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UNIT 05

Analytical Geometry

INTRODUCTION



Figure 5.1

- Analytical geometry, also called coordinate geometry, is the branch of mathematics that combines algebra and geometry to study shapes, lines and figures using a coordinate system.
- Instead of just drawing or imagining figures, we describe them with equations and points on the xy-plane, which makes it easier to calculate distances, midpoints, slopes and intersections.
- This approach allows us to solve practical problems, like finding the shortest path, checking if two lines are parallel or locating the exact position of an object.
- Analytical geometry was first developed in the 17th century by the French mathematician René Descartes, who introduced the idea of using coordinates to represent geometric figures. His work laid the foundation for modern mathematics, as analytical geometry became the link between algebra and geometry, making it possible to study space and shapes with greater precision.
- In this unit, we will study about straight lines, angles between coplanar intersecting lines, concurrency of medians, right bisectors and altitudes, area of triangular region and applications of analytical geometry to real life problems.

Content Outline

- Equation of Straight Lines
- A Linear Equation in Two Variables Represents a Straight Line
- Equations of Medians, Altitudes and Right Bisectors
- Condition of Concurrency of Three Straight Lines
- Equation of Lines Through the Point of Intersection of Two Lines

EQUATION OF STRAIGHT LINES

We have learnt that in coordinate geometry, a straight line can be represented in different forms depending on the information given.

General Form

The general form of a straight line in two variables is written as $ax + by + c = 0$, where a, b and c are real numbers and a and b are not simultaneously zero.

Slope-Intercept Form

If m is the slope and c is the y-intercept of a line, then equation of this straight line in slope-intercept form is $y = mx + c$.

Point-Slope Form

If a line passes through a point (x_1, y_1) and has slope m, then equation of straight line in point slope form is $y - y_1 = m(x - x_1)$.

Two-Point Form

If an Inclined line passes through two points (x_1, y_1) and (x_2, y_2), then equation of straight line in two-point form is

$$\frac{y-y_2}{y_2-y_1} = \frac{x-x_2}{x_2-x_1} \text{ or } \frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2} \text{ or } y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

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Intercept Form

If a and b are the x-intercept and y-intercept of the line, then equation of straight line in intercept form is $\frac{x}{a}+\frac{y}{b}=1$.

$a \neq 0,\ b \neq 0,$

Normal Form

$x\cos\theta + y\sin\theta = p$ is an equation of the straight line, where p is the perpendicular distance of the line from the origin and θ is the angle made by perpendicular with the positive x-axis, measured counter clockwise.

Example 1: Find an equation of the straight line if its slope is 3 and y-intercept is 7.

Solution:

Slope: $m = 3$

y-intercept: $c = 7$

Slope-intercept form of line:

$$y = mx + c$$

$$\boxed{y = 3x + 7}$$

Example 2: Write down an equation of the straight line passing through $(-4, 3)$ and having slope -5.

Solution:

Point on the line $= (-4, 3)$

Slope of line $m = -5$

Point-slope form of line:

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -5(x - (-4))$$

$$y - 3 = -5x - 20$$

$$\boxed{5x + y + 17 = 0}$$

Example 3: Find an equation of the straight line passing through the points $(-3, -2)$ and $(5, 4)$.

Solution:

Point on the line $= (-3, -2), (5, 4)$

Two-point form of line:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - (-2) = \frac{4 - (-2)}{5 - (-3)}(x - (-3))$$

$$y + 2 = \frac{6}{8}(x + 3)$$

$$y + 2 = \frac{3}{4}(x + 3)$$

$$4(y + 2) = 3(x + 3)$$

$$4y + 8 = 3x + 9$$

$$3x - 4y + 9 - 8 = 0$$

$$\boxed{3x - 4y + 1 = 0}$$

Example 4: Write down an equation of the straight line which cuts the x-axis at $(3, 0)$ and y-axis at $(0, -6)$.

Solution:

x-intercept: $a = 3$

y-intercept: $b = -6$

Two-intercepts form of line:

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{3} + \frac{y}{-6} = 1$$

Multiply both sides by 6

$$2x - y = 6$$

$$\boxed{2x - y - 6 = 0}$$

Example 5: The length of perpendicular from the origin to a line is 3 units and the inclination of this perpendicular is $120°$. Find the equation of the straight line.

Solution:

Here $p = 3, \theta = 120°$

Normal form of line:

$$x\cos\theta + y\sin\theta = p$$

$$x\cos 120° + y\sin 120° = 3$$

$$-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 3$$

Multiply both sides by -2

$$x - \sqrt{3}y = -6$$

$$\boxed{x - \sqrt{3}y + 6 = 0}$$

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A LINEAR EQUATION IN TWO VARIABLES REPRESENTS A STRAIGHT LINE

Theorem 1: The linear equation $ax+by+c=0$ in two variables x and y represents a straight line, where a, b and c are real numbers and a and b are not simultaneously zero.

Proof:

$$ax+by+c=0$$

Case I: When $a=0$

$$0+by+c=0$$

$$by=-c$$

$$y=-\frac{c}{b} \text{ (Horizontal line)}$$

Case-II: When $b=0$

$$ax+0+c=0$$

$$ax=-c$$

$$x=-\frac{c}{a} \text{ (Vertical line)}$$

Case-III: When $a\neq 0, b\neq 0$

$$ax+by+c=0$$

$$by=-ax-c$$

Divide both sides by 'b'.

$$\frac{by}{b}=-\frac{a}{b}x-\frac{c}{b}$$

$$y=-\frac{a}{b}x+\left(-\frac{c}{b}\right)$$

$$y=mx+c' \qquad \text{(Slope-intercept form)}$$

Where $\text{Slope}=\frac{-a}{b}$ and $y\text{-intercept}=\frac{-c}{b}$

Thus, equation $ax+by+c=0$ always represents a straight line.

Transform the General Linear Equation to Standard Forms

Theorem 2: To transform the equation $ax+by+c=0$ in the standard forms.

Proof:

$$ax+by+c=0$$

(i) Slope-intercept form: $(y=mx+c)$

$$ax+by+c=0$$

$$by=-ax-c$$

Divide both sides by b (where $b\neq 0$)

$$y=-\frac{a}{b}x-\frac{c}{b}$$

Compare it with

$$y=mx+c'$$

$$\Rightarrow \quad m=\text{slope}=\frac{-a}{b},\ c'=y\text{-intercept}=\frac{-c}{b}$$

(ii) Point-Slope Form: $(y-y_1=m(x-x_1))$

$$ax+by+c=0$$

Put $y=0 \Rightarrow ax+0+c=0 \Rightarrow x=-\frac{c}{a}$

$$\text{Point on the line}=\left(-\frac{c}{a},0\right)$$

$$\text{Slope of line}=m=-\frac{a}{b}$$

Point-slope form of line:

$$y-y_1=m(x-x_1).$$

$$\boxed{y-0=-\frac{a}{b}\left(x+\frac{c}{a}\right)}$$

(iii) Symmetric Form:

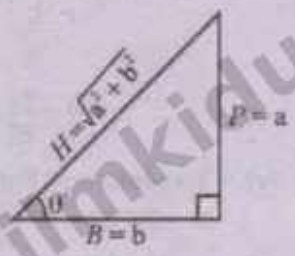
$$ax+by+c=0 \quad \ldots(1)$$

$$\text{Slope of line}=m=-\frac{a}{b}$$

$$\Rightarrow \quad \tan\theta=-\frac{a}{b}$$

$$\sin\theta=\frac{a}{\pm\sqrt{a^2+b^2}},$$

$$\cos\theta=\frac{b}{\pm\sqrt{a^2+b^2}}$$



Pythagoras theorem

$$H^2=B^2+P^2$$

$$=a^2+b^2$$

$$H=\sqrt{a^2+b^2}$$

Put $y=0$ in eq. (i), we get

$$ax+c=0 \quad \Rightarrow \quad x=-\frac{c}{a}$$

$$\text{Point on the line}=\left(-\frac{c}{a},0\right)$$

Symmetric form of line:

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \text{ (say)}$$

$$\frac{x-\left(-\frac{c}{a}\right)}{\frac{b}{\pm\sqrt{a^2+b^2}}}=\frac{y-0}{\frac{a}{\pm\sqrt{a^2+b^2}}}=r$$

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(iv) Two-point Form:

$ax + by + c = 0$

Put $y = 0$

$ax + c = 0 \Rightarrow x = -\frac{c}{a}$

$\left(-\frac{c}{a}, 0\right)$

Put $x = 0$

$by + c = 0 \Rightarrow y = -\frac{c}{b}$

$\left(0, -\frac{c}{a}\right)$

Points on the line $= \left(-\frac{c}{a}, 0\right), \left(0, -\frac{c}{b}\right)$

Two-points form of line:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{-\frac{c}{b} - 0}{0 + \frac{c}{a}}\left(x + \frac{c}{a}\right)$$

$$y = \frac{-\frac{c}{b}}{\frac{c}{a}}\left(x + \frac{c}{a}\right)$$

$$y = -\frac{a}{b}\left(x + \frac{c}{a}\right)$$

(v) Two-intercept Form:

$ax + by + c = 0$

$ax + by = -c$

Divide both sides by $-c$

$$\frac{ax}{-c} + \frac{by}{-c} = \frac{-c}{-c}$$

$$\frac{x}{\left(-\frac{c}{a}\right)} + \frac{y}{\left(-\frac{c}{b}\right)} = 1 \text{ (Two-intercept form)}$$

where x-intercept $= -\frac{c}{a}$, y-intercept $= -\frac{c}{b}$

(vi) Normal Form:

The equation $ax + by + c = 0$ ...(i)

can be written in the normal form as:

$$\frac{ax + by}{\pm\sqrt{a^2 + b^2}} = \frac{-c}{\pm\sqrt{a^2 + b^2}} \quad \text{...(ii)}$$

The sign of the radical to be such that the right hand side of (ii) is positive.

Proof:

$ax + by + c = 0$

$\Rightarrow \quad ax + by = -c$ ...(1)

Normal form of line:

$x\cos\theta + y\sin\theta = p$ ...(2)

If (1) and (2) are identical, so their coefficients are proportional

$$\frac{a}{\cos\theta} = \frac{b}{\sin\theta} = -\frac{c}{p} = k \text{(say)}$$

$\Rightarrow \frac{a}{\cos\theta} = k$ | $\frac{b}{\sin\theta} = k$ | $-\frac{c}{p} = k$

$\cos\theta = \frac{a}{k}$ | $\sin\theta = \frac{b}{k}$ | $p = -\frac{c}{k}$

As we know

$\cos^2\theta + \sin^2\theta = 1$

$$\frac{a^2}{k^2} + \frac{b^2}{k^2} = 1 \Rightarrow \frac{a^2 + b^2}{k^2} = 1 \Rightarrow k^2 = a^2 + b^2$$

$$k = \pm\sqrt{a^2 + b^2}$$

$$\Rightarrow \cos\theta = \frac{a}{\pm\sqrt{a^2 + b^2}}, \sin\theta = \frac{b}{\pm\sqrt{a^2 + b^2}}, p = \frac{-c}{\pm\sqrt{a^2 + b^2}}$$

Putting values of $\cos\theta$, $\sin\theta$ and p in eq. (2), we get

$$x\left(\frac{a}{\pm\sqrt{a^2 + b^2}}\right) + y\left(\frac{b}{\pm\sqrt{a^2 + b^2}}\right) = \frac{-c}{\pm\sqrt{a^2 + b^2}}$$

$$\frac{ax + by}{\pm\sqrt{a^2 + b^2}} = \frac{-c}{\pm\sqrt{a^2 + b^2}} \text{ (Proved)}$$

Example 6: Transform the equation $3x - 10y + 13 = 0$ into

(i) Slope-intercept form (ii) Two-intercept form (iii) Normal form

(iv) Point-slope form (v) Two-point form (vi) Symmetric form

Solution:

(i) Slope-intercept form ($y = mx + c$)

$3x - 10y + 13 = 0$

$-10y = -3x - 13$

Dividing both sides by -10

$$\frac{-10y}{-10} = \frac{-3x}{-10} - \frac{13}{-10}$$

$$y = \frac{3}{10}x + \frac{13}{10}$$

where, $m = \frac{3}{10}, c = \frac{3}{10}$

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(ii) Two-intercept form $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

$$3x-10y+13=0$$
$$3x-10y=-13$$

Dividing both sides by -13

$$\frac{3x}{-13}-\frac{10y}{-13}=\frac{-13}{-13}$$
$$-\frac{3}{13}x+\frac{10}{13}y=1$$
$$\frac{x}{-\frac{13}{3}}+\frac{y}{\frac{13}{10}}=1$$

where, $a=-\frac{13}{3}, b=\frac{13}{10}$

(iii) Normal form $(x\cos\theta+y\sin\theta=p)$

$$3x-10y+13=0$$
$$3x-10y=-13$$

Dividing both sides by $-\sqrt{a^2+b^2}=-\sqrt{3^2+(-10)^2}$

$$=-\sqrt{9+100}$$
$$=-\sqrt{109}$$

$$\frac{3x}{-\sqrt{109}}-\frac{10y}{-\sqrt{109}}=\frac{-13}{-\sqrt{109}}$$
$$-\frac{3}{\sqrt{109}}x+\frac{10}{\sqrt{109}}y=\frac{13}{\sqrt{109}}$$

where, $\cos\theta=-\frac{3}{\sqrt{109}}, \sin\theta=\frac{10}{\sqrt{109}}, p=\frac{13}{\sqrt{109}}$

(iv) Point-slope Form:

$$3x-10y+13=0$$

Put $y=0 \Rightarrow 3x=-13 \Rightarrow x=-\frac{13}{3}$

Point on the line $=\left(-\frac{13}{3},0\right)$

Slope of line $=m=\frac{-3}{-10}=\frac{3}{10}$

Point-slope form of line:

$$y-y_1=m(x-x_1)$$
$$y-0=\frac{3}{10}\left(x+\frac{13}{3}\right)$$

(v) Two-point form:

$$3x-10y+13=0$$

Put $y=0$	Put $x=0$
$3x-0+13=0$	$0-10y+13=0$
$x=-\frac{13}{3}$	$y=\frac{13}{10}$

Points on the line $=\left(-\frac{13}{3},0\right),\left(0,\frac{13}{10}\right)$

Two-point tom of line:

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$
$$y-0=\frac{\frac{13}{10}-0}{0+\frac{13}{3}}\left(x+\frac{13}{3}\right)$$

(vi) Symmetric Form:

$$3x-10y+13=0$$

Put $y=0 \Rightarrow 3x+13=0 \Rightarrow x=-\frac{13}{3}$

Point on the line $=\left(-\frac{13}{3},0\right)$

As $m=\frac{-3}{-10}=\frac{3}{10}$

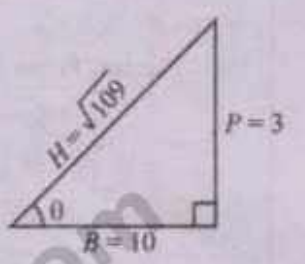
$\Rightarrow \tan\theta=\frac{3}{10}>0$ (θ lies in I-quadrant)

Pythagoras theorem

$$H^2=B^2+P^2$$
$$=100+9$$
$$H=\sqrt{109}$$

$\Rightarrow \sin\theta=\frac{3}{\sqrt{109}}$,

$\cos\theta=\frac{10}{\sqrt{109}}$



Symmetric form of a line:

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$
$$\frac{x+\frac{13}{3}}{\frac{10}{\sqrt{109}}}=\frac{y-0}{\frac{3}{\sqrt{109}}}=r$$

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Challenge!

Transform the equation $2x+5y-10=0$ into slope-intercept form, two-intercept form, normal form, point-slope form, two-point form and symmetric form.

Solution:

(i) Slope-intercept form ($y = mx + c$)

$$2x + 5y - 10 = 0$$

$$5y = -2x + 10$$

Dividing both sides by 5

$$\frac{5y}{5} = -\frac{2x}{5} + \frac{10}{5}$$

$$y = -\frac{2}{5}x + 2$$

where, $m = -\frac{2}{5}$, $c = 2$

(ii) Two-intercept form $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

$$2x + 5y - 10 = 0$$

$$2x + 5y = 10$$

Dividing both sides 10

$$\frac{2x}{10} + \frac{5y}{10} = \frac{10}{10}$$

$$\frac{1}{5}x + \frac{1}{2}y = 1$$

$$\frac{x}{5} + \frac{y}{2} = 1$$

where, $a = 5, b = 2$

(iii) Normal Form ($x\cos\theta + y\sin\theta = p$)

$$2x + 5y - 10 = 0$$

$$2x + 5y = 10$$

Dividing both sides by $\sqrt{a^2+b^2} = \sqrt{2^2+5^2} = \sqrt{29}$

$$\frac{2}{\sqrt{29}}x + \frac{5}{\sqrt{29}}y = \frac{10}{\sqrt{29}}$$

where, $\cos\theta = \frac{2}{\sqrt{29}}, \sin = \frac{5}{\sqrt{29}}, p = \frac{10}{\sqrt{29}}$

(iv) Point-slope Form:

$$2x + 5y - 10 = 0$$

Put $y = 0 \Rightarrow 2x = 10 \Rightarrow x = 5$

Point on the line = (5, 0)

$$\text{Slope of line} = m = -\frac{2}{5}$$

Point-slope form of line:

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{2}{5}(x - 5)$$

(v) Two-point form:

$$2x + 5y - 10 = 0$$

Put $y = 0$	Put $x = 0$
$2x + 0 - 10 = 0$	$0 + 5y - 10 = 0$
$x = 5$	$y = 2$

Points on the line = (5, 0), (0, 2)

Two-point form of line:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{2-0}{0-5}(x - 5)$$

(vi) Symmetric Form:

$$2x + 5y - 10 = 0$$

Put $y = 0 \Rightarrow 2x = 10 \Rightarrow x = 5$

Point on the line = (5, 0)

As $m = -\frac{2}{5}$

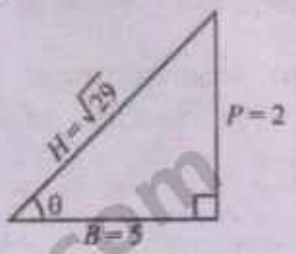
$\Rightarrow \tan\theta = -\frac{2}{5} < 0$ (θ lies in 2nd quadrant)

Pythagoras theorem

$$H^2 = B^2 + P^2$$

$$H^2 = 25 + 4$$

$$H = \sqrt{29}$$



$\Rightarrow \sin\theta = \frac{2}{\sqrt{29}}$,

$\cos\theta = -\frac{5}{\sqrt{29}}$

Symmetric form of a line:

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

$$\frac{x-5}{\frac{5}{\sqrt{29}}} = \frac{y-0}{\frac{2}{\sqrt{29}}} = r$$

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EQUATIONS OF MEDIANS, ALTITUDES AND RIGHT BISECTORS

Example 7: Triangle ABC has vertices $A(2,3)$, $B(4,5)$ and $C(-2,7)$. Find the equations of medians of the triangle ABC.

Solution:

Given that: $A(2,3)$, $B(4,5)$, $C(-2,7)$

Let D, E and F be the midpoints of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively

$$D\left(\frac{2+4}{2}, \frac{3+5}{2}\right) = D(3,4)$$

$$E\left(\frac{4+(-2)}{2}, \frac{5+7}{2}\right) = E(1,6)$$

$$F\left(\frac{2+(-2)}{2}, \frac{3+7}{2}\right) = F(0,5)$$

Remember!

A median joins the vertex of a triangle to the midpoint of the opposite side.

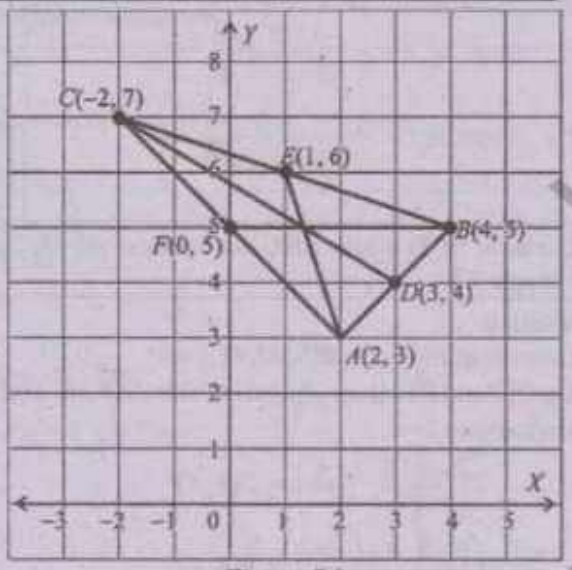

Figure 5.2

Using formula: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Equation of median $\overline{AE}$ is:

$$y - 3 = \frac{6-3}{1-2}(x-2)$$

$$y - 3 = -3(x-2)$$

$$x - 3 = -3x + 6$$

$$\boxed{3x + y - 9 = 0}$$

Equation of median $\overline{BF}$ is:

$$y - 5 = \frac{5-5}{0-4}(x-4)$$

$$y - 5 = \frac{0}{-4}(x-4)$$

$$\boxed{y - 5 = 0}$$

Equation of median $\overline{CD}$ is:

$$y - 7 = \frac{4-7}{3+2}(x+2)$$

$$y - 7 = \frac{3}{5}(x+2)$$

$$5y - 35 = -3x - 6$$

$$3x + 5y - 35 + 6 = 0$$

$$\boxed{3x + 5y - 29 = 0}$$

Example 8: $A(1,2)$, $B(5,-1)$ and $C(-2,-4)$ are the vertices of ΔABC. Find the equations of altitudes of triangle ABC.

Solution:

Given that: $A(1,2)$, $B(5,-1)$, $C(-2,-4)$

Let $\overline{AD} \perp \overline{BC}$, $\overline{BE} \perp \overline{AC}$ and $\overline{CF} \perp \overline{AB}$ be the altitudes of triangle ABC.

$$\text{Slope of } \overline{BC} = \frac{-4-(-1)}{-2-5} = \frac{-3}{-7} = \frac{3}{7}$$

$$\text{Slope of } \overline{AD} = \frac{-1}{\text{Slope of } \overline{BC}} = -\frac{7}{3}$$

Equation of altitude $\overline{AD}$ is:

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{7}{3}(x-1)$$

$$3y - 6 = -7x + 7$$

$$\boxed{7x + 3y - 13 = 0}$$

$$\text{Slope of } \overline{AC} = \frac{-4-2}{-2-1} = \frac{-6}{-3} = 2$$

$$\text{Slope of } \overline{BE} = \frac{-1}{\text{Slope of } \overline{AC}} = -\frac{1}{2}$$

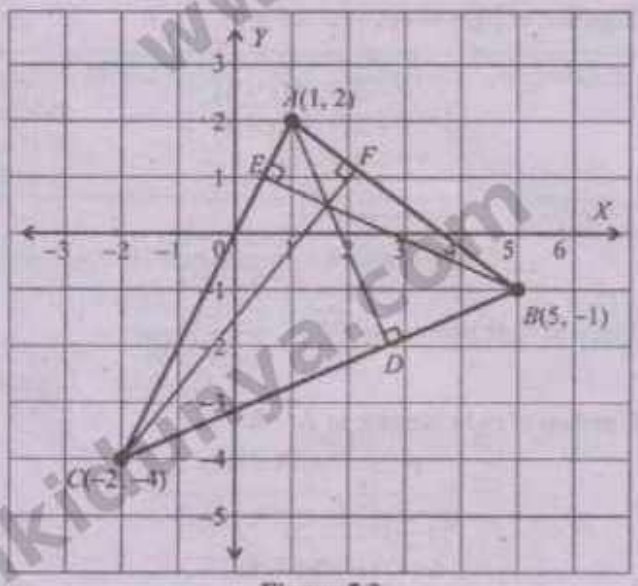

Figure 5.3

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Equation of altitude $\overline{BE}$ is:

$$y - y_1 = m(x - x_1)$$

$$y + 1 = -\frac{1}{2}(x - 5)$$

$$2y + 2 = -x + 5$$

$$\boxed{x + 2y - 3 = 0}$$

Slope of $\overline{AB} = \frac{-1-2}{5-1} = \frac{-3}{4}$

Slope of $\overline{CF} = \frac{-1}{\text{Slope of } \overline{AB}} = \frac{4}{3}$

Equation of altitude CF is:

$$y - y_1 = m(x - x_1)$$

$$y + 4 = \frac{4}{3}(x + 2)$$

$$3y + 12 = 4x + 8$$

$$\boxed{4x - 3y - 4 = 0}$$

Remember!

An altitude is the line drawn from a vertex perpendicular to the opposite side.

Remember!

If m_1 and m_2 be the slopes of perpendicular lines then $m_1 m_2 = -1$.

Example 9: Triangle ABC has vertices $A(-3,2), B(5,4)$ and $C(3,-8)$. Find the equations of right bisectors of triangle ABC.

Solution:

Given that: $A(-3, 2)$, $B(5, 4)$, $C(3, -8)$

Let D, E and F be the midpoints of sides $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively.

$$D\left(\frac{-3+5}{2}, \frac{2+4}{2}\right) = D(1, 3)$$

$$E\left(\frac{5+3}{2}, \frac{4+(-8)}{2}\right) = E(4, -2)$$

$$F\left(\frac{-3+3}{2}, \frac{2+(-8)}{2}\right) = F(0, -3)$$

Slope of $\overline{AB} = \frac{4-2}{5-(-3)} = \frac{2}{8} = \frac{1}{4}$

Slope of right bisector to $\overline{AB} = \frac{-1}{\text{Slope of } AB} = -4$

Equation of right bisector to $\overline{AB}$ is:

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -4(x - 1)$$

$$y - 3 = -4(x - 1)$$

$$y - 3 = -4x + 4$$

$$\boxed{4x + y - 7 = 0}$$

Slope of $\overline{BC} = \frac{-8-4}{3-5} = \frac{-12}{-2} = 6$

Slope of right bisector to $\overline{BC} = \frac{-1}{\text{Slope of } \overline{BC}} = -\frac{1}{6}$

Equation of right bisector to $\overline{BC}$ is:

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = -\frac{1}{6}(x - 4)$$

$$6y + 12 = -x + 4$$

$$\boxed{x + 6y + 8 = 0}$$

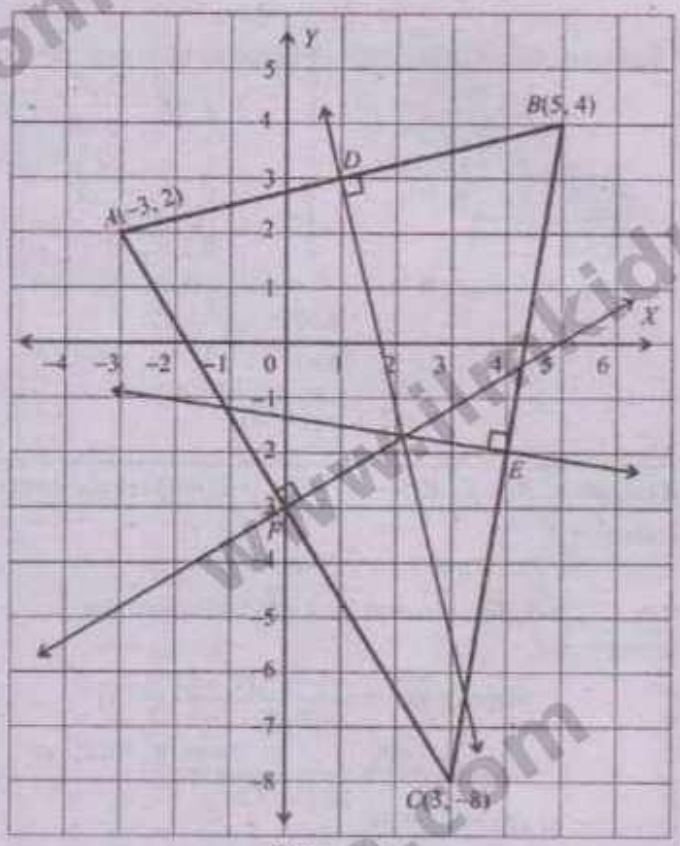

Figure 5.4

Slope of $\overline{AC} = \frac{-8-2}{3+3}$

$$= \frac{10}{6} = \frac{5}{3}$$

Slope of right bisector to $\overline{AC} = \frac{1}{\text{Slope of } \overline{AC}} = \frac{3}{5}$

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Equation of fright bisector to $\overline{AC}$ is:

$$y - y_1 = m(x - x_1)$$

$$y - (-3) = \frac{3}{5}(x - 0)$$

$$5y + 15 = 3x$$

$$\boxed{3x - 5y - 15 = 0}$$

Remember:
A perpendicular or right bisector of a triangle is a line that is perpendicular to one side of the triangle and passes through the mid point of that side.

CONDITION OF CONCURRENCY OF THREE STRAIGHT LINES

Theorem 3: Three non-parallel lines $\ell_1: a_1x + b_1y + c_1 = 0$, $\ell_2: a_2x + b_2y + c_2 = 0$, $\ell_3: a_3x + b_3y + c_3 = 0$ **are concurrent iff** $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Proof:

If the lines are concurrent, then they have a common point of intersection

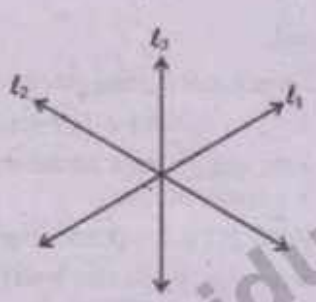
Let $P(x, y)$ be the point of intersection of ℓ_1 and ℓ_2

By cross-multiplication method

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1} \quad \text{As } \ell_1 \not\parallel \ell_2 \Rightarrow a_1b_2 - a_2b_1 \neq 0$$

$$\Rightarrow \quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \; y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \quad P(x, y) = P\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)$$



Since $P(x, y)$ lies on ℓ_3, so

$$a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

Multiply both sides by $(a_1b_2 - a_2b_1)$

$$a_3(b_1c_2 - b_2c_1) + b_3(a_2c_1 - a_1c_2) + c_3(a_1b_2 - a_2b_1) = 0$$

which is equivalent to

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Example 10: Check whether the following lines are concurrent or not. If concurrent, then find the point of concurrency:

$$3x - 2y - 2 = 0$$
$$x + y - 4 = 0$$
$$2x - y - 2 = 0$$

Solution:

Let $\ell_1: 3x - 2y - 2 = 0$, $\ell_2: x + y - 4 = 0$, $\ell_3: 2x - y - 2 = 0$

Consider:

$$\begin{vmatrix} 3 & -2 & -2 \\ 1 & 1 & -4 \\ 2 & -1 & -2 \end{vmatrix} = 3\begin{vmatrix} 1 & -4 \\ -1 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -4 \\ 2 & -2 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= 3(-2 - 4) + 2(-2 + 8) - 2(-1 - 2)$$
$$= 3(-6) + 2(6) - 2(-3)$$
$$= -18 + 12 + 6 = 0$$

So, the given lines are concurrent.

For the point of concurrency, we solve ℓ_1 and ℓ_2

$$3x - 2y - 2 = 0 \quad \ldots(1)$$
$$x + y - 4 = 0 \quad \ldots(2)$$

Eq. (1) + 2 × Eq. (2)

$$3x - 2y - 2 = 0$$
$$2x + 2y - 8 = 0$$

$$5x - 10 = 0 \Rightarrow 5x = 10 \Rightarrow x = 2$$

Eq. (2) $\Rightarrow 2 + y - 4 = 0 \Rightarrow y = 2$

Point of concurrency = (2, 2)

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Challenge!

If the lines $2x + 3y + 5 = 0$, $3x + 4y - 1 = 0$ and $kx - 7y + 6 = 0$ are concurrent, then find the value of k.

Solution:

For concurrent lines:

$$\begin{vmatrix} 2 & 3 & 5 \\ 3 & 4 & -1 \\ k & -7 & 6 \end{vmatrix} = 0$$

$$2\begin{vmatrix} 4 & -1 \\ -7 & 6 \end{vmatrix} - \begin{vmatrix} 3 & -1 \\ k & 6 \end{vmatrix} + 5\begin{vmatrix} 3 & 4 \\ k & -7 \end{vmatrix} = 0$$

$$2(24-7) - 3(18+k) + 5(-21-4k) = 0$$
$$48 - 14 - 54 - 3k - 105 - 20k = 0$$
$$-125 - 23k = 0$$
$$23k = -125$$
$$k = -\frac{125}{23}$$

EQUATION OF LINES THROUGH THE POINT OF INTERSECTION OF TWO LINES

We can find a family of lines through the point of intersection of two non-parallel lines ℓ_1 and ℓ_2.

Let $\ell_1 : a_1x + b_1y + c_1 = 0$ ... (i)

and $\ell_2 : a_2x + b_2y + c_2 = 0$ ... (ii)

For any real k, consider the equation

$$a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0 \quad \text{... (iii)}$$

This, being a linear equation, represents a straight line. For different values of k, (iii) represents different lines. Thus (iii) is a family of lines.

If (x_1, y_1) is any point lying on both (i) and (ii), then it is their point of intersection.

Since, (x_1, y_1) lies on both (i) and (ii), we have

$$a_1x_1 + b_1y_1 + c_1 = 0 \text{ and } a_2x_1 + b_2y_1 + c_2 = 0$$

From the above two equations, we note that (x_1, y_1) also lies on (iii).

Thus, (iii) is the required family of lines through the point of intersection of (i) and (ii). Since, k can assume an infinite number of values, (iii) represents an infinite number of lines.

A particular line of the family (iii) can be determined if one more condition is given.

Example 11: Find the family of lines through the point of intersection of the lines

$$2x - 3y - 14 = 0 \text{ and } 2x + y - 10 = 0$$

Find the member of the family which is:

(a) parallel to the line with slope $-\frac{1}{2}$ **(b) perpendicular to the line $2x - 3y + 2 = 0$**

Solution:

Let $\ell_1: 2x - 3y - 14 = 0$

$\ell_2: 2x + y - 10 = 0$

Family of lines through the point of intersection of ℓ_1 and ℓ_2 is:

$$\ell: \ell_1 + k\ell_2 = 0$$
$$\ell: (2x - 3y - 14) + k(2x + y - 10) = 0 \quad \text{...(1)}$$
$$2x - 3y - 14 + 2kx + ky - 10k = 0$$
$$(2 + 2k)x + (-3 + k)y + (-14 - 10k) = 0$$

$$\text{Slope of } \ell = m = \frac{2+2k}{-3+k}$$

(a) Since ℓ is the line parallel to line with slope $-\frac{1}{2}$, so

$$\frac{2+2k}{-3+k} = -\frac{1}{2}$$
$$2(2 + 2k) = 1(-3 + k)$$
$$4 + 4k = -3 + k$$
$$4k - k = -3 - 4 \Rightarrow 3k = -7 \Rightarrow k = -\frac{7}{3}$$

Put $k = -\frac{7}{3}$ in eq. (1), we get

$$(2x - 3y - 14) - \frac{7}{3}(2x + y - 10) = 0$$

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Multiply both sides by 3

$$6x - 9y - 42 - 14x - 7y + 70 = 0$$
$$-8x - 16y + 28 = 0$$
$$-4(2x + 4y - 7) = 0$$
$$\boxed{2x + 4y - 7 = 0}$$

Required member of the family.

(b) Let ℓ_3: $2x - 3y - 14 = 0$

$$\text{Slope of } \ell_3 = m' = \frac{-2}{-3} = \frac{2}{3}$$

As $\ell \perp \ell_3$, so $m \times m' = -1$

$$\Rightarrow \quad \left(\frac{2+2k}{-3+k}\right)\left(\frac{2}{3}\right) = -1$$

$$-(4 + 4k) = -1(-9 + 3k)$$
$$4 + 4k = -9 - 4 \Rightarrow k = -13$$

Put $k = -13$ in eq. (1), we get

$$(2x - 3y - 14) - 13(2x + y - 10) = 0$$
$$2x - 3y - 14 - 26x - 13y + 130 = 0$$
$$-24x - 16y + 116 = 0$$
$$-4(6x + 4y - 29) = 0$$
$$\boxed{6x + 4y - 29 = 0}$$

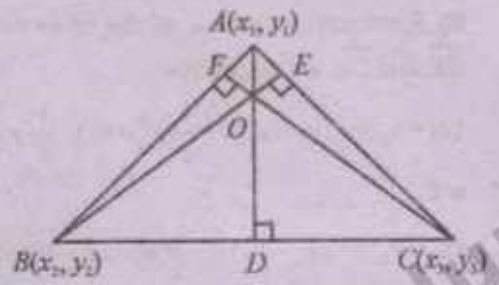
Required member of the family.

Theorem 4: Altitudes of a triangle are concurrent.

Proof:

Consider a triangle ABC with vertices $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$.

Let $\overline{AD} \perp \overline{BC}, \overline{BE} \perp \overline{AC}$ and $\overline{CF} \perp \overline{AB}$ be the altitudes of triangle ABC.



$$\text{Slope of altitude } \overline{BC} = \frac{y_3 - y_2}{x_3 - x_2}$$

$$\text{Slope of altitude } \overline{AD} = -\frac{x_3 - x_2}{y_3 - y_2} \qquad \because \overline{AD} \perp \overline{BC}$$

Equation of altitude $\overline{AD}$ is:

$$y - y_1 = m(x - x_1)$$
$$y - y_1 = -\frac{x_3 - x_2}{y_3 - y_2}(x - x_1)$$
$$(y_3 - y_2)(y - y_1) = -(x_3 - x_2)(x - x_1)$$
$$(y_3 - y_2)y - (y_3 - y_2)y_1 = -(x_3 - x_2)x + (x_3 - x_2)x_1$$
$$(x_3 - x_2)x + (y_3 - y_2)y - (x_3 - x_2)x_1 - (y_3 - y_2)y_1 = 0 \quad \ldots(1)$$

By symmetry equations of the altitudes $\overline{BE}$ and $\overline{CF}$ are respectively.

$$(x_1 - x_3)x + (y_1 - y_3)y - (x_1 - x_3)x_2 - (y_1 - y_3)y_1 = 0 \quad \ldots(2)$$

and

$$(x_2 - x_1)x + (y_2 - y_1)y - (x_2 - x_1)x_3 - (y_2 - y_1)y_3 = 0 \quad \ldots(3)$$

The three lines (1), (2) and (3) are concurrent iff

$$\begin{vmatrix} x_3 - x_2 & y_3 - y_2 & -(x_3 - x_2)x_1 - (y_3 - y_2)y_1 \\ x_1 - x_3 & y_1 - y_3 & -(x_1 - x_3)x_2 - (y_1 - y_3)y_2 \\ x_2 - x_1 & y_2 - y_1 & -(x_2 - x_1)x_3 - (y_2 - y_1)y_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x_3 - x_2 & y_3 - y_2 & -x_1x_3 + x_2x_3 - y_1y_3 + y_1y_2 \\ x_1 - x_3 & y_1 - y_3 & -x_1x_2 + x_2x_3 - y_1y_2 + y_2y_3 \\ x_2 - x_1 & y_2 - y_1 & -x_2x_3 + x_1x_3 - y_2y_3 + y_1y_3 \end{vmatrix} = 0$$

By $R_1 + (R_2 + R_3)$

$$\begin{vmatrix} 0 & 0 & 0 \\ x_1 - x_3 & y_1 - y_3 & -x_1x_2 + x_2x_3 - y_1y_2 + y_2y_3 \\ x_2 - x_1 & y_2 - y_1 & -x_2x_3 + x_1x_3 - y_2y_3 + y_1y_3 \end{vmatrix} = 0$$

$$\Rightarrow \qquad 0 = 0 \qquad \because R_1 \text{ is zero}$$

Thus, the altitudes of the triangle are concurrent.

Theorem 5: Right bisectors of a triangle are concurrent.

Proof:

Consider a triangle ABC with vertices $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$. Let D, E and F be the midpoints of sides $\overline{BC}, \overline{AC}$ and $\overline{AB}$ respectively, such that

$$D = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$

$$\text{Slope of side } \overline{BC} = \frac{y_3 - y_2}{x_3 - x_2}$$

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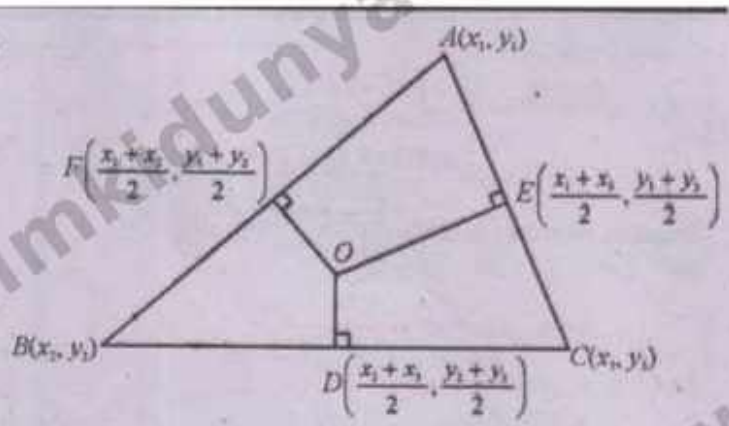
Slope of right bisector $\overline{OD} = -\frac{x_3 - x_2}{y_3 - y_2}$ $\because \overline{OD} \perp \overline{BC}$



Equation of right bisector $\overline{OD}$ is:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{y_2 + y_3}{2} = -\frac{x_3 - x_2}{y_3 - y_2}\left(x - \frac{x_2 + x_3}{2}\right)$$

$$(y_3 - y_2)\left(y - \frac{y_2 + y_3}{2}\right) = -(x_3 - x_2)\left(x - \frac{x_2 + x_3}{2}\right)$$

$$(y_3 - y_2)y - \frac{y_3^2 - y_2^2}{2} = -(x_3 - x_2)x + \frac{x_3^2 - x_2^2}{2}$$

$$(x_3 - x_2)x + (y_3 - y_2)y - \frac{1}{2}(x_3^2 - x_2^2) - \frac{1}{2}(y_3^2 - y_2^2) = 0 \quad \ldots(1)$$

By symmetry, equations of the other two right bisectors $\overline{OE}$ and $\overline{OF}$ are respectively.

$$(x_1 - x_3)x + (y_1 - y_3)y - \frac{1}{2}(x_1^2 - x_3^2) - \frac{1}{2}(y_1^2 - y_3^2) = 0 \quad \ldots(2)$$

and

$$(x_2 - x_1)x + (y_2 - y_1)y - \frac{1}{2}(x_2^2 - x_1^2) - \frac{1}{2}(y_2^2 - y_1^2) = 0 \quad \ldots(3)$$

The lines (1), (2) and (3) will be concurrent iff

$$\begin{vmatrix} x_3 - x_2 & y_3 - y_2 & -\frac{1}{2}(x_3^2 - x_2^2) - \frac{1}{2}(y_3^2 - y_2^2) \\ x_1 - x_3 & y_1 - y_3 & -\frac{1}{2}(x_1^2 - x_3^2) - \frac{1}{2}(y_1^2 - y_3^2) \\ x_2 - x_1 & y_2 - y_1 & -\frac{1}{2}(x_2^2 - x_1^2) - \frac{1}{2}(y_2^2 - y_1^2) \end{vmatrix} = 0$$

$$\begin{vmatrix} x_3 - x_2 & y_3 - y_2 & -\frac{1}{2}x_3^2 + \frac{1}{2}x_2^2 - \frac{1}{2}y_3^2 + \frac{1}{2}y_2^2 \\ x_1 - x_3 & y_1 - y_3 & -\frac{1}{2}x_1^2 + \frac{1}{2}x_3^2 - \frac{1}{2}y_1^2 + \frac{1}{2}y_3^2 \\ x_2 - x_1 & y_2 - y_1 & -\frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 - \frac{1}{2}y_2^2 + \frac{1}{2}y_1^2 \end{vmatrix} = 0$$

By $R_1 + (R_2 + R_3)$

$$\begin{vmatrix} 0 & 0 & 0 \\ x_1 - x_3 & y_1 - y_3 & -\frac{1}{2}x_1^2 + \frac{1}{2}x_3^2 - \frac{1}{2}y_1^2 + \frac{1}{2}y_3^2 \\ x_2 - x_1 & y_2 - y_1 & -\frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 - \frac{1}{2}y_2^2 + \frac{1}{2}y_1^2 \end{vmatrix} = 0$$

$\Rightarrow \quad 0 = 0 \quad \because R_1$ is zero

Thus, the right bisectors of a triangle are concurrent.

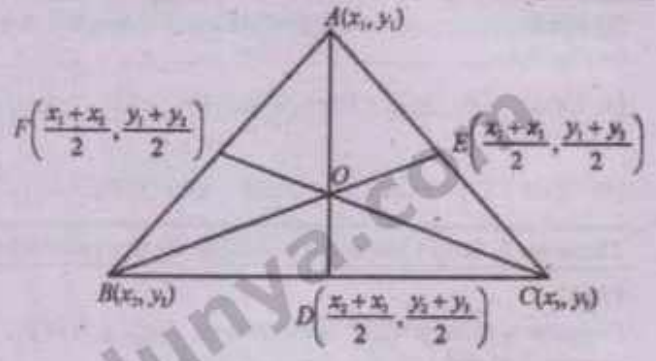
Theorem 6: Medians of the triangle are concurrent.

Proof:

Consider a triangle ABC with vertices $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$.

Let D, E and F be the midpoints of sides $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ respectively, such that

$$D = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$



Equation of the median $\overline{AD}$ is:

$$y - y_1 = \frac{\frac{y_2 + y_3}{2} - y_1}{\frac{x_2 + x_3}{2} - x_1}(x - x_1) \quad \text{(Two-points form)}$$

$$y - y_1 = \frac{\frac{y_2 + y_3 - 2y_1}{2}}{\frac{x_2 + x_3 - 2x_1}{2}}(x - x_1)$$

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$$(x_2+x_3-2x_1)(y-y_1)=(y_2+y_3-2y_1)(x-x_1)$$
$$(x_2+x_3-2x_1)y-(x_2+x_3-2x_1)y_1=(y_2+y_3-2y_1)x-(y_2+y_3-2y_1)x_1$$
$$(y_2+y_3-2y_1)x-(x_2+x_3-2x_1)y-(y_2+y_3-2y_1)x_1+(x_2+x_3-2x_1)y_1=0 \quad ...(1)$$

By symmetry, equations of the other two medians $\overline{BE}$ and $\overline{CF}$ respectively.

$$(y_3+y_1-2y_2)x-(x_3+x_1-2x_2)y-(y_3+y_1-2y_2)x_2+(x_3+x_1-2x_2)y_2=0 \quad ...(2)$$

$$(y_1+y_2-2y_3)x-(x_1+x_2-2x_3)y-(y_1+y_2-2y_3)x_3+(x_1+x_2-2x_3)y_3=0 \quad ...(3)$$

The lines (1), (2) and (3) will be concurrent iff

$$\begin{vmatrix} y_2+y_3-2y_1 & -x_2-x_3+2x_1 & -(y_2+y_3-2y_1)x_1+(x_2+x_3-2x_1)y_1 \\ y_3+y_1-2y_2 & -x_3-x_1+2x_2 & -(y_3+y_1-2y_2)x_2+(x_3+x_1-2x_2)y_2 \\ y_1+y_2-2y_3 & -x_1-x_2+2x_3 & -(y_1+y_2-2y_3)x_3+(x_1+x_2-2x_3)y_3 \end{vmatrix}=0$$

By $R_1+(R_2+R_3)$

$$\begin{vmatrix} 0 & 0 & 0 \\ y_3+y_1-2y_2 & -x_3-x_1+2x_2 & -(y_3+y_1-2y_2)x_2+(x_3+x_1-2x_2)y_2 \\ y_1+y_2-2y_3 & -x_1-x_2+2x_3 & -(y_1+y_2-2y_3)x_3+(x_1+x_2-2x_3)y_3 \end{vmatrix}=0$$

$0=0$ $\quad \because R_1$ is zero

Thus, the medians of a triangle are concurrent.

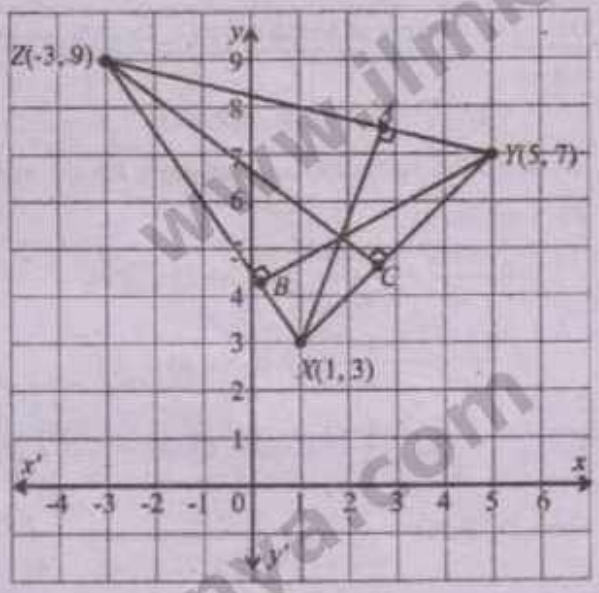
EXERCISE 5.1

Q1. Find equation of the altitude of triangle XYZ from vertex $X(1,3)$, $Y(5,7)$ and $Z(-3,9)$.

Solution:

Given that: $X(1,3), Y(5,7), Z(-3,9)$

Let $\overline{XA} \perp \overline{YZ}$, $\overline{YB} \perp \overline{XZ}$ and $\overline{ZC} \perp \overline{XY}$ be the altitudes of triangle ABC.



$$\text{Slope of side } \overline{YZ} = \frac{9-7}{-3-5}=\frac{2}{-8}=-\frac{1}{4}$$

$$\text{Slope of } \overline{XA} = -\frac{1}{\text{Slope of } \overline{YZ}}=4$$

Equation of altitude $\overline{XA}$ is:

$$y-y_1=m(x-x_1)$$
$$y-3=4(x-1)$$
$$y-3=4x-4$$
$$\boxed{4x-y-1=0}$$

$$\text{Slope of side } \overline{XZ} = \frac{9-3}{-3-1}=\frac{6}{-4}=-\frac{3}{2}$$

$$\text{Slope of } \overline{YB} = -\frac{1}{\text{Slope of } \overline{XZ}}=\frac{2}{3}$$

Equation of altitude $\overline{YB}$ is:

$$y-y_1=m(x-x_1)$$
$$y-7=\frac{2}{3}(x-5)$$
$$3y-21=2x-10$$
$$\boxed{2x-3y+11=0}$$

$$\text{Slope of side } \overline{XY} = \frac{7-3}{5-1}=\frac{4}{4}=1$$

$$\text{Slope of } \overline{ZC} = -\frac{1}{\text{Slope of } \overline{XY}}=-1$$

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Equation of altitude $\overline{ZC}$ is:

$$y - y_1 = m(x - x_1)$$
$$y - 9 = -1(x + 3)$$
$$y - 9 = -x - 3$$
$$\boxed{x + y - 6 = 0}$$

Q2. $A(-3,-7), B(4,5)$ and $C(-1,2)$ are the vertices of ΔABC. Find the equation of right bisector of side BC.

Solution:

Given that: $A(-3,-7), B(4,5), C(-1,2)$

Let D be the midpoint of side $\overline{BC}$, such that

$$D = \left(\frac{4+(-1)}{2}, \frac{5+2}{2}\right) = \left(\frac{3}{2}, \frac{7}{2}\right)$$

$$\text{Slope of } \overline{BC} = \frac{2-5}{-1-4} = \frac{-3}{-5} = \frac{3}{5}$$

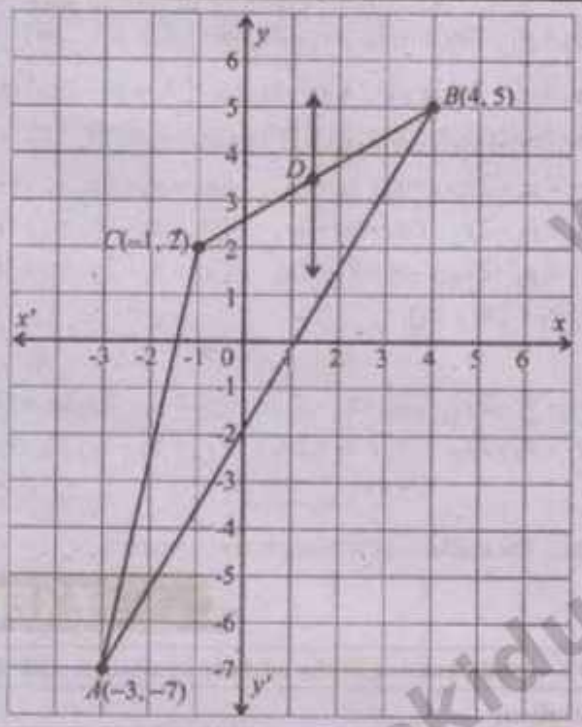
$$\text{Slope of right bisector to } \overline{BC} = -\frac{1}{\text{Slope of } \overline{BC}} = -\frac{5}{3}$$

Equation of right bisector to $\overline{BC}$ is:

$$y - y_1 = m(x - x_1)$$
$$y - \frac{7}{2} = -\frac{5}{3}x + \frac{15}{6}$$

Multiply both sides by '6'

$$6y - 21 = -10x + 15$$
$$10x + 6y - 36 = 0$$
$$2(5x + 3y - 18) = 0$$
$$\boxed{5x + 3y - 18 = 0}$$



Q3. The vertices of ΔPQR are $P(1,2), Q(5,-6)$ and $R(7,-2)$. Find the equations of all three medians.

Solution:

Given that: $P(1,2), Q(5,-6), R(7,-2)$

Let A, B and C be the midpoints of sides $\overline{PQ}, \overline{QR}$ and $\overline{PR}$ respectively.

$$A\left(\frac{1+5}{2}, \frac{2+(-6)}{2}\right) = A(3,-2)$$

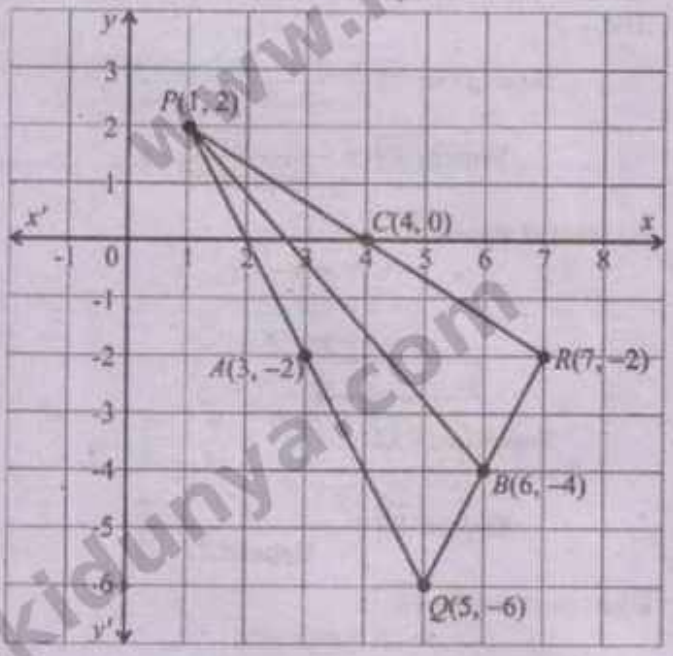
$$B\left(\frac{5+7}{2}, \frac{-6+(-2)}{2}\right) = B(6,-4)$$

$$C\left(\frac{1+7}{2}, \frac{2+(-2)}{2}\right) = C(4,0)$$

Using Formula: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Equation of median $\overline{PB}$ is:

$$y - 2 = \frac{-4-2}{6-1}(x-1)$$
$$5(y-2) = -6(x-1)$$
$$5y - 10 = -6x + 6$$
$$\boxed{6x + 5y - 16 = 0}$$



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Equation of median $\overline{QC}$ is:

$$y-(-6)=\frac{0-(-6)}{4-5}(x-5)$$
$$-1(y+6)=6(x-5)$$
$$-y-6=6x-30$$
$$\boxed{6x+y-24=0}$$

Equation of median $\overline{RA}$ is:

$$y-(-2)=\frac{-2-(-2)}{3-7}(x-7)$$
$$-4(y+2)=0(x-7)$$
$$-4(y+2)=0$$
$$\boxed{y+2=0}$$

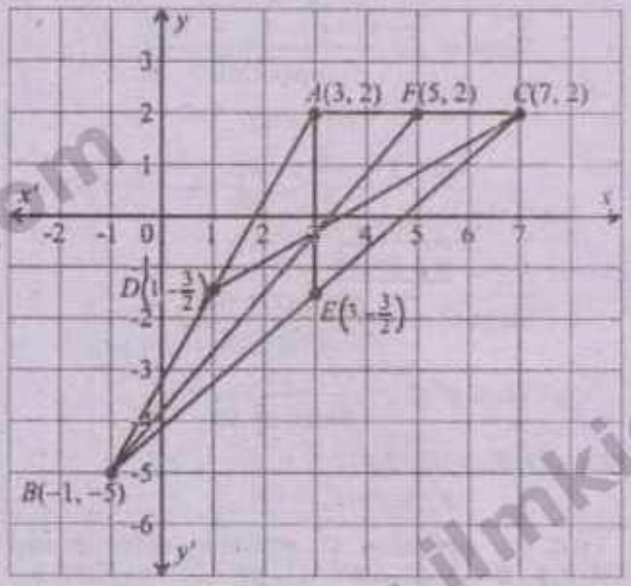
Q4. Show that the medians of the triangle with vertices $A(3,2), B(-1,-5)$ and $C(7,2)$ are concurrent.

Solution:

Given that: $A(3,2), B(-1,-5), C(7,2)$

Let D, E and F be the midpoints of the sides $\overline{AB}, \overline{BC}$ and $\overline{AC}$ respectively.

$$D\left(\frac{3+(-1)}{2},\frac{2+(-5)}{2}\right)=D\left(1,-\frac{3}{2}\right)$$
$$E\left(\frac{-1+7}{2},\frac{-5+2}{3}\right)=E\left(3,-\frac{3}{2}\right)$$
$$F\left(\frac{3+7}{2},\frac{2+2}{3}\right)=F(5,2)$$



Using Formula: $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

Equation of median $\overline{AE}$ is:

$$y-2=\frac{-\frac{3}{2}-2}{3-3}(x-3)$$
$$0(y-2)=-\frac{7}{2}(x-3)$$
$$0=-\frac{7}{2}(x-3)$$
$$\boxed{x-3=0}$$

Equation of median $\overline{BF}$ is:

$$y-(-5)=\frac{2-(-5)}{5-(-1)}(x-(-1))$$
$$y+5=\frac{7}{6}(x+1)$$
$$6y+30=7x+7$$
$$\boxed{7x-6y-23=0}$$

Equation of median $\overline{CD}$ is:

$$y-2=\frac{-\frac{3}{2}-2}{1-7}(x-7)$$
$$-6(y-2)=\frac{-7}{2}(x-7)$$
$$-12(y-2)=-7(x-7)$$
$$-12y+24=-7x+49$$
$$\boxed{7x-12y-25=0}$$

Consider: $\begin{vmatrix}1 & 0 & -3\\ 7 & -6 & -23\\ 7 & -12 & -25\end{vmatrix}=1\begin{vmatrix}-6 & -23\\ -12 & -25\end{vmatrix}-0\begin{vmatrix}7 & -23\\ 7 & -25\end{vmatrix}-3\begin{vmatrix}7 & -6\\ 7 & -12\end{vmatrix}$

$$=1(150-276)-0-3(-84+42)=1(-126)-3(-42)=-126+126=0$$

Thus, the medians of a triangle ABC are concurrent.

Q5. The vertices of a triangle are $A(2,-1), B(4,3)$ and $C(6,-1)$. Find the equations of altitudes and show that they are concurrent.

Solution:

Given that: $A(2,-1), B(4,3), C(6,-1)$

Let $\overline{AD}\perp\overline{BC}$, $\overline{BE}\perp\overline{AC}$ and $\overline{CF}\perp\overline{AB}$ be the altitudes of triangle ABC.

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Slope of $\overline{BC} = \frac{-1-3}{6-4} = -\frac{4}{2} = -2$

Slope of $\overline{AD} = \frac{-1}{\text{Slope of } \overline{BC}} = \frac{1}{2}$

Equation of altitude $\overline{AD}$ is:

$$y - y_1 = m(x - x_1)$$
$$y + 1 = \frac{1}{2}(x - 2)$$
$$2y + 2 = x - 2$$
$$\boxed{x - 2y - 4 = 0}$$

Slope of $\overline{AC} = \frac{-1-(-1)}{6-2} = \frac{0}{4} = 0$

Slope of $\overline{BE} = \frac{-1}{\text{Slope of } \overline{AC}} = \frac{1}{0}$

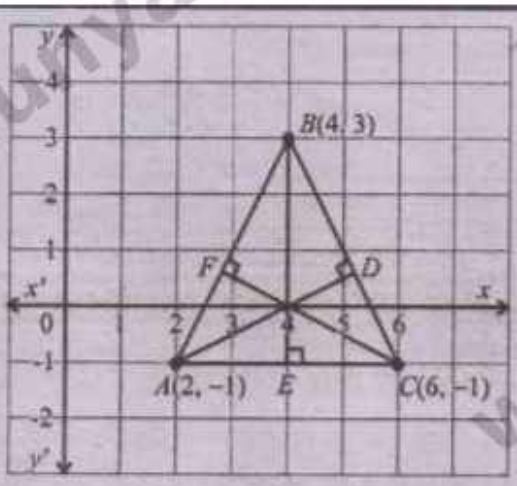
$$y - y_1 = m(x - x_1)$$
$$y - 3 = \frac{1}{0}(x - 4)$$
$$0 = -(x - 4)$$
$$\boxed{x - 4 = 0}$$

Slope of $\overline{AB} = \frac{3-(-1)}{4-2} = \frac{4}{2} = 2$

Slope of $\overline{CF} = \frac{-1}{\text{Slope of } \overline{AB}} = -\frac{1}{2}$

Equation of altitude $\overline{CF}$ is:

$$y - y_1 = m(x - x_1)$$
$$y + 1 = -\frac{1}{2}(x - 6)$$
$$2y + 2 = -x + 6$$
$$\boxed{x + 2y - 4 = 0}$$



Consider:

$$\begin{vmatrix} 1 & -2 & -4 \\ 1 & 0 & -4 \\ 1 & 2 & -4 \end{vmatrix} = 1\begin{vmatrix} 0 & -4 \\ 2 & -4 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -4 \\ 1 & -4 \end{vmatrix} - 4\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$
$$= 1(0 + 8) + 2(-4 + 4) - 4(2 - 0)$$
$$= 1(8) + 2(0) - 4(2) = 8 - 8 = 0$$

Thus, the altitudes of a triangle ABC are concurrent.

Q6. Find the equations of medians, altitudes and right bisectors of the triangle whose vertices are $A(2, 5), B(-3, 4)$ and $C(6, -8)$.

Solution:

Given that: $A(2, 5), B(-3, 4), C(6, -8)$

(I) Equations of Medians

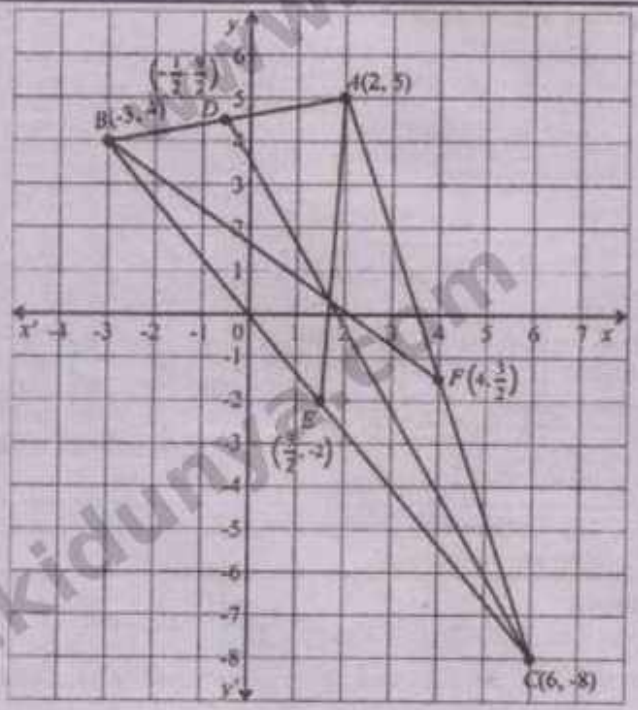
Let D, E and F be the midpoints of sides $\overline{AB}, \overline{BC}$ and $\overline{AC}$ respectively.

$$D\left(\frac{2+(-3)}{2}, \frac{5+4}{2}\right) = D\left(-\frac{1}{2}, \frac{9}{2}\right)$$
$$E\left(\frac{-3+6}{2}, \frac{4+(-8)}{2}\right) = E\left(\frac{3}{2}, -2\right)$$
$$F\left(\frac{2+6}{2}, \frac{5+(-8)}{2}\right) = F\left(4, \frac{-3}{2}\right)$$

Using Formula: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Equation of median $\overline{AE}$ is:

$$y - 5 = \frac{-2-5}{\frac{3}{2}-2}(x - 2)$$



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$$y-5=\frac{-7}{-1/2}(x-2)$$

$$y-5=14(x-2)$$

$$14x-28-y+5=0$$

$$\boxed{14x-y-23=0}$$

Equation of median $\overline{BF}$ is:

$$y-4=\frac{-\frac{3}{2}-4}{4+3}(x+3)$$

$$7(y-4)=-\frac{11}{2}(x+3)$$

$$14(y-4)=-11x-33$$

$$11x+33+14y-56=0$$

$$\boxed{11x+14y-23=0}$$

Equation of median $\overline{CD}$ is:

$$y+8=\frac{\frac{9}{2}+8}{-\frac{1}{2}-6}(x-6)$$

$$y+8=\frac{\frac{25}{2}}{-\frac{13}{2}}(x-6)$$

$$-13(y+8)=25(x-6)$$

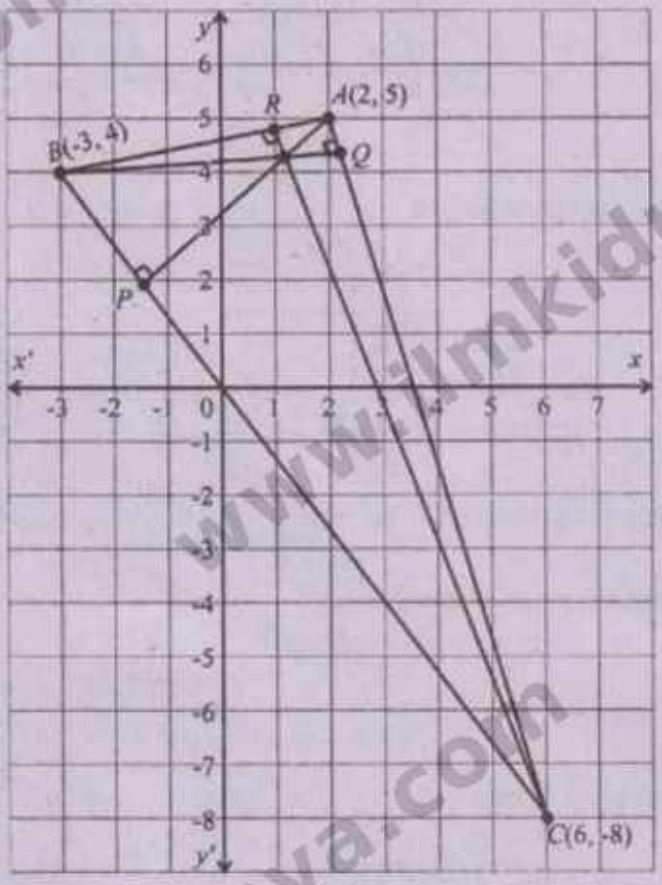
$$-13y-104=25x-150$$

$$\boxed{25x+13y-46=0}$$

(ii) Equation of altitudes

$A(2,5), B(-3,4), C(6,-8)$

Let $\overline{AP}\perp\overline{BC}, \overline{BQ}\perp\overline{AC}$ and $\overline{CR}\perp\overline{AB}$ be the altitudes of triangle ABC.



$$\text{Slope of } \overline{BC}=\frac{-8-4}{6+3}=\frac{-12}{9}=-\frac{4}{3}$$

$$\text{Slope of } \overline{AP}=\frac{-1}{\text{Slope of } \overline{BC}}=\frac{3}{4}$$

Equation of altitude $\overline{AP}$ is:

$$y-y_1=m(x-x_1)$$

$$y-5=\frac{3}{4}(x-2)$$

$$4y-20=3x-6$$

$$\boxed{3x-4y+14=0}$$

$$\text{Slope of } \overline{AC}=\frac{-8-5}{6-2}=-\frac{13}{4}$$

$$\text{Slope of } \overline{BQ}=\frac{-1}{\text{Slope of } \overline{AC}}=\frac{4}{13}$$

Equation of altitude $\overline{BQ}$ is:

$$y-y_1=m(x-x_1)$$

$$y-4=\frac{4}{13}(x+3)$$

$$13y-52=4x+12$$

$$\boxed{4x-13y+64=0}$$

$$\text{Slope of } \overline{AB}=\frac{4-5}{-3-2}=\frac{-1}{-5}=\frac{1}{5}$$

$$\text{Slope of } \overline{CR}=\frac{-1}{\text{Slope of } \overline{AB}}=-5$$

Equation of altitude $\overline{CR}$ is:

$$y-y_1=m(x-x_1)$$

$$y+8=-5(x-6)$$

$$y+8=-5x+30$$

$$\boxed{5x+y-22=0}$$

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(iii) Equations of right bisectors

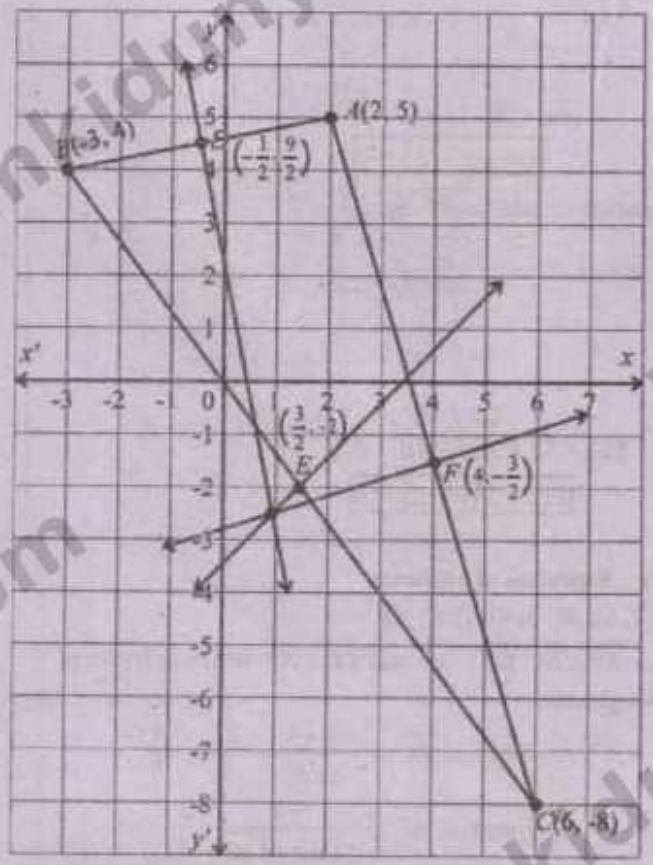
$A(2, 5), B(-3, 4), C(6, -8)$

Let D, E and F be the midpoints of sides $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively.

$$D\left(\frac{2-3}{2}, \frac{5+4}{2}\right) = D\left(-\frac{1}{2}, \frac{9}{2}\right)$$

$$E\left(\frac{-3+6}{2}, \frac{4-8}{2}\right) = E\left(\frac{3}{2}, -2\right)$$

$$F\left(\frac{2+6}{2}, \frac{5-8}{2}\right) = F\left(4, -\frac{3}{2}\right)$$



Slope of $\overline{AB} = \frac{4-5}{-3-2} = \frac{-1}{-5} = \frac{1}{5}$

Slope of right bisector to $\overline{AB} = \frac{-1}{\text{Slope of } \overline{AB}} = -5$

Equation of right bisector to $\overline{AB}$ is:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{9}{2} = -5\left(x + \frac{1}{2}\right)$$

$$y = \frac{9}{2} = -5x - \frac{5}{2}$$

Multiply both sides by 2

$$2y - 9 = -10x - 5$$

$$10x + 2y - 4 = 0$$

$$2(5x + y - 2) = 0$$

$$\boxed{5x + y - 2 = 0}$$

Slope of $\overline{BC} = \frac{-8-4}{6+3} = \frac{12}{9} = -\frac{4}{3}$

Slope of right bisector to $\overline{BC} = \frac{-1}{\text{Slope of } \overline{BC}} = \frac{3}{4}$

Equation of right bisector to $\overline{BC}$ is:

$$y - y_1 = m(x - x_1)$$

$$y + 2 =$$

$$4y + 8 = 3x - \frac{9}{2}$$

Multiply both sides by 2

$$8y + 16 = 6x - 9$$

$$\boxed{6x - 8y - 25 = 0}$$

Slope of $\overline{AC} = \frac{-8-5}{6-2} = -\frac{13}{4}$

Slope of right bisector to $\overline{AC} = \frac{-1}{\text{Slope of } \overline{AC}} = \frac{4}{3}$ $\frac{3}{4}\left(x - \frac{3}{2}\right)$

Equation of right bisector to $\overline{AC}$ is:

$$y - y_1 = m(x - x_1)$$

$$y + \frac{3}{2} = \frac{4}{13}(x - 4)$$

$$13y + \frac{39}{2} = 4x - 16$$

Multiply both sides by '2'

$$26y + 39 = 8x - 32$$

$$\boxed{8x - 26y - 71 = 0}$$

Q7. Convert each of the following equation into:

(a) $2x + 3y - 6 = 0$ **(b)** $x - 2y + 4 = 0$ **(c)** $4x + 5y - 20 = 0$ **(d)** $3x + 4y - 12 = 0$

(i) Slope-intercept form (ii) Two-point form (iii) Intercept form (iv) Point-slope form

(v) Normal form (vi) Symmetric form

Solution:

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(a) $2x+3y-6=0$

(i) Slope-intercept form ($y=mx+c$)

$$2x+3y-6=0$$
$$3y=-2x+6$$

Dividing both sides by 3

$$\frac{3y}{3}=-\frac{2x}{3}+\frac{6}{3}$$
$$y=-\frac{2}{3}x+2$$

where, $m=-\frac{2}{3}, c=2$

(ii) Two-point form

$$2x+3y-6=0$$

Put $y=0$	Put $x=0$
$2x+0-6=0$	$0+3y-6=0$
$x=\frac{6}{2}=3$	$y=\frac{6}{3}=2$

Points on the line = (3, 0), (0, 2)

Two-point form of line:

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$
$$y-0=\frac{2-0}{0-3}(x-3)$$
$$y=-\frac{2}{3}(x-3)$$

(iii) Intercept form $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

$$2x+3y-6=0$$
$$2x+3y=6$$

Dividing both sides by 6

$$\frac{2x}{6}+\frac{3y}{6}=\frac{6}{6}$$
$$\frac{x}{3}+\frac{y}{2}=1$$

where, $a=3, b=2$

(iv) Point-slope form

$$2x+3y-6=0$$

Put $y=0 \Rightarrow 2x=6 \Rightarrow x=3$

Point on the line = (3, 0)

Point of line = $m=-\frac{2}{3}$

Point-slope form of line:

$$y-y_1=m(x-x_1)$$
$$y-0=-\frac{2}{3}(x-3)$$
$$y=-\frac{2}{3}(x-3)$$

(v) Normal form ($x\cos\theta+y\sin\theta=p$)

$$2x+3y-6=0$$
$$2x+3y=6$$

Dividing both sides by $\sqrt{a^2+b^2}=\sqrt{2^2+3^2}=\sqrt{13}$

$$\frac{2}{\sqrt{13}}x+\frac{3}{\sqrt{13}}y=\frac{6}{\sqrt{13}}$$

where, $\cos\theta=\frac{2}{\sqrt{13}}, \sin\theta=\frac{3}{\sqrt{13}}, p=\frac{6}{\sqrt{13}}$

(vi) Symmetric form

$$2x+3y-6=0$$

Put $y=0 \Rightarrow 2x=6 \Rightarrow x=3$

Point on the line = (3, 0)

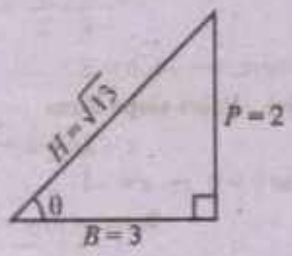
As $m=-\frac{2}{3}$

$\Rightarrow \tan\theta=-\frac{2}{3}<0$ (θ lies in II-quadrant)

Pythagoras theorem

$H^2=B^2+P^2$

$H^2=9+4 \Rightarrow H=\sqrt{13}$



$\sin\theta=\frac{2}{\sqrt{13}}, \cos\theta=\frac{3}{\sqrt{13}}$

Symmetric form of a line:

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$
$$\frac{x-3}{\frac{3}{\sqrt{13}}}=\frac{y-0}{\frac{2}{\sqrt{13}}}=r$$

(b) $x-2y+4=0$

(i) Slope-intercept form ($y=mx+c$)

$$x-2y+4=0$$
$$-2y=-x-4$$

Dividing both sides by −2

$$\frac{-2y}{2}=\frac{-x}{-2}-\frac{4}{-2}$$
$$y=\frac{1}{2}x+2$$

where, $m=\frac{1}{2}, c=2$

(ii) Two-point form

$$x-2y+4=0$$

Put $y=0$	Put $x=0$
$x+0+4=0$	$0-2y+4=0$
$x=-4$	$-2y=-4 \Rightarrow y=2$

Points on the line = (−4, 0), (0, 2)

Two-point form of line:

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

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$$y-0=\frac{2-0}{0+4}(x+4)$$

$$y=\frac{1}{2}(x+4)$$

(iii) Intercept form $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

$$x-2y+4=0$$

$$x-2y=-4$$

Dividing both sides by -4

$$\frac{x}{-4}-\frac{2y}{-4}=\frac{-4}{-4}$$

$$\frac{x}{-4}+\frac{y}{2}=1$$

where, $a=-4, b=2$

(iv) Point-slope form

$$x-2y+4=0$$

Put $y=0 \Rightarrow x=-4$

Point of line $=(-4, 0)$

Slope of line $=-\frac{1}{-2}=\frac{1}{2}$

Point-slope form of line:

$$y-y_1=m(x-x_1)$$

$$y-0=\frac{1}{2}(x+4)$$

$$y=\frac{1}{2}(x+4)$$

(v) Normal form $(x\cos\theta+y\sin\theta=p)$

$$x-2y+4=0$$

$$x-2y=-4$$

Dividing both sides by $-\sqrt{a^2+b^2}=-\sqrt{1^2+(-2)^2}=-\sqrt{5}$

$$\frac{x}{-\sqrt{5}}-\frac{2y}{-\sqrt{5}}=\frac{-4}{-\sqrt{5}}$$

$$\frac{1}{\sqrt{5}}x+\frac{2}{\sqrt{5}}y=\frac{4}{\sqrt{5}}$$

where, $\cos\theta=\frac{1}{\sqrt{5}}, \sin\theta=\frac{2}{\sqrt{5}}, p=\frac{4}{5}$

(vi) Symmetric form

$$x-2y+4=0$$

Put $y=0 \Rightarrow x-0+4=0 \Rightarrow x=-4$

Point on the line $=(-4, 0)$

As $m=-\frac{1}{-2}=\frac{1}{2}$

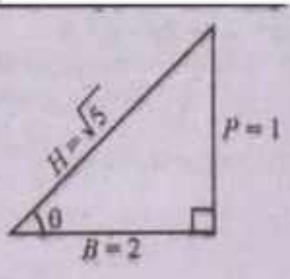
$\Rightarrow \tan\theta=\frac{1}{2}>0$ (θ lies in 1st quadrant)

Pythagoras theorem

$H^2=B^2+P^2$

$H^2=2^2+1^2 \Rightarrow H=\sqrt{5}$

$\sin\theta=\frac{1}{\sqrt{5}}, \cos\theta=\frac{2}{\sqrt{5}}$



Symmetric form of a line:

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

$$\frac{x+4}{\frac{2}{\sqrt{5}}}=\frac{y-0}{\frac{1}{\sqrt{5}}}=r$$

(c) $4x+5y-20=0$

(i) Slope-intercept form $(y=mx+c)$

$$4x+5y-20=0$$

$$5y=-4x+20$$

Dividing both sides by 5

$$\frac{5y}{5}=\frac{-4x}{5}+\frac{20}{5}$$

$$y=-\frac{4}{5}x+4$$

where, $m=-\frac{4}{5}, c=4$

(ii) Two-point form

$$4x+5y-20=0$$

Put $y=0$	Put $x=0$
$4x+0-20=0$	$0+5y-20=0$
$4x=20 \Rightarrow x=5$	$5y=20 \Rightarrow y=4$

Points on the line $=(5, 0), (0, 4)$

Two-points form of line:

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-0=\frac{4-0}{0-5}(x-5)$$

$$y=-\frac{4}{5}(x-5)$$

(iii) Intercept form $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

$$4x+5y-20=0$$

$$4x+5y=20$$

Dividing both sides by 20

$$\frac{4x}{20}+\frac{5y}{20}=\frac{20}{20}$$

$$\frac{x}{5}+\frac{y}{4}=1$$

where, $a=5, b=4$

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(iv) Point-slope form

$$4x+5y-20=0$$

Put $y=0 \Rightarrow 4x=20 \Rightarrow x=5$

Point on the line $=(5,0)$

$$\text{Slope of line} = m = -\frac{4}{5}$$

Point-slope form of line:

$$y-y_1 = m(x-x_1)$$

$$y-0 = -\frac{4}{5}(x-5)$$

$$y = -\frac{4}{5}(x-5)$$

(v) Normal form $(x\cos\theta + y\sin\theta = p)$

$$4x-5y-20=0$$

$$4x-5y=20$$

Dividing both sides by $\sqrt{a^2+b^2} = \sqrt{4^2+5^2} = \sqrt{41}$

$$\frac{4}{\sqrt{41}}x + \frac{5}{\sqrt{41}}y = \frac{20}{\sqrt{41}}$$

where, $\cos\theta = \frac{4}{\sqrt{41}}, \sin\theta = \frac{5}{\sqrt{41}}, p = \frac{20}{\sqrt{41}}$

(vi) Symmetric form

$$4x+5y-20=0$$

Put $y=0 \Rightarrow 4x=20 \Rightarrow x=5$

Point on the line $=(5,0)$

As $$m = -\frac{4}{5}$$

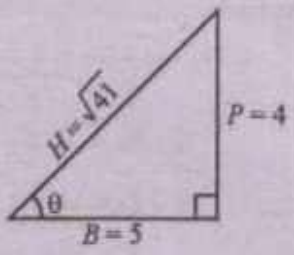
$\Rightarrow$ $$\tan\theta = -\frac{4}{5} < 0$$ (θ lies is 2nd quadrant)

Pythagoras theorem

$$H^2 = B^2 + P^2$$

$$H^2 = 5^2 + 4^2 \Rightarrow H = \sqrt{41}$$

$$\sin\theta = \frac{4}{\sqrt{41}}, \cos\theta = \frac{5}{\sqrt{41}}$$



Symmetric form of a line:

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$$

$$\frac{x-5}{\frac{5}{\sqrt{41}}} = \frac{y-0}{\frac{4}{\sqrt{41}}} = r$$

(b) $3x+4y-12=0$

(i) Slope-intercept form $(y=mx+c)$

$$3x+4y-12=0$$

$$4y = -3x+12$$

Dividing both sides by 4

$$\frac{4y}{4} = -\frac{3x}{4} + \frac{12}{4}$$

$$y = -\frac{3}{4}x+3$$

where, $m = -\frac{3}{4}, c = 3$

(ii) Two-point form

$$3x+4y-12=0$$

Put $y=0$	Put $x=0$
$3x+0-12=0$	$0+4y-12=0$
$3x=12 \Rightarrow x=4$	$4y=12 \Rightarrow y=3$

Points on the line $=(4,0),(0,3)$

Two-point form of line:

$$y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-0 = \frac{3-0}{0-4}(x-4)$$

$$y = -\frac{3}{4}(x-4)$$

(iii) Intercept form $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

$$3x+4y-12=0$$

$$3x+4y=12$$

Dividing both sides by 12

$$\frac{3x}{12} + \frac{4y}{12} = \frac{12}{12}$$

$$\frac{x}{4} + \frac{y}{3} = 1$$

where, $a=4, b=3$

(iv) Point-slope form

$$3x+4y-12=0$$

Put $y=0 \Rightarrow 3x=12 \Rightarrow x=4$

Point on the line $=(4,0)$

$$\text{Slope of line} = m = -\frac{3}{4}$$

Point-slope form of line:

$$y-y_1 = m(x-x_1)$$

$$y-0 = -\frac{3}{4}(x-4)$$

$$y = -\frac{3}{4}(x-4)$$

(v) Normal form $(x\cos\theta + y\sin\theta = p)$

$$3x+4y-12=0$$

$$3x+4y=12$$

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Dividing both sides by $\sqrt{a^2+b^2} = \sqrt{3^2+4^2} = 5$

$$\frac{3}{5}x + \frac{4}{5}y = \frac{12}{5}$$

where, $\cos\theta = \frac{3}{5}, \sin\theta \frac{4}{5} =, p = \frac{12}{5}$

(vi) Symmetric form

$$3x + 4y - 12 = 0$$

Put $y = 0 \Rightarrow 3x = 12 \Rightarrow x = 4$

Point on the line = (4, 0)

As $m = -\frac{3}{4}$

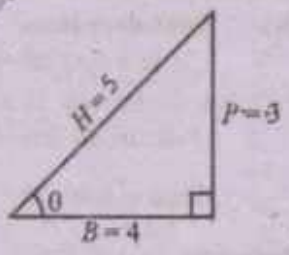
$\Rightarrow$ $\tan\theta = -\frac{3}{4} < 0$ (θ lies is 2nd quadrant)

Pythagoras theorem

$H^2 = B^2 + P^2$

$H^2 = 4^2 + 3^2 \Rightarrow H = 5$

$\sin\theta = \frac{3}{5}, \cos\theta = -\frac{4}{5}$



Symmetric form of a line:

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

$$\frac{x-4}{-\frac{4}{5}} = \frac{y-0}{\frac{3}{5}} = r$$

Q8. Find the equation of family of lines through the point of intersection of the lines $x + 2y - 3 = 0$, $3x - y + 4 = 0$. Find the member of the family which is:

(i) parallel to a line with slope $\frac{3}{2}$ **(ii) perpendicular to the line $3x + 4y = 0$**

Solution:

Let $\ell_1: x + 2y - 3 = 0$

$\ell_2: 3x - y + 4 = 0$

Family of lines through the point of intersection of ℓ_1 and ℓ_2 is:

$$\ell: \ell_1 + k\ell_2 = 0$$

$$\ell: (x + 2y - 3) + k(3x - y + 4) = 0 \quad \ldots(1)$$

$$x + 2y - 3 + 3kx - ky + 4k = 0$$

$$(1 + 3k)x + (2 - k)y + (-3 + 4k) = 0$$

$$\text{Slope of } \ell = m = -\frac{1+3k}{2-k}$$

(i) Since ℓ is the line parallel to line with slope $\frac{3}{2}$, so

$$\frac{1+3k}{2-k} = \frac{3}{2}$$

$$-2(1 + 3k) = 3(2 - k)$$

$$-2 - 6k = 6 - 3k$$

$$-2 - 6 = -3k + 6k$$

$$3k = -8 \Rightarrow k = -\frac{8}{3}$$

Put $k = \frac{-8}{3}$ in eq. (1), we get

$$(x + 2y - 3) - \frac{8}{3}(3x - y + 4) = 0$$

Multiply both sides by 3

$$3x + 6y - 9 - 24x + 8y - 32 = 0$$

$$-21x + 14y - 41 = 0$$

$$\boxed{21x - 14y + 41 = 0}$$ Multiply by -1

(ii) Let $\ell_3: 3x + 4y = 0$

$$\text{Slope of } \ell_3 = m' = -\frac{3}{4}$$

As $\ell \perp \ell_3$, so $m \times m' = -1$

$$\Rightarrow \quad \frac{1+3k}{2-k} \times -\frac{3}{4} = -1$$

$$3 + 9k = -1(8 - 4k) \Rightarrow 3 + 9k = -8 + 4k$$

$$9k - 4k = -8 - 3 \Rightarrow 5k = -11 \Rightarrow k = -\frac{11}{5}$$

Put $k = -\frac{11}{5}$ in eq. (1), we get

$$(x + 2y - 3) - \frac{11}{5}(3x - y + 4) = 0$$

Multiply both sides by 5

$$5x + 10y - 15 - 33x + 11y - 44 = 0$$

$$-28x + 21y - 59 = 0$$

$$\boxed{28x - 21y + 59 = 0}$$ Multiply by -1

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Q9. Find the family of lines through the point of intersection of the lines $2x - 3y + 5 = 0, x + 4y - 7 = 0$. Find the member of the family which is:

(i) parallel to $y = 2x + 1$ **(ii) perpendicular to $y = x$**

Solution:

Let $\ell_1: 2x - 3y + 5 = 0$

$\ell_2: x + 4y - 7 = 0$

Family of lines through the point of intersection of ℓ_1 and ℓ_2 is:

$$\ell: \ell_1 + k\ell_2 = 0$$

$$\ell: (2x - 3y + 5) + k(x + 4y - 7) = 0 \quad \ldots(1)$$

$$2x - 3y + 5 + kx + 4ky - 7k = 0$$

$$(2 + k)x + (-3 + 4k)y + (5 - 7k) = 0$$

$$\text{Slope of } \ell = m = \frac{2+k}{-3+4k}$$

(i) Let $\ell_3: y = 2x + 1 \Rightarrow 2x - y + 1 = 0$

$$\text{Slope of } \ell_3 = m_3 = -\frac{2}{-1} = 2$$

As $\ell \parallel \ell_3$, so $m = m_3$

$$\frac{2+k}{-3+4k} = 2$$

$$-(2 + k) = 2(-3 + 4k)$$

$$-2 - k = -6 + 8k$$

$$-2 + 6 = 8k + k$$

$$9k = 4 \Rightarrow k = \frac{4}{9}$$

Put $k = \frac{4}{9}$ in eq. (1), we get

$$(2x - 3y + 5) + \frac{4}{9}(x + 4y - 7) = 0$$

Multiply both sides by 9

$$18x - 27y + 45 + 4x + 16y - 28 = 0$$

$$\boxed{22x - 11y + 17 = 0}$$

(ii) Let $\ell_4: y = x \Rightarrow x - y = 0$

$$\text{Slope of } \ell_4 = m_4 = \frac{-1}{-1} = 1$$

As $\ell \parallel \ell_4$, so $m \times m_4 = -1$

$$\frac{2+k}{-3+4k} \times 1 = -1$$

$$2 + k = -3 + 4k \Rightarrow 2 + 3 = 4k - k$$

$$3k = 5 \Rightarrow k = \frac{5}{3}$$

Put $k = \frac{5}{3}$ in eq. (1), we get

$$(2x - 3y + 5) + \frac{5}{3}(x + 4y - 7) = 0$$

Multiply both sides by 3

$$6x - 9y + 15 + 5x + 20y - 35 = 0$$

$$\boxed{11x + 11y - 20 = 0}$$

Q10. Check whether the following lines are concurrent or not. If concurrent, then find the point of concurrency $x + 2y - 7 = 0; 3x - y - 4 = 0; 2x + 35y - 145 = 0$

Solution:

Let $\ell_1: x + 2y - 7 = 0$

$\ell_2: 3x - y - 4 = 0$

$\ell_3: 2x + 35y - 145 = 0$

Consider:

$$\begin{vmatrix} 1 & 2 & -7 \\ 3 & -1 & -4 \\ 2 & 35 & -145 \end{vmatrix} = 1\begin{vmatrix} -1 & -4 \\ 35 & -145 \end{vmatrix} - 2\begin{vmatrix} 3 & -4 \\ 2 & -145 \end{vmatrix} - 7\begin{vmatrix} 3 & -1 \\ 2 & 35 \end{vmatrix}$$

$$= 1(145 + 140) - 2(-435 + 8) - 7(105 + 2)$$

$$= 1(285) - 2(-427) - 7(107)$$

$$= 285 + 854 - 749$$

$$= 390 \neq 0$$

Thus, the lines are not concurrent.

Q11. Find the condition that the lines $y = m_1x + c_1; y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent.

Solution:

Let $\ell_1: y = m_1x + c_1 \Rightarrow m_1x - y + c_1 = 0$

$\ell_2: y = m_2x + c_2 \Rightarrow m_2x - y + c_2 = 0$

$\ell_3: y = m_3x + c_3 \Rightarrow m_3x - y + c_3 = 0$

If the lines are concurrent, then

$$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix} = 0$$

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Expanding by R_1, we get

$$m_1\begin{vmatrix}-1 & c_2\\ -1 & c_3\end{vmatrix}-(-1)\begin{vmatrix}m_2 & c_2\\ m_3 & c_3\end{vmatrix}+c_1\begin{vmatrix}m_2 & -1\\ m_3 & -1\end{vmatrix}=0$$

$$m_1(-c_3+c_2)+1(m_2c_3-m_3c_2)+c_1(-m_2+m_3)=0$$

$$-m_1c_3+m_1c_2+m_2c_3-m_3c_2-m_2c_1+m_3c_1=0$$

$$m_1c_2-m_2c_1+m_2c_3-m_3c_2+m_3c_1-m_1c_3=0$$

(Required condition)

Q12. Find the value of k if the lines $x+y-4=0$, $2x-y+1=0$ and $kx+2y-10=0$ are concurrent.

Solution:

Let $\ell_1: x+y-4=0$

$\ell_2: 2x-y+1=0$

$\ell_3: kx+2y-10=0$

If the lines are concurrent, then

$$\begin{vmatrix}1 & 1 & -4\\ 2 & -1 & 1\\ k & 2 & -10\end{vmatrix}=0$$

Expanding by R_1, we have

$$1\begin{vmatrix}-1 & 1\\ 2 & -10\end{vmatrix}-1\begin{vmatrix}2 & 1\\ k & -10\end{vmatrix}-4\begin{vmatrix}2 & -1\\ k & 2\end{vmatrix}=0$$

$$1(10-2)-1(-20-k)-4(4+k)=0$$

$$8+20+k-16-4k=0$$

$$-3k+12=0$$

$$\Rightarrow 3k=12 \Rightarrow k=4$$

Q13. If the lines $2x+3y-7=0$, $4x-y-5=0$ and $6x+ay-9=0$ meet at a point, then find the value of a.

Solution:

Let $\ell_1: 2x+3y-7=0$

$\ell_2: 4x-y-5=0$

$\ell_3: 6x+ay-9=0$

If the lines meet at a point (concurrent), then

$$\begin{vmatrix}2 & 3 & -7\\ 4 & -1 & -5\\ 6 & a & -9\end{vmatrix}=0$$

Expanding by R_1, we have

$$2\begin{vmatrix}-1 & -5\\ a & -9\end{vmatrix}-3\begin{vmatrix}4 & -5\\ 6 & -9\end{vmatrix}-7\begin{vmatrix}4 & -1\\ 6 & a\end{vmatrix}=0$$

$$2(9+5a)-3(-36+30)-7(4a+6)=0$$

$$18+10a-3(-6)-28a-42=0$$

$$-18a-6=0 \Rightarrow 18a=-6 \Rightarrow a=\frac{-6}{18}$$

$$\boxed{a=-\frac{1}{3}}$$

Content Outline

- Angle Between Two Lines
- Area of a Triangular Region Whose Vertices are Given

ANGLE BETWEEN TWO LINES

Let ℓ_1 and ℓ_2 be two intersecting lines ($\ell_1 \not\parallel \ell_2$), which meet at a point P. At the point P two supplementary angles are formed by the lines ℓ_1 and ℓ_2.

Unless $\ell_1 \perp \ell_2$, one of the two angles is acute. The angle from ℓ_1 to ℓ_2 is the angle θ through which ℓ_1 is rotated counterclockwise about the point P so that it coincides with ℓ_2.

In the figure below, θ is the angle of intersection. It is clear that the inclination of a line is the angle measured in the counterclockwise direction from the positive x-axis to the line.

Theorem 7: Let ℓ_1 and ℓ_2 be two non-vertical lines such that they are not perpendicular to each other. If m_1 and m_2 are the slopes of ℓ_1 and ℓ_2 respectively, then the angle θ from ℓ_1 to ℓ_2 is given by $\tan\theta=\frac{m_2-m_1}{1+m_1m_2}$.

Proof:

Sum of interior angles of a triangle = π

$\Rightarrow \quad \alpha_1+\theta+\pi-\alpha_2=\pi$

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$$\alpha_1 + \theta + \pi - \alpha_2 = 0$$

$$\theta = \alpha_2 - \alpha_1$$

By taking 'tan' on both sides

$$\tan\theta = \tan(\alpha_2 - \alpha_1)$$

$$\tan\theta = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_2 . \tan\alpha_1}$$

$$\boxed{\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}} \quad \because \begin{cases} \text{Slope of } \ell_1 = m_1 \\ \text{Slope of } \ell_2 = m_2 \end{cases}$$

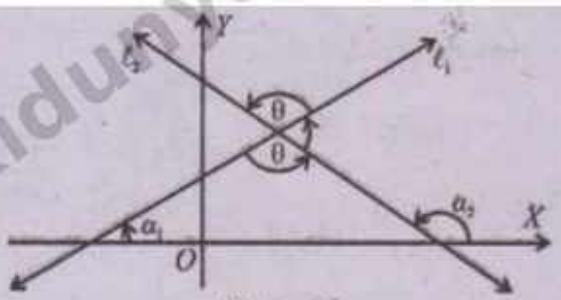

Figure 5.8

Corollary 1:

$\ell_1 \parallel \ell_2$ if and only if $m_1 = m_2$

Proof:

$\ell_1 \parallel \ell_2 \Leftrightarrow \theta = 0$

$\Leftrightarrow \quad \tan\theta = \tan(0)$

$\Leftrightarrow \frac{m_2 - m_1}{1 + m_2 m_1} = 0$

$\Leftrightarrow \quad m_2 - m_1 = 0$

$\Leftrightarrow \quad m_1 = m_2$

Corollary 2:

$\ell_1 \perp \ell_2$ if and only if $1 + m_1 m_2 = 0$

Proof:

$\ell_1 \perp \ell_2 \Leftrightarrow \theta = 90°$

$\Leftrightarrow \quad \tan\theta = \tan 90°$

$\Leftrightarrow \frac{m_2 - m_1}{1 + m_2 m_1} = \infty$

$\Leftrightarrow \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{1}{0}$

$\Leftrightarrow \boxed{1 + m_2 m_1 = 0}$

Example 12: Find the angle from the line with slope $-\frac{7}{3}$ to the line with slope $\frac{5}{2}$.

Solution:

Here $m_1 = -\frac{7}{3}$ and $m_2 = \frac{5}{2}$

Let θ be the required angle, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$\tan\theta = \frac{\frac{5}{2} - \left(-\frac{7}{3}\right)}{1 + \left(\frac{5}{3}\right)\left(-\frac{7}{3}\right)}$$

$$\tan\theta = \frac{\frac{15 + 14}{6}}{\frac{6 - 35}{6}}$$

$$\tan\theta = \frac{29}{-29} = -1 \qquad (\theta \text{ lies in } 2^{nd} \text{ quadrant})$$

$\Rightarrow \quad \theta = 180° - \tan^{-1}(1)$

$\theta = 180° - 45°$

$\boxed{\theta = 135°}$

Example 13: Find the angles of the triangle whose vertices are $A(1,2)$, $B(7,3)$ and $C(4,9)$.

Solution:

Given that: $A(1,2)$, $B(7,3)$, $C(4,9)$

Slope of $\overline{AB} = m_c = \frac{3-2}{7-1} = \frac{1}{6}$

Slope of $\overline{BC} = m_a = \frac{9-3}{4-7} = \frac{6}{-3} = -2$

Slope of $\overline{AC} = m_b = \frac{9-2}{4-1} = \frac{7}{3}$

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The angle A measured from $\overline{AB}$ to $\overline{AC}$

$$\tan A = \frac{m_b - m_c}{1 + m_b m_c} = \frac{\frac{7}{3} - \frac{1}{6}}{1 + \left(\frac{7}{3}\right)\left(\frac{1}{6}\right)} = \frac{\frac{14-1}{6}}{\frac{18+7}{18}}$$

$$\tan A = \frac{13}{6} \times \frac{18}{25} = \frac{39}{25}$$

$$\Rightarrow \quad A = \tan^{-1}\left(\frac{39}{25}\right) \Rightarrow \boxed{A = 57.34^\circ}$$

The angle B is measured from $\overline{BC}$ to $\overline{AB}$

$$\tan B = \frac{m_c - m_a}{1 + m_c m_a} = \frac{\frac{1}{6} - (-2)}{1 + \left(\frac{1}{6}\right)(-2)} = \frac{\frac{1+12}{6}}{\frac{6-2}{6}}$$

$$\tan B = \frac{13}{4} \Rightarrow B = \tan^{-1}\left(\frac{13}{4}\right) \Rightarrow \boxed{B = 72.90^\circ}$$

The angle C is measured from $\overline{AC}$ to $\overline{BC}$

$$\tan C = \frac{m_a - m_b}{1 + m_a m_b} = \frac{-2 - \frac{7}{3}}{1 + (-2)\left(\frac{7}{3}\right)} = \frac{\frac{-6-7}{3}}{\frac{3-14}{3}}$$

$$\tan C = \frac{-13}{-11} \Rightarrow C = \tan^{-1}\left(\frac{13}{11}\right) \Rightarrow \boxed{C = 49.76^\circ}$$

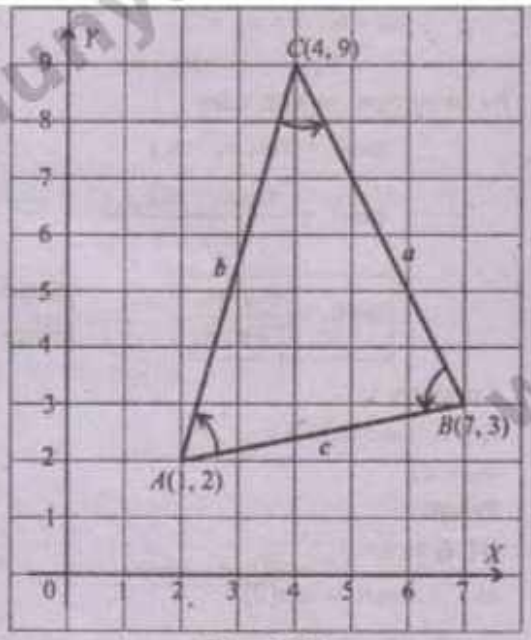

Figure 5.9

AREA OF A TRIANGULAR REGION WHOSE VERTICES ARE GIVEN

Theorem: To find the area of triangular region whose vertices are: $P(x_1, y_1), Q(x_2, y_2)$ and $R(x_3, y_3)$.

Proof:

Given that: $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$

Draw perpendicular $\overline{PL}, \overline{QN}$ and $\overline{RM}$ on x axis.

Area of triangular region PQR = Area of trapezoidal region $PLMR$ + Area of trapezoidal region $RMNQ$ − Area of trapezoidal region $PLNQ$

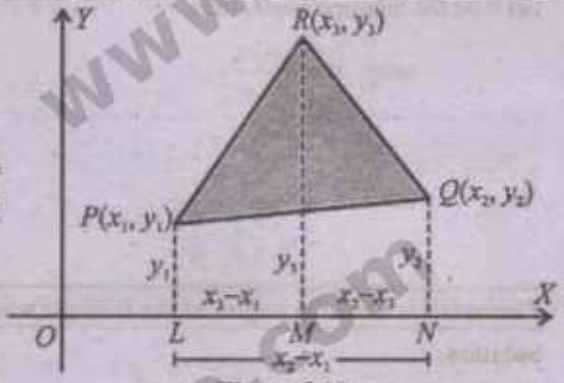

Figure 5.10

$$= \frac{1}{2}(|\overline{PL}| + |\overline{RM}|)(|\overline{LM}|) + \frac{1}{2}(|\overline{RM}| + |\overline{QN}|)(|\overline{MN}|) - \frac{1}{2}(|\overline{PL}| + |\overline{QN}|)(|\overline{LN}|)$$

$$= \frac{1}{2}[(y_1 + y_3)(x_3 - x_1) + (y_3 + y_2)(x_2 - x_3) - (y_1 + y_2)(x_2 - x_1)]$$

$$= \frac{1}{2}(x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3 + x_2y_3 - x_3y_3 + x_2y_2 - x_3y_2 - x_2y_1 + x_1y_1 - x_2y_2 + x_1y_2)$$

$$= \frac{1}{2}(x_3y_1 - x_1y_3 + x_2y_3 - x_3y_2 - x_2y_1 + x_1y_2)$$

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$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

By taking transpose, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$ (Proved) $\because |A| = |A^t|$

Corollary: If the points P, Q and R are collinear, then $\Delta = 0$

Note

A trapezium is a quadrilateral having two parallel and two non-parallel sides.

Area of trapezoidal region $= \frac{1}{2}$ (sum of || sides) (distance between || sides).

Note

Although the formula for finding area of triangle is proven for the vertices taken in Q-1. It holds for the vertices taken everywhere on xy-plane. Since the area is always non-negative, the absolute value of the determinant is considered as the area.

Example 14: Find the area of the region bounded by the triangle with vertices $A(a, b)$, $B(-a, b)$ and $C(0, -b)$.

Solution:

Given that: $A(a, b)$, $B(-a, b)$, $C(0, b)$

Area of triangular region is

$$\Delta = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ -a & b & 1 \\ 0 & -b & 1 \end{vmatrix}$$

Expanding by R_1, we get

$$= \frac{1}{2}\left\{a\begin{vmatrix} b & 1 \\ -b & 1 \end{vmatrix} - b\begin{vmatrix} -a & 1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} -a & b \\ 0 & -b \end{vmatrix}\right\}$$

$$= \frac{1}{2}\{a(b+b) - b(-a-0) + 1(ab-0)\}$$

$$= \frac{1}{2}(2ab + ab + ab)$$

$$= \frac{1}{2}(4ab)$$

$$\Delta = 2ab$$

Example 15: Find the area of the region bounded by the triangle with vertices $A(1, 2)$, $B(3, 6)$ and $C(5, 10)$.

Solution:

Given that: $A(1, 2)$, $B(3, 6)$, $C(5, 10)$

Area of triangle region ABC is

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 5 & 10 & 1 \end{vmatrix}$$

Taking 2 common from C_2

$$\Delta = \frac{1}{2}(2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3 & 1 \\ 5 & 5 & 1 \end{vmatrix}$$

$= 1(0)$ $\because C_1$ and C_2 are identical

$\Delta = 0$

Thus, the given points are collinear.

EXERCISE 5.2

Q1. Find the angle measured from the line ℓ_1 to the line ℓ_2, where

(i) ℓ_1: Joining $(2, 7)$ and $(-4, 5)$

ℓ_2: Joining $(-7, 2)$ and $(1, 3)$

Also find the acute angle in each case.

(ii) ℓ_1 : Joining $(1, 2)$ and $(3, 5)$

ℓ_2: Joining $(2, -1)$ and $(4, 3)$

Solution:

(i) ℓ_1: Joining $(2, 7)$ and $(-4, 5)$

Slope of ℓ_1: $m_1 = \frac{5-7}{-4-2} = \frac{-2}{-6} = \frac{1}{3}$

ℓ_2: Joining $(-7, 2)$ and $(1, 3)$

Slope of $\ell_2 = m_2 = \frac{3-2}{1+7} = \frac{1}{8}$

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Let θ be the angle from ℓ_1 to ℓ_2, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{\frac{1}{8} - \frac{1}{3}}{1 + \frac{1}{8}\cdot\frac{1}{3}} = \frac{\frac{3-8}{24}}{\frac{24+1}{24}} = -\frac{5}{25}$$

$$\tan\theta = \frac{-1}{5} \qquad (\theta \text{ lies in } 2^{nd} \text{ quadrant})$$

$$\theta = 180° - \tan^{-1}\left(\frac{1}{5}\right)$$

$$\theta = 180° - 11.31°$$

$$\boxed{\theta = 168.69°}$$

Acute angle:

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right| = \frac{1}{5}$$

$$\theta = \tan^{-1}\left(\frac{1}{5}\right)$$

$$\boxed{\theta = 11.31°}$$

(ii) ℓ_1: Joining (1, 2) and (3, 5)

Slope of $\ell_1 = m_1 = \frac{5-2}{3-1} = \frac{3}{2}$

ℓ_2: Joining (2, −1) and (4, 3)

Slope of $\ell_2 = m_2 = \frac{3+1}{4-2} = \frac{4}{2} = 2$

Let θ be the angle from ℓ_1 to ℓ_2, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{2 - \frac{3}{2}}{1 + 2\left(\frac{3}{2}\right)} = \frac{\frac{4-3}{2}}{1+3} = \frac{1}{8}$$

$$\theta = \tan^{-1}\left(\frac{1}{8}\right) \Rightarrow \boxed{\theta = 7.13°} \text{ Acute angle}$$

Q2. Find the angle from the line with slope m_1 to the line with slope m_2, where

(i) $m_1 = 2, m_2 = -\frac{1}{2}$

Solution:

Here $m_1 = 2, m_2 = \frac{-1}{2}$

Let θ be the required angle, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{\frac{-1}{2} - 2}{1 + \left(\frac{-1}{2}\right)(2)} = \frac{\frac{-1-4}{2}}{1-1} = \frac{-5}{0} = \infty$$

$$\theta = \tan^{-1}(\infty) \Rightarrow \boxed{\theta = 90°}$$

(ii) $m_1 = 3, m_2 = -2$

Solution:

Here $m_1 = 3, m_2 = -2$

Let θ be the required angle, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{-2-3}{1 + (-2)(3)}$$

$$= \frac{-5}{-5} = 1$$

$$\theta = \tan^{-1}(1) \Rightarrow \boxed{\theta = 45°}$$

Q3. Find the angle from line ℓ_1 to the line ℓ_2, where

(i) $\ell_1 : 2x + y - 3 = 0; \ell_2 : x + y - 5 = 0$

Solution:

ℓ_1: $2x + y - 3 = 0 \Rightarrow$ Slope of $\ell_1 = m_1 = \frac{-2}{1} = -2$

ℓ_2: $x + y - 5 = 0 \Rightarrow$ Slope of $\ell_2 = m_2 = \frac{-1}{1} = -1$

Let θ be the angle from ℓ_1 to ℓ_2, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$\tan\theta = \frac{-1-(-2)}{1+(-1)(-2)} = \frac{-1+2}{1+2} = \frac{1}{3}$$

$$\theta = \tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \boxed{\theta = 18.43°}$$

(ii) $\ell_1 : 2x - 3y + 2 = 0; \ell_2 : x + y - 4 = 0$

Solution:

ℓ_1: $2x - 3y + 2 = 0 \Rightarrow$ Slope of $\ell_1 = m_1 = \frac{2}{-3} = \frac{2}{3}$

ℓ_2: $x + y - 4 = 0 \Rightarrow$ Slope of $\ell_2 = m_2 = \frac{1}{1} = -1$

Let θ be the angle from ℓ_1 to ℓ_2, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

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$$\tan\theta = \frac{-1-\frac{2}{3}}{1+(-1)\left(\frac{2}{3}\right)} = \frac{\frac{-3-2}{3}}{\frac{3-2}{3}} = \frac{-5}{1}$$

$\tan\theta = -5$ (θ lies in 2nd quadrant)

$\theta = 180° - \tan^{-1}(5) = 180° - 78.69°$

$\boxed{\theta = 101.31°}$

Q4 The angle from the line $ax - y + 2 = 0$ to the line $x + y - 4 = 0$ is 30°. Find the value of a.

Solution:

Let ℓ_1: $ax - y + 2 = 0 \Rightarrow$ Slope of $\ell_1 = m_1 = -\frac{a}{-1} = a$

ℓ_2: $x + y - 4 = 0 \Rightarrow$ Slope of $\ell_2 = m_2 = -\frac{1}{1} = -1$

Let θ be the angle from ℓ_1 to ℓ_2, then

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$$

$$\tan 30° = \left|\frac{-1-a}{1+(-1)a}\right| \qquad \because \theta = 30°$$

$$\frac{1}{\sqrt{3}} = \left|\frac{-(1+a)}{1-a}\right|$$

$$\frac{1+a}{1-a} = \frac{1}{\sqrt{3}}$$

Squaring both sides

$$\left(\frac{1+a}{1-a}\right)^2 = \left(\frac{1}{\sqrt{3}}\right)^2$$

$$\frac{1+a^2+2a}{1+a^2-2a} = \frac{1}{3}$$

$$3 + 3a + 6a = 1 + a^2 - 2a$$

$$3a^2 - a^2 + 6a + 2a + 3 - 1 = 0$$

$$2a^2 + 8a + 2 = 0$$

$$2(a^2 + 4a + 1) = 0$$

$$a^2 + 4a + 1 = 0$$

Here $A = 1, B = 4, C = 1$

By using quadratic formula

$$a = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = \frac{2(-2 \pm \sqrt{3})}{2}$$

$$a = -2 \pm \sqrt{3}$$

$\Rightarrow$ $a = -2 + \sqrt{3}, -2 - \sqrt{3}$

Q5. Find the interior angles of the triangle whose vertices are:

(i) $A(2,3), B(6,3), C(4,7)$

Solution:

Given that: $A(2,3), B(6,3), C(4,7)$

Slope of $\overline{AB} = m_c = \frac{3-3}{6-2} = \frac{0}{4} = 0$

Slope of $\overline{BC} = m_a = \frac{7-3}{4-6} = \frac{4}{-2} = -2$

Slope of $\overline{AC} = m_b = \frac{7-3}{4-2} = \frac{4}{2} = 2$

The angle A is measured from $\overline{AB}$ to $\overline{AC}$

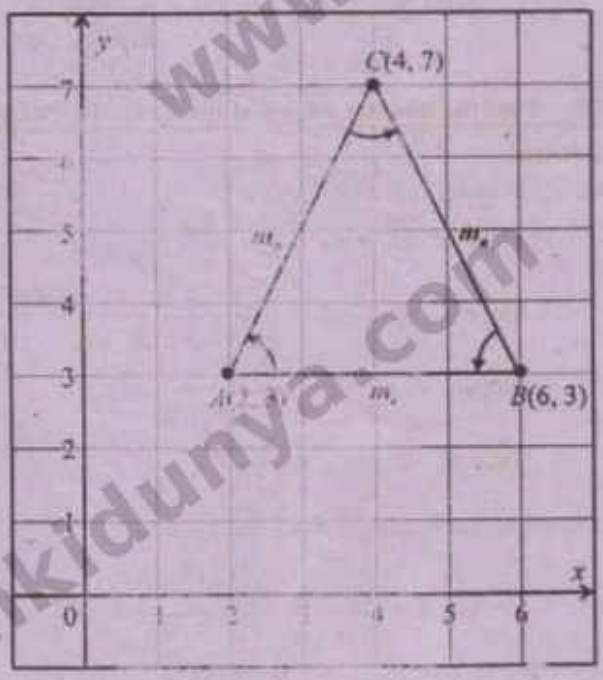
$$\tan A = \frac{m_b - m_c}{1 + m_b m_c} = \frac{2-0}{1+2(0)} = \frac{2}{1} = 2$$

$A = \tan^{-1}(2) \Rightarrow \boxed{A = 63.43°}$

The angle B is measured from $\overline{BC}$ to $\overline{AB}$

$$\tan B = \frac{m_c - m_a}{1 + m_c m_a} = \frac{0-(-2)}{1+(0)(-2)} = \frac{2}{1} = 2$$

$B = \tan^{-1}(2) \Rightarrow \boxed{B = 63.43°}$



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The angle C is measured from $\overline{AC}$ to $\overline{BC}$

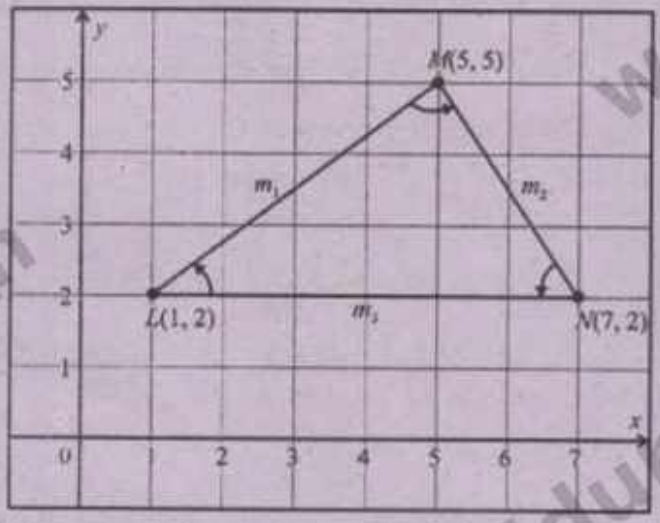
$$\tan C = \frac{m_a - m_b}{1 + m_a m_b} = \frac{-2-2}{1+(-2)(-2)} = \frac{-4}{-3} = \frac{4}{3}$$

$$C = \tan^{-1}\left(\frac{4}{3}\right) \Rightarrow \boxed{C = 53.13^\circ}$$

(ii) $L(1,2), M(5,5), N(7,2)$

Solution:

Given that: $L(1,2)$, $M(5,5)$, $N(7,2)$



$$\text{Slope of } \overline{LM} = m_1 = \frac{5-2}{5-1} = \frac{3}{4}$$

$$\text{Slope of } \overline{MN} = m_2 = \frac{2-5}{7-5} = -\frac{3}{2}$$

$$\text{Slope of } \overline{LN} = m_3 = \frac{2-2}{7-1} = \frac{0}{6} = 0$$

The angle L is measured from $\overline{LN}$ to $\overline{LM}$

$$\tan L = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{\frac{3}{4} - 0}{1 + \frac{3}{4}(0)} = \frac{3}{4}$$

$$L = \tan^{-1}\left(\frac{3}{4}\right) \Rightarrow \boxed{L = 36.87^\circ}$$

The angle M is measured from $\overline{LM}$ to $\overline{MN}$

$$\tan M = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-\frac{3}{2} - \frac{3}{4}}{1 + \left(-\frac{3}{2}\right)\left(\frac{3}{4}\right)} = \frac{\frac{-6-3}{4}}{\frac{8-9}{8}}$$

$$\tan M = -\frac{9}{4} \times \frac{8}{-1} \Rightarrow \tan M = 18$$

$$M = \tan^{-1}(18) \Rightarrow \boxed{M = 86.82^\circ}$$

The angle N is measured from $\overline{MN}$ to $\overline{LN}$

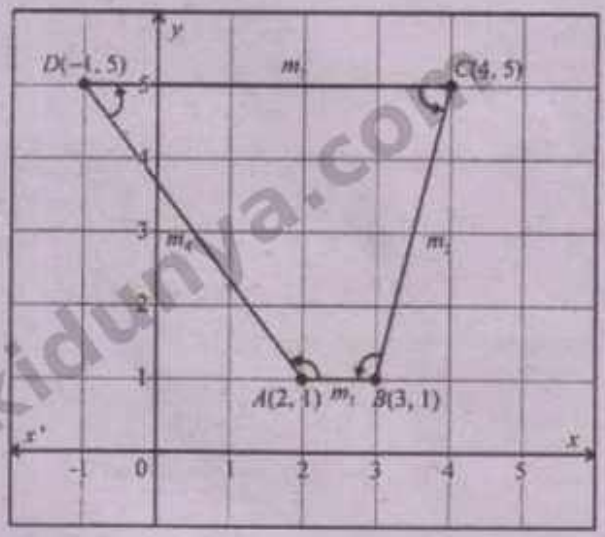
$$\tan N = \frac{m_3 - m_2}{1 + m_3 m_2} = \frac{0 - \left(-\frac{3}{2}\right)}{1 + 0\left(-\frac{3}{2}\right)} = \frac{\frac{3}{2}}{1} = \frac{3}{2}$$

$$N = \tan^{-1}\left(\frac{3}{2}\right) \Rightarrow \boxed{N = 56.31^\circ}$$

Q6. Find the interior angles of the quadrilateral whose vertices are $A(2,1), B(3,1), C(4,5)$ and $D(-1,5)$.

Solution:

Given that: $A(2,1)$, $B(3,1)$, $C(4,5)$, $D(-1,5)$



$$\text{Slope of } \overline{AB} = m_1 = \frac{1-1}{3-2} = \frac{0}{1} = 0$$

$$\text{Slope of } \overline{BC} = m_2 = \frac{5-1}{4-3} = \frac{4}{1} = 4$$

$$\text{Slope of } \overline{CD} = m_3 = \frac{5-5}{-1-4} = \frac{0}{-5} = 0$$

$$\text{Slope of } \overline{AD} = m_4 = \frac{5-1}{-1-2} = -\frac{4}{3}$$

The angle A is measured from $\overline{AB}$ to $\overline{AD}$

$$\tan A = \frac{m_4 - m_1}{1 + m_4 m_1} = \frac{-\frac{4}{3} - 0}{1 + \left(-\frac{4}{3}\right)(0)} = -\frac{4}{3}$$

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$$A = 180° - \tan^{-1}\left(\frac{4}{3}\right)$$

$$\Rightarrow \quad = 180° - 53.13° \Rightarrow \boxed{A = 126.87°}$$

The angle B is measured from $\overline{BC}$ to $\overline{AB}$

$$\tan B = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{0-4}{1+0(4)} = -4$$

$$B = 180° - \tan^{-1}(4)$$

$$= 180° - 75.96°$$

$$\boxed{B = 104.04°}$$

The angle C is measured from $\overline{CD}$ to $\overline{BC}$

$$\tan C = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{4-0}{1+4(0)} = 4$$

$$C = \tan^{-1}(4) \Rightarrow \boxed{C = 75.96°}$$

The angle D is measured from $\overline{AD}$ to $\overline{CD}$

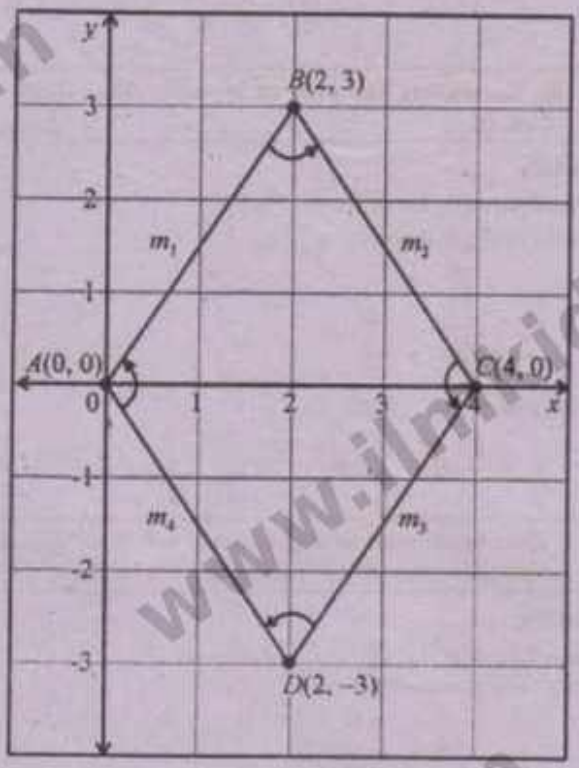
$$\tan D = \frac{m_3 - m_4}{1 + m_3 m_4} = \frac{0-\left(-\frac{4}{3}\right)}{1+0\left(-\frac{4}{3}\right)} = \frac{4}{3}$$

$$D = \tan^{-1}\left(\frac{4}{3}\right) \Rightarrow \boxed{D = 53.13°}$$

Q7. Find the interior angles of the kite whose vertices are $A(0,0), B(2,3), C(4,0)$ and $D(2,-3)$.

Solution:

Given that: $A(0, 0)$, $B(2, 3)$, $C(4, 0)$, $D(2, -3)$



$$\text{Slope of } \overline{AB} = m_1 = \frac{3-0}{2-0} = \frac{3}{2}$$

$$\text{Slope of } \overline{BC} = m_2 = \frac{0-3}{4-3} = -\frac{3}{2}$$

$$\text{Slope of } \overline{CD} = m_3 = \frac{-3-0}{2-4} = \frac{-3}{-2} = \frac{3}{2}$$

$$\text{Slope of } \overline{AD} = m_4 = \frac{-3-0}{2-0} = -\frac{3}{2}$$

The angle A is measured from $\overline{AD}$ to $\overline{AB}$

$$\tan A = \frac{m_1 - m_4}{1 + m_1 m_4} = \frac{\frac{3}{2} - \left(-\frac{3}{2}\right)}{1 + \left(\frac{3}{2}\right)\left(-\frac{3}{2}\right)} = \frac{\frac{3+3}{2}}{\frac{4-9}{4}}$$

$$\tan A = \frac{6}{2} \times \frac{4}{-5} = -\frac{12}{5}$$

$$A = 180° - \tan^{-1}\left(\frac{12}{5}\right)$$

$$\Rightarrow \quad = 180° - 67.38° \Rightarrow \boxed{A = 112.62°}$$

The angle B is measured from $\overline{AB}$ to $\overline{BC}$

$$\tan B = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{\frac{-3}{2} - \frac{3}{2}}{1 + \left(\frac{-3}{2}\right)\left(\frac{3}{2}\right)} = \frac{\frac{-3-3}{2}}{\frac{4-9}{4}}$$

$$\tan B = \frac{-6}{2} \times \frac{4}{-5} = \frac{12}{5}$$

$$B = \tan^{-1}\left(\frac{12}{5}\right) \Rightarrow \boxed{B = 67.38°}$$

The angle C is measured from $\overline{BC}$ to $\overline{CD}$

$$\tan C = \frac{m_3 - m_2}{1 + m_3 m_2} = \frac{\frac{3}{2} - \left(-\frac{3}{2}\right)}{1 + \left(\frac{3}{2}\right)\left(-\frac{3}{2}\right)} = \frac{\frac{3+3}{2}}{\frac{4-9}{4}}$$

$$\tan C = \frac{6}{2} \times \frac{4}{-5} = -\frac{12}{5}$$

$$C = 180° - \tan^{-1}\left(\frac{12}{5}\right)$$

$$\Rightarrow \quad C = 180° - 67.38° \Rightarrow \boxed{C = 112.62°}$$

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The angle D is measured from $\overline{CD}$ to $\overline{AD}$

$$\tan D = \frac{m_4 - m_3}{1 + m_4 m_3} = \frac{-\frac{3}{2} - \frac{3}{2}}{1 + \left(-\frac{3}{2}\right)\left(\frac{3}{2}\right)} = \frac{\frac{-3-3}{2}}{\frac{4-9}{4}}$$

$$= -\frac{6}{2} \times \frac{4}{-5} = \frac{12}{5}$$

$$D = \tan^{-1}\left(\frac{12}{5}\right) \Rightarrow \boxed{D = 67.38^\circ}$$

Q8. Find the area of the triangle formed by the points $A(-3,2), B(4,-1)$ and $C(1,5)$.

Solution:

Given that: $A(-3, 2), B(4, -1), C(1, 5)$

Area of triangular region ABC is

$$\Delta = \frac{1}{2}\begin{vmatrix} -3 & 2 & 1 \\ 4 & -1 & 1 \\ 1 & 5 & 1 \end{vmatrix}$$

Expanding by R_1, we get

$$\Delta = \frac{1}{2}\left\{-3\begin{vmatrix} -1 & 1 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & -1 \\ 1 & 5 \end{vmatrix}\right\}$$

$$= \frac{1}{2}\{-3(-1-5) - 2(4-1) + (20+1)\}$$

$$= \frac{1}{2}\{-3(-6) - 2(3) + 1(21)\}$$

$$= \frac{1}{2}(18 - 6 + 21) = \frac{1}{2}(33)$$

$\boxed{\Delta = 16.5}$ sq. unit

Q9. By computing the area of triangle ABC check whether the points are collinear where $A(0,4), B(3,2)$ and $C(6,0)$.

Solution:

Given that: $A(0, 4), B(3, 2), C(6, 0)$

Area of triangular region is

$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & 4 & 1 \\ 3 & 2 & 1 \\ 6 & 0 & 1 \end{vmatrix}$$

Expanding by R_1, we get

$$= \frac{1}{2}\left\{0\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - 4\begin{vmatrix} 3 & 1 \\ 6 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 6 & 0 \end{vmatrix}\right\}$$

$$= \frac{1}{2}\{0 - 4(3-6) + 1(0-12)\}$$

$$= \frac{1}{2}\{0 - 4(-3) + 1(-12)\}$$

$$\Delta = \frac{1}{2}(12 - 12) = 0$$

Since $\Delta = 0$, so the points A, B and C are collinear.

Q10. The coordinates of the points are $A(0,\lambda), B(-2,1)$ and $C(-3,-2)$. By computing the area bounded by ABC, find the value of λ if points are collinear.

Solution:

Given that: $A(0, \lambda), B(-2, 1), C(-3, -2)$

Area of triangular region is

$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & \lambda & 1 \\ -2 & 1 & 1 \\ -3 & -2 & 1 \end{vmatrix}$$

Expanding by R_1, we get

$$\Delta = \frac{1}{2}\left\{0\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} - \lambda\begin{vmatrix} -2 & 1 \\ -3 & 1 \end{vmatrix} + 1\begin{vmatrix} -2 & 1 \\ -3 & -2 \end{vmatrix}\right\}$$

$$= \frac{1}{2}\{0 - \lambda(-2+3) + 1(4+3)\}$$

$$\Delta = \frac{1}{2}\{-\lambda + 7\}$$

If the points are collinear, then

$$\Delta = 0$$

$$\Rightarrow \quad \frac{1}{2}(-\lambda + 7) = 0$$

$$-\lambda + 7 = 0 \Rightarrow \boxed{\lambda = 7}$$

Content Outline

- Homogeneous Equation of the Second Degree in Two Variables
- Acute Angle between the Lines
- Application of Analytical Geometry to Real Life

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HOMOGENEOUS EQUATION OF THE SECOND DEGREE IN TWO VARIABLES

We know that if a graph is a straight line in a plane, then its equation is a linear equation in two variables x and y. Conversely, the graph of any linear equation in x and y is a straight line.

Suppose we have two straight lines represented by

$$a_1x + b_1y + c_1 = 0 \quad \ldots(i)$$

And

$$a_2x + b_2y + c_2 = 0 \quad \ldots(ii)$$

Multiplying equations (i) and (ii), we have

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0 \quad \ldots(iii)$$

It is a second degree equation in x and y.

Equation (iii) is called **joint equation** of the pair of lines (i) and (ii). On the other hand, given an equation of second degree in x and y, say

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots(iv)$$

where $a^2 + h^2 + b^2 \neq 0$ represents equations of a pair of lines if (iv) can be resolved into two linear factors. In this section, we shall study special joint equations of pairs of lines which pass through the origin.

Let $y = m_1x$ and $y = m_2x$ be two lines passing through the origin. Their joint equation is:

$$(y - m_1x)(y - m_2x) = 0$$

Or

$$y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0 \quad \ldots(v)$$

Equation (v) is a special type of a second degree homogeneous equation.

Homogeneous Equation

Definition: An equation of the form $f(x, y) = 0$ is called a homogeneous equation of degree n if the function $f(x, y)$ is homogeneous of degree n, that is, $f(kx, ky) = k^n f(x, y)$ for some real number k.

For example, in equation (v) above if we replace x and y by kx and ky respectively, we have:

$$f(kx, ky) = k^2y^2 - k^2(m_1 + m_2)xy + k^2m_1m_2x^2$$
$$= k^2[y^2 - (m_1 + m_2)xy + m_1m_2x^2] = 0 \Rightarrow f(kx, ky) = k^2f(x, y)$$

Thus (v) is a homogeneous equation of degree 2.

Definition: A general second degree homogeneous equation can be written as:

$$ax^2 + 2hxy + by^2 = 0$$

provided a, h and b are not simultaneously zero.

Theorem 8: If $b \neq 0$, then every homogeneous second degree equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through the origin. The lines are

(i) real and distinct, if $h^2 > ab$ (ii) real and coincident, if $h^2 = ab$ (iii) imaginary, if $h^2 < ab$

Proof:

Given that: $ax^2 + 2hxy + by^2 = 0 \quad \ldots(1)$

$abx^2 + 2bhxy + b^2y^2 = 0$ Multiply by 'b'

$(by)^2 + (2hx)by + abx^2 = 0$

Add and subtract $(hx)^2$

$$(by)^2 + (2hx)by + (hx)^2 - (hx)^2 + abx^2 = 0$$
$$(by + hx)^2 - x^2(h^2 - ab) = 0$$
$$(by + hx)^2 - (x\sqrt{h^2 - ab})^2 = 0$$
$$(by + hx + x\sqrt{h^2 - ab})(by + hx - x\sqrt{h^2 - ab}) = 0$$

$$by + x(h + \sqrt{h^2 - ab}) = 0 \quad \ldots(2)$$

and $by + x(x - \sqrt{h^2 - ab}) = 0 \quad \ldots(3)$

As $O(0, 0)$ satisfy (2) and (3), so these are required pair of lines passing through origin.

From eq. (2) and eq. (3), it is clear that these lines are

(i) Real and distinct if $h^2 - ab > 0 \Rightarrow h^2 > ab$

(ii) Real and coincident if $h^2 - ab = 0 \Rightarrow h^2 = ab$

(iii) Imaginary, if $h^2 - ab < 0 \Rightarrow h^2 = ab$

Critical Thinking

Theorem 8 is also true when $b = 0$. You can prove it with the help of your instructor. You may face several cases.

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Example 16: Find an equation of each of the lines represented by $3x^2 - 10xy + 3y^2 = 0$

Solution:

$$\begin{aligned} 3x^2 - 10xy + 3y^2 &= 0 \\ 3x^2 - 9xy - xy + 3y^2 &= 0 \\ 3x(x-3y) - y(x-3y) &= 0 \end{aligned}$$

$$\Rightarrow (x-3y)(3x-y) = 0$$

$\Rightarrow$ $x - 3y = 0$ and $3x - y = 0$ are required lines.

ACUTE ANGLE BETWEEN THE LINES

Theorem: The measure of angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by: $\tan\theta = \dfrac{2\sqrt{h^2 - ab}}{a+b}$

Proof:

Given that: $ax^2 + 2hxy + by^2 = 0$

$\Rightarrow$ $by^2 + 2hxy + ax^2 = 0$

$b^2y^2 + (2hx)by + abx^2 = 0$ $\because$ Multiply by 'b'

Add and subtract h^2x^2

$$b^2y^2 + (2hx)by + h^2x^2 - h^2x^2 + abx^2 = 0$$

$$(by + hx)^2 - (h^2 - ab)x^2 = 0$$

$$(by + hx)^2 - (\sqrt{h^2 - ab}\,x)^2 = 0$$

$$(by + hx + \sqrt{h^2 - ab}\,x)(by + hx - \sqrt{h^2 - ab}\,x) = 0$$

$\Rightarrow$ $by + (h + \sqrt{h^2 - ab})x = 0$ and $by + (h - \sqrt{h^2 - ab})x = 0$

Let $\ell_1: (h + \sqrt{h^2 - ab})x + by = 0 \Rightarrow m_1 = \dfrac{h + \sqrt{h^2 - ab}}{b}$

$\ell_2: (h - \sqrt{h^2 - ab})x + by = 0 \Rightarrow m_2 = \dfrac{h - \sqrt{h^2 - ab}}{b}$

$$\Rightarrow m_1 + m_2 = \frac{h + \sqrt{h^2 - ab}}{b} - \frac{h - \sqrt{h^2 - ab}}{b} = \frac{-h - \sqrt{h^2 - ab} - h + \sqrt{h^2 - ab}}{b} = -\frac{2h}{b}$$

$$\text{and } m_1 m_2 = \left(\frac{h + \sqrt{h^2 - ab}}{b}\right)\left(\frac{h - \sqrt{h^2 - ab}}{b}\right) = \frac{h^2 - (\sqrt{h^2 - ab})^2}{b^2}$$

$$= \frac{h^2 - h^2 + ab}{b^2} = \frac{ab}{b^2} = \frac{a}{b}$$

Let θ be the angle between the lines ℓ_1 and ℓ_2, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1 - m_2)^2}}{1 + m_1 m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{\left(\frac{-2h}{b}\right)^2 - 4\frac{a}{b}}}{1 + \frac{a}{b}} = \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{\frac{b+a}{b}} = \frac{\sqrt{\frac{4h^2 - 4ab}{b^2}}}{\frac{a+b}{b}}$$

$$= \frac{2\sqrt{h^2 - ab}}{b} \cdot \frac{b}{a+b} = \frac{2\sqrt{h^2 - ab}}{a+b}, \text{ provided } h^2 - ab \geq 0$$

Critical Thinking

Formula (iv) is also true when $b = 0$. You can prove it with the help of your instructor. You may face several cases.

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For Acute Angle between the Lines:

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a+b|}$$

Special Cases:

- If two lines are parallel, then $\theta = 0$, so that $\tan\theta = 0$ this implies $h^2 - ab = 0$, which is the condition for the lines to be coincident.
- If the lines are perpendicular, then $\theta = 90°$, so that $\tan\theta$ is not defined. This implies $a + b = 0$.
- Hence, the condition for (i) to represent a pair of perpendicular lines is that sum of the coefficients of x^2 and y^2 is 0.

Example 17: Find measure of the angle between the lines represented by: $5x^2 + 6xy + y^2 = 0$.

Solution:

$$5x^2 + 6xy + y^2 = 0$$

Compare it with

$$ax^2 + 2hxy + by^2 = 0$$

$$\Rightarrow \quad a = 5,\ 2h = 6 \Rightarrow h = 3,\ b = 1$$

Let θ be the angle between the given lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\tan\theta = \frac{2\sqrt{3^2 - (5)(1)}}{5+1}$$

$$= \frac{2\sqrt{9-5}}{6} = \frac{\sqrt{4}}{3} = \frac{2}{3}$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right) \Rightarrow \boxed{\theta = 33.69°}$$

APPLICATION OF ANALYTICAL GEOMETRY TO REAL LIFE

Analytical geometry connects mathematics to the real life. It allows us to analyze positions, distances, slopes and angles making it essential in aviation, astronomy, engineering and many fields of science and technology.

Example 18: Planet A is at $(1, 0)$ AU and planet B is at $(-0.5, 0.866)$ AU. Find the distance between them. Write answer in km .

Solution:

Given that: $A(1, 0)$, $B(-0.5, 0.866)$

$$\text{Distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\Rightarrow \quad d = \sqrt{(1-(-0.5))^2 + (0-0.866)^2}$$

$$= \sqrt{(1.5)^2 + (-0.866)^2}$$

$$= \sqrt{2.25 + 0.75}$$

$$= \sqrt{3}$$

$$d \approx 1.732 \text{ AU}$$

$$d = 1.732 \times 149.6 \quad \because |AU| = 149.6 \text{ million km}$$

$$\boxed{d = 259.1 \text{ million km}}$$

Remember

An astronomical unit (AU) is a unit of length and exactly equal to 149597870700 m.

Example 19: Aircraft flies from $A(100, 200)$km to $B(500, 800)$km. Find slope of the track and straight line distance.

Solution:

Given that: $A(100, 200)$, $B(500, 800)$

$$\text{Slope of track} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{800-200}{500-100} = \frac{600}{400} = \frac{3}{2} = 1.5$$

$$\text{Distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\text{Distance} = \sqrt{(100-500)^2 + (200-800)^2}$$

$$= \sqrt{(-400)^2 + (-600)^2}$$

$$= \sqrt{160000 + 360000}$$

$$= \sqrt{520000}$$

$$= 721.11 \text{ km}$$

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EXERCISE 5.3

Find the equation of lines represented by each of the following and also find measure of the angle between them (Problem 1-5):

Q1. $6x^2 - 5xy - 4y^2 = 0$

Solution:

$$6x^2 - 5xy - 4y^2 = 0$$
$$6x^2 - 8xy + 3xy - 4y^2 = 0$$
$$2x(3x - 4y) + y(3x - 4y) = 0$$
$$(3x - 4y)(2x + y) = 0$$

$\Rightarrow$ $3x - 4y = 0$ and $2x + y = 0$ are required lines.

For angle:

$$6x^2 - 5xy - 4y^2 = 0$$

Compare it with

$$ax^2 + 2hxy + by^2 = 0$$

$\Rightarrow$ $a = 6, 2h = -5 \Rightarrow h = \frac{-5}{2}, b = -4$

Let θ be the angle between given lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\tan\theta = \frac{2\sqrt{\left(\frac{-5}{2}\right)^2 - (6)(-4)}}{6 + (-4)} = \frac{2\sqrt{\frac{25}{4} + 24}}{2} = \frac{\sqrt{25 + 96}}{4}$$

$$\tan\theta = \sqrt{\frac{121}{4}} = \frac{11}{2}$$

$$\theta = \tan^{-1}\left(\frac{11}{2}\right) \Rightarrow \boxed{\theta = 79.7^\circ}$$

Q2. $2x^2 - 7xy + 3y^2 = 0$

Solution:

$$2x^2 - 7xy + 3y^2 = 0$$
$$2x^2 - 6xy - xy + 3y^2 = 0$$
$$2x(x - 3y) - y(x - 3y) = 0$$
$$(x - 3y)(2x - y) = 0$$

$\Rightarrow$ $x - 3y = 0$ and $2x - y = 0$ are required lines.

For angle:

$$2x^2 - 7xy + 3y^2 = 0$$

Compare it with

$$ax^2 + 2hxy + by^2 = 0$$

$\Rightarrow$ $a = 2, 2h = -7 \Rightarrow h = \frac{-7}{2}, b = 3$

Let θ be the angle between given lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\tan\theta = \frac{2\sqrt{\left(\frac{-7}{2}\right)^2 - (2)(3)}}{2 + 3} = \frac{2\sqrt{\frac{49}{4} - 6}}{5}$$

$$\tan\theta = \frac{2\sqrt{\frac{49 - 24}{4}}}{5} = \frac{2}{5}\sqrt{\frac{25}{4}} = \frac{2}{5}\cdot\frac{5}{2} = 1$$

$$\theta = \tan^{-1}(1) \Rightarrow \boxed{\theta = 45^\circ}$$

Q3. $5x^2 + 6xy + y^2 = 0$

Solution:

$$5x^2 + 6xy + y^2 = 0$$
$$5x^2 + 5xy + xy + y^2 = 0$$
$$5x(x + y) + y(x + y) = 0$$
$$(x + y)(5x + y) = 0$$

$\Rightarrow$ $x + y = 0$ and $5x + y = 0$ are required lines

For Angle:

$$5x^2 + 6xy + y^2 = 0$$

Compare it with

$$ax^2 + 2hxy + by^2 = 0$$

$\Rightarrow$ $a = 5, 2h = 6 \Rightarrow h = 3, b = 1$

Let θ be the angle between given lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\tan\theta = \frac{2\sqrt{3^2 - (5)(1)}}{5 + 1} = \frac{2\sqrt{9 - 5}}{6} = \frac{\sqrt{4}}{3} = \frac{2}{3}$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right) \Rightarrow \boxed{\theta = 33.7^\circ}$$

Q4. $x^2 + 2xy\sec\alpha + y^2 = 0$

Solution:

$$x^2 + 2xy\sec\alpha + y^2 = 0$$
$$x^2 + (2y\sec\alpha)x + y^2 = 0 \quad \text{(Quadratic in 'x')}$$

Here $A = 1, B = 2y\sec\alpha, C = y^2$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$
$$= \frac{-2y\sec\alpha \pm \sqrt{(2y\sec\alpha)^2 - 4(1)(y^2)}}{2(1)}$$
$$= \frac{-2y\sec\alpha \pm \sqrt{4y^2\sec^2\alpha - 4y^2}}{2}$$

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$$x = \frac{-2y\sec\alpha \pm \sqrt{4y^2(\sec^2\alpha - 1)}}{2}$$

$$= \frac{-2y\sec\alpha \pm 2y\sqrt{\tan^2\alpha}}{2} \quad \because \sec^2\alpha - 1 = \tan^2\alpha$$

$$= \frac{2y(-\sec\alpha \pm \tan\alpha)}{2}$$

$$= y\left(-\frac{1}{\cos\alpha} \pm \frac{\sin\alpha}{\cos\alpha}\right)y$$

$$x = \frac{1}{\cos\alpha}(-1 \pm \sin\alpha)y$$

$$x\cos\alpha = y(-1 \pm \sin\alpha)$$

Required lines are

$x\cos\alpha = y(-1+\sin\alpha)$ and $x\cos\alpha = y(-1-\sin\alpha)$

$\Rightarrow x\cos\alpha + y(1-\sin\alpha) = 0$ and $x\cos\alpha + y(1+\sin\alpha) = 0$

For Angle:

$$x^2 + (2\sec\alpha)xy + y^2 = 0$$

Compare it with

$$ax^2 + 2hxy + by^2 = 0$$

$\Rightarrow \quad a = 1, 2h = 2\sec\alpha \Rightarrow b = \sec\alpha, b = 1$

Let θ be the angle between given lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\tan\theta = \frac{2\sqrt{\sec^2\alpha - (1)(1)}}{1+1} = \frac{2\sqrt{\sec^2\alpha - 1}}{2} = \sqrt{\tan^2\alpha}$$

$\tan\theta = \tan\alpha \Rightarrow \boxed{\theta = \alpha}$

Q5. $7x^2 - 8xy + y^2 = 0$

Solution:

$$7x^2 - 8xy + y^2 = 0$$

$$yx^2 - 7xy - xy + y^2 = 0$$

$$7x(x-y) - y(x-y) = 0$$

$$(x-y)(7x-y) = 0$$

$\Rightarrow \quad x - y = 0$ and $7x - y = 0$ are required lines.

For Angle:

$$7x^2 - 8xy + y^2 = 0$$

Compare it with

$$ax^2 + 2hxy + by^2 = 0$$

$\Rightarrow \quad a = 7, 2h = -8 \Rightarrow h = -4, b = 1$

Let θ be the angle between given lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\tan\theta = \frac{2\sqrt{(-4)^2 - (7)(1)}}{7+1} = \frac{2\sqrt{16-7}}{8} = \frac{\sqrt{9}}{4} = \frac{3}{4}$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) \Rightarrow \boxed{\theta = 36.9^\circ}$$

Q6. Find the value of k such that the lines represented by the equation $(k-8)x^2 - 9xy + 3ky^2 = 0$ are perpendicular to each other.

Solution:

Given that: $(k-8)x^2 - 9xy + 3ky^2 = 0$

Compare it with

$$ax^2 + 2hxy + by^2 = 0$$

$\Rightarrow \quad a = k-8, b = 3k$

Since the given lines are perpendicular to each other, so

$$a + b = 0$$

$$k - 8 + 3k = 0$$

$$4k = 8$$

$$k = \frac{8}{4} \Rightarrow \boxed{k = 2}$$

Q7. Prove that the lines represented by the equation $9y^2 + 6xy + x^2 = 0$ are coincident.

Solution:

Given lines: $9y^2 + 6xy + x^2 = 0$

$\Rightarrow \quad x^2 + 6xy + 9y^2 = 0$

Compare it with

$$ax^2 + 2hxy + by^2 = 0$$

$\Rightarrow \quad a = 1, 2h = 6 \Rightarrow h = 3, b = 9$

Consider $h^2 - ab = 3^2 - (1)(9)$

$= 9 - 9 = 0$

Thus, the given lines are coincident.

Q8. Find a joint equation of the lines through the origin and perpendicular to the lines $3x^2 + 5xy - 2y^2 = 0$.

Solution:

Given line: $3x^2 + 5xy - 2y^2 = 0$

$$3x^2 + 6xy - xy - 2y^2 = 0$$

$$3x(x+2y) - y(x+2y) = 0$$

$$(x+2y)(3x-y) = 0$$

$\Rightarrow \quad l_1: x + 2y = 0 \Rightarrow$ Slope of $l_1 = m_1 = -\frac{1}{2}$

$l_2: 3x - y = 0 \Rightarrow$ Slope of $l_2 = m_2 = -\frac{3}{-1} = 3$

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Slopes of lines perpendicular to given lines are 2 and $-\frac{1}{3}$

Equations of lines passing through origin

$y = 2x$

$2x - y = 0$

$y = \frac{-1}{3}x \quad \because y = mx$

$3y = -x$

$x + 3y = 0$

Joint equation: $(2x - y)(x + 3y) = 0$

$2x^2 + 6xy - xy - 3y^2 = 0$

$\boxed{2x^2 + 5xy - 3y^2 = 0}$

Q9. $A(-2, -1)$ and $B(3, 4)$ are two points on the solar panel roof. Find slope and tilt angle.

Solution:

Given points: $A(-2, -1)$, $B(3, 4)$

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope} = \frac{4-(-1)}{3-(-2)} = \frac{5}{5}$$

$\boxed{\text{Slope} = 1}$

As $\tan\theta = 1$, where θ is tild angle.

$\Rightarrow \quad \theta = \tan^{-1}(1)$

$\boxed{\theta = 45^\circ}$

Q10. The paths of two rays from a star are represented by the equation: $2x^2 + 5xy - 3y^2 = 0$
Find the equations of the two lines and determine the angle between the rays.

Solution:

Given lines: $2x^2 + 5xy - 3y^2 = 0$

$2x^2 + 6xy - xy - 3y^2 = 0$

$2x(x + 3y) - y(x + 3y) = 0$

$(x + 3y)(2x - y) = 0$

$\Rightarrow \quad x + 3y = 0$ and $2x - y = 0$ are required lines

For Angle:

$2x^2 + 5xy - 3y^2 = 0$

Compare it with

$ax^2 + 2hxy + by^2 = 0$

$\Rightarrow \quad a = 2, \; 2h = 5 \Rightarrow h = \frac{5}{2}, \; b = -3$

Let θ be the angle between given lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$= \frac{2\sqrt{\left(\frac{5}{2}\right)^2 - (2)(-3)}}{2 + (-3)} = \frac{2\sqrt{\frac{25}{4} + 6}}{-1}$$

$$= -2\sqrt{\frac{25 + 24}{4}} = -2\sqrt{\frac{49}{4}} = -2\left(\frac{7}{2}\right) = -7$$

$\theta = 180^\circ - \tan^{-1}(7) = 180^\circ - 81.87^\circ$

$\boxed{\theta = 98.13^\circ}$

Acute Angle:

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = 7$$

$\theta = \tan^{-1}(7) \quad \Rightarrow \boxed{\theta = 81.87^\circ}$

Q11. The trajectories of two projectiles in a physics experiment are modeled by the homogeneous equation $9x^2 - 12xy + 4y^2 = 0$. Prove that this represents two coincident lines and find angle between them.

Solution:

Given lines: $9x^2 - 12xy + 4y^2 = 0$

Compare it with

$ax^2 + 2hxy + by^2 = 0$

$\Rightarrow \quad a = 9, \; 2h = -12 \Rightarrow h = -6, \; b = 4$

Consider $h^2 - ab = (-b)^2 - (9)(4)$

$= 36 - 36$

$= 0$

Thus, the lines are coincident.

Let θ be the angle between the given lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$= \frac{2\sqrt{(-6)^2 - 9(4)}}{2(9)}$$

$$= \frac{2\sqrt{36 - 36}}{18} = \frac{2\sqrt{0}}{9} = 0$$

$\theta = \tan^{-1}(0) \Rightarrow \boxed{\theta = 0^\circ}$

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Q12. The stresses in a metal plate are represented by the homogeneous equation $3x^2 - 8xy - 3y^2 = 0$. Show that this equation represents a pair of straight lines through the origin. Find:

(i) the equations of lines represented by this equation.

(ii) angle between them.

Solution:

(i) the equations of lines represented by this equation.

Given line: $3x^2 - 8xy - 3y^2 = 0$

$$3x^2 - 9xy + xy - 3y^2 = 0$$

$$3x(x-3y) + y(x-3y) = 0$$

$$(x-3y)(3x+y) = 0$$

$\Rightarrow$ $x - 3y = 0$ and $3x + y = 0$ is required pair of lines.

At $O(0, 0)$

$0 - 3(0) = 0$

$0 = 0$ (True)

$3(0) + 0 = 0$

$0 = 0$ (True)

Thus, the given equation represents a pair of lines through origin $O(0,0)$.

(ii) angle between them.

Solution:

$$3x^2 - 8xy - 3y^2 = 0$$

Compare it with

$$ax^2 + 2hxy + by^2 = 0$$

$\Rightarrow$ $a = 3, 2h = -8 \Rightarrow h = -4, b = -3$

Let θ be the angle between given lines

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{(-4)^2 - (3)(-3)}}{3+(-3)} = \frac{2\sqrt{16+9}}{0} = \infty$$

$\theta = \tan^{-1}(\infty) \Rightarrow \boxed{\theta = 90^\circ}$

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Multiple Choice Questions (MCQs)

1. If m is the slope and c is the y-intercept, the slope-intercept form of line is:

A) $y = mx + c$ B) $x = my + c$ C) $y = \frac{m}{x} + c$ D) $y = x + \frac{c}{m}$

2. The equation of a line through (x_1, y_1) with slope m is:

A) $y - y_1 = m(x - x_1)$ B) $y + y_1 = m(x + x_1)$ C) $y - x_1 = m(y - y_1)$ D) $y_1 - y = m(x_1 + x)$

3. The slope of the line through (x_1, y_1) and (x_2, y_2) is:

A) $\frac{x_2 - x_1}{y_2 - y_1}$ B) $\frac{y_2 - y_1}{x_2 - x_1}$ C) $\frac{x_1 + x_2}{y_1 + y_2}$ D) $x_1 y_1 - x_2 y_2$

4. If a line has x-intercept a and y-intercept b, its intercept form is:

A) $\frac{x}{a} + \frac{y}{b} = 1$ B) $ax + by = 1$ C) $x + \frac{b}{y} = a$ D) $\frac{x}{a} - \frac{y}{b} = 1$

5. The normal form of a straight line is:

A) $x\cos\theta + y\sin\theta = p$ B) $x\sin\theta + y\cos\theta = p$ C) $x\cos\theta - y\sin\theta = p$ D) $x + y = p\cos\theta$

6. Slope of the line $y = 3x + 7$ is:

A) 7 B) 3 C) -3 D) $\frac{1}{3}$

7. The line through $(-3, -2)$ and $(5, 4)$ has slope:

A) $\frac{3}{4}$ B) $\frac{4}{3}$ C) $\frac{-3}{4}$ D) $\frac{-4}{3}$

8. For $ax + by + c = 0$ with $b \neq 0$, the slope is:

A) $\frac{a}{b}$ B) $\frac{-a}{b}$ C) $\frac{-b}{a}$ D) $\frac{b}{a}$

9. For $ax + by + c = 0$ with $a \neq 0$, the x-intercept is:

A) $\frac{c}{a}$ B) $\frac{-c}{a}$ C) $\frac{a}{c}$ D) $\frac{-a}{c}$

10. The condition for two non-vertical lines to be perpendicular is:

A) $m_1 = m_2$ B) $m_1 m_2 = 1$ C) $m_1 m_2 = -1$ D) $m_1 + m_2 = 1$

11. The equation $ax + by + c = 0$ represents a straight line when

A) $a = b = 0$ B) a and b are not both zero
C) c must be zero D) $a = b = c = 0$

12. If $a = 0$ and $b \neq 0$ in $ax + by + c = 0$, the equation represents:

A) a horizontal line B) a vertical line C) a parabola D) a circle

13. If the slope of BC is $\frac{3}{5}$, the slope of its right bisector is

A) $\frac{3}{5}$ B) $\frac{5}{3}$ C) $\frac{-5}{3}$ D) $\frac{-3}{5}$

14. A family of lines through the intersection of two non-parallel lines is formed by:

A) $\ell_1 + k\ell_2 = 0$ B) $\ell_1 - \ell_2 = 1$ only C) $k \cdot \ell_1 \ell_2 = 0$ D) $\frac{\ell_1}{k} + \ell_2 = 0$

15. The altitudes of a triangle are:

A) always concurrent B) always parallel
C) never concurrent D) concurrent only for right triangles

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16. **The right bisectors of a triangle are:**
A) always concurrent B) always parallel
C) never concurrent D) equal to the medians

17. **The medians of a triangle are:**
A) concurrent B) parallel
C) perpendicular in every triangle D) equal to the sides

18. **If two lines are parallel, the angle between them is:**
A) 0 B) $\frac{\pi}{4}$ C) $\frac{\pi}{2}$ D) π

19. **If two lines are perpendicular, the angle between them is**
A) 0 B) $\frac{\pi}{4}$ C) $\frac{\pi}{2}$ D) π

20. **Three points are collinear if the area of the triangle formed by them is:**
A) 1 B) -1 C) 0 D) π

21. **A homogeneous equation of degree n satisfies:**
A) $f(kx, ky) = k^n f(x, y)$ B) $f(kx, ky) = kf(x, y)$
C) $f(kx, ky) = f(x, y) + k$ D) $f(kx, ky) = 0$ always

22. **The homogeneous second-degree equation $ax^2 + 2hxy + by^2 = 0$ represents:**
A) a pair of straight lines through the origin B) a circle only
C) a single horizontal line D) a parabola only

23. **The lines represented by $ax^2 + 2hxy + by^2 = 0$ are real and distinct when:**
A) $h^2 < ab$ B) $h^2 > ab$ C) $h^2 = ab$ D) $ab = 0$ only

24. **The lines represented by $ax^2 + 2hxy + by^2 = 0$ are real and coincident when**
A) $h^2 > ab$ B) $h^2 = ab$ C) $h^2 < ab$ D) $h = 0$ only

25. **For $A(-3, -7)$, $B(4, 5)$, $C(-1, 2)$, the midpoint of BC is:**
A) $\left(\frac{3}{2}, \frac{7}{2}\right)$ B) $\left(\frac{5}{2}, \frac{3}{2}\right)$ C) $(3, 7]$ D) $\left(\frac{-3}{2}, \frac{-7}{2}\right)$

26. **For $A(-3, -7)$, $B(4, 5)$, $C(-1, 2)$, the right bisector of BC is:**
A) $5x + 3y - 18 = 0$ B) $3x + 5y - 18 = 0$ C) $5x - 3y - 18 = 0$ D) $3x - 5y + 18 = 0$

27. **For the triangle with $A(-2, -1)$, $B(4, 3)$, $C(6, -2)$, the slope of the altitude from A is:**
A) $\frac{2}{5}$ B) $\frac{-2}{5}$ C) $\frac{5}{2}$ D) $\frac{-5}{2}$

28. **For lines of slopes $m_1 = 3$ and $m_2 = -2$, the acute angle between them is:**
A) 0° B) 45° C) 90° D) 135°

29. **For $5x^2 + 6xy + y^2 = 0$, the pair of lines is:**
A) real and distinct B) real and coincident C) imaginary D) parallel

30. **For $x^2 + 2xy + y^2 = 0$, the pair of lines is:**
A) real and distinct B) real and coincident C) imaginary D) perpendicular

ANSWER KEY

1.	A	2.	A	3.	B	4.	A	5.	A	6.	B	7.	A	8.	B	9.	B	10.	C
11.	B	12.	A	13.	C	14.	A	15.	A	16.	A	17.	A	18.	A	19.	C	20.	C
21.	A	22.	A	23.	B	24.	B	25.	A	26.	A	27.	A	28.	B	29.	A	30.	B

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