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UNIT 09

Inverse Trigonometric Functions and Their Graphs

INTRODUCTION

- In earlier classes, we studied trigonometric functions and their real-life applications.
- Now, we extend that knowledge by exploring inverse trigonometric functions which help us to find angles when function values are known. This is essential in solving equations and modeling periodic behavior in calculus of inverse trigonometric functions.
- In this unit, we will define inverse trigonometric functions, their domains and ranges, examine their graphs within the given interval and understand their key properties and uses.

Content Outline

- Inverse Trigonometric Functions and their Graph
  - Inverse of Sine Function
  - Inverse of Cosine Function
  - Inverse of Tangent Function
  - Inverse of Cosecant Function
  - Inverse of Secant Function
  - Inverse of Cotangent Function

INVERSE TRIGONOMETRIC FUNCTIONS AND THEIR GRAPH

- We have studied about inverse of a function. We learned that only a one-to-one and onto function will have an inverse. If a function is not one-to-one, it may be possible to restrict its domain to make it one-to-one so that its inverse can be found.

Recall!

Let f be a bijective function from a set X to a set Y. The inverse function of f, written as f^{-1}, is a bijective function from a set Y back to set X such that $y \in Y$

$$x = f^{-1}(y) \quad \Leftrightarrow \quad y = f(x)$$

Here domain of f^{-1} = range of f and range of f^{-1} = domain of f. This mean that f^{-1} reverses the correspondence of f and their composition gives:

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x, \forall x \in X$$

$$\text{and } (f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y, \forall y \in Y$$

- The equation $\sin 90° = 1$ clearly shows that $90°$ is the angle whose sine is 1. Sometimes, we know the value of sine, cosine or tangent, but not the angle itself. In such cases, we use the inverse trigonometric functions to find the angle.

  For example, the inverse of $\sin 90° = 1$ can be written as $90° = \sin^{-1}(1)$ or $90° = \arcsin(1)$, and read it as $90°$ is equal to sine inverse of 1 or arc sin of 1. Generally, the equation $x = \sin y$ gives value of x for a given angle of y. To find the angle y when x is known, we use the inverse function, that is,

$$y = \sin^{-1} x$$

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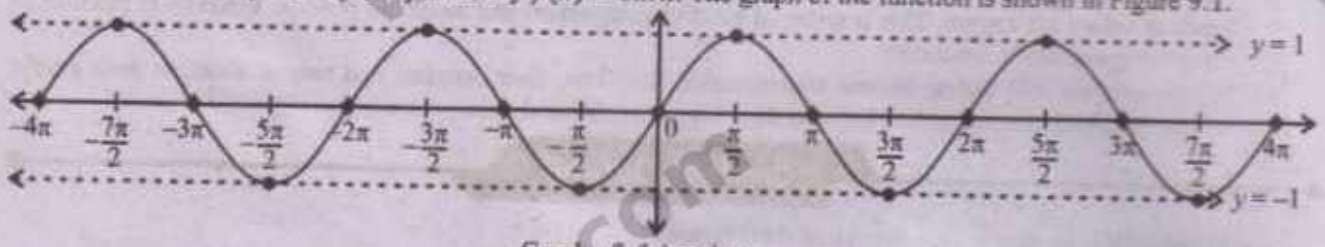
- This means y is an angle whose sine is x and the symbol $\sin^{-1}x$ is read as sine inverse of x or arcsine of x. In other words, $y = \sin^{-1} x$ iff $x = \sin y$. Similarly, we can write the other equations as $y = \cos^{-1} x$ iff $x = \cos y$ and $y = \tan^{-1} x$ iff $x = \tan y$, etc.
- We know that trigonometric functions sine, cosine and tangent are periodic, meaning they repeat their values at regular intervals. Due to this periodicity, trigonometric functions are generally not bijective (they are not both one-to-one and within their entire domain). As a result, their inverses do not naturally exist without restrictions.
- However, by carefully restricting their domains and codomains, we can make these functions bijective that allows us to define their inverses, known as inverse trigonometric functions. Next, we discuss these functions one by one in details.

Inverse of Sine Function

Consider the function $f: \mathbb{R} \to [-1,1]$ define by $f(x) = \sin x$. The graph of the function is shown in Figure 9.1.



Graph of $f(x) = \sin x$

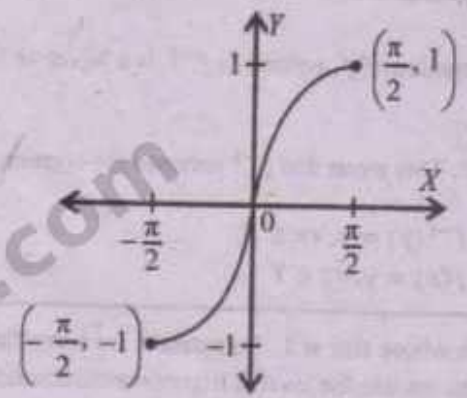
Figure 9.1

Clearly the graph of sine function is not one-to-one because it touches the line $y = 1$ and $y = -1$ at many points due to the domain of sine function which is the set of all real numbers. However, if we restrict the domain of $f(x) = \sin x$ to the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then the function with restricted domain i.e., $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$ becomes one-one and onto (bijective) that is invertible and is called **principal sine function**. Then the inverse sine function is defined by

$$y = \sin^{-1} x \text{ iff } x = \sin y$$

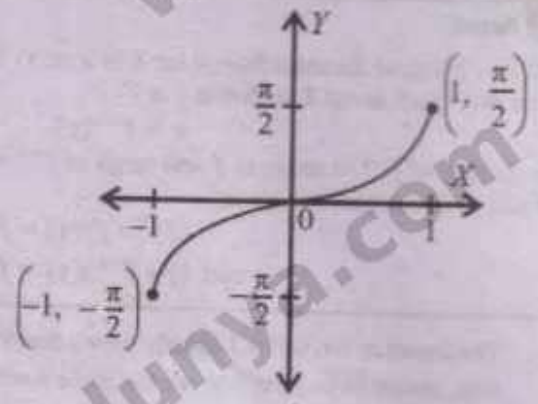
where the domain of $\sin^{-1} x$ is $[-1,1]$ and range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The graph of the function $\sin: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$ is shown in Figure 9.2(a). and graph of $\sin^{-1}: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is shown in Figure 9.2(b).



Graph of $y = \sin x$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

Figure 9.2(a)



Graph of $y = \sin^{-1} x$ from -1 to 1

Figure 9.2(b)

The graph of $\sin^{-1} x$ is also obtain by reflecting the graph of principal sine function $y = \sin x$ along the line $y = x$ as a mirror image shown in Figure 9.3. Observe that the graph of $y = \sin x$ is symmetric about the line $y = x$.

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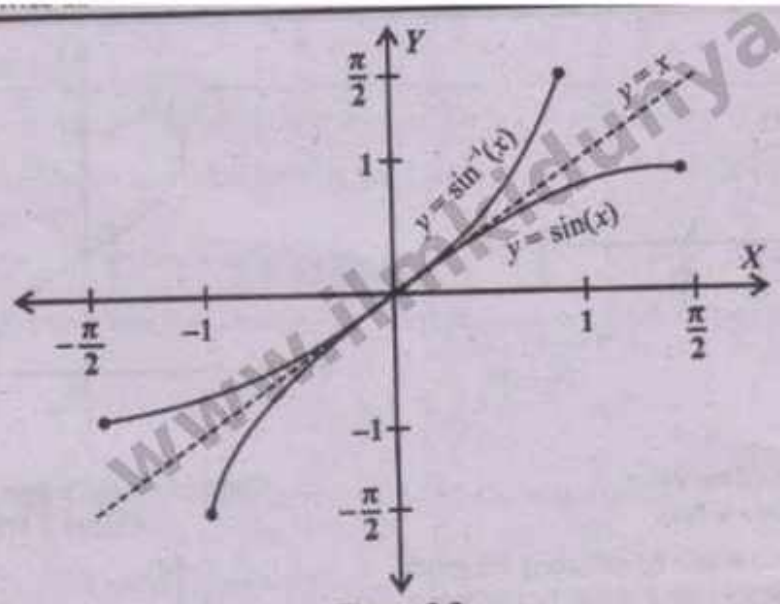


Figure 9.3

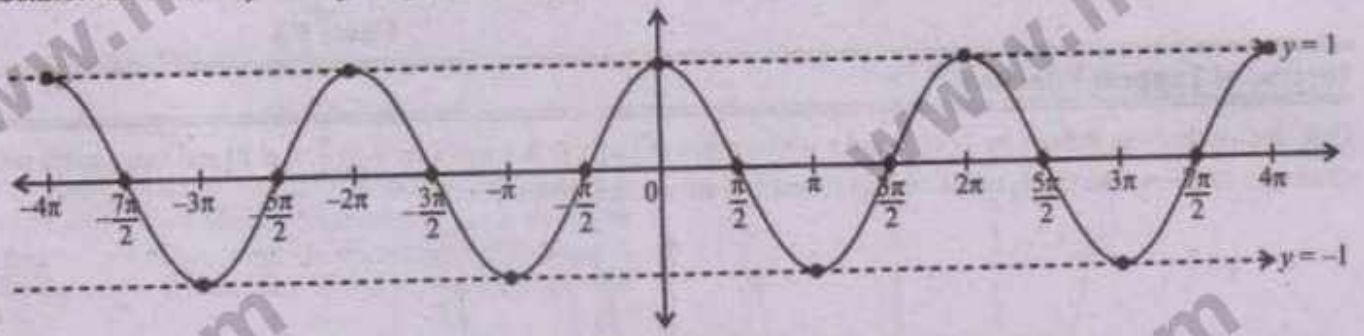
The sine function can be restricted to any one of the infinitely many intervals $\left[-\frac{3\pi}{2},-\frac{\pi}{2}\right],\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\left[\frac{\pi}{2},\frac{3\pi}{2}\right],\left[\frac{3\pi}{2},\frac{5\pi}{2}\right]$ or generally $\left[n\pi-\frac{\pi}{2},n\pi+\frac{\pi}{2}\right]$, $n \in Z$ with range $[-1,1]$. The length of each interval π and the sine function is one-to-on in each interval. However, unless otherwise stated, $y = \sin^{-1}x$ is always considered to be a function with the domain $[-1,1]$ and principal range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Note

$\sin^{-1}x \neq (\sin x)^{-1}$ or $\frac{1}{\sin x}$

Inverse of Cosine Function

Consider the function $f: \mathbb{R} \to [-1,1]$ define by $f(x) = \cos x$. The graph of the function is shown in Figure 9.4.



Graph of $f(x) = \cos x$

Figure 9.4

Like the graph of sine function, the graph of cosine function is not bijective because again it fails the horizontal line test. It touches the line $y = 1$ and $y = -1$ at multiple points. However, if we carefully restrict the domain of $f(x) = \cos x$ to the closed interval $[0,\pi]$, then the function with restricted domain i.e., $f: [0,\pi] \to [-1,1]$ becomes one-one and onto that is invertible and is called **principal cosine function**. The inverse cosine function is defined by

$$y = \cos^{-1}x \text{ iff } x = \cos y$$

The domain of $\cos^{-1}x$ is $[-1,1]$ and range of $\cos^{-1}x$ is $[0,\pi]$.
The graph of the function cos: $[0,\pi] \to [-1,1]$ is shown in Figure 9.5(a) and graph of $\cos^{-1}: [-1,1] \to [0,\pi]$ is shown in Figure 9.5(b).

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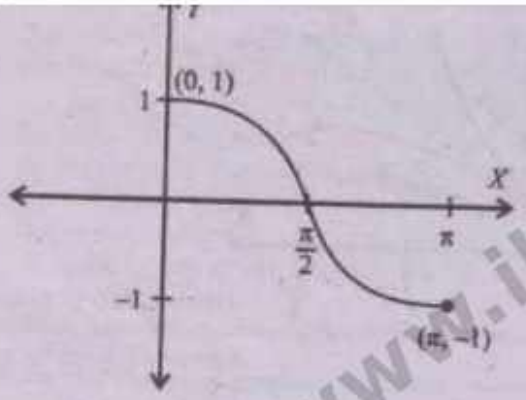
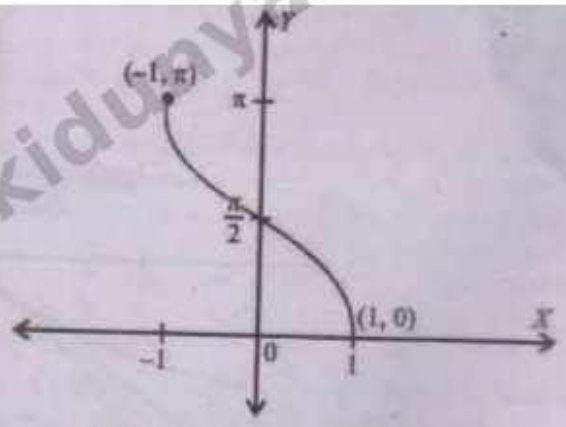


Graph of $y = \cos x$ from 0 to π

Figure 9.5(a)



Graph of $y = \cos^{-1} x$ from -1 to 1

Figure 9.5(b)

The graph of $\cos^{-1} x$ is also obtain by reflecting the graph of principal cosine function $y = \cos x$ along the line $y = x$ as a mirror image shown in Figure 9.6. Observe that the graph of $y = \cos x$ is symmetric about the line $y = x$.

Like, sine function, the cosine function with range $[-1,1]$ can be restricted to any one of the infinitely many intervals $[-2\pi, -\pi], [-\pi, 0], [0, \pi], [\pi, 2\pi]$ or generally $[n\pi, n\pi + \pi], n \in \mathbb{Z}$ to make the cosine function one-to-one. If no interval is specified, then $\cos^{-1} x$ is considered to be a function with domain $[-1,1]$ and principal range $[0, \pi]$.

Note

$\cos^{-1} x \neq (\cos x)^{-1}$ or $\frac{1}{\cos x}$

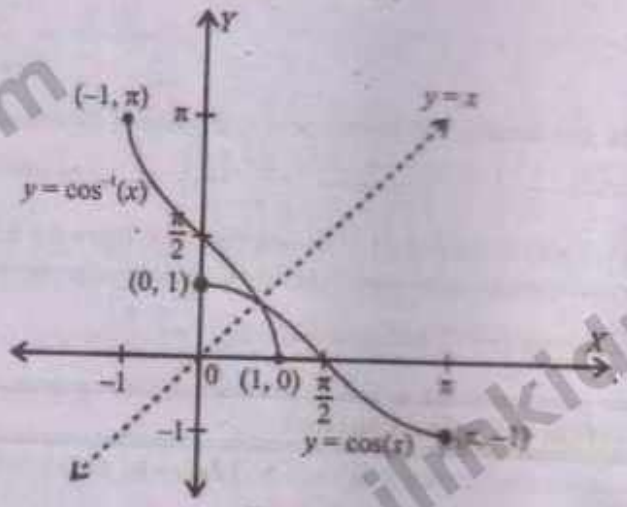
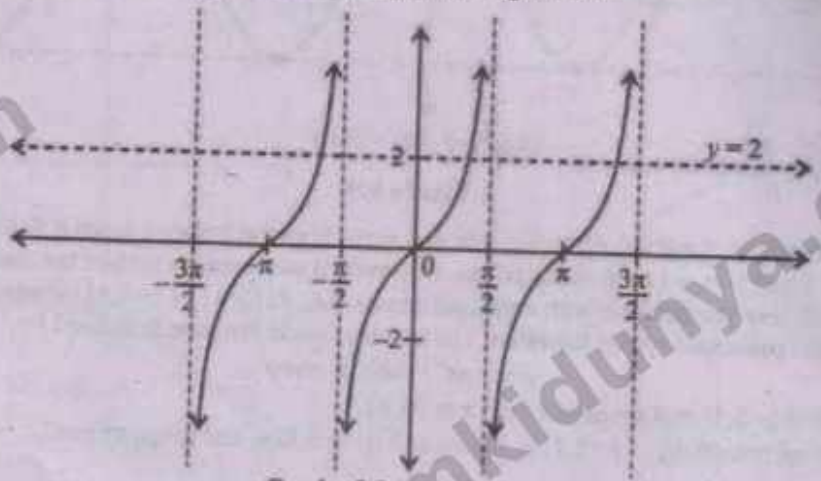


Figure 9.6

Inverse of Tangent Function

Consider the function defined by $f(x) = \tan x$ with domain as $\left\{x \mid x \in \mathbb{R} \wedge x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$ and range as the set of all real numbers. The graph of the tangent function is shown in Figure 9.7.



Graph of $f(x) = \tan x$

Figure 9.7

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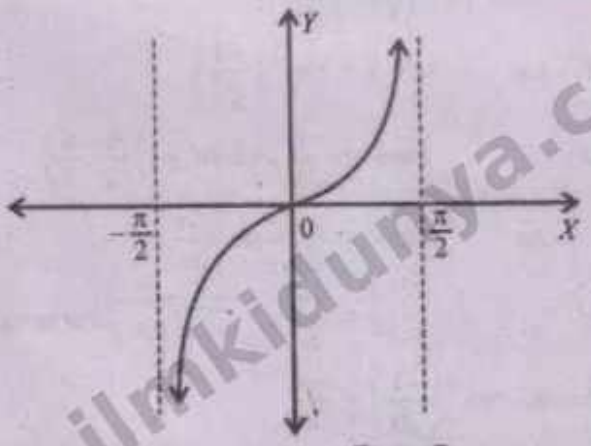
Clearly the graph of tangent function is not bijective because the line $y = 2$ intersect the graph at multiple points. However, if we restrict the domain of $f(x) = \tan x$ to the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the function with restricted domain i.e. $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ becomes one-one and onto that is invertible. This function with domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and range $(-\infty, \infty)$ is called **principal tangent function**. The inverse tangent function is defined by $y = \tan^{-1} x$ iff $x = \tan y$.

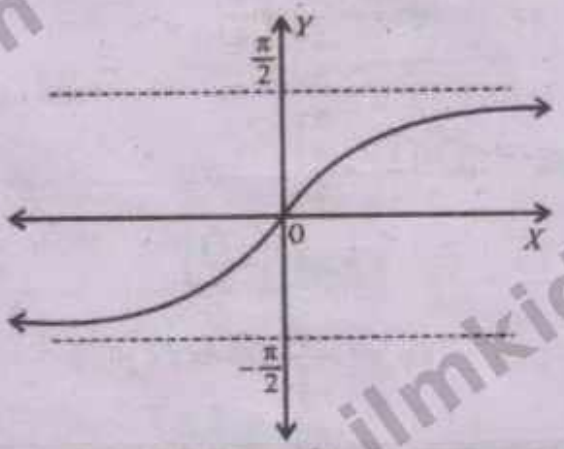
The domain of $\tan^{-1} x$ is $(-\infty, \infty)$ and range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The graph of the function $\tan: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is shown in Figure 9.8(a) and graph of $\tan^{-1}: (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is shown in Figure 9.8(b).



Graph of $y = \tan x$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

Figure 9.8(a)



Graph of $y = \tan^{-1} x$

Figure 9.8(b)

The graph of $\tan^{-1} x$ is also obtain by reflecting the graph of principal tangent function $y = \tan x$ along the line $y = x$ as a mirror image shown in Figure 9.9. Observe that the graph of $y = \tan x$ is symmetric about the line $y = x$.

Like sine and cosine functions, the domain of tangent function can be restricted to any one of the infinitely many intervals $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ or generally $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right), n \in \mathbb{Z}$ with range $\mathbb{R}$ to make the tangent function one-to-one. If no interval is specified then $\tan^{-1} x$ is a function with domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

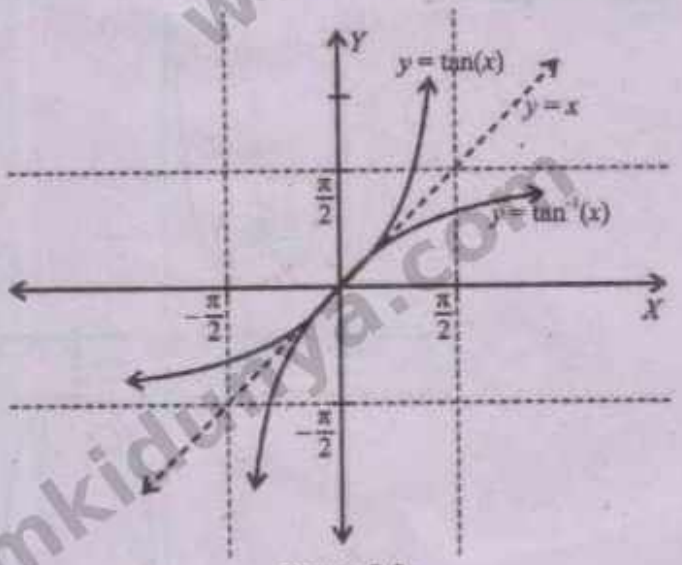


Figure 9.9

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Like, sine function, the cosine function with range $[-1,1]$ can be restricted to any one of the infinitely many intervals $[-2\pi,-\pi]$, $[-\pi,0]$, $[0,\pi]$, $[\pi,2\pi]$ or generally $[n\pi,n\pi+\pi], n\in\mathbb{Z}$ to make the cosine function one-to-one. If no interval is specified, then $\cos^{-1}x$ is considered to be a function with domain $[-1,1]$ and principal range $[0,\pi]$.

Remember!

$\tan^{-1}x \neq (\tan x)^{-1}$, or $\dfrac{1}{\tan x}$

Example 1: Find the principal value of: (i) $\sin^{-1}\left(\frac{1}{2}\right)$ (ii) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution:

(i) Let $y=\sin^{-1}\left(\frac{1}{2}\right)$ ...(1)

$\Rightarrow \sin y=\frac{1}{2}$, where $y\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

As $\sin y=\frac{1}{2}=\sin\frac{\pi}{6}$

$\Rightarrow y=\frac{\pi}{6}$ Put in eq. (1)

Hence, $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

(ii) Let $y=\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ ...(1)

$\Rightarrow \cos y=\frac{-\sqrt{3}}{2}$, where $y\in[0,\pi]$

As $\cos y=\frac{-\sqrt{3}}{2}=\cos\frac{5\pi}{6}$

$\Rightarrow y=\frac{5\pi}{6}$ Put in eq. (1)

Hence, $\sin^{-1}\left(\frac{1}{2}\right)=\frac{5\pi}{2}$

(iii) Let $y=\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ ...(1)

$\Rightarrow \tan y=\frac{-1}{\sqrt{3}}$, where $y\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

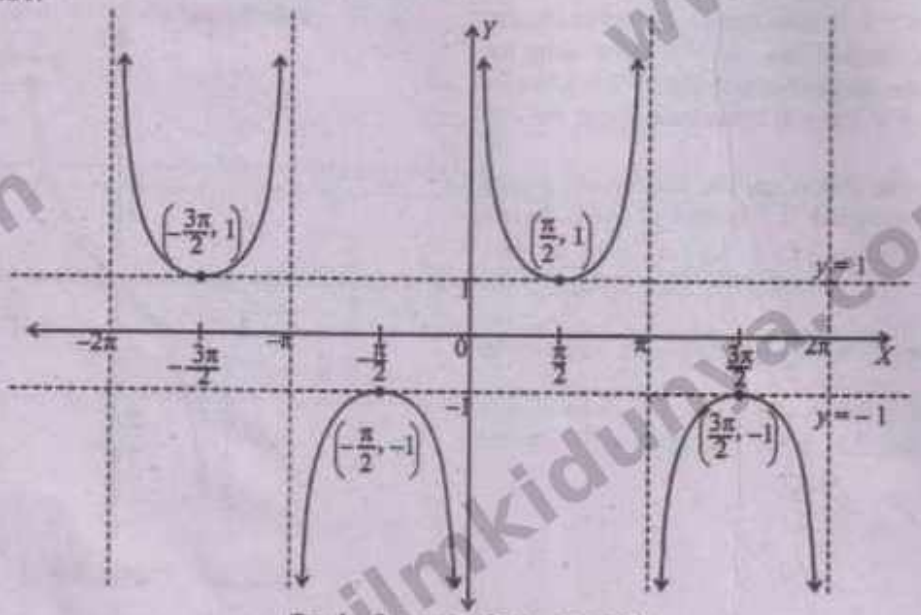
As $\tan y=\frac{-1}{\sqrt{3}}=\tan\left(\frac{-\pi}{6}\right)$

$\Rightarrow y=\frac{-\pi}{6}$ Put in eq. (1)

Hence, $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)=\frac{-\pi}{6}$

Inverse of Cosecant Function

The graph of the function $f(x)=\csc x$ with domain $\{x \mid x\in\mathbb{R}\wedge x\neq n\pi, n\in\mathbb{Z}\}$ and range is the set $\mathbb{R}-(-1,1)$ is shown in Figure 9.10.



Graph of $y=\csc x$ from -2π to 2π

Figure 9.10

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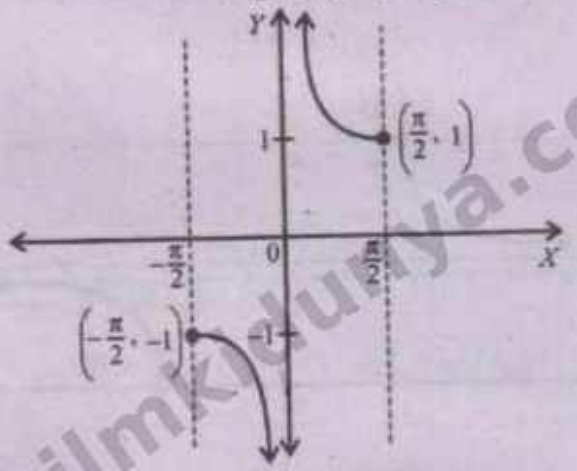
Clearly from the graph it follows that cosecant function is not bijective because the line $y = 1$ and $y = -1$ touches the graph at many points. However, if we restrict the domain of $f(x) = \csc x$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, then the function with restricted domain i.e. $f: \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \to (-\infty, 1) \cup (1, \infty)$ becomes one-one and onto that is invertible and is called **principal cosecant function**. The inverse cosecant function is defined by

$$y = \csc^{-1} x \text{ iff } x = \csc y$$

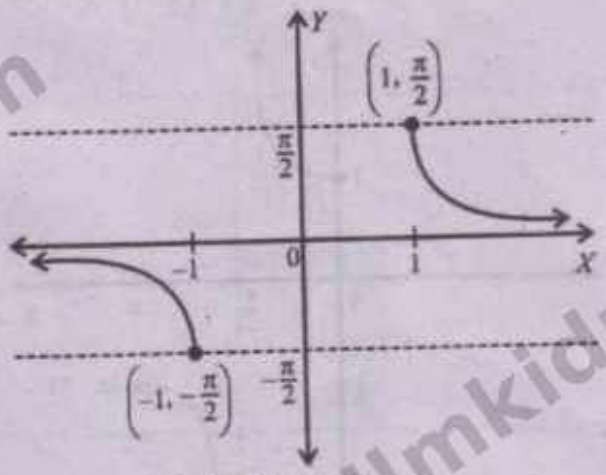
The domain of $\csc^{-1} x = \mathbb{R} - (-1, 1)$ and range of $\csc^{-1} x = \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

The graph of the function $\csc: \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \to (-\infty, -1) \cup (1, \infty)$ is shown in Figure 9.11(a) and graph of $\csc^{-1}: (-\infty, -1) \cup (1, \infty) \to \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ is shown in Figure 9.11(b).



Graph of $y = \csc x$ from -2π to 2π

Figure 9.11(a)

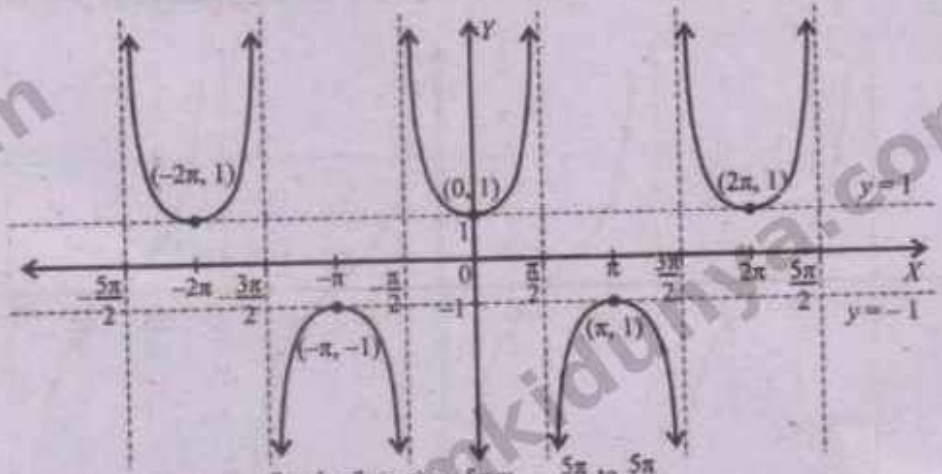


Graph of $y = \csc^{-1} x$

Figure 9.11(b)

Inverse of Secant Function

The graph of the function $f(x) = \sec x$ with domain $\left\{x \mid x \in \mathbb{R} \wedge x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$ and range $\mathbb{R} - (1,1)$ is shown in Figure 9.12.



Graph of $y = \sec x$ from $-\frac{5\pi}{2}$ to $\frac{5\pi}{2}$

Figure 9.12

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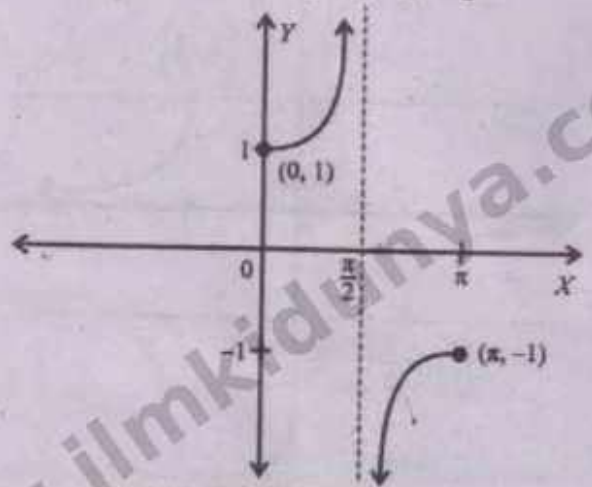
Clearly from the graph it follows that secant function is not bijective because the line $y = 1$ and $y = -1$ touches the graph at many points. However, if we restrict the domain of $f(x) = \sec x$ to the interval $[0, \pi] - \left\{\frac{\pi}{2}\right\}$, then the function with restricted domain i.e. $f: \left[0, \frac{\pi}{2}, 0\right) \cup \left(\frac{\pi}{2}, \pi\right] \to (-\infty, -1] \cup [1, \infty)$ becomes one-one and onto (bijective) that is invertible and is called **principal secant function**. The inverse secant function is defined by

$$y = \sec^{-1} x \text{ if } x = \sec y$$

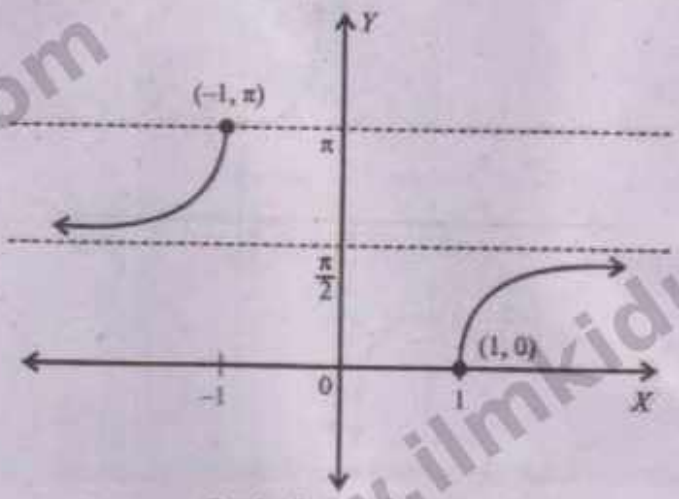
The domain of $\sec^{-1} x$ is $\mathbb{R} - (-1, 1)$ and range of $\sec^{-1} x$ is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

The graph of the function $\sec: \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \to (-\infty, -1] \cup [1, \infty)$ is shown in Figure 9.13(a) and graph of $\sec^{-1}: (-\infty, -1] \cup [1, \infty) \to \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ is shown in Figure 9.13(b).



Graph of $y = \sec x$ from 0 to π

Figure 9.13(a)

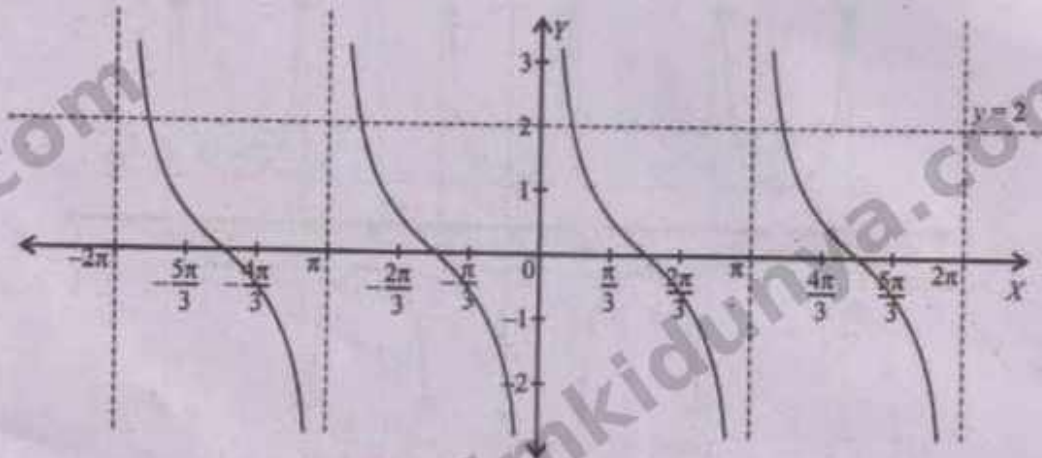


Graph of $y = \sec^{-1} x$

Figure 9.13(b)

Inverse of Cotangent Function

The graph of the function $f(x) = \cot x$ with domain $\{x \mid x \in \mathbb{R} \wedge x \neq n\pi, n \in \mathbb{Z}\}$ and range $(-\infty, \infty)$ is shown in Figure 9.14.



Graph of $y = \cot x$ from -2π to 2π

Figure 9.14

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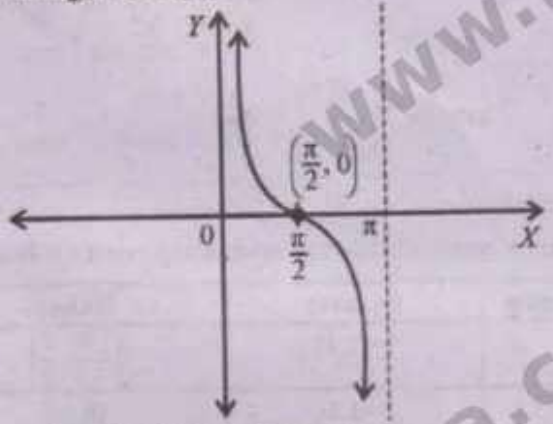
Clearly from the graph it follows that cotangent function is not bijective because the line $y = 2$ intersects the graph at many points. So, we restrict the domain of $f(x) = \cot x$ to the interval $(0, \pi)$, then the function with restricted domain i.e., $f: (0, \pi) \to (-\infty, \infty)$ becomes one-one and onto that is invertible and is called **principal cotangent function**. The inverse $\cot x$ function is defined by

$$y = \cot^{-1} x \text{ iff } x = \cot y$$

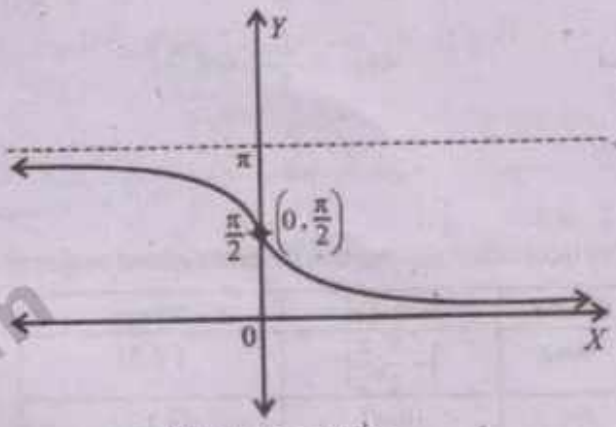
The domain of $\cot^{-1} x$ is $(-\infty, \infty)$ and range of $\cot^{-1} x$ is $(0, \pi)$.

The graph of the function $\cot: (0, \pi) \to (-\infty, \infty)$ is shown in Figure 15(a) and graph of $\cot^{-1}: (-\infty, \infty) \to (0, \pi)$ is shown in Figure 9.15(b).



Graph of $y = \cot x$ from 0 to π

Figure 9.15(a)



Graph of $y = \cot^{-1} x$

Figure 9.15(b)

Example 2: Find value of:

(i) $\sec[\tan^{-1}(-\sqrt{3})]$ **(ii)** $\cot\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$ **(iii)** $\csc\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$

Solution:

(i) $\sec[\tan^{-1}(-\sqrt{3})] = ?$

Let $y = \tan^{-1}(-\sqrt{3})$ ...(1)

$\Rightarrow$ $\tan y = -\sqrt{3}$, where $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

As $\tan y = -\sqrt{3} = \tan\left(-\frac{\pi}{3}\right)$

$\Rightarrow$ $y = -\frac{\pi}{3}$ Put in eq. (1)

$\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

Now,

$\sec[\tan^{-1}(-\sqrt{3})] = \sec\left(-\frac{\pi}{3}\right)$

$\sec[\tan^{-1}(-\sqrt{3})] = \sec\frac{\pi}{3}$ $\because \sec(-\theta) = \sec\theta$

$\sec[\tan^{-1}(-\sqrt{3})] = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}}$

$\sec[\tan^{-1}(-\sqrt{3})] = 2$

(ii) $\cot\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = ?$

Let $y = \cos^{-1}\left(-\frac{1}{2}\right)$ ...(1)

$\Rightarrow$ $\cos y = -\frac{1}{2}$, where $y \in [0, \pi]$

As $\cos y = -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow$ $y = \frac{2\pi}{3}$ Put in eq. (1)

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

Now,

$\cot\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = \cot\left(\frac{2\pi}{3}\right)$

$= \frac{1}{\tan\left(\frac{2\pi}{3}\right)} = \frac{1}{-\sqrt{3}}$

$\cot\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = -\frac{1}{\sqrt{3}}$

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(iii) $\csc\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] = ?$

Let $y = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ ...(1)

$\Rightarrow \sin y = \frac{\sqrt{3}}{2}$, where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

As $\sin y = \frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right)$

$\Rightarrow y = \frac{\pi}{3}$ Put in eq. (1)

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

Now,

$\csc\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] = \csc\left(\frac{\pi}{3}\right)$

$= \frac{1}{\sin\frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2}}$

$\csc\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{2}{\sqrt{3}}$

The table below summarizes the domains and ranges of the principal trigonometric and inverse trigonometric functions.

Function	Domain	Range
$\sin x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1,1]$
$\cos x$	$[0,\pi]$	$[-1,1]$
$\tan x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(-\infty, \infty)$
$\csc x$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$	$(-\infty, -1] \cup [1, \infty)$
$\sec x$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$	$(-\infty, -1] \cup [1, \infty)$
$\cot x$	$(0,\pi)$	$(-\infty, \infty)$

Function	Domain	Range
$\sin^{-1} x$	$[-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\cot^{-1} x$	$(-\infty, \infty)$	$(0,\pi)$

Example 3 Find the domain of the following functions

(i) $f(x) = \cos^{-1}(3x-1)$ (ii) $g(x) = \sin^{-1}\left(\frac{x^2-5}{4}\right)$ (iii) $h(x) = \sec^{-1}(\ln x)$

Solution:

(i) $f(x) = \cos^{-1}(3x-1)$

For Domain:

$-1 \le 3x-1 \le 1$

$0 \le 3x \le 2$ Add 1 to all sides

$\frac{0}{3} \le \frac{3x}{3} \le \frac{2}{3}$ Divide all sides by 3

$0 \le x \le \frac{2}{3}$

$\Rightarrow$ Domain $= \left[0, \frac{2}{3}\right]$

(ii) $g(x) = \sin^{-1}\left(\frac{x^2-5}{4}\right)$

For Domain:

$-1 \le \frac{x^2-5}{4} \le 1$

$-4 \le x^2 - 5 \le 4$ Multiply all sides by 4

$1 \le x^2 \le 9$ Add 5 to all sides

$\Rightarrow 1 \le x^2$ and $x^2 \le 9$

$\sqrt{x^2} \ge \sqrt{1}$ | $\sqrt{x^2} \le \sqrt{9}$

$|x| \ge 1$ | $|x| \le 3$

$\pm x \ge 1$ | $\Rightarrow -3 \le x \le 3$

$\Rightarrow -x \ge 1$ or $x \ge 1$ | $\Rightarrow x \in [-3,3]$

$x \le -1$

$\Rightarrow x \in (-\infty, -1] \cup [1, \infty)$

By taking intersection of both results

$x \in [-3, -1] \cup [1, 3]$

$\Rightarrow$ Domain $= [-3, -1] \cup [1, 3]$

(iii) $h(x) = \sec^{-1}(\ln x)$

For Domain:

$\ln x \le -1$ or $\ln x \ge 1$

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$\ln x \le -\ln e$

$\ln x \le \ln e^{-1}$

$x \le e^{-1}$

$x \le \frac{1}{e}$

$\ln x \ge \ln e$

$x \ge e$

$\Rightarrow \quad x \in \left(0, \frac{1}{e}\right] \cup [e, \infty)$ $\because \ln x$ is defined for $x \in (0, \infty)$

Domain $= \left(0, \frac{1}{e}\right] \cup [e, \infty)$

Example 4: Prove that:

(i) $\csc^{-1} x = \sin^{-1} \frac{1}{x}, x \in (-\infty, -1] \cup [1, \infty)$ (ii) $\sec^{-1} x = \cos^{-1} \frac{1}{x}, x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\cot^{-1} x = \tan^{-1} \frac{1}{x}, x \in (0, \infty)$

Solution:

(i) Let $y = \csc^{-1} x$ ...(1)

$\Rightarrow \quad \csc y = x$

$\frac{1}{\sin y} = x$

$\sin y = \frac{1}{x}$

$y = \sin^{-1}\left(\frac{1}{x}\right)$ Put in eq. (1)

Hence, $\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

(ii) Let $y = \sec^{-1} x$ ...(1)

$\Rightarrow \quad \sec y = x$

$\frac{1}{\cos y} = x$

$\cos y = \frac{1}{x}$

$y = \cos^{-1}\left(\frac{1}{x}\right)$ Put in eq. (1)

Hence, $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

(iii) Let $y = \cot^{-1} x$ ...(1)

$\Rightarrow \quad \cot y = x$

$\frac{1}{\tan y} = x$

$\tan y = \frac{1}{x}$

$y = \tan^{-1}\left(\frac{1}{x}\right)$ $\because x \in (0, \infty)$ and $y \in \left(0, \frac{\pi}{2}\right)$

Put in eq. (1)

Hence, $\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$

Example 5: Prove that

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, x \in [-1, 1]$ (ii) $\csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}, x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in (-\infty, \infty)$

Solution:

(i) Let $\sin^{-1} x = y$ where $x \in [-1, 1], y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad x = \sin y$

$x = \cos\left(\frac{\pi}{2} - y\right)$ $\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\cos^{-1} x = \frac{\pi}{2} - y$ $\because y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \frac{\pi}{2} - y \in [0, \pi]$

$y + \cos^{-1} x = \frac{\pi}{2}$

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ (Proved)

(ii) Let $\csc^{-1} x = y$

where $x \in (-\infty, -1] \cup [1, \infty), y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\Rightarrow \quad x = \csc y$

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$$x = \sec\left(\frac{\pi}{2} - y\right) \qquad \because \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\sec^{-1} x = \frac{\pi}{2} - y$$

$$\because y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \Rightarrow \frac{\pi}{2} - y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

$$y + \sec^{-1} x = \frac{\pi}{2}$$

$$\csc^{-1} x + \sec^{-1} x = \frac{\pi}{2} \text{ (Proved)}$$

(iii) Let $\tan^{-1} x = y$ where $x \in (-\infty, \infty), y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow \quad x = \tan y$$

$$x = \cot\left(\frac{\pi}{2} - y\right) \qquad \because \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\cot^{-1} x = \frac{\pi}{2} - y$$

$$\because y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \frac{\pi}{2} - y \in (0, \pi)$$

$$y + \cot^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \text{ (Proved)}$$

Example 6: Show that $\cos^{-1}\frac{7}{25} = \csc^{-1}\frac{25}{24}$

Solution:

Let $\alpha = \cos^{-1}\frac{7}{25}$ ...(1)

$$\cos\alpha = \frac{7}{25}$$

Note that: $\cos\alpha = \frac{7}{25} > 0 \Rightarrow \alpha \in \left(0, \frac{\pi}{2}\right)$ (1st Quadrant)

As we know

$$\sin\alpha = +\sqrt{1 - \cos^2\alpha}$$

$$= \sqrt{1 - \left(\frac{7}{25}\right)^2} = \sqrt{\frac{625 - 49}{625}} = \sqrt{\frac{576}{625}}$$

$$\sin\alpha = \frac{24}{25}$$

$$\csc\alpha = \frac{25}{24}$$

$$\Rightarrow \quad \alpha = \csc^{-1}\frac{25}{24}$$ Put in eq. (1)

Hence, $\cos^{-1}\frac{7}{25} = \csc^{-1}\frac{25}{24}$ (Proved)

Example 7: Without using calculator find the value of $\sin\left[\frac{1}{2}\tan^{-1}\frac{4}{3}\right]$.

Solution:

$$\sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right) = ?$$

Let $\alpha = \tan^{-1}\frac{4}{3}$ ...(1)

$$\tan\alpha = \frac{4}{3}$$

Note that: $\tan\alpha = \frac{4}{3} > 0 \Rightarrow \alpha \in \left(0, \frac{\pi}{2}\right)$ (1st Quadrant)

Here $P = 4, B = 3$

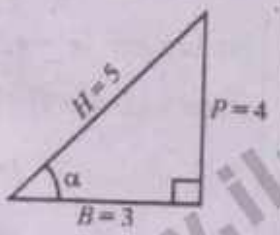
By Pythagoras theorem:

$$H^2 = B^2 + P^2$$

$$H^2 = 3^2 + 4^2$$

$$H^2 = 25 \Rightarrow H = 5$$

$$\Rightarrow \sin\alpha = \frac{4}{5}, \cos\alpha = \frac{3}{5}$$



Now,

$$\sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right) = \sin\frac{\alpha}{2}$$ Using (1)

$$= \sqrt{\frac{1 - \cos\alpha}{2}}$$

$$= \sqrt{\frac{1 - \frac{3}{5}}{2}}$$

$$= \sqrt{\frac{5 - 3}{10}}$$

$$= \sqrt{\frac{2}{10}}$$

$$\sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right) = \frac{1}{\sqrt{5}}$$

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EXERCISE 9.1

Q1. Find the principal value of the following:

(i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution:

Let $\alpha = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ ...(1)

$\Rightarrow \sin\alpha = \frac{\sqrt{3}}{2}$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

As $\sin\alpha = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$

$\Rightarrow \alpha = \frac{\pi}{3}$ Put in eq. (1)

Hence, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

(ii) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution:

Let $\alpha = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ ...(1)

$\Rightarrow \cos\alpha = -\frac{\sqrt{2}}{2}$ where $\alpha \in [0, \pi]$

As $\cos\alpha = -\frac{1}{\sqrt{2}} = \cos\frac{3\pi}{4}$

$\Rightarrow \alpha = \frac{3\pi}{4}$ Put in eq. (1)

Hence, $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

(iii) $\tan^{-1}(-\sqrt{3})$

Solution:

Let $\alpha = \tan^{-1}(-\sqrt{3})$ ...(1)

$\Rightarrow \tan\alpha = -\sqrt{3}$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

As $\tan\alpha = -\sqrt{3} = \tan\left(-\frac{\pi}{3}\right)$

$\Rightarrow \alpha = -\frac{\pi}{3}$ Put in eq. (1)

Hence, $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

(iv) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution:

Let $\alpha = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ ...(1)

$\Rightarrow \sec\alpha = \frac{2}{\sqrt{3}}$ where $\alpha \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\frac{1}{\cos\alpha} = \frac{2}{\sqrt{3}}$

As $\cos\alpha = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$

$\Rightarrow \alpha = \frac{\pi}{6}$ Put in eq. (1)

Hence, $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$

(v) $\csc^{-1}(-\sqrt{2})$

Solution:

Let $\alpha = \csc^{-1}(-\sqrt{2})$ ...(1)

$\Rightarrow \csc\alpha = -\sqrt{2}$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\frac{1}{\sin\alpha} = -\sqrt{2}$

As $\sin\alpha = -\frac{1}{\sqrt{2}} = \sin\left(-\frac{\pi}{4}\right)$

$\Rightarrow \alpha = -\frac{\pi}{4}$ Put in eq. (1)

Hence, $\csc^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$

(vi) $\sin^{-1}\left[\cos\left(\frac{3\pi}{4}\right)\right]$

Solution:

Let $\alpha = \sin^{-1}\left[\cos\left(\frac{3\pi}{4}\right)\right]$ ...(1)

$\Rightarrow \sin\alpha = \cos\left(\frac{3\pi}{4}\right)$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin\alpha = -\frac{1}{\sqrt{2}}$

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As $\sin\alpha = -\frac{1}{\sqrt{2}} = \sin\left(-\frac{\pi}{4}\right)$

$\Rightarrow \quad \alpha = -\frac{\pi}{4}$ Put in eq. (1)

Hence, $\sin^{-1}\left[\cos\left(\frac{3\pi}{4}\right)\right] = -\frac{\pi}{4}$

(vii) $\tan^{-1}\left[\cot\left(\frac{\pi}{6}\right)\right]$

Solution:

Let $\alpha = \tan^{-1}\left[\cot\left(\frac{\pi}{6}\right)\right]$ ...(1)

$\Rightarrow \quad \tan\alpha = \frac{1}{\tan\frac{\pi}{6}}$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\tan\alpha = \frac{1}{\frac{1}{\sqrt{3}}}$

As $\tan\alpha = \sqrt{3} = \tan\frac{\pi}{3}$

$\Rightarrow \quad \alpha = \frac{\pi}{3}$ Put in eq. (1)

Hence, $\tan^{-1}\left[\cot\left(\frac{\pi}{6}\right)\right] = \frac{\pi}{3}$

(viii) $\sec^{-1}\left[2\cos\left(\frac{3\pi}{4}\right)\right]$

Solution:

Let $\alpha = \sec^{-1}\left[2\cos\left(\frac{3\pi}{4}\right)\right]$ ...(1)

$\Rightarrow \quad \sec\alpha = 2\cos\left(\frac{3\pi}{4}\right)$ where $\alpha \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$

$\sec\alpha = 2\left(\frac{-1}{\sqrt{2}}\right)$

$\frac{1}{\cos\alpha} = -\sqrt{2}$

As $\cos\alpha = \frac{-1}{\sqrt{2}} = \cos\frac{3\pi}{4}$

$\Rightarrow \quad \alpha = \frac{3\pi}{4}$ Put in eq. (1)

Hence, $\sec^{-1}\left[2\cos\left(\frac{3\pi}{4}\right)\right] = \frac{3\pi}{4}$

(ix) $\csc^{-1}\left(\sqrt{2}\cos\frac{\pi}{4}\right)$

Solution:

Let $\alpha = \csc^{-1}\left[\sqrt{2}\cos\frac{\pi}{4}\right]$ ...(1)

$\Rightarrow \quad \csc\alpha = \sqrt{2}\cos\frac{\pi}{4}$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\csc\alpha = \sqrt{2}\cdot\frac{1}{\sqrt{2}}$

$\frac{1}{\sin\alpha} = 1$

As $\sin\alpha = 1 = \sin\frac{\pi}{2}$

$\Rightarrow \quad \alpha = \frac{\pi}{2}$ Put in eq. (1)

Hence, $\csc^{-1}\left[\sqrt{2}\cos\frac{\pi}{4}\right] = \frac{\pi}{2}$

(x) $\tan^{-1}\left[3\cos\left\{3\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}\right]$

Solution:

$\tan^{-1}\left[3\cos\left\{3\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}\right] = ?$

Let $\alpha = \cos^{-1}\frac{\sqrt{3}}{2}$ ...(1)

$\Rightarrow \quad \cos\alpha = \frac{\sqrt{3}}{2}$ where $\alpha \in [0,\pi]$

As $\cos\alpha = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$

$\Rightarrow \quad \alpha = \frac{\pi}{6}$ Put in eq. (1)

$\boxed{\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}}$

Let $\beta = \tan^{-1}\left[3\cos\left\{3\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}\right]$ ...(2)

$\Rightarrow \quad \tan\beta = 3\cos\left\{3\left(\frac{\pi}{6}\right)\right\}$ $\because \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

$\tan\beta = 3\cos\frac{\pi}{2}$

$\tan\beta = 3(0)$

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As $\tan\beta = 0 = \tan(0)$

$\Rightarrow \quad \beta = 0$ Put in eq. (2)

Hence, $\tan^{-1}\left[3\cos\left\{3\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}\right] = 0$

(xi) $\csc^{-1}\left[\frac{4}{3}\cos\left\{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right\}\right]$

Solution:

$$\csc^{-1}\left[\frac{4}{3}\cos\left\{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right\}\right] = ?$$

Let $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ ...(1)

$\Rightarrow \quad \tan\alpha = \frac{1}{\sqrt{3}}$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

As $\tan\alpha = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\Rightarrow \quad \alpha = \frac{\pi}{6}$ Put in eq. (1)

$\boxed{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}}$

Let $\beta = \csc^{-1}\left[\frac{4}{3}\cos\left\{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right\}\right]$ ...(2)

$\csc\beta = \frac{4}{3}\cos\left(\frac{\pi}{6}\right)$ $\because \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\csc\beta = \frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)$

$\frac{1}{\sin\beta} = \frac{2}{\sqrt{3}}$

As $\sin\beta = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$

$\Rightarrow \quad \beta = \frac{\pi}{3}$ Put in eq. (2)

Hence, $\csc^{-1}\left[\frac{4}{3}\cos\left\{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right\}\right] = \frac{\pi}{3}$

Q2. Find the domain of the following:

(i) $f(x) = \sin^{-1}(5x-3)$

Solution:

$$f(x) = \sin^{-1}(5x-3)$$

For Domain:

$$-1 \le 5x-3 \le 1$$

$$-1+3 \le 5x \le 1+3 \quad \text{Add 3 to all sides}$$

$$2 \le 5x \le 4$$

$$\frac{2}{5} \le x \le \frac{4}{5} \quad \text{Divide all sides by 5}$$

$\Rightarrow$ Domain $= \left[\frac{2}{5}, \frac{4}{5}\right]$

(ii) $g(x) = \cos^{-1}\sqrt{x-3}$

Solution:

$$g(x) = \cos^{-1}\sqrt{x-3}$$

For Domain:

$$-1 \le \sqrt{x-3} \le 1$$

since a principal square root is always non-negative (≥ 0), so

$$0 \le \sqrt{x-3} \le 1$$

$$0 \le x-3 \le 1 \quad \text{Square all sides}$$

$$0+3 \le x \le 1+3 \quad \text{Add 3 to all sides}$$

$$3 \le x \le 4$$

$\Rightarrow$ Domain $= [3, 4]$

(iii) $h(x) = \csc^{-1}(x^2-3)$

Solution:

$$h(x) = \csc^{-1}(x^2-3)$$

For Domain:

$x^2-3 \le -1$ or $x^2-3 \ge 1$

$x^2 \le -1+3$ $\quad$ $x^2 \ge 1+3$

$x^2 \le 2$ $\quad$ $x^2 \ge 4$

$\sqrt{x^2} \le \sqrt{2}$ $\quad$ $\sqrt{x^2} \ge \sqrt{4}$

$|x| \le \sqrt{2}$ $\quad$ $|x| \ge 2$

$\Rightarrow -\sqrt{2} \le x \le \sqrt{2}$ $\quad$ $\Rightarrow x \le -2$ or $x \ge 2$

$\Rightarrow x \in [-\sqrt{2}, \sqrt{2}]$ $\quad$ $\Rightarrow x \in (-\infty, -2] \cup [2, \infty)$

By taking union of both results

Domain $= (-\infty, -2] \cup [-\sqrt{2}, \sqrt{2}] \cup [2, \infty)$

(iv) $k(x) = \sin^{-1}(\ln x)$

Solution:

$$k(x) = \sin^{-1}(\ln x)$$

For Domain:

$$-1 \le \ln x \le 1, \text{ where } x \in (0, \infty)$$

$$-\ln e \le \ln x \le \ln e \quad \because \ln e = 1$$

$$\ln e^{-1} \le \ln x \le \ln e$$

$\Rightarrow \quad e^{-1} \le x \le e$

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$$\frac{1}{e} \le x \le e$$

$$\Rightarrow \quad \text{Domain} = \left[\frac{1}{e}, e\right]$$

(v) $p(x) = \tan^{-1}\sqrt{4-x^2}$

Solution:

$$p(x) = \tan^{-1}\sqrt{4-x^2}$$

For Domain:

$$-\infty < \sqrt{4-x^2} < \infty$$

Since a principal square root is always non-negative (≥ 0), so

$$\sqrt{4-x^2} \ge 0$$
$$4 - x^2 \ge 0 \qquad \text{Squaring both sides}$$
$$x^2 \le 4$$
$$\sqrt{x^2} \le \sqrt{4}$$
$$|x| \le 2$$
$$\Rightarrow \quad -2 \le x \le 2$$

$$\text{Domain} = [-2, 2]$$

(vi) $q(x) = \sec^{-1}(x+2)$

Solution:

$$q(x) = \sec^{-1}(x+2)$$

For Domain:

$$x + 2 \le -1 \quad \text{or} \quad x + 2 \ge 1$$
$$x \le -1 - 2 \quad | \quad x \ge 1 - 2$$
$$x \le -3 \quad | \quad x \ge -1$$
$$\Rightarrow \quad x \in (-\infty, -3] \quad | \quad \Rightarrow \quad x \in [-1, \infty)$$

By taking union of both results

$$\text{Domain} = (-\infty, -3] \cup [-1, \infty)$$

(vii) $u(x) = \sec^{-1}\left(\frac{3x-1}{2}\right)$

Solution:

$$u(x) = \sec^{-1}\left(\frac{3x-1}{2}\right)$$

For Domain:

$$\frac{3x-1}{2} \le -1 \quad \text{or} \quad \frac{3x-1}{2} \ge 1$$
$$3x - 1 \le -2 \quad | \quad 3x - 1 \ge 2$$
$$3x \le -2 + 1 \quad | \quad 3x \ge 2 + 1$$
$$3x \le -1 \quad | \quad 3x \ge 3$$
$$x \le -\frac{1}{3} \quad | \quad x \ge \frac{3}{3}$$
$$\Rightarrow \quad x \in \left(-\infty, \frac{-1}{3}\right] \quad | \quad x \ge 1$$
$$\Rightarrow \quad x \in [1, \infty)$$

By taking union of both results

$$\text{Domain} = \left(-\infty, \frac{-1}{3}\right] \cup [1, \infty)$$

(viii) $v(x) = \sin^{-1}(\ln x) + \csc^{-1}(e^x)$

Solution:

$$v(x) = \sin^{-1}(\ln x) + \csc^{-1}(e^x)$$

For Domain of $\sin^{-1}(\ln x)$

$$-1 \le \ln x \le 1$$

where $x \in (0, \infty)$

$$-\ln e \le \ln x \le \ln e$$
$$\ln e^{-1} \le \ln x \le \ln e$$
$$\Rightarrow \quad e^{-1} \le x \le e$$
$$\frac{1}{e} \le x \le e$$
$$\Rightarrow \quad x \in \left[\frac{1}{e}, e\right]$$

For Domain of $\csc^{-1}(e^x)$

$$e^x \le -1 \text{ or } e^x \ge 1$$

Since $e^x > 0 \ \forall x \in R$, so

Take: $e^x \ge 1$

$$e^x \ge e^0$$
$$x \ge 0$$
$$\Rightarrow \quad x \in [0, \infty)$$

By taking intersection of both results

$$\text{Domain} = \left[\frac{1}{e}, e\right] \cap [0, \infty)$$

$$\text{Domain} = \left[\frac{1}{e}, e\right]$$

Q3. Show that:

(i) $\cos^{-1}\frac{8}{17} = \sin^{-1}\frac{15}{17}$

Solution:

Let $\alpha = \cos^{-1}\frac{8}{17}$ (1)

$$\Rightarrow \quad \cos\alpha = \frac{8}{17}$$

Note that: $\cos\alpha = \frac{8}{17} > 0 \Rightarrow \alpha \in \left(0, \frac{\pi}{2}\right)$ (1st Quadrant)

As we know

$$\sin\alpha = +\sqrt{1-\cos^2\alpha}$$

$$\sin\alpha = \sqrt{1-\left(\frac{8}{17}\right)^2} = \sqrt{\frac{289-64}{289}} = \sqrt{\frac{225}{289}}$$

$$\sin\alpha = \frac{15}{17}$$

$$\Rightarrow \quad \alpha = \sin^{-1}\left(\frac{15}{17}\right) \qquad \text{Put in eq. (1)}$$

Hence, $\cos^{-1}\frac{8}{17} = \sin^{-1}\frac{15}{17}$ (Proved)

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(ii) $\csc^{-1}\frac{5}{4}=\cos^{-1}\frac{3}{5}$

Solution:

Let $\alpha=\csc^{-1}\frac{5}{4}$ ...(1)

$\Rightarrow \csc\alpha=\frac{5}{4}\Rightarrow\sin\alpha=\frac{4}{5}$

Note that: $\sin\alpha=\frac{4}{5}>0\Rightarrow\alpha\in\left(0,\frac{\pi}{2}\right)$ (1st Quadrant)

Here $P=4, H=5$

By Pythagoras theorem

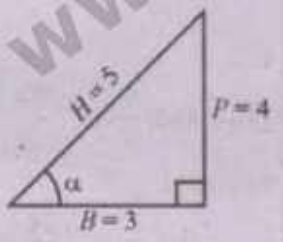
$H^2=B^2+P^2$

$5^2=B^2+4^2$

$B^2=25-16$

$B^2=9$

$B=3$



$\Rightarrow \cos\alpha=\frac{B}{H}=\frac{3}{5}$

$\Rightarrow \alpha=\cos^{-1}\frac{3}{5}$ Put in eq. (1)

Hence $\csc^{-1}\frac{5}{4}=\cos^{-1}\frac{3}{5}$ (Proved)

(iii) $2\tan^{-1}\frac{3}{4}=\tan^{-1}\frac{24}{7}$

Solution:

Let $\alpha=\tan^{-1}\frac{3}{4}$ ...(1)

$\Rightarrow \tan\alpha=\frac{3}{4}$

Note that: $\tan\alpha=\frac{3}{4}>0\Rightarrow\alpha\in\left(0,\frac{\pi}{2}\right)$ (1st Quadrant)

As we know

$$\tan2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\tan2\alpha=\frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}=\frac{\frac{3}{2}}{\frac{16-9}{16}}$$

$$\tan2\alpha=\frac{3}{2}\times\frac{16}{7}$$

$$\tan2\alpha=\frac{24}{7}$$

$$2\alpha=\tan^{-1}\frac{24}{7}$$

$$\boxed{2\tan^{-1}\frac{3}{4}=\tan^{-1}\frac{24}{7}}$$ Using eq. (1)

(iv) $2\sin^{-1}\frac{7}{25}=\sin^{-1}\frac{336}{625}$

Solution:

Let $\alpha=\sin^{-1}\frac{7}{25}$ ...(1)

$\Rightarrow \sin\alpha=\frac{7}{25}$

Note that: $\sin\alpha=\frac{7}{25}>0\Rightarrow\alpha\in\left(0,\frac{\pi}{2}\right)$ (1st Quadrant)

As we know

$$\cos\alpha=+\sqrt{1-\sin^2\alpha}$$

$$\cos\alpha=\sqrt{1-\left(\frac{7}{25}\right)^2}=\sqrt{\frac{625-49}{625}}=\sqrt{\frac{576}{625}}$$

$$\cos\alpha=\frac{24}{25}$$

Using: $\sin2\alpha=2\sin\alpha\cdot\cos\alpha$

$$=2\left(\frac{7}{25}\right)\left(\frac{24}{25}\right)$$

$$\sin2\alpha=\frac{336}{625}$$

$$2\alpha=\sin^{-1}\left(\frac{336}{625}\right)$$

$$\boxed{2\sin^{-1}\frac{7}{25}=\sin^{-1}\frac{336}{625}}$$ Using eq. (1)

Q4. Without using calculator find the value of the following:

(i) $\cos\left(\frac{1}{2}\cot^{-1}\frac{5}{12}\right)$

Solution:

$$\cos\left(\frac{1}{2}\cot^{-1}\frac{5}{12}\right)=?$$

Let $\alpha=\cot^{-1}\frac{5}{12}$ ...(1)

$\Rightarrow \cot\alpha=\frac{5}{12}\Rightarrow\tan\alpha=\frac{12}{5}$

Note that: $\cot\alpha=\frac{5}{12}>0\Rightarrow\alpha\in\left(0,\frac{\pi}{2}\right)$ (1st Quadrant)

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Here $P = 12, B = 5$

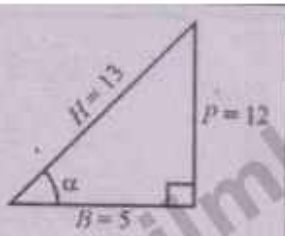
By Pythagoras theorem

$H^2 = B^2 + P^2$

$H^2 = 5^2 + 12^2$

$H^2 = 25 + 144 = 169$

$H^2 = 169 \Rightarrow H = 13$



$$\Rightarrow \quad \cos\alpha = \frac{5}{13}$$

Now, $\cos\left(\frac{1}{2}\cot^{-1}\frac{5}{12}\right) = \cos\frac{\alpha}{2}$ Using (1)

$$= +\sqrt{\frac{1+\cos\alpha}{2}}$$

$$= \sqrt{\frac{1+\frac{5}{13}}{2}} = \sqrt{\frac{13+5}{13(2)}} = \sqrt{\frac{18}{13(2)}}$$

$$= \sqrt{\frac{9}{13}}$$

$$\cos\left(\frac{1}{2}\cot^{-1}\frac{5}{12}\right) = \frac{3}{\sqrt{13}}$$

(ii) $\sec\left(\frac{1}{2}\sin^{-1}\frac{7}{25}\right)$

Solution:

$$\sec\left(\frac{1}{2}\sin^{-1}\frac{7}{25}\right) = ?$$

Let $\alpha = \sin^{-1}\frac{7}{25}$ ...(1)

$$\Rightarrow \quad \sin\alpha = \frac{7}{25}$$

Note that: $\sin\alpha = \frac{7}{25} > 0 \Rightarrow \alpha \in \left(0, \frac{\pi}{2}\right)$ (1st Quadrant)

As we know

$$\cos\alpha = +\sqrt{1-\sin^2\alpha}$$

$$= \sqrt{1-\left(\frac{7}{25}\right)^2} = \sqrt{\frac{625-49}{626}} = \sqrt{\frac{576}{625}}$$

$$\cos\alpha = \frac{24}{25}$$

Now, $\sec\left(\frac{1}{2}\sin^{-1}\frac{7}{25}\right) = \sec\frac{\alpha}{2}$ Using (1)

$$= \frac{1}{\cos\frac{\alpha}{2}} = \frac{1}{+\sqrt{\frac{1-\cos\alpha}{2}}}$$

$$\sec\left(\frac{1}{2}\sin^{-1}\frac{7}{25}\right) = \sqrt{\frac{2}{1+\cos\alpha}} = \sqrt{\frac{2}{1+\frac{24}{25}}}$$

$$= \sqrt{\frac{2}{\frac{25+24}{25}}} = \sqrt{\frac{2(25)}{49}}$$

$$= \sec\left(\frac{1}{2}\sin^{-1}\frac{7}{25}\right)$$

$$= \frac{\sqrt{50}}{7} = \frac{5\sqrt{2}}{7}$$

(iii) $\tan\left(\frac{1}{2}\cot^{-1}\frac{24}{7}\right)$

Solution:

$$\tan\left(\frac{1}{2}\cot^{-1}\frac{24}{7}\right) = ?$$

Let $\alpha = \cot^{-1}\frac{24}{7}$ ...(1)

$$\Rightarrow \quad \cot\alpha = \frac{24}{7} \Rightarrow \tan\alpha = \frac{7}{24}$$

Note that: $\cot\alpha = \frac{24}{7} > 0 \Rightarrow \alpha \in \left(0, \frac{\pi}{2}\right)$ (1st Quadrant)

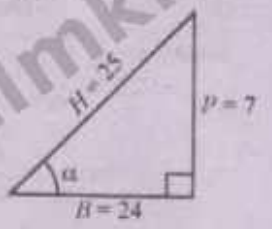
Here $P = 7, B = 24$

By Pythagoras theorem

$H^2 = B^2 + P^2$

$H^2 = 24^2 + 7^2 = 576 + 49$

$H^2 = 625 \Rightarrow H = 25$



$$\Rightarrow \quad \cos\alpha = \frac{24}{25}$$

Now,

$$\tan\left(\frac{1}{2}\cot^{-1}\frac{24}{7}\right) = \tan\frac{\alpha}{2}$$ Using (1)

$$= +\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$= \sqrt{\frac{1-\frac{24}{25}}{1+\frac{24}{25}}}$$

$$= \sqrt{\frac{\frac{25-24}{25}}{\frac{25+24}{25}}} = \sqrt{\frac{1}{49}}$$

$$\tan\left(\frac{1}{2}\cot^{-1}\frac{24}{7}\right) = \frac{1}{7}$$

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(iv) $\tan\left(\frac{1}{2}\sin^{-1}\frac{\sqrt{3}}{2}\right)$

Solution:

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{\sqrt{3}}{2}\right)=?$$

Let $\alpha=\sin^{-1}\frac{\sqrt{3}}{2}$ ...(1)

$\Rightarrow \sin\alpha=\frac{\sqrt{3}}{2}$ where $\alpha\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

As $\sin\alpha=\frac{\sqrt{3}}{2}=\sin\frac{\pi}{3}$

$\Rightarrow \alpha=\frac{\pi}{3}$ Put in eq. (1)

$\Rightarrow \boxed{\sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{3}}$

Now,

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{\sqrt{3}}{2}\right)=\tan\left(\frac{1}{2}\times\frac{\pi}{3}\right) \quad \because \sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{3}$$

$$=\tan\frac{\pi}{6}$$

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{\sqrt{3}}{2}\right)=\frac{1}{\sqrt{3}}$$

(v) $\cot\left(\frac{1}{2}\sec^{-1}\frac{2\sqrt{6}}{3}\right)$

Solution:

$$\cot\left(\frac{1}{2}\sec^{-1}\frac{2\sqrt{6}}{3}\right)=?$$

Let $\alpha=\sec^{-1}\frac{2\sqrt{6}}{3}$ ...(1)

$\Rightarrow \sec\alpha=\frac{2\sqrt{6}}{3}$

$$\cos\alpha=\frac{3}{2\sqrt{6}}=\frac{3}{2\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}=\frac{3\sqrt{6}}{12}$$

$$\cos\alpha=\frac{\sqrt{6}}{4}$$

As $\sec\alpha=\frac{2\sqrt{6}}{3}>0\Rightarrow\alpha\in\left(0,\frac{\pi}{2}\right)$ (1st Quadrant)

Now, $\cot\left(\frac{1}{2}\sec^{-1}\frac{2\sqrt{6}}{3}\right)=\cot\frac{\alpha}{2}$ Using (1)

$$=+\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$$

$$=\sqrt{\frac{1+\frac{\sqrt{6}}{4}}{1-\frac{\sqrt{6}}{4}}}=\sqrt{\frac{\frac{4+\sqrt{6}}{4}}{\frac{4-\sqrt{6}}{4}}}$$

$$=\sqrt{\frac{4+\sqrt{6}}{4-\sqrt{6}}}$$

$$=\sqrt{\frac{4+\sqrt{6}}{4-\sqrt{6}}\times\frac{4+\sqrt{6}}{4+\sqrt{6}}}$$

$$=\sqrt{\frac{(4+\sqrt{6})^2}{(4)^2-(\sqrt{6})^2}}$$

$$=\sqrt{\frac{16+6+8\sqrt{6}}{16-6}}$$

$$=\sqrt{\frac{2(11+4\sqrt{6})}{10}}$$

$$\cot\left(\frac{1}{2}\sec^{-1}\frac{2\sqrt{6}}{3}\right)=\sqrt{\frac{11+4\sqrt{6}}{5}}$$

(vi) $\csc\left(\frac{1}{2}\cot^{-1}\frac{4}{3}\right)$

Solution:

$$\csc\left(\frac{1}{2}\cot^{-1}\frac{4}{3}\right)=?$$

Let $\alpha=\cot^{-1}\frac{4}{3}$ ...(1)

$\Rightarrow \cot\alpha=\frac{4}{3}\Rightarrow\tan\alpha=\frac{3}{4}$

Note that: $\cot\alpha=\frac{4}{3}>0\Rightarrow\alpha\in\left(0,\frac{\pi}{2}\right)$ (1st Quadrant)

Here $P=3, H=4$

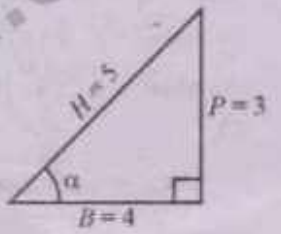
By Pythagoras theorem

$H^2=B^2+P^2$

$H^2=4^2+3^2$

$H^2=16+9$

$H^2=25\Rightarrow H=5$



$\Rightarrow \cos\alpha=\frac{4}{5}$

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Now, $\csc\left(\frac{1}{2}\cot^{-1}\frac{4}{3}\right) = \csc\frac{\alpha}{2}$ Using (1)

$= \frac{1}{\sin\frac{\alpha}{2}} = +\sqrt{\frac{2}{1-\cos\alpha}}$

$= \sqrt{\frac{2}{1-\frac{4}{5}}} = \sqrt{\frac{2}{\frac{5-4}{5}}} = \sqrt{\frac{10}{1}}$

$\csc\left(\frac{1}{2}\cot^{-1}\frac{4}{3}\right) = \sqrt{10}$

SUM AND DIFFERENCE FORMULAS FOR INVERSE TRIGONOMETRIC FUNCTIONS

In this section, we will derive the addition and subtraction formulas for inverse trigonometric functions. These formulas can be derived by applying the fundamental theorem of trigonometry along with algebraic manipulation.

Formula (i): $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ **provided** $\sin^{-1}x + \sin^{-1}y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Proof:

Let $\alpha = \sin^{-1}x$

$\Rightarrow \sin\alpha = x,\ \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

As $\cos\alpha \geq 0\ \forall\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so

$\cos\alpha = +\sqrt{1-\sin^2\alpha}$

$\cos\alpha = \sqrt{1-x^2}$

Let $\beta = \sin^{-1}y$

$\Rightarrow \sin\beta = y,\ \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

As $\cos\beta \geq 0\ \forall\beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so

$\cos\beta = +\sqrt{1-\sin^2\beta}$

$\cos\beta = \sqrt{1-y^2}$

As we know

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\alpha + \beta = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

Thus, $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

Result: Prove that $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

Proof:

As we know

$\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

Put $y = x$, we get

$\sin^{-1}x + \sin^{-1}x = \sin^{-1}(x\sqrt{1-x^2} + x\sqrt{1-x^2})$

$2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

Formula (ii): $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$ provided $\sin^{-1}x - \sin^{-1}y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Proof:

Let $\alpha = \sin^{-1}x$

$\Rightarrow \sin\alpha = x,\ \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

As $\cos\alpha \geq 0\ \forall\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so

$\cos\alpha = +\sqrt{1-\sin^2\alpha}$

$\cos\alpha = \sqrt{1-x^2}$

Let $\beta = \sin^{-1}y$

$\Rightarrow \sin\beta = y,\ \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

As $\cos\beta \geq 0\ \forall\beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so

$\cos\beta = +\sqrt{1-\sin^2\beta}$

$\cos\beta = \sqrt{1-y^2}$

As we know

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\alpha - \beta = \sin^{-1}(x\sqrt{1-y^2} - \sqrt{1-x^2}\,y)$

Thus, $\sin^{-1}x - \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$

Formula (iii): $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{(1-x^2)(1-y^2)}\right)$ **provided** $\cos^{-1}x + \cos^{-1}y \in [0, \pi]$

Proof:

Let $\alpha = \cos^{-1}x$

$\Rightarrow \cos\alpha = x,\ \alpha \in [0, \pi]$

As $\sin\alpha \geq 0\ \forall\alpha \in [0, \pi]$, so

$\sin\alpha = +\sqrt{1-\cos^2\alpha}$

$\sin\alpha = \sqrt{1-x^2}$

Let $\beta = \cos^{-1}y$

$\Rightarrow \cos\beta = y,\ \beta \in [0, \pi]$

As $\sin\beta \geq 0\ \forall\beta \in [0, \pi]$, so

$\sin\beta = +\sqrt{1-\cos^2\beta}$

$\sin\beta = \sqrt{1-y^2}$

As we know

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\alpha + \beta = \cos^{-1}(xy - \sqrt{1-x^2}\cdot\sqrt{1-y^2})$

Thus, $\cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{(1-x^2)(1-y^2)})$

Result: Prove that $2\cos^{-1}x = \cos^{-1}(2x^2-1)$

Proof:

As we know

$\cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{(1-x^2)(1-y^2)})$

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Put $y = x$, we get

$$\cos^{-1}x + \cos^{-1}x = \cos^{-1}(x.x - \sqrt{(1-x^2)(1-x^2)})$$

$$2\cos^{-1}x = \cos^{-1}(x^2 - \sqrt{(1-x^2)^2})$$

$$2\cos^{-1}x = \cos^{-1}(x^2 - 1 + x^2)$$

$$2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$$

Formula (iv): $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{(1-x^2)(1-y^2)}\right)$ **provided** $\cos^{-1}x - \cos^{-1}y \in [0, \pi]$

Proof:

Let $\alpha = \cos^{-1}x$

$\Rightarrow \cos\alpha = x,\ \alpha \in [0, \pi]$

As $\sin\alpha \geq 0\ \forall \alpha \in [0, \pi]$, so

$$\sin\alpha = +\sqrt{1-\cos^2\alpha}$$

$$\sin\alpha = \sqrt{1-x^2}$$

Let $\beta = \cos^{-1}y$

$\Rightarrow \cos\beta = y,\ \beta \in [0, \pi]$

As $\sin\beta \geq 0\ \forall \beta \in [0, \pi]$, so

$$\sin\beta = +\sqrt{1-\cos^2\beta}$$

$$\sin\beta = \sqrt{1-y^2}$$

As we know

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\alpha - \beta = \cos^{-1}(x.y + \sqrt{1-x^2}\sqrt{1-y^2})$$

Thus, $\cos^{-1}x - \cos^{-1}y = \cos^{-1}(xy + \sqrt{(1-x^2)(1-y^2)})$

Formula (v): $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ **provided** $\tan^{-1}x + \tan^{-1}y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Proof:

Let $\alpha = \tan^{-1}x$

$\Rightarrow \tan\alpha = x,\ \alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Let $\beta = \tan^{-1}y$

$\Rightarrow \tan\beta = y,\ \beta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

As we know

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\alpha + \beta = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Thus, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Result: Prove that $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$

Proof:

As we know

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Put $y = x$, we get

$$\tan^{-1}x + \tan^{-1}x = \tan^{-1}\left(\frac{x+x}{1-x.x}\right)$$

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Formula (vi): $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ **provided** $\tan^{-1}x - \tan^{-1}y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Proof:

Let $\alpha = \tan^{-1}x$

$\Rightarrow \tan\alpha = x,\ \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Let $\beta = \tan^{-1}y$

$\Rightarrow \tan\beta = y,\ \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

As we know

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\alpha - \beta = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

Thus, $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Remember:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

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Example 8: Prove that $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}, x \in [0, 1]$.

Solution:

Let $\alpha = \cos^{-1}x$ ...(1)

$\Rightarrow$ $\cos\alpha = x$ where $x \in [0,1] \Rightarrow \alpha \in \left[0, \frac{\pi}{2}\right]$

As $\sin\alpha \geq 0 \quad \forall \alpha \in \left[0, \frac{\pi}{2}\right]$, so

$= +\sqrt{1-\cos^2\alpha}$

$= \sqrt{1-x^2}$

$\Rightarrow$ $\alpha = \sin^{-1}\sqrt{1-x^2}$ Put in eq. (1)

Hence, $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$

Example 9: Prove that $\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{12}{13}\right)=\frac{\pi}{2}$.

Solution:

$$\text{L.H.S} = \sin^{-1}\frac{5}{13}+\sin^{-1}\frac{12}{13}$$

Using formula: $\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$

$$\text{L.H.S} = \sin^{-1}\left(\frac{5}{13}\sqrt{1-\left(\frac{12}{13}\right)^2}+\frac{12}{13}\sqrt{1-\left(\frac{5}{13}\right)^2}\right)$$

$$= \sin^{-1}\left(\frac{5}{13}\sqrt{1-\frac{144}{169}}+\frac{12}{13}\sqrt{1-\frac{25}{169}}\right) = \sin^{-1}\left(\frac{5}{13}\sqrt{\frac{169-144}{169}}+\frac{12}{13}\sqrt{\frac{169-25}{169}}\right)$$

$$= \sin^{-1}\left(\frac{5}{13}\sqrt{\frac{25}{169}}+\frac{12}{13}\sqrt{\frac{144}{169}}\right) = \sin^{-1}\left(\frac{5}{13}\cdot\frac{5}{13}+\frac{12}{13}\cdot\frac{12}{13}\right) = \sin^{-1}\left(\frac{25}{169}+\frac{144}{169}\right) = \sin^{-1}\left(\frac{169}{169}\right)$$

$$= \sin^{-1}(1)$$

$$= \frac{\pi}{2} = \text{R.H.S (Proved)}$$

Example 10: Prove that $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)=\frac{\pi}{4}$.

Solution:

$$\text{L.H.S} = \tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\frac{x}{y}\cdot\frac{x-y}{x+y}}\right) = \tan^{-1}\left(\frac{\frac{x^2+xy-yx+y^2}{y(x+y)}}{\frac{yx+y^2+x^2-xy}{y(x+y)}}\right) = \tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right) = \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S}$$

Thus, $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)=\frac{\pi}{4}$

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Example 11: Solve the equation $\tan^{-1}\left(\frac{x-1}{x+1}\right)+\tan^{-1}\left(\frac{2x}{x+1}\right)=\frac{\pi}{4}$.

Solution:

$$\tan^{-1}\left(\frac{x-1}{x+1}\right)+\tan^{-1}\left(\frac{2x}{x+1}\right)=\frac{\pi}{4} \quad ...(1)$$

Using formula: $\tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$$\tan^{-1}\left(\frac{\frac{x-1}{x+1}+\frac{2x}{x+1}}{1-\frac{x-1}{x+1}\cdot\frac{2x}{x+1}}\right)=\frac{\pi}{4}$$

$$\frac{\frac{x-1+2x}{x+1}}{\frac{(x+1)^2-2x(x-1)}{(x+1)^2}}=\tan\frac{\pi}{4}$$

$$\frac{3x-1}{x+1}\cdot\frac{(x+1)^2}{x^2+1+2x-2x^2+2x}=1$$

$$\frac{(3x-1)(x+1)}{-x^2+4x+1}=1$$

$$3x^2+2x-1=-x^2+4x+1$$

$$4x^2-2x-2=0$$

$$2x^2-x-1=0 \quad \text{Divide by '2'}$$

$$2x^2-2x+x-1=0$$

$$2x(x-1)+1(x-1)=0$$

$$(x-1)(2x+1)=0$$

Either $x-1=0$ or $2x+1=0$

$\Rightarrow$ $x=1$ or $x=\frac{-1}{2}$

Checking:

When $x=1$

$$\text{Eq. (1)} \Rightarrow \tan^{-1}\left(\frac{1-1}{1+1}\right)+\tan^{-1}\left(\frac{2(1)}{1+1}\right)$$

$$=\tan^{-1}(0)+\tan^{-1}(1)=\frac{\pi}{4} \text{ (True)}$$

When $x=\frac{-1}{2}$

$$\text{Eq. (1)} \Rightarrow \tan^{-1}\left(\frac{\frac{-1}{2}-1}{\frac{-1}{2}+1}\right)+\tan^{-1}\left(\frac{2\left(\frac{-1}{2}\right)}{\frac{-1}{2}+1}\right)$$

$$=\tan^{-1}(-3)+\tan^{-1}(-2)\neq\frac{\pi}{4}$$

Hence, $x=1$ is the only solution.

EXERCISE 9.2

Prove the following:

Q1. $\sin^{-1}\frac{7}{25}+\sin^{-1}\frac{15}{17}=\cos^{-1}\frac{87}{425}$

Solution:

Let $\alpha=\sin^{-1}\frac{7}{25}$

$\Rightarrow \sin\alpha=\frac{7}{25}$

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{7}{25}\right)^2}=\sqrt{1-\frac{49}{625}}=\sqrt{\frac{625-49}{625}}=\sqrt{\frac{576}{25}}$$

Let $\beta=\sin^{-1}\frac{15}{17}$

$\Rightarrow \sin\beta=\frac{15}{17}$

$$\cos\beta=\sqrt{1-\sin^2\beta}=\sqrt{1-\left(\frac{15}{17}\right)^2}=\sqrt{1-\frac{225}{289}}=\sqrt{\frac{289-225}{289}}=\sqrt{\frac{64}{289}}$$

$\cos\alpha=\frac{24}{25}$ | $\cos\beta=\frac{8}{17}$

As we know

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\alpha+\beta=\cos^{-1}\left(\frac{24}{25}\cdot\frac{8}{17}-\frac{7}{25}\cdot\frac{15}{17}\right)$$

$$\alpha+\beta=\cos^{-1}\left(\frac{192}{425}-\frac{105}{425}\right)$$

$$\alpha+\beta=\cos^{-1}\left(\frac{192-105}{425}\right)$$

$$\sin^{-1}\frac{7}{25}+\sin^{-1}\frac{15}{17}=\cos^{-1}\frac{87}{425}$$

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Q2. $\cos^{-1}\frac{7}{25}-\cos^{-1}\frac{5}{13}=\cos^{-1}\frac{323}{325}$

Solution:

$$\text{L.H.S} = \cos^{-1}\frac{7}{25}-\cos^{-1}\frac{5}{13}$$

Using Formula: $\cos^{-1}x-\cos^{-1}y=\cos^{-1}(xy+\sqrt{(1-x^2)(1-y^2)})$

$$\text{L.H.S} = \cos^{-1}\left(\frac{7}{25}\cdot\frac{5}{13}+\sqrt{\left(1-\left(\frac{7}{25}\right)^2\right)\left(1-\left(\frac{5}{13}\right)^2\right)}\right)$$

$$= \cos^{-1}\left(\frac{35}{325}+\sqrt{\left(1-\frac{49}{625}\right)\left(1-\frac{25}{169}\right)}\right)$$

$$= \cos^{-1}\left(\frac{35}{325}+\sqrt{\left(\frac{625-49}{625}\right)\left(\frac{169-25}{169}\right)}\right)$$

$$= \cos^{-1}\left(\frac{35}{325}+\sqrt{\frac{576}{625}\cdot\frac{144}{169}}\right)$$

$$= \cos^{-1}\left(\frac{35}{325}+\frac{24}{25}\cdot\frac{12}{13}\right)$$

$$= \cos^{-1}\left(\frac{35}{325}+\frac{288}{325}\right)$$

$$= \cos^{-1}\left(\frac{323}{325}\right) = \text{R.H.S (Proved)}$$

Q3. $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$

Solution:

$$\text{L.H.S} = \left[\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}\right]-\tan^{-1}\frac{8}{19}$$

Using formula: $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\cdot\frac{3}{5}}\right)-\tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left(\frac{\frac{15+12}{20}}{\frac{20-9}{20}}\right)-\tan^{-1}\frac{8}{19}$$

$$\text{L.H.S} = \tan^{-1}\frac{27}{11}-\tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\cdot\frac{8}{19}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{513-88}{209}}{\frac{209+216}{209}}\right)$$

$$= \tan^{-1}\left(\frac{225}{225}\right) = \tan^{-1}(1)$$

$$= \frac{\pi}{4} = \text{R.H.S (Proved)}$$

Q4. $\sin^{-1}\frac{12}{13}-\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{33}{65}$

Solution:

$$\text{L.H.S} = \sin^{-1}\frac{12}{13}-\sin^{-1}\frac{3}{5}$$

Using Formula: $\sin^{-1}x-\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}-y\sqrt{1-x^2})$

$$\text{L.H.S} = \sin^{-1}\left(\frac{12}{13}\sqrt{1-\left(\frac{3}{5}\right)^2}-\frac{3}{5}\sqrt{1-\left(\frac{12}{13}\right)^2}\right)$$

$$= \sin^{-1}\left(\frac{12}{13}\sqrt{1-\frac{9}{25}}-\frac{3}{5}\sqrt{1-\frac{144}{169}}\right)$$

$$= \sin^{-1}\left(\frac{12}{13}\sqrt{\frac{25-9}{25}}-\frac{3}{5}\sqrt{\frac{169-144}{169}}\right)$$

$$= \sin^{-1}\left(\frac{12}{13}\sqrt{\frac{16}{25}}-\frac{3}{5}\sqrt{\frac{25}{169}}\right)$$

$$= \sin^{-1}\left(\frac{12}{13}\cdot\frac{4}{5}-\frac{3}{5}\cdot\frac{5}{13}\right) = \sin^{-1}\left(\frac{48}{65}-\frac{15}{65}\right)$$

$$= \sin^{-1}\frac{33}{65} = \text{R.H.S (Proved)}$$

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Q5. $\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{63}{65}$

Solution:

Let $\alpha=\sin^{-1}\frac{4}{5}$

$\Rightarrow \sin\alpha=\frac{4}{5}$

$\cos\alpha=\sqrt{1-\sin^2\alpha}$

$=\sqrt{1-\left(\frac{4}{5}\right)^2}$

$=\sqrt{1-\frac{16}{25}}$

$=\sqrt{\frac{25-16}{25}}$

$=\sqrt{\frac{9}{25}}$

$\cos\alpha=\frac{3}{5}$

Let $\beta=\cos^{-1}\frac{12}{13}$

$\Rightarrow \cos\beta=\frac{12}{13}$

$\sin\beta=\sqrt{1-\cos^2\beta}$

$=\sqrt{1-\left(\frac{12}{13}\right)^2}$

$=\sqrt{1-\frac{144}{169}}$

$=\sqrt{\frac{169-144}{169}}$

$=\sqrt{\frac{25}{169}}$

$\sin\beta=\frac{5}{13}$

As we know

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\alpha+\beta=\sin^{-1}\left(\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}\right)$

$\alpha+\beta=\sin^{-1}\left(\frac{48}{65}+\frac{15}{65}\right)$

$\alpha+\beta=\sin^{-1}\left(\frac{48+15}{65}\right)$

$\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{63}{65}$

Q6. $\sin^{-1}\frac{3}{5}+2\tan^{-1}\frac{1}{4}=\cos^{-1}\frac{36}{85}$

Solution:

$\sin^{-1}\frac{3}{5}+2\tan^{-1}\frac{1}{4}=\cos^{-1}\frac{36}{85}$ ...(1)

Consider $2\tan^{-1}\frac{1}{4}=\tan^{-1}\left(\frac{2\left(\frac{1}{4}\right)}{1-\left(\frac{1}{4}\right)^2}\right)$

$\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$=\tan^{-1}\left(\frac{\frac{1}{2}}{1-\frac{1}{16}}\right)=\tan^{-1}\left(\frac{\frac{1}{2}}{\frac{16-1}{16}}\right)$

$=\tan^{-1}\left(\frac{1}{2}\cdot\frac{16}{15}\right)=\tan^{-1}\frac{8}{15}$

From eq. (1) we have

$\sin^{-1}\frac{3}{5}+\tan^{-1}\frac{8}{15}=\cos^{-1}\frac{36}{85}$

Let $\alpha=\sin^{-1}\frac{3}{5}$

$\Rightarrow \sin\alpha=\frac{3}{5}$

$\cos\alpha=\sqrt{1-\sin^2\alpha}$

$\cos\alpha=\sqrt{1-\left(\frac{3}{5}\right)^2}$

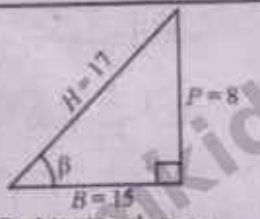
$=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{25-9}{25}}$

$=\sqrt{\frac{16}{25}}$

$\cos\alpha=\frac{4}{5}$

$\beta=\tan^{-1}\frac{8}{15}$

$\Rightarrow \tan\beta=\frac{8}{15}$



Pythagoras theorem

$H^2=B^2+P^2$

$H^2=15^2+8^2$

$H^2=225+64$

$H^2=289 \Rightarrow H=17$

$\sin\beta=\frac{P}{H}=\frac{8}{17}$

$\cos\beta=\frac{B}{H}=\frac{15}{17}$

As we know

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\alpha+\beta=\cos^{-1}\left(\frac{4}{5}\cdot\frac{15}{17}-\frac{3}{5}\cdot\frac{8}{17}\right)$

$\alpha+\beta=\cos^{-1}\left(\frac{60}{85}-\frac{24}{85}\right)$

$\alpha+\beta=\cos^{-1}\left(\frac{60-24}{85}\right)$

$\sin^{-1}\frac{3}{5}+\tan^{-1}\frac{8}{15}=\cos^{-1}\frac{36}{85}$

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Q7. $\sin^{-1}\frac{1}{\sqrt{2}}+\sin^{-1}\frac{1}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{10}}=\frac{\pi}{2}$

Solution:

$$\text{L.H.S} = \left(\sin^{-1}\frac{1}{\sqrt{2}}+\sin^{-1}\frac{1}{\sqrt{5}}\right)+\sin^{-1}\frac{1}{\sqrt{10}}$$

Using Formula: $\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$

$$\text{L.H.S} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1-\left(\frac{1}{\sqrt{5}}\right)^2}+\frac{1}{\sqrt{5}}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)}\right)+\sin^{-1}\frac{1}{\sqrt{10}}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{5}}+\frac{1}{\sqrt{5}}\sqrt{1-\frac{1}{2}}\right)+\sin^{-1}\frac{1}{\sqrt{10}}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{5-1}{5}}+\frac{1}{\sqrt{5}}\sqrt{\frac{2-1}{2}}\right)+\sin^{-1}\frac{1}{\sqrt{10}}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{4}{5}}+\frac{1}{\sqrt{5}}\sqrt{\frac{1}{2}}\right)+\sin^{-1}\frac{1}{\sqrt{10}}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\cdot\frac{2}{\sqrt{5}}+\frac{1}{\sqrt{5}}\cdot\frac{1}{\sqrt{2}}\right)+\sin^{-1}\frac{1}{\sqrt{10}}$$

$$= \sin^{-1}\left(\frac{2}{\sqrt{10}}+\frac{1}{\sqrt{10}}\right)+\sin^{-1}\frac{1}{\sqrt{10}}$$

$$= \sin^{-1}\frac{3}{\sqrt{10}}+\sin^{-1}\frac{1}{\sqrt{10}}$$

$$= \sin^{-1}\left(\frac{3}{\sqrt{10}}\sqrt{1-\left(\frac{1}{\sqrt{10}}\right)^2}+\frac{1}{\sqrt{10}}\sqrt{1-\left(\frac{3}{\sqrt{10}}\right)^2}\right)$$

$$= \sin^{-1}\left(\frac{3}{\sqrt{10}}\sqrt{1-\frac{1}{10}}+\frac{1}{\sqrt{10}}\sqrt{1-\frac{9}{10}}\right)$$

$$= \sin^{-1}\left(\frac{3}{\sqrt{10}}\sqrt{\frac{10-1}{10}}+\frac{1}{\sqrt{10}}\sqrt{\frac{10-9}{10}}\right)$$

$$= \sin^{-1}\left(\frac{3}{\sqrt{10}}\sqrt{\frac{9}{10}}+\frac{1}{\sqrt{10}}\sqrt{\frac{1}{10}}\right)$$

$$= \sin^{-1}\left(\frac{3}{\sqrt{10}}\cdot\frac{3}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cdot\frac{1}{\sqrt{10}}\right)$$

$$= \sin^{-1}\left(\frac{9}{10}+\frac{1}{10}\right)$$

$$= \sin^{-1}\left(\frac{10}{10}\right) = \sin^{-1}(1)$$

$$= \frac{\pi}{4} = \text{R.H.S (Proved)}$$

Q8. $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$

Solution:

$$\text{L.H.S} = \tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}$$

Using Formula: $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$\text{L.H.S} = \tan^{-1}\frac{1}{7}+\tan^{-1}\left(\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right)$$

$$= \tan^{-1}\frac{1}{7}+\tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)$$

$$= \tan^{-1}\frac{1}{7}+\tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right)$$

$$= \tan^{-1}\frac{1}{7}+\tan^{-1}\frac{3}{4}$$

Using Formula: $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\cdot\frac{3}{4}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4+21}{28}}{\frac{28-3}{28}}\right) = \tan^{-1}\left(\frac{25}{25}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4} = \text{R.H.S (Proved)}$$

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Q9. $\tan^{-1}\frac{10}{3}-\tan^{-1}\frac{7}{13}+\tan^{-1}\frac{9}{5}-\tan^{-1}\frac{2}{7}=\frac{\pi}{2}$

Solution:

$$L.H.S=\left(\tan^{-1}\frac{10}{3}-\tan^{-1}\frac{7}{13}\right)+\left(\tan^{-1}\frac{9}{5}-\tan^{-1}\frac{2}{7}\right)$$

Using Formula: $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$$L.H.S=\tan^{-1}\left(\frac{\frac{10}{3}-\frac{7}{13}}{1+\frac{10}{3}\cdot\frac{7}{13}}\right)+\tan^{-1}\left(\frac{\frac{9}{5}-\frac{2}{7}}{1+\frac{9}{5}\cdot\frac{2}{7}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{130-21}{39}}{\frac{39+70}{39}}\right)+\tan^{-1}\left(\frac{\frac{63-10}{35}}{\frac{25+18}{35}}\right)$$

$$=\tan^{-1}\left(\frac{109}{109}\right)+\tan^{-1}\left(\frac{53}{53}\right)$$

$$=\tan^{-1}(1)+\tan^{-1}(1)=\frac{\pi}{4}+\frac{\pi}{4}$$

$$=\frac{\pi+\pi}{4}=\frac{2\pi}{4}=\frac{\pi}{2}=R.H.S\ (Proved)$$

Q10. $\sec^2\left(\tan^{-1}\frac{1}{4}\right)+\csc^2\left(\cot^{-1}\frac{2}{5}\right)=\frac{889}{400}$

Solution:

$$L.H.S=\sec^2\left(\tan^{-1}\frac{1}{4}\right)+\csc^2\left(\cot^{-1}\frac{2}{5}\right)$$

Using Formula: $\sec^2\theta=1+\tan^2\theta, \csc^2\theta=1+\cot^2\theta$

$$L.H.S=\left[1+\tan^2\left(\tan^{-1}\frac{1}{4}\right)\right]+\left[1+\cot^2\left(\cot^{-1}\frac{2}{5}\right)\right]$$

$$=2+\left(\tan\left(\tan^{-1}\frac{1}{4}\right)\right)^2+\left(\cot\left(\cot^{-1}\frac{2}{5}\right)\right)^2$$

$$=2+\left(\frac{1}{4}\right)^2+\left(\frac{2}{5}\right)^2 \quad \because f(f^{-1}(x))=x\ \forall x\in \text{dom}f^{-1}$$

$$=2+\frac{1}{16}+\frac{4}{25}$$

$$=\frac{800+25+64}{400}=\frac{889}{400}=R.H.S\ (Proved)$$

Q11. $\tan^2(\sec^{-1}3)+\cot^2(\csc^{-1}4)=23$

Solution:

$$L.H.S=\tan^2(\sec^{-1}3)+\cot^2(\csc^{-1}4)$$

Using Formula: $\tan^2\theta=\sec^2\theta-1,\ \cot^2\theta=\csc^2\theta-1$

$$L.H.S=[\sec^2(\sec^{-1}3)-1]+[\csc^2(\csc^{-1}4)-1]$$

$$=(\sec(\sec^{-1}3))^2+(\csc(\csc^{-1}4))^2-2$$

$$=(3)^2+(4)^2-2 \quad \because f(f^{-1}(x))=x\ \forall x\in \text{dom}f^{-1}$$

$$=9+16-2$$

$$=23=R.H.S\ (Proved)$$

Q12. $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right), -1\le x\le 1$

Solution:

Let $\alpha=\tan^{-1}x$ ...(1)

$\Rightarrow \tan\alpha=x$

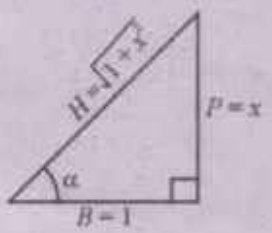
Here $P=x, B=1$

By Pythagoras theorem

$$H^2=B^2+P^2$$

$$H^2=1^2+x^2$$

$$H=\sqrt{1+x^2}$$



$$\Rightarrow \sin\alpha=\frac{P}{H}=\frac{x}{\sqrt{1+x^2}},\ \cos\alpha=\frac{B}{H}=\frac{1}{\sqrt{1+x^2}}$$

As we know

$$\sin 2\alpha=2\sin\alpha.\cos\alpha$$

$$2\alpha=\sin^{-1}\left(2.\frac{x}{\sqrt{1+x^2}}.\frac{1}{\sqrt{1+x^2}}\right)$$

$$2\alpha=\sin^{-1}\left(\frac{2x}{\left(\sqrt{1+x^2}\right)^2}\right)$$

$$2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right) \quad \text{Using (1)}$$

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Q13. $\tan(\sin^{-1}x)=\frac{x}{\sqrt{1-x^2}}, -1<x<1$

Solution:

Let $\alpha = \sin^{-1}x$ ...(1)

$\Rightarrow \sin\alpha = x$

Using: $\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-x^2}$

As we know

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$\tan\alpha = \frac{x}{\sqrt{1-x^2}}$$

$$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$$ Using (1)

Q14. $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution:

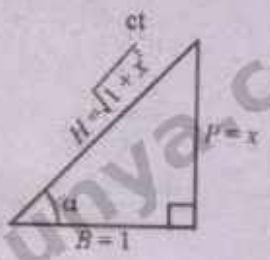
Let $\alpha = \tan^{-1}x$ ...(1)

$\Rightarrow \tan\alpha = x$

Here $P = x, B = 1$

By Pythagoras theorem

$$H^2 = B^2 + P^2$$
$$H^2 = 1^2 + x^2$$
$$H = \sqrt{1+x^2}$$



$$\Rightarrow \sin\alpha = \frac{x}{\sqrt{1+x^2}}$$

$$\cos\alpha = \frac{1}{\sqrt{1+x^2}}$$

As we know

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\cos 2\alpha = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 - \left(\frac{x}{\sqrt{1+x^2}}\right)^2$$

$$2\alpha = \cos^{-1}\left(\frac{1}{1+x^2} - \frac{x^2}{1+x^2}\right)$$

$$2\alpha = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$ Using (1)

Q15. $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

Solution:

Let $\alpha = \sin^{-1}x$ ...(1)

$\Rightarrow \sin\alpha = x$

Using: $\cos\alpha = \sqrt{1-\sin^2\alpha}$

$$\cos\alpha = \sqrt{1-x^2}$$

$$\Rightarrow \cos(\sin^{-1}x) = \sqrt{1-x^2}$$ Using (1)

Q16. $\tan^{-1}x + \cot^{-1}(x+1) = \tan^{-1}(x^2+x+1)$

Solution:

L.H.S $= \tan^{-1}x + \cot^{-1}(x+1)$

$$= \tan^{-1}x + \tan^{-1}\left(\frac{1}{x+1}\right) \quad \because \cot^{-1}\left(\frac{B}{P}\right) = \tan^{-1}\left(\frac{P}{B}\right)$$

Using Formula: $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\text{L.H.S} = \tan^{-1}\left(\frac{x+\frac{1}{x+1}}{1-x\cdot\frac{1}{x+1}}\right)$$

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{x(x+1)+1}{x+1}}{\frac{x+1-x}{x+1}}\right)$$

$$= \tan^{-1}\left(\frac{x^2+x+1}{1}\right)$$

$$= \tan^{-1}(x^2+x+1)$$

= R.H.S (Proved)

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Q17. $2\tan^{-1}\left(\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)=\cos^{-1}\left(\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}\right)$

Solution:

$$2\tan^{-1}\left(\tan\frac{\theta}{2}.\tan\frac{\phi}{2}\right)=\cos^{-1}\left(\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}\right)$$

$$\Rightarrow \quad 2\tan^{-1}x=\cos^{-1}\left(\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}\right) \text{ where } x=\tan\frac{\theta}{2}\tan\frac{\phi}{2}$$

Let $\alpha=\tan^{-1}x$ ...(1)

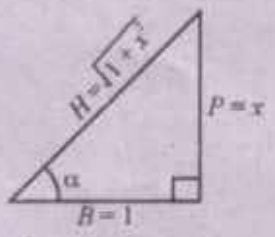
$\Rightarrow \quad \tan\alpha=x$

Here $P=x, B=1$

By Pythagoras theorem

$$H^2=B^2+P^2$$

$$H^2=1^2+x^2 \Rightarrow H=\sqrt{1+x^2}$$



$$\Rightarrow \quad \sin\alpha=\frac{x}{\sqrt{1+x^2}}, \cos\alpha=\frac{1}{\sqrt{1+x^2}}$$

As we know

$$\cos2\alpha=\cos^2\alpha-\sin^2\alpha$$

$$\cos2\alpha=\left(\frac{1}{\sqrt{1+x^2}}\right)^2-\left(\frac{x}{\sqrt{1+x^2}}\right)^2$$

$$2\alpha=\cos^{-1}\left(\frac{1}{1+x^2}-\frac{x^2}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \qquad \text{Using (1)}$$

Put $x=\tan\frac{\theta}{2}.\tan\frac{\phi}{2}$, we get

$$2\tan^{-1}\left(\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)=\cos^{-1}\left(\frac{1-\tan^2\frac{\theta}{2}\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\theta}{2}\tan^2\frac{\phi}{2}}\right)$$

Using Formula: $\tan^2\frac{\theta}{2}=\frac{1-\cos\theta}{1+\cos\theta}$

$$2\tan^{-1}\left(\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)=\cos^{-1}\left(\frac{1-\left(\frac{1-\cos\theta}{1+\cos\theta}\right)\left(\frac{1-\cos\phi}{1+\cos\phi}\right)}{1+\left(\frac{1-\cos\theta}{1+\cos\theta}\right)\left(\frac{1-\cos\phi}{1+\cos\phi}\right)}\right)$$

$$=\cos^{-1}\left(\frac{\frac{(1+\cos\theta)(1+\cos\phi)-(1-\cos\theta)(1-\cos\phi)}{(1+\cos\theta)(1+\cos\phi)}}{\frac{(1+\cos\theta)(1+\cos\phi)+(1-\cos\theta)(1-\cos\phi)}{(1+\cos\theta)(1+\cos\phi)}}\right)$$

$$=\cos^{-1}\left(\frac{1+\cos\phi+\cos\theta+\cos\theta\cos\phi-1+\cos\phi+\cos\theta-\cos\theta\cos\phi}{1+\cos\phi+\cos\theta+\cos\theta\cos\phi+1-\cos\phi-\cos\theta+\cos\theta\cos\phi}\right)$$

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$$2\tan^{-1}\left(\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right) = \cos^{-1}\left(\frac{2\cos\theta+2\cos\phi}{2+2\cos\theta\cos\phi}\right) = \cos^{-1}\left(\frac{2(\cos\theta+\cos\phi)}{2(1+\cos\theta\cos\phi)}\right)$$

$$= \cos^{-1}\left(\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}\right) \text{ (Proved)}$$

Q18. $\cot^{-1}\left(\frac{\cos x}{1-\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}$

Solution:

$$\text{L.H.S} = \cot^{-1}\left(\frac{\cos x}{1-\sin x}\right)$$

$$= \tan^{-1}\left(\frac{1-\sin x}{\cos x}\right) \quad \because \cot^{-1}\left(\frac{B}{P}\right)=\tan^{-1}\left(\frac{P}{B}\right)$$

$$= \tan^{-1}\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$$

Divide up and down by $\cos\frac{x}{2}$

$$= \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\frac{x}{2}}{1+\tan\frac{\pi}{4}\tan\frac{x}{2}}\right) \quad \because \tan\frac{\pi}{4}=1$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) \quad \because \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$= \frac{\pi}{4}-\frac{x}{2}$$

= R.H.S (Proved)

Q19. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then show that $x+y+z=xyz$

Solution:

Given: $(\tan^{-1}x+\tan^{-1}y)+\tan^{-1}z=\pi$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right)+\tan^{-1}z=\pi$$

$$\tan^{-1}\left(\frac{\frac{x+y}{1-xy}+z}{1-\left(\frac{x+y}{1-xy}\right)(z)}\right)=\pi$$

$$\frac{\frac{x+y+z(1-xy)}{1-xy}}{\frac{1-xy-z(x+y)}{1-xy}} = \tan\pi$$

$$\frac{x+y+z-xyz}{1-xy-zx-zy}=0$$

$$\Rightarrow \quad x+y+z-xyz=0$$

$$x+y+z=xyz \text{ (Proved)}$$

Q20. Find the value of x if, $\tan^{-1}\left(\frac{2x-1}{3x-1}\right)+\tan^{-1}\left(\frac{2x+1}{3x+1}\right)=\frac{\pi}{4}$

Solution:

Given: $\tan^{-1}\left(\frac{2x-1}{3x-1}\right)+\tan^{-1}\left(\frac{2x+1}{3x+1}\right)=\frac{\pi}{4}$

Using Formula: $\tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$$\tan^{-1}\left(\frac{\frac{2x-1}{3x-1}+\frac{2x+1}{3x+1}}{1-\frac{2x-1}{3x-1}\frac{2x+1}{3x+1}}\right)=\frac{\pi}{4}$$

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$$\Rightarrow \frac{\frac{(2x-1)(3x+1)+(2x+1)(3x-1)}{(3x-1)(3x+1)}}{\frac{(3x-1)(3x+1)-(2x-1)(2x+1)}{(3x-1)(3x+1)}} = \tan\left(\frac{\pi}{4}\right)$$

$$\frac{(6x^2-x-1)+(6x^2+x-1)}{((3x)^2-1^2)-((2x)^2-1^2)} = 1$$

$$\frac{6x^2-x-1+6x^2+x-1}{9x^2-1-4x^2+1} = 1$$

$$12x^2-2 = 5x^2$$

$$12x^2-5x^2 = 2$$

$$7x^2 = 2$$

$$x^2 = \frac{2}{7} \Rightarrow \sqrt{x^2} = \sqrt{\frac{2}{7}}$$

$$x = \pm\sqrt{\frac{2}{7}}$$

Q21. If $\sin^{-1}\frac{2p}{1+p^2}+\cos^{-1}\frac{1-q^2}{1+q^2}=\tan^{-1}\frac{2y}{1-y^2}$, **show that** $y=\frac{p+q}{1-pq}$

Solution:

Given: $\sin^{-1}\frac{2p}{1+p^2}+\cos^{-1}\frac{1-q^2}{1+q^2}=\tan^{-1}\frac{2y}{1-y^2}$ ...(1)

Put $p=\tan\alpha$; $q=\tan\beta$; $y=\tan\gamma$

$\Rightarrow$ $\alpha=\tan^{-1}p$; $\beta=\tan^{-1}q$; $\gamma=\tan^{-1}y$

Eq. (1) $\Rightarrow$ $\sin^{-1}\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right)+\cos^{-1}\left(\frac{1-\tan^2\beta}{1+\tan^2\beta}\right)$

$$= \tan^{-1}\left(\frac{2\tan\gamma}{1-\tan^2\gamma}\right)$$

$$\sin^{-1}(\sin 2\alpha)+\cos^{-1}(\cos 2\beta)=\tan^{-1}(\tan 2\gamma)$$

$\Rightarrow$ $2\alpha+2\beta=2\gamma$

Dividing both sides by '2'

$$\alpha+\beta=\gamma$$

$\Rightarrow$ $\tan^{-1}p+\tan^{-1}q=\tan^{-1}y$

$$\tan^{-1}\left(\frac{p+q}{1-pq}\right)=\tan^{-1}y$$

On comparing, we have

$$y=\frac{p+q}{1-pq}$$ (Proved)

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Multiple Choice Questions (MCQs)

1. **An inverse function of a function f is defined when f is:**

   A) one-to-one only
   B) one-to-one and onto (bijective)
   C) constant
   D) periodic only

2. **The principal domain selected for $\sin x$ to define $\sin^{-1} x$ is:**

   A) $[-\pi, \pi]$
   B) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
   C) $[0, 2\pi]$
   D) R

3. **The graph of $y = \sin^{-1} x$ is obtained from the principal $y = \sin x$ graph by reflection in**

   A) the x-axis
   B) the y-axis
   C) the line $y = x$
   D) the line $y = -x$

4. **The range of $\cos^{-1} x$ is:**

   A) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
   B) $[0, \pi]$
   C) $[-1, 1]$
   D) $(-\infty, \infty)$

5. **The principal domain of $\tan x$ is:**

   A) $[0, \pi]$
   B) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
   C) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
   D) R

6. **The principal domain of $\csc x$ is:**

   A) $\left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
   B) $[0, \pi]$
   C) R
   D) $[-1, 1]$

7. **The range of $\cot^{-1} x$ is:**

   A) $(0, \pi)$
   B) R
   C) $[0, \pi]$
   D) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

8. **$\sin^{-1}\left(\frac{1}{2}\right)$ equals:**

   A) $\frac{\pi}{3}$
   B) $\frac{\pi}{6}$
   C) $\frac{-\pi}{6}$
   D) $\frac{5\pi}{6}$

9. **$\cot\left[\cos^{-1}\left(\frac{-1}{2}\right)\right]$ equals:**

   A) $-\sqrt{3}$
   B) $-\frac{1}{\sqrt{3}}$
   C) $\sqrt{3}$
   D) $\frac{1}{\sqrt{3}}$

10. **For $x > 0$, the inverse cotangent identity shown is**

    A) $\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$
    B) $\cot^{-1} x = \sin^{-1} x$
    C) $\cot^{-1} x = \cos^{-1} x$
    D) $\cot^{-1} x = \frac{1}{\tan^{-1} x}$

11. **For $x \in [-1, 1]$, $\sin^{-1} x + \cos^{-1} x$ equals:**

    A) 0
    B) $\frac{\pi}{2}$
    C) π
    D) x

12. **For $x \in (-\infty, \infty)$, $\tan^{-1} x + \cot^{-1} x$ equals:**

    A) 0
    B) $\frac{\pi}{4}$
    C) $\frac{\pi}{2}$
    D) π

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13. For $y = \tan x$, vertical asymptotes occur at:

A) $x = n\pi$ B) $x = \frac{(2n+1)\pi}{2}$ C) $x = \frac{\pi}{4} + n\pi$ D) $x = 2n\pi$

14. If $\sin^{-1} a = 0$, then value of a is:

A) $\frac{\pi}{2}$ B) π C) 0 D) 0 & π

15. $\cos^{-1} A - \cos^{-1} B =$

A) $\cos^{-1}(AB + \sqrt{(1-A^2)(1-B^2)})$ B) $\cos^{-1}(AB - \sqrt{(1-A^2)(1-B^2)})$

C) $\cos^{-1}(\sqrt{1-A^2} + \sqrt{1-B^2})$ D) $\cos^{-1}(\sqrt{1-A^2} - \sqrt{1-B^2})$

16. $2\cos^{-1} A$ is equal to:

A) $\sin^{-1}(2A^2-1)$ B) $\sin^{-1}(A^2-2)$ C) $\cos^{-1}(2A^2-1)$ D) $\cos^{-1}(A^2-2)$

17. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} =$

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$

18. $\sin^{-1} A - \sin^{-1} B =$

A) $\sin^{-1}(A-B)$ B) $\sin^{-1}(A\sqrt{1-A^2} + A\sqrt{1-B^2})$

C) $\sin^{-1}(A\sqrt{1-B^2})$ D) $\sin^{-1}(A\sqrt{1-B^2} - B\sqrt{1-A^2})$

19. The graph of $y = \sin^{-1} x$ is along the:

A) x-axis B) y-axis C) Both A and B D) none of these

20. The value of $\sin\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$ is:

A) $\frac{\sqrt{3}}{2}$ B) $\frac{1}{\sqrt{2}}$ C) $\frac{1}{2}$ D) $\frac{1}{\sqrt{3}}$

ANSWER KEY

1.	B	2.	B	3.	C	4.	B	5.	C	6.	A	7.	A	8.	B	9.	B	10.	A
11.	B	12.	C	13.	B	14.	D	15.	A	16.	C	17.	B	18.	D	19.	B	20.	C

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