www.ilmkidunya.com

UNIT 03

Integration

INTRODUCTION

- In previous units, we studied differentiation and learned how to determine rates of change, slopes of curves, maxima and minima, and solutions to various real-world problems.
- In this unit, we will study integration, which is the reverse process of differentiation and an important branch of calculus used to find accumulation, area, volume, and total change.
- We will learn the concepts of anti-derivatives, indefinite and definite integrals, their notation, properties, and fundamental rules, including the power rule and the Fundamental Theorem of Calculus.
- We will also study the relationship between definite integrals and the net area under a curve, and use integration to find areas bounded by curves and lines, as well as volumes of solids of revolution.
- Furthermore, we will explore important integration techniques such as substitution, integration by parts, trigonometric substitution, partial fractions, and integration involving exponential and logarithmic functions.
- The concepts of integration have wide applications in mathematics, physics, engineering, economics, biology, and technology, including calculating the volume of containers, population growth, consumer and producer surplus, distance traveled, velocity, drug dosage, moment of inertia.
- By the end of this unit, we will develop both conceptual understanding and practical problem-solving skills related to integration and its real-life applications.

Content Outline

➢ Antiderivatives and The Indefinite Integral

ANTIDERIVATIVES AND THE INDEFINITE INTEGRAL

Definition: A function F is called an **antiderivative** of a given function f on an interval I if $F'(x) = f(x)$ for all x in I, and the process of recovering a function F from its derivative f is called **antidifferentiation**.

To illustrate, let $f(x) = 2x$.

If $F(x) = x^2$ then $F'(x) = 2x = f(x)$. Moreover, the function $G(x) = x^2 + 10$ also satisfies $G'(x) = 2x = f(x)$. Therefore, if $F(x)$ is any antiderivatives of $f(x)$ on a given interval, then for any value of c, the function $F(x) + c$ is also an antiderivative of $f(x)$ on that interval.

Definition: Let F be an antiderivative of f on an interval I. The set (or family) of all antiderivatives of f is called the **indefinite integral** of $f(x)$ with respect to x, denoted by $\int f(x)dx$ and defined as $\int f(x)dx = F(x) + c$, where $F'(x) = f(x)$ and c is an arbitrary constant.

In the notation $\int f(x)dx = F(x) + c$,

(i) The symbol $\int$ is called the **integral sign**.

(ii) $\int f(x)dx$ is called the **indefinite integral** of $f(x)$.

(iii) The function $f(x)$ inside the integral sign is the **integrand**.

(iv) The symbol dx is called the **differential** of x. It indicates the **variable of integration**—the variable with respect to which the integration is performed.

(v) $F(x)$ is an antiderivative of $f(x)$.

(vi) c is the constant of integration.

www.ilmkidunya.com

www.ilmkidunya.com

Basic Rules of Integration

Rule-1: $\int kf(x)dx = k\int f(x)dx$ (Integration of function with constant (k)).

Rule-2: $\int[f(x)\pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$ (The Sum and Difference Rule).

Rule-3: $\int x^n dx = \frac{x^{n+1}}{n+1}+c$, where $n \neq -1$ (The Power Rule).

$\int[f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}+c$, where $n \neq -1$. (The General Power Rule)

Rule-4: $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$

Rule-5: If f' is the derivative of f, and $\int f'(x)dx = f(x)+c$, then for any constants $a \neq 0$ and b,

$\int f'(ax+b)dx = \frac{f(ax+b)}{a}+c$. (The Linear Substitution Rule).

Some Useful Integration Formulas

From the above Definitions, we obtain integration formulas directly from differentiation formulas, as shown in the following table:

Derivative $\frac{d}{dx}(f(x))$	Indefinite Integral $\int \frac{d}{dx}(f(x))dx = f(x)+c$
$\frac{d}{dx}(c)=0$	$\int 0dx = 0$
$\frac{d}{dx}(x)=1$	$\int 1dx = \int dx = x+c$
$\frac{d}{dx}(x^n)=nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1}+c, n\neq -1$
$\frac{d}{dx}(\ln\|x\|)=\frac{1}{x}, x\neq 0$	$\int \frac{1}{x}dx = \ln\|x\|+c, x\neq 0$
$\frac{d}{dx}(e^x)=e^x$	$\int e^x dx = e^x+c$
$\frac{d}{dx}(a^x)=a^x \ln a,\ a>0, a\neq 1$	$\int a^x dx = \frac{a^x}{\ln a}+c, a>0, a\neq 1$
$\frac{d}{dx}(\sin x)=\cos x$	$\int \cos x dx = \sin x+c$
$\frac{d}{dx}(\cos x)=-\sin x$	$\int \sin x dx = -\cos x+c$
$\frac{d}{dx}(\tan x)=\sec^2 x$	$\int \sec^2 x dx = \tan x+c$
$\frac{d}{dx}(\cot x)=-\csc^2 x$	$\int \csc^2 x dx = -\cot x+c$
$\frac{d}{dx}(\sec x)=\sec x\tan x$	$\int \sec x\tan x dx = \sec x+c$
$\frac{d}{dx}(\csc x)=-\csc x\cot x$	$\int \csc x\cot x dx = -\csc x+c$
$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x+c$

www.ilmkidunya.com

www.ilmkidunya.com

$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int\frac{1}{x^2\sqrt{x^2-1}}dx=\sec^{-1}\lvert x\rvert+c$
$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$	$\int\frac{1}{1+x^2}dx=\tan^{-1}x+c$

Example 1: Evaluate the following integrals: (i) $\int x^{99}dx$ **(ii)** $\int -98x^{-99}dx$

Solution:

(i) $\int x^{99}dx$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$

$=\frac{x^{99+1}}{99+1}+c$

$=\frac{x^{100}}{100}+c$

(ii) $-98\int x^{-99}dx$

$=-98\int x^{-99}dx$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$

$=-98\frac{x^{-99+1}}{-99+1}+c=\frac{-98}{-98}x^{-98}+c$

$=\frac{1}{x^{98}}+c$

Example 2: Evaluate the integral: $\int(8x^7-6x^5+4x^3-2x-3)dx$

Solution:

$\int(8x^7-6x^5+4x^3-2x-3)dx$

$=\int 8x^7dx-\int 6x^5dx+\int 4x^3dx-\int 2xdx-\int 3dx$

$=8\int x^7dx-6\int x^5dx+4\int x^3dx-2\int xdx-3\int 1dx$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$

$=8\left(\frac{x^{7+1}}{7+1}\right)-6\left(\frac{x^{5+1}}{5+1}\right)+4\left(\frac{x^{3+1}}{3+1}\right)-2\left(\frac{x^{1+1}}{1+1}\right)-3x+c$

$=8\left(\frac{x^8}{8}\right)-6\left(\frac{x^6}{6}\right)+4\left(\frac{x^4}{4}\right)-2\left(\frac{x^2}{2}\right)-3x+c$

$=x^8-x^6+x^4-x^2-3x+c$

Example 3: Evaluate the integrals: (i) $\int\frac{7}{\sqrt{x}}dx$ **(ii)** $\int\frac{(1+x^2)^2}{x^4}dx$

Solution:

(i) $\int\frac{7}{\sqrt{x}}dx$

$=7\int x^{\frac{-1}{2}}dx$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$

$=7\cdot\frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}+c$

$=7\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c$

$=14\sqrt{x}+c$

(ii) $\int\frac{(1+x^2)^2}{x^4}dx$

$=\int\frac{(1)^2+2(1)(x^2)+(x^2)^2}{x^4}dx$

$=\int\frac{1+2x^2+x^4}{x^4}dx=\int\left(\frac{1}{x^4}+\frac{2x^2}{x^4}+\frac{x^4}{x^4}\right)dx$

$=\int(x^{-4}+2x^{-2}+1)dx$

$=\int x^{-4}dx+2\int x^{-2}dx+\int 1dx$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$

$=\frac{x^{-4+1}}{-4+1}+2\cdot\frac{x^{-2+1}}{-2+1}+x+c$

$=\frac{x^{-3}}{-3}+2\cdot\frac{x^{-1}}{-1}+x+c=-\frac{1}{3x^3}-\frac{2}{x}+x+c$

www.ilmkidunya.com

www.ilmkidunya.com

Example 4: Evaluate: (i) $\int (5x^2+7x-3)^7(10x+7)dx$ **(ii)** $\int\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)dx$

Solution:

(i) $\int (5x^2+7x-3)^7(10x+7)dx$

Let $f(x)=5x^2+7x-3$

$\Rightarrow \quad f'(x)=5(2x)+7(1)-0=10x+7$

Using: $\int [f(x)]^n \cdot f'(x)dx=\frac{[f(x)]^{n+1}}{n+1}+c; n\neq -1$

$=\frac{(5x^2+7x-3)^{7+1}}{7+1}+c$

$=\frac{1}{8}(5x^2+7x-3)^8+c$

(ii) $\int\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)dx$

Let $f(x)=x+\frac{1}{x}=x+x^{-1}$

$\Rightarrow \quad f'(x)=1+(-1x^{-2})=1-\frac{1}{x^2}$

Using: $\int [f(x)]^n \cdot f'(x)dx=\frac{[f(x)]^{n+1}}{n+1}+c; n\neq -1$

$=\frac{\left(x+\frac{1}{x}\right)^{1+1}}{1+1}+c=\frac{1}{2}\left(x+\frac{1}{x}\right)^2+c$

Example 5: Evaluate $\int \tan x\,dx$

Solution:

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$

$= -\int \frac{-\sin x}{\cos x}dx$

Let $f(x)=\cos x$

$\Rightarrow \quad f'(x)=-\sin x$

Using: $\int \frac{f'(x)}{f(x)}dx=\ell n|f(x)|+c$

$=-\ell n|\cos x|+c$

or $=\ell n(\cos x)^{-1}+c$

$=\ell n\left(\frac{1}{\cos x}\right)+c=\ell n|\sec x|+c$

Example 6: Evaluate $\int \sec x dx$

Solution:

$\int \sec x dx$

Multiply up and down by $(\sec x+\tan x)$

$=\int \frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)}dx$

$=\int \frac{\sec^2 x+\sec x\cdot\tan x}{\sec x+\tan x}dx$

Let $f(x)=\sec x+\tan x$

$f'(x)=\sec x\tan x+\sec^2 x$

Using: $\int \frac{f'(x)}{f(x)}dx=\ell n|f(x)|+c$

$\int \sec x\,dx = \ell n|\sec x+\tan x|+c$

Example 7: Evaluate $\int \tan^2 x dx$

Solution:

$\int \tan^2 x\,dx$

$=\int(\sec^2 x-1)dx = \int \sec^2 x\,dx-\int 1dx$

$=\tan x-x+c$

Example 8: Evaluate $\int \cos^2 x dx$

Solution:

$\int \cos^2 x\,dx$

$=\int \frac{1+\cos 2x}{2}dx=\int\left(\frac{1}{2}+\frac{\cos 2x}{2}\right)dx$

$=\frac{1}{2}\int 1dx+\frac{1}{2}\int \cos 2x\,dx=\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+c$

Example 9: Evaluate $\int \frac{\tan x-\sec x}{1+\tan^2 x}dx$

Solution:

$\int \frac{\tan x-\sec x}{1+\tan^2 x}dx$

$=\int \frac{\tan x-\sec x}{\sec^2 x}dx \quad \because 1+\tan^2\theta=\sec^2\theta$

$=\int\left(\frac{\tan x}{\sec^2 x}-\frac{\sec x}{\sec^2 x}\right)dx=\int \frac{\sin x}{\cos x}\cdot\cos^2 x\,dx-\int \frac{1}{\sec x}dx$

$=\int \sin x\cdot\cos x\,dx-\int \cos x\,dx=\frac{(\sin x)^{1+1}}{1+1}-\sin x+c$

$=\frac{1}{2}\sin^2 x-\sin x+c$

www.ilmkidunya.com

www.ilmkidunya.com

Example 10: Evaluate $\int \sin 3x \cos 7x\, dx$

Solution:

$$\int \sin 3x \cos 7x\, dx$$

$$= \frac{1}{2}\int 2\sin 3x \cdot \cos 7x\, dx$$

Formula: $2\sin\alpha\cos\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$

$$= \frac{1}{2}\int (\sin(3x+7x)+\sin(3x-7x))dx$$

$$= \frac{1}{2}\int (\sin 10x + \sin(-4x))dx$$

$$= \frac{1}{2}\left[\int \sin 10x\, dx - \int \sin 4x\, dx\right]$$

$$= \frac{1}{2}\left[\frac{-\cos 10x}{10} - \frac{-\cos 4x}{4}\right] + c$$

$$= -\frac{\cos 10x}{20} + \frac{\cos 4x}{8} + c$$

Example 11: Evaluate $\int \frac{\sin(x+a)}{\sin(x+b)}dx$

Solution:

$$\int \frac{\sin(x+a)}{\sin(x+b)}dx$$

$$= \int \frac{\sin[(x+b)+(a-b)]}{\sin(x+b)}dx$$

Formula: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \int \frac{\sin(x+b)\cos(a-b)+\cos(x+b)\sin(a-b)}{\sin(x+b)}dx$$

$$= \int \left[\frac{\sin(x+b)\cos(a-b)}{\sin(x+b)} + \frac{\cos(x+b)\sin(a-b)}{\sin(x+b)}\right]dx$$

$$= \int \left[\cos(a-b) + \sin(a-b)\cdot\frac{\cos(x+b)}{\sin(x+b)}\right]dx$$

$$= \cos(a-b)\int 1dx + \sin(a-b)\cdot\int \frac{\cos(x+b)}{\sin(x+b)}dx$$

$$= \cos(a-b)\cdot x + \sin(a-b)\cdot \ln|\sin(x+b)| + c$$

Example 12: Evaluate $\int \frac{a^{2x}-b^{2x}}{(ab)^x}dx$ **where** a **and** b **are distinct and have the same sign.**

Solution:

$$\int \frac{a^{2x}-b^{2x}}{(ab)^x}dx$$

$$= \int \frac{a^{2x}-b^{2x}}{a^x b^x}dx = \int \left[\frac{a^{2x}}{a^x b^x} - \frac{b^{2x}}{a^x b^x}\right]dx$$

$$= \int \left[\frac{a^{2x-x}}{b^x} - \frac{b^{2x-x}}{a^x}\right]dx$$

$$= \int \left[\frac{a^x}{b^x} - \frac{b^x}{a^x}\right]dx$$

$$= \int \left(\frac{a}{b}\right)^x dx - \int \left(\frac{b}{a}\right)^x dx$$

Formula: $\int a^x dx = \frac{a^x}{\ln a} + c$

$$= \frac{\left(\frac{a}{b}\right)^x}{\ln\left(\frac{a}{b}\right)} - \frac{\left(\frac{b}{a}\right)^x}{\ln\left(\frac{b}{a}\right)} + c$$

EXERCISE 3.1

Q1. Find all the antiderivatives for each derivative.

(i) $\frac{dy}{dx} = 100x^4$

Solution:

$$\frac{dy}{dx} = 100x^4$$

Integrate w.r.t. 'x'

$$\int \frac{dy}{dx}dx = \int (100x^4)dx$$

$$y = 100\int x^4 dx$$

Using: $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

$$y = 100\cdot\frac{x^{4+1}}{4+1} + c$$

$$= 100\cdot\frac{x^5}{5} + c$$

$$y = 20x^5 + c$$

www.ilmkidunya.com

www.ilmkidunya.com

(ii) $\frac{dy}{dt} = 2 + 24t^3 - 10t^4$

Solution:

$$\frac{dy}{dt} = 2 + 24t^3 - 10t^4$$

Integrate w.r.t. 't'

$$\int \frac{dy}{dt} dt = \int (2 + 24t^3 - 10t^4) dt$$

$$y = \int 2dt + \int 24t^3 dt - \int 10t^4 dt$$

$$y = 2\int 1dt + 24\int t^3 dt - 10\int t^4 dt$$

Using: $\int t^n dt = \frac{t^{n+1}}{n+1} + c, n \neq -1$

$$y = 2t + 24 \cdot \frac{t^{3+1}}{3+1} - 10 \cdot \frac{t^{4+1}}{4+1} + c$$

$$= 2t + 24 \cdot \frac{t^4}{4} - 10 \cdot \frac{t^5}{5} + c$$

$$= 2t + 6t^4 - 2t^5 + c$$

Q2. Find each indefinite integral. (Check answer by differentiating)

(i) $\int (1 + 2t) dt$

Solution:

$$\int (1 + 2t) dt$$

$$= \int 1dt + \int 2t\, dt$$

$$= t + 2\int t^1 dt$$

$$= t + 2\frac{t^{1+1}}{1+1} + c = t + 2 \cdot \frac{t^2}{2} + c$$

$$= t + t^2 + c$$

Check:

$$\frac{d}{dt}(t + t^2 + c)$$

$$= 1 + 2t + 0$$

$$= 1 + 2t$$

(ii) $\int (3u^2 + 4u) du$

Solution:

$$\int (3u^2 + 4u) du$$

$$= \int 3u^2 du + \int 4u\, du = 3\int u^2 du + 4\int u^1 du$$

Using: $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

$$= 3 \cdot \frac{u^{2+1}}{2+1} + 4 \cdot \frac{u^{1+1}}{1+1} + c$$

$$= 3\frac{u^3}{3} + 4 \cdot \frac{u^2}{2} + c$$

$$= u^3 + 2u^2 + c$$

Check:

$$\frac{d}{du}(u^3 + 2u^2 + c)$$

$$= \frac{d}{du}(u^3) + 2\frac{d}{du}(u^2) + \frac{d}{du}(c)$$

$$= 3u^2 + 2(2u) + 0 = 3u^2 + 4u$$

(iii) $\int 100x^8 dx$

Solution:

$$\int 100x^8 dx$$

$$= 100\int x^8 dx$$

Using: $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

$$= 100 \cdot \frac{x^{8+1}}{8+1} + c$$

$$= \frac{100}{9}x^9 + c$$

Check:

$$\frac{d}{dx}\left(\frac{100}{9}x^9 + c\right)$$

$$= \frac{100}{9}\frac{d}{dx}(x^9) + \frac{d}{dx}(c)$$

$$= \frac{100}{9}(9x^8) + 0$$

$$= 100x^8$$

(iv) $\int (2x^{-2} + 3x^{-3}) dx$

Solution:

$$\int (2x^{-2} + 3x^{-3}) dx$$

$$= 2\int x^{-2} dx + 3\int x^{-3} dx$$

Using: $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

$$= 2 \cdot \frac{x^{-2+1}}{-2+1} + 3 \cdot \frac{x^{-3+1}}{-3+1} + c$$

$$= 2\frac{x^{-1}}{-1} + 3 \cdot \frac{x^{-2}}{-2} + c$$

$$= -\frac{2}{x} - \frac{3}{2x^2} + c$$

www.ilmkidunya.com

www.ilmkidunya.com

Check:

$$\frac{d}{dx}\left(\frac{-2}{x}-\frac{3}{2x^2}+c\right)$$

$$=-2\frac{d}{dx}(x^{-1})-\frac{3}{2}\frac{d}{dx}(x^{-2})+\frac{d}{dx}(c)$$

$$=-2(-1x^{-2})-\frac{3}{2}(-2x^{-3})+0$$

$$=2x^{-2}+3x^{-3}$$

Q3. Evaluate the following:

(i) $\int(3x+3^x+x^3)dx$

Solution:

$$\int(3x+3^x+x^3)dx$$

$$=\int 3x\,dx+\int 3^x dx+\int x^3 dx$$

$$=3\int x\,dx+\int 3^x dx+\int x^3 dx$$

$$=3\frac{x^{1+1}}{1+1}+\frac{3^x}{\ln 3\cdot 1}+\frac{x^{3+1}}{3+1}+c$$

$$=\frac{3}{2}x^2+\frac{3^x}{\ln 3}+\frac{x^4}{4}+c$$

(ii) $\int x^2\left(x+\frac{1}{x}\right)dx$

Solution:

$$\int x^2\left(x+\frac{1}{x}\right)dx$$

$$=\int\left(x^3+\frac{x^2}{x}\right)dx$$

$$=\int x^3dx+\int x^1dx$$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c;n\neq -1$

$$=\frac{x^{3+1}}{3+1}+\frac{x^{1+1}}{1+1}+c$$

$$=\frac{x^4}{4}+\frac{x^2}{2}+c$$

(iii) $\int\left(\sqrt{y}+\frac{1}{y}\right)^2dy$

Solution:

$$\int\left(\sqrt{y}+\frac{1}{y}\right)^2dy$$

$$=\int\left[(\sqrt{y})^2+2(\sqrt{y})\left(\frac{1}{y}\right)+\left(\frac{1}{y}\right)^2\right]dy$$

$$=\int\left[y+2y^{\frac{1}{2}-1}+\frac{1}{y^2}\right]dy$$

$$=\int y^1dy+2\int y^{\frac{-1}{2}}dy+\int y^{-2}dy$$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c,n\neq -1$

$$=\frac{y^{1+1}}{1+1}+2\cdot\frac{y^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}+\frac{y^{-2+1}}{-2+1}+c$$

$$=\frac{y^2}{2}+2\cdot\frac{y^{1/2}}{\frac{1}{2}}+\frac{y^{-1}}{-1}+c$$

$$=\frac{y^2}{2}+4\sqrt{y}-\frac{1}{y}+c$$

(iv) $\int\frac{\sqrt{x}(x+2)}{x}dx$

Solution:

$$\int\frac{\sqrt{x}(x+2)}{x}dx$$

$$=\int\frac{x\sqrt{x}+2\sqrt{x}}{x}dx$$

$$=\int\left[\frac{x\sqrt{x}}{x}+\frac{2\sqrt{x}}{x}\right]dx$$

$$=\int\left[\sqrt{x}+2x^{\frac{1}{2}-1}\right]dx$$

$$=\int x^{\frac{1}{2}}dx+2\int x^{\frac{-1}{2}}dx$$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c,n\neq -1$

$$=\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+2\cdot\frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}+c$$

$$=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+2\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$=\frac{2}{3}x^{\frac{3}{2}}+4\sqrt{x}+c$$

www.ilmkidunya.com

www.ilmkidunya.com

(v) $\int \frac{x+1}{\sqrt{x+2}}dx$

Solution:

$$\int \frac{x+1}{\sqrt{x+2}}dx$$

$$= \int \frac{(x+2)-1}{\sqrt{x+2}}dx = \int\left[\frac{x+2}{\sqrt{x+2}} - \frac{1}{\sqrt{x+2}}\right]dx$$

$$= \int[(x+2)^{\frac{1}{2}} - (x+2)^{\frac{-1}{2}}]dx$$

$$= \int(x+2)^{\frac{1}{2}} \cdot 1dx - \int(x+2)^{\frac{-1}{2}} \cdot 1dx$$

Using: $\int[f(x)]^n \cdot f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c; n \neq -1$

$$= \frac{(x+2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(x+2)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + c$$

$$= \frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2}{3}(x+2)^{\frac{3}{2}} - 2\sqrt{x+2} + c$$

(vi) $\int \frac{(e^x + e^{-x})^3}{e^{3x}}dx$

Solution:

$$\int \frac{(e^x + e^{-x})^3}{e^{3x}}dx$$

$$= \int\left(\frac{e^x + e^{-x}}{e^x}\right)^3 dx$$

$$= \int\left(\frac{e^x}{e^x} + \frac{e^{-x}}{e^x}\right)^3 dx = \int(1 + e^{-2x})^3 dx$$

Formula: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$= \int(1^3 + 3(1)^2 \cdot e^{-2x} + 3(1)(e^{-2x})^2 + (e^{-2x})^3)dx$$

$$= \int(1 + 3e^{-2x} + 3e^{-4x} + e^{-6x})dx$$

$$= \int 1dx + 3\int e^{-2x}dx + 3\int e^{-4x}dx + \int e^{-6x}dx$$

$$= x + 3\frac{e^{-2x}}{-2} + 3\frac{e^{-4x}}{-4} + \frac{e^{-6x}}{-6} + c$$

$$= x - \frac{3}{2}e^{-2x} - \frac{3}{4}e^{-4x} - \frac{1}{6}e^{-6x} + c$$

(vii) $\int \frac{1}{\sqrt{x+1} + \sqrt{x}}dx$

Solution:

$$\int \frac{1}{\sqrt{x+1} + \sqrt{x}}dx$$

$$= \int \frac{1}{\sqrt{x+1} + \sqrt{x}} \times \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}}dx$$

$$= \int \frac{\sqrt{x+1} - \sqrt{x}}{(\sqrt{x+1})^2 - (\sqrt{x})^2}dx$$

$$= \int \frac{\sqrt{x+1} - \sqrt{x}}{x+1-x}dx$$

$$= \int(\sqrt{x+1} - \sqrt{x})dx$$

$$= \int(x+1)^{\frac{1}{2}}dx - \int x^{\frac{1}{2}}dx$$

$$= \frac{(x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}} + c$$

(viii) $\int 7^{\ln x}dx$

Solution:

$$\int 7^{\ln x}dx$$

As we know: $a^{\ln b} = b^{\ln a}$

$$\int 7^{\ln x}dx = \int x^{\ln 7}dx$$

$$= \frac{x^{\ln 7+1}}{\ln 7+1} + c$$

$$= \frac{x^{\ln 7 + \ln e}}{\ln 7 + \ln e} + c \qquad \because \ln e = 1$$

$$= \frac{x^{\ln 7e}}{\ln 7e} + c$$

(ix) $\int 2^{\log_4 x}dx$

Solution:

$$\int 2^{\log_4 x}dx$$

$$= \int 2^{\log_{2^2} x}dx$$

www.ilmkidunya.com

www.ilmkidunya.com

$= \int 2^{\frac{1}{2}\log_2 x} dx$ $\quad \because \log_{a^n} x = \frac{1}{n}\log_a x$

$= \int 2^{\log_2 x^{\frac{1}{2}}} dx$ $\quad \because b\log a = \log(a^b)$

$= \int 2^{\log_2 \sqrt{x}} \cdot dx$

$= \int \sqrt{x}\, dx$ $\quad \because a^{\log_a(x)} = x$

Using: $\int x^n dx = \frac{x^{n+1}}{n+1} + c; n \neq -1$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{3/2} + c$

$= \frac{2}{3}x^{\frac{3}{2}} + c$

(x) $\int \frac{a^x + a^{-x}}{a^x - a^{-x}} dx \begin{pmatrix} a>0 \\ a\neq 1 \end{pmatrix}$

Solution:

$\int \frac{a^x + a^{-x}}{a^x - a^{-x}} dx$

Multiply and divide by $\ln a$

$= \frac{1}{\ln a}\int \frac{a^x \cdot \ln a + a^{-x} \cdot \ln a}{a^x - a^{-x}} dx$

Let $f(x) = a^x - a^{-x}$

$\Rightarrow f'(x) = a^x \cdot \ln a \cdot 1 - a^{-x} \cdot \ln a(-1)$

$= a^x \ln a + a^x \ln a$

Using: $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

$= \frac{1}{\ln a}\ln|a^x - a^{-x}| + c$

(xi) $\int \frac{2^{2x}+1}{2^x} dx$

Solution:

$\int \frac{2^{2x}+1}{2^x} dx$

$= \int \left(\frac{2^{2x}}{2^x} + \frac{1}{2^x}\right) dx$

$= \int (2^{2x-x} + 2^{-x}) dx = \int 2^x dx + \int 2^{-x} dx$

Using: $\int a^x dx = \frac{a^x}{\ln a \cdot (1)} + c$

$= \frac{2^x}{\ln 2 \cdot (1)} + \frac{2^{-x}}{\ln 2(-1)} + c = \frac{2^x}{\ln 2} - \frac{2^{-x}}{\ln 2} + c$

(xii) $\int \frac{x^2+1}{x^3 - x} dx$

Solution:

$\int \frac{x^2+1}{x^3 - x} dx$

Divide up and down by x^2

$= \int \frac{\frac{x^2}{x^2} + \frac{1}{x^2}}{\frac{x^3}{x^2} - \frac{x}{x^2}} dx = \int \frac{1 + \frac{1}{x^2}}{x - \frac{1}{x}} dx$

Let $f(x) = x - \frac{1}{x} = x - x^{-1}$

$f'(x) = 1 - (-1x^{-2}) = 1 + \frac{1}{x^2}$

Using: $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

$= \ln\left|x - \frac{1}{x}\right| + c$

Q4. Evaluate the following:

(i) $\int \csc x\, dx$

Solution:

Multiply up and down by $(\csc x - \cot x)$

$= \int \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x} dx$

$= \int \frac{\csc^2 x - \csc x \cdot \cot x}{\csc x - \cot x} dx$

Let $f(x) = \csc x - \cot x$

$f'(x) = -\csc x \cdot \cot x - (-\csc^2 x)$

$= \csc^2 x - \csc x \cdot \cot x$

Using: $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

$\int \csc x\, dx = \ln|\csc x - \cot x| + c$

(ii) $\int \cot^2 x\, dx$

Solution:

$\int \cot^2 x\, dx$

$= \int (\csc^2 x - 1) dx = \int \csc^2 x dx - \int 1 dx$

$= -\cot x - x + c$

www.ilmkidunya.com

www.ilmkidunya.com

(iii) $\int \sec^4 x\,dx$

Solution:

$$\int \sec^4 x\,dx$$
$$= \int \sec^2 x \cdot \sec^2 x\,dx$$
$$= \int (1+\tan^2 x)\cdot \sec^2 x\,dx$$
$$= \int (\sec^2 x + \tan^2 x \cdot \sec^2 x)dx$$
$$= \int \sec^2 x\,dx + \int (\tan x)^2 \cdot \sec^2 x\,dx$$

Using: $\int [f(x)]^n \cdot f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1$

$$= \tan x + \frac{1}{3}\tan^3 x + c$$

(iv) $\int \sin x \cdot \cos 7x\,dx$

Solution:

$$\int \sin x \cdot \cos 7x\,dx$$
$$= \frac{1}{2}\int 2\sin x \cdot \cos 7x\,dx$$

Formula: $2\sin\alpha \cdot \cos\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$

$$= \frac{1}{2}\int [\sin(x+7x)+\sin(x-7x)]dx$$
$$= \frac{1}{2}\int \sin 8x\,dx + \frac{1}{2}\int \sin(-6x)dx$$
$$= \frac{1}{2}\int \sin 8x\,dx - \frac{1}{2}\int \sin 6x\,dx \qquad \because \sin(-\theta) = -\sin\theta$$
$$= \frac{1}{2}\left(\frac{-\cos 8x}{8}\right) - \frac{1}{2}\left(\frac{-\cos 6x}{6}\right) + c$$
$$= \frac{\cos 6x}{12} - \frac{\cos 8x}{16} + c$$

(v) $\int \frac{\cos^3 x - \sin^3 x}{\cos x - \sin x}dx$

Solution:

$$\int \frac{\cos^3 x - \sin^3 x}{\cos x - \sin x}dx$$

Formula: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$$= \int \frac{(\cos x - \sin x)(\cos^2 x + \sin x \cdot \cos x + \sin^2 x)}{(\cos x - \sin x)}dx$$
$$= \int (\cos^2 x + \sin^2 x + \sin x \cos x)dx$$
$$= \int (1+\sin x \cos x)dx = \int 1dx + \int \sin x \cdot \cos x\,dx$$
$$= x + \frac{1}{2}\int 2\sin x \cos x\,dx = x + \frac{1}{2}\int \sin 2x\,dx$$
$$= x + \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) + c$$
$$= x - \frac{1}{4}\cos 2x + c$$

(vi) $\int \frac{\cos 2x}{\sin x + \cos x}dx$

Solution:

$$\int \frac{\cos 2x}{\sin x + \cos x}dx$$
$$= \int \frac{\cos^2 x - \sin^2 x}{\sin x + \cos x}dx \qquad \because \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{\sin x + \cos x}dx$$
$$= \int (\cos x - \sin x)dx$$
$$= \int \cos x\,dx - \int \sin x\,dx = \sin x - (-\cos x) + c$$
$$= \sin x + \cos x + c$$

(vii) $\int \frac{1}{1+\cos x}dx$

Solution:

$$\int \frac{1}{1+\cos x}dx$$
$$= \int \frac{1}{2\cos^2\frac{x}{2}}dx \qquad \because \begin{cases} \cos 2x = 2\cos^2 x - 1 \\ \cos x = 2\cos^2\frac{x}{2} - 1 \\ 1+\cos x = 2\cos^2\frac{2}{2} \end{cases}$$
$$= \frac{1}{2}\int \sec^2\frac{x}{2}dx = \frac{1}{2}\cdot\frac{\tan\frac{x}{2}}{\frac{1}{2}} + c$$
$$= \tan\frac{x}{2} + c$$

(viii) $\int \frac{1-\cos 2x}{1+\cos 2x}dx$

Solution:

$$\int \frac{1-\cos 2x}{1+\cos 2x}dx$$
$$= \int \frac{2\sin^2 x}{2\cos^2 x}dx \qquad \because \begin{cases} \cos 2\alpha = 1 - 2\sin^2\alpha \\ \cos 2\alpha = 2\cos^2\alpha - 1 \end{cases}$$
$$= \int \tan^2 x\,dx = \int (\sec^2 x - 1)dx$$
$$= \int \sec^2 x\,dx - \int 1dx$$
$$= \tan x - x + c$$

www.ilmkidunya.com

www.ilmkidunya.com

(ix) $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$

Solution:

$$\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \theta - 1)}{\cos x - \cos \theta} dx$$

$$= \int \frac{2\cos^2 x - 1 - 2\cos^2 \theta + 1}{\cos x - \cos \theta} dx$$

$$= \int \frac{2\cos^2 x - 2\cos^2 \theta}{\cos x - \cos \theta} dx = \int \frac{2(\cos^2 x - \cos^2 \theta)}{\cos x - \cos \theta} dx$$

$$= 2\int \frac{(\cos x - \cos \theta)(\cos x + \cos \theta)}{\cos x - \cos \theta} dx$$

$$= 2\int (\cos x + \cos \theta) dx = 2\int \cos x \, dx + 2\cos \theta \int 1 dx$$

$$= 2\sin x + 2\cos \theta \cdot x + c$$

$$= 2(\sin x + x\cos \theta) + c$$

Content Outline

- Method of Substitution or Change of Variable
- Trigonometric Substitution

METHOD OF SUBSTITUTION OR CHANGE OF VARIABLE

The method stems from reversing the Chain Rule for derivatives. Recall that for the differentiable functions $y = F(u)$ and $u = g(x)$ the Chain Rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

In terms of indefinite integrals, this becomes $\int F'(g(x))g'(x) = F(g(x)) + c$

Theorem 3.1: Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f ($F' = f$) on I, then $\int f(g(x)) \cdot g'(x) dx = F(g(x)) + c$.

Letting $u = g(x)$ gives $du = g'(x)dx$ and $\int f(u)du = F(u) + c$.

The equation $\int f(u)du = F(u) + c$ has the same form as the basic antiderivative formula, $\int f(x)dx = F(x) + c$, but the independent variable is now u.

The process of replacing x with a new variable $u = g(x)$ and dx with the corresponding du is called a **change of variable** or **the method of substitution**.

Recall!

The Rule 4: $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

Let $u = f(x)$. Then $du = f'(x)dx$.

Substituting into the given integral, we have $\int \frac{f'(x)}{f(x)} dx = \int \frac{1}{u} du$

We know from basic integration rules that $\int \frac{1}{u} du = \ln|u| + c$

Finally, substituting $u = f(x)$, we obtain $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

www.ilmkidunya.com

www.ilmkidunya.com

Example 13: Evaluate $\int (5x^5+5)^3 x^4\,dx$

Solution:

$$\int (5x^5+5)^3 x^4\,dx$$

Multiply and divide by 25

$$= \frac{1}{25}\int (5x^5+5)^3 \cdot 25x^4\,dx$$

Let $f(x) = 5x^5+5$

$\Rightarrow$ $f'(x) = 25x^4$

Using: $\int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c; n \neq -1$

$$= \frac{1}{25}\frac{(5x^5+5)^{3+1}}{3+1} + c$$

$$= \frac{1}{100}(5x^5+5)^4 + c$$

Example 14: Integrate $\int \sqrt{x}\sqrt{1+\sqrt{x}}\,dx$

Solution:

$$\int \sqrt{x}\sqrt{1+\sqrt{x}}\,dx$$

Let $u = 1+\sqrt{x} \Rightarrow \sqrt{x} = u-1 \Rightarrow x = (u-1)^2$

$dx = d(u-1)^2 \Rightarrow dx = 2(u-1)^1 du$

$$\int \sqrt{x}\cdot\sqrt{1+\sqrt{x}}\,dx = \int (u-1)\cdot\sqrt{u}\cdot 2(u-1)du$$

$$= 2\int (u-1)^2 \cdot \sqrt{u}\,du$$

$$= 2\int (u^2-2u+1)\cdot\sqrt{u}\,du$$

$$= 2\int (u^2\cdot u^{\frac{1}{2}} - 2u\cdot u^{\frac{1}{2}} + u^{\frac{1}{2}})du$$

$$= 2\left(\int u^{\frac{5}{2}}du - 2\int u^{\frac{3}{2}}du + \int u^{\frac{1}{2}}du\right)$$

Using: $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

$$= 2\left(\frac{u^{\frac{5}{2}+1}}{\frac{5}{2}+1} - 2\cdot\frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$= 2\left(\frac{u^{\frac{7}{2}}}{\frac{7}{2}} - 2\cdot\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$= 2\left(\frac{2}{7}u^{\frac{7}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + c$$

$$= \frac{4}{7}u^{\frac{7}{2}} - \frac{8}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + c$$

Put $u = 1+\sqrt{x}$

$$= \frac{4}{7}(1+\sqrt{x})^{\frac{7}{2}} - \frac{8}{5}(1+\sqrt{x})^{\frac{5}{2}} + \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + c$$

Example 15: Evaluate $\int \frac{x}{\sqrt{1+x^2}+1+x^2}\,dx$

Solution:

$$\int \frac{x}{\sqrt{1+x^2}+1+x^2}\,dx$$

Let $u = \sqrt{1+x^2} \Rightarrow u^2 = 1+x^2 \Rightarrow d(u^2) = d(1+x^2)$

$2u\,du = 2x\,dx \Rightarrow x\,dx = u\,du$

$$\int \frac{x}{\sqrt{1+x^2}+1+x^2}\,dx = \int \frac{u\,du}{u+u^2}$$

$$= \int \frac{u}{u(1+u)}du$$

$$\int \frac{x}{\sqrt{1+x^2}+1+x^2}\,dx = \int \frac{1}{1+u}du$$

Using: $\int \frac{f'(u)}{f(u)}du = \ln|f(u)| + c$

$$= \ln|1+u| + c$$

Put $u = \sqrt{1+x^2}$

$$= \ln|1+\sqrt{1+x^2}| + c$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 16: Evaluate $\int \frac{\sin x}{e^{\cos x}} dx$

Solution:

$$\int \frac{\sin x}{e^{\cos x}} dx$$

Let $u = \cos x \Rightarrow du = d(\cos x)$

$du = -\sin x\, dx \Rightarrow \sin x\, dx = -du$

$$\int \frac{\sin x}{e^{\cos x}} dx = \int \frac{-du}{e^u}$$

$$\int \frac{\sin x}{e^{\cos x}} dx = -\int e^{-u} du = \frac{e^{-u}}{-1} + c$$

$$= e^{-u} + c$$

Put $u = \cos x$

$$= e^{-\cos x} + c$$

Example 17: Evaluate $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$

Solution:

$$\int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^{\tan^{-1} x} \cdot \frac{1}{1+x^2} dx$$

Let $u = \tan^{-1} x \Rightarrow du = d(\tan^{-1} x)$

$$du = \frac{1}{1+x^2} dx$$

$$\int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^u du$$

$$= e^u + c$$

Put $u = \tan^{-1} x$

$$\int \frac{e^{\tan^{-1} x}}{1+x^2} dx = e^{\tan^{-1} x} + c$$

Example 18: Evaluate $\int \frac{e^{2x} + 2e^x}{(e^x+1)^2} dx$

Solution:

$$\int \frac{e^{2x} + 2e^x}{(e^x+1)^2} dx = \int \frac{e^x + 2}{(e^x+1)^2} \cdot e^x dx$$

$$= \int \frac{(e^x+1)+1}{(e^x+1)^2} \cdot e^x dx$$

Let $u = e^x + 1 \Rightarrow du = d(e^x+1) \Rightarrow du = e^x dx$

$$\int \frac{e^{2x} + 2e^x}{(e^x+1)^2} dx = \int \frac{u+1}{u^2} du$$

$$= \int \left(\frac{u}{u^2} + \frac{1}{u^2}\right) du = \int \left(\frac{1}{u} + u^{-2}\right) du$$

$$\int \frac{e^{2x} + 2e^x}{(e^x+1)^2} dx = \int \frac{1}{u} du + \int u^{-2} du$$

$$= \ln|u| + \frac{u^{-2+1}}{-2+1} + c$$

$$= \ln|u| + \frac{u^{-1}}{-1} + c$$

$$= \ln|u| - \frac{1}{u} + c$$

Put $u = e^x + 1$

$$= \ln|e^x+1| - \frac{1}{e^x+1} + c$$

Example 19: Evaluate $\int \frac{e^x \ln(1+e^x)}{1+e^x} dx$

Solution:

$$\int \frac{e^x \ln(1+e^x)}{1+e^x} dx = \int \ln(1+e^x) \cdot \frac{e^x}{1+e^x} dx$$

Let $u = \ln(1+e^x) \Rightarrow du = d(\ln(1+e^x))$

$$du = \frac{1}{1+e^x} e^x dx = \frac{e^x}{1+e^x} dx$$

$$\int \frac{e^x \cdot \ln(1+e^x)}{1+e^x} dx = \int u \cdot du$$

$$= \frac{u^{1+1}}{1+1} + c = \frac{1}{2} u^2 + c$$

Put $u = \ln(1+e^x)$

$$= \frac{1}{2} [\ln(1+e^x)]^2 + c$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 20: Evaluate $\int \frac{\ell n(1+x)-\ell n\, x}{x(1+x)}dx$

Solution:

$$\int \frac{\ell n(1+x)-\ell n\, x}{x(1+x)}dx$$

Let $u=\ell n(1+x)-\ell n\, x \Rightarrow du=d(\ell n(1+x)-\ell n\, x)$

$$du=\left(\frac{1}{1+x}-\frac{1}{x}\right)dx \Rightarrow du=\frac{x-1-x}{x(1+x)}dx$$

$$du=\frac{-1}{x(1+x)}dx \Rightarrow -du=\frac{1}{x(1+x)}dx$$

$$\int \frac{\ell n(1+x)-\ell n\, x}{x(1+x)}dx = \int u(-du)$$

$$\int \frac{\ell n(1+x)-\ell n\, x}{x(1+x)}dx = -\int u\,du$$

$$= -\frac{u^{1+1}}{1+1}+c$$

$$= -\frac{1}{2}u^2+c$$

Put $u=\ell n(1+x)-\ell n\, x$

$$= -\frac{1}{2}[\ell n(1+x)-\ell n\, x]^2+c$$

Example 21: Evaluate $\int \frac{x^2\tan^{-1}(x^3)}{1+x^6}dx$

Solution:

$$\int \frac{x^2\tan^{-1}(x^3)}{1+x^6}dx = \int \tan^{-1}(x^3)\cdot\frac{x^2}{1+x^6}dx$$

Let $u=\tan^{-1}(x^3) \Rightarrow du=d(\tan^{-1}x^3)$

$$du=\frac{1}{1+(x^3)^2}\cdot(3x^2)dx \Rightarrow \frac{1}{3}du=\frac{x^2}{1+x^6}dx$$

$$\int \frac{x^2\tan^{-1}(x^3)}{1+x^6}dx = \int u\cdot\frac{1}{3}du$$

$$\int \frac{x^2\tan^{-1}(x^3)}{1+x^6}dx = \frac{1}{3}\int u\,du$$

$$= \frac{1}{3}\frac{u^{1+1}}{1+1}+c = \frac{1}{6}u^2+c$$

Put $u=\tan^{-1}(x^3)$

$$= \frac{1}{6}[\tan^{-1}(x^3)]^2+c$$

TRIGONOMETRIC SUBSTITUTION

Integrals involving the radicals $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, and $\sqrt{a^2+x^2}$ (where $a>0$) can often be evaluated by making a substitution involving a trigonometric function. The objective of trigonometric substitution is to eliminate the radical in the integrand. We do this by using the Pythagorean identities:

$$\sin^2\theta+\cos^2\theta=1 \quad \text{and} \quad \sec^2\theta-\tan^2\theta=1$$

Case 1: Integrals involving $\sqrt{a^2-x^2}$ (where, $a>0$) can frequently be reduced to a simpler form by using the substitution $x=a\sin\theta$, where $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Case 2: Integrals involving $\sqrt{a^2+x^2}$ (where, $a>0$) can frequently be reduced to a simpler form by using the substitution $x=a\tan\theta$, where $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Case 3: Finally, integrals involving $\sqrt{x^2-a^2}$ (where, $a>0$) can frequently be reduced to a simpler form by using the substitution $x=a\sec\theta$, where $\theta\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$.

www.ilmkidunya.com

www.ilmkidunya.com

Some Special Integrals as Formulas

Prove that: (i) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$, where $x \neq \pm a$

(ii) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$, where $|x| < a > 0$

Proof: (i)

$$\text{L.H.S} = \int \frac{1}{a^2-x^2}dx$$

$$= \int \frac{1}{(a-x)(a+x)}dx$$

$$= \frac{1}{2a}\int \frac{2a}{(a-x)(a+x)}dx \quad \text{Multiply and divide by } 2a$$

$$= \frac{1}{2a}\int \frac{(a-x)+(a+x)}{(a-x)(a+x)}dx$$

$$= \frac{1}{2a}\int \left[\frac{(a-x)}{(a-x)(a+x)} + \frac{(a+x)}{(a-x)(a+x)}\right]dx$$

$$= \frac{1}{2a}\int \left[\frac{1}{a+x} + \frac{1}{a-x}\right]dx$$

$$= \frac{1}{2a}\left[\int \frac{1}{a+x}dx - \int \frac{-1}{a-x}dx\right]$$

Using: $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

$$= \frac{1}{2a}(\ln|a+x| - \ln|a-x|) + c$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c = \text{R.H.S (Proved)}$$

Proof: (ii)

$$\text{L.H.S} = \int \frac{1}{\sqrt{a^2-x^2}}dx \text{, where } a > 0, \text{ domain} = (-a, a)$$

Let $x = a\sin\theta$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow dx = a\cos\theta\, d\theta$$

$$\text{L.H.S} = \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}} \cdot a\cos\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{a^2(1-\sin^2\theta)}} a\cos\theta\, d\theta$$

$$= \frac{a}{a}\int \frac{1}{\sqrt{\cos^2\theta}} \cdot \cos\theta\, d\theta$$

$$= \int \frac{1}{\cos\theta} \cdot \cos\theta\, d\theta \qquad \because \cos\theta > 0 \text{ on } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$= \int 1 d\theta$$

$$= \theta + c$$

$$= \sin^{-1}\frac{x}{a} + c \qquad \text{As } x = a\sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{a}$$

$$= \text{R.H.S (Proved)}$$

Challenge!

Prove the given formulae:

(i) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$, where $a \neq 0, x \neq \pm a$

(ii) $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$, where $a \neq 0$

(iii) $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x + \sqrt{a^2+x}\right| + c$, where $a \neq 0$

(iv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + c$, where $|x| > a > 0$

(v) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$, where $|x| > a > 0$

www.ilmkidunya.com

www.ilmkidunya.com

(i) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$, where $a \neq 0, x \neq \pm a$

Proof:

$$\text{L.H.S} = \int \frac{1}{x^2-a^2}dx$$

$$= \int \frac{1}{(x-a)(x+a)}dx$$

$$= \frac{1}{2a}\int \frac{2a}{(x-a)(x+a)}dx \quad \text{Multiply and divide by } 2a$$

$$= \frac{1}{2a}\int \frac{(x+a)-x+a}{(x-a)(x+a)}dx$$

$$= \frac{1}{2a}\int\left[\frac{x+a}{(x-a)(x+a)} - \frac{x-a}{(x-a)(x+a)}\right]dx$$

$$= \frac{1}{2a}\left[\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx\right]$$

Using: $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

$$= \frac{1}{2a}(\ln|x-a| - \ln|x+a|) + c$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c = \text{R.H.S (Proved)}$$

(ii) $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$, where $a \neq 0$

Proof:

$$\text{L.H.S} = \int \frac{1}{a^2+x}dx$$

Let $x = a\tan\theta \quad \Rightarrow dx = a\sec^2\theta\, d\theta$

$$\text{L.H.S} = \int \frac{1}{a^2+a^2\tan^2\theta}\cdot a\sec^2\theta\, d\theta$$

$$= \int \frac{1}{a^2(1+\tan^2\theta)}\cdot a\sec^2\theta\, d\theta$$

$$= \frac{1}{a}\int \frac{1}{\sec^2\theta}\cdot \sec^2\theta\, d\theta \quad \because 1+\tan^2\theta = \sec^2\theta$$

$$= \frac{1}{a}\int 1 d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c \quad \begin{cases} \text{As } x = a\tan\theta \\ \Rightarrow \tan\theta = \frac{x}{a} \Rightarrow \theta = \tan^{-1}\frac{x}{a} \end{cases}$$

$$= \text{R.H.S (Proved)}$$

(iii) $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x+\sqrt{a^2+x}\right| + c$, where $a \neq 0$

Proof:

$$\text{L.H.S} = \int \frac{1}{\sqrt{a^2+x^2}}dx$$

Let $x = a\tan\theta \quad \Rightarrow dx = a\sec^2\theta\, d\theta$

$$\text{L.H.S} = \int \frac{1}{\sqrt{a^2+a^2\tan^2\theta}}\cdot a\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{a^2(1+\tan^2\theta)}}a\sec^2\theta\, d\theta$$

$$= \frac{a}{a}\int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}d\theta$$

$$= \int \frac{\sec^2\theta}{\sec\theta}d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta+\tan\theta| + c_1 = \ln|\sqrt{\sec^2\theta}+\tan\theta| + c_1$$

$$= \ln|\sqrt{1+\tan^2\theta}+\tan\theta| + c_1$$

$$= \ln\left|\sqrt{1+\frac{x^2}{a^2}}+\frac{x}{a}\right| + c_1 \quad \because x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a}$$

$$= \ln\left|\sqrt{\frac{a^2+x^2}{a^2}}+\frac{x}{a}\right| + c_1 = \ln\left|\frac{\sqrt{a^2+x^2}}{a}+\frac{x}{a}\right| + c_1$$

$$= \ln\left|\frac{\sqrt{a^2+x^2}+x}{a}\right| + c_1$$

$$= \ln|x+\sqrt{a^2+x^2}| - \ln|a| + c_1$$

$$= \ln|x+\sqrt{a^2+x^2}| + c, \text{ where } c = -\ln|a| + c_1$$

$$= \text{R.H.S (Proved)}$$

(iv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c$, where $|x| > a > 0$

Proof:

$$\text{L.H.S} = \int \frac{1}{\sqrt{x^2-a^2}}dx$$

Let $x = a\sec\theta \quad \Rightarrow dx = a\sec\theta\tan\theta\, d\theta$

$$\text{L.H.S} = \int \frac{1}{\sqrt{a^2\sec^2\theta-a^2}}\cdot a\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{a^2(\sec^2\theta-1)}}\cdot a\sec\theta\cdot\tan\theta\, d\theta$$

$$= \frac{a}{a}\int \frac{1}{\sqrt{\tan^2\theta}}\cdot \sec\theta\cdot\tan\theta\, d\theta$$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \int \frac{\sec\theta\tan\theta}{\tan\theta}d\theta$$
$$= \int \sec\theta\, d\theta$$
$$= \ln|\sec\theta+\tan\theta|+c_1$$
$$= \ln|\sec\theta+\sqrt{\tan^2\theta}|+c_1$$
$$= \ln|\sec\theta+\sqrt{\sec^2\theta-1}|+c_1$$
$$= \ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|+c_1 \quad \because x=a\sec\theta \Rightarrow \sec\theta=\frac{x}{a}$$
$$= \ln\left|\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right|+c_1$$
$$= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+c_1$$
$$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+c_1$$
$$= \ln|x+\sqrt{x^2-a^2}|-\ln a+c_1$$
$$= \ln|x+\sqrt{x^2-a^2}|+c, \text{ where } c=-\ln a+c_1$$
= R.H.S (Proved)

(v) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c$, where $|x|>a>0$

Proof:

L.H.S $= \int \frac{1}{x\sqrt{x^2-a^2}}dx$

Let $x=a\sec\theta \Rightarrow x=a\sec\theta\tan\theta\, d\theta$

L.H.S $= \int \frac{1}{a\sec\theta\sqrt{a^2\sec^2\theta-a^2}}\cdot a\sec\theta\tan\theta\, d\theta$
$$= \frac{a}{a}\int \frac{\sec\theta\cdot\tan\theta}{\sec\theta\sqrt{a^2(\sec^2\theta-1)}}d\theta$$
$$= \int \frac{\tan\theta}{a\sqrt{\tan^2\theta}}d\theta \quad \because 1+\tan^2\theta=\sec^2\theta$$
$$= \frac{1}{a}\int \frac{\tan\theta}{\tan\theta}d\theta = \frac{1}{a}\int 1\, d\theta$$
$$= \frac{1}{a}\theta+c \quad \left\{\begin{array}{l}\text{As } x=a\sec\theta \Rightarrow \sec\theta=\frac{x}{a} \\ \Rightarrow \theta=\sec^{-1}\left(\frac{|x|}{a}\right)\end{array}\right.$$
$$= \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right)+c$$
= R.H.S (Proved)

Example 22: Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}dx$

Solution:

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{\sqrt{3^2-x^2}}{x^2}dx$$

Let $x=3\sin\theta \Rightarrow dx=3\cos\theta\, d\theta$

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\cdot 3\cos\theta\, d\theta$$
$$= \int \frac{\sqrt{9(1-\sin^2\theta)}}{3\sin^2\theta}\cdot\cos\theta\, d\theta$$
$$= \int \frac{3\sqrt{\cos^2\theta}}{3\sin^2\theta}\cdot\cos\theta\, d\theta$$
$$= \int \frac{\cos\theta}{\sin^2\theta}\cdot\cos\theta\, d\theta$$
$$= \int \frac{\cos^2\theta}{\sin^2\theta}d\theta = \int \cot^2\theta\, d\theta$$
$$= \int (\csc^2\theta-1)d\theta \quad \because 1+\cot^2\theta=\csc^2\theta$$
$$= -\cot\theta-\theta+c = \frac{\cos\theta}{\sin\theta}-\theta+c$$
$$= \frac{\sqrt{\cos^2\theta}}{\sin\theta}-\theta+c$$
$$= \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}-\theta+c$$

As $x=3\sin\theta \Rightarrow \sin\theta=\frac{x}{3} \Rightarrow \theta=\sin^{-1}\frac{x}{3}$

$$= \frac{\sqrt{1-\left(\frac{x}{3}\right)^2}}{\frac{x}{3}}-\sin^{-1}\left(\frac{x}{3}\right)+c$$
$$= \frac{\sqrt{\frac{9-x^2}{9}}}{\frac{x}{3}}-\sin^{-1}\left(\frac{x}{3}\right)+c$$
$$= \frac{1}{3}\sqrt{9-x^2}\cdot\frac{3}{x}-\sin^{-1}\left(\frac{x}{3}\right)+c$$
$$= \frac{\sqrt{9-x^2}}{x}-\sin^{-1}\left(\frac{x}{3}\right)+c$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 23: Evaluate $\int \frac{1}{x\sqrt{x^4+1}}dx, x\in(-\infty,0)\cup(0,\infty)$

Solution:

$$\int \frac{1}{x\sqrt{x^4+1}}dx = \frac{1}{2}\int \frac{2x}{x^2\sqrt{(x^2)^2+1^2}}dx$$

Let $x^2 = 1\cdot\tan\theta \Rightarrow 2x\,dx = \sec^2\theta\,d\theta$

$$\int \frac{1}{x\sqrt{x^4+1}}dx = \frac{1}{2}\int \frac{\sec^2\theta\,d\theta}{\tan\theta\sqrt{\tan^2\theta+1}}$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\tan\theta\sqrt{\sec^2\theta}}d\theta$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\frac{\sin\theta}{\cos\theta}\cdot\sec\theta}d\theta$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\sin\theta\cdot\frac{1}{\cos\theta}}d\theta$$

$$= \frac{1}{2}\int \csc\theta\cdot\frac{\sec\theta}{\sec\theta}d\theta$$

$$= \frac{1}{2}\int \csc\theta\,d\theta$$

$$= \frac{1}{2}\ln|\csc\theta-\cot\theta|+c$$

$$= \frac{1}{2}\ln|\sqrt{\csc^2\theta}-\cot\theta|+c$$

$$= \frac{1}{2}\ln|\sqrt{1+\cot^2\theta}-\cot\theta|+c$$

$$= \frac{1}{2}\ln\left|\sqrt{1+\left(\frac{1}{x^2}\right)^2}-\frac{1}{x^2}\right|+c \quad \left\{\begin{array}{l}\text{As } x^2=\tan\theta\\ \Rightarrow \cot\theta=\frac{1}{x^2}\end{array}\right.$$

$$= \frac{1}{2}\ln\left|\sqrt{\frac{x^4+1}{x^4}}-\frac{1}{x^2}\right|+c$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{x^4+1}}{x^2}-\frac{1}{x^2}\right|+c$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{x^4+1}-1}{x^2}\right|+c$$

Example 24: Evaluate $\int \frac{\sqrt{x^2-9}}{x}dx, (x>3)$

Solution:

$$\int \frac{\sqrt{x^2-9}}{x}dx = \int \frac{\sqrt{x^2-3^2}}{x}dx$$

Let $x = 3\sec\theta \Rightarrow dx = 3\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{x^2-9}}{x}dx = \int \frac{\sqrt{9\sec^2\theta-9}}{3\sec\theta}\cdot 3\sec\theta\cdot\tan\theta\,d\theta$$

$$= \int \sqrt{9(\sec^2\theta-1)}\cdot\tan\theta\,d\theta$$

$$= \int 3\sqrt{\tan^2\theta}\cdot\tan\theta\,d\theta$$

$$= 3\int \tan\theta\cdot\tan\theta\,d\theta$$

$$= 3\int \tan^2\theta\,d\theta$$

$$= 3\int(\sec^2\theta-1)d\theta$$

$$= 3(\tan\theta-\theta)+c$$

As $x = 3\sec\theta \Rightarrow \sec\theta = \frac{x}{3} \Rightarrow \theta = \sec^{-1}\left(\frac{x}{3}\right)$

$$= 3\left[\sqrt{\tan^2\theta}-\sec^{-1}\left(\frac{x}{3}\right)\right]+c$$

$$= 3\left[\sqrt{\sec^2\theta-1}-\sec^{-1}\left(\frac{x}{3}\right)\right]+c$$

$$= 3\left[\sqrt{\frac{x^2}{9}-1}-\sec^{-1}\left(\frac{x}{3}\right)\right]+c \quad \because \sec\theta = \frac{x}{3}$$

$$= 3\left[\sqrt{\frac{x^2-9}{9}}-\sec^{-1}\left(\frac{x}{3}\right)\right]+c$$

$$= \frac{3\sqrt{x^2-9}}{3}-3\sec^{-1}\left(\frac{x}{3}\right)+c$$

$$= \sqrt{x^2-9}-3\sec^{-1}\left(\frac{x}{3}\right)+c$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 25: Evaluate $\int \frac{(x+1)e^x}{1+x^2e^{2x}}dx$

Solution:

$$\int \frac{(x+1)e^x}{1+x^2e^{2x}}dx = \int \frac{(x+1)e^x}{1+(xe^x)^2}dx$$

Let $u = xe^x \Rightarrow du = d(xe^x)$

$du = (xe^x + e^x \cdot 1)dx \Rightarrow du = (x+1)e^x dx$

$$\int \frac{(x+1)e^x}{1+x^2e^{2x}}dx = \int \frac{1}{1+u^2}du$$
$$= \tan^{-1}u + c$$
$$= \tan^{-1}(xe^x) + c \quad \because u = xe^x$$

Example 26: Evaluate $\int \frac{1}{x\sqrt{1-x^3}}dx, x \in (-\infty, 0) \cup (0, 1)$.

Solution:

$$\int \frac{1}{x\sqrt{1-x^3}}dx = \int \frac{x^2}{x^3\sqrt{1-x^3}}dx$$

Let $u = \sqrt{1-x^3} \Rightarrow u^2 = 1-x^3 \Rightarrow x^3 = 1-u^2$

$d(x^3) = d(1-u^2) \Rightarrow 3x^2dx = -2u\,du \Rightarrow x^2dx = \frac{-2}{3}u\,du$

$$\int \frac{1}{x\sqrt{1-x^3}}dx = \int \frac{\frac{-2}{3}u\,du}{(1-u^2)\cdot u}$$
$$= \frac{2}{3}\int \frac{1}{u^2-1^2}du$$

Formula: $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

$$= \frac{2}{3}\cdot\frac{1}{2(1)}\ln\left|\frac{u-1}{u+1}\right| + c$$
$$= \frac{1}{3}\ln\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + c \quad \because u = \sqrt{1-x^3}$$

Example 27: Evaluate $\int \frac{2x}{\sqrt{x^4+9}}dx$

Solution:

$$\int \frac{2x}{\sqrt{x^4+9}}dx = \int \frac{2x}{\sqrt{(x^2)^2+9}}dx$$

Put $x^2 = u \Rightarrow 2x\,dx = du$

$$\int \frac{2x}{\sqrt{x^4+9}}dx = \int \frac{1}{\sqrt{u^2+9}}du$$
$$= \int \frac{1}{\sqrt{u^2+3^2}}du$$

Formula: $\int \frac{1}{\sqrt{a^2+x^2}}dx = \ln|x + \sqrt{a^2+x^2}| + c$

$$= \ln|u + \sqrt{u^2+3^2}| + c$$
$$= \ln|x^2 + \sqrt{(x^2)^2+9}| + c \quad \because u = x^2$$
$$= \ln|x^2 + \sqrt{x^4+9}| + c$$

Example 28: Evaluate $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Solution:

$$\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{1}{(x-2)\sqrt{x^2-4x+4-1}}dx$$
$$= \int \frac{1}{(x-2)\sqrt{(x-2)^2-1}}dx$$

Put $u = x-2 \Rightarrow du = dx$

$$= \int \frac{1}{u\sqrt{u^2-1^2}}du$$

Formula: $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + c$

$$= \frac{1}{1}\sec^{-1}\left|\frac{u}{1}\right| + c$$
$$= \sec^{-1}|u| + c$$
$$= \sec^{-1}|x-2| + c \quad \because u = x-2$$

www.ilmkidunya.com

www.ilmkidunya.com

EXERCISE 3.2

Q1. Evaluate the following integrals:

(i) $\int \frac{ax+b}{\sqrt{ax^2+2bx+c}}dx$

Solution:

$$\int \frac{ax+b}{\sqrt{ax^2+2bx+c}}dx$$

Multiply and divide by '2'

$$=\frac{1}{2}\int (ax^2+2bx+c)^{-\frac{1}{2}}(2ax+2b)dx$$

Let $f(x)=ax^2+2bx+c$

$f'(x)=2ax+2b(1)+0=2ax+2b$

Using: $\int [f(x)]^n \cdot f'(x)dx=\frac{[f(x)]^{n+1}}{n+1}+c; n\neq -1$

$$=\frac{1}{2}\cdot\frac{(ax^2+2bx+c)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c'$$

$$=\frac{1}{2}\cdot\frac{2}{1}(ax^2+2bx+c)^{\frac{1}{2}}+c'$$

$$=\sqrt{ax^2+2bx+c}+c'$$

(ii) $\int x\sqrt{1+\sqrt{x}}\,dx$

Solution:

$$\int x\sqrt{1+\sqrt{x}}\,dx$$

Let $u=1+\sqrt{x} \Rightarrow \sqrt{x}=u-1 \Rightarrow x=(u-1)^2$

$\Rightarrow dx=2(u-1)^1 du$

$$\int x\sqrt{1+\sqrt{x}}dx = \int (u-1)^2\cdot\sqrt{u}\cdot 2(u-1)du$$

$$=2\int \sqrt{u}(u-1)^3 du$$

$$=2\int \sqrt{u}(u^3-3u^2(1)+3u(1)^2-1^3)du$$

$$=2\int \sqrt{u}(u^3-3u^2+3u-1)du$$

$$=2\int (u^{\frac{7}{2}}-3u^{\frac{5}{2}}+3u^{\frac{3}{2}}-u^{\frac{1}{2}})du$$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$

$$=2\left[\frac{u^{\frac{7}{2}+1}}{\frac{7}{2}+1}-3\frac{u^{\frac{5}{2}+1}}{\frac{5}{2}+1}+3\frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1}-\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]+c$$

$$=2\left[\frac{2}{9}u^{\frac{9}{2}}-3\left(\frac{2}{7}\right)u^{\frac{7}{2}}+3\left(\frac{2}{5}\right)u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}\right]+c$$

$$=\frac{4}{9}u^{\frac{9}{2}}-\frac{12}{7}u^{\frac{7}{2}}+\frac{12}{5}u^{\frac{5}{2}}-\frac{4}{3}u^{\frac{3}{2}}+c$$

Put $u=1+\sqrt{x}$

$$=\frac{4}{9}(1+\sqrt{x})^{\frac{9}{2}}-\frac{12}{7}(1+\sqrt{x})^{\frac{7}{2}}+\frac{12}{5}(1+\sqrt{x})^{\frac{5}{2}}-\frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}}+c$$

(iii) $\int \frac{x}{1+x\tan x}dx$

Solution:

$$\int \frac{x}{1+x\tan x}dx$$

$$=\int \frac{x}{1+x\frac{\sin x}{\cos x}}dx$$

$$=\int \frac{x}{\frac{\cos x+x\sin x}{\cos x}}dx$$

$$=\int \frac{x\cos x}{\cos x+x\sin x}dx$$

Let $f(x)=\cos x+x\sin x$

$\Rightarrow f'(x)=-\sin x+x\cos x+\sin x\cdot 1=x\cos x$

Using: $\int \frac{f'(x)}{f(x)}dx=\ln|f(x)|+c$

$$=\ln|\cos x+x\sin x|+c$$

(iv) $\int \frac{x}{(a+bx)^3}dx$

Solution:

$$\int \frac{x}{(a+bx)^3}dx$$

Multiply and divide by 'b'

$$=\frac{1}{b}\int \frac{bx}{(a+bx)^3}dx$$

$$=\frac{1}{b}\int \frac{(a+bx)-a}{(a+bx)^3}dx$$

$$=\frac{1}{b}\int \left[\frac{a+bx}{(a+bx)^3}-\frac{a}{(a+bx)^3}\right]dx$$

$$=\frac{1}{b}\int [(a+bx)^{-2}-a(a+bx)^{-3}]dx$$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \frac{1}{b}\int (a+bx)^{-2}dx - \frac{a}{b}\int (a+bx)^{-3}dx$$

$$= \frac{1}{b}\cdot\frac{1}{b}\int (a+bx)^{-2}\cdot b\,dx - \frac{a}{b}\cdot\frac{1}{b}\int (a+bx)^{-3}\cdot b\,dx$$

Using: $\int [f(x)]^n \cdot f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c; n \neq -1$

$$= \frac{1}{b^2}\cdot\frac{(a+bx)^{-2+1}}{-2+1} - \frac{a}{b^2}\frac{(a+bx)^{-3+1}}{-3+1} + c$$

$$= \frac{1}{b^2}\left[\frac{(a+bx)^{-1}}{-1} - \frac{a(a+bx)^{-2}}{-2}\right] + c$$

$$= -\frac{1}{b^2}\left[\frac{1}{a+bx} - \frac{a}{2(a+bx)^2}\right] + c$$

$$= -\frac{1}{b^2}\left[\frac{2(a+bx)-a}{2(a+bx)^2}\right] + c$$

$$= -\frac{1}{b^2}\left[\frac{2a+2bx-a}{2(a+bx)^2}\right] + c$$

$$= -\frac{a+2bx}{2b^2(a+bx)^2} + c$$

(v) $\int x^5\sqrt{x^3+1}\,dx$

Solution:

$$\int x^5\sqrt{x^3+1}dx = \int x^3\sqrt{x^3+1}\cdot x^2 dx$$

Let $u = x^3+1 \Rightarrow x^3 = u-1$

$\Rightarrow 3x^2dx = du - 0 \Rightarrow x^2dx = \frac{1}{3}du$

$$\int x^5\sqrt{x^3+1}\,dx = \int (u-1)\cdot\sqrt{u}\cdot\frac{1}{3}du$$

$$= \frac{1}{3}\int (u^{\frac{3}{2}} - u^{\frac{1}{2}})du$$

Using: $\int x^n dx = \frac{x^{n+1}}{n+1} + c; n \neq -1$

$$= \frac{1}{3}\left(\frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$= \frac{1}{3}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + c$$

$$= \frac{2}{15}u^{\frac{5}{2}} - \frac{2}{9}u^{\frac{3}{2}} + c$$

Put $u = x^3+1$

$$= \frac{2}{15}(x^3+1)^{\frac{5}{2}} - \frac{2}{9}(x^3+1)^{\frac{3}{2}} + c$$

(vi) $\int \frac{1}{1+\sqrt{2+x}}dx$

Solution:

$$\int \frac{1}{1+\sqrt{2+x}}dx$$

Let $u = \sqrt{2+x} \Rightarrow u^2 = 2+x$

$d(u^2) = d(2+x) = 2u\,du = 0 + dx \Rightarrow dx = 2u\,du$

$$\int \frac{1}{1+\sqrt{2+x}}dx = \int \frac{1}{1+u}(2u\,du)$$

$$= 2\int \frac{u}{1+u}du$$

$$= 2\int \frac{(1+u)-1}{1+u}du$$

$$= 2\int \left[\frac{1+u}{1+u} - \frac{1}{1+u}\right]du$$

$$= 2\int 1du - 2\int \frac{1}{1+u}du$$

$$= 2u - 2\ell n|1+u| + c$$

Put $u = \sqrt{2+x}$

$$= 2\sqrt{2+x} - 2\ell n|1+\sqrt{2+x}| + c$$

(vii) $\int \frac{\sec^2 x}{\tan x\cdot \ell n\tan x}dx$

Solution:

$$\int \frac{\sec^2 x}{\tan x\cdot \ell n\tan x}dx$$

$$= \int \frac{\sec^2 x \big/ \tan x}{\ell n(\tan x)}dx$$

Let $f(x) = \ell n(\tan x)$

$\Rightarrow f'(x) = \frac{1}{\tan x}\cdot\sec^2 x$

Using: $\int \frac{f'(x)}{f(x)}dx = \ell n|f(x)| + c$

$= \ell n|\ell n(\tan x)| + c$

(viii) $\int \frac{\sin x - \cos x}{\sqrt{1+\sin 2x}}dx$

Solution:

$$\int \frac{\sin x - \cos x}{\sqrt{1+\sin 2x}}dx$$

$$= \int \frac{\sin x - \cos x}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x}}dx$$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2}} dx$$

$$= -\int \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

Let $f(x) = \sin x + \cos x$

$\Rightarrow f'(x) = \cos x - \sin x$

Using: $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

$= -\ln|\sin x + \cos x| + c$

(ix) $\int \frac{\sin 2x}{\sqrt{1+\sin^2 x}} dx$

Solution:

$$\int \frac{\sin 2x}{\sqrt{1+\sin^2 x}} dx$$

$$= \int (1+\sin^2 x)^{-\frac{1}{2}} \cdot \sin 2x \, dx$$

Let $f(x) = 1 + \sin^2 x$

$\Rightarrow f'(x) = 0 + 2\sin x \cos x = \sin 2x$

Using: $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1$

$$= \frac{(1+\sin^2 x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{(1+\sin^2 x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{1+\sin^2 x} + c$$

(x) $\int \frac{\ln \sin x}{\tan x} dx$

Solution:

$$\int \frac{\ln \sin x}{\tan x} dx$$

$$= \int \ln(\sin x) \cdot \frac{1}{\tan x} dx$$

Let $f(x) = \ln(\sin x)$

$\Rightarrow f'(x) = \frac{1}{\sin x} \cdot \cos x = \cot x = \frac{1}{\tan x}$

Using: $\int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c; n \neq -1$

$$= \frac{[\ln(\sin x)]^{1+1}}{1+1} + c$$

$$= \frac{1}{2}[\ln(\sin x)]^2 + c$$

(xi) $\int \frac{e^{2x}}{(2+e^{2x})^2} dx$

Solution:

$$\int \frac{e^{2x}}{(2+e^{2x})^2} dx$$

$$= \int (2+e^{2x})^{-2} \cdot e^{2x} dx$$

Multiply and divide by '2'

$$= \frac{1}{2}\int (2+e^{2x})^{-2} \cdot 2e^{2x} dx$$

Let $f(x) = 2 + e^{2x}$

$\Rightarrow f'(x) = 0 + e^{2x} \cdot 2 = 2e^{2x}$

Using: $\int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c; n \neq -1$

$$= \frac{1}{2} \cdot \frac{(2+e^{2x})^{-2+1}}{-2+1} + c$$

$$= \frac{1}{2} \cdot \frac{(2+e^{2x})^{-1}}{-1} + c$$

$$= -\frac{1}{2(2+e^{2x})} + c$$

(xii) $\int \frac{e^{2x}-1}{e^{2x}+1} dx$

Solution:

$$\int \frac{e^{2x}-1}{e^{2x}+1} dx$$

Divide up and down by e^x

$$= \int \frac{\frac{e^{2x}}{e^x} - \frac{1}{e^x}}{\frac{e^{2x}}{e^x} + \frac{1}{e^x}} dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

Let $f(x) = e^x + e^{-x}$

$\Rightarrow f'(x) = e^x + e^{-x} \cdot (-1) = e^x - e^{-x}$

Using: $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

$= \ln(e^x + e^{-x}) + c$

$= \ln\left(e^x + \frac{1}{e^x}\right) + c = \ln\left(\frac{e^{2x}+1}{e^x}\right) + c$

$= \ln(e^{2x}+1) - \ln e^x + c$

$= \ln(e^{2x}+1) - x \ln e + c$

$= \ln(e^{2x}+1) - x + c \quad \because \ln e = 1$

www.ilmkidunya.com

www.ilmkidunya.com

Q2. Evaluate the following integrals:

(i) $\int \frac{e^x}{9-e^{2x}}dx$

Solution:

$$\int \frac{e^x}{9-e^{2x}}dx = \int \frac{e^x}{9-(e^x)^2}dx$$

Let $u = e^x \Rightarrow du = e^x dx$

$$\int \frac{e^x}{9-e^{2x}}dx = \int \frac{du}{9-u^2}$$

$$= \int \frac{1}{3^2-u^2}du$$

Formula: $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c$

$$= \frac{1}{2(3)}\ln\left|\frac{3+u}{3-u}\right|+c$$

Put $u = e^x$, we get

$$= \frac{1}{6}\ln\left|\frac{3+e^x}{3-e^x}\right|+c$$

(ii) $\int \frac{\cos x}{\sin^2 x-4}dx$

Solution:

$$\int \frac{\cos x}{\sin^2 x-4}dx = \int \frac{\cos x}{(\sin x)^2-4}dx$$

Let $u = \sin x \Rightarrow du = \cos x\,dx$

$$\int \frac{\cos x}{\sin^2 x-4}dx = \int \frac{1}{u^2-4}du$$

$$= \int \frac{1}{u^2-2^2}du$$

Formula: $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$

$$= \frac{1}{2(2)}\ln\left|\frac{u-2}{u+2}\right|+c$$

Put $u = \sin x$, we get

$$= \frac{1}{4}\ln\left|\frac{\sin x-2}{\sin x+2}\right|+c$$

(iii) $\int \frac{e^x}{1+e^{2x}}dx$

Solution:

$$\int \frac{e^x}{1+e^{2x}}dx = \int \frac{e^x}{1+(e^x)^2}dx$$

Let $u = e^x \Rightarrow du = e^x dx$

$$\int \frac{e^x}{1+e^{2x}}dx = \int \frac{du}{1+u^2}$$

$$= \int \frac{1}{1+u^2}du$$

$$= \tan^{-1}(u)+c$$

Put $u = e^x$, we get

$$= \tan^{-1}(e^x)+c$$

(iv) $\int \frac{1}{x\sqrt{4-(\ln x)^2}}dx$

Solution:

$$\int \frac{1}{x\sqrt{4-(\ln x)^2}}dx = \int \frac{1}{\sqrt{4-(\ln x)^2}}\cdot\frac{1}{x}dx$$

Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$\int \frac{1}{x\sqrt{4-(\ln x)^2}}dx = \int \frac{1}{\sqrt{4-u^2}}du$$

$$= \int \frac{1}{\sqrt{2^2-u^2}}du$$

Formula: $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+c$

$$= \sin^{-1}\left(\frac{u}{2}\right)+c$$

Put $u = \ln x$, we get

$$= \sin^{-1}\left(\frac{\ln x}{2}\right)+c$$

(v) $\int \frac{1}{9x^2-25}dx$

Solution:

$$\int \frac{1}{9x^2-25}dx = \int \frac{1}{(3x)^2-25}dx$$

Let $u = 3x \Rightarrow du = 3dx \Rightarrow dx = \frac{1}{3}du$

$$\int \frac{1}{9x^2-25}dx = \int \frac{\frac{1}{3}du}{u^2-25}$$

$$= \frac{1}{3}\int \frac{1}{u^2-5^2}du$$

www.ilmkidunya.com

www.ilmkidunya.com

Formula: $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

$$= \frac{1}{3}\cdot\frac{1}{2(5)}\ln\left|\frac{u-5}{u+5}\right| + c$$

Put $u = 3x$, we get

$$= \frac{1}{30}\ln\left|\frac{3x-5}{3x+5}\right| + c$$

(vi) $\int \frac{dx}{x\sqrt{3x^2-12}}$

Solution:

$$\int \frac{1}{x\sqrt{3x^2-12}}dx = \int \frac{1}{x\sqrt{(\sqrt{3}x)^2-12}}dx$$

Let $u = \sqrt{3}x \Rightarrow du = \sqrt{3}dx \Rightarrow dx = \frac{1}{\sqrt{3}}du$

$$\int \frac{1}{x\sqrt{3x^2-12}}dx = \int \frac{1}{\frac{1}{\sqrt{3}}u\sqrt{u^2-12}}\cdot\frac{1}{\sqrt{3}}du$$

$$= \int \frac{1}{u\sqrt{u^2-(2\sqrt{3})^2}}du$$

Formula: $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + c$

$$= \frac{1}{2\sqrt{3}}\sec^{-1}\left|\frac{u}{2\sqrt{3}}\right| + c$$

Put $u = \sqrt{3}x$, we get

$$= \frac{1}{2\sqrt{3}}\sec^{-1}\left|\frac{\sqrt{3}x}{2\sqrt{3}}\right| + c$$

$$= \frac{1}{2\sqrt{3}}\sec^{-1}\left|\frac{x}{2}\right| + c$$

(vii) $\int \frac{x^2}{x^2+a^2}dx$

Solution:

$$\int \frac{x^2}{x^2+a^2}dx$$

Let $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$

$$\int \frac{x^2}{x^2+a^2}dx = \int \frac{a^2\tan^2\theta}{a^2\tan^2\theta + a^2}\cdot a\sec^2\theta\, d\theta$$

$$= \int \frac{a^3\tan^2\theta\cdot\sec^2\theta}{a^2(\tan^2\theta+1)}d\theta$$

$$= \int \frac{a\tan^2\theta\cdot\sec^2\theta}{\sec^2\theta}d\theta$$

$$= a\int \tan^2\theta\, d\theta$$

$$= a\int (\sec^2\theta - 1)d\theta$$

$$= a(\tan\theta - \theta) + c$$

$$= a\left(\frac{x}{a} - \tan^{-1}\frac{x}{a}\right) + c$$

$$= x - a\tan^{-1}\left(\frac{x}{a}\right) + c$$

As $x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right)$

(viii) $\int \frac{1}{x[1+(\ln x)^2]}dx$

Solution:

$$\int \frac{1}{x[1+(\ln x)^2]}dx = \int \frac{1}{[1+(\ln x)^2]}\cdot\frac{1}{x}dx$$

Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$\int \frac{1}{x[1+(\ln x)^2]}dx = \int \frac{1}{1+u^2}du$$

$$= \tan^{-1}(u) + c$$

Put $u = \ln x$, we get

$$= \tan^{-1}(\ln x) + c$$

(ix) $\int \frac{1}{x^2\sqrt{x^2+4}}dx$

Solution:

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \int \frac{1}{x^2\sqrt{x^2+2^2}}dx$$

Put $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \int \frac{1}{(2\tan\theta)^2\sqrt{4\tan^2\theta+4}}\cdot 2\sec^2\theta\, d\theta$$

$$= \int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4(\tan^2\theta+1)}}d\theta$$

$$= \int \frac{\sec^2\theta}{2\tan^2\theta\cdot 2\sqrt{\sec^2\theta}}d\theta$$

$$= \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\cdot\sec\theta}d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta$$

$$= \frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}d\theta = \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}d\theta$$

www.ilmkidunya.com

www.ilmkidunya.com

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta = \frac{1}{4}\int (\sin\theta)^{-2}\cos\theta\, d\theta$$

Using: $\int [f(\theta)]^n \cdot f'(\theta)d\theta = \frac{[f(\theta)]^{n+1}}{n+1} + c; n \neq -1$

$$= \frac{1}{4}\cdot\frac{(\sin\theta)^{-2+1}}{-2+1} + c$$

$$= -\frac{1}{4}(\sin\theta)^{-1} + c$$

$$= -\frac{1}{4\sin\theta} + c$$

$$= -\frac{1}{4}\cdot\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos\theta} + c$$

$$= -\frac{1}{4}\cot\theta\cdot\sec\theta + c$$

$$= -\frac{1}{4}\cdot\frac{1}{\tan\theta}\cdot\sqrt{\sec^2\theta} + c$$

$$= -\frac{1}{4}\cdot\frac{1}{\tan\theta}\cdot\sqrt{1+\tan^2\theta} + c$$

$$= -\frac{1}{4}\cdot\frac{1}{x/2}\sqrt{1+\frac{x^2}{4}} + c \qquad \text{As } x = 2\tan\theta \Rightarrow \tan\theta = \frac{x}{2}$$

$$= -\frac{1}{4}\cdot\frac{2}{x}\sqrt{\frac{4+x^2}{4}} + c$$

$$= -\frac{1}{2x}\cdot\frac{1}{2}\sqrt{x^2+4} + c$$

$$= -\frac{\sqrt{x^2+4}}{4x} + c$$

(x) $\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$

Solution:

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx$$

Put $x = 1\tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$= \int \frac{1}{(1+\tan^2\theta)^{\frac{3}{2}}}\cdot\sec^2\theta\, d\theta$$

$$= \int \frac{1}{(\sec^2\theta)^{\frac{3}{2}}}\cdot\sec^2\theta\, d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^3\theta}d\theta = \int \frac{1}{\sec\theta}d\theta$$

$$= \int \cos\theta\, d\theta = \sin\theta + c$$

$$= \frac{\sin\theta}{\cos\theta}\cdot\cos\theta + c = \tan\theta\cdot\frac{1}{\sec\theta} + c$$

$$= \frac{\tan\theta}{\sqrt{\sec^2\theta}} + c = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} + c$$

$$= \frac{x}{\sqrt{1+x^2}} + c \qquad \because \tan\theta = x$$

(xi) $\int \frac{x^2+1}{x\sqrt{x^2-9}}dx$

Solution:

$$\int \frac{x^2+1}{x\sqrt{x^2-9}}dx$$

$$= \int\left[\frac{x^2}{x\sqrt{x^2-9}} + \frac{1}{x\sqrt{x^2-9}}\right]dx$$

$$= \int\left[\frac{x}{\sqrt{x^2-9}} + \frac{1}{x\sqrt{x^2-3^2}}\right]dx$$

$$= \frac{1}{2}\int (x^2-9)^{-\frac{1}{2}}(2x)dx + \int \frac{1}{x\sqrt{x^2-3^2}}dx$$

Using in 2nd integral: $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + c$

$$= \frac{1}{2}\cdot\frac{(x^2-9)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{1}{3}\sec^{-1}\left|\frac{x}{3}\right| + c$$

$$= \frac{1}{2}\cdot\frac{(x^2-9)^{\frac{1}{2}}}{\frac{1}{2}} + \frac{1}{3}\sec^{-1}\left|\frac{x}{3}\right| + c$$

$$= \sqrt{x^2-9} + \frac{1}{3}\sec^{-1}\left|\frac{x}{3}\right| + c$$

(xii) $\int \sqrt{\frac{1+x}{1-x}}dx$

Solution:

$$\int \sqrt{\frac{1+x}{1-x}}dx$$

$$= \int \sqrt{\frac{1+x}{1-x}\cdot\frac{1+x}{1+x}}dx = \int \sqrt{\frac{(1+x)^2}{1^2-x^2}}dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}}dx = \int\left[\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right]dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}dx + \frac{1}{-2}\int (1-x^2)^{-\frac{1}{2}}\cdot(-2x)dx$$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \sin^{-1}x - \frac{1}{2}\cdot\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \sin^{-1}x - \frac{1}{2}\cdot\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = \sin^{-1}x - \sqrt{1-x^2} + c$$

(xiii) $\int \frac{\sqrt{a^2-x^2}}{x}dx \quad (a>0)$

Solution:

$$\int \frac{\sqrt{a^2-x^2}}{x}dx$$

Put $x = a\sin\theta \quad \Rightarrow dx = a\cos\theta\, d\theta$

$$= \int \frac{\sqrt{a^2-a^2\sin^2\theta}}{a\sin\theta}\cdot a\cos\theta\, d\theta$$

$$= \int \frac{\sqrt{a^2(1-\sin^2\theta)}}{\sin\theta}\cdot\cos\theta\, d\theta$$

$$= a\int \frac{\sqrt{\cos^2\theta}}{\sin\theta}\cdot\cos\theta d\theta$$

$$= a\int \frac{\cos\theta}{\sin\theta}\cdot\cos\theta\, d\theta = a\int \frac{\cos^2\theta}{\sin\theta}d\theta$$

$$= a\int \frac{1-\sin^2\theta}{\sin\theta}d\theta = a\int\left(\frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}\right)d\theta$$

$$= a\int(\csc\theta - \sin\theta)d\theta = a\int\csc\theta\, d\theta - a\int\sin\theta\, d\theta$$

$$= a\,\ell n|\csc\theta - \cot\theta| + a\cos\theta + c$$

$$= a\,\ell n\left|\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right| + a\cos\theta + c$$

$$= a\,\ell n\left|\frac{1-\sqrt{\cos^2\theta}}{\sin\theta}\right| + a\sqrt{\cos^2\theta} + c$$

$$= a\,\ell n\left|\frac{1-\sqrt{1-\sin^2\theta}}{\sin\theta}\right| + a\sqrt{1-\sin^2\theta} + c$$

$$= a\,\ell n\left|\frac{1-\sqrt{1-\frac{x^2}{a^2}}}{\frac{x}{a}}\right| + a\sqrt{1-\frac{x^2}{a^2}} + c \quad \begin{cases}\text{As } x = a\sin\theta \\ \Rightarrow \sin\theta = \frac{x}{a}\end{cases}$$

$$= a\,\ell n\left|\frac{1-\sqrt{\frac{a^2-x^2}{a^2}}}{\frac{x}{a}}\right| + a\sqrt{\frac{a^2-x^2}{a^2}} + c$$

$$= a\,\ell n\left|\frac{1-\frac{\sqrt{a^2-x^2}}{a}}{\frac{x}{a}}\right| + \frac{a}{a}\sqrt{a^2-x^2} + c$$

$$= a\,\ell n\left|\frac{a-\sqrt{a^2-x^2}}{a}\cdot\frac{a}{x}\right| + \sqrt{a^2-x^2} + c$$

$$= a\,\ell n\left|\frac{a-\sqrt{a^2-x^2}}{x}\right| + \sqrt{a^2-x^2} + c$$

(xiv) $\int \frac{\tan^3 x}{\sec^4 x + \tan^4 x}dx$

Solution:

$$\int \frac{\tan^3 x}{\sec^4 x + \tan^4 x}dx$$

$$= \int \frac{\frac{\sin^3 x}{\cos^3 x}}{\frac{1}{\cos^4 x} + \frac{\sin^4 x}{\cos^4 x}}dx = \int \frac{\frac{\sin^3 x}{\cos^3 x}}{\frac{1+\sin^4 x}{\cos^4 x}}dx$$

$$= \int \frac{\sin^3 x}{\cos^3 x}\cdot\frac{\cos^4 x}{1+\sin^4 x}dx$$

Multiply and divide by 4

$$= \frac{1}{4}\int \frac{4\sin^3 x\cdot\cos x}{1+\sin^4 x}dx$$

Let $f(x) = 1+\sin^4 x$

$\Rightarrow \quad f'(x) = 0 + 4\sin^3 x\cos x = 4\sin^3 x\cos x$

Using: $\int \frac{f'(x)}{f(x)}dx = \ell n|f(x)| + c$

$$= \frac{1}{4}\ell n|1+\sin^4 x| + c$$

(xv) $\int \frac{1}{\sqrt{x^2+2x+10}}dx$

Solution:

$$\int \frac{1}{\sqrt{x^2+2x+10}}dx$$

$$= \int \frac{1}{\sqrt{(x^2+2x+1)+9}}dx = \int \frac{1}{\sqrt{3^2+(x+1)^2}}dx$$

Formula: $\int \frac{1}{\sqrt{a^2+x^2}}dx = \ell n|x + \sqrt{a^2+x^2}| + c$

$$= \ell n|(x+1) + \sqrt{3^2+(x+1)^2}| + c$$

$$= \ell n|x+1+\sqrt{x^2+2x+10}| + c$$

www.ilmkidunya.com

www.ilmkidunya.com

Content Outline

- Integration by Parts

INTEGRATION BY PARTS

The rule that corresponds to the Product Rule for differentiation is called **integration by parts**. Integration by parts is a technique for evaluating integrals of the form $\int f(x)g(x)dx$, where f is differentiable and g has an antiderivative.

Product Rule in Integral Form

Since g has an antiderivative, there exists some function h such that $h'(x) = g(x)$.

Applying product rule of differentiation.

$$\frac{d}{dx}[f(x)h(x)] = f(x)h'(x) + h(x)f'(x)$$

Integrating both sides with respect to x gives:

$$f(x)h(x) = \int [f(x)h'(x) + h(x)f'(x)]dx$$

Splitting the integral, we have:

$$f(x)\,h(x) = \int f(x)h'(x)dx + \int h(x)f'(x)dx$$

Rearranging terms, we obtain:

$$\int f(x)h'(x)dx = f(x)h(x) - \int h(x)f'(x)dx \quad \ldots(1)$$

Since $h'(x) = g(x)$, $h(x) = \int g(x)dx$.

Substituting into (1), we get:

$$\int f(x)g(x)dx = f(x)\left(\int g(x)dx\right) - \int\left[\left(\int g(x)dx(f'(x))\right)\right]dx \quad \ldots(2)$$

LIATE: A Useful Mnemonic for Integration by Parts

LIATE is a widely used guideline for integration by parts. It helps you choose first function whose type comes first in LIATE.

L → Logarithmic functions
I → Inverse trigonometric functions
A → Algebraic functions
T → Trigonometric functions
E → Exponential functions

Remember: LIATE is just a guideline, not an absolute rule – there are cases where a different choice works better.

Note: Although, $\int g(x)dx$ denote the integral of $h(x)$. Here we use it as the antiderivative of $g(x)$, that is, $h(x)$.

Example 29: Evaluate $\int x\sin nx\,dx, n \neq 0$.

Solution:

$$\int x\sin nx\,dx$$

We take: 1st function = x, 2nd function = $\sin nx$

Integrating by parts

$$= x\cdot\int \sin nx\,dx - \int\left\{\frac{d}{dx}(x)\cdot\int \sin nx\,dx\right\}dx$$

$$= x\left(\frac{-\cos nx}{n}\right) - \int 1\left(\frac{-\cos nx}{n}\right)dx$$

$$= \frac{-x\cos nx}{n} + \frac{1}{n}\int \cos nx\,dx$$

$$= \frac{-x\cos nx}{n} + \frac{1}{n}\cdot\frac{\sin nx}{n} + c$$

$$= \frac{-x\cos nx}{n} + \frac{\sin nx}{n^2} + c$$

Example 30: Evaluate $\int xe^{ax}dx, a \neq 0$

Solution:

We take: 1st function = x, 2nd function = e^{ax}

Integrating by parts

$$= x\int e^{ax}dx - \int\left\{\frac{d}{dx}(x)\cdot\int e^{ax}dx\right\}dx$$

$$= x\cdot\frac{e^{ax}}{a} - \int 1\cdot\frac{e^{ax}}{a}\cdot dx$$

$$= \frac{x}{a}\cdot e^{ax} - \frac{1}{a}\int e^{ax}\cdot dx$$

$$= \frac{x}{a}\cdot e^{ax} - \frac{1}{a}\cdot\frac{e^{ax}}{a} + c = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right) + c$$

$$= e^{ax}\left(\frac{ax-1}{a^2}\right) + c$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 31: Evaluate $\int \frac{x}{\cos^2 x} dx$

Solution:

$$\int \frac{x}{\cos^2 x} dx = \int x \cdot \sec^2 x \, dx$$

We take: 1st function = x, 2nd function = $\sec^2 x$

Integrating by parts

$$= x\int \sec^2 x \, dx - \int \left\{ \frac{d}{dx}(x) \int \sec^2 x \, dx \right\} dx$$

$$= x\tan x - \int 1 \cdot \tan x \, dx$$

$$= x\tan x - \int \tan x \, dx$$

$$= x\tan x - (-\ell n|\cos x|) + c$$

$$= x\tan x + \ell n|\cos x| + c$$

Example 32: Evaluate $\int \ell n(\sin x) \cdot \sin 2x \, dx$

Solution:

$$\int \ell n(\sin x) \cdot \sin 2x \, dx = \int \ell n(\sin x) \cdot 2\sin x \cos x \, dx$$

Let $u = \sin x \Rightarrow du = \cos x \, dx$

$$= \int \ell n\, u \cdot 2u \, du$$

$$= 2\int \ell n\, u \cdot u \, du$$

We take: 1st function = $\ell n\, u$, 2nd function = u

Integrating by parts

$$= 2\left[\ell n\, u \cdot \int u \, du - \int \left\{ \frac{d}{du}(\ell n\, u) \cdot \int u \, du \right\} du \right]$$

$$= 2\left[\ell n\, u \cdot \frac{u^2}{2} - \int \frac{1}{u} \cdot \frac{u^2}{2} du \right]$$

$$= 2\left[\frac{u^2}{2} \cdot \ell n\, u - \frac{1}{2}\int u \, du \right]$$

$$= u^2 \ell n\, u - \frac{u^2}{2} + c = u^2\left[\ell n\, u - \frac{1}{2} \right] + c$$

$$= \sin^2 x \left[\ell n(\sin u) - \frac{1}{2} \right] + c \qquad \because u = \sin x$$

Challenge!

Show that: $\int e^{g(x)}[f(x)g'(x) + f'(x)]dx = e^{g(x)}f(x) + c$

Solution:

L.H.S $= \int e^{g(x)}[f(x)g'(x) + f'(x)]dx$

$$= \int e^{g(x)} \cdot f(x)g'(x)dx + \int e^{g(x)} \cdot f'(x)dx$$

We take: 1st function = $e^{g(x)}$, 2nd function = $f'(x)$

Integrating 2nd integral by parts

$$= \int e^{g(x)} \cdot f(x)g'(x)dx + e^{g(x)} \int f'(x)dx - \int \left\{ \frac{d}{dx}(e^{g(x)}) \int f'(x)dx \right\} dx$$

$$= \int e^{g(x)} \cdot f(x)g'(x)dx + e^{g(x)} \cdot f(x) - \int e^{g(x)} \cdot g'(x) \cdot f(x)dx = e^{g(x)} \cdot f(x) + c = \text{R.H.S (Proved)}$$

Example 33: Evaluate $\int e^{x^2}(\cos x + 2x\sin x)dx$

Solution:

$$\int e^{x^2}(\cos x + 2x\sin x)dx = \int e^{x^2} \cdot \cos x \, dx + \int e^{x^2} \cdot 2x\sin x \, dx$$

Integrating the first integral by parts

$$\int e^{x^2}(\cos x + 2x\sin x)dx = e^{x^2}\int \cos x \, dx - \int \left\{ \frac{d}{dx}(e^{x^2}) \int \cos x \, dx \right\} dx + \int e^{x^2} \cdot 2x\sin x \, dx$$

$$= e^{x^2} \cdot \sin x - \int e^{x^2} \cdot 2x \cdot \sin x \, dx + \int e^{x^2} \cdot 2x\sin x \, dx$$

$$= e^{x^2} \cdot \sin x + c$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 34: Evaluate $\int \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)dx$

Solution:

$$\int \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)dx$$

Put $x = \tan\theta \quad \Rightarrow dx = \sec^2\theta\, d\theta$

$$= \int \sin^{-1}\left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\right)\sec^2\theta\, d\theta$$

$$= \int \sin^{-1}\left(\frac{\tan\theta}{\sqrt{\sec^2\theta}}\right)\sec^2\theta\, d\theta$$

$$= \int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\sec^2\theta\, d\theta$$

$$= \int \sin^{-1}\left(\frac{\sin\theta}{\cos\theta}\cdot\cos\theta\right)\sec^2\theta\, d\theta$$

$$= \int \sin^{-1}(\sin\theta)\sec^2\theta\, d\theta = \int \theta\cdot\sec^2\theta\, d\theta$$

We take: 1st function = θ, 2nd function = $\sec^2\theta$

Integrating by parts

$$= \theta\cdot\int\sec^2\theta\, d\theta - \int\left\{\frac{d}{d\theta}(\theta)\cdot\int\sec^2\theta\, d\theta\right\}d\theta$$

$$= \theta\cdot\tan\theta - \int 1\cdot\tan\theta\, d\theta$$

$$= \theta\cdot\tan\theta - \ell n|\sec\theta| + c$$

$$= \theta\cdot\tan\theta - \ell n(\sqrt{\sec^2\theta}) + c$$

$$= \theta\cdot\tan\theta - \ell n(\sqrt{1+\tan^2\theta}) + c$$

$$= \tan^{-1}x\cdot x - \frac{1}{2}\ell n(1+x^2) + c \qquad \text{As } x = \tan\theta \Rightarrow \theta = \tan^{-1}x$$

$$= x\tan^{-1}x - \frac{1}{2}\ell n(1+x^2) + c$$

Example 35: Show that $\int e^{ax}\sin bx\, dx = \frac{1}{\sqrt{a^2+b^2}}e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

Solution:

Let $I = \int e^{ax}\sin bx\, dx \quad \ldots(1)$

We take: 1st function = e^{ax}, 2nd function = $\sin bx$

Integrating by parts

$$I = e^{ax}\int\sin bx\, dx - \int\left\{\frac{d}{dx}(e^{ax})\int\sin bx\, dx\right\}dx$$

$$= e^{ax}\left(\frac{-\cos bx}{b}\right) - \int e^{ax}\cdot a\left(\frac{-\cos bx}{b}\right)dx$$

$$= \frac{-e^{ax}\cdot\cos bx}{b} + \frac{a}{b}\int e^{ax}\cdot\cos bx\, dx$$

(1st, 2nd)

Again, integrating by parts:

$$= \frac{-e^{ax}\cdot\cos bx}{b} + \frac{a}{b}\left[e^{ax}\cdot\frac{\sin bx}{b} - \int e^{ax}\cdot a\cdot\frac{\sin bx}{b}dx\right]$$

$$= \frac{-e^{ax}\cdot\cos bx}{b} + \frac{ae^{ax}\cdot\sin bx}{b^2} - \frac{a^2}{b^2}\int e^{ax}\sin bx\, dx$$

$$I = \frac{e^{ax}}{b}\left\{-\cos bx + \frac{a}{b}\sin bx\right\} - \frac{a^2}{b^2}I + c_1 \quad \text{Using (1)}$$

$$I + \frac{a^2}{b^2}I = \frac{e^{ax}}{b}\left\{\frac{-b\cos bx + a\sin bx}{b}\right\} + c_1$$

$$I\left(\frac{b^2+a^2}{b^2}\right) = \frac{e^{ax}}{b^2}(\sin bx\cdot a - \cos bx\cdot b) + c_1$$

$$I = \frac{b^2}{a^2+b^2}\cdot\frac{e^{ax}}{b^2}(\sin bx\cdot a - \cos bx\cdot b) + \frac{b^2c_1}{a^2+b^2}$$

Let $a = r\cos\theta$ and $b = r\sin\theta$

$$I = \frac{e^{ax}}{a^2+b^2}(\sin bx\cdot r\cos\theta - \cos bx\cdot r\sin\theta) + c \qquad \text{where } c = \frac{b^2c_1}{a^2+b^2}$$

$$= \frac{e^{ax}}{a^2+b^2}\cdot r(\sin bx\cdot\cos\theta - \cos bx\cdot\sin\theta) + c$$

$$I = \frac{re^{ax}}{a^2+b^2}\sin(bx-\theta) + c \quad \ldots(2)$$

As $a = r\cos\theta, b = r\sin\theta$

$$\Rightarrow \quad a^2+b^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta)$$

$$= r^2(1) = r^2$$

$$\Rightarrow \quad r^2 = a^2+b^2 \Rightarrow r = \sqrt{a^2+b^2}$$

and $\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \tan\theta = \frac{b}{a} \quad \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$

Putting values in (2), we get

$$I = \frac{re^{ax}}{r^2}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

$$= \frac{e^{ax}}{r}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

$$I = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cdot\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c \text{ (Proved)}$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 36: Evaluate $\int x\sin^2 x\,dx$

Solution:

$$\int x\cdot\sin^2 x\,dx$$

$$=\int x\cdot\left(\frac{1-\cos 2x}{2}\right)dx$$

$$=\frac{1}{2}\int(x-x\cos 2x)dx$$

$$=\frac{1}{2}\int x^1dx-\frac{1}{2}\int \underset{1st}{x}\cdot\underset{2nd}{\cos 2x}\,dx$$

Integrating the 2nd integral by parts

$$=\frac{1}{2}\frac{x^2}{2}-\frac{1}{2}\left[x\cdot\frac{\sin 2x}{2}-\int 1\cdot\frac{\sin 2x}{2}dx\right]$$

$$=\frac{x^2}{4}-\frac{x}{4}\sin 2x+\frac{1}{4}\int\sin 2x\,dx$$

$$=\frac{x^2}{4}-\frac{x}{4}\sin 2x+\frac{1}{4}\cdot\left(\frac{-\cos 2x}{2}\right)+c$$

$$=\frac{x^2}{4}-\frac{x\cdot\sin 2x}{4}-\frac{\cos 2x}{8}+c$$

A Special Integral with Proof

Question: Prove that: $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c$

Proof:

Let $I=\int\sqrt{x^2+a^2}dx$ ...(1)

$$=\int\sqrt{x^2+a^2}\cdot 1dx$$

We take: 1st function = $\sqrt{x^2+a^2}$, 2nd function = 1
Integrating by parts

$$I=\sqrt{x^2+a^2}\cdot\int 1dx-\int\left\{\frac{d}{dx}(x^2+a^2)^{\frac{1}{2}}\cdot\int 1dx\right\}dx$$

$$=\sqrt{x^2+a^2}\cdot x-\int\frac{1}{2}(x^2+a^2)^{-\frac{1}{2}}\cdot(2x)\cdot x\,dx=x\sqrt{x^2+a^2}-\int\frac{x^2}{\sqrt{x^2+a^2}}dx$$

$$=x\sqrt{x^2+a^2}-\int\frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}dx=x\sqrt{x^2+a^2}-\int\left(\frac{x^2+a^2}{\sqrt{x^2+a^2}}-\frac{a^2}{\sqrt{x^2+a^2}}\right)dx$$

$$I=x\sqrt{x^2+a^2}-\int\sqrt{x^2+a^2}dx+a^2\int\frac{1}{\sqrt{x^2+a^2}}dx$$

$$I=x\sqrt{x^2+a^2}-I+a^2\ln|x+\sqrt{x^2+a^2}|+c_1 \quad \text{Using (1)}$$

$$2I=x\sqrt{x^2+a^2}+a^2\ln\left|x+\sqrt{x^2+a^2}\right|+c_1$$

Dividing both sides by '2'

$$I=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c \text{ where } c=\frac{c_1}{2}$$

Challenge!

Prove that: (i) $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c, a>0$ and $|x|\le a$

(ii) $\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+c, a>0$ and $|x|\ge a$

www.ilmkidunya.com

www.ilmkidunya.com

(i) $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c, a>0 \text{ and } |x|\le a$

Proof:

Let $I=\int\sqrt{a^2-x^2}\,dx$ ...(1)

$=\int\sqrt{a^2-x^2}\cdot 1\,dx$

We take: 1st function = $\sqrt{a^2-x^2}$, 2nd function = 1

Integrating by parts:

$$I=\sqrt{a^2-x^2}\int 1\,dx-\int\left\{\frac{d}{dx}(a^2-x^2)^{\frac{1}{2}}\cdot\int 1\,dx\right\}dx$$

$$=\sqrt{a^2-x^2}\cdot x-\int\frac{1}{2}(a^2-x^2)^{\frac{-1}{2}}(-2x)^2\cdot x\,dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{-x}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\int\left(\frac{a^2-x^2}{\sqrt{a^2-x^2}}-\frac{a^2}{\sqrt{a^2-x^2}}\right)dx$$

$$I=x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\,dx+a^2\int\frac{1}{\sqrt{a^2-x^2}}dx$$

$$I=x\sqrt{a^2-x^2}-I+a^2\sin^{-1}\left(\frac{x}{a}\right)+c^2 \quad \text{Using (1)}$$

$$2I=x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)+c_1$$

Dividing both sides by '2'

$$I=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c, \text{ where } c=\frac{c_1}{2}$$

(ii) $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+c, a>0 \text{ and } |x|\ge a$

Proof:

Let $I=\int\sqrt{x^2-a^2}\,dx$ ...(1)

$=\int\sqrt{x^2-a^2}\cdot 1\,dx$

We take: 1st function = $\sqrt{x^2-a^2}$, 2nd function = 1

$$I=\sqrt{x^2-a^2}\int 1\,dx-\int\left\{\frac{d}{dx}(x^2-a^2)^{\frac{1}{2}}\cdot\int 1\,dx\right\}dx$$

$$=\sqrt{x^2-a^2}\cdot x-\int\frac{1}{2}(x^2-a^2)^{\frac{-1}{2}}\cdot(2x)\cdot x\,dx$$

$$=x\sqrt{x^2-a^2}-\int\frac{x^2}{\sqrt{x^2-a^2}}dx$$

$$=x\sqrt{x^2-a^2}-\int\frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}dx$$

$$=x\sqrt{x^2-a^2}-\int\left(\frac{x^2-a^2}{\sqrt{x^2-a^2}}+\frac{a^2}{\sqrt{x^2-a^2}}\right)dx$$

$$I=x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}\,dx-a^2\int\frac{1}{\sqrt{x^2-a^2}}dx$$

$$I=x\sqrt{x^2-a^2}-I-a^2\ln|x+\sqrt{x^2-a^2}|+c_1$$

$$2I=x\sqrt{x^2-a^2}-a^2\ln|x+\sqrt{x^2-a^2}|+c_1$$

Dividing both sides by '2'

$$I=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|+c \quad \text{where } c=\frac{c_1}{2}$$

Example 37: Evaluate $\int\sqrt{x^2-4x+13}\,dx$

Solution

$\int\sqrt{x^2-4x+13}\,dx$

Add and subtract 2^2

$=\int\sqrt{(x^2-4x+2^2)+13-2^2}\,dx=\int\sqrt{(x-2)^2+9}\,dx$

Let $u=x-2 \Rightarrow du=dx$

$=\int\sqrt{u^2+3^2}\,du$

Formula: $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c$

$$=\frac{u}{2}\sqrt{u^2+3^2}+\frac{3^2}{2}\ln|u+\sqrt{u^2+3^2}|+c$$

Put $u=x-2$, we get

$$=\frac{x-2}{2}\sqrt{(x-2)^2+9}+\frac{9}{2}\ln\left|(x-2)+\sqrt{(x-2)^2+9}\right|+c$$

$$=\frac{x-2}{2}\sqrt{x^2+4-4x+9}+\frac{9}{2}\ln\left|(x-2)+\sqrt{x^2+4-4x+9}\right|+c$$

$$=\frac{x-2}{2}\sqrt{x^2-4x+13}+\frac{9}{2}\ln\left|(x-2)+\sqrt{x^2-4x+13}\right|+c$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 38: Evaluate $\int\sqrt{8x-x^2-7}\,dx, |x-4|\le 3$

Solution:

$$\int\sqrt{8x-x^2-7}dx$$

Add and subtract 4^2

$$=\int\sqrt{-x^2+8x-4^2+4^2-7}\,dx=\int\sqrt{-(x^2-8x+4^2)+16-7}\,dx$$

$$=\int\sqrt{-(x-4)^2+9}\,dx=\int\sqrt{3^2-(x-4)^2}\,dx$$

Let $u=x-4 \Rightarrow du=dx$

$$=\int\sqrt{3^2-u^2}\,du$$

Formula: $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

$$=\frac{u}{2}\sqrt{3^2-u^2}+\frac{3^2}{2}\sin^{-1}\left(\frac{u}{3}\right)+c$$

Put $u=x-4$, we get

$$=\frac{x-4}{2}\sqrt{9-(x-4)^2}+\frac{9}{2}\sin^{-1}\left(\frac{x-4}{3}\right)+c$$

$$=\frac{x-4}{2}\sqrt{8x-x^2-7}+\frac{9}{2}\sin^{-1}\left(\frac{x-4}{3}\right)+c$$

Challenge!

Evaluate $\int\sqrt{\frac{5-x}{x-1}}\,dx, (1<x\le 5)$

Solution:

$$\int\sqrt{\frac{5-x}{x-1}}\,dx=\int\sqrt{\frac{5-x}{x-1}\cdot\frac{5-x}{5-x}}\,dx$$

$$=\int\sqrt{\frac{(5-x)^2}{5x-x^2-5+x}}\,dx=\int\frac{5-x}{\sqrt{-x^2+6x-5}}\,dx$$

$$=\int\frac{5-x}{\sqrt{-x^2+6x-3^2+3^2-5}}\,dx=\int\frac{5-x}{\sqrt{-(x^2-6x+3^2)+4}}\,dx$$

$$=\int\frac{5-x}{\sqrt{4-(x-3)^2}}\,dx=\int\frac{2-(x-3)}{\sqrt{4-(x-3)^2}}\,dx$$

$$=\int\left[\frac{2}{\sqrt{4-(x-3)^2}}-\frac{x-3}{\sqrt{4-(x-3)^2}}\right]dx$$

$$=2\int\frac{1}{\sqrt{2^2-(x-3)^2}}dx-\frac{1}{-2}\int(4-(x-3)^2)^{-\frac{1}{2}}(-2)(x-3)dx$$

Formula: $\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\frac{x}{a}+c$

$$=2\sin^{-1}\left(\frac{x-3}{2}\right)+\frac{1}{2}\frac{(4-(x-3)^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$=2\sin^{-1}\left(\frac{x-3}{2}\right)+\frac{1}{2}\cdot\frac{(4-(x-3)^2)^{\frac{1}{2}}}{1/2}+c$$

$$=2\sin^{-1}\left(\frac{x-3}{2}\right)+\sqrt{2^2-(x-3)^2}+c$$

$$=2\sin^{-1}\left(\frac{x-3}{2}\right)+\sqrt{(2-x+3)(2+x-3)}+c$$

$$=2\sin^{-1}\left(\frac{x-3}{2}\right)+\sqrt{(5-x)(x-1)}+c$$

Example 39: Evaluate $\int\sqrt{x^2-8x+7}\,dx, |x-4|\ge 3$

Solution:

$$\int\sqrt{x^2-8x+7}\,dx$$

Add and subtract 4^2

$$=\int\sqrt{(x^2-8x+4^2)+7-4^2}\,dx$$

$$=\int\sqrt{(x-4)^2-9}\,dx$$

Put $u=x-4 \Rightarrow du=dx$

$$=\int\sqrt{u^2-3^2}\,du$$

Formula: $\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|+c$

$$=\frac{u}{2}\sqrt{u^2-3^2}-\frac{3^2}{2}\ln|u+\sqrt{u^2-3^2}|+c$$

Put $u=x-4$, we get

$$=\frac{x-4}{2}\sqrt{(x-4)^2-9}-\frac{9}{2}\ln|x-4+\sqrt{(x-4)^2-9}|+c$$

$$=\frac{x-4}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\ln|x-4+\sqrt{x^2-8x+7}|+c$$

www.ilmkidunya.com

www.ilmkidunya.com

EXERCISE 3.3

Evaluate the following integrals:

Q1. $\int \ln x\,dx$

Solution:

$$\int \ln x\,dx = \int \ln x \cdot 1\,dx$$

We take: 1st function = $\ln x$, 2nd function = 1

Integrating by parts

$$= \ln x \cdot \int 1dx - \int\left\{\frac{d}{dx}(\ln x)\cdot\int 1dx\right\}dx$$

$$= \ln x \cdot x - \int \frac{1}{x}\cdot x\,dx$$

$$= x\ln x - \int 1dx$$

$$= x\ln x - x + c$$

Q2. $\int x^7 \cdot \ln x\,dx$

Solution:

$$\int x^7 \cdot \ln x\,dx$$

$$= \int \ln x \cdot x^7 dx$$

We take: 1st function = $\ln x$, 2nd function = x^7

Integrating by parts

$$= \ln x \cdot \int x^7 dx - \int\left\{\frac{d}{dx}(\ln x)\cdot\int x^7 dx\right\}dx$$

$$= \ln x \cdot \frac{x^8}{8} - \int \frac{1}{x}\cdot\frac{x^8}{8}dx$$

$$= \frac{x^8}{8}\ln x - \frac{1}{8}\int x^7 dx = \frac{x^8}{8}\ln x - \frac{1}{8}\cdot\frac{x^8}{8} + c$$

$$= \frac{x^8}{8}\ln x - \frac{x^8}{64} + c$$

Q3. $\int x\cos 2x\,dx$

Solution:

$$\int x\cos 2x\,dx$$

We take: 1st function = x, 2nd function = $\cos 2x$

Integrating by parts

$$= x\int \cos 2x\,dx - \int\left\{\frac{d}{dx}(x)\cdot\int \cos 2x\,dx\right\}dx$$

$$= x\cdot\frac{\sin 2x}{2} - \int 1\cdot\frac{\sin 2x}{2}dx$$

$$= \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx$$

$$= \frac{x}{2}\sin 2x - \frac{1}{2}\cdot\left(\frac{-\cos 2x}{2}\right) + c$$

$$= \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + c$$

Q4. $\int \frac{x}{\sin^2 x}dx$

Solution:

$$\int \frac{x}{\sin^2 x}dx$$

We take: 1st function = x, 2nd function = $\csc^2 x$

Integrating by parts

$$= x\int \csc^2 x\,dx - \int\left\{\frac{d}{dx}(x)\cdot\int \csc^2 x\,dx\right\}dx$$

$$= x\{-\cot x\} - \int 1\cdot(-\cot x)dx$$

$$= -x\cot x + \int \cot x\,dx$$

$$= -x\cot x + \ln|\sin x| + c$$

Q5. $\int xe^{2x}dx$

Solution:

$$\int xe^{2x}dx$$

We take: 1st function = x, 2nd function = e^{2x}

Integrating by parts

$$= x\int e^{2x}dx - \int\left\{\frac{d}{dx}(x)\cdot\int e^{2x}dx\right\}dx$$

www.ilmkidunya.com

www.ilmkidunya.com

$$= x\cdot\frac{e^{2x}}{2}-\int 1\cdot\frac{e^{2x}}{2}dx$$

$$= \frac{x}{2}e^{2x}-\frac{1}{2}\int e^{2x}dx$$

$$= \frac{x}{2}e^{2x}-\frac{1}{2}\cdot\frac{e^{2x}}{2}+c$$

$$= \frac{x}{2}e^{2x}-\frac{1}{4}e^{2x}+c$$

Q6. $\int x^2 e^{ax}\,dx$

Solution:

$$\int x^2 e^{ax}dx$$

We take: 1st function = x^2, 2nd function = e^{ax}
Integrating by parts

$$= x^2\int e^{ax}dx-\int\left\{\frac{d}{dx}(x^2)\cdot\int e^{ax}dx\right\}dx$$

$$= x^2\cdot\frac{e^{ax}}{a}-\int 2x\cdot\frac{e^{ax}}{a}dx$$

$$= \frac{x^2}{a}e^{ax}-\frac{2}{a}\int_{\text{1st}\ \text{2nd}} x\cdot e^{ax}dx$$

Again, integrating by parts

$$= \frac{x^2}{a}e^{ax}-\frac{2}{a}\left[x\cdot\frac{e^{ax}}{a}-\int 1\cdot\frac{e^{ax}}{a}dx\right]$$

$$= \frac{x^2}{a}e^{ax}-\frac{2x}{a^2}e^{ax}-\frac{2}{a^2}\int e^{ax}dx$$

$$= \frac{x^2}{a}e^{ax}-\frac{2x}{a^2}e^{ax}-\frac{2}{a^2}\cdot\frac{e^{ax}}{a}+c$$

$$= \frac{x^2}{a}e^{ax}-\frac{2x}{a^2}e^{ax}-\frac{2}{a^3}e^{ax}+c$$

Q7. $\int x\tan^2 x\,dx$

Solution:

$$\int x\tan^2 x\,dx$$

$$= \int x(\sec^2 x-1)dx$$

$$= \int x\cdot\sec^2 x\,dx-\int x\,dx$$

We take: 1st function = x, 2nd function = $\sec^2 x$
Integrating the first integral by parts

$$= x\int\sec^2 x\,dx-\int\left\{\frac{d}{dx}(x)\cdot\int\sec^2 x\,dx\right\}dx-\frac{x^2}{2}$$

$$= x\cdot\tan x-\int 1\cdot\tan x\,dx-\frac{x^2}{2}$$

$$= x\tan x-\int\tan x\,dx-\frac{x^2}{2}$$

$$= x\tan x-\ln|\sec x|-\frac{x^2}{2}+c$$

or $$= x\tan x+\ln|\cos x|-\frac{x^2}{2}+c$$

Q8. $\int \ln(\sin x)\cdot\cos x\,dx$

Solution:

$$\int \ln(\sin x)\cdot\cos x\,dx$$

Let $u=\sin x \Rightarrow du=\cos x\,dx$

$$= \int \ln u\,du$$

$$= \int \ln u\cdot 1\,du$$

We take: 1st function = $\ln u$, 2nd function = 1
Integrating by parts

$$= \ln u\cdot\int 1du-\int\left\{\frac{d}{du}(\ln u)\cdot\int 1du\right\}du$$

$$= \ln u\cdot u-\int\frac{1}{u}\cdot u\,du = u\ln u-\int 1du$$

$$= u\ln u-u+c$$

Put $u=\sin x$, we get

$$= \sin x\cdot\ln(\sin x)-\sin x+c$$

Q9. $\int e^x \ln(1+e^x)dx$

Solution:

$$\int e^x \ln(1+e^x)dx$$

$$= \int \ln(1+e^x)\cdot e^x dx$$

Let $u=1+e^x \Rightarrow du=e^x dx$

$$= \int \ln u\,du$$

$$= \int \ln u\cdot 1\,du$$

We take: 1st function = $\ln u$, 2nd function = 1

www.ilmkidunya.com

www.ilmkidunya.com

Integrating by parts

$= \ln u \cdot \int 1du - \int \left\{ \frac{d}{du}(\ln u) \cdot \int 1du \right\} du$

$= \ln u \cdot u - \int \frac{1}{u} \cdot u\, du$

$= u \ln u - \int 1du$

$= u \ln u - u + c_1$

Put $u = 1 + e^x$, we get

$= (1+e^x)\ln(1+e^x) - (1+e^x) + c_1$

$= (1+e^x)\ln(1+e^x) - e^x - 1 + c_1$

$= (1+e^x)\ln(1+e^x) - e^x + c$, where $c = -1 + c$

Q10. $\int (2x^3 + 2x)e^{x^2} dx$

Solution:

$\int (2x^3 + 2x)e^{x^2} dx$

$= \int 2x(x^2+1)e^{x^2} dx$

$= \int (x^2+1)e^{x^2} \cdot 2x\, dx$

Let $u = x^2 \quad \Rightarrow du = 2x\, dx$

$= \int (u+1)e^u du = \int u \cdot e^u du + \int e^u du$

We take: 1st function = u, 2nd function = e^u

$= u\int e^u du - \int \left\{ \frac{d}{du}(u) \int e^u du \right\} du + \frac{e^u}{1}$

$= u \cdot e^u - \int 1 \cdot e^u du + e^u$

$= u \cdot e^u - e^u + e^u + c = u \cdot e^u + c$

Put $u = x^2$, we get

$= x^2 \cdot e^{x^2} + c$

Q11. $\int e^x \tan^{-1}(e^x) dx$

Solution:

$\int e^x \tan^{-1}(e^x) dx$

$= \int \tan^{-1}(e^x) \cdot e^x dx$

Let $u = e^x \quad \Rightarrow du = e^x dx$

$= \int \tan^{-1} u \cdot du = \int \tan^{-1} u \cdot 1du$

We take: 1st function = $\tan^{-1} u$, 2nd function = 1

$= \tan^{-1} u \int 1du - \int \left\{ \frac{d}{du}(\tan^{-1} u) \int 1du \right\} du$

$= \tan^{-1} u \cdot u - \int \frac{1}{1+u^2} \cdot u\, du = u \cdot \tan^{-1} u - \frac{1}{2}\int \frac{2u}{1+u^2} du$

Using: $\int \frac{f'(u)}{f(u)} du = \ln|f(u)| + c$

$= u \cdot \tan^{-1} u - \frac{1}{2}\ln(1+u^2) + c$

Put $u = e^x$, we get

$= e^x \cdot \tan^{-1}(e^x) - \frac{1}{2}\ln(1+e^{2x}) + c$

Q12. $\int \frac{x}{1+\cos x} dx$

Solution:

$\int \frac{x}{1+\cos x} dx$

$= \int \frac{x}{2\cos^2 \frac{x}{2}} dx$

$= \frac{1}{2}\int x \cdot \sec^2 \frac{x}{2} dx$

We take: 1st function = x, 2nd function = $\sec^2 \frac{x}{2}$

Integrating by parts

$= \frac{1}{2}\left[x \cdot \int \sec^2 \frac{x}{2} dx - \int \left\{ \frac{d}{dx}(x) \cdot \int \sec^2 \frac{x}{2} dx \right\} dx \right]$

$= \frac{1}{2}\left[x \cdot \frac{\tan \frac{x}{2}}{\frac{1}{2}} - \int 1 \cdot \frac{\tan \frac{x}{2}}{\frac{1}{2}} dx \right]$

$= x \cdot \tan \frac{x}{2} - \int \tan \frac{x}{2} dx = x \cdot \tan \frac{x}{2} - \ln \left| \frac{\sec \frac{x}{2}}{\frac{1}{2}} \right| + c$

$= x \cdot \tan \frac{x}{2} - 2\ln \left| \sec \frac{x}{2} \right| + c$

or $= x \tan \frac{x}{2} + 2\ln \left| \cos \frac{x}{2} \right| + c$

www.ilmkidunya.com

www.ilmkidunya.com

Q13. $\int \frac{x}{1-\sin x}dx$

Solution:

$$\int \frac{x}{1-\sin x}dx$$

$$= \int \frac{x}{1-\sin x}\times\frac{1+\sin x}{1+\sin x}dx$$

$$= \int \frac{x+x\sin x}{1^2-\sin^2 x}dx$$

$$= \int \frac{x+x\sin x}{\cos^2 x}dx$$

$$= \int\left(\frac{x}{\cos^2 x}+\frac{x\sin x}{\cos x\cdot\cos x}\right)dx$$

$$= \int_{1st\ 2nd} x\cdot\sec^2 x\,dx + \int_{1st\ \ 2nd} x\cdot(\sec x\tan x)dx$$

Integrating by parts

$$= \left[x\int\sec^2 x\,dx - \int\left\{\frac{d}{dx}(x)\int\sec^2 x\,dx\right\}dx\right] + \left[x\cdot\int\sec x\tan x\,dx - \int\left\{\frac{d}{dx}(x)\int\sec x\tan x\,dx\right\}dx\right]$$

$$= x\cdot\tan x - \int 1\cdot\tan x\,dx + x\cdot\sec x - \int 1\cdot\sec x\,dx$$

$$= x\cdot\tan x - \ln|\sec x| + x\sec x - \ln|\sec x+\tan x| + c$$

$$= x\cdot\tan x + x\sec x - \ln|\sec x| - \ln|\sec x+\tan x| + c$$

$$= x(\sec x+\tan x) - \ln|\sec x(\sec x+\tan x)| + c$$

$$= x(\sec x+\tan x) - \ln|\sec^2 x+\sec x\tan x| + c$$

$$= x(\sec x+\tan x) - \ln\left|\frac{1}{\cos^2 x}+\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\right| + c$$

$$= x(\sec x+\tan x) - \ln\left|\frac{1+\sin x}{\cos^2 x}\right| + c$$

$$= x(\sec x+\tan x) - \ln\left|\frac{1+\sin x}{1-\sin^2 x}\right| + c$$

$$= x(\sec x+\tan x) - \ln\left|\frac{1+\sin x}{(1+\sin x)(1-\sin x)}+c\right|$$

$$= x(\sec x+\tan x) - \ln|1-\sin x| + c$$

Q14. $\int[2\ln x+(\ln x)^2]dx$

Solution:

$$\int[2\ln x+(\ln x)^2]dx$$

Let $\ln x = u \Rightarrow x = e^u \Rightarrow dx = e^u du$

$$\int[2\ln x+(\ln x)^2]dx = \int[2u+u^2]e^u du$$

$$= \int e^u[u^2+2u]du$$

Here $g(u)=u \Rightarrow g'(u)=1$

and $f(u)=u^2 \Rightarrow f'(u)=2u$

Formula: $\int e^{g(u)}[f(u)\cdot g'(u)+f'(u)]du = e^{g(u)}f(u)+c$

$$= e^u\cdot u^2 + c$$

Put $u=\ln x$ and $e^u = x$, we get

$$\int[2\ln x+(\ln x)^2]dx = x(\ln x)^2 + c$$

Q15. $\int\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}dx$

Solution:

$$\int\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}dx = \int\tan^{-1}\sqrt{x}\cdot\frac{1}{\sqrt{x}}dx$$

Let $u=\sqrt{x} \Rightarrow du=\frac{1}{2\sqrt{x}}dx \Rightarrow 2du=\frac{1}{\sqrt{x}}dx$

$$= \int\tan^{-1}u\cdot 2du$$

$$= 2\int\tan^{-1}u\cdot 1du$$

We take: 1st function = $\tan^{-1}u$, 2nd function = 1

Integrating by parts

$$= 2\left[\tan^{-1}u\int 1du - \int\left\{\frac{d}{du}(\tan^{-1}u)\int 1du\right\}du\right]$$

$$= 2\left[\tan^{-1}u\cdot u - \int\frac{1}{1+u^2}\cdot u\,du\right]$$

$$= 2u\tan^{-1}u - \int\frac{2u}{1+u^2}du$$

Using: $\int\frac{f'(u)}{f(u)}du = \ln|f(u)|+c$

$$= 2u\tan^{-1}u - \ln(1+u^2)+c$$

Put $u=\sqrt{x}$, we get

$$= 2\sqrt{x}\tan^{-1}(\sqrt{x}) - \ln(1+x)+c$$

www.ilmkidunya.com

www.ilmkidunya.com

Q16. $\int \sqrt{x}\sin\sqrt{x}\,dx$

Solution:

$$\int \sqrt{x}\sin\sqrt{x}\,dx$$

Let $u=\sqrt{x} \Rightarrow du=\frac{1}{2\sqrt{x}}dx$

$$du=\frac{1}{2u}dx \Rightarrow 2u\,du=dx$$

$$=\int u\cdot\sin u\cdot 2u\,du = 2\int u^2\sin u\,du$$

We take: 1st function = u^2, 2nd function = $\sin u$

$$=2\left[u^2\int\sin u\,du-\int\left\{\frac{d}{du}(u^2)\int\sin u\,du\right\}du\right]$$

$$=2\left[u^2(-\cos u)-\int 2u(-\cos u)du\right]$$

$$=-2u^2\cos u+4\int_{1st}\underset{2nd}{u\cos u}\,du$$

Again, integrating by parts

$$=-2u^2\cos u+4\left[u\cdot\sin u-\int 1\cdot\sin u\,du\right]$$

$$=-2u^2\cos u+4u\sin u-4(-\cos u)+c$$

$$=-2u^2\cos u+4u\sin u+4\cos u+c$$

Put $u=\sqrt{x}$, we get

$$=-2x\cos\sqrt{x}+4\sqrt{x}\sin\sqrt{x}+4\cos\sqrt{x}+c$$

$$=(4-2x)\cos\sqrt{x}+4\sqrt{x}\sin\sqrt{x}+c$$

Q17. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$

Solution:

$$\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$$

$$=\int e^x\left[\frac{1+x^2-2x}{(1+x^2)^2}\right]dx$$

$$=\int e^x\left[\frac{1+x^2}{(1+x^2)^2}-\frac{2x}{(1+x^2)^2}\right]dx$$

$$=\int e^x\left[\frac{1}{1+x^2}-\frac{2x}{(1+x^2)^2}\right]dx$$

$$=\int\underset{1st}{(1+x^2)^{-1}}\cdot\underset{2nd}{e^x}\,dx-\int\frac{2x}{(1+x^2)^2}\cdot e^x\,dx$$

Integrating the first integral by parts

$$=(1+x^2)^{-1}\cdot e^x-\int(-1)(1+x^2)^{-2}\cdot(2x)\cdot e^x dx-\int\frac{2x\cdot e^x}{(1+x^2)^2}dx$$

$$=\frac{e^x}{1+x^2}+\int\frac{2x\cdot e^x}{(1+x^2)^2}dx-\int\frac{2x\cdot e^x}{(1+x^2)^2}dx=\frac{e^x}{1+x^2}+c$$

Alternate Method:

$$\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$$

$$=\int e^x\left[\frac{1+x^2-2x}{(1+x^2)^2}\right]dx$$

$$=\int e^x\left[\frac{1+x^2}{(1+x^2)^2}-\frac{2x}{(1+x^2)^2}\right]dx$$

$$=\int e^x\left[1\cdot\frac{1}{1+x^2}-\frac{2x}{(1+x^2)^2}\right]dx$$

Here $g(x)=x \Rightarrow g'(x)=1$

and $f(x)=(1+x^2)^{-1} \Rightarrow f'(x)=-1(1+x^2)^{-2}\cdot 2x$

Formula: $\int e^{g(x)}[f(x)\cdot g'(x)+f'(x)]dx=e^{g(x)}f(x)+c$

$$=e^x\cdot(1+x^2)^{-1}+c$$

$$=\frac{e^x}{1+x^2}+c$$

Q18. $\int\frac{e^{\tan^{-1}x}}{(1+x^2)^2}dx$

Solution:

$$\int\frac{e^{\tan^{-1}x}}{(1+x^2)^2}dx=\int\frac{e^{\tan^{-1}x}}{1+x^2}\cdot\frac{1}{1+x^2}dx$$

Let $u=\tan^{-1}x \Rightarrow du=\frac{1}{1+x^2}dx$

and $\tan u=x \Rightarrow 1+x^2=1+\tan^2 u=\sec^2 u$

$$=\int\frac{e^u}{\sec^2 u}du=\int e^u\cos^2 u\,du$$

$$=\int e^u\left(\frac{1+\cos 2u}{2}\right)du=\frac{1}{2}\int(e^u+e^u\cos 2u)du$$

$$=\frac{1}{2}\int e^u du+\frac{1}{2}\int e^u\cos 2u\,du$$

www.ilmkidunya.com

www.ilmkidunya.com

$$\int \frac{e^{\tan^{-1}x}}{(1+x^2)^2}dx = \frac{1}{2}e^u + \frac{1}{2}I \quad \ldots(1)$$

Consider $I = \int e^u \cos 2u\, du$

We take: 1st function = e^u, 2nd function = $\cos 2u$

Integrating by parts

$$I = e^u \cdot \frac{\sin 2u}{2} - \int e^u \cdot 1 \cdot \frac{\sin 2u}{2} du$$

$$= \frac{e^u}{2}\cdot \sin 2u - \frac{1}{2}\int \underset{1st}{e^u} \underset{2nd}{\sin 2u}\, du$$

Again, integrating by parts

$$= \frac{e^u}{2}\sin 2u - \frac{1}{2}\left[e^u\left(\frac{-\cos 2u}{2}\right) - \int e^u \left(\frac{-\cos 2u}{2}\right)du\right]$$

$$= \frac{e^u}{2}\sin 2u + \frac{e^u}{4}\cos 2u - \frac{1}{4}\int e^u \cos 2u\, du$$

$$I = \frac{e^u}{2}\sin 2u + \frac{e^u}{4}\cos 2u - \frac{1}{4}I + c_1 \quad \text{Using (1)}$$

$$I + \frac{1}{4}I = e^u\left\{\frac{\sin 2u}{2} + \frac{\cos 2u}{4}\right\} + c_1$$

$$\frac{5}{4}I = e^u \cdot \left\{\frac{2\sin 2u + \cos 2u}{4}\right\} + c_1$$

$$I = \frac{4}{5}\frac{e^u}{4}(2\sin 2u + \cos 2u) + \frac{4c_1}{5}$$

$$I = \frac{e^u}{5}(2\sin 2u + \cos 2u) + c, \text{ where } c = \frac{4c_1}{5}$$

Equation (1) implies

$$\int \frac{e^{\tan^{-1}x}}{(1+x^2)^2}dx = \frac{1}{2}e^u + \frac{1}{2}\cdot\frac{e^u}{5}(2\sin 2u + \cos 2u) + c$$

$$= \frac{e^u}{2}\left\{1 + \frac{1}{5}(2\sin 2u + \cos 2u)\right\} + c$$

$$= \frac{e^u}{2}\left\{\frac{5 + 2\sin 2u + \cos 2u}{5}\right\} + c$$

Formulas: $\cos 2u = \frac{1-\tan^2 u}{1+\tan^2 u}$, $\sin 2u = \frac{2\tan u}{1+\tan^2 u}$

$$= \frac{e^u}{10}\left(5 + 2\cdot\frac{2\tan u}{1+\tan^2 u} + \frac{1-\tan^2 u}{1+\tan^2 u}\right) + c$$

Put $u = \tan^{-1}x$ and $\tan u = x$, we get

$$= \frac{e^{\tan^{-1}x}}{10}\left(5 + \frac{4x}{1+x^2} + \frac{1-x^2}{1+x^2}\right) + c$$

$$= \frac{e^{\tan^{-1}x}}{10}\left(\frac{5 + 5x^2 + 4x + 1 - x^2}{1+x^2}\right) + c$$

$$= \frac{e^{\tan^{-1}x}}{10}\left(\frac{4x^2 + 4x + 6}{1+x^2}\right) + c$$

$$= \frac{e^{\tan^{-1}x}}{10}\cdot 2\left(\frac{2x^2 + 2x + 3}{1+x^2}\right) + c$$

$$= \frac{e^{\tan^{-1}x}(2x^2 + 2x + 3)}{5(1+x^2)} + c$$

Q19. $\int \sin^{-1} x\, dx$

Solution:

$$\int \sin^{-1} x\, dx$$

$$= \int \sin^{-1} x \cdot 1\, dx$$

We take: 1st function = $\sin^{-1} x$, 2nd function = 1

Integrating by parts:

$$= \sin^{-1} x \cdot \int 1 dx - \int\left\{\frac{d}{dx}(\sin^{-1} x)\cdot \int 1 dx\right\}dx$$

$$= \sin^{-1} x \cdot x - \int \frac{1}{\sqrt{1-x^2}}\cdot x\, dx$$

$$= x\sin^{-1} x + \frac{1}{2}\int (1-x^2)^{\frac{-1}{2}}\cdot(-2x)dx$$

Using: $\int [f(x)]^n \cdot f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1$

$$= x\sin^{-1} x + \frac{1}{2}\frac{(1-x^2)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + c$$

$$= x\sin^{-1} x + \frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= x\sin^{-1} x + \sqrt{1-x^2} + c$$

Q20. $\int \frac{\ln x \tan^{-1}(\ln x)}{x}dx$

Solution:

$$\int \frac{\ln x \tan^{-1}(\ln x)}{x}dx$$

$$= \int \ln x \cdot \tan^{-1}(\ln x)\cdot\frac{1}{x}dx$$

www.ilmkidunya.com

www.ilmkidunya.com

Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$= \int u \cdot \tan^{-1} u \, du$

We take: 1st function = $\tan^{-1} u$, 2nd function = u

Integrating by parts

$= \tan^{-1} u \cdot \int u\,dx - \int \left\{ \frac{d}{du}(\tan^{-1} u) \int u\,du \right\} du$

$= \tan^{-1} u \cdot \frac{u^2}{2} - \int \frac{1}{1+u^2} \cdot \frac{u^2}{2} du$

$= \frac{u^2}{2} \cdot \tan^{-1} u - \frac{1}{2}\int \frac{1+u^2-1}{1+u^2} du$

$= \frac{u^2}{2}\tan^{-1} u - \frac{1}{2}\int \left(\frac{1+u^2}{1+u^2} - \frac{1}{1+u^2} \right) du$

$= \frac{u^2}{2}\tan^{-1} u - \frac{1}{2}\int 1\,du + \frac{1}{2}\int \frac{1}{1+u^2} du$

$= \frac{u^2}{2}\tan^{-1} u - \frac{1}{2}u + \frac{1}{2}\tan^{-1} u + c$

$= \frac{1}{2}[u^2 \tan^{-1} u + \tan^{-1} u - u] + c$

$= \frac{1}{2}[(u^2+1)\tan^{-1} u - u] + c$

Put $u = \ln x$, we get

$= \frac{1}{2}[((\ln x)^2+1)\tan^{-1}(\ln x) - \ln x] + c$

Q21. $\int \csc^3 x\,dx$

Solution:

Let $I = \int \csc^3 x\,dx$ ...(1)

$= \int \csc x \cdot \csc^2 x\,dx$

We take: 1st function = $\csc x$, 2nd function = $\csc^2 x$

Integrating by parts

$I = \csc x \cdot (-\cot x) - \int (-\csc x \cdot \cot x)(-\cot x)dx$

$= -\csc x \cdot \cot x - \int \csc x \cdot \cot^2 x\,dx$

$= -\csc x \cdot \cot x - \int \csc x \cdot (\csc^2 x - 1)dx$

$I = -\csc x \cdot \cot x - \int \csc^3 x\,dx + \int \csc x\,dx$

$I = -\csc x \cdot \cot x - I + \ln|\csc x - \cot x| + c_1$ Using (1)

$2I = -\csc x \cdot \cot x + \ln|\csc x - \cot x| + c_1$

Dividing both sides by '2'

$I = \frac{1}{2}\csc x \cdot \cot x + \frac{1}{2}\ln|\csc x - \cot x| + c$, where $c = \frac{c_1}{2}$

Q22. $\int e^{5x}(x^5 + x^4 + 1)dx$

Solution:

$\int e^{5x}(x^5 + x^4 + 1)dx$

$= \frac{1}{5}\int e^{5x}(5x^5 + 5x^4 + 5)dx$

$= \frac{1}{5}\int e^{5x}[5(x^5+1) + 5x^4]dx$

Here $g(x) = 5x \Rightarrow g'(x) = 5$

and $f(x) = x^5 + 1 \Rightarrow f'(x) = 5x^4$

Formula: $\int e^{g(x)}[f(x) \cdot g'(x) + f'(x)]dx = e^{g(x)} f(x) + c$

$= \frac{1}{5}e^{5x}(x^5+1) + c$

Q23. $\int \frac{e^{\sqrt{x}}(1+\sqrt{x})}{\sqrt{x}}dx$

Solution:

$\int \frac{e^{\sqrt{x}}(1+\sqrt{x})}{\sqrt{x}}dx$

$= \int e^{\sqrt{x}}\left[\frac{1}{\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{x}}\right]dx$

$= \int e^{\sqrt{x}}\left[2\sqrt{x} \cdot \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right]dx$

Here $g(x) = \sqrt{x} \Rightarrow g'(x) = \frac{1}{2\sqrt{x}}$

and $f(x) = 2\sqrt{x} \Rightarrow f'(x) = 2 \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$

Formula: $\int e^{g(x)}[f(x) \cdot g'(x) + f'(x)]dx = e^{g(x)} f(x) + c$

$= e^{\sqrt{x}} \cdot 2\sqrt{x} + c = 2\sqrt{x} \cdot e^{\sqrt{x}} + c$

www.ilmkidunya.com

www.ilmkidunya.com

Q24. $\int e^{3x}(\sin x - 3\cos x)dx$

Solution:

$$\int e^{3x}(\sin x - 3\cos x)dx$$

$$= -\int e^{3x}(3\cos x - \sin x)dx$$

Here $g(x) = 3x \quad \Rightarrow g'(x) = 3$

and $f(x) = \cos x \quad \Rightarrow f'(x) = -\sin x$

Formula: $\int e^{g(x)}[f(x)\cdot g'(x) + f'(x)]dx = e^{g(x)}\cdot f(x) + c$

$$= -e^{3x}\cos x + c$$

Content Outline

- Integration by Using Partial Fractions

INTEGRATION BY USING PARTIAL FRACTIONS

Let $P(x)$ and $Q(x)$ be polynomial functions, where $Q(x) \neq 0$. If $Q(x)$ can be factored into linear and/or irreducible quadratic factors, then the rational function $\frac{P(x)}{Q(x)}$, can be expressed as a sum of simpler rational functions. Each of these simpler functions can be integrated using methods already known to us.

We will illustrate the following four cases, based on the nature of the factors in $Q(x)$:

Case-I: $Q(x)$ contains only non-repeated linear factors.

Case-II: $Q(x)$ contains repeated linear factors (may also include non-repeated linear factors).

Case-III: $Q(x)$ contains one or more distinct irreducible quadratic factors (linear factors may be absent or present).

Case-IV: $Q(x)$ contains at least one repeated irreducible quadratic factor (linear factors and non-repeated irreducible quadratic factors may also be present).

Case-I: Integration of $\int \frac{P(x)}{Q(x)}dx$ where $Q(x)$ contains only non-repeated linear factors

Example 40: Integrate $\int \frac{3x-2}{x(x-1)(x-2)}dx$

Solution:

$$\int \frac{3x-2}{x(x-1)(x-2)}dx$$

Let $\frac{3x-2}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$ ...(1)

Multiply each term by LCM $x(x-1)(x-2)$

$3x-2 = A(x-1)(x-2) + B(x)(x-2) + C(x)(x-1)$ ...(2)

Put $x = 0$ in eq. (2)

$3(0) - 2 = A(0-1)(0-2)$

$-2 = A(-1)(-2)$

$-2 = 2A$

$\boxed{A = -1}$

Put $x - 1 = 0 \Rightarrow x = 1$ in eq. (2)

$3(1) - 2 = B(1)(1-2)$

$3 - 2 = B(-1)$

$1 = -B \Rightarrow \boxed{B = -1}$

Put $x - 2 = 0 \Rightarrow x = 2$ in eq. (2)

$3(2) - 2 = C(2)(2-1)$

$4 = C(2)(1) \Rightarrow C = \frac{4}{2} \Rightarrow \boxed{C = 2}$

Putting values of A, B and C in eq. (1), we get

$$\frac{3x-2}{x(x-1)(x-2)} = \frac{-1}{x} + \frac{-1}{x-1} + \frac{2}{x-2}$$

$$\Rightarrow \int \frac{3x-2}{x(x-1)(x-2)}dx = -\int \frac{1}{x}dx - \int \frac{1}{x-1}dx + 2\int \frac{1}{x-2}dx$$

$$= -\ln|x| - \ln|x-1| + 2\ln|x-2| + c$$

Case-II: Integration of $\int \frac{P(x)}{Q(x)}dx$ where $Q(x)$ contains both repeated and non-repeated linear factors

Example 41: Evaluate $\int \frac{6x-10}{(x+1)(x-3)^2}dx$

Solution:

$$\int \frac{6x-10}{(x+1)(x-3)^2}dx$$

Let $\frac{6x-10}{(x+1)(x-3)^2} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$ ...(1)

www.ilmkidunya.com

www.ilmkidunya.com

Multiply each term by LCM $(x+1)(x-3)^2$

$$6x-10=A(x-3)^2+B(x+1)(x-3)+C(x+1) \quad \ldots(2)$$

Put $x+1=0 \Rightarrow x=-1$ in eq. (2)

$6(-1)-10=A(-1-3)^2$

$-16=16A$

$\boxed{A=-1}$

Put $x-3=0 \Rightarrow x=3$ in eq. (2)

$6(3)-10=C(3+1)$

$8=4C$

$\boxed{C=2}$

From eq. (2), we have

$$6x-10=A(x^2-6x+9)+B(x^2-2x-3)+Cx+C$$
$$=Ax^2-6Ax+9A+Bx^2-2Bx-3B+Cx+C$$
$$6x-10=(A+B)x^2+(-6A-2B+C)x+(9A-3B+C)$$

Comparing coefficient of x^2, we get

$A+B=0$

$-1+B=0 \Rightarrow \boxed{B=1}$

Putting values of A, B and C in eq. (1), we get

$$\frac{6x-10}{(x+1)(x-3)^2}=\frac{-1}{x+1}+\frac{1}{x-3}+\frac{2}{(x-3)^2}$$

$$\Rightarrow \int\frac{6x-10}{(x+1)(x-3)^2}dx$$
$$=-\int\frac{1}{x+1}dx+\int\frac{1}{x-3}dx+2\int(x-2)^{-2}\cdot 1dx$$
$$=-\ln|x+1|+\ln|x-3|+\frac{2(x-3)^{-2+1}}{-2+1}+c$$
$$=\ln|x-3|-\ln|x+1|+\frac{2(x-3)^{-1}}{-1}+c$$
$$=\ln\left|\frac{x-3}{x+1}\right|-\frac{2}{x-3}+c$$

Case-III: Integration of $\int\frac{P(x)}{Q(x)}dx$ where $Q(x)$ contains a linear factor and a distinct irreducible quadratic factor

Example 42: Find $\int\frac{5x^2+3x+2}{(x+1)(x^2+1)}dx$

Solution:

$$\int\frac{5x^2+3x+2}{(x+1)(x^2+1)}dx$$

Let $\frac{5x^2+3x+2}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$ $\ldots(1)$

Multiplying each term by LCM $(x+1)(x^2+1)$

$$5x^2+3x+2=A(x^2+1)+(Bx+C)(x+1) \quad \ldots(2)$$

Put $x+1=0 \Rightarrow x=-1$ in eq. (2)

$$5(-1)^2+3(-1)+2=A((-1)^2+1)$$
$$5-3+2=2A \Rightarrow A=\frac{4}{2} \Rightarrow \boxed{A=2}$$

From eq. (2), we have

$$5x^2+3x+2=Ax^2+A+Bx^2+Bx+Cx+C$$
$$5x^2+3x+2=(A+B)x^2+(B+C)x+(A+C)$$

Comparing coefficients of x^2 and x, we get

$A+B=5$

$2+B=5$

$B=5-2$

$\boxed{B=3}$

$B+C=3$

$3+C=3$

$C=3-3$

$\boxed{C=0}$

Putting values of A, B and C in eq. (1), we get

$$\frac{5x^2+3x+2}{(x+1)(x^2+1)}=\frac{2}{x+1}+\frac{3x+0}{x^2+1}$$

$$\Rightarrow \int\frac{5x^2+3x+2}{(x+1)(x^2+1)}dx=2\int\frac{1}{x+1}dx+3\int\frac{x}{x^2+1}dx$$
$$=2\ln|x+1|+\frac{3}{2}\int\frac{2x}{x^2+1}dx$$
$$=2\ln|x+1|+\frac{3}{2}\ln(x^2+1)+c$$

Case-IV: Integration of $\int\frac{P(x)}{Q(x)}dx$ where $Q(x)$ contains repeated irreducible quadratic factors

Example 43: Evaluate $\int\frac{x^3+2x+1}{(x^2+1)^2}dx$

Solution:

$$\frac{x^3+2x+1}{(x^2+1)^2}=\frac{x^3+x+x+1}{(x^2+1)^2}$$
$$=\frac{x(x^2+1)+x+1}{(x^2+1)^2}$$
$$=\frac{x(x^2+1)}{(x^2+1)^2}+\frac{x+1}{(x^2+1)^2}$$

$$\frac{x^3+2x+1}{(x^2+1)^2}=\frac{x}{x^2+1}+\frac{x}{(x^2+1)^2}+\frac{1}{(x^2+1)^2}$$

$$\Rightarrow \int\frac{x^3+2x+1}{(x^2+1)^2}dx=\int\frac{x}{x^2+1}dx+\int\frac{x}{(x^2+1)^2}dx+\int\frac{1}{(x^2+1)^2}dx$$
$$=\frac{1}{2}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\int(x^2+1)^{-2}(2x)dx+\int\frac{1}{(x^2+1)^2}dx$$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \frac{1}{2}\ell n(x^2+1) + \frac{1}{2}\cdot\frac{(x^2+1)^{-2+1}}{-2+1} + \int\frac{1}{(x^2+1)^2}dx$$

$$= \frac{1}{2}\ell n(x^2+1) - \frac{1}{2(x^2+1)} + \int\frac{1}{(x^2+1)^2}dx \quad \ldots(1)$$

Consider $\int\frac{1}{(x^2+1)^2}dx$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\int\frac{1}{(x^2+1)^2}dx = \int\frac{1}{(\tan^2\theta+1)^2}\cdot\sec^2\theta\, d\theta$$

$$= \frac{1}{(\sec^2\theta)^2}\cdot\sec^2\theta\, d\theta$$

$$= \int\frac{1}{\sec^2\theta}d\theta = \int\cos^2\theta\, d\theta$$

$$= \int\frac{1+\cos 2\theta}{2}d\theta$$

$$= \frac{1}{2}\int 1d\theta + \frac{1}{2}\int\cos 2\theta\, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\cdot\frac{\sin 2\theta}{2}$$

$$\int\frac{1}{(x^2+1)^2}dx = \frac{1}{2}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right)$$

$$= \frac{1}{2}\left(\theta + \frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta\right)$$

$$= \frac{1}{2}\left(\theta + \tan\theta\cdot\frac{1}{\sec^2\theta}\right)$$

$$= \frac{1}{2}\left(\theta + \frac{\tan\theta}{1+\tan^2\theta}\right)$$

$$\int\frac{1}{(x^2+1)^2}dx = \frac{1}{2}\left(\tan^{-1}x + \frac{x}{1+x^2}\right) \quad \begin{cases}\text{As } \tan\theta = x \\ \Rightarrow \theta = \tan^{-1}(x)\end{cases}$$

Put this in eq. (1), we get

$$\int\frac{x^3+2x+1}{(x^2+1)^2}dx$$

$$= \frac{1}{2}\ell n(x^2+1) - \frac{1}{2(x^2+1)} + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\cdot\frac{x}{1+x^2} + c$$

$$= \frac{1}{2}\ell n(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right) + c$$

$$= \frac{1}{2}\ell n(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{x-1}{2(x^2+1)} + c$$

EXERCISE 3.4

Evaluate the following integrals:

Q1. $\int\frac{5x+7}{(x+1)(x+2)}dx$

Solution:

$$\int\frac{5x+7}{(x+1)(x+2)}dx$$

Let $\frac{5x+7}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ ...(1)

Multiply each term by LCM $(x+1)(x+2)$

$$5x+7 = A(x+2) + B(x+1) \quad \ldots(2)$$

Put $x+1=0 \Rightarrow x=-1$ in eq. (2): $5(-1)+7 = A(-1+2)$

$2 = A(1)$

$\boxed{A=2}$

Put $x+2=0 \Rightarrow x=-2$ in eq. (2): $5(-2)+7 = B(-2+1)$

$-3 = B(-1)$

$\boxed{B=3}$

Putting values of A and B in eq. (1), we get

$$\frac{5x+7}{(x+1)(x+2)} = \frac{2}{x+1} + \frac{3}{x+2}$$

$$\Rightarrow \int\frac{5x+7}{(x+1)(x+2)}dx = 2\int\frac{1}{x+1}dx + 3\int\frac{1}{x+2}dx$$

$$= 2\ell n|x+1| + 3\ell n|x+2| + c$$

Q2. $\int\frac{3x+5}{x^2+4x+3}dx$

Solution:

$$\int\frac{3x+5}{x^2+4x+3}dx$$

$$\frac{3x+5}{x^2+4x+3} = \frac{3x+5}{x^2+3x+x+3}$$

$$= \frac{3x+5}{x(x+3)+1(x+3)}$$

$$\frac{3x+5}{x^2+4x+3} = \frac{3x+5}{(x+1)(x+3)}$$

Let $\frac{3x+5}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2}$ ...(1)

Multiply each term by LCM $(x+1)(x+3)$

$$3x+5 = A(x+3) + B(x+1) \quad \ldots(2)$$

www.ilmkidunya.com

www.ilmkidunya.com

Put $x+1=0 \Rightarrow x=-1$ in eq. (2)

$3(-1)+5=A(-1+3)$

$2=2A$

$\boxed{A=1}$

Put $x+3=0 \Rightarrow x=-3$ in eq. (2)

$3(-3)+5=B(-3+1)$

$-4=-2B$

$\boxed{B=2}$

Putting values of A and B in eq. (1), we get

$$\frac{3x+5}{(x+1)(x+3)}=\frac{1}{x+1}+\frac{2}{x+2}$$

$$\Rightarrow \int\frac{3x+5}{x^2+4x+3}dx=\int\frac{1}{x+1}dx+2\int\frac{1}{x+2}dx$$

$$=\ell n|x+1|+2\ell n|x+2|+c$$

Q3. $\int\frac{x^3+2x^2-x-1}{x^2-1}dx$

Solution:

$$\int\frac{x^3+2x^2-x-1}{x^2-1}dx$$

$$\frac{x^3+2x^2-x-1}{x^2-1}=Q+\frac{R}{D}$$

$$=(x+2)+\frac{1}{x^2-1}$$

$$=(x+2)+\frac{1}{(x-1)(x+1)} \quad ...(a)$$

Let $\frac{1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$ ...(1)

Multiply each term by LCM $(x-1)(x+1)$

$1=A(x+1)+B(x-1)$ ...(2)

Put $x-1=0 \Rightarrow x=1$ in eq. (2)

$1=A(1+1)$

$1=2A$

$\boxed{A=\frac{1}{2}}$

Put $x+1=0 \Rightarrow x=-1$ in eq. (2)

$1=B(-1-1)$

$1=-2B$

$\boxed{B=-\frac{1}{2}}$

Putting values of A and B in eq. (1), we get

$$\frac{1}{(x-1)(x+1)}=\frac{1}{2(x+1)}-\frac{1}{2(x+1)}$$

From eq. (a)

$$\frac{x^3+2x^2-x-1}{x^2-1}=x+2+\frac{1}{2(x-1)}-\frac{1}{2(x+1)}$$

$$\Rightarrow \int\frac{x^3+2x^2-x-1}{x^2-1}dx$$

$$=\int x\,dx+2\int 1dx+2\int 1dx+\frac{1}{2}\int\frac{1}{x-1}dx-\frac{1}{2}\int\frac{1}{x+1}dx$$

$$=\frac{x^2}{2}+2x+\frac{1}{2}\ell n|x-1|-\frac{1}{2}\ell n|x+1|+c$$

$$=\frac{x^2}{2}+2x+\frac{1}{2}(\ell n|x-1|-\ell n|x+1|)+c$$

$$=\frac{x^2}{2}+2x+\frac{1}{2}\ell n\left|\frac{x-1}{x+1}\right|+c$$

Q4. $\int\frac{x^2-12x-21}{x^2-6x-7}dx$

Solution:

$$\int\frac{x^2-12x-21}{x^2-6x-7}dx$$

$$\frac{x^2-12x-21}{x^2-6x-7}=Q+\frac{R}{D}$$

$$=1+\frac{-6x-14}{x^2-6x-7}=1+\frac{-6x-14}{x^2-7x+x-7}$$

$$=1+\frac{-6x-14}{x(x-7)+1(x-7)}$$

$$=1+\frac{-6x-14}{(x-7)(x+1)} \quad ...(a)$$

Let $\frac{-6x-14}{(x-7)(x+1)}=\frac{A}{x-7}+\frac{B}{x+1}$ ...(1)

Multiply each term by LCM $(x-7)(x+1)$

$-6x-14=A(x+1)+B(x-7)$ ...(2)

Put $x-7=0 \Rightarrow x=7$ in eq. (2)

$-6(7)-14=A(7+1)$

$-56=8A$

$\boxed{A=-7}$

Put $x+1=0 \Rightarrow x=-1$ in eq. (2)

$-6(-1)-14=B(-1-7)$

$-8=-8B$

$\boxed{B=1}$

Putting values of A and B in eq. (1), we get

$$\frac{-6x-14}{(x-7)(x+1)}=\frac{-7}{x-7}+\frac{1}{x+1}$$

From eq. (a)

$$\frac{x^2-12x-21}{x^2-6x-7}=1-\frac{7}{x-7}+\frac{1}{x+1}$$

$$\Rightarrow \int\frac{x^2-12x-21}{x^2-6x-7}dx=\int 1dx-7\int\frac{1}{x-7}dx+\int\frac{1}{x+1}dx$$

$$=x-7\ell n|x-7|+\ell n|x+1|+c$$

www.ilmkidunya.com

www.ilmkidunya.com

Q5. $\int \frac{3-16x}{4+5x-6x^2}dx$

Solution:

$$\int \frac{3-16x}{4+5x-6x^2}dx$$

$$\frac{3-16x}{4+5x-6x^2} = \frac{-(16x-3)}{-(6x^2-5x-4)}$$

$$= \frac{16x-3}{6x^2-8x+3x-4}$$

$$= \frac{16x-3}{2x(3x-4)+1(3x-4)}$$

$$\frac{3-16x}{4+5x-6x^2} = \frac{16x-3}{(2x+1)(3x-4)}$$

Let $\frac{16x-3}{(2x+1)(3x-4)} = \frac{A}{2x+1}+\frac{B}{3x-4}$ ...(1)

Multiply each term by LCM $(2x+1)(3x-4)$

$$16x-3 = A(3x-4)+B(2x+1) \quad ...(2)$$

Put $2x+1=0 \Rightarrow x=\frac{-1}{2}$ in eq. (2)

$$16\left(\frac{-1}{2}\right)-3 = A\left(\frac{-3}{4}-4\right)$$

$$-8-3 = A\left(\frac{-11}{2}\right)$$

$$-11\times\frac{2}{-11} = A$$

$$\boxed{A=2}$$

Put $3x-4=0 \Rightarrow x=\frac{4}{3}$ in eq. (2)

$$16\left(\frac{4}{3}\right)-3 = B\left(\frac{8}{3}+1\right)$$

$$\frac{64-9}{3} = B\left(\frac{8+3}{3}\right)$$

$$55 = 11B$$

$$\boxed{B=5}$$

Putting values of A and B in eq. (1), we get

$$\frac{16x-3}{(2x+1)(3x-4)} = \frac{2}{2x+1}+\frac{5}{3x-4}$$

$$\int \frac{16x-3}{(2x+1)(3x-4)}dx = \int \frac{2}{2x+1}dx+5\int \frac{1}{3x-4}dx$$

$$\int \frac{3-16x}{4+5x-6x^2}dx = \ln|2x+1|+\frac{5}{3}\int \frac{3}{3x-4}dx$$

$$= \ln|2x+1|+\frac{5}{3}\ln|3x-4|+c$$

Q6. $\int \frac{2x^2-8x+6}{x(x-1)(x-2)(x-3)}dx$

Solution:

$$\int \frac{2x^2-8x+6}{x(x-1)(x-2)(x-3)}dx = \int \frac{2(x^2-4x+3)}{x(x-1)(x-2)(x-3)}dx$$

$$= \int \frac{2(x^2-3x-x+3)}{x(x-1)(x-2)(x-3)}dx$$

$$= \int \frac{2[x(x-3)-1(x-3)]}{x(x-1)(x-2)(x-3)}dx$$

$$= \int \frac{2(x-3)(x-1)}{x(x-1)(x-2)(x-3)}dx$$

$$= \int \frac{2}{x(x-2)}dx \quad ...(a)$$

Let $\frac{2}{x(x-2)} = \frac{A}{x}+\frac{B}{x-2}$ ...(1)

Multiply each term by LCM $x(x-2)$

$$2 = A(x-2)+Bx \quad ...(2)$$

Put $x=0$ in eq. (2)

$$2 = A(0-2)$$

$$2 = -2A \Rightarrow A = -1$$

Put $x-2=0 \Rightarrow x=2$ in eq. (2)

$$2 = B(2) \Rightarrow \boxed{B=1}$$

Putting values of A and B in eq. (1), we get

$$\frac{2}{x(x-2)} = -\frac{1}{x}+\frac{1}{x-2}$$

$$\Rightarrow \int \frac{2}{x(x-2)}dx = -\int \frac{1}{x}dx+\int \frac{1}{x-2}dx$$

$$= -\ln x+\ln|x-2|+c$$

$$= \ln\left|\frac{x-2}{x}\right|+c$$

Put in eq. (a)

$$\int \frac{2x^2-8x+6}{x(x-1)(x-2)(x-3)}dx = \ln\left|\frac{x-2}{x}\right|+c$$

www.ilmkidunya.com

www.ilmkidunya.com

Q7. $\int \frac{x^2-3}{(x+2)^3}dx$

Solution:

$$\int \frac{x^2-3}{(x+2)^3}dx$$

Let $\frac{x^2-3}{(x+2)^3} = \frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3}$ ...(1)

Multiply each term by LCM $(x+2)^3$

$$x^2-3 = A(x+2)^2+B(x+2)+C \quad ...(2)$$

Put $x+2=0 \Rightarrow x=-2$ in eq. (2)

$$(-2)^2-3 = C \Rightarrow \boxed{C=1}$$

From eq. (2), we have

$$x^2-3 = A(x^2+4x+4)+Bx+2B+C$$
$$x^2-3 = Ax^2+4Ax+4A+Bx+2B+C$$
$$x^2-3 = Ax^2+(4A+B)x+(4A+2B+C)$$

Comparing coefficients of x^2 and x, we get

$\boxed{A=1}$ | $4A+B=0$, $4(1)+B=0 \Rightarrow \boxed{B=-4}$

Putting values of A, B and C in eq. (1)

$$\frac{x^2-3}{(x+2)^3} = \frac{1}{x+2}-\frac{4}{(x+2)^2}+\frac{1}{(x+2)^3}$$

$$\int \frac{x^2-3}{(x+2)^3}dx = \int \frac{1}{x+2}dx-4\int(x+2)^{-2}\cdot 1dx+\int(x+2)^{-3}\cdot 1dx$$
$$= \ell n|x+2|-\frac{4(x+2)^{-2+1}}{-2+1}+\frac{(x+2)^{-3+1}}{-3+1}+c$$
$$= \ell n|x+2|+\frac{4}{x+2}-\frac{1}{2(x+2)}+c$$

Q8. $\int \frac{e^x-1}{e^x(e^x+1)}dx$

Solution:

$$\int \frac{e^x-1}{e^x(e^x+1)}dx$$

Let $u=e^x \Rightarrow du=e^x dx \Rightarrow du=udx \Rightarrow dx=\frac{1}{u}du$

$$\int \frac{e^x-1}{e^x(e^x+1)}dx = \int \frac{u-1}{u(u+1)}\cdot\frac{1}{u}du$$
$$= \int \frac{u-1}{u^2(u+1)}du$$

Let $\frac{u-1}{u^2(u+1)} = \frac{A}{u}+\frac{B}{u^2}+\frac{C}{u+1}$ ...(1)

Multiply each term by LCM $u^2(u+1)$

$$u-1 = A(u)(u+1)+B(u+1)+C(u^2) \quad ...(2)$$

Put $u=0$ in eq. (2)

$$0-1 = B(0+1)$$
$$-1 = B$$
$$\boxed{B=-1}$$

Put $u+1=0 \Rightarrow u=-1$ in eq. (2)

$$-1-1 = C(-1)^2$$
$$-2 = C$$
$$\boxed{C=-2}$$

From eq. (2)

$$u-1 = A(u^2+u)+Bu+B+Cu^2$$
$$= Au^2+Au+Bu+B+Cu^2$$
$$u-1 = (A+C)u^2+(A+B)u+B$$

Comparing coefficients of u^2, we get

$$A+C=0 \Rightarrow A-2=0 \Rightarrow \boxed{A=2}$$

Putting values of A, B and C in eq. (1)

$$\frac{u-1}{u^2(u+1)} = \frac{2}{u}-\frac{1}{u^2}-\frac{2}{u+1}$$

$$\Rightarrow \int \frac{u-1}{u^2(u+1)}du = 2\int\frac{1}{u}du-\int u^{-2}du-2\int\frac{1}{u+1}du$$
$$= 2\ell n|u|-\frac{u^{-2+1}}{-2+1}-2\ell n|u+1|+c$$
$$= 2\ell n|u|+\frac{1}{u}-2\ell n|u+1|+c$$

Put $u=e^x$, we get

$$\int \frac{e^x-1}{e^x(e^x+1)}dx = 2\ell n(e^x)+\frac{1}{e^x}-2\ell n(e^x+1)+c$$
$$= e^{-x}+2(\ell n e^x-\ell n(e^x+1))+c$$
$$= e^{-x}+2\ell n\left(\frac{e^x}{e^x+1}\right)+c$$

Q9. $\int \frac{25(x+2)}{(x-2)(x+3)^2}dx$

Solution:

$$\int \frac{25(x+2)}{(x-2)(x+3)^2}dx$$

Let $\frac{25(x+2)}{(x-2)(x+3)^2} = \frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$ ...(1)

Multiply each term by LCM $(x-2)(x+3)^2$

$$25(x+2) = A(x+3)^2+B(x-2)(x+3)+C(x-2) \quad ...(2)$$

Put $x-2=0 \Rightarrow x=2$ in eq. (2)

Put $x+2=0 \Rightarrow x=-3$ in eq. (2)

www.ilmkidunya.com

www.ilmkidunya.com

$25(2+2) = A(2+3)^2$
$100 = 25A$
$\boxed{A = 4}$

$25(-3+2) = C(-3-2)$
$-25 = -5C$
$\boxed{C = 5}$

From eq. (2), we have

$25x + 50 = A(x^2+6x+9) + B(x^2+x-6) + Cx - 2C$

$25x + 50 = Ax^2 + 6Ax + 9A + Bx^2 + Bx - 6B + Cx - 2C$

$25x + 50 = (A+B)x^2 + (6A+B+C)x + (9A-6B-2C)$

Comparing coefficients of x^2, we get

$A + B = 0 \Rightarrow 4 + B = 0 \Rightarrow B = -4$

Putting values of A, B and C in eq. (1), we get

$$\frac{25(x+2)}{(x-2)(x+3)^2} = \frac{4}{x-2} - \frac{4}{x+3} + \frac{5}{(x+3)^2}$$

$$\Rightarrow \int \frac{25(x+2)}{(x-2)(x+3)^2}dx = 4\int \frac{1}{x-2}dx - 4\int \frac{1}{x+3}dx + 5\int (x+3)^{-2}\cdot 1dx$$

$$= 4\ln|x-2| - 4\ln|x+3| + 5\cdot\frac{(x+3)^{-2+1}}{-2+1} + c$$

$$= 4(\ln|x-2| - \ln|x+3|) + 5\cdot\frac{(x+3)^{-1}}{-1} + c$$

$$= 4\ln\left|\frac{x-2}{x+3}\right| - \frac{5}{x+3} + c$$

Q10. $\int \frac{x^3+4x^2+7x+5}{(x+1)^3(x+2)}dx$

Solution:

$$\int \frac{x^3+4x^2+7x+5}{(x+1)^3(x+2)}dx$$

Let $$\frac{x^3+4x^2+7x+5}{(x+1)^3(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x+2} \quad \ldots(1)$$

Multiply each term by LCM $(x+1)^3(x+2)$

$$x^3+4x^2+7x+5 = A(x+1)^2(x+2) + B(x+1)(x+2) + C(x+2) + D(x+1)^3 \quad \ldots(2)$$

Put $x+1=0 \Rightarrow x=-1$ in eq. (2)

$(-1)^3 + 4(-1)^2 + 7(-1) + 5 = C(-1+2)$
$-1+4-7+5 = C$
$\boxed{C = 1}$

Put $x+2=0 \Rightarrow x=-2$ in eq. (1)

$(-2)^3 + 4(-2)^2 + 7(-2) + 5 = D(-2+1)^3$
$-8+16-14+5 = -D$
$\boxed{D = 1}$

From eq. (2), we have

$x^3+4x^2+7x+5 = A(x^2+2x+1)(x+2) + B(x^2+3x+2) + Cx + 2C + D(x^3+3x^2+3x+1^3)$

$x^3+4x^2+7x+5 = A(x^3+2x^2+2x^2+4x+x+2) + Bx^2+3Bx+2B+Cx+2C+Dx^3+3Dx^2+3Dx+D$

$x^3+4x^2+7x+5 = Ax^3+4Ax^2+5Ax+2A+Bx^2+3Bx+2B+Cx+2C+Dx^3+3Dx^2+3Dx+D$

$x^3+4x^2+7x+5 = (A+D)x^3 + (4A+B+3D)x^2 + (5A+3B+C+3D)x + (2A+2B+2C+D)$

Comparing coefficients of x^3 and x^2, we get

$A + D = 1$
$A + 1 = 1$
$\boxed{A = 0}$

$4A + B + 3D = 4$
$4(0) + B + 3(1) = 4$
$B = 4 - 3 \Rightarrow \boxed{B = 1}$

Putting values of A, B, C and D in eq. (1), we get

$$\frac{x^3+4x^2+7x+5}{(x+1)^3(x+2)} = \frac{0}{x+1} + \frac{1}{(x+1)^2} + \frac{1}{(x+1)^3} + \frac{1}{x+2}$$

$$\Rightarrow \int \frac{x^3+4x^2+7x+5}{(x+1)^3(x+2)}dx = \int (x+1)^{-2}\cdot 1dx + \int (x+1)^{-3}\cdot 1dx + \int \frac{1}{x+2}dx$$

$$= \frac{(x+1)^{-2+1}}{-2+1} + \frac{(x+1)^{-3+1}}{-3+1} + \ln|x+2| + c$$

$$= \frac{(x+1)^{-1}}{-1} + \frac{(x+1)^{-2}}{-2} + \ln|x+2| + c = \ln|x+2| - \frac{1}{x+1} - \frac{1}{2(x+1)^2} + c$$

www.ilmkidunya.com

www.ilmkidunya.com

Q11. $\int \frac{x^2+2x-1}{(x-1)(x^2+1)}dx$

Solution:

$$\int \frac{x^2+2x-1}{(x-1)(x^2+1)}dx$$

Let $$\frac{x^2+2x-1}{(x-1)(x^2+1)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+1} \quad \ldots(1)$$

Multiply each term by LCM $(x-1)(x^2+1)$

$$x^2+2x-1 = A(x^2+1)+(Bx+C)(x-1) \quad \ldots(2)$$

Put $x-1=0 \Rightarrow x=1$ in eq. (2)

$$1^2+2(1)-1 = A(1^2+1) \Rightarrow 2=2A \Rightarrow \boxed{A=1}$$

From eq. (2), we have

$$x^2+2x-1 = Ax^2+A+Bx^2-Bx+Cx-C$$
$$x^2+2x-1 = (A+B)x^2+(-B+C)x+(A-C)$$

Comparing coefficients of x^2 and x, we get

$A+B=1$	$-B+C=2$
$1+B=1$	$-0+C=2$
$\boxed{B=0}$	$\boxed{C=2}$

Putting values of A, B and C in eq. (1), we get

$$\frac{x^2+2x-1}{(x-1)(x^2+1)} = \frac{1}{x-1}+\frac{0x+2}{x^2+1}$$

$$\Rightarrow \int \frac{x^2+2x-1}{(x-1)(x^2+1)} = \int \frac{1}{x-1}dx+2\int \frac{1}{1+x^2}dx$$
$$= \ln|x-1|+2\tan^{-1}x+c$$

Q12. $\int \frac{2x^3+x^2+2x+4}{(x^2+1)(x^2+4)}dx$

Solution:

$$\int \frac{2x^3+x^2+2x+4}{(x^2+1)(x^2+4)}dx$$

Let $$\frac{2x^3+x^2+2x+4}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4} \quad \ldots(1)$$

Multiply each term by LCM $(x^2+1)(x^2+4)$

$$2x^3+x^2+2x+4 = (Ax+B)(x^2+4)+(Cx+D)(x^2+1)$$
$$2x^3+x^2+2x+4 = Ax^3+4Ax+Bx^2+4B+Cx^3+Cx+Dx^2+D$$
$$2x^3+x^2+2x+4 = (A+C)x^3+(B+D)x^2+(4A+C)x+(4B+D)$$

Comparing coefficients of x^3, x^2, x and x^0, we get

$A+C=2$	$B+D=1$	$4A+C=2$	$4B+D=4$
$A=2-C \quad \ldots(2)$	$B=1-D \quad \ldots(3)$	$4(2-C)+C=2$	$4(1-D)+D=4$
		$8-4C+C=2$	$4-4D+D=4$
		$8-2=3C$	$-3D=0$
		$3C=6 \Rightarrow \boxed{C=2}$	$\boxed{D=0}$

www.ilmkidunya.com

www.ilmkidunya.com

Put $C = 2$ in eq. (2)

$A = 2 - 2 \Rightarrow \boxed{A = 0}$

Put $D = 0$ in eq. (3)

$B = 1 - 0 \Rightarrow \boxed{B = 1}$

Putting values of A, B, C and D in eq. (1), we get

$$\frac{2x^3+x^2+2x+4}{(x^2+1)(x^2+4)} = \frac{0x+1}{x^2+1} + \frac{2x+0}{x^2+4}$$

$$\Rightarrow \int \frac{2x^3+x^2+2x+4}{(x^2+1)(x^2+4)}dx = \int \frac{1}{1+x^2}dx + \int \frac{2x}{x^2+4}dx$$

$$= \tan^{-1}x + \ln(x^2+4) + c$$

Q13. $\int \frac{x^4+4x^2+2x+1}{(x^2+1)^2(x+1)}dx$

Solution:

$$\int \frac{x^4+4x^2+2x+1}{(x^2+1)^2(x+1)}dx$$

Let $$\frac{x^4+4x^2+2x+1}{(x^2+1)^2(x+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x+1} \quad \ldots(1)$$

Multiply each term by LCM $(x^2+1)^2(x+1)$

$$x^4+4x^2+2x+1 = (Ax+B)(x^2+1)(x+1) + (Cx+D)(x+1) + E(x^2+1)^2 \quad \ldots(2)$$

Put $x+1 = 0 \Rightarrow x = -1$ in eq. (2)

$$(-1)^4 + 4(-1)^2 + 2(-1) + 1 = E((-1)^2+1)^2$$

$$1+4-2+1 = E(2)^2 \quad \Rightarrow 4E = 4 \Rightarrow \boxed{E = 1}$$

From eq. (2), we have

$$x^4+4x^2+2x+1 = (Ax+B)(x^3+x^2+x+1) + Cx^2 + Cx + Dx + D + E(x^4+2x^2+1)$$

$$x^4+4x^2+2x+1 = Ax^4 + Ax^3 + Ax^2 + Ax + Bx^3 + Bx^2 + Bx + B + Cx^2 + Cx + Dx + D + Ex^4 + 2Ex^2 + E$$

$$x^4+4x^2+2x+1 = (A+E)x^4 + (A+B)x^3 + (A+B+C+2E)x^2 + (A+B+C+D)x + (B+D+E)$$

Comparing coefficients of x^4, x^3, x^2 and x, we get

$A + E = 1$

$A + 1 = 1$

$\boxed{A = 0}$

$A + B = 0$

$0 + B = 0$

$\boxed{B = 0}$

$A + B + C + 2E = 4$

$0 + 0 + C + 2(1) = 4$

$C = 4 - 2 \Rightarrow \boxed{C = 2}$

$A + B + C + D = 2$

$0 + 0 + 2 + D = 2$

$\boxed{D = 0}$

Putting values of A, B, C, D and E in eq. (1), we get

$$\frac{x^4+4x^2+2x+1}{(x^2+1)^2(x+1)} = \frac{0x+0}{x^2+1} + \frac{2x+0}{(x^2+1)^2} + \frac{1}{x+1}$$

$$\Rightarrow \int \frac{x^4+4x^2+2x+1}{(x^2+1)^2(x+1)}dx = \int (x^2+1)^{-2}(2x)dx + \int \frac{1}{x+1}dx$$

$$= \frac{(x^2+1)^{-2+1}}{-2+1} + \ln|x+1| + c$$

$$= \frac{(x^2+1)^{-1}}{-1} + \ln|x+1| + c$$

$$\int \frac{x^4+4x^2+2x+1}{(x^2+1)^2(x+1)}dx = \ln|x+1| - \frac{1}{x^2+1} + c$$

www.ilmkidunya.com

www.ilmkidunya.com

Q14. $\int \frac{x^2}{(x^2-2x+2)^2}dx$

Solution:

$$\int \frac{x^2}{(x^2-2x+2)^2}dx = \int \frac{x^2}{((x^2-2x+1)+1)^2}dx$$

$$= \int \frac{x^2}{((x-1)^2+1)^2}dx$$

Let $u = x-1 \Rightarrow x = u+1 \Rightarrow dx = du$

$$\int \frac{x^2}{(x^2-2x+2)^2}dx = \int \frac{(u+1)^2}{(u^2+1)^2}du$$

$$= \int \frac{u^2+2u}{(u^2+1)^2}du$$

$$= \int \left[\frac{u^2+1}{(u^2+1)^2} + \frac{2u}{(u^2+1)^2}\right]du$$

$$= \int \frac{1}{1+u^2}du + \int (u^2+1)^{-2}\cdot 2u\,du$$

$$= \tan^{-1}u + \frac{(u^2+1)^{-2+1}}{-2+1} + c$$

$$= \tan^{-1}u + \frac{(u^2+1)^{-1}}{-1} + c$$

Put $u = x-1$, we get

$$= \tan^{-1}(x-1) - \frac{1}{(x-1)^2+1} + c$$

$$= \tan^{-1}(x-1) - \frac{1}{x^2-2x+1+1} + c$$

$$= \tan^{-1}(x-1) - \frac{1}{x^2-2x+2} + c$$

Q15. $\int \frac{\cos x}{\sin^3 x - 1}dx$

Solution:

$$\int \frac{\cos x}{\sin^3 x - 1}dx$$

Let $u = \sin x \Rightarrow du = \cos x\,dx$

$$\int \frac{1}{u^3-1}dx = \int \frac{1}{(u-1)(u^2+u+1)}du$$

Let $\frac{1}{(u-1)(u^2+u+1)} = \frac{A}{u-1} + \frac{Bu+C}{u^2+u+1}$ ...(1)

Multiply each term by LCM $(u-1)(u^2+u+1)$

$1 = A(u^2+u+1) + (Bu+C)(u-1)$ ...(2)

Put $u-1=0 \Rightarrow u=1$ in eq. (2)

$1 = A(1^2+1+1) \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$

From eq. (2), we have

$1 = Au^2 + Au + A + Bu^2 - Bu + Cu - C$

$1 = (A+B)u^2 + (A-B+C)u + (A-C)$

Comparing coefficients of u^2 and u, we get

$A + B = 0$

$\frac{1}{3} + B = 0$

$\boxed{B = \frac{-1}{3}}$

$A - B + C = 0$

$\frac{1}{3} + \frac{1}{3} + C = 0$

$\frac{2}{3} + C = 0$

$\Rightarrow \boxed{C = \frac{-2}{3}}$

Putting values of A, B and C in eq. (1), we get

$$\frac{1}{(u-1)(u^2+u+1)} = \frac{\frac{1}{3}}{u-1} + \frac{\frac{-1}{3}u - \frac{2}{3}}{u^2+u+1}$$

$$\Rightarrow \int \frac{1}{u^3-1}du = \frac{1}{3}\int \frac{1}{u-1}du - \frac{1}{3}\int \frac{u+2}{u^2+u+1}du$$

$$= \frac{1}{3}\ln|u-1| - \frac{1}{3}\cdot\frac{1}{2}\int \frac{2u+4}{u^2+u+1}du = \frac{1}{3}\ln|u-1| - \frac{1}{6}\int \frac{(2u+1)+3}{u^2+u+1}du$$

$$= \frac{1}{3}\ln|u-1| - \frac{1}{6}\int \frac{2u+1}{u^2+u+1}du - \frac{1}{6}\int \frac{3}{u^2+u+1}du$$

$$= \frac{1}{3}\ln|u-1| - \frac{1}{6}\ln(u^2+u+1) - \frac{1}{2}\int \frac{1}{u^2+u+\left(\frac{1}{2}\right)^2+1-\left(\frac{1}{2}\right)^2}$$

www.ilmkidunya.com

www.ilmkidunya.com

$$\int \frac{1}{u^3-1}du = \frac{1}{3}\ln|u-1| - \frac{1}{6}\ln(u^2+u+1) - \frac{1}{2}\int \frac{1}{\left(u+\frac{1}{2}\right)^2+1-\frac{1}{4}}du$$

$$= \frac{1}{3}\ln|u-1| - \frac{1}{6}\ln(u^2+u+1) - \frac{1}{2}\int \frac{1}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}du = \frac{1}{3}\ln|u-1| - \frac{1}{6}\ln(u^2+u+1) - \frac{1}{2}\int \frac{1}{\left(u+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}du$$

$$= \frac{1}{3}\ln|u-1| - \frac{1}{6}\ln(u^2+u+1) - \frac{1}{2}\cdot\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{u+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$\int \frac{1}{u^3-1}du = \frac{1}{3}\ln|u-1| - \frac{1}{6}\ln(u^2+u+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right)+c$$

Put $u = \sin x$, we get

$$\int \frac{\cos x}{\sin^3 x-1}dx = \frac{1}{3}\ln|\sin x-1| - \frac{1}{6}\ln(\sin^2 x+\sin x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\sin x+1}{\sqrt{3}}\right)+c$$

Content Outline

- The Definite Integral

THE DEFINITE INTEGRAL

Definition: Let f be a continuous function on the closed interval $[a, b]$. The **definite integral** of f from a to b is denoted by $\int_a^b f(x)dx$ and is defined by the net signed area bounded by the curve $y = f(x)$ and the x-axis from the line $x = a$ to the $x = b$.

Remark I: We will study briefly the net signed area in the section 3.7.1.

Remark II: The value of a definite integral is a real number, not a family of functions, as is the case for indefinite integrals. The numbers a and b are the **limits of integration**. Here, a is the **lower limit** and b is the **upper limit**.

Remark III: We will use the fundamental theorem of calculus (stated blow) to evaluate a definite integral.

Theorem 3.2: Fundamental Theorem of Calculus (FTC)

If f is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example 44: Evaluate: $\int_{-1}^{1}(x^{\frac{1}{5}}+1)dx$

Solution:

$$\int_{-1}^{1}(x^{\frac{1}{5}}+1)dx = \left[\frac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1}+x\right]_{-1}^{1} = \left[\frac{x^{\frac{6}{5}}}{\frac{6}{5}}+x\right]_{-1}^{1} = \left[\frac{5}{6}x^{\frac{6}{5}}+x\right]_{-1}^{1}$$

$$= \left(\frac{5}{6}(1)^{\frac{6}{5}}+1\right) - \left(\frac{5}{6}(-1)^{\frac{6}{5}}+(-1)\right)$$

$$= \left(\frac{5}{6}+1\right) - \left(\frac{5}{6}-1\right) = \frac{5}{6}+1-\frac{5}{6}+1$$

$$= 2$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 45: Evaluate $\int_0^{\pi/4} \frac{\sec\theta}{\sin\theta+\cos\theta}d\theta$

Solution:

$$\int_0^{\pi/4} \frac{\sec\theta}{\sin\theta+\cos\theta}d\theta$$

$$= \int_0^{\pi/4} \frac{\sec\theta}{\cos\theta\left(\frac{\sin\theta}{\cos\theta}+1\right)}d\theta$$

$$= \int_0^{\pi/4} \frac{\sec\theta\cdot\sec\theta}{\tan\theta+1}d\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{1+\tan\theta}d\theta$$

Formula: $\int \frac{f'(x)}{f(x)}dx = \ell n|f(x)|+c$

$$= \left[\ell n|\tan\theta+1|\right]_0^{\pi/4}$$

$$= \ell n\left|\tan\frac{\pi}{4}+1\right| - \ell n|\tan 0+1|$$

$$= \ell n|1+1| - \ell n|0+1| = \ell n2 - \ell n1$$

$$= \ell n2 - 0 = \ell n2$$

Example 46: Evaluate $\int_0^{\sqrt{7}} \frac{x}{\sqrt{x^2+9}}dx$

Solution:

$$\int_0^{\sqrt{7}} \frac{x}{\sqrt{x^2+9}}dx = \frac{1}{2}\int_0^{\sqrt{7}} (x^2+9)^{\frac{-1}{2}}\cdot 2x\,dx$$

Formula: $\int [f(x)]^n \cdot f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}+c, n\neq -1$

$$= \frac{1}{2}\left[\frac{(x^2+9)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}\right]_0^{\sqrt{7}} = \frac{1}{2}\left[\frac{(x^2+9)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^{\sqrt{7}}$$

$$= \left[\sqrt{x^2+9}\right]_0^{\sqrt{7}} = \sqrt{(\sqrt{7})^2+9} - \sqrt{0^2+9}$$

$$= \sqrt{16} - \sqrt{9} = 4-3 = 1$$

Substitution in a Definite Integral (Change of variables)

Suppose that g is differentiable function on $[a,b]$. Also suppose that f is continuous on the range of g. Then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

Example 47: Evaluate $\int_0^2 2xe^{x^2}dx$

Solution:

$$\int_0^2 2xe^{x^2}dx = \int_0^2 e^{x^2}\cdot 2x\,dx$$

Let $u = x^2 \Rightarrow du = 2x\,dx$

When $x = 0 \Rightarrow u = (0)^2 = 0$

When $x = 2 \Rightarrow u = (2)^2 = 4$

$$\int_0^2 e^{x^2}\cdot 2x\,dx = \int_0^4 e^u du$$

$$= \left[e^u\right]_0^4$$

$$= e^4 - e^0 = e^4 - 1$$

Integration by Parts Formula for Definite Integrals

Let $f(x)$ and $g(x)$ be functions with continuous derivatives on the interval $[a,b]$. Then the integration by parts formula for definite integrals is:

$$\int_a^b f(x)g(x)dx = \left[f(x)\left(\int g(x)dx\right)\right]_a^b - \int_a^b \left[\left(\int g(x)dx\right)(f'(x))\right]dx$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 48: If $f(0) = f(\pi) = 0$, show that $\int_0^{\pi} f(x)\sin x\,dx = -\int_0^{\pi} f''(x)\sin x\,dx$

Solution:

$$\text{L.H.S} = \int_0^{\pi} \underset{1st}{f(x)} \underset{2nd}{\sin x}\,dx$$

Integrating by parts:

$$= [f(x)(-\cos x)]_0^{\pi} - \int_0^{\pi} f'(x)\cdot(-\cos x)\,dx$$

$$= -[f(\pi)\cdot\cos\pi - f(0)\cdot\cos 0] + \int_0^{\pi} \underset{1st}{f'(x)} + \underset{2nd}{\cos x}\,dx$$

Again, integrating by parts:

$$= -[(0)(-1)-(0)(1)] + [f'(x)\cdot\sin x]_0^{\pi} - \int_0^{\pi} f''(x)\sin x\,dx$$

$$\because f(0) = f(\pi) = 0$$

$$= 0 + [f'(\pi)\cdot\sin\pi - f'(0)\cdot\sin 0] - \int_0^{\pi} f''(x)\cos x\,dx$$

$$= f'(\pi)\cdot(0) - f'(0)\cdot(0) - \int_0^{\pi} f''(x)\cos x\,dx$$

$$= 0 - 0 - \int_0^{\pi} f''(x)\cos x\,dx$$

$$= -\int_0^{\pi} f''(x)\cos x\,dx = \text{R.H.S (Proved)}$$

Properties of the Definite Integrals

Throughout this section, assume f is a continuous function on the given interval.

Property-I: $\int_a^b f(x)dx = \int_a^b f(t)dt$

Property-II: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Property-III: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$.

This result can be generalized as follows:

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \ldots + \int_{c_{n-1}}^{c_n} f(x)dx + \int_{c_n}^{b} f(x)dx$$

where $a < c_1 < c_2 < c_3 < \cdots < c_{n-1} < c_n < b$

Property-IV: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(i) As a special case, if we take $a = 0$, we obtain:

$$\int_0^b f(x)dx = \int_0^b f(b-x)dx$$

(ii) As another special case, if we take $a = -b$, we obtain:

$$\int_{-b}^{b} f(x)dx = \int_{-b}^{b} f(-x)dx.$$

(iii) For any integrand of the form $\frac{f(x)}{f(x)+g(x)}$, if we make the substitution $x \to a+b-x$ such that $f(a+b-x) = g(x)$ and $g(a+b-x) = f(x)$, then $\int_a^b \frac{f(x)}{f(x)+g(x)}dx = \frac{b-a}{2}$.

Property-V: $\int_{-a}^{a} f(x)dx = \int_0^a (f(x)+f(-x))dx$

Try this:

Using Property-II, show that $\int_a^a f(x)dx = 0$

Proof:

Using property: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Put $b = a$, we get

$$\int_a^a f(x)dx = -\int_a^a f(x)dx$$

$$\int_a^a f(x)dx + \int_a^a f(x)dx = 0$$

$$2\int_a^a f(x)dx = 0$$

$$\Rightarrow \int_a^a f(x)dx = 0 \text{ (Proved)}$$

www.ilmkidunya.com

www.ilmkidunya.com

In particular, if $f(-x) = f(x)$ (that is, f is an even function), then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

and if $f(-x) = -f(x)$ (that is, f is an odd function), then $\int_{-a}^{a} f(x)dx = 0$

Property-VI: $\int_{0}^{a} f(x)dx = \int_{0}^{a/2} f(x) + f(a-x))dx$

In particular, $\int_{0}^{a} f(x)dx = \begin{cases} 0 & \text{if } f(a-x) = -f(x) \\ 2\int_{0}^{a/2} f(x)dx & \text{if } f(a-x) = f(x) \end{cases}$

Property-VII: If $f(x) \geq 0$ on the interval $[a, b]$, then $\int_{a}^{b} f(x)dx \geq 0$ and if $f(x) \leq 0$ on the interval $[c, d]$, then $\int_{c}^{d} f(x)dx \leq 0$.

Property-VIII: If $f(x) \leq g(x)$ on $[a, b]$, then $\int_{a}^{b} f(x)dx \leq \int_{a}^{b} g(x)dx$.

Property-IX: $\int_{a}^{b} (\alpha f_1(x) + \beta f_2(x))dx = \alpha\int_{a}^{b} f_1(x)dx + \beta\int_{a}^{b} f_2(x)dx$.

Property-X: If $f(x)$ is periodic with period T, then $\int_{a}^{b} f(x)dx = \int_{a+nT}^{b+nT} f(x)dx$, where n is an integer. In particular, $\int_{0}^{nT} f(x)dx = n\int_{0}^{T} f(x)dx$.

Example 49: Let $f(x) = \begin{cases} x^2, & \text{if } 0 \leq x < 2 \\ 4 - x & \text{if } 2 \leq x \leq 4 \end{cases}$ **Compute:** $\int_{0}^{4} f(x)dx$.

Solution:

$$\int_{0}^{4} f(x)dx$$

$$= \int_{0}^{2} f(x)dx + \int_{2}^{4} f(x)dx = \int_{0}^{2} x^2 dx + \int_{2}^{4} (4-x)dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 + \left[4x - \frac{x^2}{2}\right]_2^4$$

$$= \frac{1}{3}[(2)^3 - (0)^3] + \left[\left(4(4) - \frac{4^2}{2}\right) - \left(4(2) - \frac{2^2}{2}\right)\right]$$

$$= \frac{1}{3}(8-0) + (16-8) - (8-2)$$

$$= \frac{8}{3} + 8 - 6 = \frac{8}{3} + 2 = \frac{8+6}{3}$$

$$= \frac{14}{3}$$

Example 50: Evaluate: $\int_{-6}^{4} f(x)dx$ **where** $f(x) = |x-1| + |x+2|$.

Solution:

$$\int_{-6}^{4} f(x)dx$$

$$= \int_{-6}^{4} (|x-1| + |x+2|)dx = \int_{-6}^{4} |x-1|dx + \int_{-6}^{4} |x+2|dx$$

$$= \int_{-6}^{1} |x-1|dx + \int_{1}^{4} |x-1|dx + \int_{-6}^{-2} |x+2|dx + \int_{-2}^{4} |x+2|dx = -\int_{-6}^{1} (x-1)dx + \int_{1}^{4} (x-1)dx - \int_{-6}^{-2} (x+2)dx + \int_{-2}^{4} (x+2)dx$$

www.ilmkidunya.com

www.ilmkidunya.com

Formula: $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1$

$$= -\left[\frac{(x-1)^{1+1}}{1+1}\right]_{-6}^{1} + \left[\frac{(x-1)^{1+1}}{1+1}\right]_{1}^{4} - \left[\frac{(x+2)^{1+1}}{1+1}\right]_{-6}^{-2} + \left[\frac{(x+2)^{1+1}}{1+1}\right]_{-2}^{4}$$

$$= -\frac{1}{2}\left[(x-1)^2\right]_{-6}^{1} + \frac{1}{2}\left[(x-1)^2\right]_{1}^{4} - \frac{1}{2}\left[(x+2)^2\right]_{-6}^{-2} + \frac{1}{2}\left[(x+2)^2\right]_{-2}^{4}$$

$$= -\frac{1}{2}\left[(1-1)^2 - (-6-1)^2\right] + \frac{1}{2}\left[(4-1)^2 - (1-1)^2\right] \quad \left| \int_0^b f(\theta)d\theta = \int_0^b f(b-\theta)d\theta \right.$$

$$= -\frac{1}{2}(0-49) + \frac{1}{2}(9-0) - \frac{1}{2}(0-16) + \frac{1}{2}(36-0) = \frac{1}{2}(49+9+16+36) = \frac{1}{2}(110)$$

$$= 55$$

Example 51: Evaluate $\int_0^1 x^2(1-x)^{\frac{3}{2}}dx$.

Solution:

$$\int_0^1 x^2(1-x)^{\frac{3}{2}}dx$$

Using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$= \int_0^1 (1-x)^2(1-(1-x))^{\frac{3}{2}}dx$$

$$= \int_0^1 (1-x)^2(1-1+x)^{\frac{3}{2}}dx$$

$$= \int_0^1 (1-2x+x^2)\cdot x^{\frac{3}{2}}dx$$

$$= \int_0^1 (x^{\frac{3}{2}} - 2x^{1}\cdot x^{\frac{3}{2}} + x^2\cdot x^{\frac{3}{2}})dx = \int_0^1 (x^{\frac{3}{2}} - 2x^{\frac{5}{2}} + x^{\frac{7}{2}})dx$$

$$= \left[\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{2x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1}\right]_0^1$$

$$= \left[\frac{2}{5}x^{\frac{5}{2}} - 2\cdot\frac{2}{7}x^{\frac{7}{2}} + \frac{2}{9}x^{\frac{9}{2}}\right]_0^1$$

$$= \left[\frac{2}{5}(1)^{\frac{5}{2}} - \frac{4}{7}(1)^{\frac{7}{2}} + \frac{2}{9}(1)^{\frac{9}{2}}\right] - [0-0+0]$$

$$= \frac{2}{3} - \frac{4}{7} + \frac{2}{9} = \frac{126-180+70}{315}$$

$$= \frac{16}{315}$$

Example 52: Evaluate $\int_0^{2\pi} \cos^5 x\,dx$

Solution:

Let $I = \int_0^{2\pi} \cos^5 x\,dx$

Here $f(x) = \cos^5 x$

As $f(2\pi - x) = [\cos(2\pi - x)]^5$

$= (\cos x)^5$

$= \cos^5 x$

$= f(x)$

Using property $\int_0^a f(x)dx = 2\int_0^{a/2} f(x)dx$, if $f(a-x) = f(x)$

$\therefore \quad I = 2\int_0^{\pi} \cos^5 x\,dx$

As $f(\pi - x) = [\cos(\pi - x)]^5$

$= (-\cos x)^5$

$= -\cos^5 x = -f(x)$

So by property $\int_0^a f(x)dx = 0$ if $f(a-x) = -f(x)$

$\therefore \quad I = 2(0) = 0$

www.ilmkidunya.com

www.ilmkidunya.com

Example 53: Show that $\int_{\pi/5}^{3\pi/10} \frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2}+\sin^2 x\cos^2 x}dx = 0.$

Solution:

Let $I = \int_{\pi/5}^{3\pi/10} \frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2}+\sin^2 x\cdot\cos^2 x}dx$ ...(1)

By using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Replace x by $\frac{\pi}{5}+\frac{3\pi}{10}-x = \frac{2\pi+3\pi}{10}-x = \frac{\pi}{2}-x$

$$I = \int_{\pi/5}^{3\pi/10} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}-\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{2}+\sin^2\left(\frac{\pi}{2}-x\right)\cos^2\left(\frac{\pi}{2}-x\right)}dx$$

$$= \int_{\pi/5}^{3\pi/10} \frac{\sqrt{\cos x}-\sqrt{\sin x}}{\sqrt{2}+\cos^2 x\cdot\sin^2 x}dx$$

$$I = -\int_{\pi/5}^{3\pi/10} \frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2}+\sin^2 x\cdot\cos^2 x}dx$$

$I = -I$ Using (1)

$2I = 0$

$I = 0$

Hence, $\int_{\pi/5}^{3\pi/10} \frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2}+\sin^2 x\cdot\cos^2 x}dx = 0$

Example 54: Given $\int_{-3}^{3} \frac{f(x)}{1+e^x}dx = 4$ **and** $\int_{-3}^{3} f(x)dx = 10$. **Evaluate** $\int_{-3}^{3} \frac{f(x)}{1+e^{-x}}dx.$

Solution:

$$\int_{-3}^{3} \frac{f(x)}{1+e^{-x}}dx.$$

$$= \int_{-3}^{3} \left(\frac{1}{1+\frac{1}{e^x}}\right) f(x)dx$$

$$= \int_{-3}^{3} \left(\frac{e^x}{e^x+1}\right) f(x)dx$$

$$= \int_{-3}^{3} \left(\frac{e^x+1-1}{e^x+1}\right) f(x)dx$$

$$= \int_{-3}^{3} \left(\frac{e^x+1}{e^x+1}-\frac{1}{e^x+1}\right) f(x)dx$$

$$= \int_{-3}^{3} \left(1-\frac{1}{1+e^x}\right) f(x)dx$$

$$= \int_{-3}^{3} f(x)dx - \int_{-3}^{3} \frac{f(x)}{1+e^x}dx$$

Put $\int_{-3}^{3} f(x)dx = 10$ and $\int_{-3}^{3} \frac{f(x)}{1+e^x}dx = 4$, we get

$= 10 - 4$

$= 6$

Thus, $\int_{-3}^{3} \frac{f(x)}{1+e^{-x}}dx = 6.$

Example 55: Evaluate $\int_{-a}^{a} \frac{x^3\sin^2 x}{x^4+2x^2+1}dx.$

Solution:

$$\int_{-a}^{a} \frac{x^3\sin^2 x}{x^4+2x^2+1}dx.$$

Let $f(x) = \frac{x^3\cdot\sin^2 x}{x^4+2x^2+1}$

Replace x by $-x$, we have

$$f(-x) = \frac{(-x)^3(\sin(-x))^2}{(-x)^4+2(-x)^2+1}$$

$$f(-x) = \frac{(-x^3)(-\sin x)^2}{x^4+2x^2+1}$$

$$= \frac{x^3\cdot\sin^2 x}{x^4+2x^2+1}$$

$$= -f(x)$$

$\Rightarrow$ $f(x)$ is an odd function.

Hence, $\int_{-a}^{a} \frac{x^3\sin^2 x}{x^4+2x^2+1}dx = 0$

www.ilmkidunya.com

www.ilmkidunya.com

Example 56: Given $\int_0^2 f(x)dx = 4$ and

$f(2-x) = f(x) = f(-x)$. Evaluate $\int_1^3 f(x)dx$.

Solution:

Given: $\int_0^2 f(x)dx = 4$

$\Rightarrow \quad 2\int_0^1 f(x)dx = 4 \qquad \because f(2-x) = f(x)$

$\int_0^1 f(x)dx = \frac{4}{2} = \int_0^1 f(x)dx = 2 \qquad ...(1)$

Now, $\int_1^3 f(x)dx = ?$

Put $x = 2-t \Rightarrow dx = -dt$

When $x = 1 \Rightarrow t = 1$

When $x = 3 \Rightarrow t = -1$

$\Rightarrow \quad \int_1^3 f(x)dx = \int_1^{-1} f(2-t)(-dt)$

$= -\int_1^{-1} f(2-t)dt$

$= (-)(-)\int_{-1}^{1} f(2-t)dt = \int_{-1}^{1} f(2-x)dx$

$= \int_{-1}^{1} f(x)dx \qquad \because f(2-x) = f(x)$

$= 2\int_0^1 f(x)dx \qquad \because f(-x) = f(x)$

$= 2(2) = 4 \qquad$ Using eq. (1)

Challenge!

Given $\int_0^8 f(x)dx = 15$, $\int_0^6 f(x)dx = 12$, and $\int_2^8 f(x)dx = 10$. Find $\int_2^6 f(x)dx$.

Solution:

Given that: $\int_0^8 f(x)dx = 15$

$\Rightarrow \quad \int_0^2 f(x)dx + \int_2^8 f(x)dx = 15$

$\int_0^2 f(x)dx + 10 = 15 \qquad \because \int_2^8 f(x)dx = 10$

$\int_0^2 f(x)dx = 15 - 10$

$\int_0^2 f(x)dx = 5 \qquad ...(1)$

As $\int_0^6 f(x)dx = 12$

$\Rightarrow \quad \int_0^2 f(x)dx + \int_2^6 f(x)dx = 12$

$5 + \int_2^6 f(x)dx = 12 \qquad$ Using eq. (1)

$\int_2^6 f(x)dx = 12 - 5$

$\int_2^6 f(x)dx = 7$

EXERCISE 3.5

Q1. Evaluate the following definite integrals:

(i) $\int_1^2 (x^2 + x)dx$

Solution:

$\int_1^2 (x^2 + x)dx$

Using: $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

$= \left[\frac{x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1}\right]_1^2 = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_1^2$

$= \left(\frac{2^3}{3} + \frac{2^2}{2}\right) - \left(\frac{1^3}{3} + \frac{1^2}{2}\right)$

$= \frac{8}{3} + 2 - \frac{1}{3} - \frac{1}{2}$

$= \frac{16 + 12 - 2 - 3}{6} = \frac{23}{6}$

www.ilmkidunya.com

www.ilmkidunya.com

(ii) $\int_{-2}^{3}(4x^3+3x^2+2x)dx$

Solution:

$$\int_{-2}^{3}(4x^3+3x^2+2x)dx$$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$

$$=\left[\frac{4x^{3+1}}{3+1}+\frac{3x^{2+1}}{2+1}+\frac{2x^{1+1}}{1+1}\right]_{-2}^{3}$$

$$=\left[\frac{4x^4}{4}+\frac{3x^3}{3}+\frac{2x^2}{2}\right]_{-2}^{3}$$

$$=\left[x^4+x^3+x^2\right]_{-2}^{3}$$

$$=(3^4+3^3+3^2)-((-2)^4+(-2)^3+(-2)^2)$$

$$=(81+27+9)-(16-8+4)$$

$$=117-12$$

$$=105$$

(iii) $\int_{0}^{\pi/2}\sin^2 x dx$

Solution:

$$\int_{0}^{\pi/2}\sin^2 x dx$$

$$=\int_{0}^{\pi/2}\left(\frac{1-\cos 2x}{2}\right)dx=\frac{1}{2}\int_{0}^{\pi/2}(1-\cos 2x)dx$$

$$=\frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]_{0}^{\pi/2}$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{2}-\frac{1}{2}\sin 2\left(\frac{\pi}{2}\right)\right)-\left(0-\frac{1}{2}\sin 2(0)\right)\right]$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{2}-\frac{1}{2}\sin\pi\right)-\left(-\frac{1}{2}\sin 0\right)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{2}-\frac{1}{2}(0)+\frac{1}{2}(0)\right]$$

$$=\frac{\pi}{4}$$

(iv) $\int_{0}^{\pi/4}\frac{1}{1+\sin x}dx$

Solution:

$$\int_{0}^{\pi/4}\frac{1}{1+\sin x}dx$$

$$=\int_{0}^{\pi/4}\frac{1}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}dx$$

$$=\int_{0}^{\pi/4}\frac{1-\sin x}{1^2-\sin^2 x}dx=\int_{0}^{\pi/4}\frac{1-\sin x}{\cos^2 x}dx$$

$$=\int_{0}^{\pi/4}\left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos x\cdot\cos x}\right)dx$$

$$=\int_{0}^{\pi/4}(\sec^2 x-\sec x\cdot\tan x)dx$$

$$=\left[\tan x-\sec x\right]_{0}^{\pi/4}=\left(\tan\frac{\pi}{4}-\sec\frac{\pi}{4}\right)-(\tan 0-\sec 0)$$

$$=(1-\sqrt{2})-(0-1)=1-\sqrt{2}+1$$

$$=2-\sqrt{2}$$

(v) $\int_{0}^{\pi/2}\cos^3\theta d\theta$

Solution:

$$\int_{0}^{\pi/2}\cos^3\theta d\theta$$

$$=\int_{0}^{\pi/2}\cos^2\theta\cdot\cos\theta d\theta=\int_{0}^{\pi/2}(1-\sin^2\theta)\cdot\cos\theta d\theta$$

$$=\int_{0}^{\pi/2}(\cos\theta-\sin^2\theta\cdot\cos\theta)d\theta$$

Using: $\int[f(x)]^n\cdot f'(x)dx=\frac{[f(x)]^{n+1}}{n+1}+c,(n\neq -1)$

$$=\left[\sin\theta-\frac{\sin^3\theta}{3}\right]_{0}^{\pi/2}$$

$$=\left(\sin\frac{\pi}{2}-\frac{1}{3}\sin^3\frac{\pi}{2}\right)-\left(\sin 0-\frac{1}{3}\sin^3 0\right)$$

$$=\left(1-\frac{1}{3}(1)^3\right)-\left(0-\frac{1}{3}(0)^3\right)$$

$$=1-\frac{1}{3}=\frac{2}{3}$$

(vi) $\int_{0}^{1}\frac{\sqrt{x}}{1+x}dx$

Solution:

$$\int_{0}^{1}\frac{\sqrt{x}}{1+x}dx$$

Let $u=\sqrt{x}\Rightarrow u^2=x\rightarrow 2u\,du=dx$

www.ilmkidunya.com

www.ilmkidunya.com

When $x=0 \Rightarrow u=\sqrt{0}=0$

When $x=1 \Rightarrow u=\sqrt{1}=1$

$$\int_0^1 \frac{\sqrt{x}}{1+x}dx = \int_0^1 \frac{u}{1+u^2}\cdot 2u\,du$$

$$= 2\int_0^1 \frac{u^2}{1+u^2}du = 2\int_0^1 \frac{(1+u^2)-1}{1+u^2}du$$

$$= 2\int_0^1 \left(\frac{1+u^2}{1+u^2}-\frac{1}{1+u^2}\right)du$$

$$= 2\int_0^1 \left(1-\frac{1}{1+u^2}\right)du$$

$$= 2[u-\tan^{-1}u]_0^1$$

$$= 2[(1-\tan^{-1}(1))-(0-\tan^{-1}(0))]$$

$$= 2\left(1-\frac{\pi}{4}+0\right)$$

$$= 2-\frac{\pi}{2}$$

(vii) $\int_0^1 \frac{\ell n(1+x)}{1+x^2}dx$

Solution:

Let $I = \int_0^1 \frac{\ell n(1+x)}{1+x^2}dx$ ...(1)

Put $x=\tan\theta \Rightarrow dx=\sec^2\theta\,d\theta$

When $x=0 \Rightarrow \theta=\tan^{-1}(0)=0$

When $x=1 \Rightarrow \theta=\tan^{-1}(1)=\frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{\ell n(1+\tan\theta)}{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \frac{\ell n(1+\tan\theta)}{\sec^2\theta}\cdot\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \ell n(1+\tan\theta)d\theta$$

Using Property: $\int_a^b f(\theta)d\theta = \int_a^b f(a+b-\theta)d\theta$

$$I = \int_0^{\pi/4} \ell n\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta$$

$$= \int_0^{\pi/4} \ell n\left(1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \ell n\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta \qquad \because \tan\frac{\pi}{4}=1$$

$$= \int_0^{\pi/4} \ell n\left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \ell n\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$I = \int_0^{\pi/4} (\ell n2-\ell n(1+\tan\theta))d\theta$$

$$I = \int_0^{\pi/4} \ell n2\,d\theta - \int_0^{\pi/4} \ell n(1+\tan\theta)d\theta$$

$$I = \ell n2\cdot\int_0^{\pi/4} 1d\theta - I \qquad \text{Using (1)}$$

$$2I = \ell n2[\theta]_0^{\pi/4}$$

$$I = \frac{1}{2}\ell n2\left(\frac{\pi}{4}-0\right)$$

$$I = \frac{\pi}{8}\ell n2$$

(viii) $\int_0^1 \tan^{-1}x\,dx$

Solution:

$$\int_0^1 \tan^{-1}x\,dx = \int_0^1 \underset{1st}{\tan^{-1}x}\cdot\underset{2nd}{1}dx$$

Integrating by parts

$$\int_0^1 \tan^{-1}x\,dx = \left[\tan^{-1}x\cdot(x)\right]_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot x\,dx$$

$$= \left[x\tan^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx$$

$$= [1\cdot\tan^{-1}(1)-0\cdot\tan^{-1}(0)] - \frac{1}{2}\left[\ell n(1+x^2)\right]_0^1$$

$$= \frac{\pi}{4}-0-\frac{1}{2}[\ell n(1+1^2)-\ell n(1+0^2)]$$

$$= \frac{\pi}{4}-\frac{1}{2}(\ell n2-\ell n1)$$

$$= \frac{\pi}{4}-\frac{1}{2}\ell n2-0 \qquad \because \ell n1=0$$

$$= \frac{\pi}{4}-\frac{1}{2}\ell n2$$

www.ilmkidunya.com

www.ilmkidunya.com

(ix) $\int_0^{2\pi} \frac{\sin^2 x}{1+e^{\cos x}}dx$

Solution:

Let $I = \int_0^{2\pi} \frac{\sin^2 x}{1+e^{\cos x}}dx$

Here $f(x) = \frac{\sin^2 x}{1+e^{\cos x}}$

As $f(2\pi - x) = \frac{[\sin(2\pi - x)]^2}{1+e^{\cos(2\pi - x)}}$

$= \frac{(-\sin x)^2}{1+e^{\cos x}} = \frac{\sin^2 x}{1+e^{\cos x}} = f(x)$

So by using property:

$$\int_0^a f(x)dx = 2\int_0^{a/2} f(x)dx \text{ if } f(a-x) = f(x)$$

$\therefore$ $I = 2\int_0^{\pi} \frac{\sin^2 x}{1+e^{\cos x}}dx$ ...(1)

By using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Replace x by $0 + \pi - x = \pi - x$

$$I = 2\int_0^{\pi} \frac{\sin^2(\pi - x)}{1+e^{\cos(\pi - x)}}dx$$

$$= 2\int_0^{\pi} \frac{\sin^2 x}{1+e^{-\cos x}}dx = 2\int_0^{\pi} \frac{\sin^2 x}{1+\frac{1}{e^{\cos x}}}dx$$

$$= 2\int_0^{\pi} \frac{\sin^2 x}{\frac{e^{\cos x}+1}{e^{\cos x}}}dx = 2\int_0^{\pi} \frac{\sin^2 x \cdot e^{\cos x}}{1+e^{\cos x}}dx$$

$$= 2\int_0^{\pi} \frac{\sin^2 x\, e^{\cos x} + \sin^2 x - \sin^2 x}{1+e^{\cos x}}dx$$

$$= 2\int_0^{\pi} \left[\frac{\sin^2 x(e^{\cos x}+1)}{1+e^{\cos x}} - \frac{\sin^2 x}{1+e^{\cos x}}\right]dx$$

$$I = 2\int_0^{\pi} \sin^2 x\, dx - 2\int_0^{\pi} \frac{\sin^2 x}{1+e^{\cos x}}dx$$

$I = 2\int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right)dx - I$ Using (1)

$2I = \int_0^{\pi} (1-\cos 2x)dx$

$$= \left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \left(x - \frac{1}{2}\sin 2x\right) - \left(0 - \frac{1}{2}\sin 0\right)$$

$$= \pi - \frac{1}{2}(0) - 0$$

$2I = \pi$

$I = \frac{\pi}{2}$

Q2. Evaluate the following definite integrals:

(i) $\int_{-1}^{1} f(x)dx$, where $f(x) = \begin{cases} 5-3x & \text{if } x \le 0 \\ 5+3x & \text{if } x > 0 \end{cases}$

Solution:

$$\int_{-1}^{1} f(x)dx$$

$$= \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx = \int_{-1}^{0} (5-3x)dx + \int_{0}^{1} (5+3x)dx$$

$$= \left[5x - \frac{3x^2}{2}\right]_{-1}^{0} + \left[5x + \frac{3x^2}{2}\right]_{0}^{1}$$

$$= \left[\left(5(0) - \frac{3}{2}(0)^2\right) - \left(5(-1) - \frac{3}{2}(-1)^2\right)\right] + \left[\left(5(1) + \frac{3}{2}(1)^2\right) - \left(5(0) + \frac{3}{2}(0)^2\right)\right]$$

$$= \left[0 - \left(-5 - \frac{3}{2}\right)\right] + \left[\left(5 + \frac{3}{2}\right) - 0\right]$$

$$= \frac{13}{2} + \frac{13}{2} = \frac{13+13}{2} = \frac{26}{2} = 13$$

(ii) $\int_{-3}^{5} f(x)dx$, where $f(x) = |x-4| + |x|$

Solution:

$$\int_{-3}^{5} f(x)dx$$

$$= \int_{-3}^{5} (|x-4| + |x|)dx \quad \because f(x) = |x-4| + |x|$$

$$= \int_{-3}^{5} |x-4|dx + \int_{-3}^{5} |x|dx$$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \int_{-3}^{4}|x-4|dx+\int_{4}^{5}|x-4|dx+\int_{-3}^{0}|x|dx+\int_{0}^{5}|x|dx$$

$$= -\int_{-3}^{4}(x-4)+\int_{4}^{5}(x-4)dx-\int_{-3}^{0}x\,dx+\int_{0}^{5}x\,dx$$

Using: $\int[f(x)]^n\cdot f'(x)dx=\frac{[f(x)]^{n+1}}{n+1}+c,(n\neq -1)$

$$= -\left[\frac{(x-4)^2}{2}\right]_{-3}^{4}+\left[\frac{(x-4)^2}{2}\right]_{4}^{5}-\left[\frac{x^2}{2}\right]_{-3}^{0}+\left[\frac{x^2}{2}\right]_{0}^{5}$$

$$= -\frac{1}{2}[(4-4)^2-(-3-4)^2]+\frac{1}{2}[(5-4)^2-(4-4)^2]-\frac{1}{2}(0^2-(-3)^2)+\frac{1}{2}(5^2-0^2)$$

$$= -\frac{1}{2}(0-49)+\frac{1}{2}(1-0)-\frac{1}{2}(0-9)+\frac{1}{2}(25-0)$$

$$= \frac{1}{2}(49+1+9+25)=\frac{1}{2}(84)$$

$$= 42$$

(iii) $\int_{-3}^{4}|4-x^2|dx.$

Solution:

$$\int_{-3}^{4}|4-x^2|dx$$

$$= \int_{-3}^{-2}|4-x^2|dx+\int_{-2}^{2}|4-x^2|dx+\int_{2}^{4}|4-x^2|dx$$

$$= -\int_{-3}^{-2}(4-x^2)dx+\int_{-2}^{2}(4-x^2)dx-\int_{2}^{4}(4-x^2)dx$$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c;(n\neq -1)$

$$= -\left[4x-\frac{x^3}{3}\right]_{-3}^{-2}+\left[4x-\frac{x^3}{3}\right]_{-2}^{2}-\left[4x-\frac{x^3}{3}\right]_{2}^{4}$$

$$= -\left[\left(4(-2)-\frac{(-2)^3}{3}\right)-\left(4(-3)-\frac{(-3)^3}{3}\right)\right]+\left[\left(4(2)-\frac{2^3}{3}\right)-\left(4(-2)-\frac{(-2)^3}{3}\right)\right]-\left[\left(4(4)-\frac{4^3}{3}\right)-\left(4(2)-\frac{2^3}{3}\right)\right]$$

$$= -\left(-8+\frac{8}{3}+12-9\right)+\left(8-\frac{8}{3}+8-\frac{8}{3}\right)-\left(16-\frac{64}{3}-8+\frac{8}{3}\right)$$

$$= -\left(\frac{8}{3}-5\right)+\left(16-\frac{8}{3}-\frac{8}{3}\right)-\left(8-\frac{64}{3}+\frac{8}{3}\right)$$

$$= -\frac{8}{3}+5+16-\frac{8}{3}-\frac{8}{3}-8+\frac{64}{3}-\frac{8}{3}=13-4\left(\frac{8}{3}\right)+\frac{64}{3}$$

$$= \frac{39-32+64}{3}=\frac{71}{3}$$

Q3. Let $f(x)=\begin{cases}x^2 & \text{if } 0\le x\le 2\\ 2x+1 & \text{if } 2<x\le 4\end{cases}$, Here $g(x)$ is defined as: $g(x)=f(x)+f(4-x)$. Compute: $\int_0^4 g(x)dx$.

Solution:

$$\int_0^4 g(x)dx = \int_0^4[f(x)+f(4-x)]dx$$

$$= \int_0^4 f(x)dx+\int_0^4 f(4-x)dx \quad \ldots(1)$$

Using property: $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$$\int_0^4 f(4-x)dx = \int_0^4 f(4-(4-x))dx$$

$$= \int_0^4 f(4-4+x)dx=\int_0^4 f(x)dx$$

Eq. (1) $\Rightarrow \int_0^4 g(x)dx = \int_0^4 f(x)dx+\int_0^4 f(x)dx$

$$= 2\int_0^4 f(x)dx = 2\left[\int_0^2 f(x)dx+\int_2^4 f(x)dx\right]$$

$$= 2\left[\int_0^2 x^2dx+\int_2^4(2x+1)dx\right]$$

Using: $\int x^n dx=\frac{x^{n+1}}{n+1}+c;n\neq -1$

$$\int_0^4 g(x)dx = 2\left[\frac{x^{2+1}}{2+1}\right]_0^2+2\left[\frac{2x^{1+1}}{1+1}+x\right]_2^4$$

$$= \frac{2}{3}\left[x^3\right]_0^2+2\left[x^2+x\right]_2^4$$

$$= \frac{2}{3}[2^3-0^3]+2[(4^2+4)-(2^2+2)]$$

$$= \frac{16}{3}+2(14)=\frac{16+84}{3}=\frac{100}{3}$$

www.ilmkidunya.com

www.ilmkidunya.com

Q4. Simplify:

(i) $\int_0^1 f(x)dx - \int_0^2 f(x)dx - \int_3^1 f(x)dx + \int_3^2 f(x)dx.$

Solution:

$$\int_0^1 f(x)dx - \int_0^2 f(x)dx - \int_3^1 f(x)dx + \int_3^2 f(x)dx.$$

$$= \left(\int_0^1 f(x)dx - \int_3^1 f(x)dx\right) + \left(-\int_0^2 f(x)dx + \int_3^2 f(x)dx\right)$$

$$= \left(\int_0^1 f(x)dx - \int_3^1 f(x)dx\right) - \left(\int_0^2 f(x)dx - \int_3^2 f(x)dx\right)$$

Using property: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

$$= \left(\int_0^1 f(x)dx + \int_1^3 f(x)dx\right) - \left(\int_0^2 f(x)dx + \int_2^3 f(x)dx\right)$$

Using property: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

$$= \int_0^3 f(x)dx - \int_0^3 f(x)dx = 0$$

(ii) $\int_0^3 5f(x)dx - \int_1^3 5f(4-x)dx + \int_1^5 5f(x)dx - \int_0^5 4f(5-x)dx$

Solution:

$$\int_0^3 5f(x)dx - \int_1^3 5f(4-x)dx + \int_1^5 5f(x)dx - \int_0^5 4f(5-x)dx = 5\int_0^3 f(x)dx - 5\int_1^3 f(4-x)dx + 5\int_1^5 f(x)dx - 4\int_0^5 f(5-x)dx$$

Using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ in 2nd and 4th integrals

$$= 5\int_0^3 f(x)dx - 5\int_1^3 f(4-(4-x))dx + 5\int_1^5 f(x)dx - 4\int_0^5 f(5-(5-x))dx$$

$$= 5\int_0^3 f(x)dx - 5\int_1^3 f(4-4+x)dx + 5\int_1^5 f(x)dx - 4\int_0^5 f(5-5+x)dx = 5\int_0^3 f(x)dx - 5\int_1^3 f(x)dx + 5\int_1^5 f(x)dx - 4\int_0^5 f(x)dx$$

Using property: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

$$= 5\int_0^3 f(x)dx + 5\int_3^1 f(x)dx + 5\int_1^5 f(x)dx - 4\int_0^5 f(x)dx = 5\left\{\int_0^3 f(x)dx + \int_3^1 f(x)dx + \int_1^5 f(x)dx\right\} - 4\int_0^5 f(x)dx$$

$$= 5\int_0^5 f(x)dx - 4\int_0^5 f(x)dx \qquad \because f \text{ is continuous on } [0, 5]$$

$$= \int_0^5 f(x)dx$$

Q5. Show that $\int_0^1 x^m(1-x)^n dx = \int_0^1 x^n(1-x)^m dx$ and hence evaluate $\int_0^1 x(1-x)^{10} dx$.

Solution:

$$\int_0^1 x^m(1-x)^n dx = \int_0^1 x^n(1-x)^m dx$$

L.H.S $= \int_0^1 x^m(1-x)^n dx$

By using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Replace x by $0 + 1 - x = 1 - x$

L.H.S $= \int_0^1 (1-x)^m(1-(1-x))^n dx$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \int_0^1 (1-x)^m (1-1+x)^n dx$$

$$= \int_0^1 (1-x)^m \cdot x^n dx$$

$$= \int_0^1 x^n (1-x)^m dx$$

= R.H.S (Proved)

To Evaluate: $\int_0^1 x(1-x)^{10} dx$

$$= \int_0^1 x^{10}(1-x)^1 dx \quad \because \int_0^1 x^m (1-x)^n dx = \int_0^1 x^n (1-x)^m dx$$

$$= \int_0^1 (x^{10} - x^{11}) dx$$

Using: $\int x^n dx = \frac{x^{n+1}}{n+1} + c, (n \neq -1)$

$$= \left[\frac{x^{11}}{11} - \frac{x^{12}}{12}\right]_0^1$$

$$= \left[\frac{(1)^{11}}{11} - \frac{(1)^{12}}{12}\right] - [0-0]$$

$$= \frac{1}{11} - \frac{1}{12} = \frac{12-11}{132} = \frac{1}{132}$$

Q6. Evaluate the following definite integrals:

(i) $\int_0^4 x\sqrt{4-x}\,dx$

Solution:

$$\int_0^4 x\sqrt{4-x}\,dx$$

By using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Replace x by $0+4-x = 4-x$

$$= \int_0^4 (x-4)\sqrt{4-(4-x)}\,dx$$

$$= \int_0^4 (x-4)\sqrt{4-4+x}\,dx$$

$$= \int_0^4 (x-4)\cdot\sqrt{x}\,dx$$

$$= \int_0^4 (x\sqrt{x} - 4\sqrt{x})\,dx$$

$$= \int_0^4 (x^{\frac{3}{2}} - 4x^{\frac{1}{2}})\,dx$$

Using: $\int x^n dx = \frac{x^{n+1}}{n+1} + c, (n \neq -1)$

$$= \left[\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - 4\cdot\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^4$$

$$= \left[\frac{2}{5}x^{\frac{5}{2}} - 4\cdot\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$$

$$= \left[\frac{2}{5}(4)^{\frac{5}{2}} - \frac{8}{3}(4)^{\frac{3}{2}}\right] - [0-0]$$

$$= \frac{2}{5}(2^2)^{\frac{5}{2}} - \frac{8}{3}(2^2)^{\frac{3}{2}} = \frac{2}{5}(2^5) - \frac{8}{3}(2^3)$$

$$= \frac{64}{5} - \frac{64}{3} = \frac{320-192}{15} = \frac{128}{15}$$

(ii) $\int_0^1 \frac{x^2}{\sqrt{1-x}}dx$

Solution:

$$\int_0^1 \frac{x^2}{\sqrt{1-x}}dx$$

By using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Replace x by $0+1-x = 1-x$

$$= \int_0^1 \frac{(1-x)^2}{\sqrt{1-(1-x)}}dx$$

$$= \int_0^1 \frac{(1-x)^2}{\sqrt{1-1+x}}dx$$

$$= \int_0^1 \frac{1-2x+x^2}{\sqrt{x}}dx$$

$$= \int_0^1 \left(\frac{1}{\sqrt{x}} - \frac{2x}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right)dx$$

$$= \int_0^1 (x^{\frac{-1}{2}} - 2x^{\frac{1}{2}} + x^{\frac{3}{2}})\,dx$$

www.ilmkidunya.com

www.ilmkidunya.com

Using: $\int x^n dx = \frac{x^{n+1}}{n+1} + c, (n \neq -1)$

$$= \left[\frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} - 2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} \right]_0^1$$

$$= \left[2\sqrt{x} - 2\cdot\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} \right]_0^1$$

$$= \left[2\sqrt{1} - \frac{4}{3}(1)^{\frac{3}{2}} + \frac{2}{5}(1)^{\frac{5}{2}} \right] - [0-0+0]$$

$$= 2 - \frac{4}{3} + \frac{2}{5}$$

$$= \frac{30-20+6}{15}$$

$$= \frac{16}{15}$$

(iii) $\int_{-1}^{1} \frac{e^x - 1}{e^x + 1} dx$

Solution:

Let $f(x) = \frac{e^x - 1}{e^x + 1}$

Replace x by $-x$

$$f(-x) = \frac{e^{-x} - 1}{e^{-x} + 1}$$

$$= \frac{\frac{1}{e^x} - 1}{\frac{1}{e^x} + 1} = \frac{\frac{1-e^x}{e^x}}{\frac{1+e^x}{e^x}}$$

$$= \frac{-(e^x - 1)}{e^x + 1}$$

$$= -f(x)$$

$\Rightarrow$ $f(x)$ is an odd function.

Hence, $\int_{-1}^{1} \frac{e^x - 1}{e^x + 1} dx = 0$.

Q7. Let $f(x)$ be a function satisfying $f(x) + f(-x) = x^2 + x^3$ for all x. Evaluate the definite integral $\int_{-2}^{2} f(x)dx$.

Solution:

$$\int_{-2}^{2} f(x)dx = ?$$

Using property: $\int_{-a}^{a} f(x)dx = \int_{0}^{a} [f(x) + f(-x)]dx$

$$\Rightarrow \int_{-2}^{2} f(x)dx = \int_{0}^{2} [f(x) + f(-x)]dx$$

$$= \int_{0}^{2} x^2 dx \qquad \because f(x) + f(-x) = x^2$$

$$\int_{-2}^{2} f(x)dx = \left[\frac{x^{2+1}}{2+1} \right]_0^2$$

$$= \frac{1}{3}\left[x^3 \right]_0^2$$

$$= \frac{1}{3}(2^3 - 0^3)$$

$$= \frac{1}{3}(8) = \frac{8}{3}$$

Q8. Let $\int_{0}^{3} f(x)dx = 20$, $\int_{0}^{10} f(x)dx = 45$. Evaluate $\int_{3}^{10} f(x)dx$.

Solution:

$$\int_{3}^{10} f(x)dx = ?$$

As we know

$$\int_{3}^{10} f(x)dx = \int_{0}^{3} f(x)dx + \int_{3}^{10} f(x)dx$$

$$\Rightarrow \int_{3}^{10} f(x)dx = \int_{0}^{10} f(x)dx - \int_{0}^{3} f(x)dx$$

Put $\int_{0}^{3} f(x)dx = 20$ and $\int_{0}^{10} f(x)dx = 45$, we get

$$\int_{3}^{10} f(x)dx = 45 - 20$$

$$= 25$$

www.ilmkidunya.com

www.ilmkidunya.com

Q9. Given that: $\int_0^9 f(x)dx = 16$ and $\int_0^3 f(x)dx = 12$. Evaluate the integral $\int_3^9 (2f(x)+5)dx$.

Solution:

$$\int_3^9 (2f(x)+5)dx = 2\int_3^9 f(x)dx + 5\int_3^9 1dx$$

$$= 2\int_3^9 f(x)dx + 5[x]_3^9$$

$$= 2\int_3^9 f(x)dx + 5[9-3]$$

$$= 2\int_3^9 f(x) + 30 \quad \ldots(1)$$

As we know

$$\int_0^9 f(x)dx = \int_0^3 f(x)dx + \int_3^9 f(x)dx$$

Put $\int_0^9 f(x)dx = 16$ and $\int_0^3 f(x)dx = 12$, we get

$$16 = 12 + \int_3^9 f(x)dx$$

$$\Rightarrow \int_3^9 f(x)dx = 16 - 12$$

$$\int_3^9 f(x)dx = 4$$

Put in eq. (1), we get

$$\int_3^9 (2f(x)+5)dx = 2(4)+30$$

$$= 38$$

Q10. If $f(4-x)=f(x)$, show that $\int_0^2 f(2x)dx = \int_0^2 f(x)dx$.

Solution:

$$\text{L.H.S} = \int_0^2 f(2x)dx$$

Let $u = 2x \Rightarrow du = 2dx \Rightarrow dx = \frac{1}{2}du$

When $x = 0 \Rightarrow u = 2(0) = 0$; When $x = 2 \Rightarrow u = 2(2) = 4$

$$\text{L.H.S} = \int_0^4 f(u)\cdot\frac{1}{2}du$$

$$= \frac{1}{2}\int_0^4 f(u)du$$

$$= \frac{1}{2}\int_0^4 f(x)dx \qquad \because \int_a^b f(u)du = \int_a^b f(x)dx$$

Using property: $\int_0^a f(x)dx = \int_0^{a/2} (f(x)+f(a-x))dx$

$$\text{L.H.S} = \frac{1}{2}\int_0^{4/2} (f(x)+f(4-x))dx$$

$$= \frac{1}{2}\int_0^2 (f(x)+f(x))dx \qquad \text{Given } f(4-x)=f(x)$$

$$= \frac{1}{2}\int_0^2 2f(x)dx = \int_0^2 f(x)dx$$

= R.H.S (Proved)

Q11. Show that $\int_{-1}^{1} (1-x)f(1-x^2)dx = \int_{-1}^{1} f(1-x^2)dx$.

Solution:

$$\text{L.H.S} = \int_{-1}^{1} (1-x)f(1-x^2)dx$$

$$= \int_{-1}^{1} [f(1-x^2) - x\cdot f(1-x^2)]dx$$

$$= \int_{-1}^{1} f(1-x^2)dx - \int_{-1}^{1} x\cdot f(1-x^2)dx$$

Let $g(x) = x\cdot f(1-x^2)$

Replace x by $-x$

$$g(-x) = -x\cdot f(1-(-x)^2)$$

$$= -xf(1-x^2) = -g(x)$$

$\Rightarrow$ $g(x)$ is an odd function.

$$\text{L.H.S} = \int_{-1}^{1} f(1-x^2)dx - \int_{-1}^{1} (\text{odd function})dx$$

$$= \int_{-1}^{1} f(1-x^2)dx - 0 = \int_{-1}^{1} f(1-x^2)dx = \text{R.H.S}$$

www.ilmkidunya.com

www.ilmkidunya.com

Q12. If $f(a+x)+f(a-x)=0$, show that $\int_0^{2a} f(x)dx=0$.

Solution:

L.H.S $= \int_0^{2a} f(x)dx = 0$

Using property: $\int_0^a f(x)dx = \int_0^{a/2}(f(x)+f(a-x))dx$

L.H.S $= \int_0^{2a/2}(f(x)+f(2a-x))dx$

$= \int_0^a (f(x)+(2a-x))ax$

Given: $f(a+x)+f(a-x)=0$

Replace x by $x-a$

$f(a+x-a)+f(a-x+a) = 0$

$f(x)+f(2a-x) = 0$

$f(2a-x) = -f(x)$ Put in L.H.S

L.H.S $= \int_0^a (f(x)-f(x))dx$

$= \int_0^a f(x)dx - \int_0^a f(x)dx$

$= 0$

= R.H.S (Proved)

Q13. Show that $\int_{-a}^{a} \frac{f(x)}{1+e^x}dx = \int_0^a f(x)dx$, **where** $f(x)$ **is an even function. Hence or otherwise evaluate** $\int_{-a}^{a} \frac{x^2}{1+e^x}dx$.

Solution:

To show: $\int_{-a}^{a} \frac{f(x)}{1+e^x}dx = \int_0^a f(x)dx$

Let $I = \int_{-a}^{a} \frac{f(x)}{1+e^x}dx$ ...(1)

Using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Replace x by $a+(-a)-x=-x$

$I = \int_{-a}^{a} \frac{f(-x)}{1+e^{-x}}dx$

$= \int_{-a}^{a} \frac{f(x)}{1+\frac{1}{e^x}}dx$ $\because f(x)$ is even function $\Rightarrow f(-x)=f(x)$

$= \int_{-a}^{a} \frac{f(x)}{\frac{e^x+1}{e^x}}dx$

$= \int_{-a}^{a} \frac{e^x \cdot f(x)}{1+e^x}dx$

$= \int_{-a}^{a} \frac{(e^x f(x)+f(x))-f(x)}{1+e^x}dx$

$= \int_{-a}^{a} \left[\frac{(e^x+1)\cdot f(x)}{1+e^x} - \frac{f(x)}{1+e^x}\right]dx$

$I = \int_{-a}^{a} f(x)dx - \int_{-a}^{a} \frac{f(x)}{1+e^x}dx$

$I = \int_{-a}^{a} f(x)dx - I$ Using (1)

$2I = 2\int_0^a f(x)$ $\because$ For even function $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

$I = \frac{2}{2}\int_0^a f(x)dx$

$\Rightarrow \int_{-a}^{a} \frac{f(x)}{1+e^x}dx = \int_0^a f(x)dx$ (Proved) ...(2)

To Evaluate: $\int_{-a}^{a} \frac{x^2}{1+e^x}dx$

Here $f(x) = x^2$

Replace x by $-x$

$f(-x) = (-x)^2$

$= x^2 = f(x)$

$\Rightarrow$ $f(x)$ is an even function.

$\int_{-a}^{a} \frac{x^2}{1+e^x}dx = \int_0^a x^2 dx$ Using (2)

$= \left[\frac{x^3}{3}\right]_0^a$

$= \frac{a^3}{3} - 0 = \frac{a^3}{3}$

www.ilmkidunya.com

www.ilmkidunya.com

Q14. Given the function $f(x)$ satisfies the following conditions: (i) $\int_0^4 f(x)dx = 12$ (ii) $f(x+2) = f(x)$

Evaluate $\int_{-1}^{5} f(x)dx$.

Solution:

Given conditions:

(i) $\int_0^4 f(x)dx = 12$ (ii) $f(x+2) = f(x)$

Condition (ii) $\Rightarrow f(x)$ is a periodic function of period 2, so

$$\int_0^4 f(x)dx = \int_0^{2(2)} f(x)dx$$

$$= 2\int_0^2 f(x)dx \quad \because \int_0^{nT} f(x)dx = n\int_0^2 f(x)dx$$

$$\Rightarrow \quad 12 = 2\int_0^2 f(x)dx \qquad \text{Using (i)}$$

$$\boxed{\int_0^2 f(x)dx = 6} \qquad \text{(Value of single period)}$$

Now, $\int_{-1}^{5} f(x)dx = \int_{-1}^{1} f(x)dx + \int_1^3 f(x)dx + \int_3^5 f(x)dx$

$\because$ Length $= 5 - (-1) = 6$

$= 6 + 6 + 6 = 18$

Q15. Show that:

(i) $\int_{\pi/6}^{\pi/3} \ell n(\tan x)dx = 0$

Solution:

Let $I = \int_{\pi/6}^{\pi/3} \ell n(\tan x)dx$ ...(1)

Using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Replace x by $\frac{\pi}{3} + \frac{\pi}{6} - x = \frac{2\pi+\pi}{6} - x = \frac{\pi}{2} - x$

$$= \int_{\pi/6}^{\pi/3} \ell n\left(\tan\left(\frac{\pi}{2} - x\right)\right)dx$$

$$= \int_{\pi/6}^{\pi/3} \ell n(\cot x)dx$$

$$= \int_{\pi/6}^{\pi/3} \ell n\left(\frac{1}{\tan x}\right)dx$$

$$= \int_{\pi/6}^{\pi/3} \ell n(\tan x)^{-1}dx$$

$$I = -\int_{\pi/6}^{\pi/3} \ell n(\tan x)dx \quad \because \ell n(a^b) = b\,\ell n\,a$$

$I = -I$ Using (1)

$2I = 0$

$I = 0$

Hence, $\int_{\pi/6}^{\pi/3} \ell n(\tan x)dx = 0$ (Proved)

(ii) $\int_0^{\pi} \frac{\sin x \cos x}{1 + e^{\sin x}}dx = 0$

Solution:

Let $I = \int_0^{\pi} \frac{\sin x \cos x}{1 + e^{\sin x}}dx$ ...(1)

Using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Replace x by $0 + \pi - x = \pi - x$

$$I = \int_0^{\pi} \frac{\sin(\pi - x)\cdot\cos(\pi - x)}{1 + e^{\sin(\pi - x)}}dx$$

$$= \int_0^{\pi} \frac{(\sin x)(-\cos x)}{1 + e^{\sin x}}dx \quad \because \begin{cases} \sin(\pi - x) = \sin x \\ \cos(\pi - x) = -\cos x \end{cases}$$

$$I = -\int_0^{\pi} \frac{\sin x \cos x}{1 + e^{\sin x}}dx$$

$I = -I$ Using (1)

$2I = 0$

$I = 0$

Hence, $\int_0^{\pi} \frac{\sin x \cos x}{1 + e^{\sin x}}dx = 0$ (Proved)

(iii) $\int_0^{\pi} x(\pi - x)\cos(\pi - x)dx = 0$

Solution:

Let $I = \int_0^{\pi} x(\pi - x)\cos(\pi - x)dx = 0$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \int_0^{\pi} x(\pi-x)(-\cos x)dx \qquad \because \cos(\pi-x) = -\cos x$$

$$I = -\int_0^{\pi} x(\pi-x)\cos x\,dx \qquad \ldots(1)$$

Using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Replace x by $0+\pi-x = \pi-x$

$$I = -\int_0^{\pi} (\pi-x)(\pi-\pi+x)\cos(\pi-x)dx$$

$$= -\int_0^{\pi} (\pi-x)x\cdot(-\cos x)dx \qquad \because \cos(\pi-x) = -\cos x$$

$$I = -\left[-\int_0^{\pi} x(\pi-x)\cos x\,dx\right]$$

$I = -I$ Using (1)

$2I = 0$

$\Rightarrow \quad I = 0$

Hence, $\int_0^{\pi} x(\pi-x)\cos(\pi-x)dx = 0$ (Proved)

(iv) $\int_1^3 \frac{(x-2)^3+4(x-2)}{1+e^{(x-2)^2}}dx = 0$

Solution:

L.H.S $= \int_1^3 \frac{(x-2)^3+4(x-2)}{1+e^{(x-2)^2}}dx$

Let $u = x-2 \Rightarrow du = dx$

When $x = 1 \Rightarrow u = 1-2 = -1$

When $x = 3 \Rightarrow u = 3-2 = 1$

L.H.S $= \int_{-1}^{1} \frac{u^3+4u}{1+e^{u^2}}du$

Let $f(u) = \frac{u^3+4u}{1+e^{u^2}}$

Replace u by $-u$

$$f(-u) = \frac{(-u)^3+4(-u)}{1+e^{(-u)^2}}$$

$$= \frac{-u^3-4u}{1+e^{u^2}} = -\left(\frac{u^3+4u}{1+e^{u^2}}\right) = -f(u)$$

$\Rightarrow$ $f(u)$ is an odd function.

L.H.S $= \int_{-1}^{1} (\text{odd function})du$

$= 0 =$ R.H.S (Proved)

Q16. Evaluate the following definite integrals:

(i) $\int_1^3 \frac{x^2}{x^2+(4-x)^2}dx$

Solution:

Let $I = \int_1^3 \frac{x^2}{x^2+(4-x)^2}dx \qquad \ldots(1)$

Using property: $\int_a^b f(x)dx = \int_a^b (a+b-x)dx$

Replace x by $1+3-x = 4-x$

$$= \int_1^3 \frac{(4-x)^2}{(4-x)^2+(4-4+x)}dx$$

$$= \int_1^3 \frac{(4-x)^2}{(4-x)^2+x^2}dx$$

$$= \int_1^3 \frac{[(4-x)^2+x^2]-x^2}{(4-x)^2+x^2}dx$$

$$= \int_1^3 \left[\frac{(4-x)^2+x^2}{(4-x)^2+x^2} - \frac{x^2}{(4-x)^2+x^2}\right]dx$$

$$I = \int_1^3 \left[1 - \frac{x^2}{(4-x)^2+x^2}\right]dx$$

$$I = \int_1^3 1dx - \int_1^3 \frac{x^2}{(4-x)^2+x^2}dx$$

$I = [x]_1^3 - I$ Using (1)

$2I = 3-1$

$2I = 2$

$I = \frac{2}{2}$

$I = 1$

Hence, $\int_1^3 \frac{x^2}{x^2+(4-x)^2}dx = 1$

(ii) $\int_0^{\ln 2} \frac{e^x}{e^x+2e^{-x}}dx$

Solution:

$$\int_0^{\ln 2} \frac{e^x}{e^x+2e^{-x}}dx$$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \int_0^{\ln 2} \frac{e^x}{e^x + \frac{2}{e^x}}dx$$

$$= \int_0^{\ln 2} \frac{e^x}{\frac{e^{2x}+2}{e^x}}dx$$

$$= \int_0^{\ln 2} \frac{e^x \cdot e^x}{e^{2x}+2}dx$$

$$= \int_0^{\ln 2} \frac{e^{2x}}{e^{2x}+2}dx$$

$$= \frac{1}{2}\int_0^{\ln 2} \frac{2e^{2x}}{e^{2x}+2}dx$$

Using: $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

$$= \frac{1}{2}\left[\ln(e^{2x}+2)\right]_0^{\ln 2}$$

$$= \frac{1}{2}\left[\ln(e^{2\ln 2}+2) - \ln(e^0+2)\right]$$

$$= \frac{1}{2}\left[\ln(e^{\ln 2^2}+2) - \ln(1+2)\right]$$

$$= \frac{1}{2}[\ln(4+2) - \ln 3] \qquad \because e^{\log_e a} = a$$

$$= \frac{1}{2}\ln\left(\frac{6}{3}\right) = \frac{1}{2}\ln 2$$

(iii) $\int_1^3 \frac{\ln(x+2)}{\ln(6-x)+\ln(x+2)}dx$

Solution:

Let $I = \int_1^3 \frac{\ln(x+2)}{\ln(6-x)+\ln(x+2)}dx$ ...(1)

Using property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Replace x by $1+3-x = 4-x$

$$I = \int_1^3 \frac{\ln(4-x+2)}{\ln(6-(4-x))+\ln(4-x+2)}dx$$

$$I = \int_1^3 \frac{\ln(6-x)}{\ln(2+x)+\ln(6-x)}$$ ...(2)

Equation (1) + Equation (2)

$$I + I = \int_1^3 \left[\frac{\ln(x+2)}{\ln(6-x)+\ln(x+2)} + \frac{\ln(6-x)}{\ln(2+x)+\ln(6-x)}\right]dx$$

$$2I = \int_1^3 \frac{\ln(x+2)+\ln(6-x)}{\ln(2+x)+\ln(6-x)}dx$$

$$I = \frac{1}{2}\int_1^3 1dx$$

$$= \frac{1}{2}[x]_1^3 = \frac{1}{2}(3-1) = \frac{1}{2}(2)$$

$$I = 1$$

(iv) $\int_0^{\pi} \frac{e^{\sin x} - e^{\cos x}}{e^{\sin x} + e^{\cos x}}(\sin x + \cos x)dx$

Solution:

$$\int_0^{\pi} \frac{e^{\sin x} - e^{\cos x}}{e^{\sin x} + e^{\cos x}}(\sin x + \cos x)dx$$

Multiply up and down by $e^{-\cos x}$

$$= \int_0^{\pi} \frac{e^{\sin x}\cdot e^{-\cos x} - e^{\cos x}\cdot e^{-\cos x}}{e^{\sin x}\cdot e^{-\cos x} + e^{\cos x}\cdot e^{-\cos x}}(\sin x + \cos x)dx$$

$$= \int_0^{\pi} \frac{e^{\sin x - \cos x} - 1}{e^{\sin x - \cos x} + 1}(\sin x + \cos x)dx$$

Let $u = \sin x - \cos x \Rightarrow du = (\cos x + \sin x)dx$

When $x = 0 \Rightarrow u = \sin 0 - \cos 0 = -1$

When $x = \pi \Rightarrow u = \sin\pi - \cos\pi = 0 - (-1) = 1$

$$= \int_{-1}^{1} \frac{e^u - 1}{e^u + 1}du$$

Here $f(u) = \frac{e^u - 1}{e^u + 1}$

Replace u by $-u$

$$f(-u) = \frac{e^{-u} - 1}{e^{-u} + 1} = \frac{\frac{1}{e^u} - 1}{\frac{1}{e^u} + 1} = \frac{\frac{1 - e^u}{e^u}}{\frac{1 + e^u}{e^u}}$$

$$= \frac{-(e^u - 1)}{e^u + 1} = -f(u)$$

$\Rightarrow$ $f(u)$ is an odd function.

$$= \int_{-1}^{1} (\text{odd function})du$$

$$= 0$$

www.ilmkidunya.com

www.ilmkidunya.com

Content Outline

- Application of Definite Integrals
  - Area Under the Curve
  - Consumer and Producer Surplus
  - Solids of Revolution

APPLICATION OF DEFINITE INTEGRALS

Area Under the Curve

If f is continuous and $f(x) \geq 0$ over the interval $[a, b]$, then the area A of the region under the curve of f from a to b is

$$A = \int_a^b f(x)dx.$$

Notes:

1. If $f(x) \leq 0$ on the interval $[a, b]$, then the area bounded by the curve $y = f(x)$ and x-axis is given by $\left|\int_a^b f(x)dx\right|$ or equivalently $\int_a^b |f(x)|dx$ or equivalently $-\int_a^b f(x)dx$.

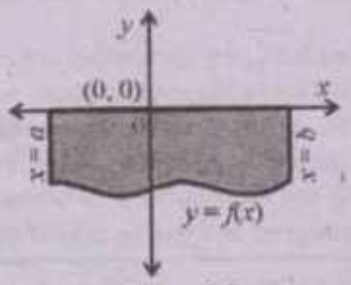


Figure 3.1

2. If $f(x) \geq 0$ on the interval $[a, c]$ and $f(x) \leq 0$ on the interval $[c, b]$, then the total area bounded by the curve $y = f(x)$ and x-axis on the interval $[a, b]$ is

$$\int_a^c f(x)dx + \left|\int_c^b f(x)dx\right|$$

or equivalently $\int_a^c f(x)dx + \int_c^b |f(x)|dx$

or equivalently $\int_a^c f(x)dx - \int_c^b f(x)dx$

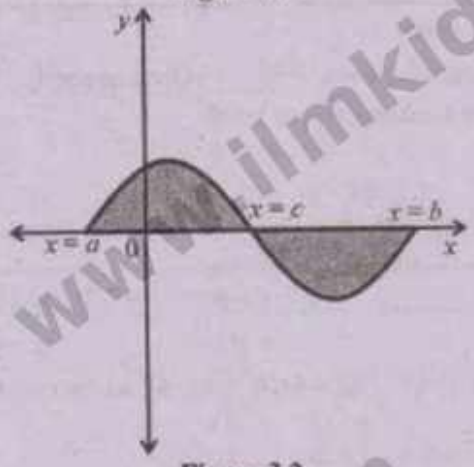


Figure 3.2

3. If $f(x)$ changes sign at $c_1, c_2, c_3, \ldots, c_n$ in (a, b) where $c_1 < c_2 < \cdots < c_n$ are the roots of $f(x) = 0$.

$$\text{Total Area} = \int_a^b |f(x)|dx = \left|\int_a^{c_1} f(x)dx\right| + \left|\int_{c_1}^{c_2} f(x)dx\right| + \ldots + \left|\int_{c_n}^b f(x)dx\right|$$

4. The area bounded by the curve $x = f(y)$, the y-axis, the horizontal lines $y = c$ and $y = d (d > c)$ is given by $\int_c^d f(y)dy$.

5. **Area between two curves**

If f and g are continuous and $f(x) \geq g(x)$ over the interval $[a, b]$, then the area A of the region bounded by the graphs of f, g, $x = a$, and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]dx$$

The above result does not require $f(x)$ or $g(x)$ to remain positive over the interval $[a, b]$.

www.ilmkidunya.com

www.ilmkidunya.com

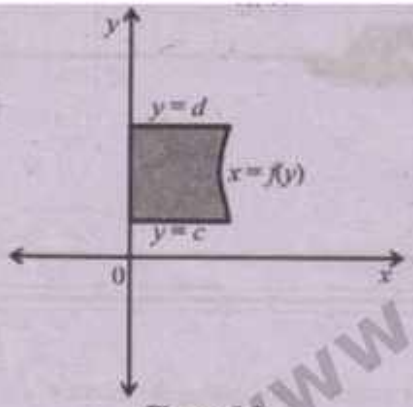


Figure 3.3

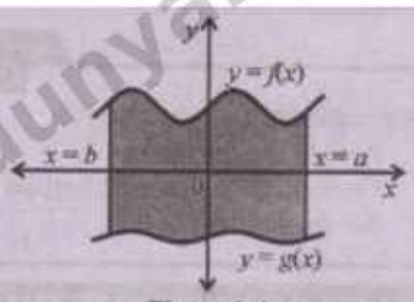


Figure 3.4

Net Signed Area

The definite integral $\int_a^b f(x)dx$ represents the net signed area between the curve and the x-axis from $x = a$ to $x = b$. Its sign depends entirely on whether the curve crosses the x-axis.

- If the curve lies entirely above the x-axis, the integral directly gives the net signed area, which is **positive.**
- If the curve lies entirely below the x-axis, the integral directly gives the net signed area, which is **negative.**
- If the curve crosses the x-axis, the net signed area is found by **algebraically adding** the positive area above the x-axis and the negative area below the x-axis.
- Net signed area will be **zero** if the total positive area above the axis exactly equals the total negative area below it.

Example 57: Find the area bounded by the parabola $y = x^2 - 5x + 4$ and the x-axis.

Solution:

Parabola: $y = x^2 - 5x + 4$ (Open upward)

For x-intercepts, put $y = 0$

$$x^2 - 5x + 4 = 0$$
$$x^2 - 4x - x + 4 = 0$$
$$x(x-4) - 1(x-4) = 0$$
$$(x-1)(x-4) = 0$$
$$\Rightarrow \quad x - 1 = 0 \quad ; \quad x - 4 = 0$$
$$x = 1 \quad ; \quad x = 4$$

As $y = x^2 - 5x + 4 \leq 0$ in $[1, 4]$, so the curve is below the x-axis.

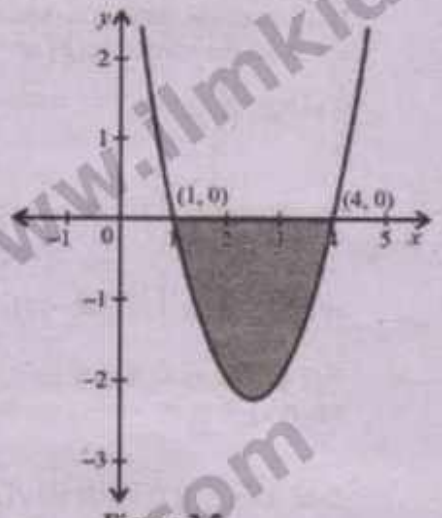


Figure 3.5

$$\text{Area} = -\int_1^4 y\,dx$$
$$= -\int_1^4 (x^2 - 5x + 4)dx$$
$$= -\left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_1^4$$
$$= -\left[\left(\frac{4^3}{3} - \frac{5(4)^2}{2} + 4(4)\right) - \left(\frac{1^3}{3} - \frac{5(1)^2}{2} + 4(1)\right)\right]$$
$$= -\left(\frac{64}{3} - \frac{80}{2} + 16 - \frac{1}{3} + \frac{5}{2} - 4\right)$$
$$= -\left(\frac{64}{3} - \frac{1}{3} + \frac{5}{2} - 28\right)$$
$$= -\left(\frac{128 - 2 + 15 - 168}{6}\right)$$
$$= -\left(\frac{-27}{6}\right)$$

Area $= \frac{9}{2}$ square units

www.ilmkidunya.com

www.ilmkidunya.com

Example 58: Find the area bounded by the curves $y = 13 - x^2$ and $y = 2x + 5$.

Solution:

1st curve: $y = 13 - x^2$

2nd curve: $y = 2x + 5$

For points of intersection of the curves

$$13 - x^2 = 2x + 5$$
$$0 = 2x + 5 - 13 + x^2$$
$$x^2 + 2x - 8 = 0$$
$$x^2 + 4x - 2x - 8 = 0$$
$$x(x+4) - 2(x+4) = 0$$
$$(x-2)(x+4) = 0$$
$$\Rightarrow \quad x - 2 = 0 \quad ; \quad x + 4 = 0$$
$$x = 2 \quad ; \quad x = -4$$

Determine which curve is on the top:

Take $x = 0 \in [-4, 2]$

$$y(0) = 13 - 0^2 = 13$$
$$y(0) = 2(0) + 5 = 5$$

$\Rightarrow$ Upper curve: $y = 13 - x^2$

and Lower curve: $y = 2x + 5$

$$\text{Area} = \int_{-4}^{2} [\text{upper} - \text{lower}]\,dx$$
$$= \int_{-4}^{2} [(13 - x^2) - (2x + 5)]\,dx$$
$$= \int_{-4}^{2} (13 - x^2 - 2x - 5)\,dx$$
$$= \int_{-4}^{2} (-x^2 - 2x + 8)\,dx$$
$$= \left[\frac{-x^3}{3} - \frac{2x^2}{2} + 8x\right]_{-4}^{2}$$
$$= \left(\frac{-2^3}{3} - 2^2 + 8(2)\right) - \left(\frac{(-4)^3}{3} - (-4)^2 + 8(-4)\right)$$
$$= \frac{-8}{3} - 4 + 16 - \frac{64}{3} + 16 + 32$$
$$= \frac{-8}{3} - \frac{64}{3} + 60 = \frac{-8 - 64 + 180}{3}$$
$$= \frac{108}{3}$$

Area = 36 square units

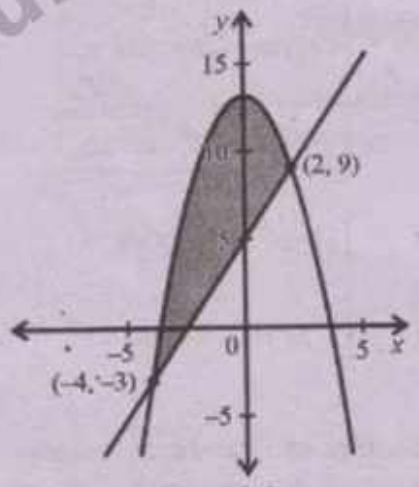


Figure 3.6

Example 59: Find the area of the region bounded by the parabolas $y = 4 - x^2$ and $y = x^2 - 2x$.

Solution:

1st curve: $y = 4 - x^2$

2nd curve: $y = x^2 - 2x$

For points of intersection of the curves

$$4 - x^2 = x^2 - 2x$$
$$0 = x^2 - 2x + x^2 - 4$$
$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0 \qquad \text{Divide by '2'}$$
$$x^2 - 2x + x - 2 = 0$$
$$x(x-2) + 1(x-2) = 0$$
$$(x+1)(x-2) = 0$$
$$\Rightarrow \quad x + 1 = 0 \quad ; \quad x - 2 = 0$$
$$x = -1 \quad ; \quad x = 2$$

Determine which curve is on the top:

Take $x = 0 \in [-1, 2]$

$$y(0) = 4 - 0^2 = 4$$
$$y(0) = 0^2 - 2(0) = 0$$

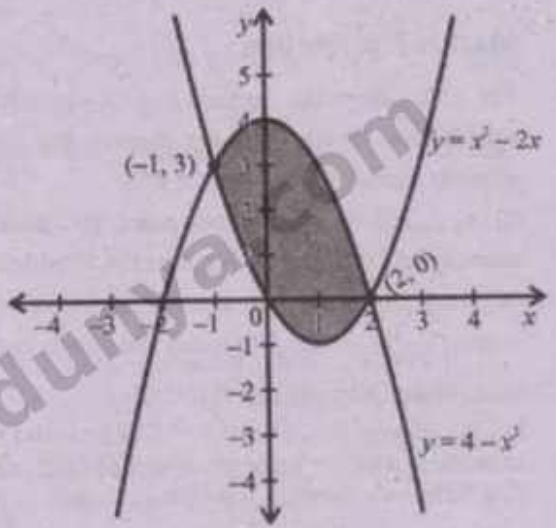


Figure 3.7

www.ilmkidunya.com

www.ilmkidunya.com

$\Rightarrow$ Upper curve: $y = 4 - x^2$

and Lower curve: $y = x^2 - 2x$

$$\text{Area} = \int_{-1}^{2} [\text{Upper} - \text{Lower}]\,dx$$

$$= \int_{-1}^{2} [(4 - x^2) - (x^2 - 2x)]\,dx$$

$$= \int_{-1}^{2} (4 - x^2 - x^2 + 2x)\,dx$$

$$= \int_{-1}^{2} (-2x^2 + 2x + 4)\,dx$$

$$= \left[\frac{-2x^3}{3} + \frac{2x^2}{2} + 4x\right]_{-1}^{2}$$

$$= \left(\frac{-2(2)^3}{3} + (2)^2 + 4(2)\right) - \left(\frac{-2(-1)^3}{3} + (-1)^2 + 4(-1)\right)$$

$$= \frac{-16}{3} + 4 + 8 - \frac{2}{3} - 1 + 4$$

$$= \frac{-16}{3} - \frac{2}{3} + 15 = \frac{-16 - 2 + 45}{3}$$

$$= \frac{27}{3}$$

Area = 9 square units.

Consumer and Producer Surplus

One place in economics where area between two curves is important is in the calculation of a quantity called consumer surplus and a quantity called producer surplus.

Demand Function

The demand curve for a commodity shows the quantity consumers are willing to buy at any given price. Let p be the price that a consumer is willing to pay for a quantity x of a certain commodity. Then p and x are related to each other through the demand function $p = D(x)$. The graph of the demand function generally slopes downwards, as the demand for a commodity decreases when its price.

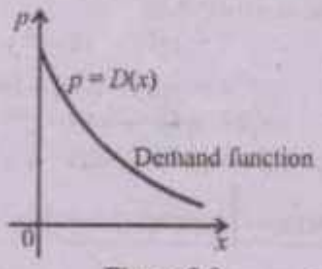


Figure 3.8

Supply Function

Let p be the price at which a producer is willing to sell a quantity x of a certain commodity. Then p and x are related to each other through the supply function $p = S(x)$. The graph of the supply function generally slopes upwards, as the supply of a commodity increases when its price increases; that is, with an increase in price, the producer is willing to supply more. [Figure 3.9]

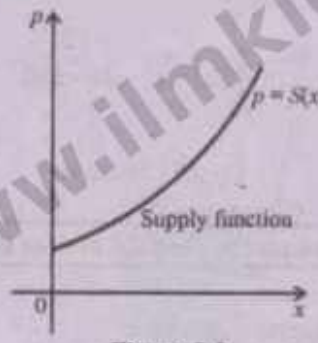


Figure 3.9

Market Equilibrium

The point where the demand and supply curves intersect is called the market equilibrium. At this point the quantity that consumers want to buy equals the quantity that producers want to sell.

If (x_e, p_e) is the equilibrium point, the quantity x_e is called the equilibrium quantity and the price p_e is called the equilibrium price.

Algebraically to find the equilibrium point, we solve the system of equations formed by the demand equation and supply equation.

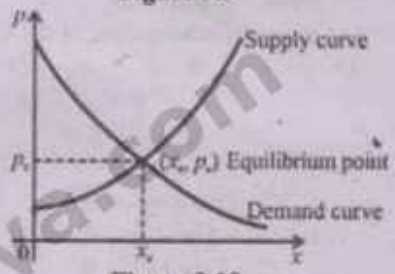


Figure 3.10

Consumer Surplus (CS)

Let the demand function be $p = D(x)$, and let x_e be the quantity demanded when the market price is p_e. However, some consumers who are unaware of equilibrium price are willing to pay more than p_e for the quantity x_e of the commodity. The difference between what a consumer is willing to pay and what they actually pay is a benefit to the consumer.

www.ilmkidunya.com

www.ilmkidunya.com

Thus, consumer surplus can be defined as:

Consumer surplus is the total benefit gained by consumers when they purchase an item at the equilibrium price, rather than at the higher price they would have been willing to pay.

Geometrically, it is the area under the demand curve and above the horizontal line drawn through the equilibrium point.

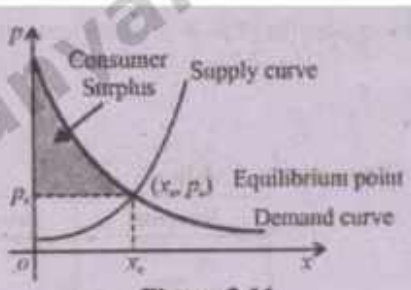


Figure 3.11

$$CS = \begin{pmatrix}\text{Area under demand function}\\ \text{from } x=0 \text{ to } x=x_e\end{pmatrix} - \begin{pmatrix}\text{Area under equilibrium price}\\ \text{from } x=0 \text{ to } x=x_e\end{pmatrix}$$

$$= \int_0^{x_e} (\text{Demand function})dx - \int_0^{x_e} (\text{Equilibrium price})dx = \int_0^{x_e} D(x)dx - \int_0^{x_e} p_e dx = \int_0^{x_e} (D(x) - p_e)dx$$

Producer Surplus (PS)

Let the supply function be $p = S(x)$, and let x_e be the quantity supplied when the market price is p_e. However, some producers are willing to accept less than p_e for the quantity x_e of the commodity. The difference between what producers actually receive and what they would have been willing to accept is a benefit to the producer.

Thus, producer surplus can be defined as:

Producer surplus is the total benefit gained by producers when they sell an item at the equilibrium price, rather than at the lower price they would have been willing to accept.

Geometrically, it is the area above the supply curve and below the horizontal line drawn through the equilibrium point.

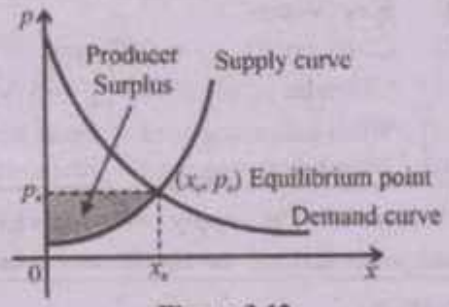


Figure 3.12

$$PS = \begin{pmatrix}\text{Area under equilibrium price}\\ \text{from } x=0 \text{ to } x=x_e\end{pmatrix} - \begin{pmatrix}\text{Area above supply function}\\ \text{from } x=0 \text{ to } x=x_e\end{pmatrix}$$

$$= \int_0^{x_e} (\text{Equilibrium price})dx - \int_0^{x_e} (\text{Supply function})dx = \int_0^{x_e} p_e dx - \int_0^{x_e} S(x)dx = \int_0^{x_e} (p_e - S(x))dx$$

Consumer Surplus (CS) and Producer Surplus (PS) Formulae

1. With Respect to Quantity (x)

Let:

- $p = D(x)$ = demand price function
- $p = S(x)$ = supply price function
- x_e = equilibrium quantity
- p_e = equilibrium price $= D(x_e) = S(x_e)$

Consumer Surplus (CS) = area between demand curve and equilibrium price line, from $x = 0$ to $x = x_e$: $CS = \int_0^{x_e} (D(x) - p_e)dx$

Producer Surplus (PS) = area between supply curve and equilibrium price line, from $x = 0$ to $x = x_e$: $PS = \int_0^{x_e} (p_e - S(x))dx$

Total surplus (TS): $TS = CS + PS = \int_0^{x_e} D(x) - S(x)dx$

2. With Respect to Price (p)

Let:

- $x_d(p) = D^{-1}(p)$ = quantity demanded at price p
- $X_s(p) = S^{-1}(p)$ = quantity supplied at price p
- P_{max} = choke price (demand = 0 for higher prices)
- P_{min} = minimum price (supply = 0 for lower prices)
- P_e = equilibrium price

Consumer Surplus (CS) = area between p_e and p_{max}, under demand curve $x_d(p)$:

$$CS = \int_{p=p_e}^{p_{max}} x_d(p)dp$$

www.ilmkidunya.com

www.ilmkidunya.com

Producer Surplus (PS) = area between p_{min} and p_e, under supply curve $x_s(p)$:

$$PS = \int_{p=p_{min}}^{p_e} x_s(p)dp$$

3. Summary Table

Surplus	Integration w.r.t. Quantity	Integration w.r.t. price
CS	$\int_0^{x_e} (D(x)-p_e)dx$	$\int_{p=p_e}^{p_{max}} x_d(p)dp$
PS	$\int_0^{x_e} (p_e-S(x))dx$	$\int_{p=p_{min}}^{p_e} x_s(p)dp$

4. Key Notes

(i) Choke price $p_{max} = D(0)$ (price when $x = 0$ in demand).
(ii) Minimum supply price $p_{min} = S(0)$ (price when $x = 0$ in supply)
(iii) When integrating w.r.t. x, limits are quantities, integrand is price difference.
(iv) When integrating w.r.t. p, limits are prices, integrand is quantity.

Example 60: The supply curve for a good is $p_s = \sqrt{16+x}$. The demand curve is linear with vertical intercept 73 and passes through the equilibrium point. If the quantity sold is $x_e = 33$, find the equation of the demand curve (p_d).

Solution:

Supply curve: $p_s = \sqrt{16+x}$

At $x_e = 33$:

$$p_e = \sqrt{16+x_e} = \sqrt{16+33} = \sqrt{49} = 7$$

$\Rightarrow$ Equilibrium point $= (x_e, p_e) = (33, 7)$

Required demand curve

$p_d = mx + c$, where c = vertical intercept

$p_d = mx + 73$ ...(1)

Since, it passes through (33, 7), so

$7 = m(33) + 73 \Rightarrow 7 - 73 = 33m$

$-66 = 3m \quad \Rightarrow m = -2$ Put in eq. (1)

$\boxed{p_d = -2x + 73}$

Example 61: If the supply function for a particular commodity is $25p = (x+10)^2$ and the market price is *Rs.* 64, find the producer's surplus by two methods:

(i) Integrate with respect to x **(ii) Integrate with respect to p**

Solution:

Supply Function: $25p = (x+10)^2 \Rightarrow p = \frac{(x+10)^2}{25}$

At $p_e = 64$:

$$25p_e = (x_e+10)^2$$

$$\Rightarrow 25(64) = (x_e+10)^2$$

$$\sqrt{(x_e+10)^2} = \sqrt{1600}$$

$x_e + 10 = 40$ (Take +ve root)

$x_e = 30$

Method (i): Integrate w.r.t. 'x'

$$PS = \int_0^{x_e} (p - p(x))dx$$

$$= \int_0^{30} \left[64 - \frac{(x+10)^2}{25}\right]dx$$

$$= \left[64x - \frac{1}{25}\frac{(x+10)^{2+1}}{2+1}\right]_0^{30}$$

$$= \left[64x - \frac{1}{75}(x+10)^3\right]_0^{30}$$

$$= \left[64(30) - \frac{1}{75}(30+10)^3\right] - \left[0 - \frac{1}{75}(10)^3\right]$$

$$= 1920 - \frac{1}{75}(40)^3 + \frac{1}{75}(10)^3$$

$$= 1920 + \frac{1}{75}(-64000+1000)$$

$$= 1920 - 840$$

$$= \text{Rs. } 1080$$

Method (ii): Integrate w.r.t. 'p'

As $25p = (x+10)^2$

$\Rightarrow \sqrt{25p} = \sqrt{(x+10)^2}$ Taking square root

$5\sqrt{p} = x + 10$

$x = 5\sqrt{p} - 10$

www.ilmkidunya.com

www.ilmkidunya.com

Find p_{min} when $x = 0$:

$$0 = 5\sqrt{p} - 10 \Rightarrow 5\sqrt{p} = 10 \Rightarrow \sqrt{p} = 2 \Rightarrow p = 2$$

$$\Rightarrow p_{min} = 4$$

$$PS = \int_{p_{min}}^{p_1} x_s(p)\,dp$$

$$= \int_4^{64} (5\sqrt{p} - 10)\,dp$$

$$= \left[\frac{5p^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 10p\right]_4^{64}$$

$$= \left[5\cdot\frac{2}{3}p^{\frac{3}{2}} - 10p\right]_4^{64}$$

$$= \left[\frac{10}{3}(64)^{\frac{3}{2}} - 10(64)\right] - \left[\frac{10}{3}(4)^{\frac{3}{2}} - 10(4)\right]$$

$$= \left[\frac{10}{3}(8^2)^{\frac{3}{2}} - 640\right] - \left[\frac{10}{3}(2^2)^{\frac{3}{2}} - 40\right]$$

$$= \frac{5120}{3} - 640 - \frac{80}{3} + 40$$

$$= \frac{5120 - 80}{3} - 600$$

$$= \frac{5040}{3} - 600$$

$$= 1680 - 600$$

$$= \text{Rs. } 1080$$

Hence, both methods give P_S = Rs. 1080

Solids of Revolution

The volume of an object plays an important role in many problems in the physical sciences. In this section, we consider methods for computing volumes of solid of revolution.

If a region in a plane is revolved about a line in the plane, the resulting solid is a solid of revolution, and we say that the solid is generated by the region. The line about which we revolve is an axis of revolution. In particular, if the region shown in Figure 3.13(a) is revolved about the x-axis, we obtain the solid illustrated in Figure 3.13(b).

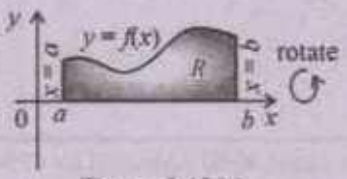


Figure 3.13(a)

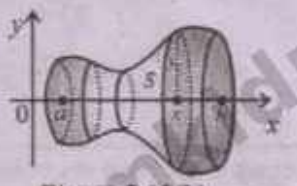


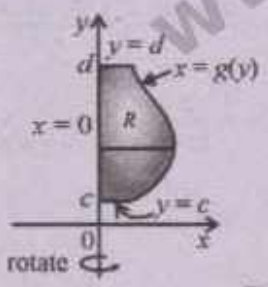
Figure 3.13(b)

If a plane perpendicular to the x-axis intersects the solid shown in Figure 3.13(b), a circular section is obtained and volume in this case can be found by disk method which is explained below.

Let f be continuous and non-negative on $[a, b]$, and let R be the region bounded by the graph of $y = f(x)$, the x-axis and the vertical lines $x = a$, and $x = b$. The volume V of the solid of revolution generated by revolving R about the x-axis is $V = \int_a^b \pi[f(x)]^2\,dx$.

Similarly, let g be continuous and nonnegative on $[c, d]$, and let R be the region bounded by the graph of $x = g(y)$, the y-axis and the horizontal lines $y = c$, and $y = d$. The volume V of the solid of revolution generated by revolving R about the y-axis is

$$V = \int_c^d \pi[g(y)]^2\,dy$$



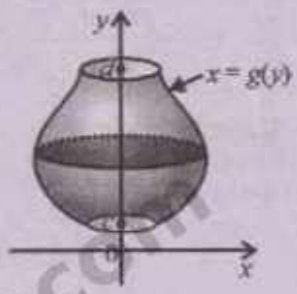


Figure 3.14

Volume of Solid of Revolution by Cross-sectional Method

Let S be a solid that lies between the planes P_a and P_b. If the cross-sectional area of S in the plane P_x is $A(x)$, where A is a continuous function, then volume of S is

$$V = \int_a^b A(x)\,dx$$

www.ilmkidunya.com

www.ilmkidunya.com

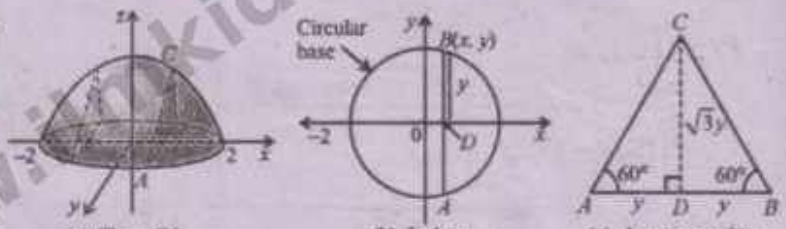
Example 62: A solid has a circular base of radius 2. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

Solution:

Consider a circle with centre $O(0, 0)$ and radius 2.

$$x^2 + y^2 = 2^2$$
$$y^2 = 4 - x^2$$
$$\sqrt{y^2} = \sqrt{4-x^2}$$
$$y = +\sqrt{4-x^2} \quad \ldots(1) \text{ (Take +ve root)}$$

Since, the cross-sections perpendicular to base are equilateral triangles, so



(a) The solid (b) Its base (c) A cross-section

Figure 3.15

From triangle ABC, we get

Base of triangle $= |AB| = 2y$

$$\tan 60° = \frac{CD}{AD} \Rightarrow \sqrt{3} = \frac{CD}{y} \Rightarrow CD = \sqrt{3}y$$

Height of triangle $= CD = \sqrt{3}y$

Cross-sectional Area: $A(x) = \frac{1}{2}$(Base)(Height)

$$A(x) = \frac{1}{2}(2y)(\sqrt{3}y)$$
$$= \sqrt{3}y^2 = \sqrt{3}(4-x^2) \quad \text{Using (1)}$$

$$\text{Volume} = \int_{-2}^{2} A(x)dx$$
$$= \int_{-2}^{2} \sqrt{3}(4-x^2)dx = \sqrt{3}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$
$$= \sqrt{3}\left[\left(4(2) - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)\right]$$
$$= \sqrt{3}\left[8 - \frac{8}{3} + 8 - \frac{8}{3}\right] = \sqrt{3}\left[16 - \frac{16}{3}\right]$$
$$= \sqrt{3}\left(\frac{32}{3}\right) = \frac{32\sqrt{3}}{3} \text{ cubic units}$$

Example 63: Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 64$, and $x = 0$ around the y-axis.

Solution:

Curve: $y = x^3$

Since the axis of rotation is the y-axis, so

$$y^{\frac{1}{3}} = (x^3)^{\frac{1}{3}}$$
$$\Rightarrow \quad x = y^{\frac{1}{3}}$$

Upper limit: $y = 64$

Lower limit: $y = (0)^3 = 0$ $\quad \because x = 0$

As we know

$$V = \int_c^d \pi[g(y)]^2 dy$$
$$= \int_0^{64} \pi(y^{\frac{1}{3}})^2 dy$$
$$= \pi\int_0^{64} y^{\frac{2}{3}} dy$$
$$= \pi\left[\frac{y^{\frac{2}{3}+1}}{\frac{2}{3}+1}\right]_0^{64}$$

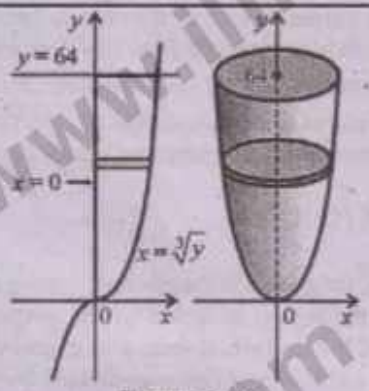


Figure 3.16

$$= \pi \cdot \frac{3}{5}\left[y^{\frac{5}{3}}\right]_0^{64}$$
$$= \frac{3\pi}{5}[(64)^{\frac{5}{3}} - (0)^{\frac{5}{3}}] = \frac{3\pi}{5}(4^3)^{\frac{5}{3}}$$
$$= \frac{3\pi}{5}(4^5) = \frac{3\pi}{5}(1024)$$
$$= \frac{3072\pi}{5} \text{ cubic units}$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 64: A cone is formed by rotating the straight line $y = 2x$ around the x-axis from $x = 0$ to $x = 5$ cm. Find the volume of the cone.

Solution:

Given: $y = 2x \Rightarrow f(x) = 2x$

Limits: $x = 0$ to $x = 5$

As we know

$$V = \int_a^b \pi[f(x)]^2 dx$$

$$= \int_0^5 \pi[2x]^2 dx$$

$$= \int_0^5 \pi(4x^2) dx$$

$$V = 4\pi \int_0^5 x^2 dx$$

$$= 4\pi \left[\frac{x^3}{3}\right]_0^5$$

$$= \frac{4\pi}{3}(5^3 - 0^3) = \frac{4\pi}{3}(120)$$

$$V = \frac{500\pi}{3} \approx 523.6 \text{ cm}^3$$

Example 65: A funnel is designed by rotating the line $y = 3x + 1$ around the x-axis from $x = 0$ to $x = 4$ cm. Find the volume of the funnel.

Solution:

Given: $y = 3x + 1 \Rightarrow f(x) = 3x + 1$

Limits: $x = 0$ to $x = 4$

As we know

$$V = \int_a^b \pi[f(x)]^2 dx$$

$$= \pi \int_0^4 (3x+1)^2 dx$$

$$= \pi \int_0^4 (9x^2 + 6x + 1) dx$$

$$V = \pi \left[\frac{9x^3}{3} + \frac{6x^2}{2} + x\right]_0^4$$

$$= \pi \left[3x^3 + 3x^2 + x\right]_0^4$$

$$= \pi[(3(4)^3 + 3(4)^2 + 4) - (0+0+0)]$$

$$= \pi(192 + 48 + 4)$$

$$V = 244\pi \text{ cm}^3$$

EXERCISE 3.6

Q1. Find the area of the region that is above the x-axis, but below the curve $y = (x-2)(3-x)$.

Solution:

Parabola: $y = (x-2)(3-x)$ (downward open)

For x-intercepts, put $y = 0$

$$(x-2)(3-x) = 0$$

$$\Rightarrow \quad x - 2 = 0 \quad ; \quad 3 - x = 0$$

$$x = 2 \quad ; \quad x = 3$$

As $y = (x-2)(3-x) \geq 0$ in $[2, 3]$, so the curve is above the x-axis.

$$\text{Area} = \int_2^3 y\, dx$$

$$= \int_2^3 (x-2)(3-x) dx$$

$$= \int_2^3 (-x^2 + 5x - 6) dx$$

$$= \left[\frac{-x^3}{3} + \frac{5x^2}{2} - 6x\right]_2^3$$

$$= \left(\frac{-3^3}{3} + \frac{5}{2}(3)^2 - 6(3)\right) - \left(\frac{-2^3}{3} + \frac{5}{2}(2)^2 - 6(2)\right)$$

$$= -9 + \frac{45}{2} - 18 + \frac{8}{3} - 10 + 12$$

$$= \frac{45}{2} + \frac{8}{3} - 25$$

$$= \frac{135 + 16 - 150}{6}$$

$$\text{Area} = \frac{1}{6} \text{ square units}$$

www.ilmkidunya.com

www.ilmkidunya.com

Q2. Find the total area and the net signed area between the graph of $y = x^2 - 4$ and the x-axis over the interval (i) $[-3, -1]$ (ii) $[-3, 3]$.

Solution:

Parabola: $y = x^2 - 4$ (Opens upward)

For x-intercepts, put $y = 0$

$$x^2 - 4 = 0$$

$$(x+2)(x-2) = 0$$

$$\Rightarrow \quad x+2 = 0 \quad ; \quad x-2 = 0$$

$$x = -2 \quad ; \quad x = 2$$

(i) For interval $[-3, -1]$

$$\text{Net Signed Area} = \int_{-3}^{-1} y\,dx$$

$$= \int_{-3}^{-1} (x^2 - 4)dx$$

$$= \left[\frac{x^3}{3} - 4x\right]_{-3}^{-1}$$

$$= \left(\frac{(-1)^3}{3} - 4(-1)\right) - \left(\frac{(-3)^3}{3} - 4(-3)\right)$$

$$= \frac{-1}{3} + 4 + 9 - 12$$

$$= \frac{-1}{3} + 1$$

$$= \frac{-1+3}{3}$$

$$= \frac{2}{3} \text{ sq. units}$$

As $y \geq 0$ in $[-3, -2]$ and $y \leq 0$ in $[-2, -1]$, so

$$\text{Total Area} = \int_{-3}^{-2} y\,dx - \int_{-2}^{-1} y\,dx$$

$$= \int_{-3}^{-2} (x^2 - 4)dx - \int_{-2}^{-1} (x^2 - 4)dx$$

$$= \left[\frac{x^3}{3} - 4x\right]_{-3}^{-2} - \left[\frac{x^3}{3} - 4x\right]_{-2}^{-1}$$

$$= \left[\left(\frac{-8}{3} + 8\right) - (-9 + 12)\right] - \left[\left(\frac{-1}{3} + 4\right) - \left(\frac{-8}{3} + 8\right)\right]$$

$$= \frac{16}{3} - 3 - \frac{11}{3} + \frac{16}{3} = \frac{16 - 9 - 11 + 16}{3}$$

$$= 4 \text{ sq. units}$$

(ii) For interval $[-3, 3]$

$$\text{Net Signed Area} = \int_{-3}^{3} y\,dx$$

$$= \int_{-3}^{3} (x^2 - 4)dx$$

$$= \left[\frac{x^3}{3} - 4x\right]_{-3}^{3}$$

$$= (9 - 12) - (-9 + 12)$$

$$= -3 - 3$$

$$= -6 \text{ sq. units}$$

As $y \geq 0$ in $[-3, -2]$, $y \leq 0$ in $[-2, 2]$ and $y \geq 0$ in $[2, 3]$, so

$$\text{Total Area} = \int_{-3}^{-2} y\,dx - \int_{-2}^{2} y\,dx + \int_{2}^{3} y\,dx$$

$$= \int_{-3}^{-2} (x^2 - 4)dx - \int_{-2}^{2} (x^2 - 4)dx + \int_{2}^{3} (x^2 - 4)dx$$

$$= \left[\frac{x^3}{3} - 4x\right]_{-3}^{-2} - \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} + \left[\frac{x^3}{3} - 4x\right]_{2}^{3}$$

$$= \left[\left(\frac{-8}{3} + 8\right) - (-9 + 12)\right] - \left[\left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right)\right] + \left[(9 - 12) - \left(\frac{8}{3} - 8\right)\right]$$

$$= \frac{16}{3} - 3 + \frac{16}{3} + \frac{16}{3} - 3 + \frac{16}{3}$$

$$= -6 + \frac{64}{3}$$

$$= \frac{-18 + 64}{3}$$

$$= \frac{46}{3} \text{ sq. units}$$

www.ilmkidunya.com

www.ilmkidunya.com

Q3. Sketch the region bounded by the given curves and find the area of the region.

(i) $y = 2x, y = x^2$

Solution:

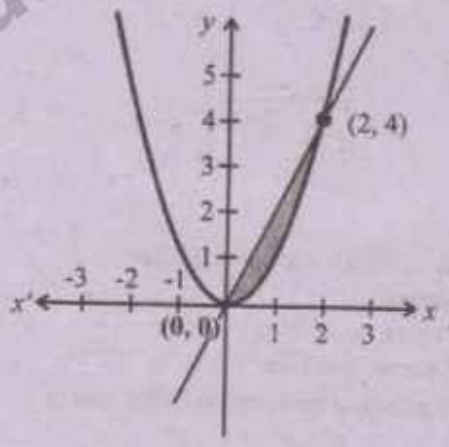
1st curve: $y = 2x$

2nd curve: $y = x^2$

For points of intersection of the curves

$$2x = x^2$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$\Rightarrow x = 0 \; ; \; x - 2 = 0 \Rightarrow x = 2$$

x	0	0.5	1	1.5	2
$y = 2x$	0	1	2	3	4
$y = x^2$	0	0.25	1	2.25	4



Determine which curve is on the top:

Take $x = 1 \in [0, 2]$

$y(1) = 2(1) = 2$

$y(1) = (1)^2 = 1$

$\Rightarrow$ Upper curve: $y = 2x$

and Lower curve: $y = x^2$

$$\text{Area} = \int_0^2 [\text{Upper} - \text{Lower}]\,dx$$

$$= \int_0^2 [2x - x^2]\,dx$$

$$= \left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_0^2$$

$$= \left[(2)^2 - \frac{2^3}{3}\right] - (0 - 0)$$

$$= 4 - \frac{8}{3} = \frac{12 - 8}{3}$$

$$= \frac{4}{3} \text{ sq. unit}$$

(ii) $y = \sqrt{x}, y = x^2$

Solution:

1st curve: $y = \sqrt{x}$

2nd curve: $y = x^2$

For points of intersection of the curves

$$\sqrt{x} = x^2$$
$$(\sqrt{x})^2 = (x^2)^2 \qquad \text{Squaring both sides}$$
$$x = x^4$$
$$x^4 - x = 0$$
$$x(x^3 - 1) = 0$$
$$\Rightarrow \quad x = 0$$

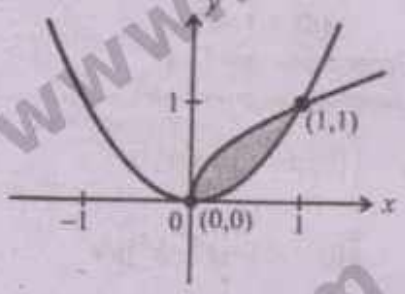
and $x^3 - 1 = 0$

$(x-1)(x^2 + x + 1) = 0$

$x - 1 = 0 \; ; \; x^2 + x + 1 = 0$ (Imaginary roots)

$x = 1$

x	0	0.2	0.4	0.6	0.8	1
$y = \sqrt{x}$	0	0.45	0.63	0.77	0.89	1
$y = x^2$	0	0.04	0.16	0.36	0.64	1



Determine which curve is on the top:

Take $x = 0.2 \in [0, 1]$

$y(0.2) = \sqrt{0.2} = 0.45$

$y(0.2) = (0.2)^2 = 0.04$

$\Rightarrow$ Upper curve: $y = \sqrt{x}$

Lower curve: $y = x^2$

$$\text{Area} = \int_0^1 [\text{Upper} - \text{Lower}]\,dx$$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \int_0^1 (x^{\frac{1}{2}} - x^2)dx$$

$$= \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{2+1}}{2+1}\right]_0^1$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1$$

$$= \frac{1}{3}\left[(2(1)^{\frac{3}{2}} - (1)^3) - (0-0)\right]$$

$$= \frac{1}{3}(2-1) = \frac{1}{3} \text{ square units}$$

(iii) $y = x^4 - x^2, y = 1 - x^2$

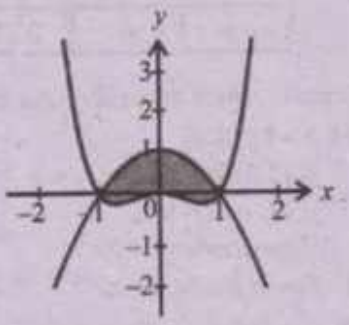
Solution:

1st curve: $y = x^4 - x^2$

2nd curve: $y = 1 - x^2$

For points of intersection of the curves

$$x^4 - x^2 = 1 - x^2$$
$$x^4 - 1 = 0$$
$$(x^2)^2 - 1^2 = 0$$
$$(x^2 - 1)(x^2 + 1) = 0$$
$$\Rightarrow \quad x^2 - 1 = 0 \quad ; \quad x^2 + 1 = 0$$
$$x^2 = 1 \quad ; \quad x^2 = -1$$
$$x = \pm 1 \quad ; \quad x = \pm i \text{ (Imaginary roots)}$$



x	-1	-0.5	0	0.5	1
$y = x^4 - x^2$	0	-0.19	0	-0.19	0
$y = 1 - x^2$	0	0.75	1	0.75	0

Determine which curve is on the top:

Take $x = 0 \in [-1, 1]$

$$y(0) = 0^4 - 0^2 = 0$$
$$y(0) = 1 - 0^2 = 1$$

$\Rightarrow$ Upper curve: $y = 1 - x^2$

Lower curve: $y = x^4 - x^2$

$$\text{Area} = \int_{-1}^{1} [\text{Upper} - \text{Lower}]dx$$

$$= \int_{-1}^{1} [(1 - x^2) - (x^4 - x^2)]dx$$

$$= \int_{-1}^{1} (1 - x^2 - x^4 + x^2)dx$$

$$= \int_{-1}^{1} (1 - x^4)dx$$

$$= \left[x - \frac{x^5}{5}\right]_{-1}^{1}$$

$$= \left(1 - \frac{(1)^5}{5}\right) - \left(-1 - \frac{(-1)^5}{5}\right)$$

$$= 1 - \frac{1}{5} + 1 - \frac{1}{5} = 2 - \frac{2}{5}$$

$$\text{Area} = \frac{8}{5} \text{ square units}$$

(iv) $y = x^2$ and $y = |x|$.

Solution:

1st curve: $y = x^2$

2nd curve: $y = |x|$

For points of intersection of the curves

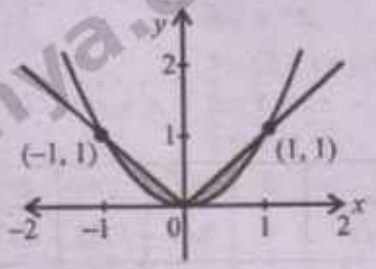
$$x^2 = |x|$$

$x^2 = +x$ if $x \geq 0$

$$x^2 - x = 0$$
$$x(x - 1) = 0$$
$$\Rightarrow \quad x = 0 \; ; \; x - 1 = 0$$
$$x = 1$$

$x^2 = -x$ if $x < 0$

$$x^2 + x = 0$$
$$x(x + 1) = 0$$
$$\Rightarrow \quad x = 0 \; ; \; x + 1 = 0$$
$$x = -1$$



www.ilmkidunya.com

www.ilmkidunya.com

x	-1	-0.5	0	0.5	1
$y = x^2$	1	0.25	0	0.25	1
$y = \|x\|$	1	0.5	0	0.5	1

Determine which curve is on the top:

Take $x = 0.5 \in [-1, 1]$

$y(0.5) = (0.5)^2 = 0.25$

$y(0.5) = |0.5| = 0.5$

$\Rightarrow$ Upper curve: $y = |x|$

Lower curve: $y = x^2$

Since both the functions are even (symmetric about y-axis).

Therefore

$$\text{Area} = 2\int_0^1 [\text{Upper} - \text{Lower}]\,dx$$

$$= 2\int_0^1 (|x| - x^2)\,dx$$

$$= 2\int_0^1 (x - x^2)\,dx \qquad \because x \geq 0$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right) - (0-0)\right]$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = 2\left(\frac{3-2}{6}\right) = \frac{1}{3} \text{ sq. units}$$

Q4. Find the area of the region bounded by $y = |x-3|$, $x = 2$, $x = 4$ and the x-axis.

Solution:

Curve: $y = |x-3|$

$$y = \begin{cases} +(x-3) \text{ if } x-3 \geq 0 \Rightarrow x \geq 3 \\ -(x-3) \text{ if } x-3 < 0 \Rightarrow x < 3 \end{cases}$$

Limits: $x = 2$ to $x = 4$

$$\text{Area} = \int_2^4 y\,dx$$

$$= \int_2^4 |x-3|\,dx$$

$$= \int_2^3 |x-3|\,dx + \int_3^4 |x-3|\,dx$$

$$= -\int_2^3 (x-3)\,dx + \int_3^4 (x-3)\,dx$$

$$= -\left[\frac{(x-3)^{1+1}}{1+1}\right]_2^3 + \left[\frac{(x-3)^{1+1}}{1+1}\right]_3^4$$

$$= \frac{-1}{2}\left[(3-3)^2 - (2-3)^2\right] + \frac{1}{2}\left[(4-3)^2 - (3-3)^2\right]$$

$$= \frac{-1}{2}(0-1) + \frac{1}{2}(1-0)$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \text{ sq. units}$$

Q5. Find the area bounded by the parabola $y = x^2 - 6x + 8$, $x = 2$, $x = 4$ and the x-axis.

Solution:

Curve: $y = x^2 - 6x + 8$

Limits: $x = 2$ to $x = 4$

As $y = x^2 - 6x + 8 \leq 0$ in $[2, 4]$, so the curve is below the x-axis.

$$\text{Area} = -\int_2^4 y\,dx$$

$$= -\int_2^4 (x^2 - 6x + 8)\,dx$$

$$= -\left[\frac{x^3}{3} - \frac{6x^2}{2} + 8x\right]_2^4$$

$$= -\left[\frac{x^3}{3} - 3x^2 + 8x\right]_2^4$$

$$= -\left[\left(\frac{4^3}{3} - 3(4)^2 + 8(4)\right) - \left(\frac{2^3}{3} - 3(2)^2 + 8(2)\right)\right]$$

$$= -\left[\frac{64}{3} - 48 + 32 - \frac{8}{3} + 12 - 16\right]$$

$$= -\left(\frac{64}{3} - \frac{8}{3} - 20\right)$$

$$= -\left(\frac{64 - 8 - 60}{3}\right) = \frac{4}{3} \text{ sq. units}$$

www.ilmkidunya.com

www.ilmkidunya.com

Q6. Find the area bounded by the parabola $x = 9 - y^2$, and the y-axis.

Solution:

Parabola: $x = 9 - y^2$

For intercepts, put $x = 0$

$$9 - y^2 = 0$$
$$(3+y)(3-y) = 0$$
$$\Rightarrow \quad 3+y=0 \quad ; \quad 3-y=0$$
$$y=-3 \quad ; \quad y=3$$

As $x = 9 - y^2 \geq 0$ in $[-3, 3]$, so the curve is right the y-axis.

$$\text{Area} = \int_{-3}^{3} x\,dy$$

$$= \int_{-3}^{3} (9-y^2)\,dy$$

$$= \left[9y - \frac{y^3}{3}\right]_{-3}^{3}$$

$$= \left(9(3) - \frac{3^3}{3}\right) - \left(9(-3) - \frac{(-3)^3}{3}\right)$$

$$= 27 - 9 + 27 - 9 = 54 - 18$$

Area = 36 square units

Q7. Find the ratio of the areas of the regions bounded by the parabola $y = x^2 - 4x + 3$ and the coordinate axes.

Solution:

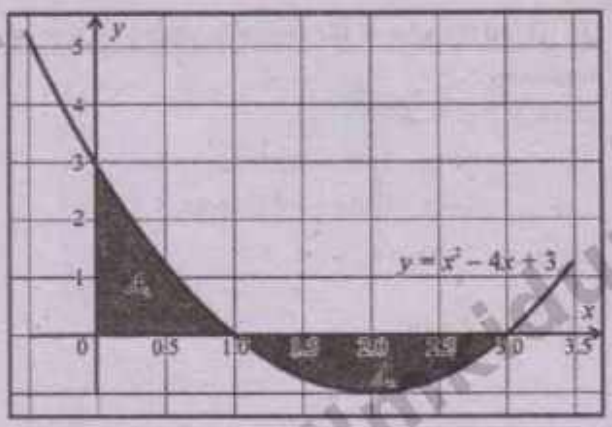
Parabola: $y = x^2 - 4x + 3$ (Opens downward)

For x-intercepts, put $y = 0$

$$x^2 - 4x + 3 = 0$$
$$x^2 - 3x - x + 3 = 0$$
$$x(x-3) - 1(x-3) = 0$$
$$(x-1)(x-3) = 0$$
$$\Rightarrow x-1=0 \; ; \; x-3=0$$
$$x=1 \; ; \quad x=3$$

For y-intercepts, put $x = 0$

$$y = 0^2 - 4(0) + 3$$
$$y = 0 - 0 + 3$$
$$y = 3$$



As $y = x^2 - 4x + 3 \geq 0$ in $[0, 1]$, so the curve is above the x-axis.

$$A_1 = \int_0^1 y\,dx$$

$$= \int_0^1 (x^2 - 4x + 3)dx = \left[\frac{x^3}{3} - \frac{4x^2}{2} + 3x\right]_0^1$$

$$= \left(\frac{1}{3} - \frac{4}{2} + 3\right) - (0 - 0 + 3)$$

$$= \frac{1}{3} - 2 + 3$$

$$= \frac{1}{3} + 1 = \frac{4}{3} \text{ sq. unit}$$

As $y = x^2 - 4x + 3 \leq 0$ in $[1, 3]$, so the curve is below the x-axis.

$$A_2 = -\int_1^3 y\,dx$$

$$= -\int_1^3 (x^2 - 4x + 3)dx = -\left[\frac{x^3}{3} - \frac{4x^2}{2} + 3x\right]_1^3$$

$$= -\left[\left(\frac{3^3}{3} - 2(3)^2 + 3(3)\right) - \left(\frac{1^3}{3} - 2(1)^2 + 3(1)\right)\right]$$

$$= -\left[(9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3\right)\right]$$

$$= -\left[0 - \frac{1}{3} - 1\right] = -\left(\frac{-4}{3}\right) = \frac{4}{3}$$

Ratio $= A_1 : A_2$

$$= \frac{4}{3} : \frac{4}{3} = \frac{4/3}{4/3} = \frac{1}{1} = 1:1$$

Q8. For the function $f(x) = 2x$ over the interval $[-3, 3]$:

(a) Find the net signed area

Solution:

Function: $f(x) = 2x; x \in [-3, 3]$

$$\text{Net Signed Area} = \int_{-3}^{3} f(x)dx$$

$$= \int_{-3}^{3} 2x\,dx = \left[\frac{2x^2}{2}\right]_{-3}^{3}$$

$$= \left[x^2\right]_{-3}^{3}$$

$$= 3^2 - (-3)^2 = 0$$

www.ilmkidunya.com

www.ilmkidunya.com

(b) Find the total (physical) area

Solution:

As $f(x)=2x\le 0$ in $[-3, 0]$ and $f(x)=2x\ge 0$ in $[0, 3]$, so

$$\text{Total Area} = -\int_{-3}^{0} f(x)dx + \int_{0}^{3} f(x)dx$$

$$= -\int_{-3}^{0} 2x\,dx + \int_{0}^{3} 2x\,dx$$

$$= -\left[\frac{2x^2}{2}\right]_{-3}^{0} + \left[\frac{2x^2}{2}\right]_{0}^{3}$$

$$= -\left[x^2\right]_{-3}^{0} + \left[x^2\right]_{0}^{3}$$

$$= -(0^2-(-3)^2)+(3^2-0^2)$$

$$= 9+9$$

$$= 18 \text{ sq. unit}$$

Q9. Find the area of the region bounded by the curve $y=(x-1)(x+1)(x-2)$ and the x-axis.

Solution:

Curve: $y=(x-1)(x+1)(x-2)$

For x-intercepts, put $y=0$

$(x-1)(x+1)(x-2)=0$

$$\Rightarrow \quad x-1=0 \;\Big|\; x+1=0 \;\Big|\; x-2=0$$
$$x=1 \;\Big|\; x=-1 \;\Big|\; x=2$$

As $y\ge 0$ in $[-1, 1]$, so the curve is above the x-axis and $y\le 0$ in $[1, 2]$, so the curve is below the x-axis.

$$y=(x-1)(x+1)(x-2)$$

$$=(x^2-1)(x-2)$$

$$y=x^3-2x^2-x+2$$

$$\text{Area} = \int_{-1}^{1} y\,dx - \int_{1}^{2} y\,dx$$

$$= \int_{-1}^{1}(x^3-2x^2-x+2)dx - \int_{1}^{2}(x^3-2x^2-x+2)dx$$

$$= \left[\frac{x^4}{4}-\frac{2x^3}{3}-\frac{x^2}{2}+2x\right]_{-1}^{1} - \left[\frac{x^4}{4}-\frac{2x^3}{3}-\frac{x^2}{2}+2x\right]_{1}^{2}$$

$$= \left[\left(\frac{1}{4}-\frac{2}{3}-\frac{1}{2}+2\right)-\left(\frac{1}{4}+\frac{2}{3}-\frac{1}{2}-2\right)\right]$$

$$-\left[\left(\frac{16}{4}-\frac{16}{3}-\frac{4}{2}+4\right)-\left(\frac{1}{4}-\frac{2}{3}-\frac{1}{2}+2\right)\right]$$

$$= \left(\frac{1}{4}-\frac{2}{3}-\frac{1}{2}+2-\frac{1}{4}-\frac{2}{3}+\frac{1}{2}+2\right)-$$

$$\left(4-\frac{16}{3}-2+4-\frac{1}{4}+\frac{2}{3}+\frac{1}{2}-2\right)$$

$$= \left(\frac{-2}{3}-\frac{2}{3}+4\right)-\left(4-\frac{16}{3}+\frac{2}{3}-\frac{1}{4}+\frac{1}{2}\right)$$

$$= \frac{-4}{3}+4-4+\frac{14}{3}+\frac{1}{4}-\frac{1}{2} = \frac{-4}{3}+\frac{14}{3}+\frac{1}{4}-\frac{1}{2}$$

$$= \frac{10}{3}+\frac{1}{4}-\frac{1}{2} = \frac{40+3-6}{12}$$

$$= \frac{37}{12} \text{ sq. units}$$

Q10. Find the area of the region bounded by the line $y=x+1$ and the parabola $y^2=2x+10$.

Solution:

1st curve: $y=x+1 \quad \Rightarrow x=y-1$

2nd curve: $y^2=2x+10 \Rightarrow 2x=y^2-10 \Rightarrow x=\frac{1}{2}(y^2-10)$

For points of intersection of the curves

$$\frac{1}{2}(y^2-10)=y-1$$

$$y^2-10=2y-2$$

$$y^2-2y-8=0$$

$$y^2-4y+2y-8=0$$

$$y(y-4)+2(y-4)=0$$

$$(y+2)(y-4)=0$$

$$\Rightarrow \quad y+2=0 \;;\; y-4=0$$
$$y=-2 \;;\; y=4$$

Determine which curve is on the right:

Take $y=0\in[-2,4]$

$$x(0)=0-1=-1$$

$$x(0)=\frac{1}{2}(0^2-10)=-5$$

$\Rightarrow$ Right curve: $x=y-1$

Left curve: $x=\frac{1}{2}(y^2-10)$

$$\text{Area} = \int_{-2}^{4}(\text{Right}-\text{Left})dy$$

$$= \int_{-2}^{4}\left[(y-1)-\frac{1}{2}(y^2-10)\right]dy$$

$$= \int_{-2}^{4}\left(y-1-\frac{1}{2}y^2+5\right)dy$$

$$= \int_{-2}^{4}\left(\frac{-1}{2}y^2+y+4\right)dy$$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \left[\frac{-1}{2}\cdot\frac{y^3}{3}+\frac{y^2}{2}+4y\right]_{-2}^{4}$$

$$= \left(\frac{-1}{6}(4)^3+\frac{4^2}{2}+4(4)\right)-\left(\frac{-1}{6}(-2)^3+\frac{(-2)^2}{2}+4(-2)\right)$$

$$= \left(\frac{-64}{6}+8+16\right)-\left(\frac{8}{6}+2-8\right)$$

$$= \frac{-32}{3}+24-\frac{4}{3}+6$$

$$= \frac{-32}{3}-\frac{4}{3}+30 = \frac{-36}{3}+30 = -12+30$$

$= 18$ sq. units

Q11. The demand function is $p = 30 - 2x - x^2$ and supply function is $p = -6 + 4x + x^2$. Find consumer's surplus and producer's surplus.

Solution:

Demand Function: $p = 30 - 2x - x^2$

Supply Function: $p = -6 + 4x + x^2$

For equilibrium point, put

$$30 - 2x - x^2 = -6 + 4x + x^2$$

$$0 = -6 + 4x + x^2 + x^2 + 2x - 30$$

$$2x^2 + 6x - 36 = 0$$

$$x^2 + 3x - 18 = 0 \qquad \text{Divide by '2'}$$

$$x^2 + 6x - 3x - 18 = 0$$

$$x(x+6) - 3(x+6) = 0$$

$$(x-3)(x+6) = 0$$

$$x - 3 = 0 \;;\; x + 6 = 0$$

$$x = 3 \;;\; x = -6 \text{ (Not possible)}$$

At $x = 3$:

$$p = 30 - 2(3) - 3^2 = 30 - 6 - 9 = 15$$

$\Rightarrow$ Equilibrium point $= (x_e, p_e) = (3, 15)$

Consumer's surplus is

$$CS = \int_0^{x_e} (D(x) - p_e)\,dx$$

$$= \int_0^3 ((30 - 2x - x^2) - 15)\,dx$$

$$= \int_0^3 (-x^2 - 2x + 15)\,dx$$

$$= \left[\frac{-x^3}{3}-\frac{2x^2}{2}+15x\right]_0^3$$

$$= \left(\frac{-3^3}{3}-3^2+15(3)\right)-(0-0+0)$$

$$CS = -9 - 9 + 45 = 27$$

Producer's surplus is

$$PS = \int_0^{x_e} (p_e - S(x))\,dx$$

$$= \int_0^3 (15 - (-6 + 4x + x^2))\,dx$$

$$= \int_0^3 (-x^2 - 4x + 21)\,dx$$

$$= \left[\frac{-x^3}{3}-\frac{4x^2}{2}+21x\right]_0^3 = \left[\frac{-x^3}{3}-2x^2+21x\right]_0^3$$

$$= \left(\frac{-3^3}{3}-2(3)^2+21(3)\right)-(0-0+0)$$

$$= -9 - 18 + 63$$

$$PS = 36$$

Q12. At equilibrium, the quantity sold is $x_e = 4$ and the price is $p_e = 16$. Consumer's surplus is 16 and producer's surplus is 8. Both the demand and supply curves are linear. Find the demand function $D(x)$ and the supply function $S(x)$.

Solution:

Let Demand Function: $D(x) = ax + b$

Supply Function: $S(x) = cx + d$

Since both the functions pass through the equilibrium point $(x_e, p_e) = (4, 16)$, so

$$D(4) = 4a + b$$

$$16 = 4a + b$$

$$b = 16 - 4a \ldots(1)$$

$$S(4) = 4c + d$$

$$16 = 4c + d$$

$$d = 16 - 4c \ldots(2)$$

Given that:

$$CS = 16$$

$$\int_0^{x_e} (D(x) - p_e)\,dx = 16$$

$$\int_0^4 (ax + b - 16)\,dx = 16$$

$$PS = 8$$

$$\int_0^{x_e} (p_e - S(x))\,dx = 8$$

$$\int_0^4 (16 - cx - d)\,dx = 8$$

www.ilmkidunya.com

www.ilmkidunya.com

$\left[\frac{ax^2}{2}+bx-16x\right]_0^4=16$

$\left(\frac{4^2}{2}a+4b-16(4)\right)-0=16$

Using eq. (1), we get

$8a+4(16-4a)-64=16$

$\left[16x-\frac{cx^2}{2}-dx\right]_0^4=8$

$\left[16(4)-\frac{4^2}{2}c-4d\right]-0=8$

Using eq. (2), we get

$64-8c-4(16-4c)=8$

$8a+64-16a-64=16$

$-8a=16$

$a=-2$

Put this in eq. (1)

$b=16-4(-2)=24$

$\Rightarrow D(x)=-2x+24$

$64-8c-64+16c=8$

$8c=8$

$c=1$

Put this in eq. (2)

$d=16-4(1)=12$

$\Rightarrow S(x)=1x+12$

Q13. The supply function is $p=\ell n(x+2)$ and the quantity sold is $x_e=2$. Consumer's surplus is $6-2\ell n\,2$. Find:

(a) Equilibrium price **(b) Linear demand function $p=a-bx$.**

Solution:

Supply Function: $p=\ell n(x+2)$

Quantity sold: $x_e=2$, CS $=6-2\ell n2$

(a) Equilibrium price: $p_e=\ell n(x_e+2)$

$p_e=\ell n(2+2)$

$p_e=\ell n4=\ell n2^2$

$p_e=2\ell n2$

(b) Linear Demand Function: $p=a-bx$

Since, it passes through equilibrium point (x_e, p_e), so

$p_e=a-bx_e$

$2\ell n2=a-b(2)$

$a-2b=2\ell n2$ ...(1)

As CS $=6-2\ell n2$

$\Rightarrow \int_0^{x_e}(D(x)-p_e)dx=6-2\ell n2$

$\int_0^2(a-bx-2\ell n2)dx=6-2\ell n2$

$\left[ax-\frac{bx^2}{2}-(2\ell n2)x\right]_0^2=6-2\ell n2$

$2a-2b-4\ell n2=6-2\ell n2$

$2a-2b=6-2\ell n2+4\ell n2$

$2(a-b)=6+2\ell n2$

$a-b=3+\ell n2$ ...(2)

Equation (1) – Equation (2)

$a-2b=2\ell n2$

$\pm a\mp b=\pm3\pm\ell n2$

$-b=-3+\ell n2$

$\Rightarrow b=3-\ell n2$ Put in eq. (2)

$a-(3-\ell n2)=3+\ell n2$

$a-3+\ell n2=3+\ell n2$

$a=6$

$\Rightarrow$ Linear Demand Function is

$p=a-bx$

$p=6-(3-\ell n2)x$

Q14. If the demand function for a particular commodity is $p=100-2x$ and the market price is $p_e=$ Rs. 64, find the consumer's surplus using two methods:

(a) Integrate with respect to x. **(b) Integrate with respect to p.**

Solution:

Demand Function: $p=100-2x$

At $p_e=64$:

$p_e=100-2x_e$

$\Rightarrow 64=100-2x_e$

$2x_e=100-64 \Rightarrow 2x_e=36$

$x_e=18$

Method (i): Integrate w.r.t 'x'

$CS=\int_0^{x_e}(D(x)-p_e)dx$

$=\int_0^{18}(100-2x-64)dx$

$=\int_0^{18}(-2x+36)dx$

$=\left[\frac{-2x^2}{2}+36x\right]_0^{18}$

$=[-x^2+36x]_0^{18}$

$=(-(18)^2+36(18))-0=-324+648$

CS = Rs. 324

Method (ii): Integrate w.r.t 'p'

As $p=100-2x$

$2x=100-p$

www.ilmkidunya.com

www.ilmkidunya.com

$$x = 50 - \frac{1}{2}p \qquad \text{Divide by '2'}$$

Find p_{max}, when $x = 0$:

$$0 = 50 - \frac{1}{2}p \Rightarrow \frac{1}{2}p = 50 \Rightarrow p = 100$$

$$\Rightarrow \quad p_{max} = 100$$

$$CS = \int_{P_e}^{P_{max}} x_d(p)\,dp$$

$$= \int_{64}^{100}\left(50 - \frac{1}{2}p\right)dp$$

$$= \left[50p - \frac{1}{2}\cdot\frac{p^2}{2}\right]_{64}^{100} = \left[50p - \frac{1}{4}p^2\right]_{64}^{100}$$

$$= \left(50(100) - \frac{1}{4}(100)^2\right) - \left(50(64) - \frac{1}{4}(64)^2\right)$$

$$= (5000 - 2500) - (3200 - 1024)$$

$$= 2500 - 2176$$

$$CS = \text{Rs. } 324$$

Hence, both methods give CS = Rs. 324

Q15. Find the volume of the solid obtained by rotating the region bounded by the given curves about the indicated axis.

(i) $y = x^2, x = 2, y = 0$; x-axis

Solution:

Given curve: $y = x^2 \Rightarrow f(x) = x^2$

As $y = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$

Limits: $x = 0$ to $x = 2$

As we know

$$V = \int_a^b \pi[f(x)]^2\,dx$$

$$= \int_0^2 \pi(x^2)^2\,dx$$

$$= \pi\int_0^2 x^4\,dx$$

$$= \pi\left[\frac{x^5}{5}\right]_0^2 = \frac{\pi}{5}[x^5]_0^2$$

$$= \frac{\pi}{5}(2^5 - 0) = \frac{\pi}{5}(32)$$

$$V = \frac{32\pi}{5} \text{ cubic units}$$

(ii) $x + y = 2, x = 0, y = 0$; x-axis

Solution:

Given curve: $x + y = 2 \Rightarrow y = 2 - x$

As $y = 0 \Rightarrow x + 0 = 2 \Rightarrow x = 2$

Limits: $x = 0$ to $x = 2$

As we know

$$V = \int_a^b \pi[f(x)]^2\,dx$$

$$= \int_0^2 \pi(2 - x)^2\,dx$$

$$V = -\pi\int_0^2 (2 - x)^2 \cdot (-1)\,dx$$

$$= -\pi\left[\frac{(2 - x)^{2+1}}{2+1}\right]_0^2$$

$$= \frac{-\pi}{3}\left[(2 - x)^3\right]_0^2$$

$$= \frac{-\pi}{3}\left[(2 - 2)^3 - (2 - 0)^3\right]$$

$$= \frac{-\pi}{3}(0 - 8)$$

$$V = \frac{8\pi}{3} \text{ cubic units}$$

(iii) $y = \sqrt{x - 5}, x = 6, x = 8, y = 0$; x-axis

Solution:

Given curve: $y = \sqrt{x - 5} \Rightarrow f(x) = \sqrt{x - 5}$

Limits: $x = 6$ to $x = 8$

As we know

$$V = \int_a^b \pi[f(x)]^2\,dx$$

$$= \pi\int_6^8 (\sqrt{x - 5})^2\,dx$$

$$= \pi\int_6^8 (x - 5)\cdot 1\,dx$$

$$= \pi\left[\frac{(x - 5)^{1+1}}{1+1}\right]_6^8$$

$$= \frac{\pi}{2}\left[(x - 5)^2\right]_6^8$$

www.ilmkidunya.com

www.ilmkidunya.com

$$V = \frac{\pi}{2}\left[(8-5)^2-(6-5)^2\right]$$

$$= \frac{\pi}{2}(9-1) = \frac{\pi}{2}(8)$$

$$V = 4\pi \text{ cubic units}$$

(iv) $y = x^2 - 2x, y = 0$; x-axis

Solution:

Given curve: $y = x^2 - 2x$

As $y = 0 \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0$

$\Rightarrow \quad x = 0; x-2 = 0 \Rightarrow x = 2$

Limits: $x = 0$ to $x = 2$

As we know

$$V = \int_a^b \pi[f(x)]^2 dx$$

$$V = \pi\int_0^2 (x^2-2x)^2 dx$$

$$= \pi\int_0^2 (x^4 - 4x^3 + 4x^2)dx$$

$$= \pi\left[\frac{x^5}{5} - \frac{4x^4}{4} + \frac{4x^3}{3}\right]_0^2 = \pi\left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_0^2$$

$$= \pi\left[\left(\frac{2^5}{5} - 2^4 + \frac{4}{3}(2)^3\right) - 0\right]$$

$$= \pi\left(\frac{32}{5} - 16 + \frac{32}{3}\right) = \pi\left(\frac{96-240+16}{15}\right)$$

$$V = \frac{16\pi}{15} \text{ cubic units}$$

Content Outline

- Applying Concepts of Integration to Real-Life World Problems
  - Moment of Inertia
  - Distance, Velocity, Acceleration
  - Average Value of a Function
  - Drug Dosage Required by Integrating Concentration

APPLYING CONCEPTS OF INTEGRATION TO REAL-LIFE WORLD PROBLEMS

Moment of Inertia

Definition: The moment of inertia of a single particle of mass m about an axis AB is defined to be mr^2, where r is the distance from the particle to the axis AB. For a system of particles $m_1, m_2, \ldots, m_n$ placed at distances $r_1, r_2, \ldots, r_n$ respectively from a given axis, the moment of inertia of the system defined as

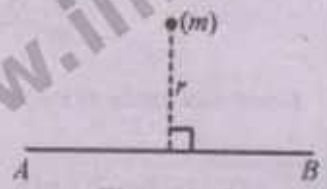


Figure 3.17

$$m_1r_1^2 + m_2r_2^2 + \ldots + m_nr_n^2 = \sum_{i=1}^{n} m_i r_i^2$$

The moment of inertia of a rigid body with continuous mass distribution matter is:

$$\lim_{\delta m \to 0} \sum r^2 \cdot \delta m = \int r^2 dm$$

the integral being taken all over the entire body.

Note: If the moment of inertia of a body of mass M about any axis be MR^2, then R is called the radius of gyration of the body about that axis.

Example 66: A uniform Rod of Length $2a$:

Find the moment of inertia of a uniform rod of length $2a$ and mass M about a line through one end and perpendicular to the rod.

Solution:

Let M be the mass of a uniform rod AB of length $2a$, then

$$\text{Mass per unit length} = \frac{M}{2a}$$

Let AY be a line through A and perpendicular to AB.

Consider a small element PQ of length δx at a distance x from the line AY.

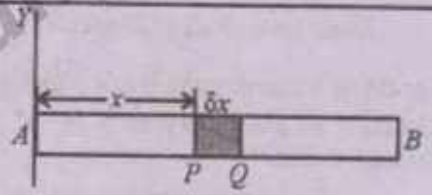


Figure 3.18

www.ilmkidunya.com

www.ilmkidunya.com

δm = Mass of an element PQ
= (Mass per unit length) $\times \delta x$
$= \frac{M}{2a} \cdot \delta x$

Moment of inertia of the element PQ about the line AY is:

$$x^2 \delta m = x^2 \cdot \frac{M}{2a} \cdot \delta x$$

Moment of inertia of whole rod about $AY = \int_0^{2a} x^2 \frac{M}{2a} dx$

$$= \frac{M}{2a} \int_0^{2a} x^2 dx$$

$$= \frac{M}{2a} \left[\frac{x^3}{3}\right]_0^{2a} = \frac{M}{6a} \left[x^3\right]_0^{2a}$$

$$= \frac{M}{6a} [(2a)^3 - (0)^3]$$

$$= \frac{M}{6a} (8a^3 - 0) = \frac{4}{3} Ma^2 \text{ units}$$

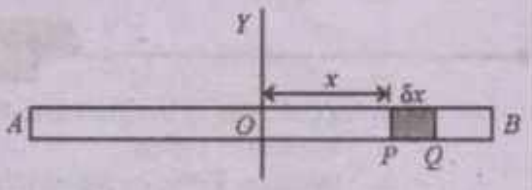
Challenge!

Prove that the moment of inertia of a uniform rod of length $2a$ and mass M about a line through the middle point and perpendicular to the rod is $\frac{1}{3}Ma^2$.

Solution:

Let M be the mass of a uniform rod AB of length $2a$, then

$$\text{Mass per unit length} = \frac{M}{2a}$$



Let OY be the line through middle point O and perpendicular to AB.
Consider a small element PQ of length δx at a distance x from the line OY.

δm = Mass of an element PQ
= (Mass per unit length) $\times \delta x$
$= \frac{M}{2a} \cdot \delta x$

Moment of inertia of the element PQ about the line OY is

$$x^2 \delta m = x^2 \cdot \frac{M}{2a} \cdot \delta x$$

Moment of inertia of whole rod about $OY = \int_{-a}^{a} x^2 \cdot \frac{M}{2a} dx$

$$= \frac{M}{2a} \int_{-a}^{a} x^2 dx$$

Since $f(x) = x^2$ is an even function, so

$$= \frac{M}{2a} \cdot 2 \int_0^a x^2 dx$$

$$= \frac{M}{a} \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{M}{3a} \left[x^3\right]_0^a$$

$$= \frac{M}{3a} [a^3 - 0]$$

$$= \frac{1}{3} Ma^2 \text{ (Proved)}$$

Example 67: A Uniform Rectangular Lamina:

Find the moment of inertia of a uniform rectangular lamina about a line through the centre and parallel to a side.

Solution:

Let M be the mass of a rectangular lamina $ABCD$ with centre O, such that
$AB = 2a$, $BC = 2b$
Total Area = $(2a)(2b) = 4ab$

$$\text{Mass per unit area (Density)} = \frac{M}{4ab}$$

Let OX be a line through O and parallel to side AB.
Consider an elementary strip PQ of breadth δx perpendicular to OX at a distance x from O.

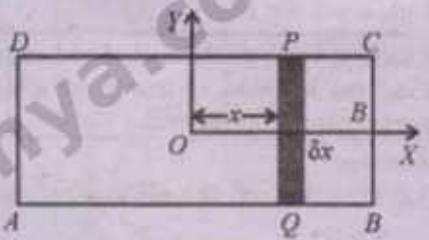


Figure 3.19

δx = Mass of elementary strip
= Density $\times$ Area of strip

www.ilmkidunya.com

www.ilmkidunya.com

$$\delta x = \frac{M}{4ab}\cdot(2b\cdot\delta x) = \frac{M}{2a}\delta x$$

Moment of inertia of this strip about OX is

$$\delta m\cdot\frac{b^2}{3} = \frac{M}{2a}\cdot\delta x\cdot\frac{b^2}{3}$$

Moment of inertia of rectangular lamina about

$$OX = \int_{-a}^{a}\frac{M}{2a}\cdot\frac{b^2}{3}dx$$

$$OX = \frac{Mb^2}{6a}\int_{-a}^{a}1\,dx$$

$$= \frac{Mb^2}{6a}[x]_{-a}^{a}$$

$$= \frac{Mb^2}{6a}[a-(-a)]$$

$$= \frac{Mb^2}{6a}(2a) = \frac{1}{3}Mb^2 \text{ units}$$

Distance, Velocity and Acceleration

Suppose a particle is moving along a straight line with position function $s(t)$ velocity function $v(t)$, and acceleration function $a(t)$, Since $s'(t) = v(t)$, the Fundamental theorem of Calculus gives:

$$\int_{t_1}^{t_2}v(t)dt = \int_{t_1}^{t_2}s'(t)dt = [s(t)]_{t_1}^{t_2} = s(t_2)-s(t_1) = \text{displacement} \qquad ...(1)$$

Similarly, since $v'(t) = a(t)$, the Fundamental Theorem of Calculus gives:

$$\int_{t_1}^{t_2}a(t)dt = v(t_2)-v(t_1) = \text{velocity} \qquad ...(2)$$

If we want to calculate the distance traveled during the time interval $[t_1, t_2]$, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore, total distance traveled $= \int_{t_1}^{t_2}|v(t)|dt$.

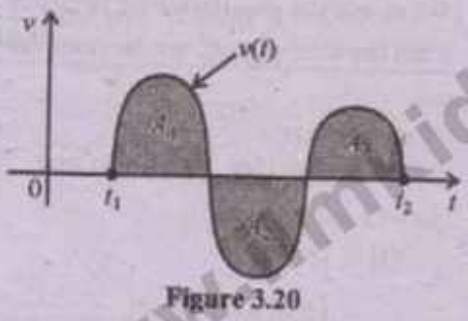


Figure 3.20

Figure 3.20 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.

Displacement $= \int_{t_1}^{t_2}v(t)dt = A_1 - A_2 + A_3$ Distance $= \int_{t_1}^{t_2}|v(t)|dt = A_1 + A_2 + A_3$

Example 68: A particle moves along a straight line with acceleration $a(t) = 6t - 18(\text{m/s}^2)$ and initial velocity $v(0) = 15$ m/s.

(a) Find the velocity function $v(t)$. **(b) Find the displacement of the particle during $0 \leq t \leq 3$.**

(c) Find the total distance traveled during $0 \leq t \leq 3$.

Solution:

(a) $a(t) = 6t - 18$; $v(0) = 15$ m/s

As we know

$$v(t) = \int a(t)dt$$

$$= \int(6t-18)dt$$

$$= 6\frac{t^2}{2} - 18t + c$$

$$= 3t^2 - 18t + c \qquad ...(1)$$

Use the initial condition: $v(0) = 15$

$$\Rightarrow \quad v(0) = 3(0)^2 - 18(0) + c$$

$$15 = 0 - 0 + c \Rightarrow c = 15 \qquad \text{Put in eq. (1)}$$

$$\boxed{v(t) = 3t^2 - 18t + 15}$$

(b) As we know

$$s = \int_0^3 v(t)dt$$

$$= \int_0^3(3t^2 - 18t + 15)dt$$

$$= \left[3\frac{t^3}{3} - 18\frac{t^2}{2} + 15t\right]_0^3$$

$$= \left[t^3 - 9t^2 + 15t\right]_0^3 = (3^3 - 9(3)^2 + 15(3)) - 0$$

$$= 27 - 81 + 45$$

$$s = -9m$$

This means that the particle moved $9m$ to the left.

www.ilmkidunya.com

www.ilmkidunya.com

(c) $v(t) = 3t^2 - 18t + 15$

As $v(t) \geq 0$ on the interval $[0, 1]$ and $v(t) \leq 0$ on $[1, 3]$, so

$$\text{Total distance traveled} = \int_0^3 |v(t)|\,dt$$

$$= \int_0^1 |v(t)|\,dt + \int_1^3 |v(t)|\,dt = \int_0^1 v(t)\,dt - \int_1^3 v(t)\,dt = \int_0^1 (3t^2 - 18t + 15)\,dt - \int_1^3 (3t^2 - 18t + 15)\,dt$$

$$= \left[3\frac{t^3}{3} - 18\frac{t^2}{2} + 15t\right]_0^1 - \left[3\frac{t^3}{3} - 18\frac{t^2}{2} + 15t\right]_1^3 = \left[t^3 - 9t^2 + 15t\right]_0^1 - \left[t^3 - 9t^2 + 15t\right]_1^3$$

$$= [(1^3 - 9(1)^2 + 15(1)) - 0] - [(3^3 - 9(3)^2 + 15(3)) - (1^3 - 9(1)^2 + 15(1))]$$

$$= (1 - 9 + 15) - [(27 - 81 + 45) - (1 - 9 + 15)] = 7 - (-9 - 7) = 7 + 16$$

$$= 23\ m$$

Example 69: Growth rate of population: A town's population growth rate (in people per year) is given by

$$P'(t) = 5000 + 200t$$

where t is the number of years after 2020. If the population in 2020 ($t = 0$) was 100,000:

(a) Find the population function $P(t)$

(b) What will the population be in 2030?

(c) Find the total population increase from 2022 to 2028.

Solution:

Population growth rate is

$$P'(t) = 5000 + 200t,\ P(0) = 100{,}000$$

(a) As we know

$$P(t) = \int P'(t)\,dt$$

$$= \int (5000 + 200t)\,dt$$

$$= 5000t + 200\frac{t^2}{2} + c$$

$$P(t) = 5000t + 100t^2 + c \quad \ldots(1)$$

Use the initial condition: $P(0) = 100000$

$$\Rightarrow \quad P(0) = 5000(0) + 100(0)^2 + c$$

$$100000 = c$$ Put in eq. (1)

$$\boxed{P(t) = 100000 + 5000t + 100t^2} \quad \ldots(2)$$

(b) For the year 2030:

$$t = 2030 - 2020 = 10 \text{ years}$$

Put $t = 10$ in eq. (2), we get

$$P(10) = 100{,}000 + 5000(10) + 100(10)^2$$

$$= 100{,}000 + 50000 + 10000$$

$$= 160{,}000 \text{ people}$$

(c) For the year 2022: $t_1 = 2022 - 2020 = 2$

For the year 2028: $t_2 = 2028 - 2020 = 8$

Increase $= P(8) - P(2)$

$$= [100000 + 5000(8) + 100(8^2)] - [100000 + 5000(2) + 100(2)^2]$$

$$= 100000 + 40000 + 6400 - 100000 - 10000 - 400$$

$$= 46400 - 10400 = 36000 \text{ people}$$

Average Value of a Function

Definition: If f is continuous on $[a, b]$, then its average value on $[a, b]$ (also called its mean value) is

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 70: Find the average value of the function $f(x) = x^2 + 2$ over the interval $[-1, 2]$.

Solution:

$f(x) = x^2 + 2; [-1, 2]$

Here $a = -1$ and $b = 2$

As we know

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{1}{2-(-1)}\int_{-1}^{2}(x^2+2)dx$$

$$= \frac{1}{2+1}\left[\frac{x^{2+1}}{2+1}+2x\right]_{-1}^{2} = \frac{1}{3}\left[\frac{1}{3}x^3+2x\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[\left(\frac{1}{3}(2)^3+2(2)\right)-\left(\frac{1}{3}(-1)^3+2(-1)\right)\right]$$

$$= \frac{1}{3}\left[\frac{8}{3}+4+\frac{1}{3}+2\right] = \frac{1}{3}\left[\frac{9}{3}+6\right] = \frac{1}{3}(3+6)$$

$f_{av} = 3$

Example 71: A temperature sensor moves slowly along a 10-meter track, continuously measuring temperature. The temperature at position x is given by

$$T(x) = 20 + x, \text{ where } T(x) \text{ is in } °C$$

Find the average temperature measured by the sensor as it moves from $x = 0$ to $x = 10$ meters.

Solution:

$T(x) = 20 + x$; Limits: $x = 0$ to $x = 10$

$$\text{Average temperature} = \frac{1}{b-a}\int_a^b T(x)dx$$

$$= \frac{1}{10-0}\int_0^{10}(20+x)dx = \frac{1}{10}\left[20x+\frac{x^2}{2}\right]_0^{10}$$

$$= \frac{1}{10}\left[\left(20(10)+\frac{10^2}{2}\right)-0\right]$$

$$= \frac{1}{10}(200+50) = \frac{1}{10}(250) = 25°C$$

Drug Dosage Required by Integrating Concentration

Core Concept: Drug Concentration and Integration

Think of fighting an infection as a **battle** between the drug and the bacteria.

Drug Concentration $C(t)$: This is like the "number of soldiers" in your bloodstream at time t. Right after a dose, you have many soldiers. Over time, the body removes them (metabolism/excretion), so the number decreases.

Minimum Inhibitory Concentration (MIC): This is the **minimum number of soldiers** needed to hold the battlefield and prevent the bacteria from growing. If the number of soldiers drops below this, the infection can start winning again.

Area Under the Curve (AUC): If you imagine a graph of "number of soldiers" (concentration) vs. time, the area under that curve represents the total fighting effort over time.

A short time with many soldiers might give the same total "fighting effort" as a long time with fewer soldiers.

This "total fighting effort" (AUC) is what matters for killing the bacteria effectively.

Why Integrate?

Integration is just a way to **add up** the total "soldier-hours" (concentration × time).

If $C(t)$ is concentration at time t, then:

$$\text{AUC} = \int_{t_1}^{t_2} C(t)dt$$

This AUC (Area Under the Curve) measures **total drug exposure**.

Doctors know from experiments:

- If AUC is too small → drug won't work well.
- If AUC is too large → risk of side effects.

So they calculate the AUC to decide the right dose and frequency.

www.ilmkidunya.com

www.ilmkidunya.com

Example 72: Basic AUC Calculation

A patient is given an intravenous dose of an antibiotic. The concentration in the bloodstream over time is modeled by:

$$C(t) = 8e^{-0.15t}(\text{mg/L}) \text{ where } t \text{ is in hours.}$$

(a) Find the total drug exposure (AUC) from $t = 0$ to $t = 6$ hours.

(b) What is the AUC from $t = 0$ to infinity?

(c) If the minimum required total exposure over the first 6 hours is 40mghr/L, is the drug exposure from this dose sufficient for a 6 -hour treatment?

Solution:

(a) $C(t) = 8e^{-0.15t}$; Limits: $t = 0$ to $t = 6$

As we know

$$AUC_{a\to b} = \int_a^b C(t)dt$$

$$AUC_{0\to 6} = \int_0^6 8e^{-0.15t}dt$$

$$= 8\left[\frac{e^{-0.15t}}{-0.15}\right]_0^6 = -\frac{8}{0.15}[e^{-0.15(6)} - e^{-0.15(0)}]$$

$$= -\frac{8}{0.15}(e^{-0.9} - e^0)$$

$$= -53.333(0.4066 - 1)$$

$$AUC_{0\to 6} = -53.333(-0.5934)$$

$$\approx 31.65 mg \cdot h/L$$

(b) $AUC_{0\to\infty} = \int_0^\infty 8e^{-0.15t}dt$

$$= 8\left[\frac{e^{-0.15t}}{-0.15}\right]_0^\infty$$

$$= \frac{8}{0.15}[e^{-\infty} - e^0]$$

$$= -53.333(0 - 1)$$

$$= 53.33 mg \cdot h/L$$

(c) Required total exposure over first 6 hours $= 40 mg \cdot h/L$

Actual AUC from $t = 0$ to $t = 6 = 31.65\ mg \cdot h/L$

$\Rightarrow$ $31.65 < 40$

Conclusion: No

The drug exposure from this dose over 6 hours is insufficient to meet the required minimum of $40\ mg \cdot h/L$.

EXERCISE 3.7

Q1. Find the moment of inertia I of a uniform rectangular plate $ABCD$ about the edge AD when $AD = 3, AB = 8$, $BC = 3, CD = 8$ and the mass per unit area is 4.

Solution:

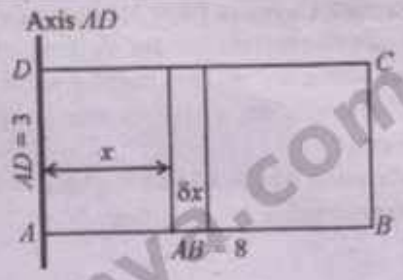
Given that:

$AD = 3, AB = 8$

Mass per unit area (Density) = 4

Axis of rotation = Edge AD

Consider a small elementary strip parallel to the axis AD of breadth δx at a distance x from AD



δm = Mass of elementary strip

= Density × Area of strip

$= 4 \times (3 \cdot \delta x) = 12\ \delta x$

Moment of inertia of this strip about AD is

$\delta I = x^2 \cdot \delta m$

$= x^2 \cdot 12\,\delta x = 12x^2\delta x$

Total moment of inertia:

$$I = \int_0^8 12x^2 dx$$

$$= 12\left[\frac{x^3}{3}\right]_0^8 = 4\left[x^3\right]_0^8 = 4(8^3 - 0^3)$$

$I = 2048$ units

www.ilmkidunya.com

www.ilmkidunya.com

Q2. Find the moment of inertia of a uniform triangular lamina about its base.

Solution:

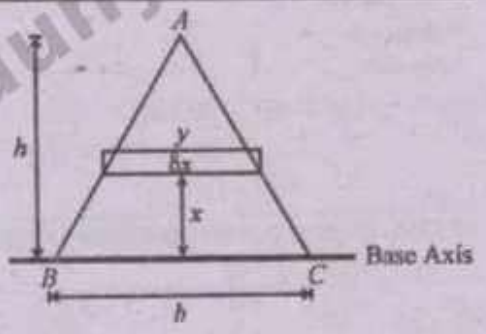
Let M be the mass of a triangular lamina ABC, such that

Base of triangle = b

Height of triangle = h

Total Area = $\frac{1}{2}(b)(h) = \frac{1}{2}bh$

Mass per unit area (Density) = $\frac{M}{\frac{1}{2}bh} = \frac{2M}{bh}$



Consider an elementary horizontal strip parallel to the base at a distance x from it, with a thickness of δx and a length of y.

Using similar triangles property:

$$\frac{y}{b} = \frac{h-x}{h} \Rightarrow y = \frac{b}{h}(h-x) \quad \ldots(1)$$

δm = Mass of elementary strip

= Density × Area of strip

$$= \frac{2M}{bh}\cdot(y\cdot\delta x)$$

$$= \frac{2M}{bh}\cdot\frac{b}{h}(h-x)\cdot\delta x \quad \text{Using (1)}$$

$$\delta m = \frac{2M}{h^2}(h-x)\delta x$$

Moment of inertia of this strip about base is

$$\delta I = x^2\cdot\delta m = x^2\cdot\frac{2M}{h^2}(h-x)\delta x$$

Total moment of inertia:

$$I = \int_0^h \frac{2M}{h^2}x^2(h-x)dx$$

$$= \frac{2M}{h^2}\int_0^h (hx^2 - x^3)dx$$

$$= \frac{2M}{h^2}\left[h\frac{x^3}{3} - \frac{x^4}{4}\right]_0^h$$

$$= \frac{2M}{h^2}\left[\left(h\cdot\frac{h^3}{3} - \frac{h^4}{4}\right) - 0\right]$$

$$= \frac{2M}{h^2}\left(\frac{4h^4 - 3h^4}{12}\right) = \frac{M}{6h^2}(h^4)$$

$$I = \frac{1}{6}Mh^2 \text{ units}$$

Q3. A particle moves along a straight line with acceleration $a(t) = 12t - 36$ (m/s^2) and initial velocity $v(0) =$ 30 m/s.

(a) Find the velocity function $v(t)$.

Solution:

$a(t) = 12t - 36$; $v(0) = 30$ m/s

As we know

$$v(t) = \int a(t)dt$$

$$= \int(12t - 36)dt$$

$$= 12\frac{t^2}{2} - 36t + c$$

$$= 6t^2 - 36t + c \quad \ldots(1)$$

Use the initial condition: $v(0) = 30$

$$\Rightarrow \quad v(0) = 6(0)^2 - 36(0) + c$$

$$30 = 0 - 0 + c \Rightarrow c = 30 \quad \text{Put in eq. (1)}$$

$$\boxed{v(t) = 6t^2 - 36t + 30}$$

(b) Find the displacement of the particle during the time period $1 \le t \le 6$.

Solution:

$v(t) = 6t^2 - 36t + 30$; $1 \le t \le 6$

As we know

$$s = \int_1^6 v(t)dt$$

$$= \int_1^6 (6t^2 - 36t + 30)dt = \left[6\frac{t^3}{3} - 36\frac{t^2}{2} + 30t\right]_1^6$$

$$= \left[2t^3 - 18t^2 + 30t\right]_1^6$$

$$= (2(6)^3 - 18(6)^2 + 30(6)) - (2(1)^3 - 18(1)^2 + 30(1))$$

$$= (2(216) - 18(36) + 180) - (2 - 18 + 30)$$

$$= 432 - 648 + 180 - 14$$

$$s = -50 \text{ m}$$

www.ilmkidunya.com

www.ilmkidunya.com

(c) Find the distance traveled during this time period.

Solution:

Note that:

$$v(t) = 6(t^2-6t+5)$$
$$= 6(t^2-5t-t+5) = 6(t(t-5)-1(t-5))$$
$$= 6(t-1)(t-5)$$

To locate where the velocity changes direction. Put $v(t)=0$

$$6(t-1)(t-5) = 0$$
$$\Rightarrow \quad t-1=0 \;;\; t-5=0$$
$$t=1 \;;\; t=5$$

As $v(t) \le 0$ on the interval $[1, 5]$ and $v(t) \ge 0$ on $[5, 6]$, so

Total distance traveled $= \int_1^6 |v(t)|\,dt$

$$= \int_1^5 |v(t)|\,dt + \int_5^6 |v(t)|\,dt = -\int_1^5 v(t)dt + \int_5^6 v(t)dt$$
$$= -\int_1^5 (6t^2-36t+30)dt + \int_5^6 (6t^2-36t+30)dt$$
$$= -\left[6\frac{t^3}{3}-36\frac{t^2}{2}+30t\right]_1^5 + \left[6\frac{t^3}{3}-36\frac{t^2}{2}+30t\right]_5^6$$
$$= -\left[2t^3-18t^2+30t\right]_1^5 + \left[2t^3-18t^2+30t\right]_5^6$$
$$= -(250-450+150)+(2-18+30)+$$
$$(432-648+180)-(250-450+150)$$
$$= 50+14-36+50 = 78 \text{ m}$$

Q4. A particle moving along a coordinate line at time $t = 0$ is at a position 2 cm from the origin and traveling at a velocity of 5 cm/sec. If the acceleration of the particle is given by: $a(t) = 3 - 4(t+2)^{-3}$. Find the velocity and the position of the particle as functions of t.

Solution:

Given that: $a(t)=3-4(t+2)^{-3}, v(0)=5, S(0)=2$

As we know

$$v(t) = \int a(t)dt$$
$$= \int [3-4(t+2)^{-3}]dt$$
$$= 3t - 4\frac{(t+2)^{-2}}{-2} + c$$
$$= 3t + 2(t+2)^{-2} + c$$
$$v(t) = 3t + \frac{2}{(t+2)^2} + c \quad \ldots(1)$$

Use the initial condition: $v(0)=5$

$$v(0) = 3(0) + \frac{2}{(0+2)^2} + c$$
$$5 = 0 + \frac{2}{4} + c \Rightarrow c = 5 - \frac{1}{2}$$
$$c = \frac{9}{2}$$

Put in eq. (1)

$$\boxed{v(t) = 3t + \frac{2}{(t+2)^2} + \frac{9}{2}} \text{ cm/s}$$

As we know

$$s(t) = \int v(t)dt$$
$$= \int \left[3t + 2(t+2)^{-2} + \frac{9}{2}\right]dt$$
$$= 3\frac{t^2}{2} + \frac{2(t+2)^{-1}}{-1} + \frac{9}{2}t + c_1$$
$$s(t) = \frac{3}{2}t^2 - \frac{2}{t+2} + \frac{9}{2}t + c_1 \quad \ldots(2)$$

Use the initial condition: $s(0)=2$

$$s(0) = \frac{3}{2}(0)^2 - \frac{2}{0+2} + \frac{9}{2}(0) + c_1$$
$$2 = 0 - 1 + 0 + c_1 \Rightarrow c_1 = 3$$

Put in eq. (2)

$$\boxed{s(t) = \frac{3}{2}t^2 + \frac{9}{2}t - \frac{2}{t+2} + 3} \text{ cm}$$

Q5. A projectile is launched vertically upward from an initial height of s_0 feet with an initial velocity of v_0 ft/s. Neglecting air resistance, show that its height $s(t)$ above the ground at time t is given by $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$ where, $g > 0$ is the magnitude of the acceleration due to gravity.

Solution:

Since the projectile is launched vertically upward, therefore

$a(t) = -g$, where $g > 0$

Also, given $s(0) = s_0$ and $v(0) = v_0$

As we know

$$v(t) = \int a(t)dt$$
$$= -\int g\,dt$$
$$= -g\int 1\,dt$$
$$v(t) = -gt + c \quad \ldots(1)$$

Use the initial condition: $v(0) = v_0$

$$v(0) = -g(0) + c$$

www.ilmkidunya.com

www.ilmkidunya.com

$v_0 = -0 + c \Rightarrow c = v_0$ Put in eq. (1)

$\boxed{v(t) = -gt + v_0}$

As we know

$$s(t) = \int v(t)dt$$

$$= \int(-gt + v_0)dt$$

$$s(t) = -g\frac{t^2}{2} + v_0 t + c_1 \quad ...(2)$$

Use the initial condition: $s(0) = s_0$

$$s(0) = \frac{g}{2}(0)^2 + v_0(0) + c_1$$

$s_0 = -0 + 0 + c_1 \Rightarrow c_1 = s_0$ Put in eq. (2)

$$\boxed{s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0}$$ (Proved)

Q6. In a bacterial culture, the growth rate is proportional to the number of bacteria present. Initially there are 200 bacteria. After 2 hours, there are 1800 bacteria.

(a) Find the number of bacteria at time t hours.

Solution:

Let N be the number of bacteria present at time t (hours)

Given that:

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = kN \text{ where } k \text{ is a constant of proportionality}$$

$$\frac{1}{N}dN = kdt$$

Integrating both sides:

$$\int\frac{1}{N}dN = k\int 1dt$$

$$\ln N = kt + c_1$$

In exponential form

$$N = e^{kt+c_1}$$

$$N = e^{kt} \cdot e^{c_1}$$

$$N = c \cdot e^{kt} \quad ...(1) \text{ where } c = e^{c_1}$$

Use the initial condition: $N = 200, t = 0$

$$200 = c \cdot e^{k(0)} \Rightarrow 200 = c \cdot e^0$$

$c = 200$ Put in eq. (1)

$$N = 200e^{kt} \quad ...(2)$$

Use the other condition: $N = 1800, t = 2$

$$1800 = 200e^{k(2)}$$

$$\frac{1800}{200} = e^{2k}$$

$$9 = e^{2k}$$

$\ln 9 = \ln(e^{2k})$ Taking $\ln$ on both sides

$\ln 3^2 = 2k \cdot \ln e$

$2\ln 3 = 2k(1)$ $\because \ln e = 1$

$k = \ln 3$ Put in eq. (2)

$$N = 200e^{\ln 3(t)}$$

$$N = 200 \cdot e^{t \cdot \ln 3} = 200e^{\ln 3^t}$$

$$N = 200 \cdot 3^t \quad \because a^{\log_a x} = x$$

(b) How many bacteria will there be after 4 hours?

Solution:

$N = ?, t = 4$ hours

$N = 200(3^4)$

$N = 200(81)$

$N = 16200$ bacteria

Q7. The rate of population growth of a small city (in people per year) is modeled by $P'(t) = 800 + 60t$, where t is the number of years after 2025. If the population in 2025($t = 0$) was 25,000 people:

(a) Find the population function $P(t)$.

Solution:

Given that:

$P'(t) = 800 + 60t, P(0) = 25000$

$P'(t) = 800 + 60t$

Integrating both sides

$$\int P'(t)dt = \int(800 + 60t)dt$$

$$P(t) = 800t + 60\frac{t^2}{2} + c$$

$$P(t) = 800t + 30t^2 + c \quad ...(1)$$

Use the initial condition: $P(0) = 25000$

$$P(0) = 800(0) + 30(0)^2 + c$$

$25000 = 0 + 0 + c \Rightarrow c = 25000$ Put in eq. (1)

$$\boxed{P(t) = 30t^2 + 800t + 25000}$$

(b) Estimate the population in 2035.

Solution:

$P = ?, t = 10$ years

$P(10) = 30(10)^2 + 800(10) + 25000$

$= 3000 + 8000 + 2500 = 36000$ people

(c) Calculate the total population growth from 2028 to 2033.

Solution:

Population $= P(8) - P(3)$

$= [30(8)^2 + 800(8) + 25000] - [30(3)^2 + 800(3) + 25000]$

$= 33320 - 27670 = 5650$ people

www.ilmkidunya.com

www.ilmkidunya.com

such that the average value of $f(x) = 2 + 4x - 3x^2$ on the interval $[2, b]$ equals -7.

Solution:

Given that:

$$f(x) = 2 + 4x - 3x^2; \text{Interval} = [2, b], f_{avg} = -7$$

As we know

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

$$-7 = \frac{1}{b-2}\int_2^b (2 + 4x - 3x^2)dx$$

$$-7(b-2) = \int_2^b (2 + 4x - 3x^2)dx$$

$$-7b + 14 = \left[2x + 4\frac{x^2}{2} - 3\frac{x^3}{3}\right]_2^b$$

$$-7b + 14 = \left[2x + 2x^2 - x^3\right]_2^b$$

$$-7b + 14 = (2b + 2b^2 - b^3) - (2(2) + 2(2)^2 - 2^3)$$

$$-7b + 14 = 2b + 2b^2 - b^3 - 4$$

$$b^3 - 2b^2 - 2b + 4 - 7b + 14 = 0$$

$$b^3 - 2b^2 - 9b + 18 = 0$$

$$b^2(b-2) - 9(b-2) = 0$$

$$(b-2)(b^2 - 9) = 0$$

Either $b - 2 = 0$ or $b^2 - 9 = 0$

$b = 2$ or $b^2 = 9 \Rightarrow b = \pm 3$

Since b is on the interval $[2, b]$, so

$\boxed{b = 3}$

Q9. Let f be continuous, $\int_0^2 f(x)dx = 5$. If the average value of f on $[0, 4]$ is twice the average value on $[0, 2]$, find $\int_2^4 f(x)dx$.

Solution:

$$\int_0^2 f(x)dx = 5; \int_2^4 f(x)dx = ?$$

Given that:

$$f_{avg} \text{ on } [0, 4] = 2 \times (f_{avg} \text{ on } [0, 2])$$

Using: $f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$

$$\Rightarrow \frac{1}{4-0}\int_0^4 f(x)dx = 2 \times \frac{1}{2-0}\int_0^2 f(x)dx$$

$$\frac{1}{4}\int_0^4 f(x)dx = \frac{2}{2}(5) \qquad \because \int_0^2 f(x)dx = 5$$

$$\int_0^4 f(x)dx = 5 \times 4$$

$$\int_0^4 f(x)dx = 20$$

$$\Rightarrow \int_0^2 f(x)dx + \int_2^4 f(x)dx = 20$$

$$5 + \int_2^4 f(x)dx = 20$$

$$\int_2^4 f(x)dx = 20 - 5$$

$$\int_2^4 f(x)dx = \boxed{15}$$

Q10. Let f be continuous, and suppose: $\int_1^3 f(x)dx = 8$. If the average value of f on $[1, 4]$ is three times the average value on $[3, 4]$, find $\int_3^4 f(x)dx$.

Solution:

$$\int_1^3 f(x)dx = 8; \int_3^4 f(x)dx = ?$$

Given that:

$$f_{avg} \text{ on } [1, 4] = 3 \times (f_{avg} \text{ on } [3, 4])$$

Using: $f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$

$$\frac{1}{4-1}\int_1^4 f(x)dx = 3 \times \frac{1}{4-3}\int_3^4 f(x)dx$$

www.ilmkidunya.com

www.ilmkidunya.com

$$\frac{1}{3}\int_1^4 f(x)dx = 3\int_3^4 f(x)dx$$

$$\frac{1}{3}\left\{\int_1^3 f(x)dx + \int_3^4 f(x)dx\right\} = 3\int_3^4 f(x)dx$$

$$\frac{1}{3}\left\{8 + \int_3^4 f(x)dx\right\} = 3\int_3^4 f(x)dx \qquad \because \int_1^3 f(x)dx = 8$$

$$\frac{8}{3} + \frac{1}{3}\int_3^4 f(x)dx = 3\int_3^4 f(x)dx$$

$$\frac{8}{3} = 3\int_3^4 f(x)dx - \frac{1}{3}\int_3^4 f(x)dx$$

$$\frac{8}{3} = \left(3 - \frac{1}{3}\right)\int_3^4 f(x)dx$$

$$\frac{8}{3} = \frac{8}{3}\int_3^4 f(x)dx$$

$$\frac{8}{3} \times \frac{3}{8} = \int_3^4 f(x)dx$$

$$\boxed{\int_3^4 f(x)dx = 1}$$

Q11. Let f be continuous and suppose: $\int_0^2 f(x)dx = 10$ and $\int_2^4 f(x)dx = 6$. If the average value of f on $[0,3]$ equals the average value on $[3,4]$, find $\int_3^4 f(x)dx$.

Solution:

$$\int_0^2 f(x)dx = 10, \int_2^4 f(x)dx = 6, \int_3^4 f(x)dx = ?$$

Given that:

$$f_{avg} \text{ on } [0,3] = f_{avg} \text{ on } [3,4]$$

Using: $f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$

$$\Rightarrow \quad \frac{1}{3-0}\int_0^3 f(x)dx = \frac{1}{4-3}\int_3^4 f(x)dx$$

$$\frac{1}{3}\left\{\int_0^2 f(x)dx + \int_2^3 f(x)dx\right\} = \int_3^4 f(x)dx$$

$$\frac{1}{3}\left\{10 + \int_2^3 f(x)dx\right\} = \int_3^4 f(x)dx$$

$$\int_3^4 f(x)dx = \frac{10}{3} + \frac{1}{3}\int_2^3 f(x)dx \qquad \ldots(1)$$

As $$\int_2^4 f(x)dx = \int_2^3 f(x)dx + \int_3^4 f(x)dx$$

$$\Rightarrow \quad 6 - \int_3^4 f(x)dx = \int_2^3 f(x)dx \qquad \text{Put in eq. (1)}$$

$$\int_3^4 f(x)dx = \frac{10}{3} + \frac{1}{3}\left(6 - \int_3^4 f(x)dx\right)$$

$$\int_3^4 f(x)dx = \frac{10}{3} + 2 - \frac{1}{3}\int_3^4 f(x)dx$$

$$\int_3^4 f(x)dx + \frac{1}{3}\int_3^4 f(x)dx = \frac{10+6}{3}$$

$$\left(1 + \frac{1}{3}\right)\int_3^4 f(x)dx = \frac{16}{3}$$

$$\frac{4}{3}\int_3^4 f(x)dx = \frac{16}{3}$$

$$\int_3^4 f(x)dx = \frac{16}{3} \times \frac{3}{4}$$

$$\boxed{\int_3^4 f(x)dx = 4}$$

www.ilmkidunya.com

www.ilmkidunya.com

Q12. A patient receives an IV bolus dose of a drug. The concentration in the blood is modeled by:

$C(t) = 12e^{-0.3t}$ (mg/L) **where t is the time in hours since that dose was taken.**

The minimum effective concentration (MEC) is 3mg/L. The drug is considered effective if total *AUC* in a 24 -hour period is at least 150mg · h/L.

(a) Find the time T (in hours) when the concentration reaches the MEC.

(b) Calculate the AUC from $t = 0$ to $t = T$ for a single dose.

(c) Determine if one dose provides enough total AUC over 24 hours. If not, how many doses (given at intervals of T hours) are needed in 24 hours to meet the requirement?

Solution:

(a) $C(t) = 12e^{-0.3t} mg/L; MEC = 3 mg/L$

Since, the concentration reaches the *MEC*, therefore

$$C(t) = 3$$

$$\Rightarrow \quad 12e^{-0.3t} = 3$$

$$e^{-0.3t} = \frac{3}{12}$$

$$e^{-0.3t} = \frac{1}{4}$$

$$\Rightarrow \quad e^{0.3t} = 4$$

Taking $\ln$ on both sides

$$\ln(e^{0.3t}) = \ln 4$$

$$0.3t \cdot \ln e = \ln 4$$

$$0.3t = \ln 4 \qquad \because \ln e = 1$$

$$t = \frac{\ln 4}{0.3}$$

$$\boxed{t \approx 4.62 \text{ hours}}$$

(b) $$AUC_{0\to T} = \int_0^T C(t)dt$$

$$= \int_0^T 12e^{-0.3t}dt$$

$$AUC_{0\to T} = 12\left[\frac{e^{-0.3t}}{-0.3}\right]_0^T$$

$$= \frac{12}{0.3}\left[e^{-0.3t}\right]_0^T$$

$$= -40(e^{-0.3T} - e^0)$$

$$= -40\left(\frac{1}{4} - 1\right) \qquad \because e^{-0.3T} = \frac{1}{4}$$

$$= -40\left(-\frac{3}{4}\right)$$

$$AUC_{0\to T} = 30 mg \cdot h/L$$

(c) One dose provides $AUC = 30 mg \cdot h/L$

Required *AUC* over 24 hours $= 150 mg \cdot h/L$

Number of doses required $= \frac{150}{30} = 5$

Since doses are given at every $T = 4.62$ hours

$$\Rightarrow \quad 5T = 5(4.62) = 23.1 \text{ hours}$$

Hence, one dose is not enough and 5 doses are needed in 24 hours

www.ilmkidunya.com

www.ilmkidunya.com

Multiple Choice Questions (MCQs)

1. A function F is called an antiderivative of f on an interval I if:
A) $F'(x) = f(x)$ for all x in I B) $F(x) = f'(x)$ for all x in I C) $F(x) = 0$ for all x in I D) $F'(x) = 1$ for all x in I

2. The integral $\int \frac{1}{x} dx, x \neq 0$ is:
A) $\ln|x| + c$ B) $x \ln x - c$ C) $\frac{1}{x} + c$ D) $\ln x^2 + c$

3. The integral $\int \sec^2 x \, dx$ is:
A) $\sec x + c$ B) $\tan x + c$ C) $\cot x + c$ D) $-\tan x + c$

4. The integral $\int \csc x \cot x \, dx$ is:
A) $\csc x + c$ B) $-\csc x + c$ C) $\cot x + c$ D) $-\cot x + c$

5. The integral $\int \frac{1}{1+x^2} dx$ is:
A) $\sin^{-1} x + c$ B) $\tan^{-1} x + c$ C) $\sec^{-1} x + c$ D) $\cot^{-1} x + c$

6. If $u = g(x)$, then du equals:
A) $g(x)dx$ B) $g'(x)dx$ C) $g'(u)dx$ D) $\frac{dx}{g'(x)}$

7. For $\int (5x^5 + 5)^3 \cdot x^4 \, dx$, the substitution used is:
A) $u = x^4$ B) $u = 5x^5 + 5$ C) $u = (5x^5 + 5)^3$ D) $u = 5x^4$

8. The substitution $x = a \tan\theta$ is used for integrals involving:
A) $\sqrt{a^2 - x^2}$ B) $\sqrt{a^2 + x^2}$ C) $\sqrt{x^2 - a^2}$ D) $a^2 - x^2$

9. The integral $\int \frac{1}{\sqrt{x^2 + a^2}} dx$ is:
A) $\sin^{-1}\left(\frac{x}{a}\right) + c$ B) $\ln|x + \sqrt{x^2 + a^2}| + c$ C) $\tan^{-1}\left(\frac{x}{a}\right) + c$ D) $\sec^{-1}\left(\frac{x}{a}\right) + c$

10. If $f(x) \leq 0$ on [a, b], the area bounded by $y = f(x)$ and the x-axis is:
A) $\int_a^b f(x)dx$ B) $-\int_a^b f(x)dx$ C) 0 always D) $\int_a^b f'(x)dx$

11. The fundamental theorem of calculus states that if $F'(x) = f(x)$ on $[a, b]$, then $\int_a^b f(x)dx =$
A) $F(b) - F(a)$ B) $F(a) - F(b)$ C) $F(a) + F(b)$ D) $F'(b) - F'(a)$

12. The integral property $\int_a^a f(x)dx =$
A) 1 B) a C) 0 D) $f(a)$

13. If f is an even function, then $\int_{-a}^{a} f(x)dx =$
A) 0 B) $2\int_0^a f(x)dx$ C) $\frac{1}{2}\int_0^a f(x)dx$ D) $-2\int_0^a f(x)dx$

14. If f is an odd function, then $\int_{-a}^{a} f(x)dx =$
A) 0 B) $2\int_0^a f(x)dx$ C) always positive D) always negative

15. The area between two curves $f(x)$ and $g(x)$, with $f(x) \geq g(x)$ on $[a, b]$, is
A) $\int_a^b [f(x) + g(x)]dx$ B) $\int_a^b [f(x) - g(x)]dx$ C) $\int_a^b f(x).g(x)dx$ D) $\int_a^b [g(x) - f(x)]dx$

www.ilmkidunya.com

www.ilmkidunya.com

16. For a demand function $p = D(x)$, the equilibrium point (x_e, p_e) occurs where:
A) $D(x_e)=0$ B) $D(x_e)=S(x_e)$ C) $x_e=0$ D) $p_e=0$

17. For integration with respect to quantity, total surplus is:
A) $\int_0^{x_e}[D(x)-S(x)]dx$ B) $\int_0^{p_e}[D(x)+S(x)]dx$ C) $\int_0^{x_e}[D(x)+S(x)]dx$ D) $\int_0^{p_e}[D(x)-S(x)]dx$

18. If f is continuous and $f(x)\geq 0$ on $[a, b]$, the area under the curve is:
A) $\int_a^b f(x)dx$ B) $-\int_a^b f(x)dx$ C) $\int_b^a f(x)dx$ D) $f(a)+f(b)$

19. The volume generated by revolving $y = f(x), f(x) \geq 0$, about the x-axis is:
A) $\int_a^b f(x)dx$ B) $\pi\int_a^b[f(x)]^2dx$ C) $2\pi\int_a^b f(x)dx$ D) $\int_a^b[f(x)]^3dx$

20. The moment of inertia of a single particle of mass m at distance r from an axis is:
A) mr B) mr^2 C) $\frac{m}{r^2}$ D) $\frac{r^2}{m}$

21. For a particle moving on a line with position s(t), the velocity is:
A) $s(t)$ B) $s'(t)$ C) $\int s(t)dt$ D) $s''(t)$

22. The average value of a continuous function f on [a, b] is:
A) $(b-a)\int_a^b f(x)dx$ B) $\frac{1}{b-a}\int_a^b f(x)dx$ C) $\int_a^b f'(x)dx$ D) $f(a)+f(b)$

23. The integral $\int(3u^2+4u)du$ is
A) u^3+2u^2+c B) $3u^3+2u^2+c$ C) u^3+4u^2+c D) u^2+2u^3+c

24. For $\int \ln(\sin x)\cos x\,dx$, a suitable substitution is:
A) $u=\sin x$ B) $u=\cos x$ C) $u=\ln x$ D) $u=x\cos x$

25. The value of $\int_1^2(x^2+x)dx$ is:
A) $\frac{23}{6}$ B) $\frac{17}{3}$ C) $\frac{19}{3}$ D) 7

26. If $\int_0^3 f(x)dx=20$ and $\int_0^{10} f(x)dx=45$, then $\int_3^{10} f(x)dx=$
A) 25 B) 65 C) 15 D) 20

27. For $f(x) = 2x$ over $[-3, 3]$, the net signed area is:
A) -18 B) 0 C) 9 D) 18

28. For $f(x) = 2x$ over $[-3, 3]$, the total physical area is:
A) 0 B) 9 C) 18 D) 36

29. For demand $p = 30 - 2x - x^2$ and supply $p = -6 + 4x + x^2$, equilibrium occurs when:
A) $30-2x-x^2=-6+4x+x^2$ B) $30-2x-x^2=0$
C) $-6+4x+x^2=0$ only D) $x=0$

30. A uniform rectangular plate $ABCD$ has $AD = 3$, $AB = 8$, and mass per unit area 4. Its total mass is:
A) 11 B) 24 C) 32 D) 96

ANSWER KEY

1.	A	2.	A	3.	B	4.	B	5.	B	6.	B	7.	B	8.	B	9.	B	10.	B
11.	A	12.	C	13.	B	14.	A	15.	B	16.	B	17.	A	18.	A	19.	B	20.	B
21.	B	22.	B	23.	A	24.	A	25.	A	26.	A	27.	B	28.	C	29.	A	30.	C

www.ilmkidunya.com