\sin^2 x - 3\cos^2 x = 0$

Solution:

$$\sin^2 x - 3\cos^2 x = 0$$
$$\sin^2 x - 3(1 - \sin^2 x) = 0$$
$$\sin^2 x - 3 + 3\sin^2 x = 0$$
$$4\sin^2 x = 3 \Rightarrow \sin^2 x = \frac{3}{4} \Rightarrow \sin x = \pm\frac{\sqrt{3}}{2}$$

When $\sin x = \frac{\sqrt{3}}{2}$:

Reference angle: $x_R = \frac{\pi}{3}$

As sine is +ve in I and II quadrants, so

I-Quad.	II-Quad.
$x = x_R$	$x = \pi - x_R$
$x = \frac{\pi}{3}$	$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

As 2π is the period of $\sin x$, so

General values of $x = \frac{\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi, n \in Z$

When $\sin x = -\frac{\sqrt{3}}{2}$:

Reference angle: $x_R = \frac{\pi}{3}$

As sine is –ve in III and IV quadrants, so

III-Quad.	IV-Quad.
$x = \pi + x_R$	$x = 2\pi - x_R$
$x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$	$x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

As 2π is the period of $\sin x$, so

General values of $x = \frac{4\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in Z$

General solution $= \left\{\frac{\pi}{3} + 2n\pi\right\} \cup \left\{\frac{2\pi}{3} + 2n\pi\right\} \cup \left\{\frac{4\pi}{3} + 2n\pi\right\} \cup \left\{\frac{5\pi}{3} + 2n\pi\right\}, n \in Z$

Q7. $\sqrt{2}\cos x \sin x - \sin x = 0$

Solution:

$$\sqrt{2}\cos x \sin x - \sin x = 0$$
$$\sin x(\sqrt{2}\cos x - 1) = 0$$

Either $\sin x = 0$ or $\sqrt{2}\cos x - 1 = 0$

$\Rightarrow \quad \cos x = \frac{1}{\sqrt{2}}$

When $\sin x = 0$:

$\Rightarrow \quad x = 0, \pi$ (Quadrental angles)

General values of $x = n\pi, n \in Z$

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When $\cos = \frac{1}{\sqrt{2}}$:

Reference angle: $x_R = \frac{\pi}{4}$

As cosine is +ve I and IV quadrants, so

I-Quad.	IV-Quad.
$x = x_R$	$x = 2\pi - x_R$
$x = \frac{\pi}{4}$	$x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

As 2π is the period of $\cos x$, so

General values of $x = \frac{\pi}{4} + 2n\pi, \frac{7\pi}{4} + 2n\pi, n \in Z$

General solution $= \left\{n\pi\right\} \cup \left\{\frac{\pi}{4} + 2n\pi\right\} \cup \left\{\frac{7\pi}{4} + 2n\pi\right\}, n \in Z$

Q8. $3\cot^2 x + 3\csc^2 x = 5$

Solution:

$$3\cot^2 x + 2\csc^2 x = 5$$

$$3\frac{\cos^2 x}{\sin^2 x} + 3\left(\frac{1}{\sin^2 x}\right) = 5$$

Multiply both sides by $\sin^2 x$

$$3(1 - \sin^2 x) + 3 = 5\sin^2 x$$

$$3 + 3 = 5\sin^2 x + 3\sin^2 x \Rightarrow 8\sin^2 x = 6$$

$$\sin^2 x = \frac{6}{8} \Rightarrow \sin^2 x = \frac{3}{4} \Rightarrow \sin x = \pm\frac{\sqrt{3}}{2}$$

When $\sin x = \frac{\sqrt{3}}{2}$:

Reference angle: $x_R = \frac{\pi}{3}$

As sine is +ve in I and II quadrants, so

I-Quad.	II-Quad.
$x = x_R$	$x = \pi - x_R$
$x = \frac{\pi}{3}$	$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

As 2π is the period of $\sin x$, so

General values of $x = \frac{\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi, n \in Z$

When $\sin x = -\frac{\sqrt{3}}{2}$:

Reference angle: $x_R = \frac{\pi}{3}$

As sine is –ve in III and IV quadrants, so

III-Quad.	IV-Quad.
$x = \pi + x_R$	$x = 2\pi - x_R$
$x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$	$x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

As 2π is the period of $\sin x$, so

General values of $x = \frac{4\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in Z$

General solution $= \left\{\frac{\pi}{3} + 2n\pi\right\} \cup \left\{\frac{2\pi}{3} + 2n\pi\right\} \cup \left\{\frac{4\pi}{3} + 2n\pi\right\} \cup \left\{\frac{5\pi}{3} + 2n\pi\right\}, n \in Z$

Q9. $\sin 2x - \sqrt{3}\cos x = 0$

Solution:

$$\sin 2x - \sqrt{3}\cos x = 0$$

$$2\sin x\cos x - \sqrt{3}\cos x = 0$$

$$\cos x(2\sin x - \sqrt{3}) = 0$$

Either $\cos x = 0$ or $2\sin x - \sqrt{3} = 0$

$$\Rightarrow \sin x = \frac{\sqrt{3}}{2}$$

When $\cos x = 0$:

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ (Quadrental angle)

As 2π is the period of $\cos x$, so

General value of $x = \frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi, n \in Z$

or $x = (2n+1)\frac{\pi}{2} = \frac{\pi}{2} + n\pi$,

When $\sin x = \frac{\sqrt{3}}{2}$:

Reference angle: $x_R = \frac{\pi}{3}$

As sine is +ve in I and II quadrants, so

I-Quad.	II-Quad.
$x = x_R$	$x = \pi - x_R$
$x = \frac{\pi}{3}$	$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

As 2π is the period of $\sin x$, so

General values of $x = \frac{\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi, n \in Z$

General solution $= \left\{\frac{\pi}{2} + 2n\pi\right\} \cup \left\{\frac{3\pi}{3} + 2n\pi\right\} \cup \left\{\frac{\pi}{3} + 2n\pi\right\} \cup \left\{\frac{2\pi}{3} + 2n\pi\right\}, n \in Z$

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OR $= \left\{\frac{\pi}{2}+n\pi\right\} \cup \left\{\frac{\pi}{3}+2n\pi\right\} \cup \left\{\frac{2\pi}{3}+2n\pi\right\}, n \in Z$

Q10. $\cos 2x = \cos x$

Solution:

$$\cos 2x = \cos x$$
$$2\cos^2 x - 1 - \cos x = 0$$
$$2\cos^2 - \cos x - 1 = 0$$
$$2\cos^2 x - 2\cos x + \cos x - 1 = 0$$
$$2\cos x(\cos x - 1) + 1(\cos x - 1) = 0$$
$$(\cos x - 1)(2\cos x + 1) = 0$$

Either $\cos x - 1 = 0$ or $2\cos x + 1 = 0$

$\cos x = 1$ or $\cos x = \frac{-1}{2}$

When $\cos x = 1$:

$\Rightarrow \quad x = 0$ (Quadrental angle)

As 2π is the period of $\cos x$, so

General value of $x = 0 + 2n\pi = 2n\pi, n \in Z$

When $\cos x = \frac{-1}{2}$:

Reference angle: $x_R = \frac{\pi}{3}$

As cosine is –ve in II and III quadrants, so

II-Quad.	III-Quad.
$x = \pi - x_R$	$x = \pi + x_R$
$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$	$x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

As 2π is the period of $\cos x$, so

General values of $x = \frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi, n \in Z$

General solution $= \{2n\pi\} \cup \left\{\frac{2\pi}{3}+2n\pi\right\} \cup \left\{\frac{4\pi}{3}+2n\pi\right\}, n \in Z$

Q11. $\tan^2 x - (\sqrt{3}+1)\tan x = -\sqrt{3}$

Solution:

$$\tan^2 x - (\sqrt{3}+1)\tan x = -\sqrt{3}$$
$$\tan^2 x - \sqrt{3}\tan x - \tan x + \sqrt{3} = 0$$
$$\tan x(\tan x - \sqrt{3}) - 1(\tan x - \sqrt{3}) = 0$$
$$(\tan x - 1)(\tan x - \sqrt{3}) = 0$$

Either $\tan x - 1 = 0$ or $\tan x - \sqrt{3} = 0$

$\tan x = 1$ or $\tan x = \sqrt{3}$

When $\tan x = 1$:

Reference angle: $x_R = \frac{\pi}{4}$

As $\tan x$ is +ve in I quadrant, so

$x = x_R = \frac{\pi}{4}$

As π is the period of $\tan x$, so

General value of $x = \frac{\pi}{4} + n\pi, n \in Z$

When $\tan x = \sqrt{3}$:

Reference angle: $x_R = \frac{\pi}{3}$

As tan is +ve in I quadrant, so

$x = x_R = \frac{\pi}{3}$

As π is the period of $\tan x$, so

$x = x_R = \frac{\pi}{3}$

As π is the period of $\tan x$, so

General value of $x = \frac{\pi}{3} + n\pi, n \in Z$

General solution of $= \left\{\frac{\pi}{4}+n\pi\right\} \cup \left\{\frac{\pi}{3}+n\pi\right\}, n \in Z$

Q12. $\tan x = \sin 2x$

Solution:

$$\tan x = \sin 2x$$
$$\frac{\sin x}{\cos x} = 2\sin x \cos x$$
$$\sin x = 2\sin x \cos^2 x$$
$$2\sin x \cos^2 x - \sin x = 0$$
$$\sin x(2\cos^2 x - 1) = 0$$

Either $\sin x = 0$ or $2\cos^2 x - 1 = 0$

$\Rightarrow \cos^2 x = \frac{1}{2} \Rightarrow \cos x = \pm\frac{1}{\sqrt{2}}$

When $\sin x = 0$:

$\Rightarrow \quad x = 0, \pi$ (Quadrental angles)

General value of $x = n\pi, n \in Z$

When $\cos x = \pm\frac{1}{\sqrt{2}}$:

Reference angle: $x_R = \frac{\pi}{4}$

As cosine is +ve in I and IV quadrants and –ve in II and III

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quadrants, so

I-Quad.	IV-Quad.
$x = x_R$	$x = 2\pi - x_R$
$x = \frac{\pi}{4}$	$x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$
II-Quad.	**III-Quad.**
$x = \pi - x_R$	$x = \pi - x_R$
$x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$	$x = \pi - \frac{\pi}{4} = \frac{5\pi}{4}$

As 2π is the period of $\cos x$, so

General values of $x = \frac{\pi}{4} + 2n\pi, \frac{3\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi, \frac{7\pi}{4} + 2n\pi, n \in Z$

OR $\quad x = (2n+1)\frac{\pi}{4} = \frac{n\pi}{2} + \frac{\pi}{4}$

General solution $= \{n\pi\} \cup \left\{\frac{\pi}{4} + \frac{n\pi}{2}\right\}, n \in Z$

Q13. $\sin 3x = \cos 2x$

Solution:

$$\sin 3x = \cos 2x$$

$$\sin 3x = \sin\left(\frac{\pi}{2} - 2x\right)$$

$$\sin 3x - \sin\left(\frac{\pi}{2} - 2x\right) = 0$$

Using Formula: $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$2\cos\left(\frac{3x + \frac{\pi}{2} - 2x}{2}\right)\sin\left(\frac{3x - \frac{\pi}{2} + 2x}{2}\right) = 0$$

$$2\cos\left(\frac{x + \frac{\pi}{2}}{2}\right)\sin\left(\frac{5x - \frac{\pi}{2}}{2}\right) = 0$$

$$\cos\left(\frac{x}{2} + \frac{\pi}{4}\right)\sin\left(\frac{5x}{2} - \frac{\pi}{4}\right) = 0$$

Either $\cos\left(\frac{x}{2} + \frac{\pi}{4}\right) = 0$ or $\sin\left(\frac{5x}{2} - \frac{\pi}{4}\right) = 0$

When $\cos\left(\frac{x}{2} + \frac{\pi}{4}\right) = 0$:

$\Rightarrow \quad \frac{x}{2} + \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}$ (Quadrental angles)

General value: $\frac{x}{2} + \frac{\pi}{4} = (2n+1)\frac{\pi}{2}, n \in Z$

$$\Rightarrow \quad \frac{x}{2} = n\pi + \frac{\pi}{2} - \frac{\pi}{4}$$

$$\frac{x}{2} = n\pi + \frac{2\pi - \pi}{4}$$

$$x = 2\left(n\pi + \frac{\pi}{4}\right)$$

$$x = \frac{\pi}{2} + 2n\pi$$

When $\sin\left(\frac{5x}{2} - \frac{\pi}{4}\right) = 0$:

$\Rightarrow \quad \frac{5x}{2} - \frac{\pi}{4} = 0, \pi$ (Quadrental angles)

General value: $\frac{5x}{2} - \frac{\pi}{4} = n\pi, n \in Z$

$$\frac{5x}{2} = \frac{\pi}{4} + n\pi$$

$$x = \frac{2}{5}\left(\frac{\pi}{4} + n\pi\right)$$

$$x = \frac{\pi}{10} + \frac{2n\pi}{5}$$

General solution $= \left\{\frac{\pi}{10} + \frac{2n\pi}{5}\right\} \cup \left\{\frac{\pi}{2} + 2n\pi\right\}, n \in Z$

Q14. $\sqrt{2\sin^2 x + 1} = \sin x + \cos x$

Solution:

$\sqrt{2\sin^2 x + 1} = \sin x + \cos x \quad \ldots(1)$

Squaring both sides

$$(\sqrt{2\sin^2 x + 1})^2 = (\sin x + \cos x)^2$$

$$2\sin^2 x + 1 = \sin^2 x + \cos^2 x + 2\sin x \cos x$$

$$2\sin^2 x + 1 = 1 + 2\sin x \cos x \quad \because \sin^2 x + \cos^2 x = 1$$

$$2\sin^2 x - 2\sin x \cos x = 0$$

$$2\sin(\sin x - \cos x) = 0$$

Either $2\sin x = 0$ or $\sin x - \cos x = 0$

$\sin x = 0$ or $\sin x = \cos x$

$\Rightarrow \frac{\sin x}{\cos x} = 1$

$\Rightarrow \tan x = 1$

When $\sin x = 0$:

$\Rightarrow x = 0, \pi$ (Quadrental angles)

Check:

Put $x = 0$ in eq. (1)

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$\sqrt{2\sin^2 0+1}=\sin 0+\cos 0 \Rightarrow 1=1$ (True)

$\Rightarrow$ $x=0$ is a root

Put $x=\pi$ in eq. (1)

$\sqrt{2\sin^2 \pi+1}=\sin\pi+\cos\pi \Rightarrow 1=-1$ (False)

$\Rightarrow$ $x=\pi$ is an extraneous root

As 2π is the period of $\sin x$, so

General value of $x=0+2n\pi=2n\pi, n\in Z$

When $\tan x=1$:

Reference angle: $x_R=\frac{\pi}{4}$

As tan is +ve in I quadrant, so

$$x=x_R=\frac{\pi}{4}$$

Check:

Put $x=\frac{\pi}{4}$ in eq. (1)

$$\sqrt{2\sin^2\frac{\pi}{4}+1}=\sin\frac{\pi}{4}+\cos\frac{\pi}{4} \Rightarrow \sqrt{2}=\sqrt{2} \text{ (True)}$$

$\Rightarrow$ $x=\frac{\pi}{4}$ is a root

As π is the period of $\tan x$, so

General value of $x=\frac{\pi}{4}+n\pi, n\in Z$

General solution $=\{2n\pi\}\cup\left\{\frac{\pi}{4}+n\pi\right\}, n\in Z$

Q15. $\sin^3 x\cos x+\cos^3 x\sin x=\frac{1}{2}$

Solution:

$$\sin^3 x\cos x+\cos^3 x\sin x = \frac{1}{2}$$

$$\sin x\cos x(\sin^2 x+\cos^2 x) = \frac{1}{2}$$

$$\sin x\cos x(1) = \frac{1}{2}$$

$$2\sin x\cos x = 1 \quad \text{Multiply by '2'}$$

$$\sin 2x = 1$$

$\Rightarrow$ $2x=\frac{\pi}{2}$ (Quadrental angle)

As 2π is the period of sine, so

General value: $2x=\frac{\pi}{2}+2n\pi, n\in Z$

$\Rightarrow$ $x=\frac{\pi}{4}+n\pi$

General solution $=\left\{\frac{\pi}{4}+n\pi\right\}, n\in Z$

Q16. $\sin^3 2x\cos 2x-\cos^3 2x\sin 2x=\frac{\sqrt{2}}{8}$

Solution:

$$\sin^3 2x\cdot\cos 2x-\cos^3 2x\cdot\sin 2x = \frac{\sqrt{2}}{8}$$

$$-\sin 2x\cos 2x(-\sin^2 2x+\cos^2 2x) = \frac{\sqrt{2}}{8}$$

Multiply both sides by '2'

$$-2\sin 2x.\cos 2x(\cos^2 2x-\sin^2 2x) = \frac{\sqrt{2}}{4}$$

$$\sin 2(2x).\cos 2(2x) = -\frac{\sqrt{2}}{4}$$

$$\sin 4x.\cos 4x = -\frac{\sqrt{2}}{4}$$

Multiply both sides by '2'

$$2\sin 4x\cos 4x = -\frac{\sqrt{2}}{2}$$

$$\sin 2(4x) = -\frac{1}{\sqrt{2}}$$

$$\sin 8x = -\frac{1}{\sqrt{2}}$$

Reference angle: $(8x)_R=\frac{\pi}{4}$

As sine is –ve in III and IV quadrants, so

III-Quad.	IV-Quad.
$8x=\pi+(8x)_R$	$8x=2\pi-(8x)_R$
$=\pi+\frac{\pi}{4}$	$=2\pi-\frac{\pi}{4}$
$=\frac{5\pi}{4}$	$=\frac{7\pi}{4}$

As 2π is the period of sine, so

General values : $8x=\frac{5\pi}{4}+2n\pi, \frac{7\pi}{4}+2n\pi$

Divide each term by 8

$$x=\frac{5\pi}{32}+\frac{n\pi}{4}, \frac{7\pi}{32}+\frac{n\pi}{4}, n\in Z$$

General solution $=\left\{\frac{5\pi}{32}+\frac{n\pi}{4}\right\}\cup\left\{\frac{7\pi}{32}+\frac{n\pi}{4}\right\}, n\in Z$

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Q17. $\sin 3x\cos x+\cos 3x\sin x=\frac{1}{\sqrt{2}}$

Solution:

$$\sin 3x\cdot\cos x+\cos 3x\cdot\sin x=\frac{1}{\sqrt{2}}$$

Using Formula: $\sin\alpha\cos\beta+\cos\alpha\sin\beta=\sin(\alpha+\beta)$

$$\sin(3x+x)=\frac{1}{\sqrt{2}}$$

$$\sin 4x=\frac{1}{\sqrt{2}}$$

Reference angle: $(4x)_R=\frac{\pi}{4}$

As sine is +ve in I and II quadrants, so

I-Quad.	II-Quad.
$4x=(4x)_R$	$4x=\pi-(4x)_R$
$4x=\frac{\pi}{4}$	$4x=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

As 2π is the period of sine, so

General values: $4x=\frac{\pi}{4}+2n\pi,\frac{3\pi}{4}+2n\pi$

Divide each term by 4

$$x=\frac{\pi}{16}+\frac{n\pi}{2},\frac{3\pi}{16}+\frac{n\pi}{2},n\in Z$$

General solution $=\left\{\frac{\pi}{16}+\frac{n\pi}{2}\right\}\cup\left\{\frac{3\pi}{16}+\frac{n\pi}{2}\right\},n\in Z$

Q18. $\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}=4$

Solution:

$$\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}=4$$

$$\frac{\sin^2 x+(1+\cos x)^2}{\sin x(1+\cos x)}=4$$

$$\sin^2 x+1+\cos^2 x+2\cos x=4\sin x(1+\cos x)$$

$$1+1+2\cos x-4\sin x(1+\cos x)=0$$

$$2+2\cos x-4\sin x(1+\cos x)=0$$

$$2(1+\cos x)-4\sin x(1+\cos x)=0$$

$$(1+\cos x)(2-4\sin x)=0$$

As $1+\cos x\neq 0\quad\Rightarrow 2-4\sin x=0$

$$4\sin x=2\quad\Rightarrow\sin=\frac{1}{2}$$

Reference angle: $x_R=\frac{\pi}{6}$

As sine is +ve in I and II quadrants, so

I-Quad	II-Quad.
$x=x_R$	$x=\pi-x_R$
$x=\frac{\pi}{6}$	$x=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

As 2π is the period of sine, so

General values of $x=\frac{\pi}{6}+2n\pi,\frac{5\pi}{6}+2n\pi,n\in Z$

General solution $=\left\{\frac{\pi}{6}+2n\pi\right\}\cup\left\{\frac{5\pi}{6}+2n\pi\right\},n\in Z$

Q19. $\frac{\tan\left(7x+\frac{\pi}{6}\right)+\tan 3x}{1-\tan\left(7x+\frac{\pi}{6}\right)\tan 3x}=1$

Solution:

$$\frac{\tan\left(7x+\frac{\pi}{6}\right)+\tan 3x}{1-\tan\left(7x+\frac{\pi}{6}\right)\tan 3x}=1$$

Using Formula: $\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\tan(\alpha+\beta)$

$$\Rightarrow\quad\tan\left(\left(7x+\frac{\pi}{6}\right)+3x\right)=1$$

$$\tan\left(10x+\frac{\pi}{6}\right)=1$$

Reference angle: $\left(10x+\frac{\pi}{6}\right)_R=\frac{\pi}{4}$

As tan is +ve in I quadrant, so

$$10x+\frac{\pi}{6}=\frac{\pi}{4}$$

As π is the period of tan, so

$$10x+\frac{\pi}{6}=\frac{\pi}{4}+n\pi,n\in Z$$

$$10x=\frac{\pi}{4}-\frac{\pi}{6}+n\pi$$

$$10x=\frac{3\pi-2\pi}{12}+n\pi$$

$$10x=\frac{\pi}{12}+n\pi$$

Divide each term by 10

$$x=\frac{\pi}{120}+\frac{n\pi}{10}$$

General solution $=\left\{\frac{\pi}{120}+\frac{n\pi}{10}\right\},n\in Z$

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Q20. $\dfrac{\cot\left(2x+\frac{\pi}{12}\right)-\cot\left(5x+\frac{\pi}{12}\right)}{1+\cot\left(2x+\frac{\pi}{12}\right)\cot\left(5x+\frac{\pi}{12}\right)}=\dfrac{1}{\sqrt{3}}$

Solution:

$$\frac{\cot\left(2x+\frac{\pi}{12}\right)-\cot\left(5x+\frac{\pi}{12}\right)}{1+\cot\left(2x+\frac{\pi}{12}\right)\cot\left(5x+\frac{\pi}{12}\right)}=\frac{1}{\sqrt{3}}$$

Taking reciprocal of both sides

$$\frac{1+\cot\left(2x+\frac{\pi}{12}\right)\cot\left(5x+\frac{\pi}{12}\right)}{\cot\left(2x+\frac{\pi}{12}\right)-\cot\left(5x+\frac{\pi}{12}\right)}=\sqrt{3}$$

Using Formula: $\dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}=\cot(\alpha-\beta)$

$$\Rightarrow \cot\left(\left(5x+\frac{\pi}{12}\right)-\left(2x+\frac{\pi}{12}\right)\right)=\sqrt{3}$$

$$\cot\left(5x+\frac{\pi}{12}-2x-\frac{\pi}{12}\right)=\sqrt{3}$$

$$\cot 3x=\sqrt{3}\Rightarrow \tan 3x=\frac{1}{\sqrt{3}}$$

Reference angle: $(3x)_R=\dfrac{\pi}{6}$

As tan is +ve in I quadrant, so

$$3x=\frac{\pi}{6}$$

As π is the period of tan, so

General value: $3x=\dfrac{\pi}{6}+n\pi, n\in Z$

Divide each term by 3

$$x=\frac{\pi}{18}+\frac{n\pi}{3}$$

General solution $=\left\{\dfrac{\pi}{18}+\dfrac{n\pi}{3}\right\}, n\in Z$

Q21. $\dfrac{\sin^3x+\cos^3x}{\sin x+\cos x}-\dfrac{\sin x}{\sqrt{1+\tan^2x}}=\dfrac{1}{2}$

Solution:

$$\frac{\sin^3x+\cos^3x}{\sin x+\cos x}-\frac{\sin x}{\sqrt{1+\tan^2x}}=\frac{1}{2}$$

$$\frac{(\sin x+\cos x)(\sin^2x-\sin x\cos x+\cos^2x)}{(\sin x+\cos x)}-\frac{\sin x}{\sqrt{\sec^2x}}=\frac{1}{2}$$

$$(\sin^2x+\cos^2x)-\sin x\cos x-\frac{\sin x}{\sec x}=\frac{1}{2}$$

$$1-\sin x\cos x-\sin x\cos x=\frac{1}{2}$$

$$-2\sin x\cos x=\frac{1}{2}-1$$

$$-\sin 2x=\frac{-1}{2}\Rightarrow \sin 2x=\frac{1}{2}$$

Reference angle: $(2x)_R=\dfrac{\pi}{6}$

As sine is +ve in I and II quadrants, so

I-Quad.	II-Quad.
$2x=\dfrac{\pi}{6}$	$2x=\pi-\dfrac{\pi}{6}$
	$2x=\dfrac{5\pi}{6}$

As 2π is the period of sine, so

General values: $2x=\dfrac{\pi}{6}+2n\pi,\dfrac{5\pi}{6}+2n\pi$

Divide each term by 2

$$x=\frac{\pi}{12}+n\pi,\frac{5\pi}{12}+n\pi, n\in Z$$

General solution $=\left\{\dfrac{\pi}{12}+n\pi\right\}\cup\left\{\dfrac{5\pi}{12}+n\pi\right\}, n\in Z$

Q22. $\sin 3x-\sin 5x+\sin 7x=0$

Solution:

$$\sin 3x-\sin 5x+\sin 7x=0$$

$$(\sin 7x+\sin 3x)-\sin 5x=0$$

Using Formula: $\sin P+\sin Q=2\sin\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right)$

$$2\sin\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right)-\sin 5x=0$$

$$2\sin 5x\cdot\cos 2x-\sin 5x=0$$

$$\sin 5x(2\cos 2x-1)=0$$

Either $\sin 5x=0$ or $2\cos 2x-1=0\Rightarrow\cos 2x=\dfrac{1}{2}$

When $\sin 5x=0$:

$\Rightarrow$ $5x=0,\pi$ (Quadrental angles)

General value: $5x=n\pi, n\in Z$ $\Rightarrow$ $x=\dfrac{n\pi}{5}$

When $\cos 2x=\dfrac{1}{2}$:

Reference angle: $(2x)_R=\dfrac{\pi}{3}$

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As cosine is +ve in I and IV quadrants, so

I-Quad.	II-Quad.
$2x = \frac{\pi}{3}$	$2x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

As 2π is the period of sine, so

General values: $2x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in Z$

Divide each term by 2

$$x = \frac{\pi}{6} + n\pi, \frac{5\pi}{6} + n\pi$$

General solution $= \left\{\frac{n\pi}{5}\right\} \cup \left\{\frac{\pi}{6} + n\pi\right\} \cup \left\{\frac{5\pi}{6} + n\pi\right\}, n \in Z$

Q23. $\cos 2x - \cos 6x = \sin 4x$

Solution:

$$\cos 2x - \cos 6x = \sin 4x$$

Using Formula: $\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

$$-2\sin\left(\frac{2x+6x}{2}\right)\sin\left(\frac{2x-6x}{2}\right) = \sin 4x$$

$$-2\sin 4x \cdot \sin(-2x) = \sin 4x$$

$$2\sin 4x \cdot \sin 2x = \sin 4x \quad \because \sin(-\theta) = -\sin\theta$$

$$2\sin 4x.\sin 2x - \sin 4x = 0$$

$$\sin 4x(2\sin 2x - 1) = 0$$

Either $\sin 4x = 0$ or $2\sin 2x - 1 = 0 \Rightarrow \sin 2x = \frac{1}{2}$

When $\sin 4x = 0$:

$\Rightarrow$ $4x = 0, \pi$ (Quadrental angles)

General value: $4x = n\pi, n \in Z$

Divide each term by 4

$$x = \frac{n\pi}{4}$$

When $\sin 2x = \frac{1}{2}$:

Reference angle: $(2x)_R = \frac{\pi}{6}$

As sine is +ve in I and II quadrants, so

I-Quad.	II-Quad.
$2x = \frac{\pi}{6}$	$2x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

As 2π is the period of sine, so

General values: $2x = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi, n \in Z$

Divide each term by 2

$$x = \frac{\pi}{12} + n\pi, \frac{5\pi}{12} + n\pi$$

General solution $= \left\{\frac{n\pi}{4}\right\} \cup \left\{\frac{\pi}{12} + n\pi\right\} \cup \left\{\frac{5\pi}{12} + n\pi\right\}, n \in Z$

Q24. $\sin x + \sin 3x + \sin 5x + \sin 7x = 0$

Solution:

$$\sin x + \sin 3x + \sin 5x + \sin 7x = 0$$

$$(\sin 7x + \sin x) + (\sin 5x + \sin 3x) = 0$$

Using Formula: $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

$$2\sin\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right) + 2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right) = 0$$

$$2\sin 4x.\cos 3x + 2\sin 4x \cdot \cos x = 0$$

$$2\sin 4x(\cos 3x + \cos x) = 0$$

Using Formula: $\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

$$2\sin 4x \cdot 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) = 0$$

$$4\sin 4x.\cos 2x.\cos x = 0$$

Either $\sin 4x = 0$ or $\cos 2x = 0$ or $\cos x = 0$

When $\sin 4x = 0$

$\Rightarrow$ $4x = n\pi \Rightarrow x = \frac{n\pi}{4}, n \in Z$

When $\cos 2x = 0$

$\Rightarrow$ $2x = (2n+1)\frac{\pi}{2} \Rightarrow x = (2n+1)\frac{\pi}{4}, n \in Z$

When $\cos x = 0$

$\Rightarrow$ $x = (2n+1)\frac{\pi}{2}, n \in Z$

General solution $= \left\{\frac{n\pi}{4}\right\} \cup \left\{(2n+1)\frac{\pi}{4}\right\} \cup \left\{(2n+1)\frac{\pi}{2}\right\}, n \in Z$

OR

$= \left\{\frac{n\pi}{4}\right\} \cup \left\{(2n+1)\frac{\pi}{2}\right\}, n \in Z \quad \because \left\{(2n+1)\frac{\pi}{4}\right\} \subset \left\{\frac{n\pi}{4}\right\}$

GRAPHICAL SOLUTION OF TRIGONOMETRIC EQUATION

- The graphical solution of trigonometric equations involves plotting the graphs of both sides of the equation and identifying their points of intersection. Each point where the curves intersect represents a solution to the equation.
- When dealing with trigonometric equations of the form $f(x) = g(x)$, it can be challenging to find exact solutions. While some equations can be solved algebraically using trigonometric identities and properties, many are too complex for such methods. In these cases, graphical methods serve as a valuable alternative.

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- By plotting both functions $y = f(x)$ and $y = g(x)$ on the coordinate plane, we can visually observe their points of intersection. The x-coordinates of these intersection points are the solutions that satisfy $f(x) = g(x)$.
- This graphical approach not only helps in determining the number of solutions but also provides deeper insight into the behavior of the functions and the nature of their interactions.

Example 4: Solve the equation $\sin x = \frac{x}{2}$ graphically over the interval $[-\pi, \pi]$

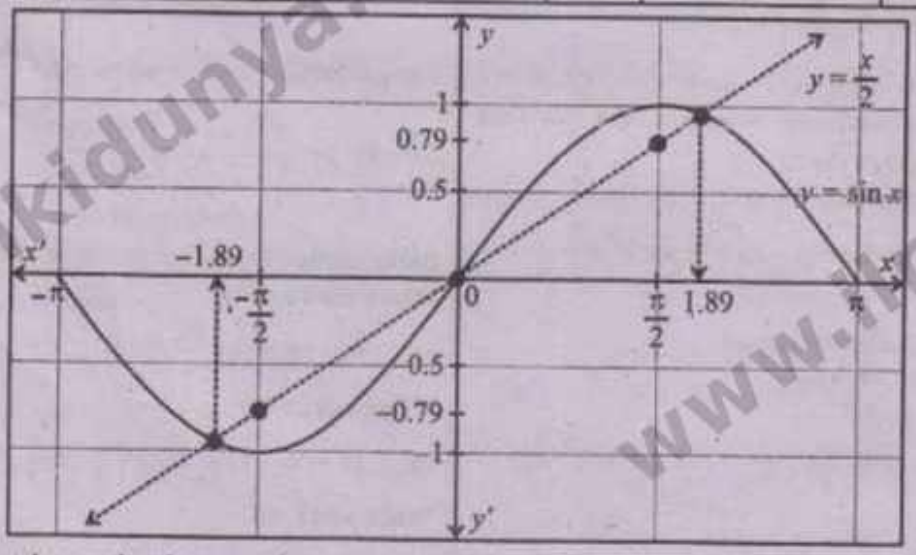
Solution:

Let $y = \sin x = \frac{x}{2}; x \in [-\pi, \pi]$

$\Rightarrow$ $y = \sin x$ and $y = \frac{x}{2}$

Table of values:

x	$-\pi$	$\frac{-\pi}{2}$	0	$\frac{\pi}{2}$	π
	−3.14	−1.57	0	1.57	3.14
$y = \sin x$	0	−1	0	1	0
$y = \frac{x}{2}$	−1.57	−0.79	0	0.79	1.57



From the graph, we observe that the curves intersect at three points in $[-\pi, \pi]$.
Required solutions are: $x = -1.89, 0, 1.89$ radians

Note

The exact values can be refined using numerical methods, but the graphical approach provides a clear visual approximation.

Example 5: Solve the equation $\cos x = \frac{x}{8}$ graphically.

Solution:

Let $y = \cos x = \frac{x}{8}$

$\Rightarrow$ $y = \cos x$ and $y = \frac{x}{8}$

Note that: As $-1 \le \cos x \le 1$

$\Rightarrow -1 \le \frac{x}{8} \le 1$

$\Rightarrow -8 \le x \le 8$

Interval $= [-8, 8]$

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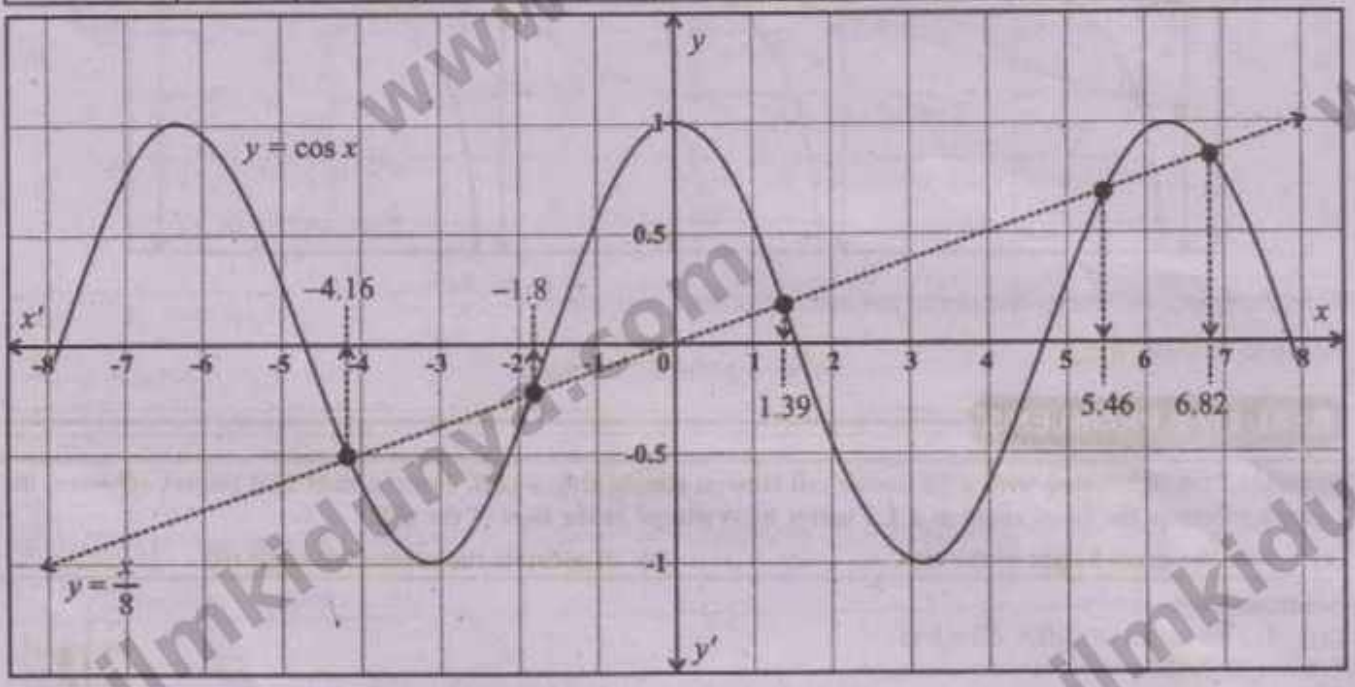
Table of values:

x	-8	-2π	$\frac{-3\pi}{2}$	$-\pi$	$\frac{-\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	8
	-8	-6.28	-4.71	-3.14	-1.57	0	1.57	3.14	4.71	6.28	8
$y=\cos x$	-0.15	1	0	-1	0	1	0	-1	0	1	-0.15
$y=\frac{x}{8}$	-1	-0.79	-0.59	-0.39	-0.20	0	0.20	0.39	0.59	0.79	1



From the graph, we observe that the curves intersect at five points in $[-8, 8]$.
Required solutions are $x = -4.16, -1.8, 1.39, 5.46, 6.82$ radians.

Example 6: Solve the equation $\tan x = \frac{x}{2}$ graphically in $\left(\frac{-3\pi}{2}, \frac{3\pi}{2}\right)$.

Solution:

Let $y = \tan x = \frac{x}{2}; x \in \left(\frac{-3\pi}{2}, \frac{3\pi}{2}\right)$

Vertical asymptotes: At $x = \pm\frac{\pi}{2} \Rightarrow x = \pm 1.57$

Table of values:

x	$\frac{-5\pi}{4}$	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$
	-3.93	-3.14	-2.36	-0.79	0	0.79	2.36	3.14	3.93
$y=\tan x$	1	0	-1	-1	0	1	-1	0	1
$y=\frac{x}{2}$	-1.96	-1.57	-1.18	-0.39	0	0.39	1.18	1.57	1.96

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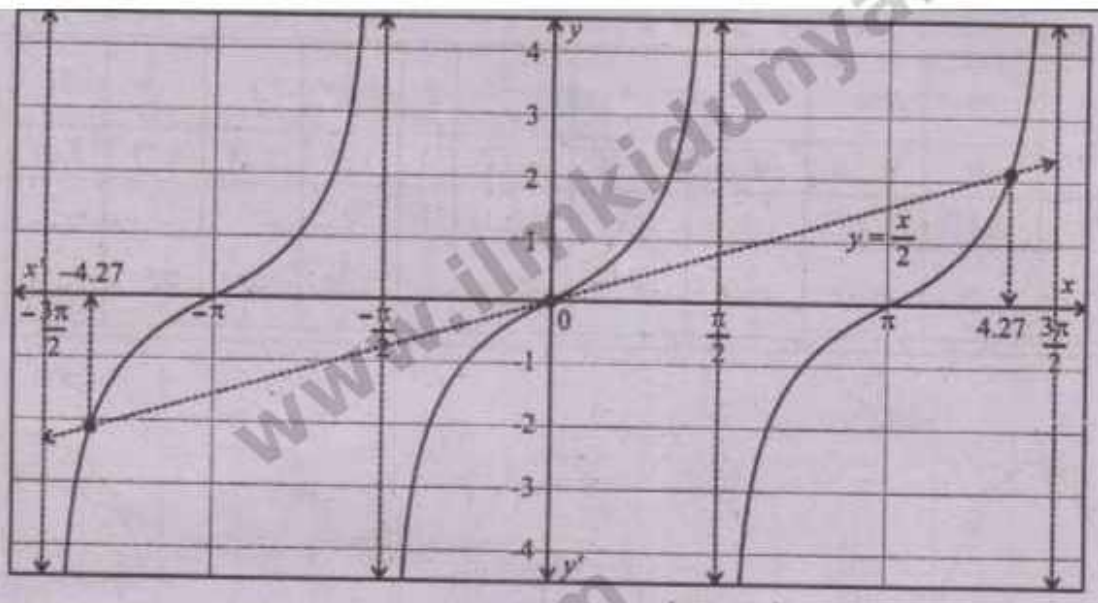

From the graph, we observe that the curves intersect at three points in $\left(\frac{-3\pi}{2}, \frac{3\pi}{2}\right)$

Required solutions are:

$$x = -4.27, 0, 4.27 \text{ radians}$$

REAL LIFE PROBLEMS

Example 7: A lighthouse with a 15 meter tall lantern stands atop a cliff. From a ship 120 meters offshore, the lantern subtends the same angle as a 1.5 meter buoy placed at the base of the cliff.

(a) Find the exact height of the cliff. **(b) Calculate the subtended angle (θ).**

Solution:

(a) Let the height of cliff = $AD = h$ m

From right ΔABD:

$$\tan\theta_1 = \frac{AD}{AB}$$

$$\tan\theta_1 = \frac{h}{120} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{h}{120}\right)$$

From right ΔABE:

$$\tan\theta = \frac{h+15}{120}$$

$$\theta = \tan^{-1}\left(\frac{h+15}{120}\right)$$

From right ΔABE:

$$\tan\theta_3 = \frac{AE}{AB}$$

$$\tan\theta_3 = \frac{1.5}{120} \Rightarrow \theta_3 = \tan^{-1}\left(\frac{1.5}{120}\right)$$

According to the given condition

Lantern angle = Buoy angle

$$\theta_2 = \theta_3$$

$$\Rightarrow \quad \theta - \theta_1 = \theta_3$$

Putting values of θ, θ_1, and θ_3, we have



Figure 10.6

$$\tan^{-1}\left(\frac{h+15}{120}\right) - \tan^{-1}\left(\frac{h}{120}\right) = \tan^{-1}\left(\frac{1.5}{120}\right)$$

Using Formula: $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

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$$\tan^{-1}\left(\frac{\frac{h+15}{120}-\frac{h}{120}}{1+\frac{h+15}{120}\cdot\frac{h}{120}}\right)=\tan^{-1}\left(\frac{1.5}{120}\right)$$

$$\Rightarrow \quad \frac{\frac{h+15-h}{120}}{\frac{14400+h^2+15h}{14400}}=\frac{1.5}{120}$$

$$\frac{15}{120}\times\frac{14400}{h^2+15h+14400}=\frac{15}{120\times 10}$$

$$(15\times 120)\times\frac{1200}{15}=h^2+15h+14400$$

$$h^2+15h+14400-144000=0$$

$$h^2+15h-129600=0$$

By quadratic formula

$$h=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-15\pm\sqrt{15^2-4(1)(-129600)}}{2(1)}$$

$$=\frac{-15\pm\sqrt{225+518400}}{2}$$

$$=\frac{-15+720.156}{2} \quad (\because \text{Height is +ve})$$

$$h=\frac{705.156}{2}\approx 352.6$$

$$\boxed{h=352.6m}$$

(b) As Lantern angle = Buoy angle

$$\Rightarrow \quad \theta_2=\theta_3=\tan^{-1}\left(\frac{1.5}{120}\right)\approx 0.72^\circ$$

Example 8: A projectile is launched from the ground with an initial velocity $v_0 = 30$ m/s. The range R of the projectile is given by the formula $R=\frac{v_0^2}{g}\sin 2\theta$, where θ is the launch angle to the horizontal and $g = 9.8$ m/s² is the acceleration due to gravity.

(a) Find the angle θ (in degrees) required for the projectile to hit a target exactly 80 meters away.

(b) If the maximum range achievable is $R_{max}=\frac{v_0^2}{g}$, determine the angle θ that gives the maximum range. Justify your answer mathematically.

Solution:

Given that: $v_0 = 30m/s$, $R=80m$, $g=9.8m/s^2$

Formula: $R=\frac{v_0^2}{g}\sin 2\theta$

$$\Rightarrow \quad 80=\frac{(30)^2}{9.8}\sin 2\theta$$

$$80\times\frac{9.8}{900}=\sin 2\theta$$

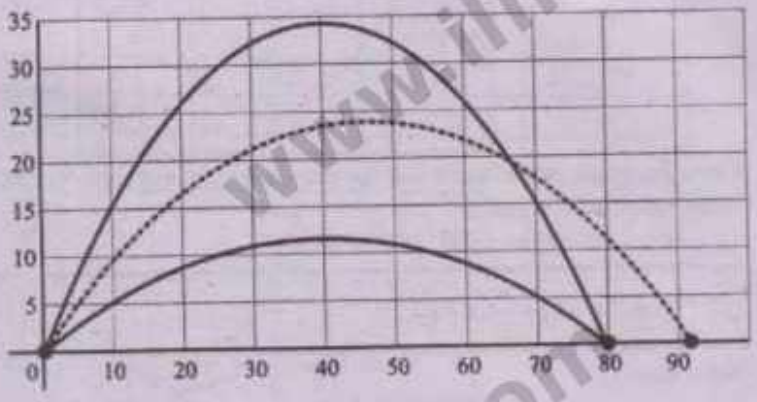
$$\sin 2\theta = 0.8711$$

Reference angle: $(2\theta)_R = 60.59^\circ$

As sine is +ve in I and II quadrants, so

I-Quad.	II-Quad.
$2\theta = (2\theta)_R$	$2\theta = 180^\circ - (2\theta)_R$
$2\theta = 60.59^\circ$	$2\theta = 180^\circ - 60.59^\circ$
$\theta = 30.29^\circ$	$2\theta = 119.41^\circ$
	$\theta = 59.71^\circ$

Thus the projectile will land exactly $80m$ away when $\theta = 30.29^\circ$ or 59.75°.



(b) For maximum range:

$\sin 2\theta = 1 \Rightarrow 2\theta = \sin^{-1}(1)$

$2\theta = 90^\circ \Rightarrow \theta = 45^\circ$

Thus, the maximum range is achieved at 45°

Now, $R_{max}=\frac{v_0^2}{g}$

$$=\frac{900}{9.8}=91.84m$$

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EXERCISE 10.2

Solve graphically:

Q1. $3\sin x = \frac{x}{2} + 1, x \in [-\pi, \pi]$

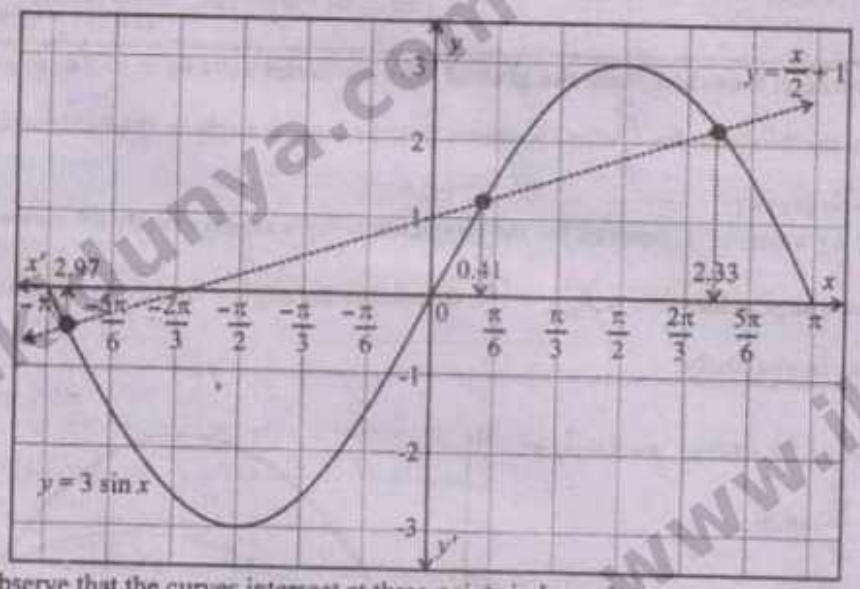
Solution:

Let $y = 3\sin x = \frac{x}{2} + 1, x \in [-\pi, \pi] \quad \Rightarrow \quad y = 3\sin x$ and $y = \frac{x}{2} + 1$

Table of values:

x	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
	−3.14	−2.62	−2.09	−1.57	−1.05	−0.52	0	0.52	1.05	1.57	2.09	2.62	3.14
$y = 3\sin x$	0	−1.52	−2.60	−3	−2.60	−1.50	0	1.50	2.60	3	2.60	1.50	0
$y = \frac{x}{2} + 1$	−0.57	−0.31	−0.05	0.22	0.48	0.74	1	1.26	1.53	1.79	2.05	2.31	2.57

Graph:



From the graph, we observe that the curves intersect at three points in $[-\pi, \pi]$

Required solutions are:

$x = -2.97, 0.41$ and 2.33 radians

Q2. $\cos(2x - 1) = \frac{x}{5}, x \in [0, \pi]$

Solution:

Let $y = \cos(2x - 1) = \frac{x}{5}, x \in [0, \pi] \quad \Rightarrow \quad y = \cos(2x - 1)$ and $y = \frac{x}{5}$

Table of values:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
	0	0.52	1.05	1.57	2.09	2.62	3.14
$y = \cos(2x - 1)$	0.54	1	0.45	−0.54	−1	−0.49	0.51
$y = \frac{x}{5}$	0	0.10	0.21	0.31	0.42	0.52	0.63

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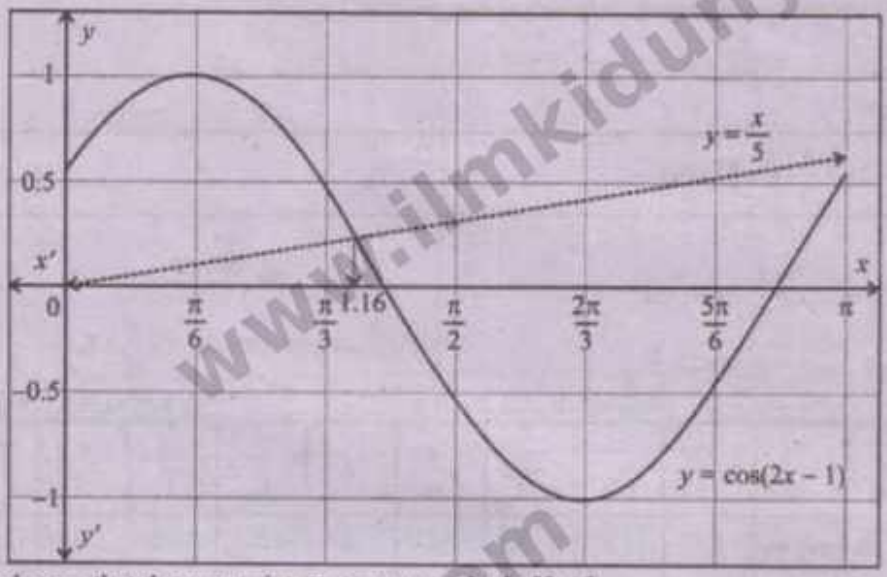
Graph:



From the graph, we observe that the curves intersect at one point in $[0, \pi]$.

Required solution is:

$x = 1.16$ radians

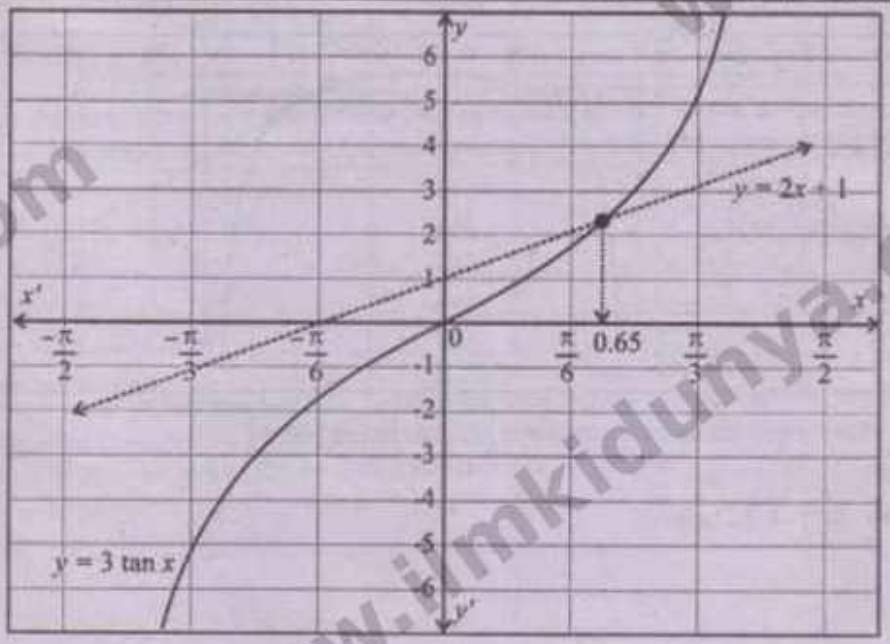
Q3. $3\tan x = 2x + 1, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution:

Let $y = 3\tan x = 2x + 1; x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ $\Rightarrow$ $y = 3\tan x$ and $y = 2x + 1$

Table of values:

x	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	−1.57	−1.05	−0.52	0	0.52	1.05	1.57
$y = 3\tan x$	$-\infty$	−5.20	−1.73	0	1.73	5.20	∞
$y = 2x + 1$	−2.14	−1.10	−0.04	1	2.04	3.10	4.14



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From the graph, we observe that the curves intersect at one point in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Required solution is

$x = 0.65$ radians

Q4. $\sec(x+3) = -2x + \frac{1}{3}, x \in [-\pi, \pi]$

Solution:

Let $y = \sec(x+3) = -2x + \frac{1}{3}; x \in [-\pi, \pi]$

$\Rightarrow$ $y = \sec(x+3)$ and $y = -2x + \frac{1}{3}$

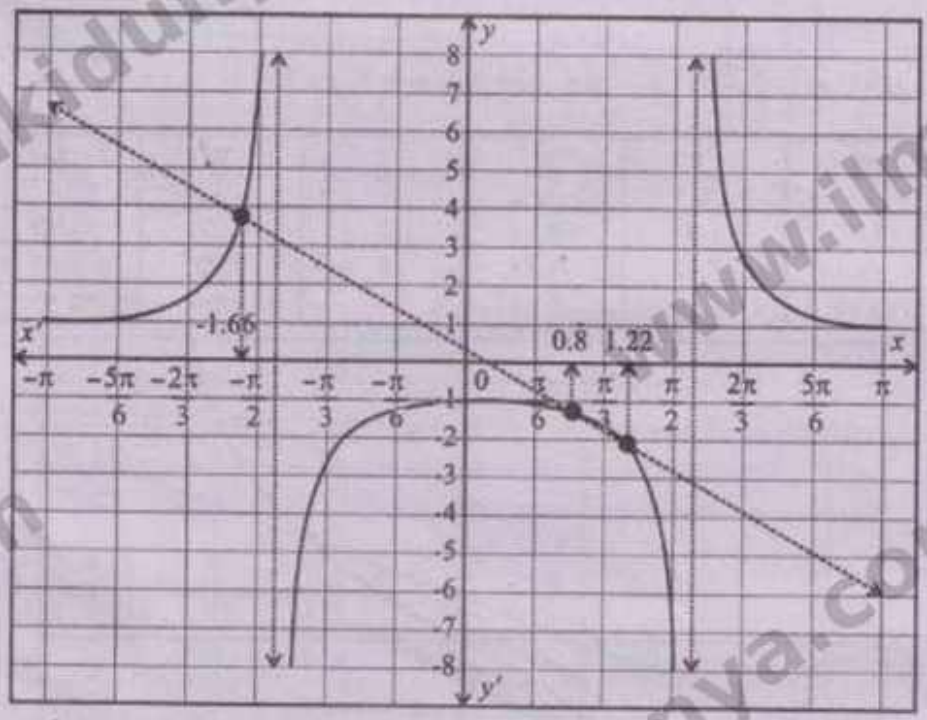
$$y = \frac{1}{\cos(x+3)}$$

Vertical asymptotes: $x = \frac{\pi}{2} - 3, x = \frac{3\pi}{2} - 3$

$\Rightarrow x = -1.43, x = 1.71$

x	$-\pi$	$\frac{5\pi}{6}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
	-3.14	-2.62	-2.09	-1.57	-1.05	-0.52	0	0.52	1.05	1.57	2.09	2.62	3.14
$y=\sec(x+3)$	1.01	1.08	1.63	7.12	-5.20	-1.27	-1.01	-1.08	-1.53	-7.20	1.63	1.26	1.01
$y=-2x+\frac{1}{3}$	6.61	5.57	4.51	3.47	2.43	1.37	0.33	-0.71	-1.77	-2.81	-3.85	-4.91	-5.95

Graph:



From the graph, we observe that the curves intersect at three points in $[-\pi, \pi]$

Required solutions are:

$x = -1.66, 0.80$ and 1.22 radians.

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Q5. $\sin 3x = \frac{x}{2}$

Solution:

Let $y = \sin 3x = \frac{x}{2}$

$\Rightarrow \quad y = \sin 3x$ and $y = \frac{x}{2}$

Note that: As $-1 \le \sin 3x \le 1$

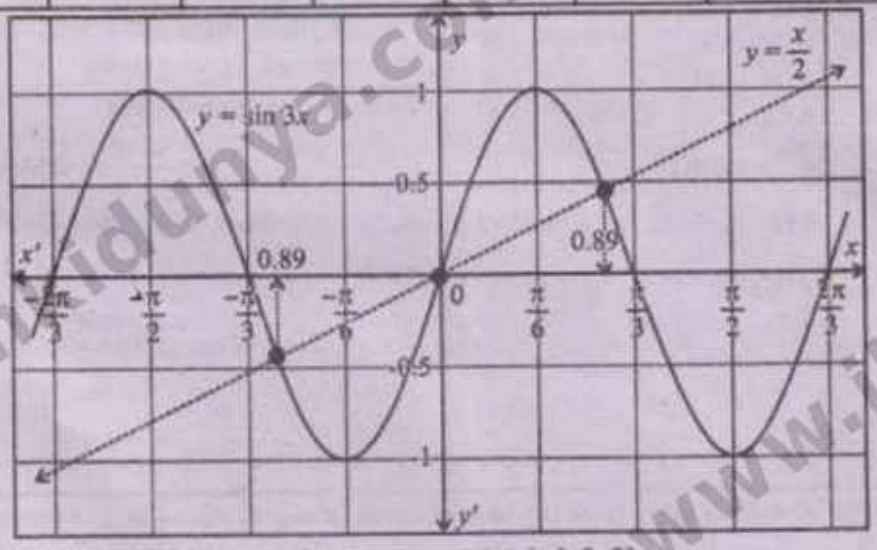
$\Rightarrow -1 \le \frac{x}{2} \le 1$

$\Rightarrow -2 \le x \le 2$

Interval = $[-2, 2]$

Table of values:

x	-2	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	2
	-2	$.57$	-1.05	-0.52	0	0.52	1.05	1.57	2
$y=\sin 3x$	0.28	1	0	-1	0	1	0	-1	-0.28
$y=\frac{x}{2}$	-1	-0.79	-0.53	-0.26	0	0.26	0.53	0.79	1



From the graph, we observe that the curves intersect at three points in $[-2, 2]$

Required solution are:

$x = -0.89, 0, 0.89$ radians

Q6. A hiker pulls a sled using a rope that exerts a force of 80 newtons over a distance of 25 meters. The total work done is 1200 joules. At what angle to the direction of motion is the force being applied?

Solution:

Given: $F = 80\,N$, $d = 25\,m$, $w = 1200\,J$

As we know

Work done $= Fd\cos\theta$

$1200 = (80)(25)\cos\theta$

$\cos\theta = \frac{1200}{(80)(25)}$

$\cos\theta = 0.6$

$\theta = \cos^{-1}(0.6)$

$\boxed{\theta = 53.13^\circ}$

Q7. A radio tower stands on flat ground with a small antenna mounted at its top. From a point 50 meters away from the base of the tower. The antenna subtends the same angle as a 1 meter vertical rod placed at the base of the tower. The total height of the tower and antenna is 10 meters.

(a) Find the height of the tower (without the antenna).

(b) Calculate the subtended angle.

Solution:

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(a) Let the height of radio tower = $AD = hm$

From right ΔABD:

$$\tan\theta_1 = \frac{AD}{AB}$$

$$\tan\theta_1 = \frac{h}{50}$$

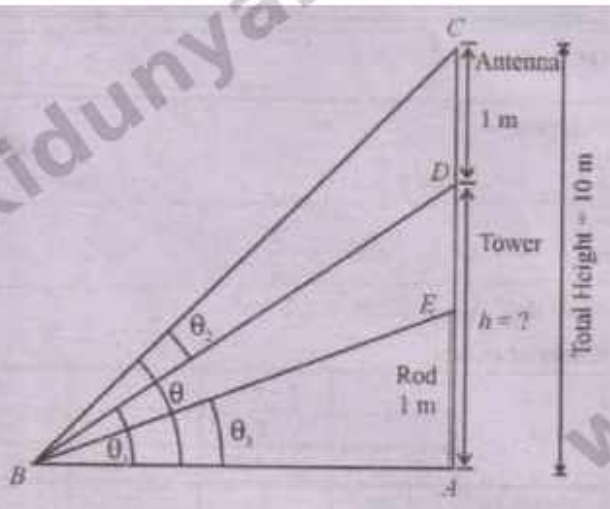
$$\Rightarrow \quad \theta_1 = \tan^{-1}\left(\frac{h}{50}\right)$$

From right ΔABC:

$$\tan\theta = \frac{AC}{AB}$$

$$\tan\theta = \frac{10}{50}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{5}\right)$$



From right ΔABE:

$$\tan\theta_3 = \frac{AE}{AB}$$

$$\tan\theta_3 = \frac{1}{50} \Rightarrow \theta_3 = \tan^{-1}\left(\frac{1}{50}\right)$$

According to the given condition

Antenna angle = Vertical rod angle

$$\theta_2 = \theta_3$$

$$\Rightarrow \quad \theta - \theta_1 = \theta_3$$

$$\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{h}{50} = \tan^{-1}\frac{1}{50}$$

Using formula $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$$\tan^{-1}\left(\frac{\frac{1}{5}-\frac{h}{50}}{1+\frac{1}{5}\cdot\frac{h}{50}}\right) = \tan^{-1}\frac{1}{50}$$

$$\Rightarrow \quad \frac{\frac{50-h}{250}}{\frac{250+h}{250}} = \frac{1}{50}$$

$$\frac{50-h}{250+h} = \frac{1}{50}$$

$$2500 - 50h = 250 + h$$

$$2500 - 250 = 50h + h$$

$$51h = 2250$$

$$h = \frac{2250}{251} \approx 8.96m$$

(b) As Antenna angle = Rod angle

$$\Rightarrow \quad \theta_2 = \theta_3$$

$$= \tan^{-1}\left(\frac{1}{50}\right)$$

$$= \tan^{-1}(0.02)$$

$$\theta_2 \approx 1.15°$$

Q8. A person standing 20 meters away from the base of a tree measures the angle of elevation to the top of the tree as 30°.

(a) Find the height of the tree.

(b) If the person walks 10 meters closer to the tree, what will be the new angle of elevation to the top of the tree?

Solution:

(a) Let the height of tree = $AC = hm$

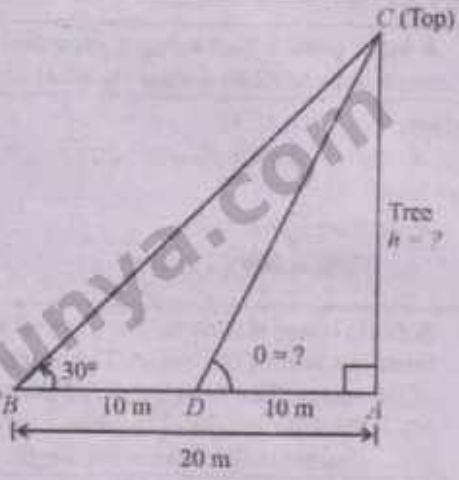
From right ΔABC, we have

$$\tan 30° = \frac{AC}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{20}$$

$$h = \frac{20}{\sqrt{3}}$$

$$\Rightarrow \quad \boxed{h \approx 11.55m}$$



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(b) Let the new angle of elevation $= \theta$

From right ΔACD, we have

$$\tan\theta = \frac{AC}{AD}$$

$$\tan\theta = \frac{h}{10}$$

$$\theta = \tan^{-1}\left(\frac{20}{\sqrt{3}/10}\right)$$

$$\theta = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

$$\Rightarrow \quad \boxed{\theta = 49.11^\circ}$$

Multiple Choice Questions (MCQs)

1. A trigonometric equation is an equation that involves one or more _____ functions and their inverses.

A) algebraic B) trigonometric C) logarithmic D) exponential

2. $\sec x + \tan x = 0$ is:

A) algebraic Equation B) exponential Equation C) radical Equation D) trigonometric Equation

3. An unknown value of the angle that satisfies a given trigonometric equation is called the _____ of the equation.

A) principal value B) period C) solution D) reference angle

4. Solutions of a trigonometric equation lying in $[0, 2\pi)$ are called the _____ solutions.

A) general B) principal C) extraneous D) graphical

5. Once a solution is found within one period, adding or subtracting multiples of the period gives the _____ solution.

A) principal B) reference C) general D) particular

6. The number of solutions of equation $\tan^2\theta = \frac{1}{3}$ are:

A) 1 B) 2 C) 4 D) infinite

7. For $\cos x = \frac{1}{2}$, the principal solutions in $[0, 2\pi]$ are:

A) $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ B) $\frac{\pi}{3}$ and $\frac{4\pi}{3}$

C) $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$ D) $\frac{\pi}{6}$ and $\frac{11\pi}{6}$

8. For $\theta = 120^\circ$, the reference angle is:

A) 30° B) 60° C) 120° D) 240°

9. For $\theta = -30^\circ$, the reference angle is:

A) -30° B) 30° C) 60° D) 150°

10. If $\cos x$ is positive, x lies in:

A) Quadrants I and II B) Quadrants II and III

C) Quadrants I and IV D) Quadrants III and IV

11. The general solution of $\cos x = \frac{1}{2}$ is:

A) $x = \frac{\pi}{3} + 2n\pi$ or $x = \frac{5\pi}{3} + 2n\pi$ B) $x = \frac{\pi}{3} + n\pi$

C) $x = \frac{2\pi}{3} + 2n\pi$ D) $x = \frac{\pi}{6} + 2n\pi$

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12. For $2\sec^2 x - \frac{8}{3} = 0$, when $\cos x = \frac{\sqrt{3}}{2}$, the reference angle is:

A) $\frac{\pi}{3}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{6}$ D) $\frac{\pi}{2}$

13. In solving $\sqrt{3}\cot x - \csc x - 1 = 0$, division by $\sin x$ requires:

A) $x = n\pi$ B) $x \neq n\pi$

C) $x = \frac{\pi}{2} + n\pi$ D) $x = \frac{n\pi}{2}$

14. Squaring both sides while solving a trigonometric equation can introduce:

A) periods B) reference angles

C) extraneous roots D) principal solutions

15. Graphical solution of $f(x) = g(x)$ involves plotting:

A) only $f(x)$ B) only $g(x)$

C) both $y = f(x)$ and $y = g(x)$ D) neither function

16. A point where the graphs $y = f(x)$ and $y = g(x)$ intersect represents:

A) a period B) a solution of $f(x) = g(x)$

C) a reference angle only D) an asymptote

17. For $\sin x = \frac{x}{2}$ over $[-\pi, \pi]$, the graphs intersect at:

A) one point B) two points

C) three points D) four points

18. The solution of $\operatorname{cosec}\theta = 2$ in the interval $[0, 2\pi]$ is:

A) $\frac{\pi}{3}$ and $\frac{\pi}{4}$ B) $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

C) $\frac{\pi}{2}$ and $\frac{2\pi}{3}$ D) $\frac{\pi}{4}$ and $\frac{3\pi}{4}$

19. For $\cos x = \frac{x}{8}$, the range $-1 \leq \cos x \leq 1$ gives:

A) $-8 \leq x \leq 8$ B) $-1 \leq x \leq 1$

C) $0 \leq x \leq 8$ D) $-4 \leq x \leq 4$

20. The general solution set of the equation $1 + \cos x = 0$ is:

A) $\{2n\pi\}\ ; n \in Z$ B) $\{n\pi\}\ ; n \in Z$

C) $\{\pi + 2n\pi\}\ ; n \in Z$ D) $\left\{\frac{\pi}{2} + 2n\pi\right\}\ ; n \in Z$

ANSWER KEY

1.	B	2.	D	3.	C	4.	B	5.	C	6.	D	7.	A	8.	B	9.	B	10.	C
11.	A	12.	C	13.	B	14.	C	15.	C	16.	B	17.	C	18.	B	19.	A	20.	C

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