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UNIT 06

Conics

INTRODUCTION

- After studying this unit, we will be able to identify conics (circle, parabola, ellipse, hyperbola); derive and apply circle equations in standard form and find circles given various conditions; determine when a line intersects or touches a circle; find tangents, normals, and tangent lengths for circles; derive and apply parabola equations, find elements, and write parabola equations from focus, vertex, directrix; find tangency conditions and equations of tangents and normals to parabolas; derive ellipse equations, convert to standard form, find their elements, find tangency conditions, and write tangent and normal equations; derive hyperbola equations, find elements from given data, find tangency conditions, and write tangent and normal equations; and apply conics to real-world problems like suspension bridges, satellite systems, electron orbits, radio systems, flashlights, and architecture.
- The Greek mathematicians **Apollonius** (260-200 B.C.) and **Pappus** (early fourth century A.D.) made significant contributions to the study of conic sections using Euclidean geometry. Here, we will employ the methods of **analytic geometry** to investigate their properties.
- Consider a circle, with its centre C. Draw a line RS that passes through C and perpendicular to the circle's plane. Choose a fixed point A on this line. Now, from A, draw every possible straight line that touch any point on the circle.
- These lines together form a surface known as a **right circular cone**. Each of these straight lines is called a **generator** (or a **ruling**) of the cone.
- The complete surface has two separate parts, called **nappes**, which meet only at the point A. This meeting point is the **vertex** of the cone and the original line RS is the **axis** of the cone.
- When you slice this cone with a plane, the intersection curve is called a **conic section** (or simply a **conic**).
- There are also special cases called **degenerate conics**. These happen when the slicing plane passes through the vertex A. Instead of getting a nice curve, the intersection might be just a single point, a single straight line, or a pair of intersecting lines.

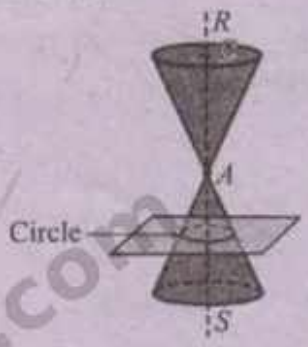

Figure 6.1(a)

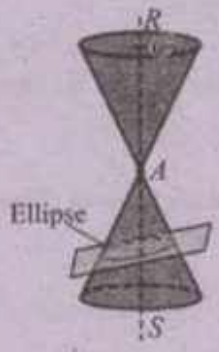

Figure 6.1(b)

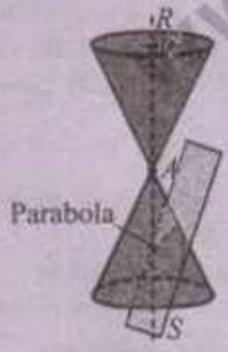

Figure 6.1(c)

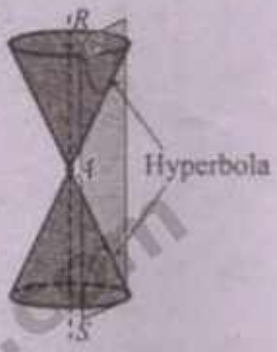

Figure 6.1(d)

Depending on the angle and position of the plane, you can get a circle, an ellipse, a parabola, or a hyperbola.

- If the cutting plane is perpendicular to the axis, the section is a circle, see Figure 6.1(a).
- If the plane is inclined and cuts only one nappe, the resulting curve is an ellipse, see Figure 6.1(b).
- If the plane is parallel to a generator and cuts only one nappe, the curve is a parabola; see Figure 6.1(c).
- If the plane is parallel to the axis and cuts both nappes, the curve is a hyperbola; see Figure 6.1(d).

We will being our study with the circle. Other conic sections will be discussed in subsequent section.

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CIRCLE

Definition: A circle is the set of all points in a plane that are at a fixed distance from a given fixed point. The fixed point is called the centre, and the fixed distance is called the radius of the circle.

Theorem 1: The equation of a circle with centre at the point (h, k) and radius r is $(x-h)^2+(y-k)^2=r^2$.

Proof:

Let Centre of circle $= C(h, k)$

Radius of circle $= r$

Let $P(x, y)$ be any point on the circle, then by definition

$$|\overline{CP}| = r$$

By distance formula

$$\sqrt{(x-h)^2+(y-k)^2} = r$$

Squaring both sides gives

$$(x-h)^2+(y-k)^2 = r^2$$

This is the required equation of circle.

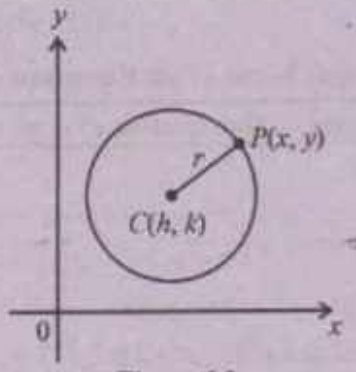

Figure 6.2

Note:

(i) Equation (1) is known as the **standard form** of the equation of a circle.

(ii) If $r = 0$, equation (1) reduces to $(x-h)^2+(y-k)^2 = 0$. This equation represents a point circle centered at (h, k).

(iii) If $(h, k) = (0, 0)$, then equation (1) reduces to $x^2+y^2=r^2$. This equation represents a circle whose centre is at origin.

Question

Question: Can you imagine, why the circle: $(x-h)^2+(y-k)^2=0$ is named as a point circle?

Ans. The equation $(x-h)^2+(y-k)^2=0$ is called a point circle because its radius is zero, meaning the entire circle shrinks into a single point.

Parametric Equations of a Circle

Let $P(x, y)$ be any point on the circle $x^2+y^2=r^2$ and let θ be the angle measured counter-clockwise from positive direction of $\overline{OP}$, then by definition:

$\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$ where $0 \le \theta < 2\pi$.

Thus, the parametric equations of the circle are:

$$x = r\cos\theta,\ y = r\sin\theta$$

The point $P(r\cos\theta, r\sin\theta)$ lies on the circle $x^2+y^2=r^2$ for all values of θ.

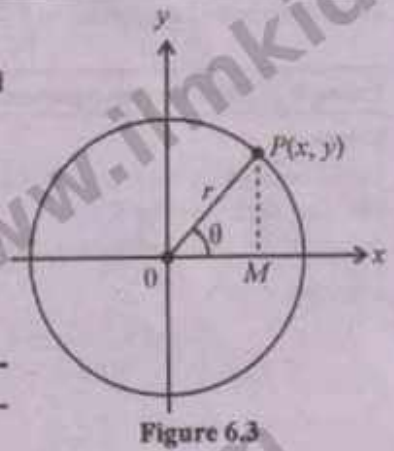

Figure 6.3

Example 1: Write an equation of the circle with centre $(-2, 1)$ and radius 5.

Solution:

Centre: $C(h, k) = (-2, 1)$

$\Rightarrow$ $h = -2,\ k = 1$

Radius: $r = 5$

Required equation of circle is:

$$(x-h)^2+(y-k)^2 = r^2$$

$$(x-(-2))^2+(y-1)^2 = (5)^2$$

$$(x+2)^2+(y-1)^2 = 25$$

$$x^2+4+4x+y^2+1-2y-25 = 0$$

$$x^2+y^2+4x-2y-20 = 0$$

Hence, the required equation is $x^2+y^2+4x-2y-20=0$.

Example 2: Prove that the locus of the point of intersection of the lines: $x\cos\theta + y\sin\theta = a$, $x\sin\theta - y\cos\theta = b$ is a circle with centre at $(0, 0)$ and radius $\sqrt{a^2+b^2}$, where θ is a parameter.

Solution:

$$x\cos\theta + y\sin\theta = a \quad \ldots(1)$$

$$x\sin\theta - y\cos\theta = b \quad \ldots(2)$$

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To find the locus, we need to eliminate the parameter θ.
Squaring and adding eq. (1) and eq. (2)

$$(x\cos\theta + y\sin\theta)^2 + (x\sin\theta - y\cos\theta)^2 = a^2 + b^2$$
$$x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta + x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta = a^2 + b^2$$
$$x^2(1) + y^2(1) = a^2 + b^2$$
$$(x-0)^2 + (y-0)^2 = (\sqrt{a^2+b^2})^2$$

Compare it with
$(x-h)^2 + (y-k)^2 = r^2$
$\Rightarrow$ Centre: $C(h,k) = C(0,0)$
Radius: $r = \sqrt{a^2+b^2}$

General Form of an Equation of a Circle

Theorem 2: The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (1) where $g, f, c \in \mathbb{R}$ represents a circle.

Proof:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$(x^2 + 2gx) + (y^2 + 2fy) = -c$$

Add $g^2 + f^2$ to both sides

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = -c + g^2 + f^2$$
$$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$$
$$(x-(-g))^2 + (y-(-f))^2 = (\sqrt{g^2+f^2-c})^2$$

Compare it with
$(x-h)^2 + (y-k)^2 = r^2$
$\Rightarrow$ Centre: $C(h,k) = C(-g,-f)$
Radius: $r = \sqrt{g^2+f^2-c}$, provided $g^2 + f^2 - c > 0$
Hence, the given equation represents a circle.

Working Rule to Find the Centre and Radius of a Circle in General Form

To find the centre and radius of a circle from its general equation, follow these steps:

Step 1: If necessary, make the coefficients of x^2 and y^2 equal to 1 by dividing the entire equation by the common coefficient of x^2 and y^2.

Step 2: Centre of the circle $= \left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$

Step 3: Radius of the circle $= \sqrt{\left(\frac{\text{coefficient of } x}{2}\right)^2 + \left(\frac{\text{coefficient of } y}{2}\right)^2 - \text{constant term}}$

Example 3: Show that the equation $5x^2 + 5y^2 + 20x + 25y + 10 = 0$ represents a circle. Also find its centre and radius.

Solution:

Given equation: $5x^2 + 5y^2 + 20x + 25y + 10 = 0$
Dividing both sides by 5

$$x^2 + y^2 + 4x + 5y + 2 = 0 \text{ (Equation of circle)}$$

Compare it with

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$\Rightarrow$ $2g = 4$; $2f = 5$; $c = 2$
$g = 2$; $f = \frac{5}{2}$

Now, $g^2 + f^2 - c = (2)^2 + \left(\frac{5}{2}\right)^2 - 2 = 4 + \frac{25}{4} - 2$

$$g^2 + f^2 - c = 2 + \frac{25}{4} = \frac{33}{4} > 0$$

So, given equation represents a real circle.

$\Rightarrow$ Centre $= (-g, -f) = \left(-2, \frac{-5}{2}\right)$

Radius $= \sqrt{g^2+f^2-c} = \sqrt{\frac{33}{4}} = \frac{\sqrt{33}}{2}$

Thus, the given equation represents a circle with centre $\left(-2, -\frac{5}{2}\right)$ and radius $\frac{\sqrt{33}}{2}$.

Example 4: A point moves so that the sum of the squares of its distances from two points $A(1,1)$ and $B(2,2)$ is equal to 4. Prove that its locus is a circle.

Solution:

Given points: $A(1,1), B(2,2)$
Let $P(x,y)$ be the moving point
By given geometric condition

$$|PA|^2 + |PB|^2 = 4$$
$$(\sqrt{(x-1)^2+(y-1)^2})^2 + (\sqrt{(x-2)^2+(y-2)^2})^2 = 4$$
$$(x-1)^2 + (y-1)^2 + (x-2)^2 + (y-2)^2 = 4$$
$$x^2 + 1 - 2x + y^2 + 1 - 2y + x^2 + 4 - 4x + y^2 + 4 - 4y - 4 = 0$$
$$2x^2 + 2y^2 - 6x - 6y + 6 = 0$$

Dividing both sides by '2'

$$x^2 + y^2 - 3x - 3y + 3 = 0 \text{ (Circle)}$$

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Equation of a Circle with Given End points of a Diameter

Question: Find an equation of the circle having the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ as a diameter.

Solution:

End points of diameter are $A(x_1, y_1)$ and $B(x_2, y_2)$

Let $P(x, y)$ be any point on the circle, such that

$m\angle APB = 90°$

Slope of $\overline{AP} = m_1 = \dfrac{y - y_1}{x - x_1}$

Slope of $\overline{BP} = m_2 = \dfrac{y - y_2}{x - x_2}$

As $\overline{AP} \perp \overline{BP}$, so $m_1 \times m_2 = -1$

$$\Rightarrow \left(\frac{y - y_1}{x - x_1}\right)\cdot\left(\frac{y - y_2}{x - x_2}\right) = -1$$

$$(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$$

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$x^2 - xx_2 - xx_1 + x_1x_2 + y^2 - yy_2 - yy_1 + y_1y_2 = 0$$

$$x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + (x_1x_2 + y_1y_2) = 0$$

This is required equation of circle.

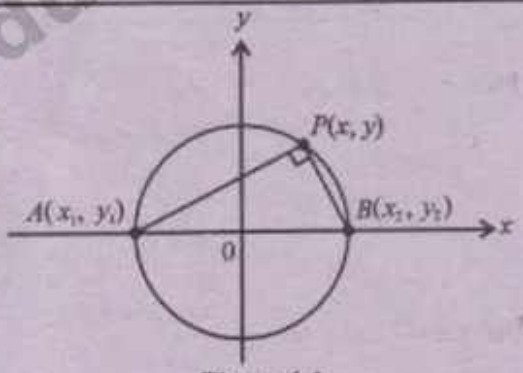

Figure 6.4

Do You Know?

A circle has the maximum area for a given perimeter among all closed shapes.

Equations of Circles Determined by Given Conditions

The general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ contains three independent constants g, f and c. These can be determined if the equation satisfies three given conditions. We will discuss four different cases.

Case-1: A Circle Passing through Three Non-collinear Points

If three non-collinear points through which a circle passes are given, we can find the three independent constants g, f and c in the general equation.

Example 5: Find the equation of the circle passing through the points $(-3, 3), (-1, 5)$ and $(5, 1)$.

Solution:

Let $A(-3, 3), B(-1, 5), C(5, 1)$

Let required equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(1)$$

Since the circle (1) passes through the points A, B and C, therefore

At $A(-3, 3)$:

$$(-3)^2 + (3)^2 + 2g(-3) + 2f(3) + c = 0$$

$$9 + 9 - 6g + 6f + c = 0$$

$$-6g + 6f + c + 18 = 0 \quad \ldots(1)$$

At $B(-1, 5)$:

$$(-1)^2 + (5)^2 + 2g(-1) + 2f(5) + c = 0$$

$$1 + 25 - 2g + 10f + c = 0$$

$$-2g + 10f + c + 26 = 0 \quad \ldots(2)$$

At $C(5, 1)$:

$$(5)^2 + (1)^2 + 2g(5) + 2f(1) + c = 0$$

$$25 + 1 + 10g + 2f + c = 0$$

$$10g + 2f + c + 26 = 0 \quad \ldots(3)$$

Equation (1) – Equation (2)

$$-6g + 6f + c + 18 = 0$$

$$-2g + 10f + c + 26 = 0$$

$$-4g - 4f - 8 = 0$$

$$\Rightarrow \quad g + f + 2 = 0 \quad \ldots(4)$$

Do You Know?

Infinitely many circles can pass through two points, but exactly one circle passes through three non-collinear points.

Alternative Method

Equation of the circle passing through three non-collinear points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by the determinant equation:

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

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Equation (2) – Equation (3)

$$-2g+10f+c+26=0$$
$$\pm 10g \pm 2f \pm c \pm 26=0$$
$$-12g+8f=0 \quad \ldots(5)$$

From eq. (4), we get

$g=-f-2$ Put this in eq. (5)

$$-12(-f-2)+8f=0$$
$$12f+24+8f=0 \Rightarrow 20f=-24$$
$$f=-\frac{24}{20} \Rightarrow f=-\frac{6}{5}$$

Put $f=-\frac{6}{5}$ in eq. (4)

$$g-\frac{6}{5}+2=0 \Rightarrow g=\frac{6}{5}-2=\frac{6-10}{5}$$
$$\boxed{g=-\frac{4}{5}}$$

Put $f=-\frac{6}{5}$ and $g=-\frac{4}{5}$ in eq. (1)

$$-6\left(-\frac{4}{5}\right)+6\left(-\frac{6}{5}\right)+c+18=0$$
$$c=-18-\frac{24}{5}+\frac{36}{5}=\frac{-90-24+36}{5}$$
$$c=\frac{78}{5}$$

From eq. (I), we have

$$x^2+y^2+2\left(-\frac{4}{5}\right)x+2\left(-\frac{6}{5}\right)-\frac{78}{5}=0$$

Multiplying both sides by 5

$$5x^2+5y^2-8x-12y-78=0$$

Case-2: A Circle Passing through Two Points and Having its Centre on a Given Line

Example 6: A circle with its centre on the y-axis passes through the origin and the point (a, b), where $b>0$. Find its equation.

Solution:

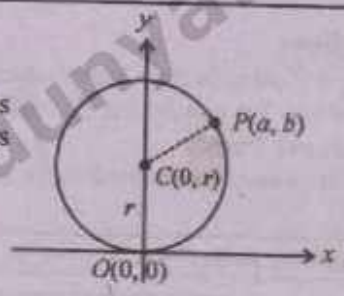
Let $P(a,b)$, $b>0$

Since the circle passes through the origin and has its centre on y-axis, so

Centre $= C(0,r)$

where r is the radius

As $|\overline{OC}|=|\overline{CP}|$



$$r=\sqrt{(a-0)^2+(b-r)^2}$$

Squaring both sides

$$r^2=a^2+(b-r)^2$$
$$r^2=a^2+b^2+r^2-2br$$
$$2br=a^2+b^2+r^2-r^2$$
$$r=\frac{a^2+b^2}{2b}, b\neq 0$$

Required equation of circle is:

$$(x-h)^2+(y-k)^2=r^2$$
$$(x-0)^2+(y-r)^2=r^2$$
$$x^2+y^2+r^2-2ry=r^2$$
$$x^2+y^2-2ry=0$$

Put $r=\frac{a^2+b^2}{2b}$, we have

$$x^2+y^2-2y\left(\frac{a^2+b^2}{2b}\right)=0$$
$$x^2+y^2-\left(\frac{a^2+b^2}{b}\right)y=0$$

Multiplying both sides by b

$$b(x^2+y^2)-(a^2+b^2)y=0$$

Case 3: A Circle Passing through Two Points and having Tangent at One of These Points

Example 7: Find the equation of the circle passing through (1, 2) and touching the line $x-2y+4=0$ at (2, 3).

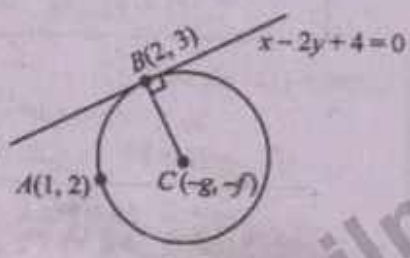
Solution:

Let $A(1,2), B(2,3)$

$\ell: x-2y+4=0$

Centre: $C(-g,-f)$

Let required equation of circle be



$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(I)$$

Since, the circle (I) passes through the points A and B, therefore

At $A(1,2)$:

$$(1)^2+(2)^2+2g(1)+2f(2)+c=0$$
$$1+4+2g+4f+c=0$$
$$2g+4f+c+5=0 \quad \ldots(1)$$

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At $B(2, 3)$:

$$(2)^2+(3)^2+2g(2)+2f(3)+c = 0$$
$$4+9+4g+6f+c = 0$$
$$4g+6f+c+13 = 0 \quad \ldots(2)$$

Equation (1) – Equation (2)

$$2g+4f+c+5 = 0$$
$$\pm 4g \pm 6f \pm c \pm 13 = 0$$
$$-2g-2f-8 = 0$$
$$\Rightarrow \quad g+f+4 = 0 \quad \ldots(3)$$

$$\text{Slope of } \ell = m_1 = \frac{1}{-2} = \frac{1}{2}$$

$$\text{Slope of } \overline{BC} = m_2 = \frac{-f-3}{-g-2}$$

As $\overline{BC} \perp \ell \Rightarrow m_1 \times m_2 = -1$

$$\frac{1}{2}\left(\frac{-f-3}{-g-2}\right) = -1$$

$$-f-3 = -1(-2g-4)$$
$$-f-3 = 2g+4$$
$$\Rightarrow \quad 2g+f+7 = 0 \quad \ldots(4)$$

Equation (4) – Equation (3)

$$2g+f+7-g-f-4 = 0$$
$$g+3 = 0 \quad \Rightarrow \boxed{g=-3}$$

Put $g = -3$ in eq. (3)

$$-3+f+4 = 0 \quad \Rightarrow \boxed{f=-1}$$

Put $g = -3$ and $f = -1$ in eq. (1)

$$2(-3)+4(-1)+c+5 = 0 \quad \Rightarrow -6-4+c+5=0$$
$$-5+c = 0 \quad \Rightarrow c=5$$

From eq. (I), we have

$$x^2+y^2+2(-3)x+2(-1)y+5 = 0$$
$$x^2+y^2-6x-2y+5 = 0$$

Case 4: A Circle Passing through Two Points and Touching a Given Line

Example 8: Find the equations of the circles passing through the points $A(3,4)$ and $B(3,-4)$ and touching the line $x+2y+9=0$.

Solution:

$A(3,4), B(3,-4)$

$\ell: x+2y+9=0$

Centre: $C(-g,-f)$

Let required equation of circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(I)$$

Since, the circle (I) passes through A and B, therefore

At $A(3,4)$:

$$(3)^2+(4)^2+2g(3)+2f(4)+c = 0$$
$$9+16+6g+8f+c = 0$$
$$6g+8f+c+25 = 0 \quad \ldots(1)$$

At $B(3,-4)$:

$$(3)^2+(-4)^2+2g(3)+2f(-4)+c = 0$$
$$9+16+6g-8f+c = 0$$
$$6g-8f+c+25 = 0 \quad \ldots(2)$$

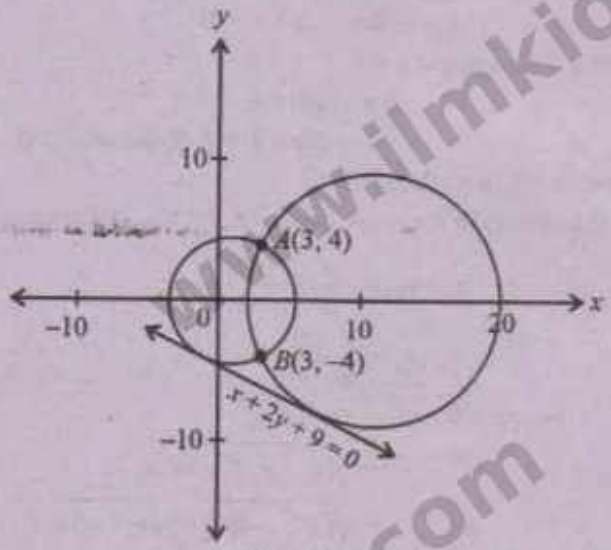

Figure 6.5

Equation (1) – Equation (2)

$$6g+8f+c+25 = 0$$
$$\pm 6g \pm 8f \pm c \pm 25 = 0$$
$$16f = 0 \quad \Rightarrow f = 0$$

As line ℓ is tangent to circle, so

Distance of c from ℓ = Radius

$$\frac{|-g-2f+9|}{\sqrt{(1)^2+(2)^2}} = |\overline{AC}|$$

$$|-(g+2f-9)| = \sqrt{5}|\overline{AC}|$$

Squaring both sides

$$(g+2f-9)^2 = 5|\overline{AC}|^2$$
$$(g+2f-9)^2 = 5[(3+g)^2+(4+f)^2]$$

Put $f = 0$, we get

$$(g+0-9)^2 = 5[(3+g)^2+(4+0)^2]$$
$$(g-9)^2 = 5(9+g^2+6g+16)$$
$$g^2+81-18g = 5g^2+30g+125$$

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$5g^2+30g+125-g^2-81+18g=0$

$4g^2+48g+44=0$

Dividing both sides by 4

$g^2+12g+11=0$

$g^2+11g+g+11=0$

$g(g+11)+1(g+11)=0$

$(g+11)(g+1)=0$

$\Rightarrow \quad g+11=0 \quad ; \quad g+1=0$

$g=-11 \quad ; \quad g=-1$

When $g=-1, f=0$:

Eq. (1) $\Rightarrow 6(-1)+8(0)+c+25=0 \Rightarrow c=-19$

From (I), required circle is

$x^2+y^2+2(-1)x+2(0)y-19=0$

$x^2+y^2-2x-19=0$

When $g=-11, f=0$:

Eq. (1) $\Rightarrow 6(-11)+8(0)+c+25=0 \Rightarrow c=41$

From (I), required circle is

$x^2+y^2+2(-11)x+2(0)y+41=0$

$x^2+y^2-22x+41=0$

Example 9: Find the equation of the circle which touches the y-axis at a distance of 3 units from the origin and cuts off an intercept of 8 units on the x-axis.

Solution:

Points of contact on y-axis = $(0, 3)(0, -3)$

Length of x-intercept = 8 units

Centre = $(\pm r, \pm 3)$ where r is the radius

Take centre = $(r, 3)$, radius = r

Equation of circle is

$(x-h)^2+(y-k)^2=r^2$

$(x-r)^2+(y-3)^2=r^2$

$x^2+r^2-2rx+y^2+9-6y-r^2=0$

$x^2+y^2-2rx-6y+9=0$

For x-intercept, put $y=0$

$x^2+0^2-2rx-6(0)+9=0$

$x^2-2rx+9=0$ (Quadratic in x)

Here $a=1, b=-2r, c=9$

Let x_1 and x_2 be the roots, then

$$x_1+x_2=\frac{b}{a}=\frac{(-2r)}{1}=2r$$

$$x_1 \cdot x_2=\frac{c}{d}=\frac{9}{1}=9$$

Intercept length $= |x_1-x_2|$

$$=\sqrt{(x_1+x_2)^2-4x_1x_2}$$

$$=\sqrt{(2r)^2-4(9)}=\sqrt{4r^2-36}$$

Given that:

Length of x-intercept = 8

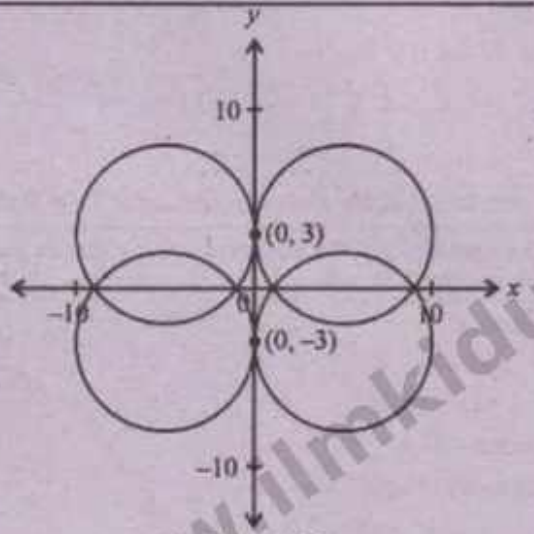

Figure 6.6

$\sqrt{4r^2-36}=8$

$\Rightarrow \quad 4r^2-36=64$

$4r^2=100 \Rightarrow r^2=25 \Rightarrow r=5$

$\Rightarrow$ Centre = $(5, 3), r=5$

One circle: $(x-h)^2+(y-k)^2=r^2$

$(x-5)^2+(y-3)^2=25$

By symmetry all the four circles are

$(x\pm5)^2+(y\pm3)^2=25$ where $\pm$ signs are independent.

EXERCISE 6.1

Q1. In each of the following, find an equation of the circle with:

(a) centre at (5, 7) and radius 2

Solution:

Centre: $C(h, k)=(5, 7) \Rightarrow h=5, k=7$

Radius: $r=2$

Required equation of circle is:

$(x-h)^2+(y-k)^2=r^2$

$(x-5)^2+(y-7)^2=(2)^2$

$x^2+25-10x+y^2+49-14y=4$

$x^2+y^2-10x-14y+74-4=0$

$x^2+y^2-10x-14y+70=0$

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(b) centre at $(a\cos\theta, a\sin\theta)$ and radius a

Solution:

Centre: $C(h,k) = (a\cos\theta, a\sin\theta)$

$\Rightarrow \quad h = a\cos\theta, k = a\sin\theta$

Radius: $r = a$

Required equation of circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x - a\cos\theta)^2 + (y - a\sin\theta)^2 = a^2$$

$$x^2 + a^2\cos^2\theta - 2ax\cos\theta + y^2 + a^2\sin^2\theta - 2ay\sin\theta - a^2 = 0$$

$$x^2 + y^2 + a^2(\cos^2\theta + \sin^2\theta) - 2ax\cos\theta - 2ay\sin\theta - a^2 = 0$$

$$x^2 + y^2 + a^2(1) - 2ax\cos\theta - 2ay\sin\theta - a^2 = 0$$

$$x^2 + y^2 - 2ax\cos\theta - 2ay\sin\theta = 0$$

(c) ends of diameter (3, 4) and (2, –7).

Solution:

Let ends of diameter = $A(3,4), B(2,-7)$

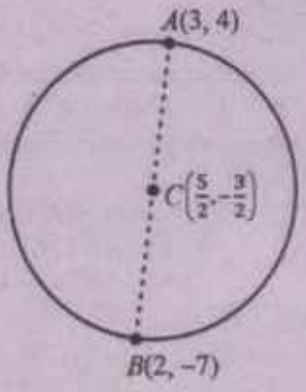
Centre = Midpoint of $\overline{AB}$

$$= C\left(\frac{3+2}{2}, \frac{4-7}{2}\right) = C\left(\frac{5}{2}, -\frac{3}{2}\right) \Rightarrow h = \frac{5}{2}, k = -\frac{3}{2}$$

Radius $= |\overline{AC}|$

$$= \sqrt{\left(3 - \frac{5}{2}\right)^2 + \left(4 + \frac{3}{2}\right)^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{11}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{121}{4}} = \sqrt{\frac{122}{4}}$$

$$\Rightarrow \qquad r = \sqrt{\frac{61}{2}}$$



Required equation of circle is:

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \left(\sqrt{\frac{61}{2}}\right)^2$$

$$x^2 + \frac{25}{4} - 5x + y^2 + \frac{9}{4} + 3y = \frac{61}{2}$$

$$x^2 + y^2 - 5x + 3y + \frac{25}{4} + \frac{9}{4} - \frac{61}{2} = 0$$

$$x^2 + y^2 - 5x + 3y + \frac{25 + 9 - 122}{4} = 0$$

$$x^2 + y^2 - 5x + 3y + \frac{-88}{4} = 0$$

$$x^2 + y^2 - 5x + 3y - 22 = 0$$

Alternate Method:

Ends of diameter = (3, 4), (2, –7)

$\Rightarrow \quad x_1 = 3, y_1 = 4$ and $x_2 = 2, y_2 = -7$

Equation of circle passing through ends of diameter is:

$$x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + (x_1x_2 + y_1y_2) = 0$$

Putting values, we have

$$x^2 + y^2 - (3+2)x - (4-7)y + ((3)(2) + (4)(-7)) = 0$$

$$x^2 + y^2 - 5x + 3y + 6 - 28 = 0$$

$$x^2 + y^2 - 5x + 3y - 22 = 0$$

Q2. Express each of the following, in the standard form and hence find its centre and radius:

(a) $2x^2 + 2y^2 - 4x - 12y + 11 = 0$

Solution:

$$2x^2 + 2y^2 - 4x - 12y + 11 = 0$$

Dividing both sides by '2'

$$x^2 + y^2 - 2x - 6y + \frac{11}{2} = 0$$

$$(x^2 - 2x) + (y^2 - 6y) = \frac{-11}{2}$$

Add 1^2 and 3^2 on both sides

$$(x^2 - 2x + 1^2) + (y^2 - 6y + 3^2) = \frac{-11}{2} + 1^2 + 3^2$$

$$(x-1)^2 + (y-3)^2 = \frac{-11 + 2 + 18}{2}$$

$$(x-1)^2 + (y-3)^2 = \frac{9}{2}$$

$$(x-1)^2 + (y-3)^2 = \left(\frac{3}{\sqrt{2}}\right)^2$$

Compare it with

$$(x-h)^2 + (y-k)^2 = r^2$$

$\Rightarrow \quad h = 1, k = 3, r = \frac{3}{\sqrt{2}}$

Centre $= (h,k) = (1,3)$

Radius $= r = \frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

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(b) $x^2+y^2-2ax+2by-2ab=0$

Solution:

$$x^2+y^2-2ax+2by-2ab=0$$
$$(x^2-2ax)+(y^2+2by)=2ab$$

Add a^2 and b^2 on both sides

$$(x^2-2ax+a^2)+(y^2+2by+b^2)=a^2+b^2+2ab$$
$$(x-a)^2+(y+b)^2=(a+b)^2$$
$$(x-a)^2+(y-(-b))^2=(a+b)^2$$

Compare it with

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow$ $h=a, k=-b, r=|a+b|$

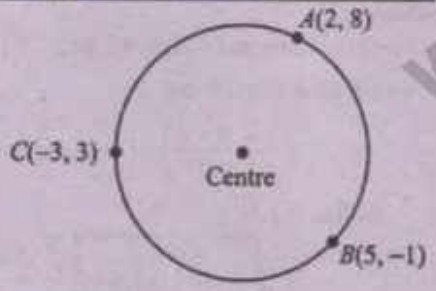
Centre $=(h,k)=(a,-b)$

Radius $=r=|a+b|$

Q3. Find an equation of circle passing through the points $(2,8)$, $(5,-1)$ and $(-3,3)$, also find the centre and radius.

Solution:

Let $A(2,8), B(5,-1), C(-3,3)$



Let required equation of circle be:

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(1)$$

Since the circle (I) passes through the points A, B and C, therefore

At $A(2,8)$:

$$(2)^2+(8)^2+2g(2)+2f(8)+c=0$$
$$4+64+4g+16f+c=0$$
$$4g+16f+c+68=0 \quad \ldots(1)$$

At $B(5,-1)$:

$$(5)^2+(-1)^2+2g(5)+2f(-1)+c=0$$
$$25+1+10g-2f+c=0$$
$$10g-2f+c+26=0 \quad \ldots(2)$$

At $C(-3,3)$:

$$(-3)^2+(3)^2+2g(-3)+2f(3)+c=0$$
$$9+9-6g+6f+c=0$$
$$-6g+6f+c+18=0 \quad \ldots(3)$$

Equation (1) – Equation (2)

$$4g+16f+c+68=0$$
$$\pm 10g \mp 2f \pm c \pm 26=0$$
$$-6g+18f+42=0$$
$$\Rightarrow \quad -g+3f+7=0 \quad \ldots(4)$$

Equation (2) – Equation (3)

$$10g-2f+c+26=0$$
$$\mp 6g \pm 6f \pm c \pm 18=0$$
$$16g-8f+8=0$$
$$\Rightarrow \quad 2g-f+1=0 \quad \ldots(5)$$

2 × Equation (4) + Equation (5)

$$-2g+6f+14=0$$
$$2g-f+1=0$$
$$5f+15=0$$
$$\Rightarrow 5f=-15 \Rightarrow \boxed{f=-3}$$

Put $f=-3$ in eq. (5)

$$2g-(-3)+1=0$$
$$2g=-4 \Rightarrow \boxed{g=-2}$$

Put $g=-2, f=-3$ in eq. (3)

$$-6(-2)+6(-3)+c+18=0$$
$$12-18+c+18=0 \Rightarrow \boxed{c=-12}$$

From eq. (I), we have

$$x^2+y^2+2(-2)x+2(-3)y-12=0$$
$$x^2+y^2-4x-6y-12=0$$

Centre $=(-g,-f)=(2,3)$

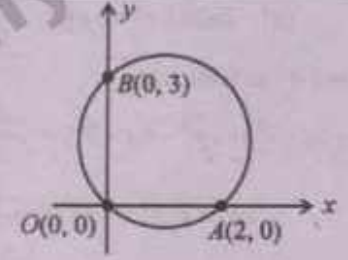
Radius $=\sqrt{g^2+f^2-c}$

$$=\sqrt{(-2)^2+(-3)^2+12}$$
$$=\sqrt{25}=5$$

Q4. Find the equation of a circle which passes through the origin and cuts off intercepts 2 and 3 on the x-axis and y-axis respectively.

Solution:

According to the statement points on the circle are $O(0,0), A(2,0), B(0,3)$



Let required equation of circle be:

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(1)$$

At $O(0,0)$:

$$0^2+0^2+2g(0)+2f(0)+c=0 \Rightarrow \boxed{c=0}$$

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At $A(2, 0)$:

$$2^2+0^2+2g(2)+2f(0)+c=0$$
$$4g+c+4=0 \Rightarrow 4g+0+4=0 \because c=0$$
$$4g=-4 \Rightarrow \boxed{g=-1}$$

At $B(0, 3)$:

$$0^2+3^2+2g(0)+2f(3)+c=0$$
$$6f+c+9=0 \Rightarrow 6f+0+9=0 \because c=0$$
$$6f=-9 \Rightarrow f=-\frac{9}{6} \Rightarrow f=-\frac{3}{2}$$

From eq. (I), we have

$$x^2+y^2+2(-1)x+2\left(-\frac{3}{2}\right)y+0=0$$
$$x^2+y^2-2x-3y=0$$

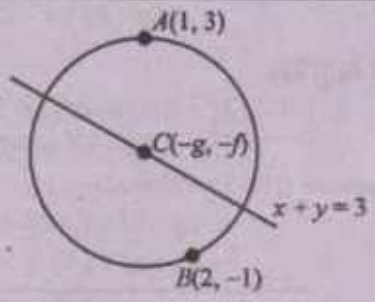
Q5. Find the equation of a circle passing through the points $(1, 3), (2, -1)$ and having its centre on the line $x + y = 3$. Does the point $(4, 1)$ lie on the circle?

Solution:

Let $A(1,3), B(2,-1)$

$\ell: x+y=3$

Centre: $C(-g, -f)$



Let required equation of circle be:

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(I)$$

Since circle (I) passes through the points A and B, therefore

At $A(1, 3)$:

$$1^2+3^2+2g(1)+2f(3)+c=0$$
$$2g+6f+c+10=0 \quad \ldots(1)$$

At $B(2, -1)$:

$$2^2+(-1)^2+2g(2)+2f(-1)+c=0$$
$$4g-2f+c+5=0 \quad \ldots(2)$$

Equation (1) − Equation (2)

$$2g+6f+c+10=0$$
$$\pm 4g \mp 2f \pm c \pm 5=0$$
$$-2g+8f+5=0 \quad \ldots(3)$$

Since the line ℓ passes through centre $C(-g, -f)$, so

$$-g-f=3$$
$$-g-f-3=0 \quad \ldots(4)$$

Equation (3) − 2 × Equation (4)

$$-2g+8f+5=0$$
$$\mp 2g \mp 2f \mp 6=0$$
$$10f+11=0 \Rightarrow \boxed{f=-\frac{11}{10}}$$

Put $f=-\frac{11}{10}$ in eq. (4)

$$-g+\frac{11}{10}-3=0 \Rightarrow g=\frac{11-30}{10} \Rightarrow \boxed{g=-\frac{19}{10}}$$

Put $g=-\frac{19}{10}, f=-\frac{11}{10}$ in eq. (2)

$$4\left(-\frac{19}{10}\right)-2\left(-\frac{11}{10}\right)+c+5=0 \Rightarrow c=\frac{76}{10}-\frac{22}{10}-5$$
$$c=\frac{76-22-50}{10}=\frac{4}{10} \Rightarrow \boxed{c=\frac{2}{5}}$$

From eq. (I), we have

$$x^2+y^2+2\left(-\frac{19}{10}\right)x+2\left(-\frac{11}{10}\right)y+\frac{2}{5}=0$$
$$x^2+y^2-\frac{19}{5}x-\frac{11}{5}y+\frac{2}{5}=0$$

Multiplying both sides by 5

$$5x^2+5y^2-19x-11y+2=0$$

Position of point (4, 1):

Put $x = 4, y = 1$ in circle

$$5(4)^2+5(1)^2-19(4)-11(1)+2=0$$
$$80+5-76-11+2=0$$
$$0=0 \text{ (True)}$$

Hence, the point (4, 1) lies on the circle.

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Q6. Find the equation of the circle passing through the point $(-3, -4)$ and touching the line $3x - y - 1 = 0$ at the point $(1, 2)$.

Solution:

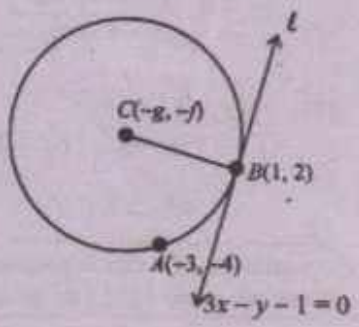
Let $A(-3, -4), B(1, 2)$

$\ell: 3x - y - 1 = 0$

Centre: $C(-g, -f)$

Let required equation of circle be:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(I)$$



Since the circle (I) passes through the points A and B, therefore

At $A(-3, -4)$:

$$(-3)^2 + (-4)^2 + 2g(-3) + 2f(-4) + c = 0$$

$$-6g - 8f + c + 25 = 0 \quad \ldots(1)$$

At $B(1, 2)$:

$$(1)^2 + (2)^2 + 2g(1) + 2f(2) + c = 0$$

$$2g + 4f + c + 5 = 0 \quad \ldots(2)$$

Equation (1) – Equation (2)

$$-6g - 8f + c + 25 = 0$$

$$\pm 2g \pm 4f \pm c \pm 5 = 0$$

$$-8g - 12f + 20 = 0$$

Dividing by -4 on both sides

$$2g + 3f - 5 = 0 \quad \ldots(3)$$

Slope of $\ell = m_1 = \frac{3}{-1} = 3$

Slope of $\overline{BC} = \frac{-f-2}{-g-1} = \frac{-(f+2)}{-(g+1)} = \frac{f+2}{g+1}$

As $\overline{BC} \perp \ell$, so $m_1 \times m_2 = -1$

$$\Rightarrow \quad 3 \times \left(\frac{f+2}{g+1}\right) = -1$$

$$3f + 6 = -g - 1 \Rightarrow g + 3f + 6 + 1 = 0$$

$$g + 3f + 7 = 0 \quad \ldots(4)$$

Equation (3) – Equation (4)

$$2g + 3f - 5 = 0$$

$$\pm g \pm 3f \pm 7 = 0$$

$$g - 12 = 0 \quad \Rightarrow \boxed{g = 12}$$

Put $g = 12$ in eq. (4)

$$12 + 3f + 7 = 0 \quad \Rightarrow 3f = -19 \Rightarrow f = -\frac{19}{3}$$

Put $g = 12, f = -\frac{19}{3}$ in eq. (2)

$$2(12) + 4\left(-\frac{19}{3}\right) + c + 5 = 0 \quad \Rightarrow c = -5 - 24 + \frac{76}{3}$$

$$c = -29 + \frac{76}{3} = \frac{-87 + 76}{3} \Rightarrow c = -\frac{11}{3}$$

From eq. (I), we have

$$x^2 + y^2 + 2(12)x + 2\left(-\frac{19}{3}\right)y - \frac{11}{3} = 0$$

Multiplying both sides by 3

$$3x^2 + 3y^2 + 72x - 38y - 11 = 0$$

Q7. Find an equation of a circle of radius a lying in the second quadrant and tangent to both the axes.

Solution:

According to the statement, we have

Radius: $r = a$

Centre $= C(-a, a)$

$\Rightarrow \quad h = -a, k = a$

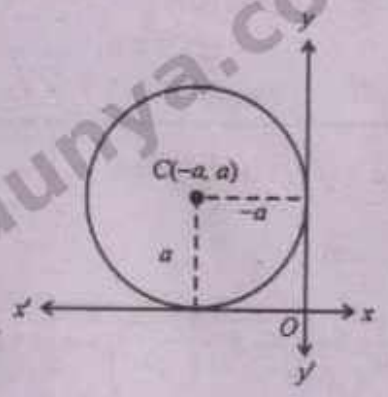
Required equation of circle is:

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x + a)^2 + (y - a)^2 = a^2$$

$$x^2 + a^2 + 2ax + y^2 + a^2 - 2ay - a^2 = 0$$

$$x^2 + y^2 + 2ax - 2ay + a^2 = 0$$



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Q8. Find the equations of the circles which touches the x-axis at the point $(5,0)$ and passes through another point $(3,2)$.

Solution:

Let $A(5,0), B(3,2)$

Since the circle touches x-axis at $(5,0)$, so

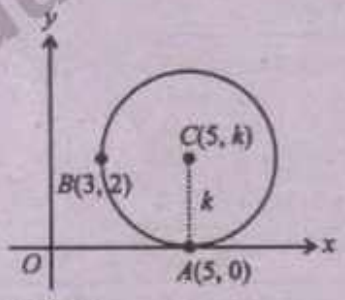
Centre: $(h,k)=(5,k)$

Radius: $r=k$

Required equation of circle is:

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-5)^2+(y-k)^2=k^2 \quad \ldots(1)$$



Since circle (1) passes through $B(3,2)$, so

$$(3-5)^2+(2-k)^2=k^2$$

$$4+4+k^2-4k-k^2=0$$

$$8=4k \Rightarrow k=2$$

From eq. (1), we have

$$(x-5)^2+(y-2)^2=2^2$$

$$x^2+25-10x+y^2+4-4y-4=0$$

$$x^2+y^2-10x-4y+25=0$$

Q9. Find the equation of the circle passing through the point of intersection of the lines $x-y-5=0$, $2x+y-4=0$ and it has its centre at $(5,0)$.

Solution:

Let Centre $=C(5,0)$

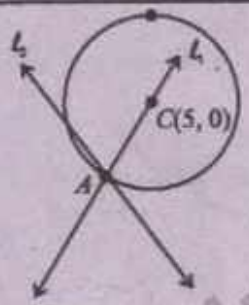
$\Rightarrow \quad h=5, k=0$

$$\ell_1: x-y-5=0 \quad \ldots(1)$$

$$\ell_2: 2x+y-4=0 \quad \ldots(2)$$

Equation (1) + Equation (2)

$$3x-9=0 \Rightarrow 3x=9 \Rightarrow x=3$$



Put $x=3$ in eq. (1)

$$3-y-5=0 \Rightarrow -y-2=0 \Rightarrow y=-2$$

Point of intersection $= A(x,y)=A(3,-2)$

Radius $= |\overline{AC}|$

$$r=\sqrt{(5-3)^2+(0+2)^2}=\sqrt{4+4}=\sqrt{8}$$

Required equation of circle is

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-5)^2+(y-0)^2=(\sqrt{8})^2$$

$$x^2+25-10x+y^2-8=0$$

$$x^2+y^2-10x+17=0$$

Q10. Find the equations of the circles touching the y-axis and also the straight line $3x-4y=0$ at the point $(8,6)$.

Solution:

Let $A(8,6)$

ℓ: $3x-4y=0$

Centre: $C(-g,-f)$

Let required equation of circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(1)$$

Since, it passes through $A(8,6)$, so

$$8^2+6^2+2g(8)+2f(6)+c=0$$

$$16g+12f+c+100=0 \quad \ldots(1)$$

Slope of $\ell=m_1=-\frac{3}{-4}=\frac{3}{4}$

Slope of $\overline{AC}=m_2=\frac{-f-6}{-g-8}=\frac{f+6}{g+8}$

As $\overline{AC}\perp\ell$, so $m_1\times m_2=-1$

$$\Rightarrow \quad \frac{3}{4}\times\frac{f+6}{g+8}=-1 \Rightarrow 3f+18=-4g-32$$

$$4g+3f+50=0 \quad \ldots(2)$$

Since, the circle touches the y-axis, so

$$r=|-g|=|g|$$

$$\Rightarrow \quad \sqrt{g^2+f^2-c}=|g|$$

Squaring both sides

$$g^2+f^2-c=g^2$$

$$c=f^2 \quad \ldots(3)$$

Put $c=f^2$ in eq. (1)

$$16g+12f+f^2+100=0 \quad \ldots(4)$$

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Eq. (4) − 4 × Eq. (2)

$$16g + 12f + 100 + f^2 = 0$$
$$\pm 16g \pm 12f \pm 200 = 0$$
$$-100 + f^2 = 0 \Rightarrow f^2 = 100 \Rightarrow f = \pm 10$$

When $f = 10$:

Eq. (3) $\Rightarrow c = 10^2 \Rightarrow \boxed{c = 100}$

Eq. (2) $\Rightarrow 4g + 3(10) + 50 = 0$

$\Rightarrow 4g = -80 \Rightarrow \boxed{g = -20}$

From eq. (I), required equation of circle is

$$x^2 + y^2 + 2(-20)x + 2(10)y + 100 = 0$$
$$x^2 + y^2 - 40x + 20y + 100 = 0$$

When $f = -10$:

Eq. (3) $\Rightarrow c = (-10)^2 \Rightarrow \boxed{c = 100}$

Eq. (2) $\Rightarrow 4g + 3(-10) + 50 = 0 \Rightarrow 4g = -20 \Rightarrow \boxed{g = -5}$

From eq. (I), required equation of circle is

$$x^2 + y^2 + 2(-5)x + 2(-10)y + 100 = 0$$
$$x^2 + y^2 - 10x - 20y + 100 = 0$$

Q11. Find the equations of the circles passing through the points $A(3, 2)$ and $B(3, -2)$ and touching the line $x + 2y + 3 = 0$.

Solution:

$A(3, 2), B(3, -2)$

$\ell: x + 2y + 3 = 0$

Centre: $C(-g, -f)$

Let required equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(I)$$

Since the circle (I) passes through the points A and B, therefore

At $A(3, 2)$:

$$3^2 + 2^2 + 2g(3) + 2f(2) + c = 0$$
$$6g + 4f + c + 13 = 0 \quad \ldots(1)$$

At $B(3, -2)$:

$$3^2 + (-2)^2 + 2g(3) + 2f(-2) + c = 0$$
$$6g - 4f + c + 13 = 0 \quad \ldots(2)$$

Equation (1) − Equation (2)

$$6g + 4f + c + 13 = 0$$
$$\pm 6g \mp 4f \pm c \pm 13 = 0$$
$$8f = 0 \Rightarrow \boxed{f = 0}$$

As line ℓ is tangent to circle, so

Distance of C from ℓ = Radius

$$\frac{|-g - 2f + 3|}{\sqrt{1^2 + 2^2}} = |\overline{AC}|$$
$$|-(g + 2f - 3)| = \sqrt{5}\,|\overline{AC}|$$

Squaring both sides

$$(g + 2f - 3)^2 = 5|\overline{AC}|^2$$
$$(g + 2f - 3)^2 = 5[(3 + g)^2 + (2 + f)^2]$$

Put $f = 0$, we get

$$(g - 3)^2 = 5[(3 + g)^2 + (2)^2]$$
$$g^2 + 9 - 6g = 5(9 + g^2 + 6g + 4)$$
$$0 = 5g^2 + 30g + 65 - g^2 - 9 + 6g$$
$$4g^2 + 36g + 56 = 0$$

Dividing both sides by 4

$$g^2 + 9g + 14 = 0$$
$$g^2 + 7g + 2g + 14 = 0$$
$$g(g + 7) + 2(g + 7) = 0$$
$$(g + 2)(g + 7) = 0$$
$$\Rightarrow \quad g + 2 = 0 \quad ; \quad g + 7 = 0$$
$$g = -2 \quad ; \quad g = -7$$

When $g = -2, f = 0$:

Eq. (1) $\Rightarrow 6(-2) + 4(0) + c + 13 = 0 \Rightarrow c = -1$

From (I), the required circle is

$$x^2 + y^2 + 2(-2)x + 2(0)y - 1 = 0$$
$$x^2 + y^2 - 4x - 1 = 0$$

When $g = -7, f = 0$:

Eq. (1) $\Rightarrow 6(-7) + 4(0) + c + 13 = 0 \Rightarrow c = 29$

From (I), the required circle is

$$x^2 + y^2 + 2(-7)x + 2(0)y + 29 = 0$$
$$x^2 + y^2 - 14x + 29 = 0$$

Q12. Find the equations of the circles of radius 5 that are tangent to the line $4x + 3y - 10 = 0$ at the point $(1, 2)$.

Solution:

Let $A(1, 2)$

$\ell: 4x + 3y - 10 = 0$

Radius = 2

Centre = $C(-g, -f)$

Let required equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(I)$$

Since, it passes through point $A(1, 2)$, so

$$1^2 + 2^2 + 2g(1) + 2f(2) + c = 0$$
$$2g + 4f + c + 5 = 0 \quad \ldots(1)$$

As radius = 5

$$\Rightarrow \quad \sqrt{g^2 + f^2 - c} = 5$$

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Squaring both sides

$$g^2+f^2-c-25=0 \quad \ldots(2)$$

Equation (1) + Equation (2)

$$\begin{array}{r} 2g+4f+c+5=0 \\ g^2+f^2-c-25=0 \\ \hline g^2+f^2+2g+4f-20=0 \end{array} \quad \ldots(3)$$

Slope of line $\ell = m_1 = -\frac{4}{3}$

Slope of $\overline{AC} = \frac{-f-2}{-g-1} = \frac{f+2}{g+1}$

As $\overline{AC} \perp \ell$, so $m_1 \times m_2 = -1$

$$\Rightarrow \quad -\frac{4}{3} \times \frac{f+2}{g+1} = -1$$

$$4f+8 = 3g+3$$

$$3g-4f-5=0 \quad \ldots(4)$$

From eq. (4), we have

$$3g = 4f+5 \Rightarrow g = \frac{4f+5}{3} \quad \ldots(5)$$

Put $g = \frac{4f+5}{3}$ in eq. (3)

$$\left(\frac{4f+5}{3}\right)^2 + f^2 + 2\left(\frac{4f+5}{3}\right) + 4f - 20 = 0$$

Multiply both sides by 9

$$16f^2+25+40f+9f^2+6(4f+5)+36f-180=0$$

$$25f^2+76f-155+24f+30=0$$

$$25f^2+100f-125=0$$

Dividing both sides by 25

$$f^2+4f-5=0$$

$$f^2+5f-f-5=0$$

$$f(f+5)-1(f+5)=0$$

$$(f-1)(f+5)=0$$

$$\Rightarrow \quad f-1=0 \quad ; \quad f+5=0$$

$$f=1 \quad ; \quad f=-5$$

When $f=1$:

Eq. (5) $\Rightarrow \quad g = \frac{4(1)+5}{3} \Rightarrow g=3$

Eq. (2) $\Rightarrow \quad 3^2+1^2-c-25=0 \Rightarrow c=-15$

From eq. (I), required circle is

$$x^2+y^2+2(3)x+2(1)y-15=0$$

$$x^2+y^2+6x+2y-15=0$$

When $f=-5$:

Eq. (5) $\Rightarrow \quad g = \frac{4(-5)+5}{3} \Rightarrow g=-5$

Eq. (2) $\Rightarrow \quad (-5)^2+(-5)^2-c-25=0 \Rightarrow c=25$

From eq. (I), required circle is

$$x^2+y^2+2(-5)x+2(-5)y+25=0$$

$$x^2+y^2-10x-10y+25=0$$

TANGENT AND NORMAL TO A CONIC

Definition: For a non-degenerated conic, if a line touches a conic at exactly one point, it is called a **tangent** to the conic, and the point is called the **point of contact.** In figure 6.7, $\overleftrightarrow{LM}$ is a tangent to the conic at the point P.

Definition: The line perpendicular to the tangent at its point of contact is called the **normal** to the conic at that point. For example, in Figure 6.7, $\overleftrightarrow{PQ} \perp \overleftrightarrow{LM}$ at the point P; thus, $\overleftrightarrow{PQ}$ is the normal to the conic at P.

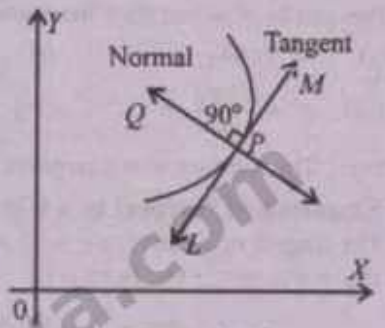

Figure 6.7

Equation of Tangent to a Circle

(a) Circle centered at origin: $x^2+y^2=r^2$

$$x^2+y^2=r^2 \quad \ldots(1)$$

Let $P(x_1, y_1)$ be a point on the circle

$$\Rightarrow \quad x_1^2+y_1^2=r^2 \quad \ldots(2)$$

Differentiating (1) w.r.t. 'x'

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(r^2)$$

$$2x+2y\frac{dy}{dx}=0 \quad \Rightarrow \quad 2y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

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Slope of tangent at $P(x_1, y_1) = P(x_1, y_1) = \left.\frac{dy}{dx}\right|_{P(x_1, y_1)}$

$$m_T = -\frac{x_1}{y_1}$$

Equation of tangent at $P(x_1, y_1)$ is:

$$y - y_1 = m_T(x - x_1)$$

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

$$yy_1 - y_1^2 = -xx_1 + x_1^2$$

$$xx_1 + yy_1 = x_1^2 + y_1^2$$

$$\boxed{xx_1 + yy_1 = r^2}$$ Using (2)

which is a required equation of tangent.

(b) General circle: $x^2 + y^2 + 2gx + 2fy + c = 0$

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(1)$$

Let $P(x_1, y_1)$ be a point on the circle

$$\Rightarrow \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \ldots(2)$$

Differentiating (1) w.r.t. 'x'

$$\frac{d}{dx}(x^2 + y^2 + 2gx + 2fy + c) = 0$$

$$2x + 2y\frac{dy}{dx} + 2g(1) + 2f\frac{dy}{dx} + 0 = 0$$

$$2\frac{dy}{dx}(y + f) = -2x - 2g$$

$$\frac{dy}{dx} = \frac{-2(x + g)}{2(y + f)}$$

$$\frac{dy}{dx} = \frac{x + g}{y + f}$$

Slope of tangent at $P(x_1, y_1) = \left.\frac{dy}{dx}\right|_{P(x_1, y_1)}$

$$m_T = -\frac{x_1 + g}{y_1 + f}$$

Equation of tangent at $P(x_1, y_1)$ is:

$$y - y_1 = m_T(x - x_1)$$

$$\Rightarrow \quad y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$

$$(y - y_1)(y_1 + f) = -(x_1 + g)(x - x_1)$$

$$yy_1 + fy - y_1^2 - fy_1 = -xx_1 + x_1^2 - gx + gx_1$$

$$xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$

Adding $(gx_1 + fy_1)$ to both sides, we get

$$xx_1 + yy_1 + gx + fy + gx_1 + fy_1 = x_1^2 + y_1^2 + gx_1 + fy_1 + gx_1 + fy_1$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = -c$$ Using (2)

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

which is the required equation of tangent.

Rule for Writing the Equation of a Tangent to a Circle

We have proved above that the equation of the tangent to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) is:

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

This can be obtained from the equation of the circle by making the following substitutions:

(i) $x^2 \to xx_1$ (ii) $y^2 \to yy_1$

(iii) $x \to \frac{x + x_1}{2}$ (iv) $y \to \frac{y + y_1}{2}$

(v) The constant term c remains unchanged.

Note

The rule holds for all second-degree curves (conics) as well.

Equation of Normal to a Circle at $P(x_1, y_1)$

The tangent to the circle $x^2 + y^2 = r^2$ at $P(x_1, y_1)$ is $xx_1 + yy_1 = r^2$. The normal is

$$xy_1 - yx_1 = 0 \text{ or } \frac{x}{x_1} = \frac{y}{y_1}.$$

Note

The centre of the circle, (0, 0), satisfies the equation of the normal, confirming that the normal to a circle always passes through its centre

---

Example 10: Find the equation of the tangent and normal to the circle $x^2 + y^2 - 4x + 6y + 12 = 0$ at the point on the circle whose ordinate is -2.

Solution:

Circle: $x^2 + y^2 - 4x + 6y + 12 = 0$

Put $y = -2$, we get

$$x^2 + (-2)^2 - 4x + 6(-2) + 12 = 0$$

$$x^2 + 4 - 4x - 12 + 12 = 0$$

$$x^2 - 4x + 4 = 0$$

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$$(x-2)^2 = 0$$
$$\Rightarrow \quad x-2=0 \quad \Rightarrow x=2$$

Point of contact = $P(2,-2)$

Equation of tangent at $P(x_1, y_1)$ is:

$$xx_1 + yy_1 - 2(x+x_1) + 3(y+y_1) + 12 = 0$$

Put $x_1 = 2, y = -2$, we get

$$x(2) + y(-2) - 2(x+2) + 3(y-2) + 12 = 0$$

$$2x - 2y - 2x - 4 + 3y - 6 + 12 = 0$$
$$y + 2 = 0$$
$$\Rightarrow \quad y = -2 \text{ (Horizontal line through } P)$$

The normal at a point P is perpendicular to the tangent.

$\Rightarrow$ Equation of normal is a vertical line through $P(2,-2)$

Normal: $x = 2$

Position of a Point with Respect to a Circle

Theorem 1: The point $P(x_1, y_1)$ lies outside, on, or inside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \gtreqless 0$

Proof:

Circle: $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre: $C(-g, -f)$

Radius: $r = \sqrt{g^2 + f^2 - c}$

The point $P(x_1, y_1)$ lies outside, on, or inside the circle according as:

$$|CP| > r, |CP| = r, \text{ or } |CP| < r$$

$$\Rightarrow \quad |CP| \gtreqless r$$

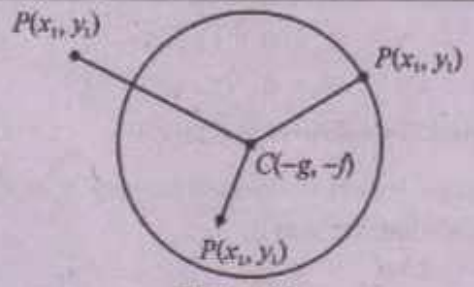


Figure 6.8

By using distance formula

$$\sqrt{(x_1+g)^2 + (y_1+f)^2} \gtreqless \sqrt{g^2 + f^2 - c}$$

Squaring both sides

$$(x_1+g)^2 + (y_1+f)^2 \gtreqless g^2 + f^2 - c$$

$$x_1^2 + g^2 + 2gx_1 + y_1^2 + f^2 + 2fy_1 \gtreqless g^2 + f^2 - c$$
$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 \gtreqless -c$$

Thus, $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \gtreqless 0$

Example 11: Determine whether the point $P(2,3)$ lies outside, on, or inside the circle:

$$x^2 + y^2 + 2x - 3y - 10 = 0$$

Solution:

Circle: $x^2 + y^2 + 2x - 3y - 10 = 0$ ...(1)

Point: $P(2,3)$

Put $x = 2, y = 3$ in L.H.S of eq. (1)

$$\text{L.H.S} = (2)^2 + (3)^2 + 2(2) - 3(3) - 10$$
$$= 4 + 9 + 4 - 9 - 10$$
$$= -2 < 0$$

Thus, the point $P(2,3)$ lies inside the circle.

Intersection of a Line and a Circle

Question: To show that a line $y = mx + c$ cuts a circle $x^2 + y^2 = a^2$ at most two points and to find the condition that the line be a tangent to circle.

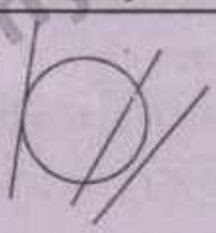
OR Discuss the nature of the points of intersection of line $y = mx + c$ and circle $x^2 + y^2 = a^2$.

Solution:

Circle: $x^2 + y^2 = a^2$ ...(1)

Line: $y = mx + c$ ...(2)

For their points of intersection, solving (1) and (2) simultaneously.



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Put $y = mx + c$ in eq. (1), we have

$$x^2 + (mx + c)^2 = a^2$$

$$x^2 + m^2x^2 + 2mcx + c^2 - a^2 = 0$$

$$(1 + m^2)x^2 + (2mc)x + (c^2 - a^2) = 0 \quad \ldots(3)$$

This being quadratic in x, gives two values of x. Thus, the line intersects the circle in at most two points.

For the nature of the points of intersection:

$$\text{Disc} = b^2 - 4ac$$

$$D = (2mc)^2 - 4(1 + m^2)(c^2 - a^2)$$

$$= 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2)$$

$$= 4(m^2c^2 - c^2 + a^2 - m^2c^2 + m^2a^2)$$

$$= 4(a^2 + m^2a^2 - c^2)$$

$$D = 4[a^2(1 + m^2) - c^2]$$

Case-I: Two distinct real points

The line $y = mx + c$ intersect the circle $x^2 + y^2 = a^2$ at two real and distinct points if

$$D > 0$$

$$4[a^2(1 + m^2) - c^2] > 0 \quad \Leftrightarrow a^2(1 + m^2) - c^2 > 0$$

$$\Leftrightarrow a^2(1 + m^2) > c^2$$

Case-II: One real point (tangent)

The line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$ at exactly one real point if

$$D = 0$$

$$4[a^2(1 + m^2) - c^2] = 0$$

$$\Leftrightarrow a^2(1 + m^2) - c^2 = 0 \Leftrightarrow a^2(1 + m^2) = c^2$$

Case-III: No real intersection

The line $y = mx + c$ cuts the circle $x^2 + y^2 = a^2$ at two imaginary points (i.e., does not cut the circle) if

$$D < 0$$

$$4[a^2(1 + m^2) - c^2] < 0$$

$$\Leftrightarrow a^2(1 + m^2) - c^2 < 0 \Leftrightarrow a^2(1 + m^2) < c^2$$

Corollary I: Condition of Tangency of a Line to a Circle

The line $y = mx + c$ is a tangent to the circle

$$x^2 + y^2 = a^2 \text{ if } D = 0$$

$$\Rightarrow \quad 4[a^2(1 + m^2) - c^2] = 0 \Rightarrow a^2(1 + m^2) = c^2$$

$$\sqrt{c^2} = \sqrt{a^2(1 + m^2)} \Rightarrow |c| = a\sqrt{1 + m^2}$$

or $c = \pm a\sqrt{1 + m^2}$

Therefore, the family of tangents to the circle as:

$$\boxed{y = mx \pm a\sqrt{1 + m^2}}$$

Note that $\frac{|c|}{\sqrt{1 + m^2}}$ is the perpendicular distance from the centre $(0, 0)$ to the line.

Hence, the condition for tangency is that:

$$|c| = a\sqrt{1 + m^2}$$

$$\frac{|c|}{\sqrt{1 + m^2}} = a \text{ (radius of circle)}$$

Corollary II: Coordinates of Point of Contact

For the tangent $y = mx + a\sqrt{1 + m^2}$ $\ldots(4)$

From eq. (3), we have

$$(1 + m^2)x^2 + (2mc)x + (c^2 - a^2) = 0$$

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2mc \pm \sqrt{0}}{2(1 + m^2)} \text{ for equal roots } D = 0$$

$$x = \frac{-m}{1 + m^2}(a\sqrt{1 + m^2}) \qquad \because c = a\sqrt{1 + m^2}$$

$$x = \frac{-am}{\sqrt{1 + m^2}} \text{ Put in eq. (4)}$$

$$y = m\left(\frac{-am^2}{\sqrt{1 + m^2}}\right) + a\sqrt{1 + m^2}$$

$$= \frac{-am^2 + a(1 + m^2)}{\sqrt{1 + m^2}} = \frac{-am^2 + a + am^2}{\sqrt{1 + m^2}}$$

$$y = \frac{a}{\sqrt{1 + m^2}}$$

$$\text{Point of contact} = \left(\frac{-am}{\sqrt{1 + m^2}}, \frac{a}{\sqrt{1 + m^2}}\right)$$

Similarly, for the tangent $y = mx - a\sqrt{1 + m^2}$

$$\text{Point of contact} = \left(\frac{am}{\sqrt{1 + m^2}}, \frac{a}{\sqrt{1 + m^2}}\right)$$

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Tangents from a Point to a Circle

Theorem 2: Two tangents can be drawn to a circle from any point $P(x_1, y_1)$. The tangents are real and distinct, coincident, or imaginary according as the point lies outside, on, or inside the circle.

Proof:

Circle: $x^2 + y^2 = a^2$

Tangent: $y = mx + a\sqrt{1+m^2} \quad \forall m$

If the tangent passes through $P(x_1, y_1)$, then:

$$y_1 = mx_1 + a\sqrt{1+m^2}$$

$$y_1 - mx_1 = a\sqrt{1+m^2}$$

Squaring both sides

$$(y_1 - mx_1)^2 = (a\sqrt{1+m^2})^2$$

$$y_1^2 - 2mx_1y_1 + m^2x_1^2 = a^2(1+m^2)$$

$$y_1^2 - 2mx_1y_1 + m^2x_1^2 - a^2 - a^2m^2 = 0$$

$$(x_1^2 - a^2)m^2 - (2x_1y_1)m + (y_1^2 - a^2) = 0$$

Disc $= b^2 - 4ac$

$$D = (-2x_1y_1)^2 - 4(x_1^2 - a^2)(y_1^2 - a^2)$$

$$= 4x_1^2y_1^2 - 4(x_1^2y_1^2 - a^2x_1^2 - a^2y_1^2 + a^4)$$

$$= 4(x_1^2y_1^2 - x_1^2y_1^2 + a^2x_1^2 + a^2y_1^2 - a^4)$$

$$= 4a^2(x_1^2 + y_1^2 - a^2)$$

The two values of m are:

(i) **Real and distinct** if

$$D > 0 \Rightarrow 4a^2(x_1^2 + y_1^2 - a^2) > 0 \Leftrightarrow x_1^2 + y_1^2 - a^2 > 0$$

$$\Leftrightarrow x_1^2 + y_1^2 > a^2 \text{ (point lies outside the circle)}$$

(ii) **Real and coincident** if

$$D = 0 \Rightarrow 4a^2(x_1^2 + y_1^2 - a^2) = 0$$

$$\Leftrightarrow x_1^2 + y_1^2 - a^2 = 0$$

$$\Leftrightarrow x_1^2 + y_1^2 = a^2 \text{ (point lies on the circle)}$$

(iii) **Imaginary** if $D < 0 \Rightarrow 4a^2(x_1^2 + y_1^2 - a^2) < 0$

$$\Leftrightarrow x_1^2 + y_1^2 - a^2 < 0$$

$$\Leftrightarrow x_1^2 + y_1^2 < a^2 \text{ (Point lies inside the circle)}$$

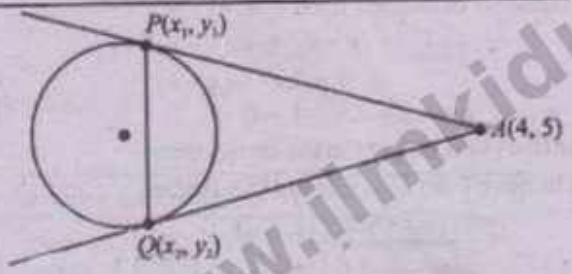
Example 12: Tangents are drawn from (4, 5) to the circle $x^2 + y^2 = 29$. Find an equation of the line joining the points of contact (The line is called the chord of contact).

Solution:

Circle: $x^2 + y^2 = 29$

Point: $A(4, 5)$

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the points of contact of the two tangents.



Equation of tangent at $P(x_1, y_1)$ is:

$$xx_1 + yy_1 = 29$$

Since it passes through $A(4, 5)$, therefore

$$4x_1 + 5y_1 = 29 \quad \ldots(1)$$

Similarly, the equation of tangent at $Q(x_2, y_2)$ passing through $A(4, 5)$ is

$$4x_2 + 5y_2 = 29 \quad \ldots(2)$$

From (1) and (2), both $P(x_1, y_1)$ and $Q(x_2, y_2)$ satisfy the equation $4x + 5y = 29$.

Hence, the equation of chord of contact is:

$$\boxed{4x + 5y = 29}$$

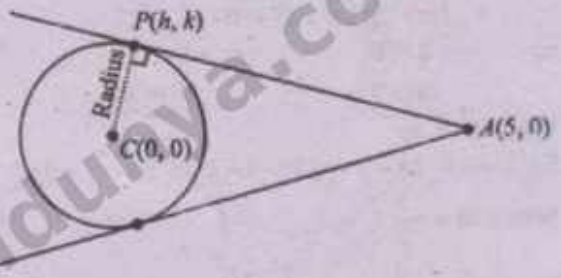
Example 13: Find a combined equation of the pair of tangents drawn from (5, 0) to the circle $x^2 + y^2 = 9$.

Solution:

Circle: $x^2 + y^2 = 9$

Point: $A(5, 0)$

Let $P(h, k)$ be any point on either of the two tangents drawn from $A(5, 0)$ to circle.



Slope of $\overline{AP} = m = \dfrac{k-0}{h-5} = \dfrac{k}{h-5}$

Equation of line $\overline{AP}$ is:

$$y - y_1 = m(x - x_1)$$

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$$y-0=\frac{k}{h-5}(x-5)$$
$$y(h-5)=k(x-5)$$
$$y(h-5)=kx-5k$$
$$kx-(h-5)y-5k=0 \quad \ldots(1)$$

If line (1) is tangent to the circle, then distance from $C(0, 0)$ to (1) = Radius

$$\frac{|0-0-5k|}{\sqrt{k^2+(h-5)^2}}=3$$
$$|5k|=3\sqrt{k^2+(h-5)^2}$$

Squaring both sides

$$25k^2=9(k^2+(h-5)^2)$$
$$25k^2=9k^2+9(h-5)^2$$
$$9(h-5)^2+9k^2-25k^2=0$$
$$9(h-5)^2-16k^2=0$$

Since (h, k) is any point on the two tangents, the combined (joint) equation of the pair of tangents is:

$$9(x-5)^2-16y^2=0$$

Example 14: Find the equations of the pair of tangents drawn from $(4, -1)$ to the circle $x^2+y^2-2x+4y=0$. Also find the points of contact.

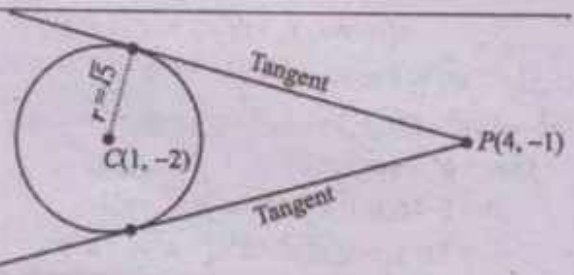
Solution:

Circle: $x^2+y^2-2x+4y=0$; $P(4, -1)$

Compare it with

$$x^2+y^2+2gx+2fy+c=0$$
$$\Rightarrow \quad 2g=-2 \;;\; 2f=4 \;;\; c=0$$
$$g=-1 \;;\; f=0$$

Centre: $C(-g, -f)=C(1, -2)$



Radius: $r=\sqrt{g^2+f^2-c}=\sqrt{(-1)^2+2^2-0}=\sqrt{5}$

Equation of tangent from $P(4, -1)$ is

$$y-y_1=m(x-x_1)$$
$$y+1=m(x-4)$$
$$mx-y-4m-1=0 \quad \ldots(1)$$

If line (1) is tangent to the circle, then

Distance from $C(1, -2)$ to (1) = Radius

$$\frac{|m(1)+2-4m-1|}{\sqrt{m^2+(-1)^2}}=\sqrt{5}$$
$$|-3m+1|=\sqrt{5}\cdot\sqrt{m^2+1}$$

Squaring both sides

$$9m^2+1-6m=5(m^2+1)$$
$$9m^2-6m+1-5m^2-5=0$$
$$4m^2-6m-4=0$$
$$2m^2-3m-2=0 \quad \text{Divide by '2'}$$
$$2m^2-4m+m-2=0$$
$$2m(m-2)+1(m-2)=0$$
$$(m-2)(2m+1)=0$$
$$\Rightarrow \quad m-2=0 \;;\; 2m+1=0$$
$$m=2 \;;\; m=-\frac{1}{2}$$

When $m=2$:

Eq. (1) $\Rightarrow 2x-y-4(2)-1=0 \Rightarrow 2x-y-9=0$

When $m=-\frac{1}{2}$:

Eq. (1) $\Rightarrow -\frac{1}{2}x-y-4\left(-\frac{1}{2}\right)-1=0$

Multiplying both sides by (-2)

$$x+2y-4+2=0 \;\Rightarrow\; x+2y-2=0$$

So, $2x-y-9=0$ and $x+2y-2=0$ are required tangents.

Points of Contact:

Circle: $x^2+y^2-2x+4y=0$ ...(2)

1st Tangent: $2x-y-9=0$ ...(3)

From eq. (3), we have

$y=2x-9$ Put in eq. (2)

$$x^2+(2x-9)^2-2x+4(2x-9)=0$$
$$x^2+4x^2+81-36x-2x+8x-18=0$$
$$5x^2-30x+45=0$$
$$x^2-6x+9=0 \quad \text{Divide by 5}$$
$$(x-3)^2=0 \Rightarrow x-3=0 \Rightarrow x=3$$

Put $x=3$ in eq. (3), we have

$$2(3)-y-9=0 \Rightarrow y=-3$$

One point of contact = $(3, -3)$

Now, circle: $x^2+y^2-2x+4y=0$ ...(2)

2nd Tangent: $x+2y-2=0$ ...(4)

From eq. (4), we have

$x=2-2y$ Put in eq. (2)

$$(2-2y)^2+y^2-2(2-2y)+4y=0$$
$$4+4y^2-8y+y^2-4+4y+4y=0$$
$$5y^2=0 \Rightarrow y=0 \quad \text{Put in eq. (4)}$$
$$x+0-2=0 \Rightarrow x=2$$

Other point of contact = $(2, 0)$

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Length of the Tangent from a Point to a Circle

Theorem 3: The length of the tangent drawn from a point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

Proof:

Circle: $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre: $C(-g, f)$

Radius: $r = \sqrt{g^2 + f^2 - c}$

Let $\overline{PT}$ be a tangent from P to the circle and T be a point of contact then $\overline{CT} \perp \overline{PT}$

From right ΔPTC, we have

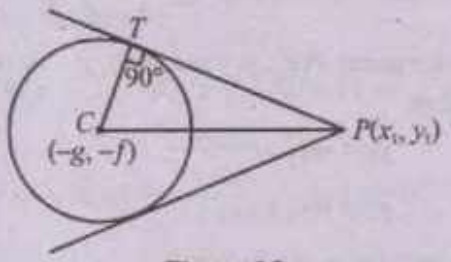


Figure 6.9

$$|\overline{PC}|^2 = |\overline{PT}|^2 + |\overline{CT}|^2 \quad \text{(Pythagoras Theorem)}$$

$$|\overline{PT}|^2 = |\overline{PC}|^2 - |\overline{CT}|^2$$

By distance formula, we get

$$|\overline{PT}|^2 = \left(\sqrt{(x_1+g)^2 + (y_1+f)^2}\right)^2 - r^2$$

$$= (x_1+g)^2 + (y_1+f)^2 - (g^2 + f^2 - c)$$

$$= x_1^2 + g^2 + 2x_1g + y_1^2 + f^2 + 2y_1f - g^2 - f^2 + c$$

$$|\overline{PT}|^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$\Rightarrow |\overline{PT}| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Rule to Find the Length of the Tangent

(i) Write the equation of the circle with coefficients of x^2 and y^2 as 1.

(ii) Make the right-hand side of the equation zero.

(iii) Substitute the coordinates of the point into the left-hand side expression.

(iv) Take the square root of the result.

Example 15: Find the length of the tangent from point $P(2, -5)$ to the circle $5x^2 + 5y^2 + 10x + 14y - 15 = 0$.

Solution:

Circle: $5x^2 + 5y^2 + 10x + 14y - 15 = 0$; $P(2, -5)$

Divide the equation by 5 to get the general form:

$$x^2 + y^2 + 2x + \frac{14}{5}y - 3 = 0$$

$$\text{Length of tangent} = \sqrt{(2)^2 + (-5)^2 + 2(2) + \frac{14}{5}(-5) - 3}$$

$$= \sqrt{4 + 25 + 4 - 14 - 3}$$

$$= \sqrt{16} = 4$$

Example 16: Length of a Chord of a Circle

Find the length of intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$.

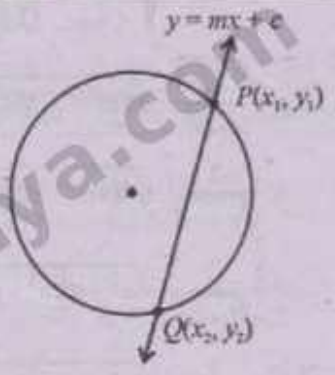
Solution:

Circle: $x^2 + y^2 = a^2$

Line: $y = mx + c$

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the points of intersection.

Put $y = mx + c$ in circle, we get



$$x^2 + (mx + c)^2 = a^2$$

$$x^2 + m^2x^2 + c^2 + 2mcx - a^2 = 0$$

$$(1 + m^2)x^2 + (2mc)x + (c^2 - a^2) = 0$$

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$\Rightarrow$ Sum of roots: $x_1 + x_2 = \dfrac{2mc}{1+m^2}$

Product of roots: $x_1 \cdot x_2 = \dfrac{c^2 - a^2}{1+m^2}$

Since the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the line, therefore

$$y_1 = mx_1 + c$$

and $$y_2 = mx_2 + c$$

Both $\Rightarrow$ $y_1 - y_2 = m(x_1 - x_2)$ ...(1)

Length of the chord $\overline{PQ}$ is:

$$|\overline{PQ}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{(x_1 - x_2)^2 + m^2(x_1 - x_2)^2} \quad \text{Using eq. (1)}$$

$$= \sqrt{(1+m^2)(x_1 - x_2)^2}$$

$$= \sqrt{1+m^2} \cdot \sqrt{x_1^2 + x_2^2 - 2x_1x_2}$$

$$= \sqrt{1+m^2} \cdot \sqrt{x_1^2 + x_2^2 + 2x_1x_2 - 4x_1x_2}$$

$$= \sqrt{1+m^2} \cdot \sqrt{(x_1 + x_2)^2 - 4x_1x_2}$$

Putting values of $x_1 + x_2$ and x_1x_2

$$= \sqrt{1+m^2} \cdot \sqrt{\left(\frac{2mc}{1+m^2}\right)^2 - 4\left(\frac{c^2 - a^2}{1+m^2}\right)}$$

$$= \sqrt{1+m^2} \cdot \sqrt{\frac{4m^2c^2}{(1+m^2)^2} - \frac{4(c^2 - a^2)}{1+m^2}}$$

$$= \sqrt{1+m^2} \cdot \sqrt{\frac{4m^2c^2 - 4(c^2 - a^2)(1+m^2)}{(1+m^2)^2}}$$

$$= \frac{\sqrt{1+m^2}}{1+m^2} \cdot \sqrt{4m^2c^2 - 4(c^2 + m^2c^2 - a^2 - a^2m^2)}$$

$$= \frac{1}{\sqrt{1+m^2}} \cdot \sqrt{4(m^2c^2 - c^2 - m^2c^2 + a^2 + a^2m^2)}$$

$$= \frac{2}{\sqrt{1+m^2}} \cdot \sqrt{a^2 + a^2m^2 - c^2}$$

$$|\overline{PQ}| = \frac{2}{\sqrt{1+m^2}} \cdot \sqrt{a^2(1+m^2) - c^2}$$

Challenge!

Show that the formula for the length of the chord PQ can be reduced to the form $2\sqrt{a^2 - d^2}$, where a is the radius of the circle and d is the length of the perpendicular from centre of the circle to the line.

Solution:

Circle: $x^2 + y^2 = a^2$ $\Rightarrow$ Centre: $C(0, 0)$, Radius $= a$

Line: $y = mx + c$ $\Rightarrow$ $mx - y + c = 0$

From Example # 16 derivation, we get

$$|\overline{PQ}| = \frac{2}{\sqrt{1+m^2}} \cdot \sqrt{a^2(1+m^2) - c^2}$$

$$= 2\sqrt{\frac{a^2(1+m^2) - c^2}{1+m^2}} = 2\sqrt{\frac{a^2(1+m^2)}{1+m^2} - \frac{c^2}{1+m^2}}$$

$$|\overline{PQ}| = 2\sqrt{a^2 - \frac{c^2}{1+m^2}} \quad \text{...(1)}$$

Distance from $C(0, 0)$ to line $mx - y + c = 0$ is:

$$d = \frac{|m(0) - 0 + c|}{\sqrt{m^2 + (-1)^2}} \quad \Rightarrow \quad d = \frac{|c|}{\sqrt{m^2 + 1}}$$

Squaring both sides

$$d^2 = \frac{c^2}{1+m^2} \quad \text{Put in eq. (1)}$$

$$|\overline{PQ}| = 2\sqrt{a^2 - d^2} \quad \text{(Proved)}$$

Challenge!

Find the length of the chord intercepted by the line $3x + 4y + 15 = 0$ and the circle $x^2 + y^2 = 25$.

Solution:

Circle: $x^2 + y^2 = 25$ $\Rightarrow$ $x^2 + y^2 = 5^2$

$\Rightarrow$ Centre: $C(0, 0)$, Radius: $a = 5$

Line: $3x + 4y + 15 = 0$

d = Distance from $C(0, 0)$ to line

$$d = \frac{|3(0) + 4(0) + 15|}{\sqrt{3^2 + 4^2}} = \frac{15}{\sqrt{25}} = \frac{15}{5} = 3$$

Length of intercepted chord $= 2\sqrt{a^2 - d^2}$

$= 2\sqrt{5^2 - 3^2} = 2\sqrt{25 - 9}$

$= 2\sqrt{16} = 2(4) = 8$ units

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EXERCISE 6.2

Q1. Find the equations of the tangent and normal to the circles:

(a) $x^2 + y^2 = 8$ **at** $(2, 2)$ **and at** $(2\sqrt{2}\cos\theta, 2\sqrt{2}\sin\theta)$.

Solution:

Circle: $x^2 + y^2 = 8$

(i) At (2, 2)

Equation of tangent at (x_1, y_1) is:

$$xx_1 + yy_1 = 8$$

Put $x_1 = 2$ and $y_1 = 2$, we get

$$x(2) + y(2) = 8$$
$$2x + 2y - 8 = 0$$

Dividing both sides by '2'

$$x + y - 4 = 0 \quad \text{(Required tangent)}$$

$$\text{Slope of tangent} = m_T = \frac{1}{1} = -1$$

$$\text{Slope of normal} = m_N = -\frac{1}{m_T} = 1$$

Equation of normal at (2, 2) is:

$$y - y_1 = m_N(x - x_1)$$
$$y - 2 = 1(x - 2)$$
$$x - 2 - y + 2 = 0$$
$$x - y = 0 \quad \text{(Required Normal)}$$

(ii) At $(2\sqrt{2}\cos\theta, 2\sqrt{2}\sin\theta)$

Equation of tangent at (x_1, y_1) is:

$$xx_1 + yy_1 = 8$$

Put $x_1 = 2\sqrt{2}\cos\theta$ and $y_1 = 2\sqrt{2}\sin\theta$, we get

$$(2\sqrt{2}\cos\theta)x + (2\sqrt{2}\sin\theta)y = 8$$

Dividing both sides by $2\sqrt{2}$

$$x\cos\theta + y\sin\theta = \frac{8}{2\sqrt{2}}$$
$$x\cos\theta + y\sin\theta = \frac{4(2)}{2\sqrt{2}}$$
$$x\cos\theta + y\sin\theta = 2\sqrt{2} \quad \text{(Required Tangent)}$$

$$\text{Slope of tangent} = m_T = -\frac{\cos\theta}{\sin\theta}$$

$$\text{Slope of normal} = m_N = -\frac{1}{m_T} = \frac{\sin\theta}{\cos\theta}$$

Equation of normal at $(2\sqrt{2}\cos\theta, 2\sqrt{2}\sin\theta)$ is:

$$y - y_1 = m_N(x - x_1)$$
$$y - 2\sqrt{2}\sin\theta = \frac{\sin\theta}{\cos\theta}(x - 2\sqrt{2}\cos\theta)$$
$$y\cos\theta - 2\sqrt{2}\sin\theta\cos\theta = x\sin\theta - 2\sqrt{2}\sin\theta\cos\theta$$
$$y\cos\theta = x\sin\theta$$
$$x\sin\theta - y\cos\theta = 0 \quad \text{(Required Normal)}$$

(b) $x^2 + y^2 - 6x - 2y - 7 = 0$ **at** $(4, 5)$

Solution:

Circle: $x^2 + y^2 - 6x - 2y - 7 = 0$; at $(4, 5)$

$$x^2 + y^2 - 3(2x) - 2y - 7 = 0$$

Equation of tangent at (x_1, y_1) is:

$$xx_1 + yy_1 - 3(x + x_1) - (y + y_1) - 7 = 0$$

Put $x_1 = 4$ and $y_1 = 5$, we get

$$4x + 5y - 3(x + 4) - (y + 5) - 7 = 0$$
$$4x + 5y - 3x - 12 - y - 5 - 7 = 0$$
$$x + 4y - 24 = 0 \text{ (Required tangent)}$$

$$\text{Slope of tangent} = m_T = -\frac{1}{4}$$

$$\text{Slope of normal} = m_N = -\frac{1}{m_T} = 4$$

Equation of normal at (4, 5) is:

$$y - y_1 = m_N(x - x_1)$$
$$y - 5 = 4(x - 4)$$
$$y - 5 = 4x - 16$$
$$4x - y - 11 = 0 \quad \text{(Required Normal)}$$

Q2. Show that the line $4x + 3y - 25 = 0$ **is tangent to the circle** $x^2 + y^2 = 25$ **and find the point of contact.**

Solution:

Line: $4x + 3y - 25 = 0$ ...(1)

Circle: $x^2 + y^2 = 25$ ...(2)

$\Rightarrow \quad x^2 + y^2 = 5^2$

Compare it with

$$x^2 + y^2 = a^2$$

$\Rightarrow$ Centre: $C(0, 0)$ and Radius: $a = 5$

Line (1) will be the tangent to circle, if

Distance from $C(0, 0)$ to line = Radius

$$\frac{|4(0) + 3(0) - 25|}{\sqrt{4^2 + 3^2}} = 5$$

$$\frac{25}{\sqrt{25}} = 5$$

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$$\frac{25}{5} = 5$$

$$5 = 5 \text{ (True)}$$

For point of contact, solving (1) and (2)

From eq. (1), we have

$$3y = 25 - 4x \Rightarrow y = \frac{25-4x}{3} \quad \ldots(3)$$

Put $y = \frac{25-4x}{3}$ in eq. (2)

$$x^2 + \left(\frac{25-4x}{3}\right)^2 = 25$$

$$x^2 + \frac{625+16x^2-200x}{9} = 25$$

$$9x^2 + 625 + 16x^2 - 200x = 225 \quad \text{Multiply by '9'}$$

$$25x^2 - 200x + 400 = 0$$

$$x^2 - 8x + 16 = 0 \quad \text{Divide by 25}$$

$$(x-4)^2 = 0 \Rightarrow x - 4 = 0 \Rightarrow x = 4$$

Put $x = 4$ in eq. (3)

$$y = \frac{25-4(4)}{3} = \frac{25-16}{3} = \frac{9}{3} = 3$$

$\Rightarrow$ Point of contact = (4, 3)

Q3. Find the equations of tangents to the circle $x^2 + y^2 = 4$,

Solution:

Circle: $x^2 + y^2 = 4$

$\Rightarrow \quad x^2 + y^2 = 2^2$

Compare it with

$$x^2 + y^2 = a^2$$

$\Rightarrow$ Centre: $C(0, 0)$; Radius: $a = 2$

Let required equation of tangent be:

$$y = mx \pm a\sqrt{1+m^2}$$

$$y = mx \pm 2\sqrt{1+m^2} \quad \ldots(1)$$

(a) which are inclined at an angle of 45° to the axis of x.

Solution:

Since the tangents are inclined at an angle of 45° with x-axis, so $\alpha = 45°$

Slope of tangent = $m = \tan\alpha = \tan 45° = 1$

Eq. (1) $\Rightarrow \quad y = 1x \pm 2\sqrt{1+1^2}$

$$x - y \pm 2\sqrt{2} = 0 \quad \text{(Required Tangents)}$$

(b) which are parallel to the line $3x + 4y + 1 = 0$.

Solution:

Line: $3x + 4y + 1 = 0 \quad \ldots(2)$

Slope of line = $-\frac{3}{4}$

Since the tangents are parallel to line (2), so

Slope of tangent = $m = -\frac{3}{4}$

Eq. (1) $\Rightarrow \quad y = -\frac{3}{4}x \pm 2\sqrt{1+\left(-\frac{3}{4}\right)^2}$

$$4y = -3x \pm 8\sqrt{1+\frac{9}{16}} \quad \text{Multiply by '4'}$$

$$4y = -3x \pm 8\sqrt{\frac{25}{16}}$$

$$3x + 4y = \pm 8\left(\frac{5}{4}\right)$$

$$3x + 4y = \pm 2(5)$$

$$3x + 4y \pm 10 = 0 \quad \text{(Required Tangents)}$$

(c) which are perpendicular to the line $6x - 8y + 3 = 0$.

Solution:

Line: $6x - 8y + 3 = 0 \quad \ldots(3)$

Slope of line = $-\frac{6}{-8} = \frac{3}{4}$

Since, the tangents are perpendicular to line (3), so

Slope of tangent = $m = -\frac{4}{3}$

Eq. (1) $\Rightarrow \quad y = -\frac{4}{3}x \pm 2\sqrt{1+\left(-\frac{4}{3}\right)^2}$

$$3y = -4x \pm 6\sqrt{1+\frac{16}{9}} \quad \text{Multiply by '3'}$$

$$4x + 3y = \pm 6\sqrt{\frac{25}{9}}$$

$$4x + 3y = \pm 6\left(\frac{5}{3}\right)$$

$$4x + 3y = \pm 2(5)$$

$$4x + 3y \pm 10 = 0 \quad \text{(Required Tangents)}$$

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Q4. Find the equations of the tangents to the circle $x^2 + y^2 - 4x - 2y - 4 = 0$, which are parallel to the line $3x - 4y + 2 = 0$.

Solution:

Circle: $x^2 + y^2 - 4x - 2y - 4 = 0$

Compare it with

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\Rightarrow \quad 2g = -4 \;;\; 2f = -2 \;;\; c = -4$$

$$g = -2 \;;\; f = -1$$

Centre: $C(-g, -f) = C(2, 1)$

Radius: $r = \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (-1)^2 + 4} = \sqrt{9} = 3$

Line: $3x - 4y + 2 = 0$

Slope of line $= -\frac{3}{-4} = \frac{3}{4}$

Since, the tangents are parallel to given line, so

Slope of tangent $= m = \frac{3}{4}$

Let required equation of tangent be:

$$y = mx + c$$

$$y = \frac{3}{4}x + c$$

$$4y = 3x + 4c \qquad \text{Multiply by '4'}$$

$$3x - 4y + 4c = 0 \qquad \ldots(1)$$

If line (1) is tangent to circle, then

Distance from $C(2, 1)$ to (1) = Radius

$$\frac{|3(2) - 4(1) + 4c|}{\sqrt{3^2 + (-4)^2}} = 3$$

$$|2 + 4c| = 3\sqrt{25}$$

$$|2 + 4c| = 15$$

$$2 + 4c = \pm 15$$

$\Rightarrow$ $2 + 4c = 15$ | $2 + 4c = -15$

$4c = 15 - 2$ | $4c = -15 - 2$

$4c = 13$ | $4c = -17$

When $4c = 13$:

Eq. (1) $\Rightarrow$ $3x - 4y + 13 = 0$

When $4c = -17$:

Eq. (1) $\Rightarrow$ $3x - 4y - 17 = 0$

Hence, $3x - 4y + 13 = 0$ and $3x - 4y - 17 = 0$ are required equations of tangents.

Q5. Find the equations of the tangents drawn from:

(a) the point $(-3, 7)$ to the circle $x^2 + y^2 = 8$.

(b) the point $(12, -4)$ to the circle $x^2 + y^2 - 4x - 2y - 20 = 0$.

Also find the points of contact.

(a) the point $(-3, 7)$ to the circle $x^2 + y^2 = 8$.

Solution:

Circle: $x^2 + y^2 = 8$; From $P(-3, 7)$

$$\Rightarrow \quad x^2 + y^2 = (\sqrt{8})^2$$

Compare it with

$$x^2 + y^2 = a^2$$

Centre: $C(0, 0)$, Radius: $a = \sqrt{8}$

Equation of tangent from $P(-3, 7)$ is:

$$y - y_1 = m(x - x_1)$$

$$y - 7 = m(x + 3)$$

$$mx - y + 3m + 7 = 0 \qquad \ldots(1)$$

If line (1) is tangent to circle, then

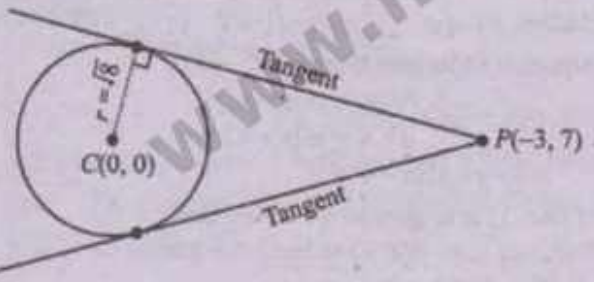
Distance from $C(0, 0)$ to line (1) = Radius

$$\frac{|m(0) - 0 + 3m + 7|}{\sqrt{m^2 + (-1)^2}} = \sqrt{8}$$

$$|3m + 7| = \sqrt{8} \cdot \sqrt{m^2 + 1}$$

Squaring both sides

$$9m^2 + 49 + 42m = 8(m^2 + 1)$$



$$9m^2 + 49 + 42m - 8m^2 - 8 = 0$$

$$m^2 + 42m + 41 = 0$$

$$m^2 + 41m + m + 41 = 0$$

$$m(m + 41) + 1(m + 41) = 0$$

$$(m + 1)(m + 41) = 0$$

$$\Rightarrow \quad m + 1 = 0 \;;\; m + 41 = 0$$

$$m = -1 \;;\; m = -41$$

When $m = -1$:

Eq. (1) $\Rightarrow$ $-1x - y + 3(-1) + 7 = 0 \Rightarrow x + y - 4 = 0$

When $m = -41$:

Eq. (1) $\Rightarrow$ $-41x - y + 3(-41) + 7 = 0 \Rightarrow 41x + y + 116 = 0$

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Points of Contact:

Circle: $x^2 + y^2 = 8$ ...(2)

1st Tangent: $x + y - 4 = 0$ ...(3)

From eq. (3), we have

$y = 4 - x$ Put in eq. (2)

$x^2 + (4-x)^2 = 8$

$x^2 + 16 + x^2 - 8x - 8 = 0$

$2x^2 - 8x + 8 = 0$

$x^2 - 4x + 4 = 0$ Divide by 2

$(x-2)^2 = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2$

Put $x = 2$ in eq. (3), we have

$2 + y - 4 = 0 \Rightarrow y = 2$

One point of contact = (2, 2)

Now, circle; $x^2 + y^2 = 8$ ...(2)

2nd Tangent: $41x + y + 116 = 0$ ...(4)

From eq. (4), we have

$y = -41x - 116$ Put in eq. (2)

$x^2 + (-41x - 116)^2 = 8$

$x^2 + 1681x^2 + 13456 + 9512x = 8$

$1682x^2 + 9512x + 13448 = 0$

$841x^2 + 4756x + 6724 = 0$

$(29x + 82)^2 = 0 \Rightarrow 29x + 82 = 0 \Rightarrow x = \frac{-82}{29}$

Put $x = \frac{82}{29}$ in eq. (4), we have

$41\left(\frac{82}{29}\right) + y + 116 = 0 \Rightarrow y = \frac{3362}{29} - 116$

$y = \frac{3362 - 3364}{29} \Rightarrow y = -\frac{2}{29}$

Other point of contact = $\left(\frac{82}{29}, -\frac{2}{29}\right)$

(b) the point (12, −4) to the circle $x^2 + y^2 - 4x - 2y - 20 = 0$.

Solution:

Circle: $x^2 + y^2 - 4x - 2y - 20 = 0$

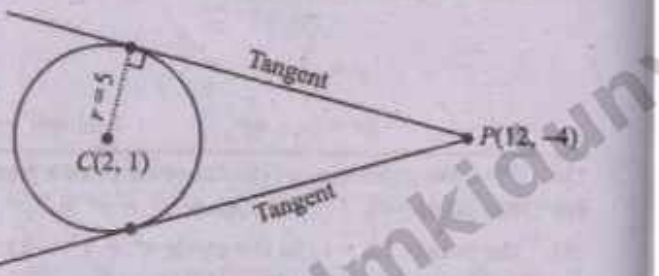
From $P(12, -4)$

Compare it with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow \quad 2g = -4 \; ; \quad 2f = -2 \; ; \quad c = -20$

$g = -2 \; ; \quad f = -1$



Centre: $C(-g, -f) = C(2, 1)$

Radius: $r = \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (-1)^2 + 20} = \sqrt{25} = 5$

Equation of tangent from $P(12, -4)$ is:

$y - y_1 = m(x - x_1)$

$y + 4 = m(x - 12)$

$mx - y - 12m - 4 = 0$ ...(1)

If line (1) is tangent to the circle, then

Distance from $C(2, 1)$ to line (1) = Radius

$\frac{|m(2) - 1 - 12m - 4|}{\sqrt{m^2 + (-1)^2}} = 5$

$|-10m - 5| = 5\sqrt{m^2 + 1}$

$|-2m - 1| = \sqrt{m^2 + 1}$ Divide by 5

Squaring both sides

$4m^2 + 1 + 4m = m^2 + 1$

$3m^2 + 4m = 0$

$m(3m + 4) = 0$

$\Rightarrow \quad m = 0; \; 3m + 4 = 0 \Rightarrow m = -\frac{4}{3}$

When $m = 0$:

Eq. (1) $\Rightarrow (0)x - y - 12(0) - 4 = 0 \Rightarrow y + 4 = 0$

When $m = -\frac{4}{3}$:

Eq. (1) $\Rightarrow -\frac{4}{3}x - y - 12\left(-\frac{4}{3}\right) - 4 = 0$

$4x + 3y - 48 + 12 = 0$ Multiply by (−3)

$4x + 3y - 36 = 0$

Points of Contact:

Circle: $x^2 + y^2 - 4x - 2y - 20 = 0$ ...(2)

1st Tangent: $y + 4 = 0$ ...(3)

From eq. (3), we have

$y = -4$ Put in eq. (2)

$x^2 + (-4)^2 - 4x - 2(-4) - 20 = 0$

$x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2$

One point of contact = (2, −4)

Now, circle: $x^2 + y^2 - 4x - 2y - 20 = 0$ ...(2)

2nd Tangent: $4x + 3y - 36 = 0$ ...(4)

From eq. (4), we have

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$3y = 36 - 4x \Rightarrow y = \frac{36-4x}{3}$

Put $y = \frac{36-4x}{3}$ in eq. (2)

$$x^2 + \left(\frac{36-4x}{3}\right)^2 - 4x - 2\left(\frac{36-4x}{3}\right) - 20 = 0$$

$$x^2 + \frac{1296 + 16x^2 - 288x}{9} - 4x - \left(\frac{72-8x}{3}\right) - 20 = 0$$

Multiply both sides by 9

$$9x^2 + 1296 + 16x^2 - 288x - 36x - 216 + 24x - 180 = 0$$

$$25x^2 - 300x + 900 = 0$$

$$x^2 - 12x + 36 = 0 \qquad \text{Divide by 25}$$

$$(x-6)^2 = 0 \Rightarrow x - 6 = 0 \Rightarrow x = 6$$

Put $x = 6$ in eq. (4), we have

$$4(6) + 3y - 36 = 0 \Rightarrow 3y = 12 \Rightarrow y = 4$$

Other point of contact = (6, 4)

Q6. A circle has the parametric equations: $x = 3 + 5\cos\theta, y = -1 + 5\sin\theta$

(a) Find the Cartesian equation of the circle.

Solution:

Parametric equations of circle are:

$x = 3 + 5\cos\theta$ | $y = -1 + 5\sin\theta$

$x - 3 = 5\cos\theta$ | $y + 1 = 5\sin\theta$

Squaring and adding both equations

$$(x-3)^2 + (y+1)^2 = 5^2\cos^2\theta + 5^2\sin^2\theta$$

$$(x-3)^2 + (y+1)^2 = 25(\cos^2\theta + \sin^2\theta)$$

$$(x-3)^2 + (y+1)^2 = 25(1)$$

$$(x-3)^2 + (y+1)^2 = 25 \text{ (Cartesian equation of circle)}$$

(b) Find the centre and radius of the circle.

Solution:

Circle: $(x-3)^2 + (y+1)^2 = 25$

$$(x-3)^2 + (y-(-1))^2 = 5^2$$

Compare it with

$$(x-h)^2 + (y-k)^2 = r^2$$

$\Rightarrow h = 3, k = -1, r = 5$

Centre: $C(h, k) = C(3, -1)$

Radius: $r = 5$

(c) Find the parametric coordinates of the point P on the circle corresponding to $\theta = \frac{\pi}{2}$.

Solution:

Parametric equations of circle are:

$x = 3 + 5\cos\theta$ and $y = -1 + 5\sin\theta$

At $\theta = \frac{\pi}{2}$:

$x = 3 + 5\cos\frac{\pi}{2}$ | $y = -1 + 5\sin\frac{\pi}{2}$

$= 3 + 5(0)$ | $= -1 + 5(1)$

$= 3 + 0$ | $= -1 + 5$

$x = 3$ | $y = 4$

Required point on circle = $P(3, 4)$

(d) Find the equation of the tangent to the circle at point P.

Solution:

Circle: $(x-3)^2 + (y+1)^2 = 25$; at $P(3, 4)$

$$x^2 - 6x + 9 + y^2 + 2y + 1 - 25 = 0$$

$$x^2 + y^2 - 3(2x) + 2y - 15 = 0$$

Equation of tangent at $P(x_1, y_1)$ is:

$$xx_1 + yy_1 - 3(x + x_1) + (y + y_1) - 15 = 0$$

Put $x_1 = 3$ and $y_1 = 4$

$$3x + 4y - 3(x+3) + (y+4) - 15 = 0$$

$$3x + 4y - 3x - 9 + y + 4 - 15 = 0$$

$$5y - 20 = 0 \Rightarrow 5(y-4) = 0$$

$y - 4 = 0$ (Required Tangent)

(e) Find the equation of the normal to the circle at point P.

Solution:

Equation of tangent at $P(3, 4)$ is:

$$y - 4 = 0$$

$y = 4$ (Horizontal line through P)

The normal at a point P is perpendicular to the tangent.

$\Rightarrow$ Equation of normal is a vertical line through $P(3, 4)$

$\Rightarrow$ Normal: $x = 3$

(f) Find the equation of the tangent to the circle at the point where $\theta = \pi$.

Solution:

Parametric equations of circle are:

$x = 3 + 5\cos\theta$ and $y = -1 + 5\sin\theta$

At $\theta = \pi$:

$x = 3 + 5\cos\pi$ | $y = -1 + 5\sin\pi$

$= 3 + 5(-1)$ | $= -1 + 5(0)$

$= 3 - 5$ | $= -1 + 0$

$x = -2$ | $y = -1$

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Point on a circle = $Q(-2,-1)$

Circle: $(x-3)^2+(y+1)^2=25$

$$x^2-6x+9+y^2+2y+1-25=0$$

$$x^2+y^2-3(2x)+2y-15=0$$

Equation of tangent at $Q(x_1, y_1)$ is:

$$xx_1+yy_1-3(x+x_1)+(y+y_1)-15=0$$

Put $x_1=-2, y=-1$

$$-2x-y-3(x-2)+(y-1)-15=0$$

$$-2x-y-3x+6+y-1-15=0$$

$$-5x-10=0 \Rightarrow 5x=-10$$

$$x=-2 \text{ (Required Tangent)}$$

PARABOLA

Definition: A Parabola is the locus of a point which moves in a plane such that its distance from a fixed point, called the focus, is equal to its distance from a given fixed straight line, called the directrix.

Derivation of the Standard Equation of Parabola

Let $F(a, 0)$ be the focus of parabola and $x=-a$ be the directrix line. Let $P(x, y)$ be any point on the parabola.

By the definition of a parabola

$$|\overline{PF}| = |\overline{PM}|$$

$$\sqrt{(x-a)^2+(y-0)^2} = \frac{|x+a|}{\sqrt{1^2+0^2}}$$

$$\sqrt{x^2+a^2-2ax+y^2} = |x+a|$$

Squaring both sides, we get

$$x^2+a^2-2ax+y^2 = x^2+a^2+2ax$$

$$y^2 = x^2+a^2+2ax-x^2-a^2+2ax$$

$$y^2 = 4ax$$

This is the standard equation of the parabola.

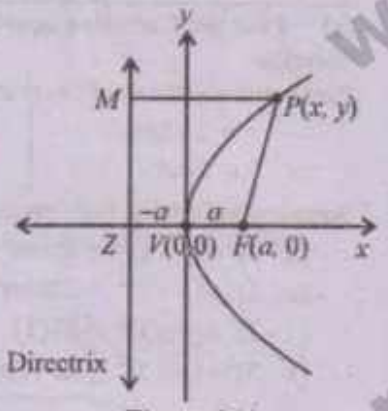

Figure 6.11

Elements of the Parabola

1. **Axis:** The straight line through the focus and perpendicular to the directrix is called axis of the parabola. In the figure, the axis is x-axis.
2. **Vertex:** The point where the axis meets the parabola is called the vertex. In the figure, V is the vertex with coordinates $(0, 0)$.
3. **Chord and Focal Chord:** A line segment joining two distinct points on a parabola is called a chord. A chord passing through the focus is called a focal chord.
4. **Focal Distance:** The distance of a point on the parabola from the focus is called its focal distance. In the figure, $|FP|$ is the focal distance.
5. **Latus Rectum:** The focal chord perpendicular to the axis of the parabola is called the latus rectum. Its equation is $x=a$. It intersects the parabola $y^2=4ax$ at points where $y^2=4a^2$ that is, $y=\pm 2a$. Thus, the end points L and L' of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$.
6. **Parametric Equations:** The equation $y^2=4ax$ can be expressed parametrically.

   Put $y=2at$ in equation $y^2=4ax$,

   we get $(2at)^2=4ax \Rightarrow 4a^2t^2=4ax \Rightarrow x=at^2$

   Hence, the parametric equations of $y^2=4ax$ are:

   $$x=at^2, y=2at$$

   The point $(at^2, 2at)$ lies on the parabola for all real t and is often referred to as the t-point.

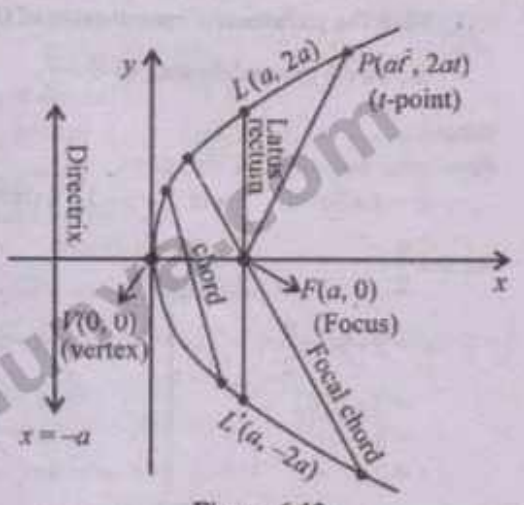

Figure 6.12

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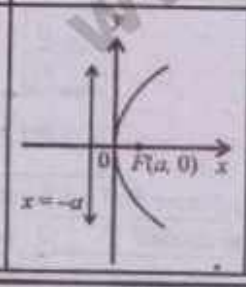
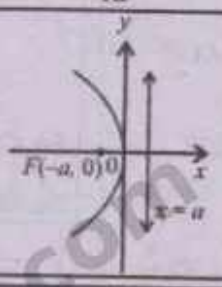
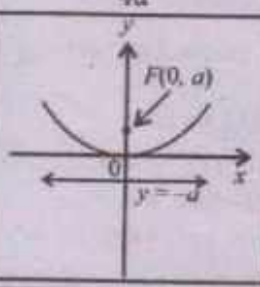
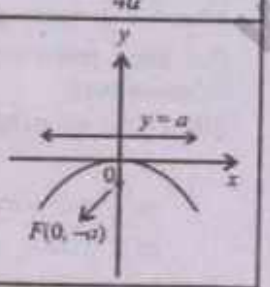
Other Standard Forms of the Parabola

By taking the directrix on the right side of the focus, or below or above the focus, we obtain three more standard forms. The table below summarizes the key features of all four forms.

Parabola in standard form	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Equation of latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Sketch of parabola in standard form				

Example 17: Analyze the parabola $x^2 - 4x - 4y - 2 = 0$ and sketch its graph.

Solution:

$$x^2 - 4x - 4y - 2 = 0$$

$$x^2 - 4x = 4y + 2$$

Add 2^2 on both sides

$$x^2 - 4x + 2^2 = 4y + 2 + 2^2$$

$$(x-2)^2 = 4y + 6$$

$$(x-2)^2 = 4\left(y + \frac{6}{4}\right)$$

$$(x-2)^2 = 4\left(y + \frac{3}{2}\right)$$

Let $x - 2 = X$ and $y + \frac{3}{2} = Y$ $\Rightarrow$ $X^2 = 4Y$

Compare it with

$$X^2 = 4aY \Rightarrow 4a = 4 \Rightarrow a = 1$$

Focus: $F(0, a)$

$\Rightarrow$ $X = 0$

$x - 2 = 0$

$x = 2$

$Y = a$

$y + \frac{3}{2} = 1$

$y = 1 - \frac{3}{2} = -\frac{1}{2}$

$\Rightarrow$ $F\left(2, -\frac{1}{2}\right)$

Vertex: $V(0, 0)$

$\Rightarrow$ $X = 0$

$x - 2 = 0$

$x = 2$

$Y = 0$

$y + \frac{3}{2} = 0$

$y = -\frac{3}{2}$

$\Rightarrow$ $V\left(2, -\frac{3}{2}\right)$

Directrix: $Y = -a$

$\Rightarrow$ $y + \frac{3}{2} = -1$

$y = -1 - \frac{3}{2}$

$y = -\frac{5}{2}$

Axis of parabola: $X = 0$

$\Rightarrow$ $x - 2 = 0$

$x = 2$

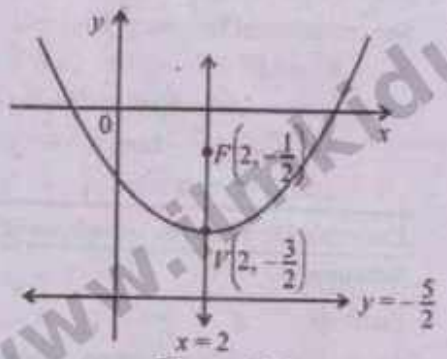


Figure 6.13

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Tangents and Normals to a Parabola (Condition of the tangency to Parabola)

Question: Show that the line $y=mx+c$ intersects the parabola $y^2=4ax$ at most two points, and find condition for tangency.

OR

Find the condition that the straight line $y=mx+c$ may be a tangent to parabola $y^2=4ax$, and hence show that the equation of any tangent to parabola is $y=mx+\frac{a}{m}(m\neq 0)$.

Solution:

Parabola: $y^2=4ax$ ...(1)

Line: $y=mx+c$ ...(2)

For their points of intersection solving (1) and (2) simultaneously

Put $y=mx+c$ in eq. (1), we have

$$(mx+c)^2=4ax$$
$$m^2x^2+c^2+2mcx-4ax=0$$
$$m^2x^2+(2mc-4a)x+c^2=0 \quad ...(3)$$

This being quadratic in x, give two values of x. Thus, the line intersects the parabola in at most two points

Here $A=m^2, B=2mc-4a, C=c^2$

For condition of tangency: Disc = 0

$$\Rightarrow \quad B^2-4AC=0$$
$$(2mc-4a)^2-4(m^2)(c^2)=0$$
$$4m^2c^2+16a^2-16mca-4m^2c^2=0$$
$$16a^2-16mca=0$$
$$16a(a-mc)=0$$
$$\Rightarrow \quad a-mc=0 \qquad \because a\neq 0 \text{ and } m\neq 0$$
$$mc=a$$
$$\boxed{c=\frac{a}{m}} \quad \text{(Required condition)}$$

Eq. (2) $\Rightarrow$ $\boxed{y=mx+\frac{a}{m}}$ (Required Tangent)

Similarly, we can find condition of tangency for other parabolas.

Parabolas	Condition of Tangency
$y^2=4ax$	$c=\frac{a}{m}$
$y^2=-4ax$	$c=-\frac{a}{m}$
$x^2=4ay$	$c=-am^2$
$x^2=-4ay$	$c=am^2$

Example 18: Find the equations of the tangent and normal to the parabola $y^2=4ax$ at the point (x_1, y_1).

Solution:

Parabola: $y^2=4ax$ ...(1) at $P(x_1, y_1)$

Since $P(x_1, y_1)$ is a point on parabola, so

$$y_1^2=4ax_1 \quad ...(2)$$

Differentiating (1) w.r.t. 'x'

$$\frac{d}{dx}(y^2)=\frac{d}{dx}(4ax)$$
$$2y\frac{dy}{dx}=4a(1)$$
$$\Rightarrow \quad \frac{dy}{dx}=\frac{2a}{y}$$

Slope of tangent at $P(x_1, y_1)=\left.\frac{dy}{dx}\right|_{P(x_1,y_1)}$

$$m_T=\frac{2a}{y_1}$$

Equation of tangent at $P(x_1, y_1)$ is:

$$y-y_1=m_T(x-x_1)$$
$$y-y_1=\frac{2a}{y_1}(x-x_1)$$

Multiplying both sides by y_1

$$yy_1-y_1^2=2ax-2ax_1$$
$$yy_1-4ax_1=2ax-2ax_1 \quad \text{Using (2)}$$
$$yy_1=2ax-2ax_1+4ax_1$$
$$yy_1=2ax+2ax_1$$
$$yy_1=2a(x+x_1)$$

This is the required equation of the tangent.

Slope of the normal $=\frac{1}{m_T}$

$$\Rightarrow \quad m_N=\frac{y_1}{2a}$$

Equation of normal at $P(x_1, y_1)$ is:

$$y-y_1=m_N(x-x_1)$$
$$\boxed{y-y_1=-\frac{y_1}{2a}(x-x_1)}$$

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EXERCISE 6.3

Q1. Find the focus, vertex and directrix of the following parabolas:

(i) $y^2 = 12x$

Solution:

$$y^2 = 12x$$

Compare it with

$$y^2 = 4ax$$

$$\Rightarrow \quad 4a = 12 \Rightarrow a = 3$$

Focus: $F(a,0) = F(3,0)$

Vertex: $V(0,0)$

Directrix: $x = -a$

$\Rightarrow \quad x = -3$

(ii) $x^2 = 8(y-1)$

Solution:

$$x^2 = 8(y-1)$$

Let $x = X$ and $y - 1 = Y$

$$X^2 = 8Y$$

Compare it with

$$X^2 = 4aY$$

$$\Rightarrow \quad 4a = 8 \Rightarrow a = 2$$

Focus: $F(0,a)$

$\Rightarrow \quad X = 0 \quad | \quad Y = a$

$x = 0 \quad | \quad y - 1 = 2$

$\quad\quad\quad

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Q2. Write the equation of the parabola with given elements.

(i) Focus (–3, 2); directrix $x = 5$

Solution:

Focus: $F(-3, 2)$

Directrix: $x = 5$

$\Rightarrow \quad x - 5 = 0$

Let $P(x, y)$ be any point on the parabola

By definition of a parabola

$$|\overline{PF}| = |\overline{PM}|$$

$$\sqrt{(x+3)^2+(y-2)^2} = \frac{|1x+0y-5|}{\sqrt{1^2+0^2}}$$

$$\sqrt{x^2+9+6x+y^2+4-4y} = |x-5|$$

Squaring both sides

$$x^2+y^2+6x-4y+13 = x^2+25-10x$$

$$y^2-4y = x^2-10x+25-x^2-6x-13$$

$$y^2-4y = -16x+12$$

$$y^2-4y+2^2 = -16x+12+2^2$$

$$(y-2)^2 = -16x+16$$

$$(y-2)^2 = -16(x-1)$$

(ii) Focus (3, 5), vertex (3, 2)

Solution:

Focus: $F(3, 5)$

Vertex: $V(3, 2) \Rightarrow h = 3, k = 2$

Since the focus is above the vertex, therefore parabola opens upward.

Equation of parabola is

$$(x-h)^2 = 4a(y-k)$$

$$(x-3)^2 = 4a(y-2) \qquad \ldots(1)$$

As $\quad a = |\overline{VF}|$

$\Rightarrow \quad a = \sqrt{(3-3)^2+(5-2)^2}$

$$a = \sqrt{0^2+3^2} = 3 \qquad \text{Put in eq. (1)}$$

$$(x-3)^2 = 4(3)(y-2)$$

$$(x-3)^2 = 12(y-2)$$

(iii) Directrix $x = 4$ vertex (6, 3)

Solution:

Directrix: $x = 4 \Rightarrow x - 4 = 0$

Vertex: $V(6, 3) \Rightarrow h = 6, k = 3$

Since the vertex is right the directrix line, therefore parabola opens to right

Equation of parabola is

$$(y-k)^2 = 4a(x-h)$$

$$(y-3)^2 = 4a(x-6) \qquad \ldots(1)$$

As $\quad a =$ Distance from $V(6, 3)$ to directrix

$$a = \frac{|6-4|}{\sqrt{1^2+0^2}} = \frac{|2|}{1} = 2$$

Put $a = 2$ in eq. (1), we get

$$(y-3)^2 = 4(2)(x-6)$$

$$(y-3)^2 = 8(x-6)$$

(iv) Axis $y = 0$ through (2, 2) and (6, –3)

Solution:

As axis of parabola is $y = 0$ (x-axis), so

Vertex: $V(h, k) = V(h, 0)$

Equation of parabola is

$$(y-k)^2 = 4a(x-h)$$

$$(y-0)^2 = 4a(x-h)$$

$$y^2 = 4a(x-h) \qquad \ldots(1)$$

Since, the parabola (1) passes through (2, 2) and (6, –3), therefore

At (2, 2):

$$2^2 = 4a(2-h)$$

$$4 = 4a(2-h)$$

$$8a-4ah-4 = 0 \qquad \ldots(2)$$

At (6, –3):

$$(-3)^2 = 4a(6-h)$$

$$9 = 24a-4ah$$

$$-24a+4ah+9 = 0 \qquad \ldots(3)$$

Equation (2) + Equation (3)

$$8a-4ah-4 = 0$$

$$-24a+4ah+9 = 0$$

$$-16a+5 = 0 \Rightarrow a = \frac{-5}{-16} \Rightarrow a = \frac{5}{16}$$

Put $a = \frac{5}{16}$ in eq. (2), we have

$$8\left(\frac{5}{16}\right)-4\left(\frac{5}{16}\right)h-4 = 0$$

$$\frac{5}{2}-\frac{5}{4}h-4 = 0$$

$$10-5h-16 = 0 \qquad \text{Multiply by 4}$$

$$5h = -6 \Rightarrow h = -\frac{6}{5}$$

Putting values in eq. (1), we get

$$y^2 = 4\left(\frac{5}{16}\right)\left(x-\left(-\frac{6}{5}\right)\right)$$

$$y^2 = \frac{5}{4}\left(x+\frac{6}{5}\right)$$

$$y^2 = \frac{5x}{4}+\frac{6}{4}$$

Multiply both sides by 4

$$4y^2 = 5x+6$$

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Q3. Find the equation of the parabola having its focus at the origin and directrix parallel to:

(i) x-axis

Solution:

Since the directrix is parallel to x-axis, so

Directrix: $y = d$

where d is the directed distance from origin to directrix

$\Rightarrow \quad y - d = 0$

Focus: $F(0, 0)$

Let $P(x, y)$ be any point on the parabola

By the definition of a parabola

$$|\overline{PF}| = |\overline{PM}|$$

$$\sqrt{(x-0)^2+(y-0)^2} = \frac{|y-d|}{\sqrt{0^2+1^2}}$$

$$\sqrt{x^2+y^2} = |y-d|$$

Squaring both sides

$$x^2+y^2 = y^2+d^2-2dy$$

$$\boxed{x^2 = -2dy+d^2}$$

(ii). y-axis

Solution:

Since the directrix is parallel to y-axis, so

Directrix: $x = d$

where d is the directed distance from origin to directrix

$\Rightarrow \quad x - d = 0$

Focus: $F(0, 0)$

Let $P(x, y)$ be any point on the parabola.

By the definition of a parabola

$$|\overline{PF}| = |\overline{PM}|$$

$$\sqrt{(x-0)^2+(y-0)^2} = \frac{|x-d|}{\sqrt{1^2+0^2}}$$

$$\sqrt{x^2+y^2} = \frac{|x-d|}{1}$$

Squaring both sides

$$x^2+y^2 = x^2+d^2-2dx$$

$$\boxed{y^2 = -2dx+d^2}$$

Q4. Show that the equation of a parabola with focus at $(a\cos\theta, a\sin\theta)$ and directrix $x\cos\theta + y\sin\theta + a = 0$ is $(x\sin\theta - y\cos\theta)^2 = 4a(x\cos\theta + y\sin\theta)$.

Solution:

Focus: $F(a\cos\theta, a\sin\theta)$

Directrix: $x\cos\theta + y\sin\theta + a = 0$

Let $P(x, y)$ be any point on the parabola

By the definition of a parabola

$$|\overline{PF}| = |\overline{PM}|$$

$$\sqrt{(x-a\cos\theta)^2+(y-a\sin\theta)^2} = \frac{|x\cos\theta+y\sin\theta+a|}{\sqrt{\cos^2\theta+\sin^2\theta}}$$

$$\sqrt{x^2+a^2\cos^2\theta-2ax\cos\theta+y^2+a^2\sin^2\theta-2ay\sin\theta} = \frac{|x\cos\theta+y\sin\theta+a|}{1}$$

$$x^2+y^2+a^2(\cos^2\theta+\sin^2\theta)-2ax\cos\theta-2ay\sin\theta = |x\cos\theta+y\sin\theta+a|^2$$

$$x^2+y^2+a^2(1)-2ax\cos\theta-2ay\sin\theta = x^2\cos^2\theta+y^2\sin^2\theta+a^2+2xy\sin\theta\cos\theta+2ay\sin\theta+2ax\cos\theta$$

$$x^2-x^2\cos^2\theta+y^2-y^2\sin^2\theta-2xy\sin\theta\cos\theta = 4ax\cos\theta+4ay\sin\theta$$

$$x^2(1-\cos^2\theta)+y^2(1-\sin^2\theta)-2xy\sin\theta\cos\theta = 4a(x\cos\theta+y\sin\theta)$$

$$x^2\sin^2\theta+y^2\cos^2\theta-2xy\sin\theta\cos\theta = 4a(x\cos\theta+y\sin\theta)$$

$$(x\sin\theta-y\cos\theta)^2 = 4a(x\cos\theta+y\sin\theta)$$

(Proved)

Q5. Prove that in a parabola, a circle described on the latus rectum as diameter touches the directrix.

Solution:

Consider a parabola $y^2 = 4ax$

Focus: $F(a, 0)$

Vertex: $V(0, 0)$

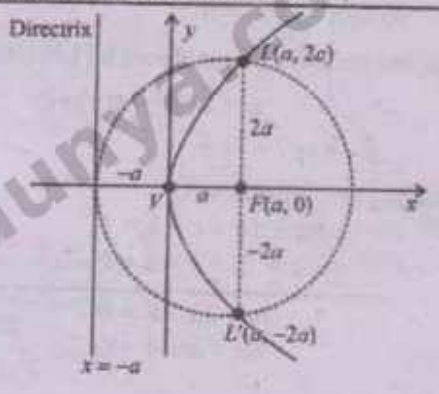
Directrix: $x = -a \Rightarrow x + a = 0$

Length of latus rectum $= 4a$

$\Rightarrow$ End of latus rectum $= L(a, 2a)$ and $L'(a, -2a)$

Since, the latus rectum $\overline{LL'}$ is the diameter of circle, therefore

Centre of circle = Midpoint of $\overline{LL'}$



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$$= C\left(\frac{a+a}{2}, \frac{2a+(-2a)}{2}\right) = C(a, 0)$$

$$\text{Radius of circle} = \frac{1}{2}|LL'| = \frac{1}{2}(4a) = 2a$$

Directrix line touches the circle iff

Distance from $C(a, 0)$ to directrix = Radius

$$\Rightarrow \quad \frac{|a+a|}{\sqrt{1^2+0^2}} = 2a$$

$$\frac{|2a|}{1} = 2a$$

$$2a = 2a \qquad \text{(True)}$$

Hence, the circle described on the latus rectum as a diameter touches the directrix.

Q6. Prove that the chord of contact of the tangents drawn from any point on the directrix of the parabola $y^2 = 4ax$ always passes through the focus.

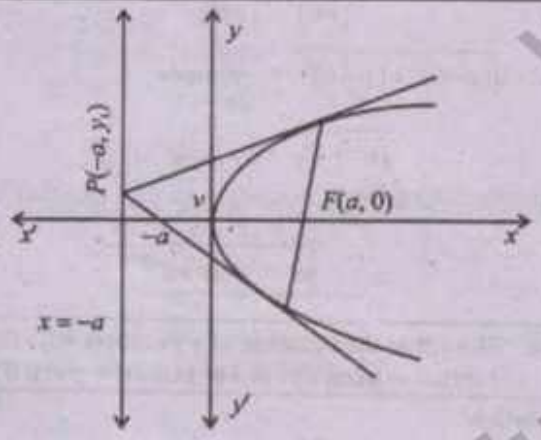
Solution:

Parabola: $y^2 = 4ax$

Focus: $F(a, 0)$

Directrix: $x = -a$

Let $P(-a, y_1)$ be any point on the parabola



Equation of chord of contact of tangents from an external point $P(x_1, y_1)$ to parabola is

$$yy_1 = 2a(x + x_1)$$

At $P(-a, y_1)$:

$$yy_1 = 2a(x + (-a))$$

$$yy_1 = 2a(x - a)$$

Check at focus $F(a, 0)$:

$$\Rightarrow \quad (0)y_1 = 2a(a - a)$$

$$0 = 2a(0)$$

$$0 = 0 \text{ (True)}$$

Hence proved.

Q7. The parabola $y = ax^2 + bx + c$ has a vertex at $(3, -2)$ and passes through the point $(1, 6)$. Find the values of a, b, and c

Solution:

Parabola: $y = ax^2 + bx + c$

Since, the vertex $(3, -2)$ lies on parabola, so

$$-2 = a(3)^2 + b(3) + c$$

$$9a + 3b + c + 2 = 0 \qquad \ldots(1)$$

Since, the parabola passes through $(1, 6)$, so

$$6 = a(1)^2 + b(1) + c$$

$$a + b + c - 6 = 0 \qquad \ldots(2)$$

Equation (1) – Equation (2)

$$\begin{array}{r} 9a + 3b + c + 2 = 0 \\ \pm\ a \pm\ b \pm\ c - 6 = 0 \\ \hline 8a + 2b + 8 = 0 \end{array}$$

$$4a + b + 4 = 0 \qquad \ldots(3)$$

As we know

$$x\text{-coordinate of vertex} = -\frac{b}{2a}$$

$$\Rightarrow \quad 3 = -\frac{b}{2a} \qquad \Rightarrow \quad b = -6a \qquad \ldots(4)$$

Put $b = -6a$ in eq. (2)

$$4a - 6a + 4 = 0 \qquad \Rightarrow \quad -2a = -4$$

$$\boxed{a = 2}$$

Put $a = 2$ in eq. (4)

$$b = -6(2) \Rightarrow \boxed{b = -12}$$

Put $a = 2$ and $b = -12$ in eq. (2)

$$2 - 12 + c - 6 = 0$$

$$c - 16 = 0 \qquad \Rightarrow \quad \boxed{c = 16}$$

Hence $a = 2$, $b = -12$ and $c = 16$

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Q8. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Solution:

$y^2 = 4ax$ at $P(at^2, 2at)$

Differentiating w.r.t. 'x'

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$$

$$2y\frac{dy}{dx} = 4a(1)$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

Slope of tangent at $P = \left.\frac{dy}{dx}\right|_P$

$$\Rightarrow \quad m_T = \frac{2a}{2at} = \frac{1}{t}$$

Slope of normal at $P = m_N = \frac{1}{m_T} = -t$

Equation of tangent at P is:

$$y - y_1 = m_T(x - x_1)$$

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$ty - 2at^2 = x - at^2 \qquad \text{Multiply by '}t\text{'}$$

$$x - ty + at^2 = 0 \qquad \text{(Required Tangent)}$$

Equation of normal at P is:

$$y - y_1 = m_N(x - x_1)$$

$$y - 2at = -t(x - at^2)$$

$$y - 2at = -tx + at^3$$

$$tx + y - at^3 - 2at = 0$$

$$tx + y - at(t^2 + 2) = 0 \qquad \text{(Required Normal)}$$

Q9. Find the equations of the tangent and normal to the parabola $x^2 - 6x - 2y + 11 = 0$ at the point $(5, 3)$.

Solution:

$x^2 - 6x - 2y + 11 = 0$ at $P(5, 3)$

Differentiating w.r.t 'x'

$$\frac{d}{dx}(x^2 - 6x - 2y + 11) = 0$$

$$2x - 6(1) - 2\frac{dy}{dx} + 0 = 0$$

$$2\frac{dy}{dx} = 2(x - 3)$$

$$\frac{dy}{dx} = x - 3$$

Slope of tangent at $P = \left.\frac{dy}{dx}\right|_P$

$$\Rightarrow \quad m_T = 5 - 3 = 2$$

Slope of normal at $P = m_N = -\frac{1}{2}$

Equation of tangent at P is:

$$y - y_1 = m_T(x - x_1)$$

$$y - 3 = 2(x - 5)$$

$$y - 3 = 2x - 10$$

$$2x - y - 7 = 0 \qquad \text{(Required Tangent)}$$

Equation of normal at P is:

$$y - y_1 = m_N(x - x_1)$$

$$y - 3 = -\frac{1}{2}(x - 5)$$

$$2y - 6 = -x + 5$$

$$x + 2y - 11 = 0 \qquad \text{(Required Normal)}$$

Q10. Find the equation of the tangent to the parabola $y^2 = 12x$ which is parallel to the line $3x - y + 4 = 0$. Also find the point of tangency.

Solution:

Parabola: $y^2 = 12x$

Compare it with

$$y^2 = 4ax$$

$$\Rightarrow \quad 4a = 12 \Rightarrow a = 3$$

Given Line: $3x - y + 4 = 0$

Slope of line $= \frac{3}{-1} = 3$

Since the tangent is parallel to given line, so

Slope of tangent $= m_T = 3$

Equation of tangent is:

$$y = mx + \frac{a}{m}$$

$$y = 3x + \frac{3}{3}$$

$$y = 3x + 1$$

$$\boxed{3x - y + 1 = 0}$$

Point of Contact:

Parabola: $y^2 = 12x$ ...(1)

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Tangent: $3x - y + 1 = 0$ ...(2)

From eq. (2), we have

$y = 3x + 1$ Put in eq. (1)

$(3x+1)^2 = 12x$

$9x^2 + 6x + 1 - 12x = 0$

$9x^2 - 6x + 1 = 0$

$(3x-1)^2 = 0$

$3x - 1 = 0 \Rightarrow x = \frac{1}{3}$ Put in eq. (2)

$3\left(\frac{1}{3}\right) - y + 1 = 0$

$1 + 1 = y \Rightarrow y = 2$

Point of contact $= \left(\frac{1}{3}, 2\right)$

ELLIPSE

Definition: An **ellipse** is the set of all points in a plane for which the sum of distances from two fixed points, called the foci (plural of focus), is constant (see Figure 14). Kepler's first law states that the orbits of the planets in the solar system are elliptical.



Figure 6.14

Derivation of the Standard Equation of an Ellipse

Let $F(c,0)$ and $F'(-c,0)$ be the foci on x-axis, such that

Centre of ellipse = Midpoint of $\overline{FF'}$

$= C\left(\frac{c+(-c)}{2}, \frac{0+0}{2}\right) = C(0,0)$

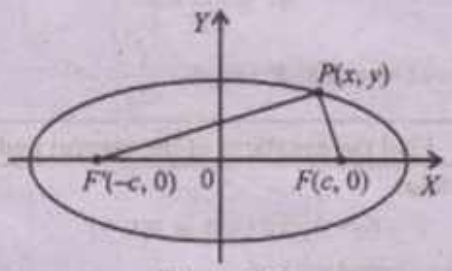

Figure 6.15

Let $P(x, y)$ be any point on the ellipse

By definition of an ellipse:

$|PF'| + |PF| = 2a$ (say), where $a > 0$

Using the distance formula

$$\sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\sqrt{x^2 + c^2 + 2cx + y^2} = 2a - \sqrt{x^2 + c^2 - 2cx + y^2}$$

Squaring both sides, we have

$$(\sqrt{x^2 + c^2 + 2cx + y^2})^2 = (2a - \sqrt{x^2 + c^2 - 2cx + y^2})^2$$

$$x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 - 4a\sqrt{x^2 + c^2 - 2cx + y^2}$$

$$4a\sqrt{x^2 + c^2 - 2cx + y^2} = 4a^2 - 4cx$$

Dividing both sides by '4'

$$a\sqrt{x^2 + c^2 - 2cx + y^2} = a^2 - cx$$

Do You Know?
The word focus means fireplace in Latin.

Again squaring both sides, we have

$$a^2(x^2 + c^2 - 2cx + y^2) = a^4 + c^2x^2 - 2a^2cx$$

$$a^2x^2 + a^2c^2 - 2a^2cx + a^2y^2 = a^4 + c^2x^2 - 2a^2cx$$

$$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2) \quad ...(1)$$

From triangle $PF'F$, we get

$|PF'| + |PF| > |F'F|$

$\Rightarrow \quad 2a > 2c \quad \because |F'F| = 2c$

$\Rightarrow \quad a > c \quad \Rightarrow a^2 - c^2 > 0$

Let $a^2 - c^2 = b^2$ Put in eq. (1)

$b^2x^2 + a^2y^2 = a^2b^2$

Dividing both sides by a^2b^2

$$\frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ Since $b^2 = a^2 - c^2$ and $a > c$ $\Rightarrow a^2 = b^2 + c^2 \Rightarrow a > b$

This is the standard equation of an ellipse.

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Key Features of the Ellipse

- **Foci:** The foci are at $(\pm c, 0)$, where $c^2 = a^2 - b^2$.
- **Centre:** The midpoint C of the segment joining the foci is called the centre. For the equation above, the centre is at the origin $(0, 0)$.
- **Vertices:** Setting $y = 0$ gives $x = \pm a$. The points $A'(-a, 0)$ and $A(a, 0)$ are the vertices. The line segment $A'A$ of length $2a$ is the major axis.
- **Co-vertices:** Setting $x = 0$ gives $y = \pm b$. The points $B'(0, -b)$ and $B(0, b)$ are called the co-vertices. The line segment $B'B$ of length $2b$ is the minor axis. Since $a > b$, the major axis is longer than the minor axis.
- **Symmetry:** The equation is unchanged if x is replaced by $-x$ and y by $-y$; hence the ellipse is symmetric about the x-axis, y-axis, and the origin.
- **Eccentricity:** The eccentricity e of an ellipse is defined as $e = \frac{c}{a}$. Since $a > c > 0$, we have $0 < e < 1$.
- **Directrices:** For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$, there are two directrices given by the vertical lines $x \pm \frac{a}{e}$.
- **Latus Rectum:** Each of the focal chords perpendicular to the major axis is called a latus rectum (plural: latera recta). The length of each latus rectum is $\frac{2b^2}{a}$.
- **Parametric Equations:** For all real θ, the point $(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Thus, the parametric equations of the ellipse are: $x = a\cos\theta, y = b\sin\theta$.
- **Foci on the y-axis:** If the foci are on the y-axis at $(0, \pm c)$, the roles of a and b are interchanged, and the standard equation becomes $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, $a > b$: with vertices $(0, \pm a)$, and covertices $(\pm b, 0)$.
- **Relation between a, b, c and e:** Since $b^2 = a^2 - c^2$, setting $c = ae \Rightarrow b^2 = a^2 - a^2e^2 = a^2(1 - e^2)$.
- **Circle as a Limiting Case:** If the foci becomes closer, then $c \to 0$ and $a \to b$, so the ellipse approaches to a circle with radius $r = a = b$, and has equation: $x^2 + y^2 = a^2$. Clearly for circle $e \to 0$.
- **Second definition of ellipse:** The ellipse can also be defined as, if a point P moves such that the ratio of its distance from a fixed point F (the focus) to its distance from a fixed straight line (the directrix) is equal to a constant e (less than 1), the locus of P is an ellipse of eccentricity e.
- That is, if $\frac{|PF|}{|PM|} = e$, where $0 < e < 1$, the locus of P is an ellipse.

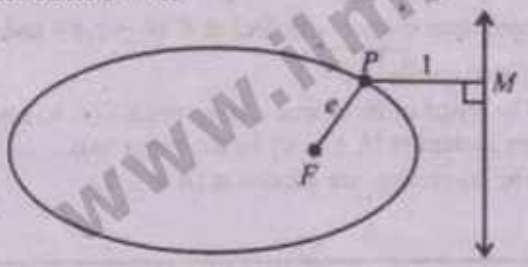

Figure 6.16

Note

The equation of ellipse can also be derived using the above definition.

Summary of Important Results

Ellipse in standard form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > 0$
Centre	$(0, 0)$	$(0, 0)$
Equation of major axis	$y = 0, -a \le x \le a$	$x = 0, -a \le y \le a$
Length of major axis	$2a$	$2a$
Equation of minor axis	$x = 0, -b \le y \le b$	$y = 0, -b \le x \le b$
Length of minor axis	$2b$	$2b$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Covertices	$(0, \pm b)$	$(\pm b, 0)$

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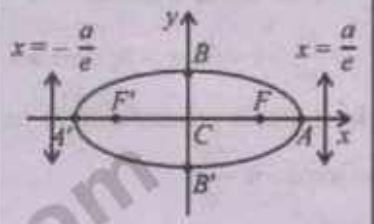
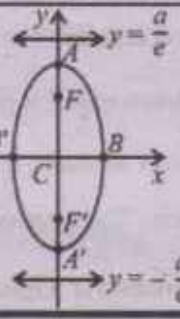
Foci	$(\pm c, 0), c = ae$	$(0, \pm c), c = ae$
Eccentricity	$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} < 1$	$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} < 1$
Equations of directrices	$y = \pm\frac{a}{e} = \pm\frac{c}{e^2}$	$y = \pm\frac{a}{e} = \pm\frac{c}{e^2}$
Equations of latera recta	$x = \pm ae$	$y = \pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Sketch of ellipse in standard form		

Note

In bothe cases of ellipses, the major axis is the longer line segment and foci lie on the major axis.

Theorem 1: (Translated forms of Horizontal Ellipses)

Proof:

The equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, with $a > b$ describes a horizontal ellipse with foci at $F'(h-c, k)$ and $F(h+c, k)$, where $c = \sqrt{a^2 - b^2}$.

The centre of the ellipse is at the point $C(h, k)$ and the vertices are located at $(h \pm a, k)$ on the major axis.

The co-vertices are located at $(h, k \pm b)$.

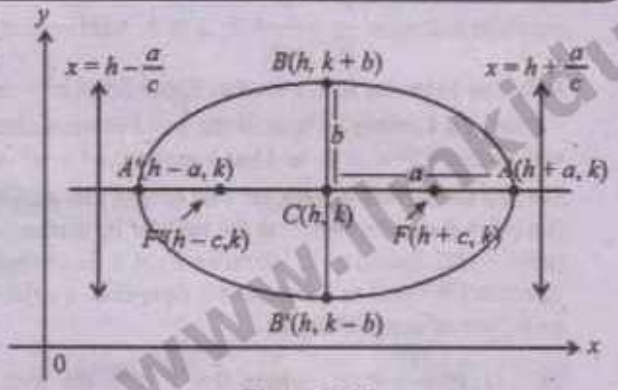


Figure 6.17

Theorem 2: (Translated forms of Vertical Ellipses)

Proof:

The equation $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, with $a > b$ describes a vertical ellipse with foci at $F'(h, k-c)$ and $F(h, k+c)$, where $c = \sqrt{a^2 - b^2}$. The centre of the ellipse is at the point $C(h, k)$ and the vertices are located at $(h, k \pm a)$ on the major axis. The co-vertices are located at $(h \pm b, k)$.

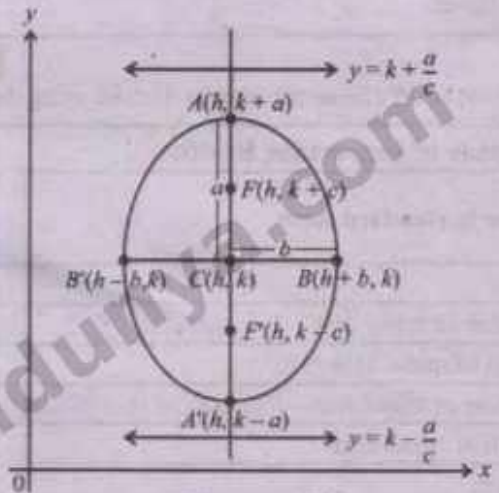


Figure 6.18

Example 19: Find an equation of the ellipse with foci (2, 1), (2, 7), major axis of length 10 and sketch its graph.

Solution:

Foci: $F'(2, 1)$ and $F(2, 7)$

Both foci have the same x-coordinate

$\Rightarrow$ Major axis is vertical.

Length of major axis = 10

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$\Rightarrow \quad 2a = 10 \Rightarrow a = \frac{10}{2} \Rightarrow a = 5$

Centre: $C\left(\frac{2+2}{2}, \frac{1+7}{2}\right) = C(2,4) \Rightarrow h = 2, k = 4$

C = Distance from $C(2,4)$ to $F(2,7)$

$= |4-7| = |-3| = 3$

As we know

$c^2 = a^2 - b^2$

$3^2 = 5^2 - b^2 \Rightarrow b^2 = 25 - 9 \Rightarrow b^2 = 16 \Rightarrow b = 4$

Required equation of ellipse is:

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-2)^2}{16} + \frac{(y-4)^2}{25} = 1$$

Sketch:

Centre: $C(h,k) = C(2,4)$

Foci: $F'(2,1), F(2,7)$

Vertices: $(h, k \pm a) = (2, 4 \pm 5) = (2,-1), (2,9)$

Co-vertices: $(h \pm b, k) = (2 \pm 4, 4) = (-2,4), (6,4)$

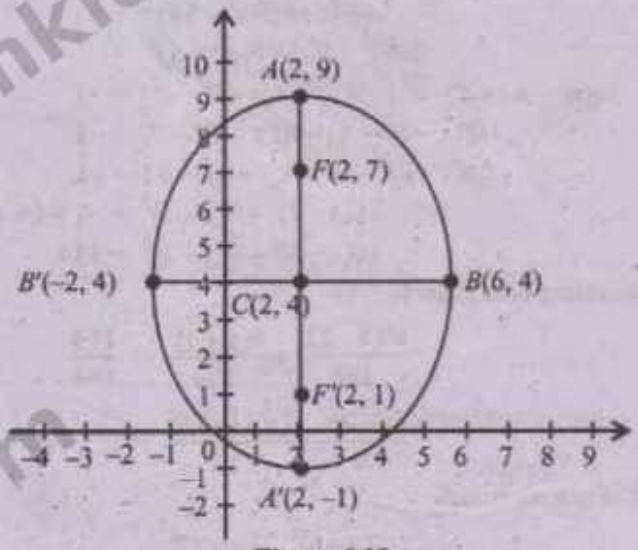

Figure 6.19

Example 20: Find an equation of the ellipse with centre (0, 0), focus (5, 0) and vertex (6, 0). Sketch its graph.

Solution:

Centre: $C(0,0)$

Focus: $F(5,0) \Rightarrow$ Other focus: $F'(-5,0)$

$\Rightarrow c = 5$

Vertex: $A(6,0) \Rightarrow$ Other vertex: $A'(-6,0)$

$\Rightarrow a = 6$

As we know

$c^2 = a^2 - b^2$

$5^2 = 6^2 - b^2 \Rightarrow b^2 = 36 - 25 \Rightarrow b^2 = 11 \Rightarrow b = \sqrt{11}$

All the points have same y-coordinate 0, so major axis is horizontal.

Required equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{6^2} + \frac{y^2}{(\sqrt{11})^2} = 1$$

$$\frac{x^2}{36} + \frac{y^2}{11} = 1$$

$$\frac{\ldots \pm abm\sqrt{a^2m^2 + b^2 - c^2} + b^2c + a^2m^2c}{b^2 + a^2m^2}$$

Sketch:

Centre: $C(0,0)$

Foci: $F(5,0)$

Ver...

...section $= \left(\frac{-a^2mc \pm ab\sqrt{a^2m^2 + b^2 - c^2}}{b^2 + a^2m^2}, \frac{b^2c \pm abm\sqrt{a^2m^2 + b^2 - c^2}}{b^2 + a^2m^2} \right)$

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Example 21: Find the centre, foci, eccentricity, vertices and directrices of the ellipse whose equation is $16x^2 + 9y^2 - 64x + 54y + 1 = 0$. Also sketch its graph.

Solution:

$$16x^2+9y^2-64x+54y+1=0$$
$$(16x^2-64x)+(9y^2+54y)=-1$$
$$16(x^2-4x)+9(y^2+6y)=-1$$
$$16(x^2-4x+2^2-2^2)+9(y^2+6y+3^2-3^2)=-1$$
$$16((x-2)^2-4)+9((y+3)^2-9)=-1$$
$$16(x-2)^2-64+9(y+3)^2-81=-1$$
$$16(x-2)^2+9(y+3)^2=-1+64+81$$
$$16(x-2)^2+9(y+3)^2=144$$

Dividing both sides by 144

$$\frac{16(x-2)^2}{144}+\frac{9(y+3)^2}{144}=\frac{144}{144}$$
$$\frac{(x-2)^2}{9}+\frac{(y-(-3))^2}{16}=1 \quad \text{(Vertical major axis)}$$

Comparing it with

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

$\Rightarrow h=2, k=-3, b^2=9 \Rightarrow b=3, a^2=16 \Rightarrow a=4.$

Centre: $C(h,k)=C(2,-3)$

As we know

$$c^2=a^2-b^2$$
$$c^2=16-9 \Rightarrow c^2=7 \Rightarrow c=\sqrt{7}$$

Sketch:

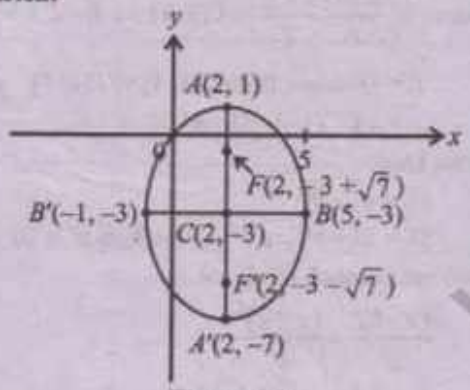

Figure 6.21

Foci: $(h, k\pm c)=(2,-3\pm\sqrt{7})$

$\Rightarrow \quad F'(2,-3-\sqrt{7}), F(2,-3+\sqrt{7})$

Eccentricity: $e=\frac{c}{a}=\frac{\sqrt{7}}{4}<1$

Vertices: $(h, k\pm a)=(2,-3\pm 4)$

$\Rightarrow \quad A'(2,-7), A(2,1)$

Directrices: $y=k\pm\frac{a}{e}=-3\pm\frac{4}{\sqrt{7}/4}=-3\pm\frac{16}{\sqrt{7}}$

Example 22: An ellipse has the following properties:

(i) Its major axis is vertical and has a length of 10 units. (ii) The distance between its foci is 6 units.

(iii) The centre of the ellipse is at the point $(3,-2)$.

Find the standard equation of this ellipse.

Solution:

Centre: $C(h,k)=C(3,-2) \Rightarrow h=3, k=-2$

Length of major axis $= 10$

$\Rightarrow \quad 2a=10 \Rightarrow a=5$

Distance between foci $= 6$

$\Rightarrow \quad 2c=6 \Rightarrow c=3$

As we know

$$c^2=a^2-b^2$$
$$3^2=5^2-b^2 \Rightarrow b^2=25-9 \Rightarrow b^2=16 \Rightarrow b=4$$

Required equation of ellipse is:

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \quad \because \text{Major axis is vertical.}$$
$$\frac{(x-3)^2}{4^2}+\frac{(y-(-2))^2}{5^2}=1$$
$$\frac{(x-3)^2}{16}+\frac{(y+2)^2}{25}=1$$

Tangents and Normals to Ellipse

Question: Prove that the condition for the line $y=mx+c$ to be tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$. Also, find the coordinates of the points of intersection.

Solution:

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Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

Line $y=mx+c$ ...(2)

For their points of intersection solving (1) and (2) simultaneously.

Put $y=mx+c$ in eq. (1), we have

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

Multiplying both sides by a^2b^2

$$b^2x^2+a^2(m^2x^2+c^2+2mcx)=a^2b^2$$

$$b^2x^2+a^2m^2x^2+a^2c^2+2a^2mcx-a^2b^2=0$$

$$(b^2+a^2m^2)x^2+(2a^2mc)+a^2(c^2-b^2)=0$$

This being quadratic in x, give two values of x. Thus, the line intersects the ellipse in at most two points.

Here $A=b^2+a^2m^2, B=2a^2mc, C=a^2(c^2-b^2)$

Tangency Condition:

For tangency condition roots will be real and equal.

$$\Rightarrow \quad D=0$$

$$B^2-4AC=0$$

$$(2a^2mc)^2-4(b^2+a^2m^2)\cdot a^2(c^2-b^2)=0$$

$$4a^4m^2c^2-4a^2(b^2+a^2m^2)(c^2-b^2)=0$$

$$4a^2[a^2m^2c^2-(b^2c^2-b^4+a^2m^2c^2-a^2m^2b^2)]=0$$

$$a^2m^2c^2-b^2c^2+b^4-a^2m^2c^2+a^2m^2b^2=0 \quad \because a\neq 0$$

$$-b^2c^2+b^4+a^2m^2b^2=0$$

$$b^2(-c^2+b^2+a^2m^2)=0$$

$$\Rightarrow \quad -c^2+b^2+a^2m^2=0 \quad \because b\neq 0$$

$$\boxed{c^2=a^2m^2+b^2} \quad \text{(Tangency condition)}$$

$$\Rightarrow \quad c=\pm\sqrt{a^2m^2+b^2} \quad \text{Put in eq. (2)}$$

$$\boxed{y=mx\pm\sqrt{a^2m^2+b^2}} \quad \text{(Tangent Line } \forall m)$$

Points of Intersection: ($D>0$)

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

$$=\frac{-2a^2mc\pm\sqrt{(2a^2mc)^2-4(b^2+a^2m^2)a^2(c^2-b^2)}}{2(b^2+a^2m^2)}$$

$$=\frac{-2a^2m\pm\sqrt{4a^4m^2c^2-4a^2(b^2c^2-b^4+a^2m^2c^2-a^2m^2b^2)}}{2(b^2+a^2m^2)}$$

$$=\frac{-2a^2mc\pm\sqrt{4a^2(a^2m^2c^2-b^2c^2+b^4-a^2m^2c^2+a^2m^2b^2)}}{2(b^2+a^2m^2)}=\frac{-2a^2mc\pm 2a\sqrt{b^2(-c^2+b^2+a^2m^2)}}{2(b^2+a^2m^2)}$$

$$x=\frac{-a^2mc\pm ab\sqrt{a^2m^2+b^2-c^2}}{b^2+a^2m^2} \quad \text{Put in eq. (2)}$$

$$y=m\left[\frac{-a^2mc\pm ab\sqrt{a^2m^2+b^2-c^2}}{b^2+a^2m^2}\right]+c$$

$$=\frac{-a^2m^2c\pm abm\sqrt{a^2m^2+b^2-c^2}+c(b^2+a^2m^2)}{b^2+a^2m^2}=\frac{-a^2m^2c\pm abm\sqrt{a^2m^2+b^2-c^2}+b^2c+a^2m^2c}{b^2+a^2m^2}$$

$$y=\frac{b^2c\pm abm\sqrt{a^2m^2+b^2-c^2}}{b^2+a^2m^2}$$

$$\text{Points of Intersection} = \left(\frac{-a^2mc\pm ab\sqrt{a^2m^2+b^2-c^2}}{b^2+a^2m^2}, \frac{b^2c\pm abm\sqrt{a^2m^2+b^2-c^2}}{b^2+a^2m^2}\right)$$

Note

If a glass of water is tilted, the rim of the water surface traces an ellipse.

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Example 23: Find the equation of the tangent and normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1).

Solution:

Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1) at $P(x_1, y_1)$

Since $P(x_1, y_1)$ lies on ellipse, so

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \quad ...(2)$$

Challenge!

Find the equation of an ellipse with centre at $(-2, 3)$, a vertical minor axis of length 8, and that touches the line $y=-x+7$.

Differentiating (1) w.r.t. 'x'

$$\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$

$$\frac{2y}{b^2}\cdot\frac{dy}{dx}=-\frac{2x}{a^2}$$

$$\frac{dy}{dx}=-\frac{2x}{a^2}\cdot\frac{b^2}{2y}$$

$$\frac{dy}{dx}=-\frac{b^2}{a^2}\cdot\frac{x}{y}$$

Slope of tangent at $P(x_1, y_1)=\left.\frac{dy}{dx}\right|_{P(x_1, y_1)}$

$$m_T=\frac{b^2x_1}{a^2y_1}$$

Equation of tangent at $P(x_1, y_1)$ is:

$$y-y_1=m_T(x-x_1)$$

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$\Rightarrow \quad \frac{y_1}{b^2}(y-y_1)=-\frac{x_1}{a^2}(x-x_1)$$

$$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=-\frac{xx_1}{a^2}+\frac{x_1^2}{a^2}$$

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

$$\boxed{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$$ Using eq. (2)

This is the required equation of tangent.

Slope of normal at $P(x_1, y_1)=-\frac{1}{m_T}$

$$\Rightarrow \quad m_N=\frac{a^2y_1}{b^2x_1}$$

Equation of normal at $P(x_1, y_1)$ is:

$$y-y_1=m_N(x-x_1)$$

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

$$b^2x_1(y-y_1)=a^2y_1(x-x_1)$$

$$b^2x_1y-b^2x_1y_1=a^2xy_1-a^2x_1y_1$$

$$a^2xy_1-b^2x_1y=a^2x_1y_1-b^2x_1y_1$$

$$a^2xy_1-b^2x_1y=x_1y_1(a^2-b^2)$$

Dividing both sides by x_1y_1: (say $x_1\neq 0, y_1\neq 0$)

$$\boxed{\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2}$$

EXERCISE 6.4

Q1. Find an equation of the ellipse with given data:

(i) Foci $(0, \pm 5)$ and minor axis of length 8.

Solution:

Foci: $(0, \pm 5) \Rightarrow c=5$

$F(0,5), F'(0,-5)$

Both foci have the same x-coordinate 0.

$\Rightarrow$ Major axis is vertical.

Centre: $C\left(\frac{0+0}{2}, \frac{5+(-5)}{2}\right)=C(0,0)$

Length of minor axis = 8

$$\Rightarrow \quad 2b=8 \Rightarrow b=\frac{8}{2} \Rightarrow b=4$$

As we know

$$c^2=a^2-b^2$$

$$5^2=a^2-4^2 \Rightarrow a^2=25+16 \Rightarrow a^2=41$$

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Required equation of ellipse is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

$$\frac{x^2}{4^2}+\frac{y^2}{41}=1$$

$$\frac{x^2}{16}+\frac{y^2}{41}=1$$

(ii) Vertices (−2, 5), (−2, −3); foci (−2, 4) and (−2, −2).

Solution:

Vertices: $A'(-2,5)$ and $A(-2,-3)$

Foci: $F'(-2,4)$ and $F(-2,-2)$

All the points have the same x-coordinate (-2)

$\Rightarrow$ Major axis is vertical.

Centre: $C\left(\frac{-2+(-2)}{2},\frac{4+(-2)}{2}\right)=C(-2,1) \Rightarrow h=-2, k=1$

a = Distance from $C(-2, 1)$ to $A'(-2, 5)$

$= |5-1| = 4$

c = Distance from $C(-2, 1)$ to $F'(-2, 4)$

$= |1-4| = |-3| = 3$

As we know

$c^2 = a^2 - b^2$

$3^2 = 4^2 - b^2 \Rightarrow b^2 = 16-9 \Rightarrow b^2 = 7$

Required equation of ellipse is:

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

$$\frac{(x-(-2))^2}{7}+\frac{(y-1)^2}{4^2}=1$$

$$\frac{(x+2)^2}{7}+\frac{(y-1)^2}{16}=1$$

(iii) Vertices $(\pm 6, 0)$, eccentricity $= \frac{2}{3}$.

Solution:

Vertices: $A'(-6,0), A(6,0) \Rightarrow a=6$

Both vertices have the same y-coordinate '0'

$\Rightarrow$ Major axis is horizontal.

Centre: $C\left(\frac{-6+6}{2},\frac{0+0}{2}\right)=C(0,0)$

Eccentricity: $e=\frac{2}{3}$

$$\Rightarrow \frac{c}{a}=\frac{2}{3} \Rightarrow \frac{c}{6}=\frac{2}{3} \Rightarrow c=\frac{2}{3}\times 6 \Rightarrow c=4$$

As we know

$c^2 = a^2 - b^2$

$4^2 = 6^2 - b^2 \Rightarrow b^2 = 36-16 = 20$

Required equation of ellipse is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{x^2}{6^2}+\frac{y^2}{20}=1$$

$$\frac{x^2}{36}+\frac{y^2}{20}=1$$

(iv) Centre $(-3, 1)$, major axis parallel to the x-axis and of length 10 units, minor axis parallel to the y-axis and of length 6 units.

Solution:

Centre: $C(-3, 1) \Rightarrow h=-3, k=1$

Given major axis is parallel to the x-axis

$\Rightarrow$ Major axis is horizontal.

Length of major axis = 10

$$\Rightarrow \quad 2a=10 \Rightarrow a=\frac{10}{2} \Rightarrow a=5$$

Length of minor axis = 6

$$\Rightarrow \quad 2b=6 \Rightarrow b=\frac{6}{2} \Rightarrow b=3$$

Required equation of ellipse is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-(-3))^2}{5^2}+\frac{(y-1)^2}{3^2}=1$$

$$\frac{(x+3)^2}{25}+\frac{(y-1)^2}{9}=1$$

Q2. Find the centre, foci, eccentricity, vertices and directrices of the ellipse whose equation is given:

(i) $4x^2 + 9y^2 = 36$

Solution:

$$4x^2+9y^2=36$$

Dividing both sides by 36

$$\frac{4x^2}{36}+\frac{9y^2}{36}=\frac{36}{36}$$

$$\frac{x^2}{9}+\frac{y^2}{4}=1 \qquad \text{(Horizontal major axis)}$$

Compare it with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$\Rightarrow \quad a^2=9 \Rightarrow a=3$ and $b^2=4 \Rightarrow b=2$

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As we know

$$c^2 = a^2 - b^2$$

$$c^2 = 9 - 4 \Rightarrow c^2 = 5 \Rightarrow c = \sqrt{5}$$

Centre: $C(0, 0)$

Foci: $(\pm c, 0) = (\pm\sqrt{5}, 0)$

Eccentricity: $e = \frac{c}{a} \Rightarrow e = \frac{\sqrt{5}}{3} < 1$

Vertices: $(\pm a, 0) = (\pm 3, 0)$

Directrices: $x = \pm\frac{a}{e} = \pm\frac{3}{\sqrt{5}/3} \Rightarrow x = \pm\frac{9}{\sqrt{5}}$

$$x = \pm\frac{9}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \Rightarrow x = \pm\frac{9\sqrt{5}}{5}$$

(ii) $4x^2 + 24x + 9y^2 - 36y + 36 = 0$

Solution:

$$4x^2 + 24x + 9y^2 - 36y + 36 = 0$$

$$(4x^2 + 24x) + (9y^2 - 36y) = -36$$

$$4(x^2 + 6x) + 9(y^2 - 4y) = -36$$

$$4(x^2 + 6x + 3^2 - 3^2) + 9(y^2 - 4y + 2^2 - 2^2) = -36$$

$$4[(x+3)^2 - 9] + 9[(y-2)^2 - 4] = -36$$

$$4(x+3)^2 - 36 + 9(y-2)^2 - 36 = -36$$

$$4(x+3)^2 + 9(y-2)^2 = -36 + 36 + 36$$

$$4(x+3)^2 + 9(y-2)^2 = 36$$

Dividing both sides by 36

$$\frac{4(x+3)^2}{36} + \frac{9(y-2)^2}{36} = \frac{36}{36}$$

$$\frac{(x-(-3))^2}{9} + \frac{(y-2)^2}{4} = 1 \text{ (Horizontal major axis)}$$

Compare it with

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\Rightarrow \quad h = -3, k = 2, a^2 = 9 \Rightarrow a = 3, b^2 = 4 \Rightarrow b = 2$$

As we know

$$c^2 = a^2 - b^2$$

$$c^2 = 9 - 4 \Rightarrow c^2 = 5 \Rightarrow c = \sqrt{5}$$

Centre: $C(h, k) = C(-3, 2)$

Foci: $(h \pm c, k) = (-3 \pm \sqrt{5}, 2)$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{5}}{3} < 0$

Vertices: $(h \pm a, k) = (-3 \pm 3, 2)$

$\Rightarrow$ $(0, 2)$ and $(-6, 2)$

Directrices: $x = h \pm \frac{a}{e} = -3 \pm \frac{3}{\sqrt{5}/3} = -3 \pm \frac{9}{\sqrt{5}}$

Q3. Find the equation of the ellipse as the locus of points $P(x, y)$ such that the sum of the distances from P to the points $(-3, 0)$ and $(3, 0)$ is 10.

Solution:

Let the given points be $F'(-3, 0)$ and $F(3, 0)$

Given condition for locus of point $P(x, y)$ is

$$|PF'| + |PF| = 10$$

By distance formula

$$\sqrt{(x+3)^2 + (y-0)^2} + \sqrt{(x-3)^2 + (y-0)^2} = 10$$

$$\sqrt{x^2 + 6x + 9 + y^2} = 10 - \sqrt{x^2 - 6x + 9 + y^2}$$

Squaring both sides

$$(\sqrt{x^2 + 6x + 9 + y^2})^2 = (10 - \sqrt{x^2 - 6x + 9 + y^2})^2$$

$$x^2 + 6x + 9 + y^2 = 100 + x^2 - 6x + 9 + y^2 - 20\sqrt{x^2 - 6x + 9 + y^2}$$

$$20\sqrt{x^2 + y^2 - 6x + 9} = x^2 + y^2 - 6x + 100 + 9 - x^2 - 6x - 9 - y^2$$

$$20\sqrt{x^2 + y^2 - 6x + 9} = -12x + 100$$

Dividing both sides by 4

$$5\sqrt{x^2 + y^2 - 6x + 9} = 25 - 3x$$

Again, squaring both sides

$$25(x^2 + y^2 - 6x + 9) = 625 + 9x^2 - 150x$$

$$25x^2 + 25y^2 - 150x + 225 - 9x^2 + 150x = 625$$

$$16x^2 + 25y^2 = 625 - 225$$

$$16x^2 + 25y^2 = 400$$

Dividing both sides by 400

$$\frac{16x^2}{400} + \frac{25y^2}{400} = \frac{400}{400}$$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1 \text{ (Required Equation of Ellipse)}$$

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Q4. Find the equation of an ellipse centered at $(5, 3)$ that is tangent to both the x-axis and y-axis.

Solution:

Centre: $C(5,3) \Rightarrow h=5, k=3$

Since, the ellipse is tangent to x-axis and y-axis, so

a = Horizontal distance from $C(5, 3)$ to y-axis
$= |h| = 5$

b = Vertical distance from $C(5, 3)$ to x-axis
$= |k| = 3$

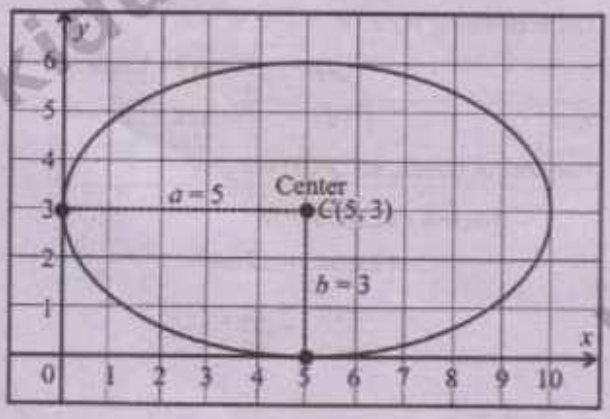
Since, $a > b$, the major axis is horizontal.

Required equation of ellipse is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-5)^2}{5^2}+\frac{(y-3)^2}{3^2}=1$$

$$\frac{(x-5)^2}{25}+\frac{(y-3)^2}{9}=1$$



Q5. What are the dimensions of the smallest rectangle aligned to the axes that completely contains the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$?

Solution:

Ellipse: $\frac{x^2}{16}+\frac{y^2}{9}=1$

Compare it with

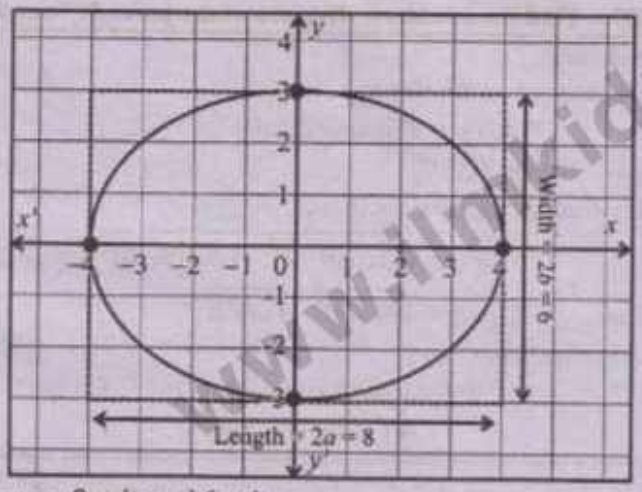
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$\Rightarrow \quad a^2=16 \Rightarrow a=4$ and $b^2=9 \Rightarrow b=3$

Centre: $C(0, 0)$

Length of rectangle = Length of major axis
$= 2a$
$= 2(4) = 8$ units

Width of rectangle = Length of minor axis
$= 2b$
$= 2(3) = 6$ units



Hence, the dimensions of smallest rectangle aligned to the axes are 8 units and 6 units.

Q6. Find the dimensions of the rectangle with greatest perimeter that can be inscribed in the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

Solution:

Ellipse: $\frac{x^2}{16}+\frac{y^2}{9}=1$

Compare it with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$\Rightarrow \quad a^2=16 \Rightarrow a=4$ and $b^2=9 \Rightarrow b=3$

Let any vertex of the inscribed rectangle in 1st quadrant be:

$$Q(x, y) = Q(a\cos\theta, b\sin\theta)$$
$$= Q(4\cos\theta, 3\sin\theta)$$

$\Rightarrow \quad x=4\cos\theta$ and $y=3\sin\theta$

Length of rectangle $= 2x = 8\cos\theta$

Width of rectangle $= 2y = 6\sin\theta$

Perimeter = 2(Length) + 2(Width)

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Let $f(\theta) = 2(8\cos\theta)+2(6\sin\theta)$

$f(\theta) = 16\cos\theta+12\sin\theta$

Differentiating w.r.t. 'θ'

$f'(\theta) = -16\sin\theta+12\cos\theta$

and $f''(\theta) = -16\cos\theta-12\sin\theta$

For critical points, set $f'(\theta)=0$

$-16\sin\theta+12\cos\theta = 0$

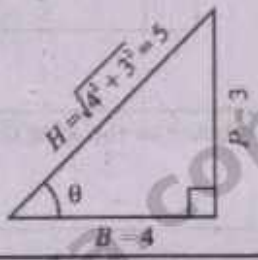
$16\sin\theta = 12\cos\theta$

$\frac{\sin\theta}{\cos\theta} = \frac{12}{16}$

$\tan\theta = \frac{3}{4}$

$\Rightarrow \sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$

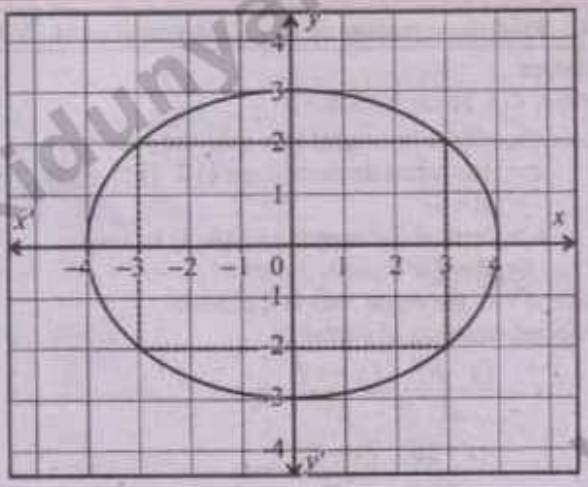
$f''(\theta) = -16\left(\frac{4}{5}\right)-12\left(\frac{3}{5}\right) < 0$



$\Rightarrow$ Perimeter will be greatest.

Length of rectangle $= 8\cos\theta = 8\left(\frac{4}{5}\right) = \frac{32}{5}$ units

Width of rectangle $= 6\sin\theta = 6\left(\frac{3}{5}\right) = \frac{18}{5}$ units



Q7. Find the minimum distance of the point $P(3\cos\theta, 2\sin\theta)$ on the ellipse from the centre of the ellipse.

Solution:

Given point on the ellipse

$P(x, y) = P(3\cos\theta, 2\sin\theta)$

$\Rightarrow$ Centre: $C(0, 0)$

By distance formula

$|\overline{CP}| = \sqrt{(3\cos\theta-0)^2+(2\sin\theta-0)^2}$

Let $d = \sqrt{9\cos^2\theta+4\sin^2\theta}$

Squaring both sides

$d^2 = 9\cos^2\theta+4\sin^2\theta$

$= 9(1-\sin^2\theta)+4\sin^2\theta$

$= 9-9\sin^2\theta+4\sin^2\theta$

$d^2 = 9-5\sin^2\theta$

For minimum value of d^2, the term $\sin^2\theta$ should be maximum.

$\Rightarrow d^2_{min} = 9-5(1)$

$d^2_{min} = 4$

$\Rightarrow d_{min} = \sqrt{4} = 2$ units

Hence, the minimum distance of the point P from the centre is 2 units.

Q8. Find the minimum distance from the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ to the line $x+y=6$.

Solution:

Given line: $x+y=6$

Slope of line $= m = -\frac{1}{1} = -1$

Ellipse: $\frac{x^2}{9}+\frac{y^2}{4} = 1$

Compare it with

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

$\Rightarrow a^2 = 9$ and $b^2 = 4$

For a line $y = mx+c$ to be tangent to an ellipse.

$c^2 = a^2m^2+b^2 = (9)(-1)^2+4$

$c^2 = 13 \Rightarrow c = \pm\sqrt{13}$

Two parallel tangent lines are:

$y = -1x \pm \sqrt{13} \Rightarrow x+y = \pm\sqrt{13}$

$x+y-\sqrt{13} = 0$ and $x+y+\sqrt{13} = 0$

Required Minimum Distance = Distance between parallel lines $x+y-6=0$ and $x+y-\sqrt{13}=0$

$$d = \frac{|c_1-c_2|}{\sqrt{a^2+b^2}}$$

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$$d = \frac{|-6-(-\sqrt{13})|}{\sqrt{(1)^2+(1)^2}}$$

$$d = \frac{|-(6-\sqrt{13})|}{\sqrt{2}}$$

$$d = \frac{6-\sqrt{13}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$d = \frac{6\sqrt{2}-\sqrt{26}}{2}$$

Q9. Find the value of k for which the line $y = 2x + k$ is tangent to the ellipse $\frac{x^2}{9} + y^2 = 1$.

Solution:

Ellipse: $\frac{x^2}{9} + \frac{y^2}{1} = 1$

Compare it with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow \quad a^2 = 9$ and $b^2 = 1$

Tangent Line: $y = 2x + k$

Compare it with

$$y = mx + c$$

$\Rightarrow \quad m = 2, c = k$

Since, the line is tangent to ellipse, so

$$c^2 = a^2m^2 + b^2$$

$$k^2 = 9(2)^2 + 1$$

$$k^2 = 9(4) + 1$$

$$k^2 = 37$$

$$\boxed{k = \pm\sqrt{37}}$$

Q10. Determine whether the line $y = 2x - 3$ is a tangent, secant, or external to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Solution:

Ellipse: $\frac{x^2}{9} + \frac{y^2}{4} = 1$ ...(1)

Line: $y = 2x - 3$ Put this in eq. (1)

$$\frac{x^2}{9} + \frac{(2x-3)^2}{4} = 1$$

Multiplying both sides by 36

$$4x^2 + 9(4x^2 + 9 - 12x) = 36$$

$$4x^2 + 36x^2 + 81 - 108x - 36 = 0$$

$$40x^2 - 108x + 45 = 0 \quad \text{(Quadratic equation)}$$

Here $A = 40, B = -108, C = 45$

For Nature of Roots:

$$\text{Disc} = B^2 - 4AC$$

$$= (-108)^2 - 4(40)(45)$$

$$= 11664 - 7200$$

$$\text{Disc} = 4464 > 0$$

$\Rightarrow$ Line intersects the ellipse at real and distinct points.

Therefore, the give line is a secant to the ellipse.

Q11. Find the equation of an ellipse with centre at $(0, 0)$, a horizontal major axis of length 8, and that touches the line $y = x + 5$.

Solution:

Centre: $C(0, 0)$

Length of major axis = 8

$$2a = 8 \Rightarrow a = \frac{8}{2} \Rightarrow a = 4$$

Tangent line: $y = 1x + 5$

Compare it with

$$y = mx + c$$

$\Rightarrow \quad m = 1, c = 5$

Since, the line is tangent to ellipse, so

$$c^2 = a^2m^2 + b^2$$

$$5^2 = (4)^2(1)^2 + b^2$$

$$25 = 16 + b^2 \Rightarrow b^2 = 25 - 16$$

$$b^2 = 9 \Rightarrow b = 3$$

Required equation of ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

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Q12. Show that the family of lines given by:

$x\cos\theta + y\sin\theta = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$ **where θ is a parameter and $a > b$, are all tangent to the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution:

Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Line: $x\cos\theta + y\sin\theta = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$

$$\Rightarrow \quad y\sin\theta = -x\cos\theta + \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$$

Dividing both sides by $\sin\theta$

$$\frac{y\sin\theta}{\sin\theta} = -\frac{x\cos\theta}{\sin\theta} + \frac{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}{\sin\theta}$$

$$y = -\frac{\cos\theta}{\sin\theta}x + \frac{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}{\sin\theta}$$

Compare it with

$$y = mx + c$$

$$\Rightarrow \quad m = -\frac{\cos\theta}{\sin\theta},\ c = \frac{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}{\sin\theta}$$

Give line will be tangent to ellipse, if

$$c^2 = a^2m^2 + b^2$$

$$\left(\frac{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}{\sin\theta}\right)^2 = a^2\left(-\frac{\cos\theta}{\sin\theta}\right)^2 + b^2$$

$$\frac{a^2\cos^2\theta + b^2\sin^2\theta}{\sin^2\theta} = a^2\cdot\frac{\cos^2\theta}{\sin^2\theta} + b^2$$

$$\frac{a^2\cos^2\theta + b^2\sin^2\theta}{\sin^2\theta} = \frac{a^2\cos^2\theta + b^2\sin^2\theta}{\sin^2\theta} \text{ (True)}$$

Hence, the family of lines are all tangent to the given ellipse.

Q13. If a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with centre C meets the major and minor axes at P and Q respectively.

Prove that $\frac{a^2}{|CP|^2}+\frac{b^2}{|CQ|^2}=1$.

Solution:

Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow$ Centre: $C(0, 0)$

Equation of any tangent to ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2}$$

If the tangent meets the x-axis, then $y = 0$

$$0 = mx + \sqrt{a^2m^2 + b^2}$$

$$mx = -\sqrt{a^2m^2 + b^2} \Rightarrow x = \frac{-\sqrt{a^2m^2 + b^2}}{m}$$

$$\Rightarrow \quad P = \left(\frac{-\sqrt{a^2m^2 + b^2}}{m}, 0\right)$$

If the tangent meets the y-axis, then $x = 0$

$$y = m(0) + \sqrt{a^2m^2 + b^2} = \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow \quad Q = (0, \sqrt{a^2m^2 + b^2})$$

By distance formula

$$|\overline{CP}|^2 = \left(\sqrt{\left(0 + \frac{\sqrt{a^2m^2 + b^2}}{m}\right)^2 - (0-0)^2}\right)^2$$

$$|\overline{CP}|^2 = \frac{a^2m^2 + b^2}{m^2}$$

$$|\overline{CQ}|^2 = \left(\sqrt{(0-0)^2 + (0 - \sqrt{a^2m^2 + b^2})^2}\right)^2 = a^2m^2 + b^2$$

$$\text{L.H.S} = \frac{a^2}{|CP|^2} + \frac{b^2}{|CQ|^2}$$

$$= \frac{a^2}{\frac{a^2m^2 + b^2}{m^2}} + \frac{b^2}{a^2m^2 + b^2}$$

$$= \frac{a^2m^2}{a^2m^2 + b^2} + \frac{b^2}{a^2m^2 + b^2}$$

$$= \frac{a^2m^2 + b^2}{a^2m^2 + b^2}$$

$$= 1$$

= R.H.S (Proved)

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HYPERBOLA

The definition of a hyperbola is similar to that of an ellipse. For an ellipse, the sum of the distances between the foci and a point on the ellipse is fixed, whereas for a hyperbola, the absolute value of the difference between these distances is fixed. One distinguishing feature of a hyperbola is that its graph has two separate branches (see Figure 21).

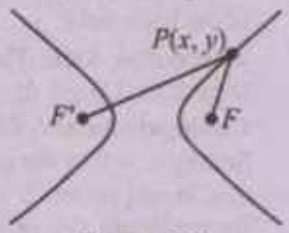

Figure 6.22

Derivation of the Standard Equation of Hyperbola

Let $F(c, 0)$ and $F'(-c, 0)$ be the foci on x-axis, such that

$$\text{Centre of ellipse} = \text{Midpoint of } \overline{FF'}$$
$$= C\left(\frac{c+(-c)}{2}, \frac{0+0}{2}\right) = C(0,0)$$

Let $P(x, y)$ be any point on the Hyperbola.

By definition of the hyperbola

$$\big||PF'|-|PF|\big| = 2a \text{ (say), where } a > 0$$

or $$|PF'|-|PF| = \pm 2a$$

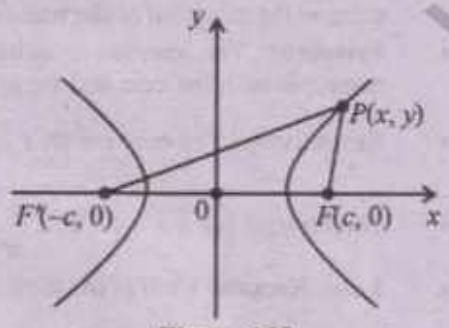

Figure 6.23

Using the distance formula

$$\sqrt{(x+c)^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} = \pm 2a$$
$$\sqrt{x^2+c^2+2cx+y^2} = \sqrt{x^2+c^2-2cx+y^2} \pm 2a$$

Squaring both sides, we have

$$\left(\sqrt{x^2+c^2+2cx+y^2}\right)^2 = \left(\sqrt{x^2+c^2-2cx+y^2} \pm 2a\right)^2$$
$$x^2+c^2+2cx+y^2 = x^2+c^2-2cx+y^2+4a^2 \pm 4a\sqrt{x^2+c^2-2cx+y^2}$$
$$\pm 4a\sqrt{x^2+c^2-2cx+y^2} = x^2+c^2-2cx+y^2+4a^2-x^2-c^2-2cx-y^2$$
$$\pm 4a\sqrt{x^2+c^2-2cx+y^2} = 4a^2-4cx$$

Dividing both sides by '4'

$$\pm a\sqrt{x^2+c^2-2cx+y^2} = a^2-cx$$

Again, squaring both sides

$$\left(\pm a\sqrt{x^2+c^2-2cx+y^2}\right)^2 = (a^2-cx)^2$$
$$a^2(x^2+c^2-2cx+y^2) = a^4+c^2x^2-2a^2cx$$
$$a^2x^2+a^2c^2-2a^2cx+a^2y^2 = a^4+c^2x^2-2a^2cx$$
$$a^2x^2-c^2x^2+a^2y^2 = a^4-2a^2cx-a^2c^2+2a^2cx$$
$$(a^2-c^2)x^2+a^2y^2 = a^4-a^2c^2$$
$$(a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2) \quad \ldots(1)$$

From triangle $PF'F$, we get

$$|PF'|-|PF| < |F'F|$$
$$\Rightarrow \quad 2a < 2c \qquad \because |F'F| = 2c$$
$$\Rightarrow \quad a < c \qquad \Rightarrow a^2-c^2 < 0 \text{ (-ve)}$$

Let $a^2-c^2 = -b^2$ Put in eq. (1)

$$-b^2x^2+a^2y = a^2(-b^2)$$

Dividing both sides by $-a^2b^2$

$$\frac{-b^2x^2}{-a^2b^2} + \frac{a^2y}{-a^2b^2} = \frac{-a^2b^2}{-a^2b^2}$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } c^2 = a^2+b^2$$

This is the standard equation of hyperbola.

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Key Features of the Hyperbola

- **Vertices:** Setting $y = 0$ in the equation of hyperbola, gives $x = \pm a$. The points $A'(-a, 0)$ and $A(a, 0)$ are the vertices of the hyperbola.
- **Transverse Axis:** The line segment $A'A$ connecting the vertices is the transverse axis (or focal axis) of the hyperbola. Length of the transverse axis is $2a$.
- **Conjugate Axis:** A line passes through the centre and perpendicular to the transverse axis is called conjugate axis. Setting $x = 0$ in the equation of hyperbola, gives $y^2 = -b^2$, which is impossible, so hyperbola does not meet y-axis in real points. If we take two points $B'(0, -b)$ and $B(0, b)$ on the y-axis, then the line segment $B'B$ is called the conjugate axis of the hyperbola. Length of the conjugate axis is $2b$.
- **Foci:** The foci are at $(\pm c, 0)$, they always lie on the transverse axis. In case of hyperbola $c^2 = a^2 + b^2$, so that, unlike the ellipse, we may have $a > b$ or $a < b$ or $a = b$.
- **Centre:** The midpoint $C(0, 0)$ of the segment joining the foci is called the centre of the hyperbola. Note that it is same as the midpoint of the transverse axis.
- **Symmetry:** The equation is unchanged if x is replaced by $-x$ and y by $-y$; hence the hyperbola is symmetric with respect to both the axis, and the origin.
- **Eccentricity:** The eccentricity e of a hyperbola is defined as $e = \frac{c}{a}$. Since $c > a$, we have $e > 1$.
- **Directrices:** For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, there are two directrices given by the vertical lines $x = \pm \frac{a}{e}$.
- **Latus Rectum:** Each of the focal chords perpendicular to the transverse axis is called a latus rectum (plural: latera recta). The length of each latus rectum is $\frac{2b^2}{a}$.
- **Parametric Equations:** For all real θ such that $\theta \neq (2n+1)\frac{\pi}{2}$ (where n is an integer), the point $(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Thus, the parametric equations of the hyperbola are: $x = a\sec\theta, y = b\tan\theta$.
- **Foci on the y-axis:** If the foci are on the y-axis at $(0, \pm c)$, the roles of x and y are interchanged, and the standard equation becomes $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ vertices $(0, \pm a)$.
- **Relation between a, b and e:**

  Since $b^2 = c^2 - a^2$, setting $c = ae \Rightarrow b^2 = a^2e^2 - a^2 \Rightarrow b^2 = a^2(e^2 - 1)$.
- **Branches:** To analyze the hyperbola further, we write $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$. This shows that $x^2 \geq a^2$ so $|x| \geq a$.

  Therefore, we have $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its branches. Note that no portion of the curve lies between $-a < x < a$, because in this interval y is imaginary.
- **Asymptotes:** An important feature of hyperbola that is not shared by ellipses is the presence of asymptotes. An asymptote of a curve $y = f(x)$ is a line $y = g(x)$ such that

  $$\lim_{x \to \pm\infty} [f(x) - g(x)] = 0$$

  For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

  $$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 \text{ or } y^2 = \frac{b^2}{a^2}x^2 - b^2.$$

  Notice that: $\lim_{x \to \pm\infty} \frac{y^2}{x^2} = \lim_{x \to \pm\infty} \left(\frac{b^2}{a^2} - \frac{b^2}{x^2}\right) = \frac{b^2}{a^2}$.

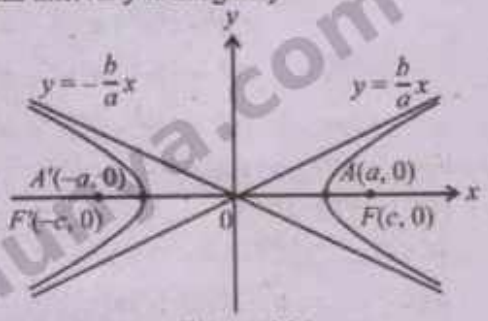

Figure 6.24

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That is, as $x \to \pm\infty, \frac{y^2}{x^2} \to \frac{b^2}{a^2}$, so that $\frac{y}{x} \to \pm\frac{b}{a}$ and so, $y = \pm\frac{b}{a}x$ are the (slant) asymptotes as shown in Figure 6.24.

The asymptotes of a curve do not meet the curve but distance of any point on the curve from any of the two lines approaches zero.

Note: The joint equation of the two asymptotes is

$$\left(y + \frac{b}{a}x\right)\left(y - \frac{b}{a}x\right) = 0, \text{ that is, } y^2 - \frac{b^2}{a^2}x^2 = 0 \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

Rectangular Hyperbola as a special case:

If in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the lengths a and b become equal then the resulting hyperbola $x^2 - y^2 = a^2$ is called rectangular hyperbola. In this case asymptotes are perpendicular to each other.

This is obtained by replacing the constant term 1 with 0 in the standard equation.

Second definition of Hyperbola:

The hyperbola can also be defined as, if a point P moves so that the ratio of its distance from a fixed point F (the focus) to its distance from a fixed straight line (the directrix) is equal to a constant, e (greater than 1), the locus of P is a hyperbola of eccentricity e.

That is, if $\frac{|PF|}{|PM|} = e$, where $e > 1$, the locus of P is a hyperbola.

Note: The equation of hyperbola can also be derived using the above definition.

Summary of Important Results

Hyperbola in standard form	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Centre	$(0,0)$	$(0,0)$
Equation of transverse axis	$y = 0, -a \le x \le a$	$x = 0, -a \le y \le a$
Length of transverse axis	$2a$	$2a$
Equation of conjugate axis	$x = 0, -b \le y \le b$	$y = 0, -b \le x \le b$
Length of conjugate axis	$2b$	$2b$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Conjugate points	$(0, \pm b)$	$(\pm b, 0)$
Foci	$(\pm c, 0), c = ae$	$(0, \pm c), c = ae$
Eccentricity	$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} > 1$	$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} > 1$
Equations of directrices	$y = \pm\frac{a}{e}$	$y = \pm\frac{a}{e}$
Equations of latera recta	$x = \pm ae$	$y = \pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Equations of asymptotes	$y = \pm\frac{b}{a}x$	$y = \pm\frac{a}{b}x$

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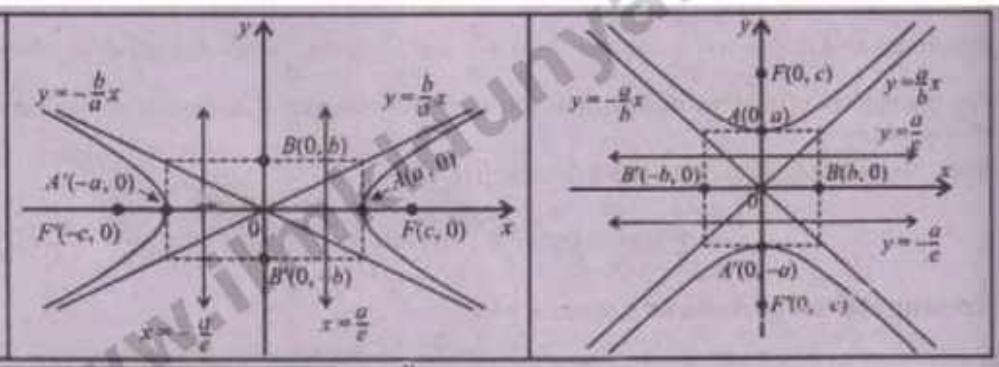
Sketch of hyperbola in standard form	

Theorem 1: Translated Forms of Horizontal Hyperbolas

Proof:

The equation $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ describes a hyperbola with foci at the points $F'(h-c,k)$ and $F(h+c,k)$, where $c=\sqrt{a^2+b^2}$.

The centre of the hyperbola is at the point $C(h,k)$ and the vertices are located at $(h\pm a,k)$.

The asymptotes are $y=\pm\frac{b}{a}(x-h)+k$.

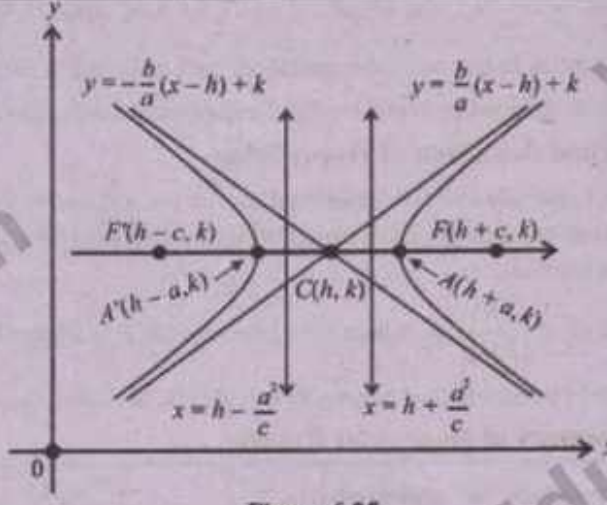


Figure 6.25

Theorem 2: Translated Forms of Vertical Hyperbolas

Proof:

The equation $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ describes a hyperbola with foci at the points $F'(h,k-c)$ and $F(h,k+c)$, where $c=\sqrt{a^2+b^2}$.

The centre of the hyperbola is at the point (h,k) and the vertices are located at $(h,k\pm a)$.

The asymptotes are $y=\pm\frac{a}{b}(x-h)+k$.

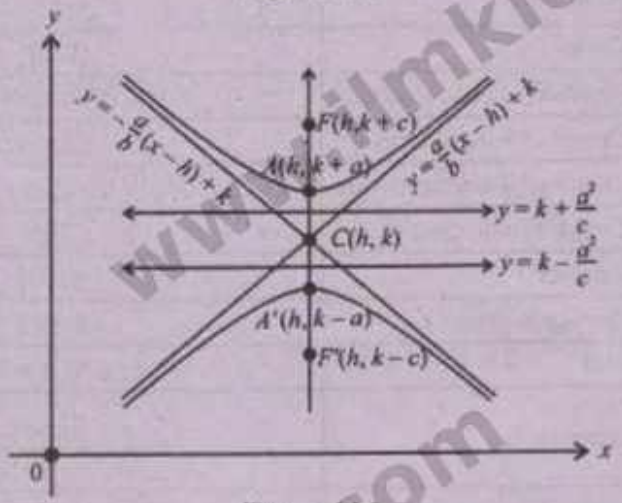


Figure 6.26

Example 24: A hyperbola has a horizontal transverse axis of length 12, centre at $(-2,5)$, and eccentricity $\frac{5}{3}$. Find the equation of the hyperbola.

Solution:

Centre: $C(-2,5) \Rightarrow h=-2, k=5$

Length of transverse axis = 12

$$2a=12 \Rightarrow a=\frac{12}{2} \Rightarrow a=6$$

Eccentricity: $e=\frac{5}{3}$

$$\Rightarrow \quad \frac{c}{a}=\frac{5}{3} \Rightarrow \frac{c}{6}=\frac{5}{3} \Rightarrow c=\frac{5}{3}\times 6 \Rightarrow c=10$$

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As we know

$$c^2 = a^2 + b^2$$

$$10^2 = 6^2 + b^2 \Rightarrow b^2 = 100 - 36 \Rightarrow b^2 = 64$$

Since the transverse axis is horizontal, therefore

Required equation of hyperbola is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-(-2))^2}{6^2} - \frac{(y-5)^2}{64} = 1$$

$$\frac{(x+2)^2}{36} - \frac{(y-5)^2}{64} = 1$$

Example 25: A hyperbola has a horizontal transverse axis, eccentricity $\frac{\sqrt{13}}{3}$ and latus rectum of length $\frac{8}{3}$. The hyperbola passes through the points (8, 2) and (2, 2). Find the equation of the hyperbola.

Solution:

Eccentricity: $e = \frac{\sqrt{13}}{3}$

$$\frac{c}{a} = \frac{\sqrt{13}}{3} \Rightarrow c = \frac{\sqrt{13}a}{3}$$

As we know

$$c^2 = a^2 + b^2$$

$$\left(\frac{\sqrt{13}a}{3}\right)^2 = a^2 + b^2 \Rightarrow b^2 = \frac{13a^2}{9} - a^2$$

$$b^2 = \frac{13a^2 - 9a^2}{9} \Rightarrow b^2 = \frac{4a^2}{9} \quad \ldots(1)$$

Length of latus rectum $= \frac{8}{3}$

$$\frac{2b^2}{a} = \frac{8}{3}$$

$$\frac{2\left(\frac{4a^2}{9}\right)}{a} = \frac{8}{3} \quad \because b^2 = \frac{4a^2}{9}$$

$$\frac{8a^2}{9a} = \frac{8}{3} \Rightarrow \frac{8a}{9} = \frac{8}{3}$$

$$a = \frac{8}{3} \times \frac{9}{8} \Rightarrow a = 3 \quad \text{Put in eq. (1)}$$

$$b^2 = \frac{4(3)^2}{9} \Rightarrow b^2 = 4$$

Since the transverse axis is horizontal, therefore

Required equation of hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-h)^2}{9} - \frac{(y-k)^2}{4} = 1 \quad \ldots(2)$$

Since the hyperbola (1) passes through (8, 2) and (2, 2), so

At (8, 2):

$$\frac{(8-h)^2}{9} - \frac{(2-k)^2}{4} = 1 \quad \ldots(3)$$

At (2, 2):

$$\frac{(2-h)^2}{9} - \frac{(2-k)^2}{4} = 1 \quad \ldots(4)$$

Equation (3) – Equation (4)

$$\frac{(8-h)^2}{9} - \frac{(2-k)^2}{4} = 1$$

$$\pm\frac{(2-h)^2}{9} \mp \frac{(2-k)^2}{4} = \pm 1$$

$$\frac{(8-h)^2}{9} - \frac{(2-h)^2}{9} = 0$$

Multiplying both sides by 9

$$(8-h)^2 - (2-h)^2 = 0$$

$$(64 + h^2 - 16h) - (4 + h^2 - 4h) = 0$$

$$64 + h^2 - 16h - 4 - h^2 + 4h = 0$$

$$-12h + 60 = 0 \Rightarrow 12h = 60$$

$h = 5$ Put in eq. (3)

$$\frac{(8-5)^2}{9} - \frac{(2-k)^2}{4} = 1$$

$$\frac{(3)^2}{9} - \frac{(2-k)^2}{4} = 1$$

$$1 - \frac{(2-k)^2}{4} = 1$$

$$-\frac{(2-k)^2}{4} = 0$$

$$(2-k)^2 = 0 \Rightarrow 2 - k = 0 \Rightarrow k = 2$$

Put $h = 5$ and $k = 2$ in eq. (2), we get

$$\frac{(x-5)^2}{9} - \frac{(y-2)^2}{4} = 1 \text{ (Required Hyperbola)}$$

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Example 26: Find an equation of the hyperbola with given data and sketch the graph.
(i) Foci: $(-3, 5 \pm 5\sqrt{2})$ (ii) Length of transverse axis = 6.

Solution:

Foci: $F(-3, 5+5\sqrt{2}), F'(-3, 5-5\sqrt{2})$

Both foci have the same x-coordinate

$\Rightarrow$ Transverse axis is vertical.

Centre: $C\left(\frac{-3+(-3)}{2}, \frac{5+5\sqrt{2}+5-5\sqrt{2}}{2}\right)$

$= C(-3, 5) \Rightarrow h = -3, k = 5$

Length of transverse axis = 6

$2a = 6 \Rightarrow a = \frac{6}{5} \Rightarrow a = 3$

c = Distance from $C(-3, 5)$ to $F(-3, 5+5\sqrt{2})$

$= |5+5\sqrt{2}-5| = 5\sqrt{2}$

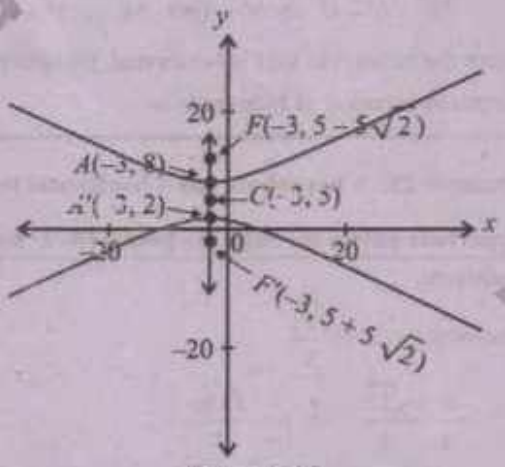

Figure 6.27

As we know

$c^2 = a^2 + b^2$

$(5\sqrt{2})^2 = 3^2 + b^2 \Rightarrow b^2 = 50 - 9 \Rightarrow b^2 = 41$

Required equation of hyperbola is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-5)^2}{3^2} - \frac{(x+3)^2}{41} = 1$$

$$\frac{(y-5)^2}{9} - \frac{(x+3)^2}{41} = 1$$

Example 27: Find an equation of the hyperbola with given data and sketch the graph.
(i) Foci (4, −3), (4, 7) and (ii) one vertex (4, 5).

Solution:

Foci: $F(4, -3), F'(4, 7)$

Centre: $C\left(\frac{4+4}{2}, \frac{-3+7}{2}\right) = C(4, 2)$

$\Rightarrow h = 4, k = 2$

Both the foci have the same x-coordinate.

$\Rightarrow$ Transverse axis is vertical.

c = Distance from $C(4, 2)$ to $F(4, -3)$

$= |2 - (-3)| = 5$

Given vertex: $A(4, 5)$

a = Distance from $C(4, 2)$ to $A(4, 5)$

$= |2 - 5| = |-3| = 3$

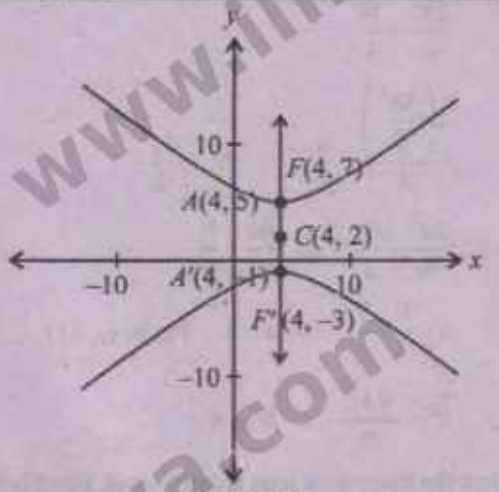

Figure 6.28

As we know

$c^2 = a^2 + b^2$

$5^2 = 3^2 + b^2 \Rightarrow b^2 = 25 - 9 \Rightarrow b^2 = 16$

Required equation of hyperbola is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-2)^2}{3^2} - \frac{(x-4)^2}{16} = 1$$

$$\frac{(y-2)^2}{9} - \frac{(x-4)^2}{16} = 1$$

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Example 28: Find the centre, foci, eccentricity, vertices and equations of the directrices of the hyperbola:

$$9x^2 - 36x - 4y^2 + 8y - 4 = 0.$$

Also sketch the graph.

Solution:

$$9x^2-36x-4y^2+8y-4=0$$
$$9(x^2-4x)-4(y^2-2y)=4$$
$$9[(x^2-4x+2^2)-2^2]-4[(y^2-2y+1^2)-1^2]=4$$
$$9[(x-2)^2-4]-4[(y-1)^2-1]=4$$
$$9(x-2)^2-36-4(y-1)^2+4=4$$
$$9(x-2)^2-4(y-1)^2=4+36-4$$
$$9(x-2)^2-4(y-1)^2=36$$

Dividing both sides by 36

$$\frac{9(x-2)^2}{36}-\frac{4(y-1)^2}{36}=\frac{36}{36}$$
$$\frac{(x-2)^2}{4}-\frac{(y-1)^2}{9}=1 \quad \text{(Horizontal transverse axis)}$$

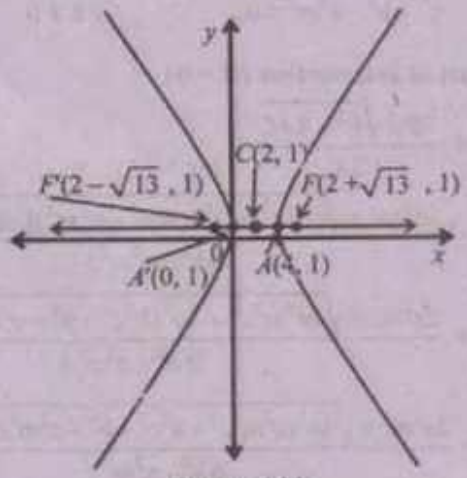

Figure 6.29

Compare it with

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$$\Rightarrow \quad h=2, k=1, a^2=4 \Rightarrow a=2, b^2=9 \Rightarrow b=3$$

As we know

$$c^2=a^2+b^2$$
$$c^2=4+9=13 \Rightarrow c=\sqrt{13}$$

Centre: $C(h,k)=C(2,1)$

Foci: $(h\pm c, k)=(2\pm\sqrt{13},1)$

$$\Rightarrow \quad F(2+\sqrt{13},1), F'(2-\sqrt{13},1)$$

Eccentricity $e=\frac{c}{a}=\frac{\sqrt{13}}{2}>1$

Vertices: $(h\pm a, k)=(2\pm 2,1)$

$$\Rightarrow \quad A(4,1), A'(0,1)$$

Directrices: $x=h\pm\frac{a}{e}=2\pm\frac{2}{\frac{\sqrt{13}}{2}}=2\pm\frac{4}{\sqrt{13}}$

$$x=2\pm\frac{4}{\sqrt{13}}\times\frac{\sqrt{13}}{\sqrt{13}} \Rightarrow x=2\pm\frac{4\sqrt{13}}{13}$$

Tangents and Normals to Hyperbola

Question: Prove that the condition for the line $y=mx+c$ to be tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $c^2=a^2m^2-b^2$. Also, find the coordinates of the points of intersection.

Solution:

Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1)

Line: $y=mx+c$ ...(2)

For their points of intersection solving (1) and (2) simultaneously.

Put $y=mx+c$ in eq. (1), we have

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

Multiplying both sides by a^2b^2

$$b^2x^2-a^2(m^2x^2+c^2+2mcx)=a^2b^2$$
$$b^2x^2-a^2m^2x^2-a^2c^2-2a^2mcx-a^2b^2=0$$
$$(b^2-a^2m^2)x-(2a^2mc)x-a^2(c^2+b^2)=0$$

This being quadratic in x, give two values of x. Thus, the line intersect the ellipse in at most two points.

Here $A=b^2-a^2m^2, B=-2a^2mc, C=-a^2(c^2+b^2)$

Tangency Condition:

For tangency condition roots will be real and equal $\Rightarrow D=0$

$$B^2-4AC=0$$
$$(-2a^2mc)^2-4(b^2-a^2m^2)[-a^2(c^2+b^2)]=0$$
$$4a^4m^2c^2+4a^2(b^2c^2+b^4-a^2m^2c^2-a^2m^2b^2)=0$$

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$$4a^2(a^2m^2c^2+b^2c^2+b^4-a^2m^2c^2-a^2m^2b^2)=0$$

$$\Rightarrow \quad b^2c^2+b^4-a^2m^2b^2=0 \quad \because a\neq 0$$

$$b^2(c^2+b^2-a^2m^2)=0$$

$$\Rightarrow \quad c^2+b^2-a^2m^2=0 \qquad \because b\neq 0$$

$$\boxed{c^2=a^2m^2-b^2} \quad \text{(Tangency condition)}$$

$$\Rightarrow \quad c=\pm\sqrt{a^2m^2-b^2} \quad \text{Put in eq. (2)}$$

$$y=mx\pm\sqrt{a^2m^2-b^2} \quad \text{(Tangent line } \forall m\text{)}$$

Points of Intersection ($D>0$)

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

$$x=\frac{2a^2mc\pm\sqrt{(-2a^2mc)^2-4(b^2-a^2m^2)[-a^2(c^2+b^2)]}}{2(b^2-a^2m^2)}$$

$$=\frac{2a^2mc\pm\sqrt{4a^4m^2c^2+4a^2(b^2c^2+b^4-a^2m^2c^2-a^2m^2b^2)}}{2(b^2-a^2m^2)}$$

$$=\frac{2a^2mc\pm\sqrt{4a^2(a^2m^2c^2+b^2c^2+b^4-a^2m^2c^2-a^2m^2b^2)}}{2(b^2-a^2m^2)}$$

$$=\frac{2a^2mc\pm 2a\sqrt{b^2c^2+b^4-a^2m^2b^2}}{2(b^2-a^2m^2)}$$

$$=\frac{a^2mc\pm a\sqrt{b^2(c^2+b^2-a^2m^2)}}{b^2-a^2m^2}$$

$$x=\frac{a^2mc\pm ab\sqrt{c^2+b^2-a^2m^2}}{b^2-a^2m^2} \quad \text{Put in eq. (2)}$$

Challenge!

Find the points of intersection between the line $2x-y-4=0$ and the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$.

$$y=m\left(\frac{a^2mc\pm ab\sqrt{c^2+b^2-a^2m^2}}{b^2-a^2m^2}\right)+c$$

$$=\frac{a^2m^2c\pm abm\sqrt{c^2+b^2-a^2m^2}+c(b^2-a^2m^2)}{b^2-a^2m^2}$$

$$=\frac{a^2m^2c\pm abm\sqrt{c^2+b^2-a^2m^2}+b^2c-a^2m^2c}{b^2-a^2m^2}$$

$$=\frac{b^2c\pm abm\sqrt{c^2+b^2-a^2m^2}}{b^2-a^2m^2}$$

Point of intersection $=\left(\frac{a^2mc\pm ab\sqrt{c^2+b^2-a^2m^2}}{b^2-a^2m^2},\frac{b^2c\pm abm\sqrt{c^2+b^2-a^2m^2}}{b^2-a^2m^2}\right)$

Example 29: Find equation of the tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1,y_1).

Solution:

Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1) at $P(x_1,y_1)$

Since $P(x_1,y_1)$ lies on ellipse, so

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1 \qquad \ldots(2)$$

Differentiating (1) w.r.t. 'x'

$$\frac{d}{dx}\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$$

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{2y}{b^2}\frac{dy}{dx}=\frac{2x}{a^2} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{2x}{a^2}\cdot\frac{b^2}{2y}$$

$$\frac{dy}{dx}=\frac{b^2}{a^2}\cdot\frac{x}{y}$$

Slope of tangent at $P(x_1,y_1)=\left.\frac{dy}{dx}\right|_{P(x_1,y_1)}$

$$m_T=\frac{b^2x_1}{a^2y_1}$$

Equation of tangent at $P(x_1,y_1)$ is:

$$y-y_1=m_T(x-x_1)$$

$$y-y_1=\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$\frac{y_1}{b^2}(y-y_1)=\frac{x_1}{a^2}(x-x_1)$$

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$$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=\frac{xx_1}{a^2}-\frac{x_1^2}{a^2}$$

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}$$

$$1=\frac{xx_1}{a^2}-\frac{yy_1}{b^2} \quad \text{Using eq. (2)}$$

$$\boxed{\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1} \quad \text{(Tangent)}$$

Slope of normal at $P(x_1, y_1)=-\frac{1}{m_T}$

$$\Rightarrow \quad m_N=\frac{a^2y_1}{b^2x_1}$$

Equation of normal at $P(x_1, y_1)$ is:

$$y-y_1=m_N(x-x_1)$$

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

$$b^2x_1(y-y_1)=-a^2y_1(x-x_1)$$

$$b^2x_1y-b^2x_1y_1=-a^2xy_1+a^2x_1y_1$$

$$a^2xy_1+b^2x_1y=a^2x_1y_1+b^2x_1y_1$$

$$a^2xy_1+b^2x_1y=x_1y_1(a^2+b^2)$$

Dividing both sides by x_1y_1: (say $x_1\neq 0, y_1\neq 0$)

$$\frac{a^2xy_1}{x_1y_1}+\frac{b^2x_1y}{x_1y_1}=a^2+b^2$$

$$\boxed{\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2} \quad \text{(Normal)}$$

Example 30: Find the equations of the tangents to the hyperbola $2x^2-3y^2=24$ that pass through the point $(2,-4)$.

Solution:

Hyperbola: $2x^2-3y^2=24$

Dividing both sides by 24

$$\frac{2x^2}{24}-\frac{3y^2}{24}=\frac{24}{24}$$

$$\frac{x^2}{12}-\frac{y^2}{8}=1$$

Compare it with

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$\Rightarrow \quad a^2=12$ and $b^2=8$

Let required equation of tangent be:

$$y=mx\pm\sqrt{a^2m^2-b^2}$$

$$y=mx\pm\sqrt{12m^2-8} \quad \ldots(1)$$

Since, the tangent lines passes through $(2,-4)$, so

$$-4=2m\pm\sqrt{12m^2-8}$$

$$-4-2m=\pm\sqrt{12m^2-8}$$

Squaring both sides

$$(-(4+2m))^2=(\pm\sqrt{12m^2-8})^2$$

$$16+4m^2+16m=12m^2-8$$

$$12m^2-8-16-4m^2-16m=0$$

$$8m^2-16m-24=0$$

Dividing both sides by 8

$$m^2-2m-3=0$$

$$m^2-3m+m-3=0$$

$$m(m-3)+1(m-3)=0$$

$$(m-3)(m+1)=0$$

Either $m-3=0$ or $m+1=0$

$\Rightarrow \quad m=3$ or $m=-1$

When $m=3$:

Eq. (1) $\Rightarrow \quad y=3x\pm\sqrt{12(3)^2-8}$

$$y=3x\pm 10$$

Check at $(2,-4)$:

$y=3x+10$	$y=3x-10$
$-4=3(2)+10$	$-4=3(2)-10$
$-4=16$ (False)	$-4=-4$ (True)

$\Rightarrow \quad 3x-y-10=0$ is a tangent.

When $m=-1$:

Eq. (1) $\Rightarrow \quad y=-1x\pm\sqrt{12(-1)^2-8}$

$$y=-x\pm 2$$

Check at $(2,-4)$:

$y=-x+2$	$y=-x-2$
$-4=-2+2$	$-4=-2-2$
$-4=0$ (False)	$-4=-4$ (True)

$\Rightarrow \quad x+y+2=0$ is a tangent.

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EXERCISE 6.5

Q1. Find an equation of the hyperbola with the given data:

(i) Centre (0, 0), Focus (0, 10), Vertex (0, 8)

Solution:

Centre: $C(0, 0)$

Focus: $F(0, 10)$ and Vertex: $A(0, 8)$

Both focus and vertex have the same x-coordinate '0'

$\Rightarrow$ Transverse axis is vertical.

c = Distance from $C(0, 0)$ to $F(0, 10)$
$= 10$

a = Distance from $C(0, 0)$ to $A(0, 8)$
$= 8$

As we know

$$c^2 = a^2 + b^2$$

$$10^2 = 8^2 + b^2 \Rightarrow b^2 = 100 - 64 \Rightarrow b^2 = 36$$

Required equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{8^2} - \frac{x^2}{36} = 1$$

$$\frac{y^2}{64} - \frac{x^2}{36} = 1$$

(ii) Foci $(0, \pm 8)$, Eccentricity = 4

Solution:

Foci: $F(0, 8)$ and $F'(0, -8)$

Both the foci have the same x-coordinate

$\Rightarrow$ Transverse axis is vertical.

Centre: $C\left(\frac{0+0}{2}, \frac{8+(-8)}{2}\right) = C(0, 0)$

c = Distance from $C(0, 0)$ to $F(0, 8)$

$c = 8$

Eccentricity: $e = 4$

$$\frac{c}{a} = 4 \Rightarrow \frac{8}{a} = 4 \Rightarrow a = \frac{8}{4} \Rightarrow a = 2$$

As we know

$$c^2 = a^2 + b^2$$

$$8^2 = 2^2 + b^2 \Rightarrow b^2 = 64 - 4 \Rightarrow b^2 = 60$$

Required equation of hyperbola is:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{2^2} - \frac{x^2}{60} = 1$$

$$\frac{y^2}{4} - \frac{x^2}{60} = 1$$

(iii) Foci $(0, \pm 10)$, Directrices $y = \pm 5$

Solution:

Foci: $F(0, 10)$ and $F'(0, -10)$

Both the foci have the same x-coordinate

$\Rightarrow$ Transverse axis is vertical.

Centre: $C\left(\frac{0+0}{2}, \frac{10+(-10)}{2}\right) = C(0, 0)$

c = Distance from $C(0, 0)$ to $F(0, 10)$
$= 10$

Directrices: $y = \pm 5 \Rightarrow \pm\frac{a}{e} = \pm 5$

$$\frac{a}{c/a} = 5 \Rightarrow \frac{a^2}{c} = 5 \Rightarrow \frac{a^2}{10} = 5 \Rightarrow a^2 = 50$$

As we know

$$c^2 = a^2 + b^2$$

$$10^2 = 50 + b^2 \Rightarrow b^2 = 100 - 50 \Rightarrow b^2 = 50$$

Required equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{50} - \frac{x^2}{50} = 1$$

Q2. Find the centre, foci, eccentricity, vertices and equations of directrices of each of the following:

(i) $\frac{x^2}{36} - \frac{y^2}{64} = 1$

Solution:

Hyperbola: $\frac{x^2}{36} - \frac{y^2}{64} = 1$

Compare it with

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$\Rightarrow$ $a^2 = 36 \Rightarrow a = 6$

and $b^2 = 64 \Rightarrow b = 8$

As we know

$$c^2 = a^2 + b^2$$

$$c^2 = 36 + 64 = 100 \Rightarrow c = 10$$

Centre: $C(0, 0)$

Foci: $(\pm c, 0) = (\pm 10, 0)$

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Eccentricity: $e=\frac{c}{a} \Rightarrow e=\frac{10}{6} \Rightarrow e=\frac{5}{3}$

Vertices: $(\pm a,0)=(\pm 6,0)$

Directrices: $x=\pm\frac{a}{e}=\pm\frac{6}{5/3} \Rightarrow x=\pm\frac{18}{5}$

(ii) $\frac{(x+3)^2}{5}-\frac{(y-2)^2}{4}=1$

Solution:

Hyperbola: $\frac{(x+3)^2}{5}-\frac{(y-2)^2}{4}=1$

Compare it with

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \Rightarrow \frac{(x-(-3))^2}{5}-\frac{(y-2)^2}{4}=1$$

$\Rightarrow \quad h=-3, k=2, a^2=5 \Rightarrow a=\sqrt{5}, b^2=4 \Rightarrow b=2$

As we know

$$c^2=a^2+b^2$$

$$c^2=5+4=9 \Rightarrow c=3$$

Centre: $C(h,k)=C(-3,2)$

Foci: $(h\pm c,k)=(-3\pm 3,2)$

$\Rightarrow \quad F(0,2)$ and $F'(-6,2)$

Eccentricity: $e=\frac{c}{a} \Rightarrow e=\frac{3}{\sqrt{5}}$

Vertices: $(h\pm a,k)=(-3\pm\sqrt{5},2)$

Directrices: $x=h\pm\frac{a}{e}=-3\pm\frac{\sqrt{5}}{3/\sqrt{5}} \Rightarrow x=-3\pm\frac{5}{3}$

$$x=-3+\frac{5}{3} \qquad x=-3-\frac{5}{3}$$

$$x=\frac{-9+5}{3} \qquad x=\frac{-9-5}{3}$$

$$x=-\frac{4}{3} \qquad x=-\frac{14}{3}$$

(iii) $\frac{y^2}{25}-\frac{x^2}{16}=1$

Solution:

Hyperbola: $\frac{y^2}{25}-\frac{x^2}{16}=1$

Compare it with

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

$\Rightarrow \quad a^2=25 \Rightarrow a=5$

and $\quad b^2=16 \Rightarrow b=4$

As we know

$$c^2=a^2+b^2$$

$$c^2=25+16=41 \Rightarrow c=\sqrt{41}$$

Centre: $C(0,0)$

Foci: $(0,\pm c)=(0,\pm\sqrt{41})$

Eccentricity: $e=\frac{c}{a} \Rightarrow e=\frac{\sqrt{41}}{5}$

Vertices: $(0,\pm a)=(0,\pm 5)$

Directrices: $y=\pm\frac{a}{e}=\pm\frac{5}{\sqrt{41}/5} \Rightarrow y=\pm\frac{25}{\sqrt{41}}$

$$y=\pm\frac{25}{\sqrt{41}}\times\frac{\sqrt{41}}{\sqrt{41}} \Rightarrow y=\pm\frac{25\sqrt{41}}{41} \Rightarrow y=\pm\frac{25}{\sqrt{41}}$$

Q3. A hyperbola has its centre at the origin, transverse axis of length 14 along the y axis, and conjugate axis of length 10 along the x-axis. Find the equation of the hyperbola.

Solution:

Centre: $C(0,0)$

Length of transverse axis = 14

$$2a=14 \Rightarrow a=7$$

Length of conjugate axis = 10

$$2b=10 \Rightarrow b=5$$

Since, the transverse axis along y-axis, so

Required equation of hyperbola is

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

$$\frac{y^2}{7^2}-\frac{x^2}{5^2}=1$$

$$\frac{y^2}{49}-\frac{x^2}{25}=1$$

Q4. A hyperbola has a horizontal transverse axis of length 8. One of its foci is at the point $(1+2\sqrt{5},3)$ and its eccentricity is $\frac{\sqrt{5}}{2}$. Find the equation of the hyperbola.

Solution:

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Length of transverse axis = 8

$2a=8 \Rightarrow a=4$

Eccentricity: $e=\frac{\sqrt{5}}{2}$

$\frac{c}{a}=\frac{\sqrt{5}}{2} \Rightarrow \frac{c}{4}=\frac{\sqrt{5}}{2} \quad \because a=4$

$\Rightarrow \quad c=2\sqrt{5}$

As we know

$c^2=a^2+b^2$

$(2\sqrt{5})^2=4^2+b^2 \Rightarrow b^2=20-16$

$b^2=4 \Rightarrow b=2$

Given Focus: $(1+2\sqrt{5},3)=(h+c,k)$

$\Rightarrow \quad h=1, k=3 \quad \because c=2\sqrt{5}$

Since, the transverse axis is horizontal, so

Required equation of hyperbola is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-1)^2}{4^2}-\frac{(y-3)^2}{2^2}=1$$

$$\frac{(x-1)^2}{16}-\frac{(y-3)^2}{4}=1$$

Q5. A hyperbola has a vertical transverse axis of length 8, centre at $(2,-1)$, and eccentricity $\frac{\sqrt{13}}{2}$. Find the equation of the hyperbola.

Solution:

Centre: $C(2,-1) \Rightarrow h=2, k=-1$

Length of transverse axis = 8

$29=8 \Rightarrow a=4 \Rightarrow a^2=16$

Eccentricity: $e=\frac{\sqrt{13}}{2}$

$\frac{c}{a}=\frac{\sqrt{13}}{2} \Rightarrow \frac{c}{4}=\frac{\sqrt{13}}{2} \quad \because a=4$

$\Rightarrow \quad c=2\sqrt{13}$

As we know

$c^2=a^2+b^2$

$(2\sqrt{13})^2=4^2+b^2 \Rightarrow b^2=52-16 \Rightarrow b^2=36$

Since, the transverse axis is vertical, so

Required equation of hyperbola is

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

$$\frac{(y+1)^2}{16}-\frac{(x-2)^2}{36}=1$$

Q6. A hyperbola has its centre at $(-1,4)$, a focus at $(5,4)$, and the corresponding directrix given by the line $x=-3$. Find the equation of the hyperbola.

Solution:

Centre: $C(-1,4) \Rightarrow h=-1, k=4$

Given focus: $F(5,4)$

Both focus and centre have the same y-coordinate

$\Rightarrow$ Transverse axis is horizontal.

c = Distance from $C(-1,4)$ to $F(5,4)$

$=|-1-5|=|-6|=6$

$\frac{a}{e}$ = Distance from $C(-1,4)$ to directrix $x=-3$

$\frac{a}{c/a}=|-1-(-3)| \quad \Rightarrow \frac{a^2}{c}=|-1+3|$

$\frac{a^2}{6}=2 \Rightarrow a^2=12$

As we know

$c^2=a^2+b^2$

$6^2=12+b^2 \Rightarrow b^2=36-12 \Rightarrow b^2=24$

Required equation of hyperbola is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x+1)^2}{12}-\frac{(y-4)^2}{24}=1$$

Q7. A hyperbola has a vertical transverse axis, eccentricity $\frac{\sqrt{19}}{4}$, and latus rectum of length $\frac{3}{2}$. The hyperbola passes through the points $(2,8)$ and $(2,0)$. Find the equation of the hyperbola.

Solution:

Eccentricity: $e=\frac{\sqrt{19}}{4}$

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$$\frac{c}{a}=\frac{\sqrt{19}}{4} \Rightarrow c=\frac{\sqrt{19}a}{4}$$

As we know

$$c^2=a^2+b^2$$

$$\left(\frac{\sqrt{19}a}{4}\right)^2=a^2+b^2 \Rightarrow b^2=\frac{19a^2}{16}-a^2$$

$$b^2=\frac{19a^2-16a^2}{16} \Rightarrow b^2=\frac{3a}{16} \quad ...(1)$$

Length of Latus rectum $=\frac{3}{2}$

$$\frac{2b^2}{a}=\frac{3}{2} \Rightarrow \frac{2}{a}\left(\frac{3a^2}{16}\right)=\frac{3}{2}$$ Using (1)

$$\frac{a^2}{a}=\frac{3}{2}\times\frac{16}{6} \Rightarrow a=4$$ Put in eq. (1)

$$b^2=\frac{3(4)^2}{16} \Rightarrow b^2=3$$

Since, the transverse axis is vertical, so
Required equation of hyperbola is

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

$$\frac{(y-k)^2}{16}-\frac{(x-h)^2}{3}=1 \quad ...(2)$$

Since, the hyperbola (1) passes through (2, 8) and (2, 0), so

At (2, 8):

$$\frac{(8-k)^2}{16}-\frac{(2-h)^2}{3}=1 \quad ...(3)$$

At (2, 0):

$$\frac{(0-k)^2}{16}-\frac{(2-h)^2}{3}=1 \quad ...(4)$$

Equation (3) – Equation (4)

$$\frac{(8-k)^2}{16}-\frac{k^2}{16}=0$$

Multiplying both sides by 16

$$(8-k)^2-k^2=0$$

$$64+k^2-16k-k^2=0$$

$$64-16k=0 \Rightarrow 16k=64 \Rightarrow k=4$$

Put $k=4$ in eq. (4)

$$\frac{4^2}{16}-\frac{(2-h)^2}{3}=1$$

$$1-\frac{(2-h)^2}{3}=1 \Rightarrow \frac{(2-h)^2}{3}=0$$

$$(2-h)^2=0 \Rightarrow 2-h=0 \Rightarrow h=2$$

Put $h=2$ and $k=4$ in eq. (2), we get

$$\frac{(y-4)^2}{16}-\frac{(x-2)^2}{3}=1$$

Q8. Find the equation of the hyperbola in standard form (centred at the origin) that passes through the points $(\sqrt{80},4)$ and $(\sqrt{32},2)$.

Solution:

Let required equation of hyperbola be:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad ...(1)$$

Since, the hyperbola (1) passes through $(\sqrt{80},4)$ and $(\sqrt{32},2)$, so

At $(\sqrt{80},4)$:

$$\frac{(\sqrt{80})^2}{a^2}-\frac{4^2}{b^2}=1$$

$$\frac{80}{a^2}-\frac{16}{b^2}=1 \quad ...(2)$$

At $(\sqrt{32},2)$:

$$\frac{(\sqrt{32})^2}{a^2}-\frac{2^2}{b^2}=1$$

$$\frac{32}{a^2}-\frac{4}{b^2}=1 ...(3)$$

Eq. (2) – 4 × Eq. (3)

$$\frac{80}{a^2}-\frac{16}{b^2}=1$$

$$\pm\frac{128}{a^2}\mp\frac{16}{b^2}=\pm4$$

$$\frac{80}{a^2}-\frac{128}{a^2}=-3$$

$$\frac{80-128}{a^2}=-3 \Rightarrow a^2=\frac{-48}{-3} \Rightarrow a^2=16$$

Put $a^2=16$ in eq. (3), we have

$$\frac{32}{16}-\frac{4}{b^2}=1$$

$$2-1=\frac{4}{b^2} \Rightarrow 1=\frac{4}{b^2} \Rightarrow b^2=4$$

Putting values of a^2 and b^2 in eq. (1), we get

$$\frac{x^2}{16}-\frac{y^2}{4}=1$$

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Q9. Find equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$.

Solution:

Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; at $P(a\sec\theta, b\tan\theta)$

Differentiating w.r.t. 'x'

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{2y}{b^2}\frac{dy}{dx}=\frac{2x}{a^2} \Rightarrow \frac{dy}{dx}=\frac{2x}{a^2}\cdot\frac{b^2}{2y}$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

Slope of tangent at $P=\left.\frac{dy}{dx}\right|_{P(a\sec\theta, b\tan\theta)}$

$$m_T=\frac{b^2(a\sec\theta)}{a^2(b\tan\theta)}=\frac{b\sec\theta}{a\tan\theta}$$

Slope of normal at $P=-\frac{1}{m_T}$

$$\Rightarrow \quad m_N=-\frac{a\tan\theta}{b\sec\theta}$$

Equation of tangent at $P(a\sec\theta, b\tan\theta)$ is

$$y-y_1=m_T(x-x_1)$$

$$y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$$

$$\frac{\tan\theta}{b}(y-b\tan\theta)=\frac{\sec\theta}{a}(x-a\sec\theta)$$

$$\frac{y\tan\theta}{b}-\tan^2\theta=\frac{x\sec\theta}{a}-\sec^2\theta$$

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=\sec^2\theta-\tan^2\theta$$

$$\boxed{\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1} \qquad \because 1+\tan^2\theta=\sec^2\theta$$

Equation of normal at $P(a\sec\theta, b\tan\theta)$ is

$$y-y_1=m_N(x-x_1)$$

$$y-b\tan\theta=-\frac{a\tan\theta}{b\sec\theta}(x-a\sec\theta)$$

$$\frac{b}{\tan\theta}(y-b\tan\theta)=-\frac{a}{\sec\theta}(x-a\sec\theta)$$

$$\frac{by}{\tan\theta}-b^2=-\frac{ax}{\sec\theta}+a^2$$

$$by\cot\theta-b^2=-ax\cos\theta+a^2$$

$$\boxed{ax\cos\theta+by\cot\theta=a^2+b^2}$$

Q10. Find the length of the latus rectum of the hyperbola $9x^2-16y^2=144$.

Solution:

Hyperbola: $9x^2-16y^2=144$

Dividing both sides by 144

$$\frac{9x^2}{144}-\frac{16y^2}{144}=\frac{144}{144}$$

$$\frac{x^2}{16}-\frac{y^2}{9}=1$$

Compare it with

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\Rightarrow \quad a^2=16 \quad \Rightarrow \quad a=4$$

and $b^2=9 \quad \Rightarrow \quad b=3$

$$\text{Length of Latus rectum}=\frac{2b^2}{a}$$

$$=\frac{2(9)}{4}$$

$$=\frac{9}{2} \text{ units}$$

Q11. Find the eccentricity of the hyperbola whose asymptotes are perpendicular.

Solution:

Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

For joint equation of two asymptotes:

Replace 1 by 0, we have

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

$$\frac{y^2}{b^2}=\frac{x^2}{a^2} \qquad \Rightarrow y^2=\frac{b}{a}$$

$$\Rightarrow \quad y=\pm\frac{b}{a}x$$

Two asymptotes are:

$$y=\frac{b}{d}x \quad \text{and} \quad y=-\frac{b^2}{a^2}x^2$$

Slopes of asymptotes are:

$$m_1=\frac{b}{a} \text{ and } m_2=-\frac{b}{a}$$

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Since, the asymptotes are perpendicular, so

$$m_1 \times m_2 = -1$$

$$\frac{b}{a} \times \frac{-b}{a} = -1$$

$$b^2 = a^2$$

Eccentricity: $e = \frac{c}{a}$

$$e = \frac{\sqrt{a^2+b^2}}{a} \quad \because c^2 = a^2 + b^2$$

$$e = \frac{\sqrt{a^2+a^2}}{a} \quad \because b^2 = a^2$$

$$e = \frac{\sqrt{2a^2}}{a} = \frac{\sqrt{2}a}{a}$$

$$\boxed{e = \sqrt{2}}$$

Q12. If the transverse axis of a hyperbola is 8 and the latus rectum is 9, find its eccentricity.

Solution:

Length of transverse axis = 8

$$2a = 8 \Rightarrow a = 4 \Rightarrow a^2 = 16$$

Length of latus rectum = 9

$$\frac{2b^2}{a} = 9 \Rightarrow b^2 = \frac{9a}{2}$$

$$b^2 = \frac{9(4)}{2} \Rightarrow b^2 = 18$$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a}$

$$e = \frac{\sqrt{16+18}}{4} = \frac{\sqrt{34}}{4}$$

Q13. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. If the eccentricity of the hyperbola is 2, find its equation.

Solution:

Given ellipse: $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$\Rightarrow \quad a^2 = 25$ and $b^2 = 9$

Using: $c^2 = a^2 - b^2$

$c^2 = 25 - 9 = 16 \Rightarrow c = 4$

Foci: $(\pm c, 0) = (\pm 4, 0)$

For Required Hyperbola:

Since hyperbola foci coincide with ellipse foci, therefore $c = 4$

Eccentricity: $e = 2 \Rightarrow \frac{c}{a} = 2$

$$\frac{4}{a} = 2 \Rightarrow a = \frac{4}{2} \Rightarrow a = 2 \Rightarrow a^2 = 4$$

Using: $c^2 = a^2 + b^2$

$$4^2 = 4 + b^2 \Rightarrow b^2 = 16 - 4 \Rightarrow b^2 = 12$$

Required equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

Q14. Find the eccentricity of the hyperbola whose latus rectum is half its transverse axis.

Solution:

Given that

Length of latus rectum = $\frac{1}{2}$(Length of transverse axis)

$$\frac{2b^2}{a} = \frac{1}{2}(2a)$$

$$\frac{2b^2}{a} = a \Rightarrow 2b^2 = a^2 \Rightarrow b^2 = \frac{a^2}{2}$$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a} \because c^2 = a^2 + b^2$

$$e = \frac{1}{a}\sqrt{a^2 + \frac{a^2}{2}}$$

$$e = \frac{1}{a}\sqrt{\frac{2a^2+a^2}{2}}$$

$$= \frac{1}{a}\sqrt{\frac{3a^2}{2}}$$

$$= \frac{1}{a} \cdot \frac{a\sqrt{3}}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$e = \frac{\sqrt{6}}{2}$$

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Q15. Find the equations of the tangents to the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$ which are parallel to the line $2x-y=5$.

Solution:

Hyperbola: $\frac{x^2}{4}-\frac{y^2}{12}=1$

$\Rightarrow \quad a^2=4$ and $b^2=12$

Line: $2x-y=5$

Slope of line $=-\frac{2}{-1}=2$

Slope of tangent $=m=2$ $\because$ line and tangent are parallel

Required equation of tangent is:

$$y=mx\pm\sqrt{a^2m^2-b^2}$$
$$y=2x\pm\sqrt{(4)(2^2)-12}$$
$$y=2x\pm\sqrt{16-12}$$
$$y=2x\pm\sqrt{4}$$
$$y=2x\pm2$$
$$2x-y\pm2=0$$

$\Rightarrow \quad 2x-y+2=0$ and $2x-y-2=0$

Q16. If the distance between the foci of a hyperbola is 16 and the distance between its vertices is 12, find its eccentricity.

Solution:

Distance between foci = 16

$2c=16 \Rightarrow c=8$

Distance between vertices = 12

$2a=12 \Rightarrow a=6$

Eccentricity: $e=\frac{c}{a}=\frac{8}{6}$

$$e=\frac{4}{3}$$

Q17. Find the equation of the hyperbola whose asymptotes are $y=\pm2x$ and which passes through the point (3, 4).

Solution:

Asymptotes: $y=\pm2x$

$\Rightarrow \quad y=2x$ and $y=-2x$

$2x-y=0$ and $2x+y=0$

Joint equation of asymptotes:

$$(2x-y)(2x+y)=0$$
$$4x^2-y^2=0$$

Let required equation of hyperbola be

$$4x^2-y^2=k \qquad \ldots(1)$$

Since, it passes through (3, 4), so

$$4(3)^2-4^2=k$$
$$k=36-16 \Rightarrow k=20$$ Put in eq. (1)
$$4x^2-y^2=20$$

Dividing both sides by 20

$$\frac{4x^2}{20}-\frac{y^2}{20}=\frac{20}{20}$$

$$\boxed{\frac{x^2}{5}-\frac{y^2}{20}=1}$$

Q18. A hyperbola for which $a=b$ is called rectangular. Show that a hyperbola is rectangular if and only if its asymptotes are perpendicular to one another.

Solution:

Consider a standard equation of hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Asymptotes: $y=\pm\frac{b}{a}x$

$\Rightarrow \quad y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$

Slopes: $m_1=\frac{b}{a}$ and $m_2=-\frac{b}{a}$

Since, the asymptotes are perpendicular, so

$$m_1\times m_2=-1$$

$$\frac{b}{a}\times\frac{-b}{a}=-1$$
$$\frac{b^2}{a^2}=1$$
$$b^2=a^2$$

$\Rightarrow \quad \boxed{b=a}$

Hence, the hyperbola is rectangular.

Converse:

As hyperbola is rectangular, so $a=b$

Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$

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$x^2 - y^2 = a^2$

Asymptotes: $y = \pm x$

$\Rightarrow$ $y = 1x$ and $y = -1x$

Slopes: $m_1 = 1$ and $m_2 = -1$

As $m_1 \times m_2 = (1)(-1) = -1$

So, the asymptotes are perpendicular to one another.

Q19. Find the equations of the tangents to the hyperbola $7x^2 - 3y^2 = 105$ that pass through the point (3, 1).

Solution:

Hyperbola: $7x^2 - 3y^2 = 105$

Dividing both sides by 105

$$\frac{7x^2}{105} - \frac{3y^2}{105} = \frac{105}{105}$$

$$\frac{x^2}{15} - \frac{y^2}{35} = 1$$

Compare it with

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$\Rightarrow$ $a^2 = 15, b^2 = 35$

Let required equation of tangent be:

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$y = mx \pm \sqrt{15m^2 - 35} \quad \ldots(1)$$

Since, the tangent line passes through (3, 1), so

$$1 = 3m \pm \sqrt{15m^2 - 35}$$

$$1 - 3m = \pm\sqrt{15m^2 - 35}$$

Square both sides

$$(1-3m)^2 = (\pm\sqrt{15m^2 - 35})^2$$

$$1 + 9m^2 - 6m = 15m^2 - 35$$

$$15m^2 - 35 - 1 - 9m^2 + 6m = 0$$

$$6m^2 + 6m - 36 = 0$$

Dividing both sides by 6

$$m^2 + m - 6 = 0$$

$$m^2 + 3m - 2m - 6 = 0$$

$$m(m+3) - 2(m+3) = 0$$

$$(m+3)(m-2) = 0$$

Either $m + 3 = 0$ or $m - 2 = 0$

$\Rightarrow$ $m = -3$ or $m = 2$

When $m = 2$:

Eq. (1) $\Rightarrow$ $y = 2x \pm \sqrt{15(2)^2 - 35}$

$$y = 2x \pm \sqrt{60 - 35}$$

$$y = 2x \pm 5$$

Check at (3, 1):

$y = 2x + 5$	$y = 2x - 5$
$1 = 2(3) + 5$	$1 = 2(3) - 5$
$1 = 6 + 5$	$1 = 6 - 5$
$1 = 11$ (False)	$1 = 1$ (True)

$\Rightarrow$ $2x - y - 5 = 0$ is a tangent.

When $m = -3$:

Eq. (1) $\Rightarrow$ $y = -3x \pm \sqrt{15(-3)^2 - 35}$

$$y = -3x \pm \sqrt{135 - 35}$$

$$y = -3x \pm 10$$

Check at (3, 1):

$y = -3x + 10$	$y = -3x - 10$
$1 = -3(3) + 10$	$1 = -3(3) - 10$
$1 = -9 + 10$	$1 = -9 - 10$
$1 = 1$ (True)	$1 = -19$ (False)

$\Rightarrow$ $3x + y - 10 = 0$ is a tangent.

Q20. Find the condition that the line $lx + my = 1$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solution:

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Line: $lx + my = 1$

$\Rightarrow$ $my = lx - 1$

Dividing both sides by m

$$\frac{my}{m} = \frac{lx}{m} - \frac{1}{m}$$

$$y = \frac{l}{m}x - \frac{1}{m}$$

Compare it with

$$y = m'x + c$$

$\Rightarrow$ $m' = \frac{l}{m}, c = -\frac{1}{m}$

Tangency Condition:

$$c^2 = a^2(m')^2 - b^2$$

$$\left(-\frac{1}{m}\right)^2 = a^2\left(\frac{l}{m}\right)^2 - b^2$$

$$\frac{1}{m^2} = \frac{a^2l^2}{m^2} - b^2$$

Multiplying both sides by m^2

$$1 = a^2l^2 - b^2m^2$$

$\Rightarrow$ $a^2l^2 - b^2m^2 = 1$ (Required condition)

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APPLYING CONCEPTS OF CONICS TO REAL LIFE WORLD PROBLEMS

Example 31: A comet has a parabolic orbit with the Earth at the focus. When the comet is h km from the earth, the line joining the comet and the Earth makes an angle of 30° with the axis of the parabola. How close will the comet come to the Earth?

Solution:

Let equation of the parabolic orbit be:

$$y^2 = 4ax \quad \ldots(1)$$

Let the focus $E(a, 0)$ be the centre of earth. The point on a parabola closest to focus is its vertex.

$\Rightarrow \quad |\overline{OE}| = a = ?$

If the comet is at $P(x, y)$, then

$$|\overline{PF}| = h \text{ km and } m\angle PEQ = 30°$$

From right ΔPEQ, we have

$$\cos 30° = \frac{x-a}{h}$$

$$\frac{\sqrt{3}}{2} = \frac{x-a}{h}$$

$$x - a = \frac{\sqrt{3}}{2}h$$

$$x = a + \frac{\sqrt{3}}{2}h$$

$$\sin 30° = \frac{y}{h}$$

$$\frac{1}{2} = \frac{y}{h}$$

$$y = \frac{h}{2}$$

Since point $P(x, y) = \left(a + \frac{\sqrt{3}}{2}h, \frac{h}{2}\right)$ lies on the parabola,

Eq. (1) $\Rightarrow \quad \left(\frac{h}{2}\right)^2 = 4a\left(a + \frac{\sqrt{3}}{2}h\right)$

$$\frac{h^2}{4} = 4a^2 + 2\sqrt{3}ah$$

$$4a^2 + 2\sqrt{3}ah - \frac{h^2}{4} = 0$$

$$4a^2 + (2\sqrt{3}h)a - \frac{h^2}{4} = 0 \qquad \text{(Quadratic in 'a')}$$

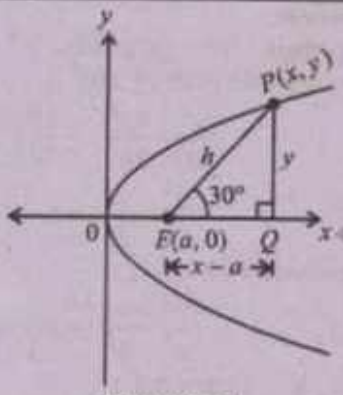

Figure 6.30

By using quadratic formula

$$a = \frac{-2\sqrt{3}h \pm \sqrt{(2\sqrt{3}h)^2 - 4(4)\left(-\frac{h^2}{4}\right)}}{2(4)}$$

$$= \frac{-2\sqrt{3}h \pm \sqrt{12h^2 + 4h^2}}{8}$$

$$= \frac{-2\sqrt{3}h \pm \sqrt{16h^2}}{8}$$

$$= \frac{-2\sqrt{3}h \pm 4h}{8} = \frac{h(-\sqrt{3} \pm 2)}{8}$$

$$a = \left(\frac{-\sqrt{3}+2}{4}\right)h \qquad \because \left(\frac{-\sqrt{3}-2}{4}\right)h < 0$$

Hence, the minimum distance between Comet and Earth is

$$a = \left(\frac{-\sqrt{3}+2}{4}\right)h \text{ km}$$

Example 32: Elliptical Electron Orbit

In the Bohr-Sommerfeld model of the atom, electrons move in elliptical orbits around the nucleus, which is located at one focus. Consider an electron in an elliptical orbit with a semi-major axis $a = 5.29 \times 10^{-11}$ m (the Bohr radius) and an eccentricity $e = 0.20$.

(a) Find the distance from the nucleus to the centre of the ellipse.

(b) Determine the minimum distance (periapsis) and maximum distance (apoapsis) of the electron from the nucleus.

Solution:

Given that: $a = 5.29 \times 10^{-11}$ m, $e = 0.20$

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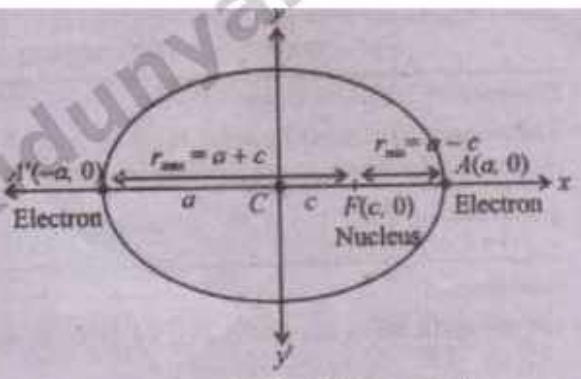
Let the nucleus be at focus $F(c, 0)$ and $A(a, 0)$, $A'(-a, 0)$ be the vertices.



(a) c = Distance from centre to focus

$= ae$ $\quad \because e = \frac{c}{a}$

$= (5.29 \times 10^{-11}) \times 0.20$

$c = 1.058 \times 10^{-11}$ m (Distance from nucleus to centre)

(b) Minimum Distance $= a - c$

$r_{min} = (5.29 \times 10^{-11}) - (1.058 \times 10^{-11})$

$= (5.29 - 1.058) \times 10^{-11}$

$= 4.232 \times 10^{-11}$ m

Maximum Distance $= a + c$

$r_{max} = (5.29 \times 10^{-11}) + (1.058 \times 10^{-11})$

$r_{max} = (5.29 + 1.058) \times 10^{-11}$

$= 6.348 \times 10^{-11}$ m

Example 33: The cable of a suspension bridge is parabolic. The roadway is 10 m, below the lowest point of the cable. The span of the bridge is 200 m, and the tops of the piers are 50 m above the roadway. Find the equation of the parabola. If the load is supported by vertical chains at intervals of 10 m, find the length of the chain that is 20 m away from either pier.

Solution:

Let BAC be the cable, vertex $A(0, 0)$ be the lowest point of the cable and $\overline{DE}$ be the roadway.

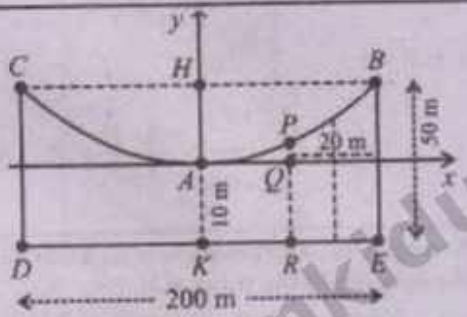


Figure 6.31

Since the parabola opens upward, therefore

$x^2 = 4ay$ ...(1)

Since $HB = 100$ m and $AH = 50 - 10 = 40$

$\Rightarrow$ $B(100, 40)$ lies on parabola (1).

$(100)^2 = 4a(40) \Rightarrow 4a = \frac{10000}{40} = 250$

Eq. (1) $\Rightarrow$ $x^2 = 250\,y$ ...(2)

If $\overline{PR}$ be the chain at a distance 20 m from the pier $\overline{BE}$, then

$AQ = 100 - 20 = 80$ m

Since $P(80, PQ)$ lies on (2), so

$(80)^2 = 250(PQ) \Rightarrow PQ = \frac{6400}{250} = 25.6$ m

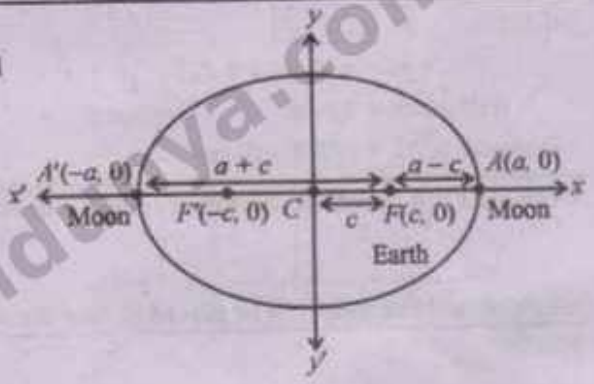
Required length of chain $= PR$

$= PQ + QR = 25.6 + 10 = 35.6$ m

Example 34: The moon revolves around Earth in an elliptical orbit, with the centre of Earth located at one focus. The major and minor axis of the orbit have lengths of 768,800 km and 67,600 km, respectively. Find the greatest and least distances (the apogee and perigee) from Earth's centre to the moon's centre.

Solution:

Let the earth be at one focus $F(c, 0)$, other focus $F'(-c, 0)$ and $A(a, 0)$, $A'(-a, 0)$ be the vertices.



Length of major axis = 768800

$2a = 768800 \Rightarrow a = 384400$

Length of minor axis = 767600

$2b = 767600 \Rightarrow b = 383800$

Greatest Distance $= a + c$

$= 384400 + 21469$

$= 405869$ km

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Least Distance $\approx a - c$

$= 384400 - 21469 = 362931$ km

Example 35: Satellite Orbit

The apogee (the point in orbit farthest from Earth) and the perigee (the point in orbit closest to Earth) of an elliptical orbit of an Earth satellite are given by A and P, respectively. Show that the eccentricity of the orbit is $e = \frac{A-P}{A+P}$.

Solution:

Let the earth be at one focus $F(c, 0)$ other focus $F'(-c, 0)$ and $A(a, 0)$, $A'(-a, 0)$ be the vertices.

Greatest Distance: $A = a + c$

Least Distance: $P = a - c$

$\Rightarrow A + P = (a + c) + (a - c)$

$= a + c + a - c$

$A + P = 2a$

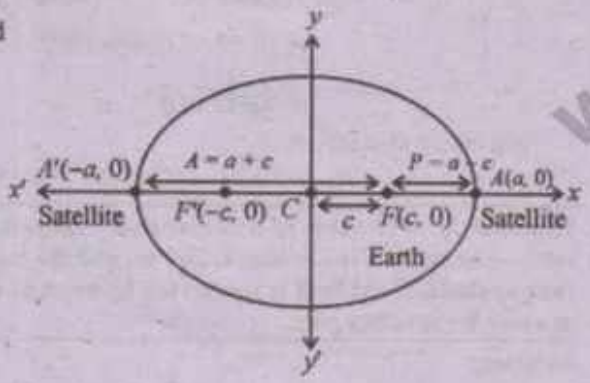
$\Rightarrow \quad a = \frac{1}{2}(A + P)$

$\Rightarrow A - P = (a + c) - (a - c)$

$= a + c - a + c$

$A - P = 2c$

$\Rightarrow \quad c = \frac{1}{2}(A - P)$



L.H.S $= e$

$= \frac{c}{a} = \frac{\frac{1}{2}(A-P)}{\frac{1}{2}(A+P)} = \frac{A-P}{A+P}$ = R.H.S (Proved)

Example 36: A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 20 cm.

(a) Find an equation of the parabola.

(b) Find the diameter of the opening $|CD|$, 20 cm from the vertex.

Solution:

(a) Let required equation of parabola be

$$y^2 = 4ax \qquad \ldots(1)$$

Since the bulb is at the focus F and the opening at the focus is 20 m, therefore

$4a = |AB|$ = Length of latus rectum = 20

Eq. (1) $\Rightarrow$ $\boxed{y^2 = 20x}$ $\qquad \ldots(2)$

(b) 20 m from vertex means $x = 20$

Eq. (2) $\Rightarrow$ $\boxed{y^2 = 20(20)}$

$y^2 = 400 \Rightarrow y = \pm 20$

Half-width = 20 cm

Diameter $|\overline{CD}| = 2(20) = 40$ cm

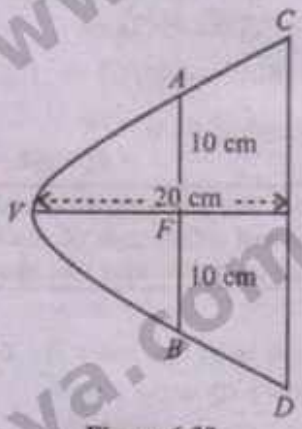


Figure 6.33

Example 37: A parabolic flashlight reflector has the shape given by the equation:

$$x = \frac{1}{20}y^2$$

where should the lightbulb be placed so that the reflected rays emerge parallel to the axis?

Solution:

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Given equation: $x = \frac{1}{20}y^2$

$\Rightarrow \quad y^2 = 20x$

Compare it with

$y^2 = 4ax$

$\Rightarrow \quad 4a = 20 \Rightarrow a = 5$

By reflective property of parabola, the lightbulb must be at focus.

Focus: $F(a, 0) = F(5, 0)$

EXERCISE 6.6

Q1. A comet has a parabolic orbit with the Sun at the focus. When the comet is 100 million km from the Sun, the line joining the Sun and the comet makes an angle of 60° with the axis of the parabola. How close will the comet get to the Sun?

Solution:

Let equation of parabolic orbit be:

$y^2 = 4ax \quad \ldots(1)$

Let the sun be at focus $F(a, 0)$ and the point on a parabola closest to focus is its vertex.

$\Rightarrow \quad |\overline{OF}| = a = ?$

If the comet is at $P(x, y)$, then

$|\overline{PF}| = 100$ million km and $m\angle PFQ = 60°$

From right ΔPFQ, we have

$\cos 60° = \frac{x-a}{100}$

$\frac{1}{2} = \frac{x-a}{100}$

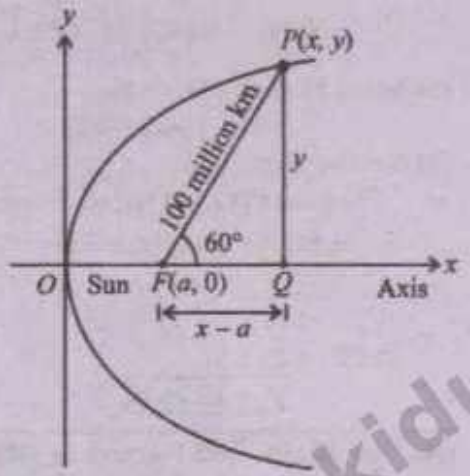
$x - a = 50$

$x = a + 50$

$\sin 30° = \frac{y}{100}$

$\frac{\sqrt{3}}{2} = \frac{y}{100}$

$y = 50\sqrt{3}$



Since, the point $P(x, y) = P(a+50, 50\sqrt{3})$ lies on the parabola, so

Eq. (1) $\Rightarrow \quad (50\sqrt{3})^2 = 4a(a+50)$

$2500(3) = 4a^2 + 200a$

$4a^2 + 200a - 7500 = 0$

Dividing both sides by '4'

$a^2 + 50a - 1875 = 0$

$a^2 + 75a - 25a - 1875 = 0$

$a(a+75) - 25(a+75) = 0$

$(a-25)(a+75) = 0$

Either $a - 25 = 0$ or $a + 75 = 0$

$a = 25$ or $a = -75 < 0$ (rejected)

Hence, the minimum distance between comet and sun is $a = 25$ million km.

Q2. Find an equation of the parabola formed by the cables of a suspension bridge whose span is ℓm and the vertical height of the supporting towers is h m.

Solution:

Let required equation of parabola be:

$x^2 = 4ay \quad \ldots(1)$

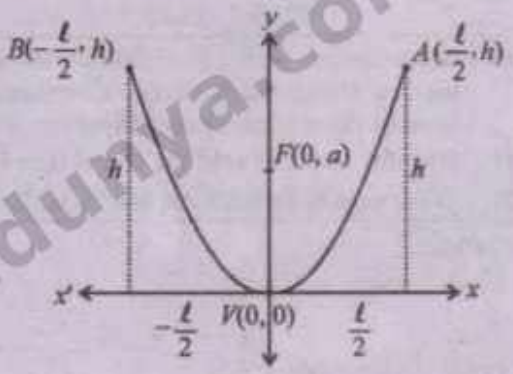
Focus: $F(0, a)$

Vertex: $V(0, 0)$

Since $A\left(\frac{\ell}{2}, h\right)$ lies on parabola, therefore

Eq. (1) $\Rightarrow \quad \left(\frac{\ell}{2}\right)^2 = 4a(h)$

$\frac{\ell^2}{4} = 4ah \Rightarrow 4a = \frac{\ell^2}{4h}$ Put in eq. (1)



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$x^2 = \left(\frac{\ell^2}{4h}\right)y$ $\Rightarrow$ $\boxed{y = \frac{4h}{\ell^2}x^2}$

Q3. A parabolic arch has 120 m base and height 25 m. Find the height of the arch at a point 42 m from the centre of the base.

Solution:

Consider a parabolic arch with centre $O(0, 0)$, such that

$$x^2 = -4a(y-k)$$

Vertex: $V(0,25) \Rightarrow k = 25$

$$\Rightarrow \quad x^2 = -4a(y-25) \qquad \ldots(1)$$

Since the point $B(60, 0)$ lies on parabola, therefore

$$(60)^2 = -4a(0-25)$$

$$3600 = 4a(25) \Rightarrow 4a = \frac{3600}{25} = 144$$

Put $4a = 144$ in eq. (1), we have

$$x^2 = -144(y-25) \qquad \ldots(2)$$

Let required height $= y_1$

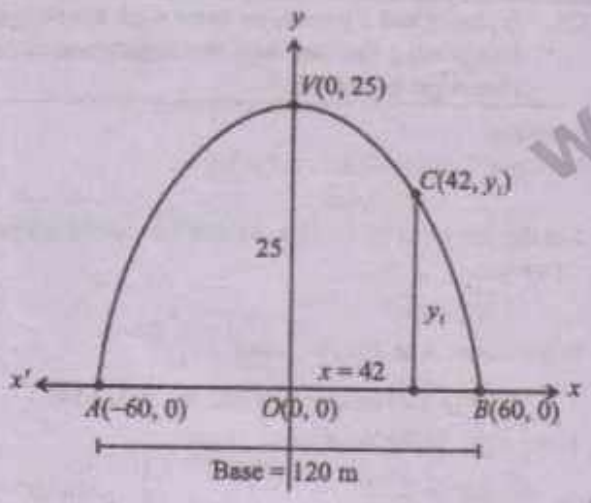
$\Rightarrow$ The point $C(42, y_1)$ lies on the parabola (2), so

$$(42)^2 = -144(y_1 - 25)$$

$$\frac{1764}{-144} = y_1 - 25$$

$$-12.25 + 25 = y_1$$

$$\Rightarrow \quad \boxed{y_1 = 12.75\ m}$$



Q4. An athletic field features an elliptical running track. The track is 120 meters long along its major axis and 80 meters wide along its minor axis.

(a) Find the distance from the centre to each focus of the ellipse.

(b) If two water fountains are placed at the foci, how far apart are they?

Solution:

Length of major axis = 120

$$2a = 120 \quad \Rightarrow a = 60$$

Length of minor axis = 80

$$2b = 80 \quad \Rightarrow b = 40$$

(a) As we know

$$c^2 = a^2 - b^2$$

$$= (60)^2 - (40)^2$$

$$= 3600 - 1600$$

$$c^2 = 2000$$

$$\sqrt{c^2} = \sqrt{2000}$$

$$c = 20\sqrt{5}\ \text{m} \quad \left(\text{Distance from centre to each focus of ellipse}\right)$$

(b) Distance between foci (Fountains) $= 2c$

$$= 2(20\sqrt{3})$$

$$= 40\sqrt{3}\ \text{m}$$

Q5. A landscaper wants to create an elliptical flower bed with a major axis of 10 meters and a minor axis of 6 meters. Using the gardener's method, two stakes are placed at the foci, and a loop of string is stretched around them to trace the ellipse.

(a) How far apart should the two stakes be placed?

(b) What should be the total length of the string loop?

Solution:

Length of major axis = 10

$$2a = 10 \Rightarrow a = 5$$

Length of minor axis = 6

$$2b = 6 \Rightarrow b = 3$$

As we know

$$c^2 = a^2 - b^2$$

$$= (5)^2 - (3)^2$$

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$= 25 - 9$

$c^2 = 16 \Rightarrow c = 4$

(a) Distance between foci (stakes) $= 2c$

$= 2(4) = 8$ m

(b) Let P be the moving tracing pin on the ellipse, then

Total Loop Length $= (|PF'| + |PF|) + |FF'|$

$= 2a + 2c$

$= 2(5) + 2(4)$

$= 10 + 8$

$= 18$ m

Q6. An architect designs a bridge with an elliptical arch. The arch has a span (width) of 24 meters and a height of 8 meters. The ellipse is centred at the origin with the major axis horizontal.

(a) Write the equation of the ellipse representing the arch.

(b) A boat needs to pass under the bridge. The boat is 6 meters wide. What is the maximum height of the boat that can pass through the arch if it must stay centred?

Solution:

(a) Total span (Length of major axis) = 24

$2a = 24 \Rightarrow a = 12$

Maximum height (Semi-minor axis) = 8

$\Rightarrow b = 8$

Since, the centre is at origin (0, 0) with a horizontal major axis, so

Required equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{12^2} + \frac{y^2}{8^2} = 1$$

$$\frac{x^2}{144} + \frac{y^2}{64} = 1 \quad \ldots(1)$$

(b) Width of boat = 6 m

Since the boat stay at centre then it extends from $x = -3$ to $x = 3$.

Let required height of boat = h m

Since point $P(3, h)$ lies on arch, therefore

Eq. (1) $\Rightarrow \frac{3^2}{144} + \frac{h^2}{64} = 1$

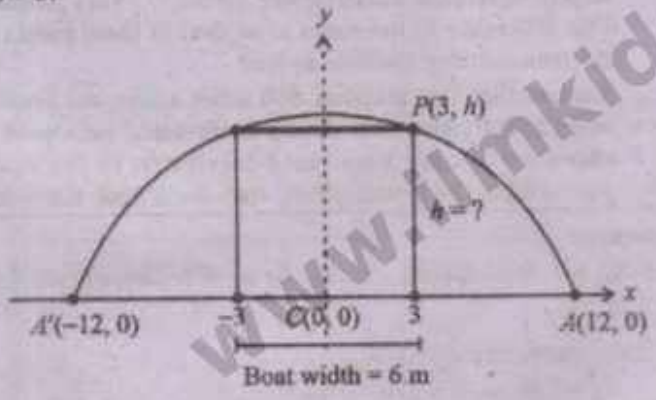
$$\frac{9}{144} + \frac{h^2}{64} = 1$$

$$\frac{1}{16} + \frac{h^2}{64} = 1 \Rightarrow \frac{h^2}{64} = 1 - \frac{1}{16}$$

$$h^2 = 64\left(\frac{16-1}{16}\right) \Rightarrow h^2 = 4(15)$$

$$h^2 = 60 \Rightarrow h = \sqrt{60}$$

$$\boxed{h = 2\sqrt{15} \text{ m}}$$



Q7. In the Bohr-Sommerfeld model, an electron moves in an elliptical orbit around a nucleus. For a hydrogen atom, the square of the orbital period T is proportional to the cube of the semi-major axis a.

Consider an electron in an elliptical orbit with a semi-major axis $a = 2.12 \times 10^{-10}$ m and an eccentricity $e = 0.30$. The constant of proportionality for hydrogen is $k = 2.96 \times 10^{-16}$ s^2/m^3.

(a) Determine the distance between the nucleus (at one focus) and the centre of the ellipse.

(b) Calculate the orbital period T of the electron.

(c) Find the maximum and minimum distances of the electron from the nucleus during its orbit.

Solution:

Given that:

$a = 2.12 \times 10^{-10}$ m, $e = 0.30$

$k = 2.96 \times 10^{-16}$ s^2/m^3

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(a) c = Distance between nucleus and centre

$= ae$ $\quad \because e = \frac{c}{a}$

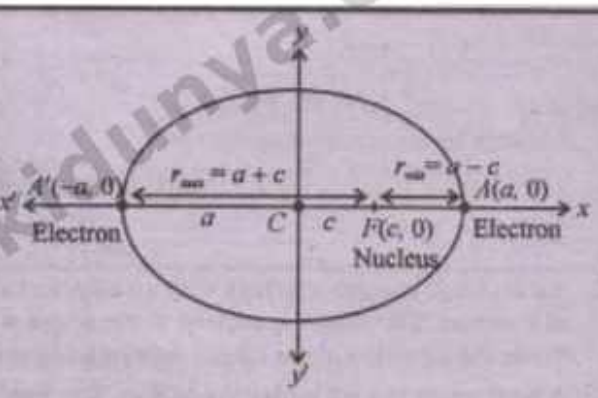
$= (2.12\times10^{-10})(0.30)$

$= (2.12\times0.30)\times10^{-10}$

$= 0.636\times10^{-10}$

$= 6.36\times10^{-1}\times10^{-10}$

$c = 6.36\times10^{-11}$ m



(b) Orbital period: T = ?

According to the given condition

$T^2 \propto a^3$

$T^2 = ka^3$

$T^2 = (2.96\times10^{-16})(2.12\times10^{-10})^3$

$= (2.96\times10^{-16})(9.528\times10^{-30})$

$= (2.96\times9.528)\times10^{-16-30}$

$T^2 = 28.20\times10^{-46}$

$\Rightarrow \quad \sqrt{T^2} = \sqrt{28.20\times10^{-46}}$

$T \approx 5.31\times10^{-23}$ seconds

(c) Max. Distance $= a + c$

$= (2.12\times10^{-10}) + (6.36\times10^{-11})$

$= 2.12\times10^{-10} + 0.636\times10^{-10}$

$= (2.12 + 0.636)\times10^{-10}$

$= 2.756\times10^{-10}$ m

Min. Distance $= a - c$

$= (2.12\times10^{-10}) - (6.36\times10^{-11})$

$= 2.12\times10^{-10} - 0.636\times10^{-10}$

$= (2.12 - 0.636)\times10^{-10}$

$= 1.484\times10^{-10}$ m

Q8. LORAN (long distance radio navigation) for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci.

Assume that two stations, 500 miles apart, are positioned on a rectangular coordinate system at $(-250, 0)$ and $(250, 0)$ and that a ship is traveling on a path with coordinates $(x, 120)$ (a horizontal line 120 miles above the x-axis). Find the x-coordinate of the position of the ship when the time difference between the pulses from the transmitting stations is 1200 microseconds (0.0012 second).

Solution:

Let the two transmitting stations be at $F'(-250, 0)$ and $F(250, 0)$.

Distance between foci = 500

$2c = 500 \Rightarrow c = 250$

If $P(x, 120)$ is the position of ship, then

By definition of hyperbola

$\big||PF'| - |PF|\big| = 2a$

$|vt_2 - vt_1| = 2a$ $\quad \because s = vt$

$2a = |v|\,|\Delta t|$

$2a = (186000)(0.0012)$

$2a = 223.2 \Rightarrow a = 111.6$

As we know

$c^2 = a^2 + b^2$

$(250)^2 = (111.6)^2 + b^2$

$\Rightarrow \quad b^2 = 62500 - 12454.56$

$b^2 = 50045.44$

Equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{12454.56} - \frac{y^2}{50045.44} = 1$$

At $P(x, 120)$:

$$\frac{x^2}{12454.56} - \frac{120^2}{50045.44} = 1$$

$$\frac{x^2}{12454.56} - 0.28774 = 1$$

$$\frac{x^2}{12454.56} = 1 + 0.28774$$

$x^2 = 12454.56\,(1.28774)$

$x^2 \approx 16038.13$

$\sqrt{x^2} = \sqrt{16038.13}$

$\boxed{x \approx 126.64 \text{ miles}}$

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Q9. A satellite orbits Earth in an elliptical path. The apogee (farthest distance from Earth) is $A = 42000$ km, and the perigee (closest distance to Earth) is $P = 30000$ km.

(a) Find the semi-major axis a of the orbit. **(b) Find the distance from the centre to each focus c.**

(c) Calculate the eccentricity e of the orbit.

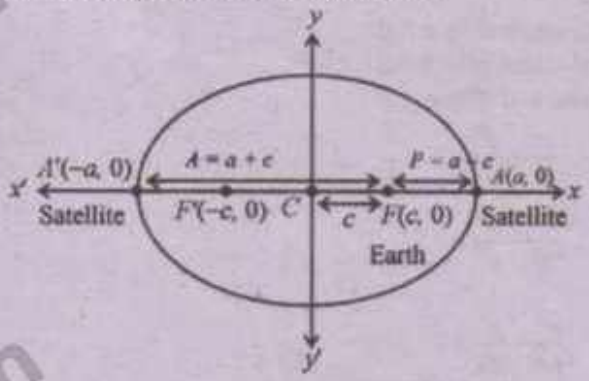
Solution:

(a) Let the earth be at one focus $F(c, 0)$, other focus $F'(-c, 0)$ and $A(a, 0)$, $A'(-a, 0)$ be the vertices.

Given that:

$$A = 42000 \Rightarrow a + c = 42000 \quad \ldots(1)$$

$$P = 30000 \Rightarrow a - c = 30000 \quad \ldots(2)$$



Equation (1) + Equation (2)

$$(a + c) + (a - c) = 42000 + 30000$$

$$2a = 72000$$

$$a = \frac{72000}{2}$$

$$\boxed{a = 36000 \text{ km}}$$

(b) Equation (1) − Equation (2)

$$(a + c) - (a - c) = 42000 - 30000$$

$$a + c - a + c = 12000$$

$$2c = 12000$$

$$c = \frac{12000}{2}$$

$$\boxed{c = 6000 \text{ km}} \quad \left(\text{Distance from centre to each focus}\right)$$

(c) Eccentricity: $e = \frac{c}{a}$

$$e = \frac{6000}{36000} \Rightarrow \boxed{e = \frac{1}{6}}$$

Q10. The end towers of a suspension bridge are 100 meters high and 540 meters apart. The suspension cable hangs from the top of the towers in the form of a parabola and the lowest point of the cable is 25 meters above the horizontal road. Find the length of the supporting vertical cable at a horizontal distance of 81 meters from the centre of the bridge.

Solution:

Let BAC be the cable, vertex $A(0, 0)$ be the lowest point of the cable and $\overline{DE}$ be the road.

Since, the parabola opens upward, therefore

$$x^2 = 4ay \quad \ldots(1)$$

Since $HB = 270$ m and $AH = 100 - 25 = 75$

$\Rightarrow$ $B(270, 75)$ lies on parabola (1)

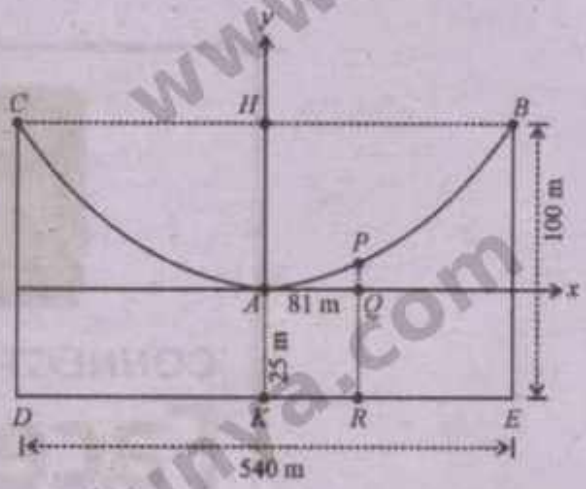
$$(270)^2 = 4a(75) \Rightarrow 4a = \frac{72900}{75} = 972$$

Eq (1) $\Rightarrow$ $x^2 = 972y \quad \ldots(2)$

If $\overline{PR}$ be the vertical supporting cable at a distance of 81 m from the centre, then $AQ = 81$ m.

Since $P(81, PQ)$ lies on (2), so

$$(81)^2 = 972(PQ) \Rightarrow PQ = \frac{6561}{972} = 6.75 \text{ m}$$



Required length of cable $= PR$

$= PQ + QR$

$= 6.75 + 25 = 31.75$ m

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Q11. An architectural arch [illegible] semi-major axis 8 meters and semi-minor axis 5 meters. A horizontal beam is to be placed across the arch at a height where the width of the arch is exactly 10 meters. Calculate the exact height above ground level where this beam should be installed.

Solution:

Since the arch in the shape of the upper half of an ellipse with:

Semi-major axis: $a = 8$

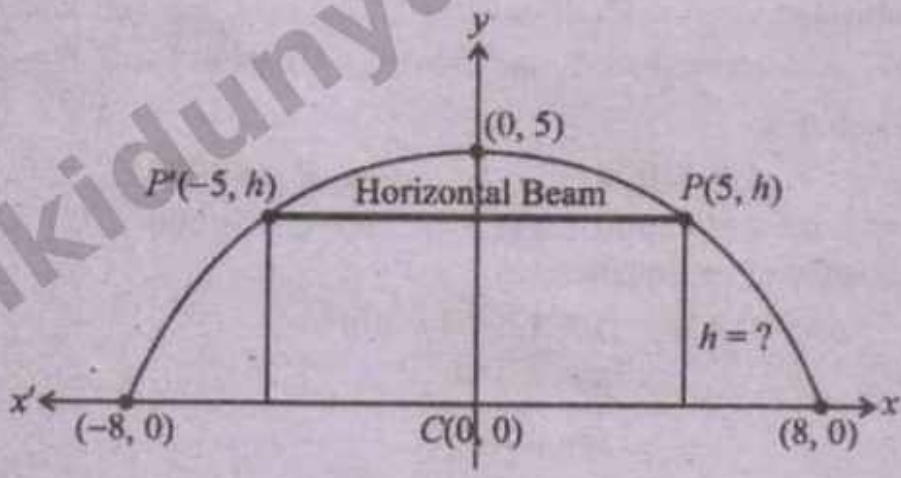
Semi-minor axis: $b = 5$

Equation of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{x^2}{8^2}+\frac{y^2}{5^2}=1$$

$$\frac{x^2}{64}+\frac{y^2}{25}=1$$



Let required height = h m

Since, the point $P(5, h)$ lies on ellipse, so

$$\frac{5^2}{64}+\frac{h^2}{25}=1$$

$$\frac{h^2}{25}=1-\frac{25}{64}$$

$$h^2 = 25\left(\frac{64-25}{64}\right)$$

$$h^2 = 25\left(\frac{39}{64}\right)$$

$$\sqrt{h^2} = \sqrt{\frac{25\times 39}{64}}$$

$$\boxed{h = \frac{5\sqrt{39}}{8}\text{ m}}$$

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Multiple Choice Questions (MCQs)

1. The fixed point of a circle is called its
   A) focus B) centre C) vertex D) directrix
2. The standard equation of a circle with centre (h, k) and radius r is:
   A) $(x-h)^2+(y-k)^2=r^2$ B) $(x+h)^2+(y+k)^2=r$ C) $x^2+y^2+2hx+2ky=0$ D) $x^2-y^2=r^2$
3. The equation $(x-h)^2+(y-k)^2=0$ represents:
   A) a real circle B) a point circle with centre (h, k)
   C) an imaginary circle D) a parabola
4. The equation $x^2+y^2=r^2$ represents a circle whose centre is:
   A) (r, r) B) $(0, 0)$ C) $(-r, 0)$ D) $(0, -r)$
5. The parametric equations of $x^2+y^2=r^2$ are:
   A) $x=r\cos\theta, y=r\sin\theta$ B) $x=r\sin\theta, y=r\cos\theta$ C) $x=r\theta, y=\frac{r}{\theta}$ D) $x=r\cos\theta, y=r\tan\theta$
6. The general form of the equation of a circle is:
   A) $x^2+y^2+2gx+2fy+c=0$ B) $x^2-y^2+2gx+2fy+c=0$
   C) $ax+by+c=0$ D) $x^2+y^2+gx+fy=1$
7. For $x^2+y^2+2gx+2fy+c=0$, the centre is:
   A) (g, f) B) $(-g, -f)$ C) $(2g, 2f)$ D) $(-2g, -2f)$
8. For $x^2+y^2+2gx+2fy+c=0$, the radius is:
   A) $\sqrt{g^2+f^2+c}$ B) $\sqrt{g^2+f^2-c}$ C) g^2+f^2-c D) $\sqrt{c-g^2-f^2}$
9. The centre of the circle $x^2+y^2-4x-12y+11=0$ is:
   A) $(-2, -6)$ B) $(2, 6)$ C) $(4, 12)$ D) $(-4, -12)$
10. The radius of $x^2+y^2-4x-12y+11=0$ is:
    A) $\sqrt{21}$ B) $\sqrt{29}$ C) 5 D) 7
11. For a circle with diameter endpoints (x_1, y_1) and (x_2, y_2), the equation can be written as
    A) $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$ B) $(x+x_1)(x+x_2)+(y+y_1)(y+y_2)=0$
    C) $x^2+y^2=x_1x_2+y_1y_2$ D) $(x-x_1)^2+(y-y_2)^2=0$
12. The angle subtended by a diameter at any point on the circle is:
    A) 0° B) 45° C) 90° D) 180°
13. The line perpendicular to a tangent at its point of contact is called the
    A) axis B) normal C) directrix D) chord
14. For the circle $x^2+y^2+2gx+2fy+c=0$, the tangent at (x_1, y_1) is:
    A) $xx_1-yy_1+g(x+x_1)+f(y+y_1)+c=0$ B) $xx_1-yy_1+g+f+c=0$
    C) $x+y=x_1+y_1$ D) $xx_1+yy_1=r$
15. For a circle $x^2+y^2=a^2$, the family of tangents with slope m is:
    A) $y=mx\pm a\sqrt{1+m^2}$ B) $y=mx\pm a$ C) $y=mx\pm a\sqrt{1-m^2}$ D) $y=mx\pm\sqrt{1+m^2}$
16. The condition for the line $y=mx+c$ to touch $x^2+y^2=a^2$ is:
    A) $c^2=a^2(1+m^2)$ B) $c^2=\frac{a^2}{1+m^2}$ C) $c=a(1+m^2)$ D) $c^2+a^2=m^2$
17. If a point lies outside a circle, the number of real tangent/s drawn from it is/are:
    A) zero B) one C) two D) infinitely many

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18. The directrix of $y^2 = 4ax$ is:
A) $x = a$ B) $x = -a$ C) $y = a$ D) $y = -a$

19. The length of latus rectum of $y^2 = 4ax$ is:
A) a B) $2a$ C) $4a$ D) $8a$

20. The parametric coordinates of a point on $y^2 = 4ax$ are
A) $(at^2, 2at)$ B) $(2at, at^2)$ C) $(at, 2at^2)$ D) $\left(\frac{a}{t}, 2at\right)$

21. The eccentricity of an ellipse is:
A) $e = \frac{a}{c}$ B) $e = \frac{c}{a}$ C) $e = \frac{b}{a}$ D) $e = \frac{a}{b}$

22. For a vertical ellipse $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$, the major axis is parallel to the:
A) x-axis B) y-axis C) line $y = x$ D) line $y = -x$

23. For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the transverse axis lies along the:
A) x-axis B) y-axis C) line $y = x$ D) directrix

24. The asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are
A) $y = \pm\frac{a}{b}x$ B) $y = \pm\frac{b}{a}x$ C) $y = \pm abx$ D) $x = \pm a$

25. The equation of the circle with centre (5, 7) and radius 2 is
A) $(x-5)^2 + (y-7)^2 = 4$ B) $(x+5)^2 + (y+7)^2 = 4$
C) $(x-5)^2 + (y-7)^2 = 2$ D) $x^2 + y^2 = 4$

26. The centre of the circle whose diameter endpoints are (3, 4) and (2, −7) is
A) $\left(\frac{5}{2}, \frac{-3}{2}\right)$ B) $\left(\frac{1}{2}, \frac{11}{2}\right)$ C) (3, −7) D) (2, 4)

27. For $x^2 - 6x - 12y + 9 = 0$, the vertex is:
A) (3, 0) B) (0, 3) C) (3, 1) D) (−3, 0)

28. An ellipse with foci $(0, \pm 5)$ and minor axis length 8 has $b =$:
A) 2 B) 4 C) 5 D) 8

29. The ellipse $4x^2 + 9y^2 = 36$ has semi-major axis:
A) 2 B) 3 C) 4 D) 6

30. For $a = 6$ and $e = \frac{2}{3}$, the focal distance c is:
A) 2 B) 4 C) 6 D) 9

ANSWER KEY

1.	B	2.	A	3.	B	4.	B	5.	A	6.	A	7.	B	8.	B	9.	B	10.	B
11.	A	12.	C	13.	B	14.	A	15.	A	16.	A	17.	C	18.	B	19.	C	20.	A
21.	B	22.	B	23.	A	24.	B	25.	A	26.	A	27.	A	28.	B	29.	B	30.	B

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