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UNIT 02

Further Differentiation

INTRODUCTION

- Differentiation, a fundamental concept in calculus introduced by Newton and Leibniz, allows us to study how quantities change and evolve.
- In our previous class, we studied the definition of differentiation along with sum, difference, product, and quotient rules, and applied them to find derivatives of polynomials and rational functions.
- Building on that foundation, we will now explore how to find the derivatives of trigonometric, inverse trigonometric, exponential, and logarithmic functions.
- We will apply differentiation to identify increasing and decreasing functions and to find equations of tangents and normals to curves at specific points. Higher-order derivatives of algebraic, implicit, parametric, and transcendental functions will also be studied.
- We will use linear approximations and differentials to estimate function values, changes, and calculate relative and percentage errors in real-life contexts like volumes of geometric shapes.
- The unit also delves into finding local and global extrema using critical points.

Content Outline

- Review of Differentiation Basics
- Derivatives of Trigonometric Functions
- The Chain Rule
- Derivatives of Inverse Trigonometric Functions

REVIEW OF DIFFERENTIATION BASICS

Definition: The derivative of the function $y = f(x)$ is denoted by $\frac{dy}{dx}$ or $f'(x)$ and is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ if exists or } \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \text{ if exists.}$$

The derivative gives the instantaneous rate of change of the function at a point.

Recall!

- **Power Rule:** $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$ where $n \in \mathbb{R}$.
- **Constant Rule:** $\frac{d}{dx}(c) = 0$, for any $c \in \mathbb{R}$.

In the following rules $u = u(x), v = v(x)$ are differentiable functions and c is any arbitrary constant:

- **Constant Multiple Rule:** $\frac{d}{dx}(cu) = c\frac{du}{dx} = cu'$.
- **Sum and Difference Rule:** $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx} = u' \pm v'$
- **Product Rule:** This rule helps to differentiate the product of two functions.

$$\frac{d}{dx}(uv) = u'v + uv'$$

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- **Quotient Rule:** This rule is used to differentiate the division of two functions.

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{u'v-uv'}{v^2}$$

where $v(x)\neq 0$ for all x in the domain.

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Question 1: Prove by definition: $\frac{d}{dx}(\sin x)=\cos x$

Solution:

Let $f(x)=\sin x \Rightarrow f(x+h)=\sin(x+h)$

By definition of derivative:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

Formula: $\sin P-\sin Q=2\cos\frac{P+Q}{2}\cdot\sin\frac{P-Q}{2}$

$$=\lim_{h\to 0}\frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$=\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$=\cos\left(\frac{2x+0}{2}\right)\cdot(1) \qquad \because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

$$f'(x)=\cos x$$

Hence, $\frac{d}{dx}(\sin x)=\cos x$

Question 2: Prove by definition: $\frac{d}{dx}(\cos x)=-\sin x$

Solution:

Let $f(x)=\cos x \Rightarrow f(x+h)=\cos(x+h)$

By definition of derivative:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

Formula: $\cos P-\cos Q=-2\sin\frac{P+Q}{2}\cdot\sin\frac{P-Q}{2}$

$$=\lim_{h\to 0}\frac{-2\sin\left(\frac{x+h+x}{2}\right)\cdot\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{-2\sin\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)}{h}$$

$$=-\lim_{h\to 0}\sin\left(\frac{2x+h}{2}\right)\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$=-\sin\left(\frac{2x+0}{2}\right)\cdot(1) \qquad \because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

$$f'(x)=-\sin x$$

Hence, $\frac{d}{dx}(\cos x)=-\sin x$

Derivative of the Other Trigonometric Functions

After finding derivatives of $\sin x$ and $\cos x$ by definition, the derivatives of the remaining trigonometric functions can be found using product rule and quotient rule of differentiation.

Example 1: Find $f'(x)$ if $f(x)=\tan x$

Solution:

$$f(x)=\tan x=\frac{\sin x}{\cos x}$$

Differentiate w.r.t. 'x', we have

$$f'(x)=\left(\frac{\sin x}{\cos x}\right)'$$

$$=\frac{\cos x(\sin x)'-\sin x(\cos x)'}{(\cos x)^2} \qquad \text{By Quotient Rule}$$

$$=\frac{\cos x\cdot\cos x-\sin x(-\sin x)}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$

$$=\frac{1}{\cos^2 x} \qquad \because \cos^2 x+\sin^2 x=1$$

$$f'(x)=\sec^2 x$$

Hence, $\frac{d}{dx}(\tan x)=\sec^2 x$

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Example 2: Find $\frac{dy}{dx}$ if $y = x^2 \csc x$

Solution:

$y = x^2 \csc x$

Differentiating w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^2 \cdot \csc x)$$

$$\frac{dy}{dx} = x^2 \cdot (\csc x)' + \csc x (x^2)'$$

$$= x^2(-\csc x \cdot \cot x) + \csc x(2x)$$

$$= -x^2 \csc x \cdot \cot x + 2x \cdot \csc x$$

$$= x \csc x(-x \cot x + 2)$$

$$\frac{dy}{dx} = x \csc x(2 - x \cot x)$$

The Derivatives of Six Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

THE CHAIN RULE

Composition of Functions

Let $g: X \to Y$, and $f: Y \to Z$, then their composition $f \circ g: X \to Z$ is defined as:

$$f \circ g(x) = f(g(x))$$

For example, consider two real valued functions $g(x) = x^2 - x$ and $f(x) = \sin x$, then

$$f \circ g(x) = f(g(x)) = f(x^2 - x) = \sin(x^2 - x)$$

Theorem 1: (The Chain Rule)

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $f \circ g(x) = f(g(x))$ is differentiable at x, an $(fog)'(x) = f'(g(x)) \cdot g'(x)$

In Leibniz's notation,

if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Critical Thinking

Suppose $f(x) = x^2$ and $g(x) = |x|$. Then the composition $(fog)(x) = x^2$ is differentiable at $x = 0$, although $g(x)$ is not differentiable at $x = 0$. Does this contradict the Chain Rule?

Ans. No this does not contradict the chain rule.

Example 3: Differentiate $y = \sin(x^2 - x)$ w.r.t 'x'.

Solution:

$y = \sin(x^2 - x)$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}\sin(x^2 - x)$$

$$\frac{dy}{dx} = \cos(x^2 - x) \cdot \frac{d}{dx}(x^2 - x)$$

$$\frac{dy}{dx} = \cos(x^2 - x)(2x - 1)$$

$$\frac{dy}{dx} = (2x - 1)\cos(x^2 - x)$$

Derivative of a Function Given in the Form of Parametric Equations

Let a function be represented by parametric equations:

$$x = f(t) \text{ and } y = g(t)$$

From $x = f(t)$, we have $t = f^{-1}(x)$

Now, by Chain Rule applying to $y = g(t)$ and $t = f^{-1}(x)$, we get:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \left(\because \frac{dt}{dx} = \frac{1}{dx/dt}\right)$$

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Example 4: Find $\frac{dy}{dx}$ if $y = a\cos t$ and $x = a\sin t$.

Solution:

$y = a\cos t$

Differentiate w.r.t. 't'

$$\frac{d}{dt}(y) = a\frac{d}{dt}(\cos t)$$

$$\frac{dy}{dt} = -a\sin t$$

$y = a\sin t$

Differentiate w.r.t. 't'

$$\frac{d}{dt}(x) = a\frac{d}{dt}(\sin t)$$

$$\frac{dx}{dt} = a\cos t$$

Apply the Chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{-a\sin t}{a\cos t} = -\frac{\sin t}{\cos t} = -\tan t$$

Implicit Differentiation

When an implicit relation cannot be easily solved for y (or when multiple explicit functions are involved), we can still compute $\frac{dy}{dx}$ using implicit differentiation. The steps are as follows:

1. Differentiate both sides of $F(x, y) = 0$ with respect to x, treating y as an implicit function of x.
2. Apply the Chain Rule to terms containing y.
3. Solve the resulting equation for $\frac{dy}{dx}$.

Example 5: Find $\frac{dy}{dx}$ if $x + y^2 + \sin(xy^2) = 0$

Solution:

Differentiate w.r.t. 'x'

$$x + y^2 + \sin(xy^2) = 0$$

$$\frac{d}{dx}(x + y^2 + \sin(xy^2)) = 0$$

$$\frac{d}{dx}(x) + \frac{d}{dx}(y^2) + \frac{d}{dx}(\sin(xy^2)) = 0$$

$$1 + 2y\frac{dy}{dx} + \cos(xy^2)\cdot\frac{d}{dx}(xy^2) = 0$$

$$1 + 2y\frac{dy}{dx} + \cos(xy^2)\left\{x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x)\right\} = 0$$

$$1 + 2y\frac{dy}{dx} + \cos(xy^2)\left\{x\cdot 2y\frac{dy}{dx} + y^2(1)\right\} = 0$$

$$1 + 2y\frac{dy}{dx} + 2xy\cos(xy^2)\frac{dy}{dx} + y^2\cos(xy^2) = 0$$

$$2y\frac{dy}{dx}\{1 + x\cos(xy^2)\} = -1 - y^2\cos(xy^2)$$

$$\frac{dy}{dx} = -\frac{1 + y^2\cos(xy^2)}{2y\{1 + x\cos(xy^2)\}}$$

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

(i) **Prove that: $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$**

Proof:

Let $y = \sin^{-1}(x)$, where $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\sin y = x \quad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{\cos^2 y}}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} \qquad \because \sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \text{Using (1)}$$

Hence, $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$

(ii) $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$, **for $x \in (-1, 1)$**

Proof:

Let $y = \cos^{-1}x$, where $x \in [-1, 1]$, $y \in [0, \pi]$

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$$\cot y = x \quad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$= -\frac{1}{\sqrt{\sin^2 y}}$$

$$= -\frac{1}{\sqrt{1-\cos^2 y}} \quad \because \sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2\theta$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad \text{Using (1)}$$

Hence, $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, **for** $x \in \mathbb{R}$

Proof:

Let $y = \tan^{-1} x$, where $x \in \mathbb{R}$, $y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\tan y = x \quad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} \quad \because \sec y \neq 0, \text{for } y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$= \frac{1}{1+\tan^2 y} \quad \because 1 + \tan^2 y = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} \quad \text{Using (1)}$$

Hence, $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(\cot^{-1} x) = \frac{1}{1+x^2}$, **for** $x \in \mathbb{R}$

Proof:

Let $y = \cot^{-1} x$, where $x \in \mathbb{R}$, $y \in (0, \pi)$

$$\cot y = x \quad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(\cot y) = \frac{d}{dx}(x)$$

$$-\csc^2 y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\csc^2 y} \quad \because \csc y \neq 0, \text{for } y \in (0, \pi)$$

$$= -\frac{1}{1+\cot^2 y} \quad \because 1 + \cot^2 y = \csc^2 y$$

$$\frac{dy}{dx} = -\frac{1}{1+x^2} \quad \text{Using (1)}$$

Hence, $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(v) **Prove that:** $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, **for** $x \in (-\infty, -1) \cup (1, \infty)$

Proof:

Let $y = \sec^{-1} x$, where $x \in (-\infty, -1) \cup (1, \infty)$, $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$$\sec y = x \quad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}(x)$$

$$\sec y \cdot \tan y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \cdot \tan y},$$

where $\sec y \neq 0$ and $\tan y \neq 0$

As $\tan y \neq 0 \Rightarrow y \neq 0, \pi \Rightarrow$ From eq. (1), $x \neq 1, -1$

$$\frac{dy}{dx} = \frac{1}{x \tan y} \quad \ldots(2) \because \sec y = x$$

Now, $$\tan y = \begin{cases} \sqrt{\tan^2 y}, y \in \left(0, \frac{\pi}{2}\right) \\ -\sqrt{\tan^2 y}, y \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

$$= \begin{cases} \sqrt{\sec^2 y - 1}, y \in \left(0, \frac{\pi}{2}\right) \\ -\sqrt{\sec^2 y - 1}, y \in \left(\frac{\pi}{2}, 0\right) \end{cases}$$

$$\tan y = \begin{cases} \sqrt{x^2-1}, x \in (1, \infty) \\ -\sqrt{x^2-1}, x \in (-\infty, -1) \end{cases} \quad \because \sec y = x$$

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Put the value of $\tan y$ in eq. (2), we get

$$\frac{dy}{dx} = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & x \in (1,\infty) \\ \frac{1}{-x\sqrt{x^2-1}}, & x \in (-\infty,-1) \end{cases}$$

By definition: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Hence, $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$,

where $x \in (-\infty,-1) \cup (1,\infty)$

(vi) Prove that: $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$,

for $x \in (-\infty,-1) \cup (1,\infty)$

Proof:

Let $y = \csc^{-1} x$, where $x \in (-\infty,-1) \cup (1,\infty)$ and $y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] - \{0\}$

$$\csc y = x \quad \ldots(1)$$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\csc y) = \frac{d}{dx}(x)$$

$$-\csc y \cdot \cot y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\csc y \cdot \cot y},$$

where $\csc y \neq 0$ and $\cot y \neq 0$

As $\cot y \neq 0 \Rightarrow y \neq -\frac{\pi}{2}, \frac{\pi}{2} \Rightarrow$ From eq. (1) $x \neq 1, -1$

$$\frac{dy}{dx} = -\frac{1}{x \cdot \cot y} \quad \ldots(2) \quad \because \csc y = x$$

Now, $$\cot y = \begin{cases} \sqrt{\cot^2 y}, y \in \left(0,\frac{\pi}{2}\right) \\ -\sqrt{\cot^2 y}, y \in \left(-\frac{\pi}{2},0\right) \end{cases}$$

$$= \begin{cases} \sqrt{\csc^2 y - 1}, y \in \left(0,\frac{\pi}{2}\right) \\ -\sqrt{\csc^2 y - 1}, y \in \left(-\frac{\pi}{2},0\right) \end{cases}$$

$$\cot y = \begin{cases} \sqrt{x^2-1}, x \in (1,\infty) \\ -\sqrt{x^2-1}, x \in (-\infty,-1) \end{cases}$$

Put the value of $\cot y$ in eq. (2), we get

$$\frac{dy}{dx} = \begin{cases} \frac{-1}{x\sqrt{x^2-1}}, x \in (1,\infty) \\ \frac{-1}{-x\sqrt{x^2-1}}, x \in (-\infty,-1) \end{cases}$$

By definition: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Hence, $\frac{d}{dx}(\csc^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$,

where $x \in (-\infty,-1) \cup (1,\infty)$

Example 6: Find $\frac{dy}{dx}$ **if** $y = x\cos^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2}, a > 0.$

Solution:

$$y = x\cos^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2}$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}\left\{x \cdot \cos^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2}\right\}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left\{x \cdot \cos^{-1}\left(\frac{x}{a}\right)\right\} - \frac{d}{dx}(a^2-x^2)^{\frac{1}{2}}$$

$$= x \cdot \frac{d}{dx}\left\{\cos^{-1}\left(\frac{x}{a}\right)\right\} + \cos^{-1}\left(\frac{x}{a}\right) \cdot \frac{d}{dx}(x) - \frac{1}{2}(a^2-x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx}(a^2-x^2)$$

$$= x \cdot \left\{\frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)\right\} + \cos^{-1}\left(\frac{x}{a}\right) \cdot (1) - \frac{1}{2\sqrt{a^2-x^2}}(-2x)$$

Recall!

$$\sqrt{x^2} = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

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$$\frac{dy}{dx} = \frac{-x}{\sqrt{\frac{a^2-x^2}{a^2}}}\cdot\frac{1}{a} + \cos^{-1}\left(\frac{x}{a}\right) + \frac{x}{\sqrt{a^2-x^2}}$$

$$= \frac{-ax}{\sqrt{a^2-x^2}}\cdot\frac{1}{a} + \cos^{-1}\left(\frac{x}{a}\right) + \frac{x}{\sqrt{a^2-x^2}}$$

$$= \frac{-x}{\sqrt{a^2-x^2}} + \cos^{-1}\left(\frac{x}{a}\right) + \frac{x}{\sqrt{a^2-x^2}}$$

$$\frac{dy}{dx} = \cos^{-1}\left(\frac{x}{a}\right)$$

EXERCISE 2.1

Q1. Derive the formula for the derivative of the following functions w.r.t. x:

(a) sec x

Solution:

Let $y = \sec x$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

By Quotient Rule:

$$\frac{dy}{dx} = \frac{\cos x\cdot\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{\cos x(0) - (-\sin x)}{\cos^2 x}$$

$$= \frac{0+\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x\cdot\cos x}$$

$$\frac{dy}{dx} = \sec x\cdot\tan x$$

Hence, $\frac{d}{dx}(\sec x) = \sec x\cdot\tan x$

(b) csc x

Solution:

Let $y = \csc x$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

By Quotient Rule:

$$\frac{dy}{dx} = \frac{\sin x\cdot\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$\frac{dy}{dx} = \frac{\sin x(0) - \cos x}{\sin^2 x}$$

$$= \frac{0-\cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin x\cdot\sin x}$$

$$\frac{dy}{dx} = -\csc x\cdot\cot x$$

Hence, $\frac{d}{dx}(\csc x) = -\csc x\cdot\cot x$

(c) cot x

Solution:

Let $y = \cot x = \frac{\cos x}{\sin x}$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

By Quotient Rule:

$$\frac{dy}{dx} = \frac{\sin x\frac{d}{dx}(\cos x) - \cos x\cdot\frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{1}{\sin^2 x} \qquad \because \sin^2 x + \cos^2 x = 1$$

$$\frac{dy}{dx} = -\csc^2 x$$

Hence, $\frac{d}{dx}(\cot x) = -\csc^2 x$

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In 2-6, differentiate with respect to the variable involved.

Q2. $y = x^2 \cos x$

Solution:

$$y = x^2 \cos x$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^2 \cdot \cos x)$$

$$\frac{dy}{dx} = x^2 \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = x^2(-\sin x) + \cos x(2x)$$

$$= -x^2 \sin x + 2x \cos x$$

$$\frac{dy}{dx} = x(2\cos x - x \sin x)$$

Q3. $y = x \sec x + \frac{1}{x}$

Solution:

$$y = x \sec x + \frac{1}{x}$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(x \sec x + \frac{1}{x}\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot \sec x) + \frac{d}{dx}(x^{-1})$$

$$= x \cdot \frac{d}{dx}(\sec x) + \sec x \cdot \frac{d}{dx}(x) + (-1x^{-2})$$

$$\frac{dy}{dx} = x(\sec x \cdot \tan x) + \sec x(1) - \frac{1}{x^2}$$

$$= x \sec x \tan x + \sec x - \frac{1}{x^2}$$

$$\frac{dy}{dx} = \sec x(x \tan x + 1) - \frac{1}{x^2}$$

Q4. $y = \frac{\sin t}{1 - \cos t}$

Solution:

$$y = \frac{\sin t}{1 - \cos t}$$

Differentiate w.r.t. 't'

$$\frac{d}{dx}(y) = \frac{d}{dt}\left(\frac{\sin t}{1 - \cos t}\right)$$

$$\frac{dy}{dx} = \frac{(1 - \cos t)\frac{d}{dt}(\sin t) - \sin t \frac{d}{dt}(1 - \cos t)}{(1 - \cos t)^2}$$

$$= \frac{(1 - \cos t)(\cos t) - \sin t(0 + \sin t)}{(1 - \cos t)^2}$$

$$= \frac{\cos t - \cos^2 t - \sin^2 t}{(1 - \cos t)^2}$$

$$= \frac{\cos t - (\cos^2 t + \sin^2 t)}{(1 - \cos t)^2}$$

$$= \frac{\cos t - 1}{(1 - \cos t)^2} \qquad \because \cos^2 t + \sin^2 t = 1$$

$$= \frac{-(1 - \cos t)}{(1 - \cos t)^2} = -\frac{1}{1 - \cos t}$$

$$\frac{dy}{dt} = \frac{1}{\cos t - 1}$$

Q5. $y = (1 + \csc t)\sin t$

Solution:

$$y = (1 + \csc t)\sin t$$

$$= \sin t + \csc t \cdot \sin t$$

$$= \sin t + \frac{1}{\sin t} \cdot \sin t$$

$$y = \sin t + 1$$

Differentiate w.r.t. 't'

$$\frac{d}{dt}(y) = \frac{d}{dt}(\sin t + 1)$$

$$\frac{dy}{dt} = \cos t + 0 = \cos t$$

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Q6. $y = \sec x + 4\sqrt{x} - 10$

Solution:

$$y = \sec x + 4\sqrt{x} - 10$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(\sec x + 4\sqrt{x} - 10)$$

$$\frac{dy}{dx} = \frac{d}{dx}(\sec x) + 4\frac{d}{dx}(x^{\frac{1}{2}}) - \frac{d}{dx}(10)$$

$$= \sec x \cdot \tan x + 4\left(\frac{1}{2}x^{\frac{-1}{2}}\right) - 0$$

$$\frac{dy}{dx} = \sec x \cdot \tan x + \frac{2}{\sqrt{x}}$$

In 7-8, find $\frac{dy}{dx}$ by making suitable substitutions and applying the Chain Rule.

Q7. $y = \sqrt{\frac{1+x}{1-x}}$

Solution:

$$y = \sqrt{\frac{1+x}{1-x}}$$

Let $y = \sqrt{u}$, where $u = \frac{1+x}{1-x}$

Differentiate w.r.t. 'u'

$$\frac{d}{dx}(y) = \frac{d}{du}(u^{\frac{1}{2}})$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{-1}{2}} = \frac{1}{2\sqrt{u}}$$

Now, $u = \frac{1+x}{1-x}$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(u) = \frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$\frac{du}{dx} = \frac{(1-x)\frac{d}{dx}(1+x) - (1+x)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$= \frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2}$$

$$\frac{du}{dx} = \frac{1-x+1+x}{(1-x)^2} = \frac{2}{(1-x)^2}$$

Apply the Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}}\cdot\frac{2}{(1-x)^2}$$

$$= \frac{1}{2\sqrt{\frac{1+x}{1-x}}}\cdot\frac{2}{(1-x)^2} \qquad \because u = \frac{1+x}{1-x}$$

$$= \frac{\sqrt{1-x}}{\sqrt{1+x}}\cdot\frac{1}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x}(1-x)^{\frac{3}{2}}}$$

Q8. $y = (\sin x + \cos x)^{3/2}$

Solution:

Let $y = u^{3/2}$, where $u = \sin x + \cos x$

Differentiate w.r.t. 'u'

$$\frac{d}{du}(y) = \frac{d}{du}(u)^{\frac{3}{2}}$$

$$\frac{dy}{du} = \frac{3}{2}u^{\frac{3}{2}-1}$$

$$\frac{dy}{du} = \frac{3}{2}u^{\frac{1}{2}}$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(u) = \frac{d}{dx}(\sin x + \cos x)$$

$$\frac{du}{dx} = \cos x - \sin x$$

Apply the Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{3}{2}u^{\frac{1}{2}}\cdot(\cos x - \sin x)$$

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{(\sin x + \cos x)}\cdot(\cos x - \sin x)$$

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In 9-12, differentiate w.r.t the variable involved.

Q9. $y = \sec(\tan x)$

Solution:

$$y = \sec(\tan x)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(\sec(\tan x))$$

$$= \sec(\tan x)\cdot\tan(\tan x)\cdot\frac{d}{dx}(\tan x)$$

$$\frac{dy}{dx} = \sec(\tan x)\cdot\tan(\tan x)\cdot\sec^2 x$$

Q10. $y = \sqrt{\sin x} + \cos\sqrt{x}$

Solution:

$$y = \sqrt{\sin x} + \cos\sqrt{x}$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(\sqrt{\sin x}) + \frac{d}{dx}(\cos\sqrt{x})$$

$$\frac{dy}{dx} = \frac{1}{2}(\sin x)^{\frac{-1}{2}}\cdot\frac{d}{dx}(\sin x) - \sin\sqrt{x}\cdot\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{\sin x}}\cdot\cos x - \sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\cdot 1$$

$$= \frac{\cos x}{2\sqrt{\sin x}} - \frac{\sin\sqrt{x}}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x}\cos x - \sqrt{\sin x}\cdot\sin\sqrt{x}}{2\sqrt{x}\cdot\sqrt{\sin x}}$$

Q11. $y = \tan^2 x + \cot(x^2)$

Solution:

$$y = \tan^2 x + \cot(x^2)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(\tan x)^2 + \frac{d}{dx}(\cot(x^2))$$

$$\frac{dy}{dx} = 2(\tan x)\cdot\frac{d}{dx}(\tan x) - \csc^2(x^2)\frac{d}{dx}(x^2)$$

$$= 2\tan x\cdot\sec^2 x - \csc^2(x^2)\cdot 2x$$

$$= 2\tan x\cdot\sec^2 x - 2x\cdot\csc^2(x^2)$$

$$\frac{dy}{dx} = 2(\tan x\cdot\sec^2 x - x\csc^2(x^2))$$

Q12. $y = \sin^3(\cos 2v)$

Solution:

$$y = \sin^3(\cos 2v)$$

Differentiate w.r.t. 'v'

$$\frac{d}{dv}(y) = \frac{d}{dv}(\sin^3(\cos 2v))$$

$$\frac{dy}{dv} = 3\sin^2(\cos 2v)\cdot\frac{d}{dv}(\sin(\cos 2v))$$

$$= 3\sin^2(\cos 2v)\cdot\cos(\cos 2v)\cdot\frac{d}{dv}(\cos 2v)$$

$$= 3\sin^2(\cos 2v)\cdot\cos(\cos 2v)\cdot(-\sin 2v)(2)$$

$$\frac{dy}{dv} = -6\sin 2v\cdot\sin^2(\cos 2v)\cdot\cos(\cos 2v)$$

Q13. Find $\frac{dy}{dx}$ for the following parametric equations:

(i) $x = \sin^2 t$ **and** $y = \cos 2t$

Solution:

$$x = \sin^2 t$$

Differentiate w.r.t. 't'

$$\frac{dx}{dt} = 2\sin t\cdot\frac{d}{dt}(\sin t) = 2\sin t\cos t = \sin 2t$$

Now, $y = \cos 2t$

Differentiate w.r.t. 't'

$$\frac{dy}{dt} = -\sin 2t\cdot(2\cdot 1)$$

$$= -2\sin 2t$$

Apply the Chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= (-2\sin 2t)\cdot\frac{1}{\sin 2t}$$

$$\frac{dy}{dx} = -2$$

(ii) $x = \sqrt{1+\sec t}$, **and** $y = \sqrt{\tan t}$

Solution:

$$x = \sqrt{1+\sec t}$$

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Differentiate w.r.t. 't'

$$\frac{dx}{dt} = \frac{1}{2}(1+\sec t)^{\frac{-1}{2}} \cdot \frac{d}{dt}(1+\sec t)$$

$$= \frac{1}{2\sqrt{1+\sec t}} \cdot (0+\sec t \cdot \tan t)$$

$$\frac{dx}{dt} = \frac{\sec t \cdot \tan t}{2\sqrt{1+\sec t}}$$

Now, $y = \sqrt{\tan t}$

Differentiate w.r.t. 't'

$$\frac{dy}{dt} = \frac{1}{2}(\tan t)^{\frac{-1}{2}} \cdot \frac{d}{dt}(\tan t)$$

$$= \frac{1}{2\sqrt{\tan t}} \cdot \sec^2 t$$

$$\frac{dy}{dt} = \frac{\sec^2 t}{2\sqrt{\tan t}}$$

Apply the Chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{\sec^2 t}{2\sqrt{\tan t}} \cdot \frac{2\sqrt{1+\sec t}}{\sec t \cdot \tan t}$$

$$\frac{dy}{dx} = \frac{\sec t\sqrt{1+\sec t}}{\tan^{3/2} t}$$

Q14. If $x = a\sec t$ and $y = b\tan t$, then show that $\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$.

Solution:

$$x = a\sec t$$

Differentiate w.r.t. 't'

$$\frac{dx}{dt} = a\frac{d}{dt}(\sec t) = a\sec t \cdot \tan t$$

Now, $y = b\tan t$

Differentiate w.r.t. 't'

$$\frac{dy}{dt} = b\frac{d}{dt}(\tan t) = b\sec^2 t$$

Apply the Chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= (b\sec^2 t)\left(\frac{1}{a\sec t \cdot \tan t}\right) = \frac{b\sec t}{a\tan t}$$

Put $x = a\sec t \Rightarrow \sec t = \frac{x}{a}$ and $y = b\tan t \Rightarrow \tan t = \frac{y}{b}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{b\left(\frac{x}{a}\right)}{a\left(\frac{y}{b}\right)} = \frac{bx}{a} \cdot \frac{b}{ay}$$

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y} \text{ (Proved)}$$

Q15. Use implicit differentiation to find $\frac{dy}{dx}$ if:

(i) $ax^2 + by^2 + 2gx + 2fy + c = 0$

Solution:

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(ax^2 + by^2 + 2gx + 2fy + c) = 0$$

$$a\frac{d}{dx}(x^2) + b\frac{d}{dx}(y^2) + 2g\frac{d}{dx}(x) + 2f\frac{d}{dx}(y) + \frac{d}{dx}(c) = 0$$

$$a(2x) + b\left(2y\frac{dy}{dx}\right) + 2g(1) + 2f\frac{dy}{dx} + 0 = 0$$

$$\frac{dy}{dx}(2by + 2f) = -2ax - 2g$$

$$\frac{dy}{dx} = \frac{-2(ax+g)}{2(by+f)}$$

$$\frac{dy}{dx} = -\frac{ax+g}{by+f}$$

(ii) $x + \tan(xy) = 0$

Solution:

$$x + \tan(xy) = 0$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(x + \tan(xy)) = 0$$

$$\frac{d}{dx}(x) + \frac{d}{dx}(\tan(xy)) = 0$$

$$1 + \sec^2(xy) \cdot \frac{d}{dx}(x \cdot y) = 0$$

$$1 + \sec^2(xy)\left(x\frac{dy}{dx} + y \cdot 1\right) = 0$$

$$1 + x\sec^2(xy)\frac{dy}{dx} + y\sec^2(xy) = 0$$

$$x\sec^2(xy)\frac{dy}{dx} = -1 - y\sec^2(xy)$$

$$\frac{dy}{dx} = -\frac{1 + y\sec^2(xy)}{x\sec^2(xy)}$$

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Q16. Differentiate w.r.t. the variables involved.

(i) $\sin^{-1}\left(\frac{x}{a}\right)$

Solution:

Let $y = \sin^{-1}\left(\frac{x}{a}\right)$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\sin^{-1}\left(\frac{x}{a}\right)\right\}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a}$$

$$= \frac{\sqrt{a^2}}{\sqrt{a^2-x^2}} \cdot \frac{1}{a}$$

$$= \frac{a}{\sqrt{a^2-x}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}$$

(ii) $\csc^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Solution:

Let $y = \csc^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\csc^{-1}\left(\frac{x^2+1}{x^2-1}\right)\right\}$$

$$= \frac{1}{\left|\frac{x^2+1}{x^2-1}\right|\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}} \cdot \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) = \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2-1)^2}}} \cdot \frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(x^2-1)}{(x^2+1)\sqrt{4(x^2)(1)}} \cdot \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} \qquad \because (a+b)^2-(a-b)^2 = 4ab$$

$$= \frac{1}{(x^2+1)\cdot 2|x|} \cdot (-4x) = \frac{2}{x^2+1} \cdot \frac{x}{|x|}$$

$$\frac{dy}{dx} = \frac{2x}{(x^2+1)|x|}$$

Point to Note

$\frac{x^2+1}{x^2-1} < -1 \; x \in (-\infty,-1)\cup(1,\infty)$

$\left|\frac{x^2+1}{x^2-1}\right| = \frac{x^2+1}{x^2-1} > 1$ for , and has no solution.

(iii) $\cot^{-1}(\sqrt{x}) + \sqrt{\cot^{-1}x}$

Solution:

Let $y = \cot^{-1}\sqrt{x} + \sqrt{\cot^{-1}x}$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(\cot^{-1}\sqrt{x} + \sqrt{\cot^{-1}x})$$

$$= \frac{d}{dx}(\cot^{-1}\sqrt{x}) + \frac{d}{dx}(\sqrt{\cot^{-1}x})$$

$$= \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x}) + \frac{1}{2}(\cot^{-1}x)^{\frac{-1}{2}} \cdot \frac{d}{dx}(\cot^{-1}x)$$

$$= \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{\cot^{-1}x}} \cdot \left(-\frac{1}{1+x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}(1+x)} - \frac{1}{2(1+x^2)\sqrt{\cot^{-1}x}}$$

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(iv) $(\tan t)^{-1} + \dfrac{1}{\tan^{-1} t}$

Solution:

Let $y = (\tan t)^{-1} + \dfrac{1}{\tan^{-1} t}$

$$y = \frac{1}{\tan t} + \frac{1}{\tan^{-1} t}$$

$$y = \cot t + \frac{1}{\tan^{-1} t}$$

Differentiate w.r.t. 't'

$$\frac{dy}{dt} = \frac{d}{dt}(\cot t) + \frac{d}{dt}(\tan^{-1} t)^{-1}$$

$$= -\csc^2 t - 1(\tan^{-1} t)^{-2} \cdot \frac{d}{dt}(\tan^{-1} t)$$

$$= -\csc^2 t - \frac{1}{(\tan^{-1} t)^2} \cdot \frac{1}{1+t^2}$$

$$\frac{dy}{dt} = -\frac{\csc^2 t(\tan^{-1} t)^2(1+t^2)+1}{(\tan^{-1} t)^2(1+t^2)}$$

Content Outline

- Derivative of Exponential Functions
- Derivative of Logarithmic Functions

DERIVATIVE OF EXPONENTIAL FUNCTIONS

Definition: A function defined by

$$f(x) = a^x, \text{ where } a > 0 \text{ and } a \neq 1$$

is called an exponential function.

- If $a = e \approx 2.71828$ then $f(x) = e^x$, is called **natural exponential function.**

Recall!

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \ell n\, a$$

Question 1: Prove that: $\dfrac{d}{dx}(a^x) = a^x \cdot \ell n\, a$, **where** $a > 0, a \neq 1$ **and** $x \in R$.

Proof:

Let $f(x) = a^x \Rightarrow f(x+h) = a^{x+h}$

By definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

$$= a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$$

$$f'(x) = a^x \cdot \ell n\, a \qquad \because \lim_{h \to 0} \frac{a^h - 1}{h} = \ell n\, a$$

Hence, $\dfrac{d}{dx}(a^x) = a^x \cdot \ell n\, a$

Challenge!

Prove that: $\dfrac{d}{dx}(e^x) = e^x$

Proof:

Let $f(x) = e^x \Rightarrow f(x+h) = e^{x+h}$

By definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x \cdot \ell n\, e \qquad \because \lim_{h \to 0} \frac{e^h - 1}{h} = \ell n\, e$$

$$f'(x) = e^x \cdot (1) \qquad \because \ell n\, e = 1$$

Hence, $\dfrac{d}{dx}(e^x) = e^x$.

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Example 7: Find $\frac{dy}{dx}$ if (i) $y = e^{\sin^2 x}$ (ii) a^{x^3}

(i) $y = e^{\sin^2 x}$

Solution:

$y = e^{\sin^2 x}$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin^2 x})$$

$$= e^{\sin^2 x} \cdot \frac{d}{dx}(\sin x)^2 = e^{\sin^2 x} \cdot 2\sin x \cdot \frac{d}{dx}(\sin x)$$

$$= e^{\sin^2 x} \cdot 2\sin x \cdot \cos x$$

$$\frac{dy}{dx} = 2\sin x \cos x \cdot e^{\sin^2 x} \text{ or } \sin 2x \cdot e^{\sin^2 x}$$

(ii) a^{x^3}

Solution:

Let $y = a^{x^3}$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(a^{x^3})$$

$$= a^{x^3} \cdot \ln a \cdot \frac{d}{dx}(x^3) = a^{x^3} \cdot \ln a(3x^2 \cdot 1)$$

$$\frac{dy}{dx} = 3x^2 \cdot a^{x^3} \ln a$$

DERIVATIVE OF LOGARITHMIC FUNCTIONS

Definition: If $a^y = x$, where $a > 0$ and $a \neq 1$, then y is called the logarithm of x to the base a, written as:

$$y = \log_a x, x > 0, y \in \mathbb{R}$$

- If $a = e$, where $e \approx 2.71828$, then the function:

$y = \log_e x = \ln x$ (whenever $e^y = x$) is called the **natural logarithm** of x.

Recall!

$$\lim_{h \to 0}\left(1 + \frac{h}{a}\right)^{\frac{e}{h}} = e$$

Question 2: Prove that: $\frac{d}{dx}(\log_a x) = \frac{1}{x \cdot \ln a}$

Proof:

Let $f(x) = \log_a x \Rightarrow f(x+h) = \log_a(x+h)$

By definition of derivative:

$$f'(x) = \lim_{h \to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0}\frac{\log_a(x+h) - \log_a x}{h}$$

$$= \lim_{h \to 0}\frac{1}{h}\log_a\left(\frac{x+h}{x}\right) \quad \because \log m - \log n = \log\left(\frac{m}{n}\right)$$

Multiply and divide by 'x'

$$= \lim_{h \to 0}\frac{1}{x} \cdot \frac{x}{h}\log_a\left(1 + \frac{h}{x}\right)$$

$$f'(x) = \frac{1}{x}\lim_{h \to 0}\log_a\left(1 + \frac{h}{x}\right)^{\frac{x}{h}} \quad \because \log(m^n) = n\log(m)$$

$$= \frac{1}{x} \cdot \log_a e \quad \because \lim_{h \to 0}\left(1 + \frac{h}{x}\right)^{\frac{x}{h}} = e$$

$$= \frac{1}{x \cdot \log_e a} \quad \because \log_a b = \frac{1}{\log_b a}$$

$$f'(x) = \frac{1}{x \ln a} \quad \because \log_e x = \ln x$$

Hence, $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$.

Example 8: Find $\frac{dy}{dx}$ if $y = \frac{a^x}{\log_a x}$.

Solution:

$$y = \frac{a^x}{\log_a x}$$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{a^x}{\log_a x}\right)$$

By Quotient Rule:

$$\frac{dy}{dx} = \frac{(\log_a x)\frac{d}{dx}(a^x) - a^x \cdot \frac{d}{dx}(\log_a x)}{(\log_a x)^2}$$

$$= \frac{(\log_a x)a^x \cdot \ln a - a^x \cdot \frac{1}{x \cdot \ln a}}{(\log_a x)^2}$$

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$$\frac{dy}{dx} = \frac{a^x\left\{(\ln a)\log_a x - \frac{1}{x\ln a}\right\}}{(\log_a x)^2} = \frac{a^x\left\{\frac{x(\ln a)^2\log_a x - 1}{x\ln a}\right\}}{(\log_a x)^2} = \frac{a^x(x(\ln a)^2\log_a x - 1)}{x\ln a(\log_a x)^2}$$

Example 9: Find $\frac{dy}{dx}$, if $y = \ln(\cos(e^x))$.

Solution:

$$y = \ln(\cos(e^x))$$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(\ln(\cos(e^x)))$$

$$= \frac{1}{\cos(e^x)}\cdot\frac{d}{dx}(\cos(e^x))$$

$$= \frac{1}{\cos(e^x)}\cdot(-\sin(e^x))\frac{d}{dx}(e^x)$$

$$= \frac{-\sin(e^x)}{\cos(e^x)}\cdot e^x$$

$$\frac{dy}{dx} = -e^x\cdot\tan(e^x)$$

Challenge!

Prove that:

Proof:

Let $f(x) = \ln x \Rightarrow f(x+h) = \ln(x+h)$

By definition of derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) \qquad \because \ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

Multiply and divide by 'x'

$$= \lim_{h\to 0}\frac{1}{x}\cdot\frac{x}{h}\ln\left(1+\frac{h}{x}\right)$$

$$= \frac{1}{x}\lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{x}{h}} \qquad \because \ln(a^b) = b\ln(a)$$

$$= \frac{1}{x}\ln e \qquad \because \lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}} = e$$

$$f'(x) = \frac{1}{x}(1) \qquad \because \ln e = 1$$

Hence, $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

EXERCISE 2.2

Q1. Find $f'(x)$ if:

(i) $f(x) = (x^3+4x)e^{\frac{1}{x}}$

Solution:

$$f(x) = (x^3+4x)\cdot e^{\frac{1}{x}}$$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}\{(x^3+4x)\cdot e^{\frac{1}{x}}\}$$

$$= (x^3+4x)\cdot\frac{d}{dx}(e^{\frac{1}{x}}) + e^{\frac{1}{x}}\cdot\frac{d}{dx}(x^3+4x)$$

$$= (x^3+4x)e^{\frac{1}{x}}\cdot\frac{d}{dx}(x^{-1}) + e^{\frac{1}{x}}(3x^2+4(1))$$

$$= e^{\frac{1}{x}}\{x(x^2+4)(-1x^{-2}) + (3x^2+4)\}$$

$$= e^{\frac{1}{x}}\left\{-x(x^2+4)\cdot\frac{1}{x^2} + (3x^2+4)\right\}$$

$$= e^{\frac{1}{x}}\left\{-\frac{1}{x}(x^2+4) + (3x^2+4)\right\}$$

$$= e^{\frac{1}{x}}\left(-x - \frac{4}{x} + 3x^2 + 4\right)$$

$$f'(x) = e^{\frac{1}{x}}\left(3x^2 - x + 4 - \frac{4}{x}\right)$$

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(ii) $f(x)=\sin(2^x)$

Solution:

$$f(x)=\sin(2^x)$$

Differentiate w.r.t. 'x'

$$f'(x)=\frac{d}{dx}(\sin(2^x))$$

$$=\cos(2^x)\cdot\frac{d}{dx}(2^x)$$

$$=\cos(2^x)\cdot 2^x\cdot \ln 2\cdot 1$$

$$f'(x)=2^x \ln 2\cdot\cos(2^x)$$

(iii) $f(x)=e^{x\cos x}$

Solution:

$$f(x)=e^{x\cos x}$$

Differentiate w.r.t. 'x'

$$f'(x)=\frac{d}{dx}(e^{x\cos x})$$

$$=e^{x\cos x}\cdot\frac{d}{dx}(x\cdot\cos x)$$

$$=e^{x\cos x}\cdot\left(x\cdot\frac{d}{dx}(\cos x)+\cos x\cdot\frac{d}{dx}(x)\right)$$

$$=e^{x\cos x}\cdot(-x\sin x+\cos x\cdot 1)$$

$$f'(x)=e^{x\cos x}(\cos x-x\sin x)$$

(iv) $f(x)=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

Solution:

$$f(x)=\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

Differentiate w.r.t. 'x'

$$f'(x)=\frac{d}{dx}\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)$$

$$=\frac{(e^x-e^{-x})\frac{d}{dx}(e^x+e^{-x})-(e^x+e^{-x})\frac{d}{dx}(e^x-e^{-x})}{(e^x-e^{-x})^2}$$

$$=\frac{(e^x-e^{-x})(e^x\cdot 1+e^{-x}(-1))-(e^x+e^{-x})(e^x\cdot 1-e^{-x}(-1))}{(e^x-e^{-x})^2}$$

$$=\frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2}$$

$$=\frac{(e^x-e^{-x})^2-(e^x+e^{-x})^2}{(e^x-e^{-x})^2}$$

$$=\frac{-4e^x\cdot e^{-x}}{(e^x-e^{-x})^2} \qquad \because (a-b)^2-(a+b)^2=-4ab$$

$$f'(x)=\frac{4}{(e^x-e^{-x})^2}$$

Q2. Find $\dfrac{dy}{dx}$ if:

(i) $y=\ln(e^x-e^{-x})$

Solution:

$$y=\ln(e^x-e^{-x})$$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx}=\frac{d}{dx}(\ln(e^x-e^{-x}))$$

$$=\frac{1}{e^x-e^{-x}}\cdot\frac{d}{dx}(e^x-e^{-x})$$

$$=\frac{1}{e^x-e^{-x}}(e^x\cdot 1-e^{-x}(-1))$$

$$\frac{dy}{dx}=\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

(ii) $y=\log_{10}\sqrt{\dfrac{1-x^2}{1+x^2}}$

Solution:

$$y=\log_{10}\sqrt{\frac{1-x^2}{1+x^2}}$$

$$=\frac{1}{2}\log_{10}\left(\frac{1-x^2}{1+x^2}\right)$$

$$y=\frac{1}{2}\{\log_{10}(1-x^2)-\log_{10}(1+x^2)\}$$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}\{\log_{10}(1-x^2)-\log_{10}(1+x^2)\}$$

$$=\frac{1}{2}\left\{\frac{d}{dx}(\log_{10}(1-x^2))-\frac{d}{dx}(\log_{10}(1+x^2))\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{(1-x^2)\ln 10}\cdot\frac{d}{dx}(1-x^2)-\frac{1}{(1+x^2)\ln 10}\cdot\frac{d}{dx}(1+x^2)\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{(1-x^2)\ln 10}(-2x)-\frac{1}{(1+x^2)\ln 10}(2x)\right\}$$

$$=\frac{-2x}{2\ln 10}\left\{\frac{1}{1-x^2}+\frac{1}{1+x^2}\right\}$$

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$$= \frac{-x}{\ln 10}\left\{\frac{1+x^2+1-x^2}{(1-x^2)(1+x^2)}\right\}$$

$$= \frac{-x}{\ln 10}\left(\frac{2}{1-x^4}\right)$$

$$\frac{dy}{dx} = \frac{-2x}{\ln 10(1-x^4)}$$

(iii) $y = \ln(x-\sqrt{x^2-1})$

Solution:

$$y = \ln(x-\sqrt{x^2-1})$$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\{\ln(x-\sqrt{x^2-1})\}$$

$$= \frac{1}{x-\sqrt{x^2-1}}\cdot\frac{d}{dx}(x-\sqrt{x^2-1})$$

$$= \frac{1}{x-\sqrt{x^2-1}}\cdot\left(1-\frac{1}{2}(x^2-1)^{\frac{-1}{2}}\cdot\frac{d}{dx}(x^2-1)\right)$$

$$= \frac{1}{x-\sqrt{x^2-1}}\cdot\left(1-\frac{1}{2\sqrt{x^2-1}}(2x)\right)$$

$$= \frac{1}{x-\sqrt{x^2-1}}\cdot\left(1-\frac{x}{\sqrt{x^2-1}}\right)$$

$$= \frac{1}{x-\sqrt{x^2-1}}\cdot\left(\frac{\sqrt{x^2-1}-x}{\sqrt{x^2-1}}\right)$$

$$= \frac{-(x-\sqrt{x^2-1})}{(x-\sqrt{x^2-1})\sqrt{x^2-1}}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{x^2-1}}$$

(iv) $y = \ln(1-\cos^2 x)$

Solution:

$$y = \ln(1-\cos^2 x)$$

$$= \ln(\sin^2 x) \qquad \because 1-\cos^2 x = \sin^2 x$$

$$y = 2\ln(\sin x)$$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = 2\frac{d}{dx}(\ln(\sin x))$$

$$= 2\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)$$

$$= 2\cdot\frac{1}{\sin x}\cos x$$

$$\frac{dy}{dx} = 2\cot x$$

(v) $y = \dfrac{\ln x}{x^2}$

Solution:

$$y = \frac{\ln x}{x^2}$$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{x^2}\right)$$

$$= \frac{x^2\frac{d}{dx}(\ln x)-\ln x\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2\cdot\frac{1}{x}-\ln x\cdot(2x)}{x^4}$$

$$= \frac{x-2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4}$$

$$\frac{dy}{dx} = \frac{1-2\ln x}{x^3}$$

(vi) $y = \dfrac{\sqrt{x^2-1}(2x+1)}{(x^3-1)^{\frac{3}{2}}}$

Solution:

$$y = \frac{\sqrt{x^2-1}(2x+1)}{(x^3-1)^{\frac{3}{2}}}$$

Taking natural log on both sides

$$\ln y = \ln\left\{\frac{\sqrt{x^2-1}(2x+1)}{(x^3-1)^{3/2}}\right\}$$

$$= \ln(\sqrt{x^2-1}\cdot(2x+1))-\ln(x^3-1)^{3/2}$$

$$= \ln(\sqrt{x^2-1})+\ln(2x+1)-\ln(x^3-1)^{3/2}$$

$$\ln y = \frac{1}{2}\ln(x^2-1)+\ln(2x+1)-\frac{3}{2}\ln(x^3-1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(\ln y) = \frac{1}{2}\frac{d}{dx}(\ln(x^2-1))+\frac{d}{dx}(\ln(2x+1))-\frac{3}{2}\frac{d}{dx}(\ln(x^3-1))$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x^2-1}\cdot(2x)+\frac{1}{2x+1}(2)-\frac{3}{2}\cdot\frac{1}{x^3-1}(3x^2)$$

$$\frac{dy}{dx} = y\left[\frac{x}{x^2-1}+\frac{2}{2x+1}-\frac{9x^2}{2(x^3-1)}\right]$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2-1}(2x+1)}{(x^3-1)^{3/2}}\left[\frac{x}{x^2-1}+\frac{2}{2x+1}-\frac{9x^2}{2(x^3-1)}\right]$$

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Content Outline

- Equations of Tangent and Normal Lines to a Curve
- Higher-Order Derivatives

EQUATIONS OF TANGENT AND NORMAL LINES TO A CURVE

Tangent Line: Suppose that x_0 is a point in the domain of the function $f(x)$. The tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is the line with equation

$$y - y_0 = m(x - x_0)$$

where $m = f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0}$, provided $f'(x_0)$ exists.

Normal Line: A line through point $(x_0, f(x_0))$ of the curve $y = f(x)$ and perpendicular to tangent line is called normal line to the curve

- If $m = f'(x_0)$ is non zero, then equation of the normal line is $y - y_0 = -\frac{1}{m}(x - x_0)$ and
- if $m = f'(x_0) = 0$, then $x = x_0$ is the equation of the normal line.

Need to Know?

Vertical Tangent

If m is tends to ∞ or $-\infty$ as $x \to x_0$, then $x = x_0$ is a vertical tangent to the curve $y = f(x)$.

Note

Slope of tangent line to a curve at a point is also known as slope of the curve at the point.

Example 10: Find the equations of the tangent and normal to the curve $y = \sec x$ at the point where $x = \frac{\pi}{4}$.

Solution:

Here $y = \sec x$ ...(1)

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \sec x \cdot \tan x$$

Slope of tangent line $= m = \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}}$

$$m = \sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)$$

$$m = (\sqrt{2})(1) = \sqrt{2}$$

Put $x = \frac{\pi}{4}$ in eq. (1), we have

$$y = \sec\frac{\pi}{4} = \sqrt{2} \Rightarrow \left(\frac{\pi}{4}, \sqrt{2}\right)$$

Equation of tangent at $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is

$$y - y_1 = m(x - x_1)$$

$$y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$$

$$y - \sqrt{2} = \sqrt{2}x - \frac{\pi\sqrt{2}}{4}$$

Multiply both sides by '4'

$$4y - 4\sqrt{2} = 4\sqrt{2}x - \pi\sqrt{2}$$

$$4\sqrt{2}x - 4y + 4\sqrt{2} - \pi\sqrt{2} = 0$$

Equation of normal at $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - \sqrt{2} = -\frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

$$\sqrt{2}(y - \sqrt{2}) = -x + \frac{\pi}{4}$$

$$\sqrt{2}y - (\sqrt{2})^2 + x - \frac{\pi}{4} = 0$$

$$x + \sqrt{2}y - 2 - \frac{\pi}{4} = 0$$

Multiply both sides by '4'

$$4x + 4\sqrt{2}y - 8 - \pi = 0$$

HIGHER-ORDER DERIVATIVES

The derivative f' of a function f is itself a function and hence may have a derivative of its own. If f' is differentiable, then its derivative is denoted by f'' and is called the second derivative of f. As long as the new function remains differentiable, we can continue the process of differentiating to obtain the third, fourth, fifth, and even higher derivatives of f. These successive derivatives are called higher-order derivatives.

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The following are the commonly used notations for higher-order derivatives:

1st derivative	2nd derivative	3rd derivative	...	nth derivative
y'	y''	y'''	...	$y^{(n)}$
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	...	$\frac{d^ny}{dx^n}$
$f'(x)$	$f''(x)$	$f'''(x)$	...	$f^{(n)}(x)$
y_1	y_2	y_3	...	y_n
Dy	D^2y	D^3y	...	D^ny

Example 11: Find the higher-order derivatives of the function $f(x)=\frac{1}{4}x^4-x^3+\frac{1}{2}x^2-3$

Solution:

$$f(x)=\frac{1}{4}x^4-x^3+\frac{1}{2}x^2-3$$

Differentiate w.r.t. 'x'

$$f'(x)=\frac{d}{dx}\left(\frac{1}{4}x^4-x^3+\frac{1}{2}x^2-3\right)$$
$$=\frac{1}{4}(4x^3)-3x^2+\frac{1}{2}(2x)-0$$
$$=x^3-3x^2+x$$
$$f''(x)=3x^2-3(2x)+1$$
$$f''(x)=3x^2-6x+1$$
$$f'''(x)=3(2x)-6(1)+0$$
$$=6x-6$$
$$f^{(4)}(x)=6(1)-0$$
$$=6$$
$$f^{(5)}(x)=0$$
$$\vdots$$
$$f^{(n)}(x)=0 \text{ (For } n\geq 5)$$

Example 12: Find all higher-order derivatives of the function $y=e^{ax}$.

Solution:

$$y=e^{ax}$$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx}=\frac{d}{dx}(e^{ax})$$
$$=e^{ax}\cdot\frac{d}{dx}(ax)=e^{ax}\cdot(a)$$
$$=ae^{ax}$$
$$\frac{d^2y}{dx^2}=a(e^{ax}\cdot a)=a^2e^{ax}$$
$$\frac{d^3y}{dx^3}=a^2(e^{ax}\cdot a)=a^3e^{ax}$$
$$\frac{d^4y}{dx^4}=a^4e^{ax}$$
$$\vdots$$
$$\frac{d^ny}{dx^n}=a^ne^{ax} \text{ (For } n\geq 1)$$
$$=a^ny \qquad \because y=e^{ax}$$

Example 13: If $x=e^{at}$ and $y=e^{2at}$, show that $x^2\frac{d^2y}{dx^2}-2y=0$.

Solution:

$$x=e^{at}$$

Differentiate w.r.t. 't'

$$\frac{dx}{dt}=\frac{d}{dt}(e^{at})$$
$$=e^{at}\cdot(a)$$
$$\frac{dx}{dt}=ax \qquad \because x=e^{at}$$

$$y=e^{2at}$$

Differentiate w.r.t. 't'

$$\frac{dy}{dt}=\frac{d}{dt}(e^{2at})$$
$$=e^{2at}\cdot(2a)$$
$$\frac{dy}{dt}=2ay \qquad \because y=e^{2at}$$

Apply the Chain rule:

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$
$$=(2ay)\left(\frac{1}{ax}\right)$$
$$\frac{dy}{dx}=\frac{2y}{x} \qquad \ldots(1)$$

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Differentiate w.r.t. 'x'

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{2y}{x}\right)$$

$$= \frac{x\cdot\frac{d}{dx}(2y) - 2y\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\left(2\frac{dy}{dx}\right) - 2y(1)}{x^2}$$

$$= \frac{2x\left(\frac{2y}{x}\right) - 2y}{x^2}$$ From (1): $\frac{dy}{dx} = \frac{2y}{x}$

$$\frac{d^2y}{dx^2} = \frac{4y-2y}{x^2}$$

$$x^2\cdot\frac{d^2y}{dx^2} = 2y$$

$$x^2\cdot\frac{d^2y}{dx^2} - 2y = 0 \text{ (Proved)}$$

Example 14: Find y_4 if $y = \ell n\sqrt{\frac{2x-3}{x+1}}$.

Solution:

$$y = \ell n\sqrt{\frac{2x-3}{x+1}}$$

$$= \frac{1}{2}\ell n\left(\frac{2x-3}{x+1}\right) \qquad \because \ell n(a^b) = b\,\ell n\,a$$

$$y = \frac{1}{2}\{\ell n(2x-3) - \ell n(x+1)\}$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{1}{2}\cdot\frac{d}{dx}\{\ell n(2x-3) - \ell n(x+1)\}$$

$$y_1 = \frac{1}{2}\left\{\frac{1}{2x-3}(2) - \frac{1}{x+1}(1)\right\}$$

$$= \frac{1}{2}\{2(2x-3)^{-1} - (x+1)^{-1}\}$$

$$y_2 = \frac{1}{2}\{-2(2x-3)^{-2}(2) + 1(x+1)^{-2}\}$$

$$y_2 = \frac{1}{2}\{-4(2x-3)^{-2} + (x+1)^{-2}\}$$

$$y_3 = \frac{1}{2}\{8(2x-3)^{-3}(2) - 2(x+1)^{-3}\}$$

$$= \frac{1}{2}\{16(2x-3)^{-3} - 2(x+1)^{-3}\}$$

$$y_4 = \frac{1}{2}\{-48(2x-3)^{-4}(2) + 6(x+1)^{-4}\}$$

$$y_4 = \frac{-48}{(2x-3)^4} + \frac{3}{(x+1)^4}$$

Challenge!

Find y_{10} in example 14.

Ans. $y_{10} = \frac{9!}{2}\left[\frac{-1024}{(2x-3)^{10}} + \frac{1}{(x+1)^{10}}\right]$

EXERCISE 2.3

Q1. Find the equation of the tangent line to the curve $y = \sin^{-1}\left(\frac{x}{2}\right) + \cos x$ at the point where $x = \frac{\pi}{2}$.

Solution:

Here, $y = \sin^{-1}\left(\frac{x}{2}\right) + \cos x$ ...(1)

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{2}\right) + \cos x\right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{1}{2} - \sin x$$

$$= \frac{1}{\sqrt{\frac{4-x^2}{4}}}\cdot\frac{1}{2} - \sin x$$

$$= \frac{2}{\sqrt{4-x^2}}\cdot\frac{1}{2} - \sin x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}} - \sin x$$

Slope of tangent line $= m = \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}}$

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$$m = \frac{1}{\sqrt{4-\left(\frac{\pi}{2}\right)^2}} - \sin\frac{\pi}{2} = \frac{1}{\sqrt{\frac{16-\pi^2}{4}}} - 1 = \frac{2}{\sqrt{16-\pi^2}} - 1$$

Put $x = \frac{\pi}{2}$ in eq. (1), we get

$$y = \sin^{-1}\left(\frac{\pi/2}{2}\right) + \cos\frac{\pi}{2} = \sin^{-1}\left(\frac{\pi}{4}\right) + 0 = \sin^{-1}\left(\frac{\pi}{4}\right)$$

$\Rightarrow$ Point on the curve is $\left(\frac{\pi}{2}, \sin^{-1}\left(\frac{\pi}{4}\right)\right)$

Equation of tangent at $\left(\frac{\pi}{2}, \sin^{-1}\left(\frac{\pi}{4}\right)\right)$ is

$$y - y_1 = m(x - x_1)$$

$$y - \sin^{-1}\left(\frac{\pi}{4}\right) = \left(\frac{2}{\sqrt{16-\pi^2}} - 1\right)\left(x - \frac{\pi}{2}\right)$$

Q2. Find a point on the curve $y = \tan^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}$, where tangent is horizontal. Also find the equations of the tangent and normal at that point.

Solution:

$$y = \tan^{-1}\left(\frac{x}{3}\right) - \frac{x}{3} \qquad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\right)$$

$$= \frac{1}{1+\left(\frac{x}{3}\right)^2}\cdot\frac{1}{3} - \frac{1}{3}$$

$$= \frac{1}{3}\cdot\frac{1}{\frac{9+x^2}{9}} - \frac{1}{3}$$

$$\frac{dy}{dx} = \frac{3}{9+x^2} - \frac{1}{3}$$

Since the tangent line is horizontal, so

Slope of tangent line $= m = 0$

$$\Rightarrow \quad \frac{dy}{dx} = 0$$

$$\frac{3}{9+x^2} - \frac{1}{3} = 0$$

$$\frac{3}{9+x^2} = \frac{1}{3}$$

$$9 = 9 + x^2 \quad \Rightarrow x^2 = 0 \quad \Rightarrow x = 0$$

Put $x = 0$ in eq. (1), we get

$$y = \tan^{-1}\left(\frac{0}{3}\right) - \frac{0}{3} = 0 - 0 = 0$$

$\Rightarrow$ Point on the curve is $(0, 0)$

Equation of tangent line at $(0, 0)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 0(x - 0)$$

$$\Rightarrow \quad \boxed{y = 0}$$

Equation of normal line at $(0, 0)$ is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - 0 = -\frac{1}{0}(x - 0)$$

$$0(y - 0) = -x$$

$$\Rightarrow \quad \boxed{x = 0}$$

Q3. Find a point on the curve $x + y - e^y = 0$ where the tangent line is vertical. Also, find the equations of the tangent and normal at that point.

Solution:

$$x + y - e^y = 0 \qquad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(x + y - e^y) = 0$$

$$1 + \frac{dy}{dx} - e^y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(1 - e^y) = -1$$

$$\frac{dy}{dx} = \frac{-1}{1-e^y}$$

$$\frac{dy}{dx} = \frac{1}{e^y - 1}$$

Since the tangent line is vertical, so

Slope of tangent line $= m = \infty$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{0}$$

$$\frac{1}{e^y - 1} = \frac{1}{0}$$

$$e^y - 1 = 0 \quad \Rightarrow e^y = 1$$

$$e^y = e^0 \quad \Rightarrow y = 0$$

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Put $y = 0$ in eq. (1), we get

$$x + 0 - e^0 = 0 \quad \Rightarrow x = e^0 \quad \Rightarrow x = 1$$

$\Rightarrow$ Point on the curve is $(1, 0)$

Equation of tangent line at $(1, 0)$ is

$$y - y_1 = m(x - x_1)$$
$$y - 0 = \infty(x - 1)$$
$$y = \frac{1}{0}(x - 1)$$
$$0 = x - 1 \quad \Rightarrow \boxed{x = 1}$$

Equation of normal line at $(1, 0)$ is

$$y - y_1 = \frac{-1}{m}(x - x_1)$$
$$y - 0 = -\frac{1}{\infty}(x - 1)$$
$$y = 0(x - 1)$$
$$\boxed{y = 0}$$

Q4. Find all points on the ellipse $x^2 - xy + y^2 = 1$ at which the tangent line is horizontal.

Solution:

$$x^2 - xy + y^2 = 1 \quad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx}(1)$$
$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$
$$2x - \left(x\frac{dy}{dx} + y \cdot 1\right) + 2y\frac{dy}{dx} = 0$$
$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(-x + 2y) = y - 2x$$
$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

Since, the tangent line is horizontal, so

Slope of tangent line $= m = 0$

$$\Rightarrow \quad \frac{dy}{dx} = 0$$
$$\frac{y - 2x}{2y - x} = 0$$
$$y - 2x = 0$$
$$y = 2x \quad \ldots(2)$$

Put $y = 2x$ in eq. (1), we get

$$x^2 - x(2x) + (2x)^2 = 1$$
$$x^2 - 2x^2 + 4x^2 = 1$$
$$3x^2 = 1 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

From eq. (2), we get

If $x = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{2}{\sqrt{3}}$

If $x = -\frac{1}{\sqrt{3}} \Rightarrow y = -\frac{2}{\sqrt{3}}$

Required points on the ellipse are

$$\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \text{ and } \left(-\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$$

Q5. Find y_2 if:

(i) $y = x^4 - 3x^2 + 4x - 5$

Solution:

$$y = x^4 - 3x^2 + 4x - 5$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^4 - 3x^2 + 4x - 5)$$
$$y_1 = 4x^3 - 3(2x) + 4(1) - 0$$
$$= 4x^3 - 6x + 4$$
$$y_2 = 4(3x^2) - 6(1) + 0$$
$$y_2 = 12x^2 - 16$$

(ii) $y = \ell n(2x + 3)$

Solution:

$$y = \ell n(2x + 3)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(\ell n(2x + 3))$$
$$y_1 = \frac{1}{2x + 3}(2 + 0)$$
$$= 2(2x + 3)^{-1}$$
$$y_2 = -2(2x + 3)^{-2} \cdot (2)$$
$$y_2 = -\frac{4}{(2x + 3)^2}$$

(iii) $y = x^3 e^{2x}$

Solution:

$$y = x^3 \cdot e^{2x}$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^3 \cdot e^{2x})$$
$$y_1 = x^3 \cdot \frac{d}{dx}(e^{2x}) + e^{2x}\frac{d}{dx}(x^3)$$
$$= x^3 \cdot e^{2x}(2) + e^{2x} \cdot (3x^2)$$
$$y_1 = e^{2x}(2x^3 + 3x^2)$$

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Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y_1) = \frac{d}{dx}\{e^{2x}(2x^3+3x^2)\}$$

$$y_2 = e^{2x}\cdot\frac{d}{dx}(2x^3+3x^2)+(2x^3+3x^2)\frac{d}{dx}(e^{2x})$$

$$= e^{2x}(6x^2+6x)+(2x^3+3x^2)e^{2x}\cdot(2)$$

$$= e^{2x}(6x^2+6x+4x^3+6x^2)$$

$$= e^{2x}(4x^3+12x^2+6x)$$

$$y_2 = 2xe^{2x}(2x^2+6x+3)$$

(iv) $x^2 - y^2 = 4$

Solution:

$$x^2 - y^2 = 4 \qquad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(x^2-y^2) = \frac{d}{dx}(4)$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$-2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{-2y}$$

$$y_1 = \frac{dy}{dx} = \frac{x}{y}$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y_1) = \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$y_2 = \frac{y\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(y)}{y^2}$$

$$= \frac{y(1) - x\frac{dy}{dx}}{y^2}$$

$$= \frac{y - x\left(\frac{x}{y}\right)}{y^2} \qquad \because \frac{dy}{dx} = \frac{x}{y}$$

$$= \frac{y^2 - x^2}{y}$$

$$= \frac{-(x^2-y^2)}{y^3}$$

$$y_2 = -\frac{4}{y^3} \qquad \text{Using (1)}$$

(v) $x^2 - xy + y^2 = 7$

Solution:

$$x^2 - xy + y^2 = 7 \qquad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(x^2-xy+y^2) = \frac{d}{dx}(7)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$2x - \left(x\frac{dy}{dx} + y\cdot 1\right) + 2y\frac{dy}{dx} = 0$$

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(-x+2y) = y - 2x$$

$$y_1 = \frac{dy}{dx} = \frac{y-2x}{2y-x}$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y_1) = \frac{d}{dx}\left(\frac{y-2x}{2y-x}\right)$$

$$y_2 = \frac{(2y-x)\frac{d}{dx}(y-2x) - (y-2x)\frac{d}{dx}(2y-x)}{(2y-x)^2}$$

$$= \frac{(2y-x)\left(\frac{dy}{dx}-2(1)\right) - (y-2x)\left(2\frac{dy}{dx}-1\right)}{(2y-x)^2}$$

$$= \frac{2y\frac{dy}{dx} - 4y - x\frac{dy}{dx} + 2x - 2y\frac{dy}{dx} + y + 4x\frac{dy}{dx} - 2x}{(2y-x)^2}$$

$$= \frac{-3y + \frac{dy}{dx}(-x+4x)}{(2y-x)^2}$$

$$= \frac{-3y + 3x\left(\frac{y-2x}{2y-x}\right)}{(2y-x)^2} \qquad \because \frac{dy}{dx} = \frac{y-2x}{2y-x}$$

$$= \frac{\frac{-3y(2y-x)+3x(y-2x)}{2y-x}}{(2y-x)^2}$$

$$= \frac{-6y^2+3xy+3xy-6x^2}{(2y-x)(2y-x)^2}$$

$$= \frac{-6x^2+6xy-6y^2}{(2y-x)^3} = \frac{-6(x^2-xy+y^2)}{(2y-x)^3}$$

$$= \frac{-6(7)}{(2y-x)^3} \qquad \text{Using (1)}$$

$$y_2 = -\frac{42}{(2y-x)^3}$$

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(vi) $x = a\cos t, y = b\sin t$

Solution:

$x = a\cos t$

Differentiate w.r.t. 't'

$$\frac{dx}{dt} = a\frac{d}{dt}(\cos t)$$

$$\frac{dx}{dt} = -a\sin t$$

$y = b\sin t$

Differentiate w.r.t. 't'

$$\frac{dy}{dt} = b\frac{d}{dt}(\sin t)$$

$$\frac{dy}{dt} = b\cos t$$

Apply the Chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$y_1 = (b\cos t)\left(-\frac{1}{a\sin t}\right)$$

$$y_1 = -\frac{b}{a}\cot t$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y_1) = -\frac{b}{a}\frac{d}{dx}(\cot t)$$

$$y_2 = -\frac{b}{a}(-\csc^2 t)\frac{dt}{dx}$$

$$= \frac{b}{a}\cdot\frac{1}{\sin^2 t}\cdot\left(-\frac{1}{a\sin t}\right) \quad \because \frac{dt}{dx} = -\frac{1}{a\sin t}$$

$$= -\frac{b}{a^2}\cdot\frac{1}{\sin^3 t}$$

$$= -\frac{b}{a^2}\cdot\frac{1}{\left(\frac{y}{b}\right)^3} \quad \because y = b\sin t \Rightarrow \sin t = \frac{y}{b}$$

$$y_2 = -\frac{b^4}{a^2y^3}$$

(vii) $x = t^2, y = 2t^3$

Solution:

$x = t^2$

Differentiate w.r.t. 'x'

$$\frac{dx}{dt} = \frac{d}{dt}(t^2)$$

$$\frac{dx}{dt} = 2t$$

$y = 2t^3$

Differentiate w.r.t. 'x'

$$\frac{dy}{dt} = 2\frac{d}{dt}(t^3)$$

$$= 2(3t^2)$$

$$\frac{dy}{dt} = 6t^2$$

Apply the Chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$y_1 = (6t^2)\left(\frac{1}{2t}\right)$$

$$y_1 = 3t$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y_1) = 3\frac{d}{dx}(t)$$

$$y_2 = 3\frac{dt}{dx}$$

$$= 3\left(\frac{1}{2t}\right) \quad \because \frac{dt}{dx} = \frac{1}{2t}$$

$$y_2 = \frac{3}{2\sqrt{x}} \quad \because x = t^2 \Rightarrow t = \sqrt{x}$$

Q6. Find y_4 and write expressions for y_{10} and y_{11} without evaluating them:

(i) $y = \sin\left(\frac{x}{2}\right)$

Solution:

$$y = \sin\left(\frac{x}{2}\right)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\sin\left(\frac{x}{2}\right)\right)$$

$$y_1 = \cos\left(\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$y_2 = -\frac{1}{2}\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}$$

$$= -\left(\frac{1}{2}\right)^2\sin\left(\frac{x}{2}\right)$$

$$y_3 = -\left(\frac{1}{2}\right)^2\cos\left(\frac{x}{2}\right)\cdot\frac{1}{2}$$

$$= -\left(\frac{1}{2}\right)^3\cos\left(\frac{x}{2}\right)$$

$$y_4 = \left(\frac{1}{2}\right)^3\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}$$

$$= \left(\frac{1}{2}\right)^4\sin\left(\frac{x}{2}\right) \text{ or } = \frac{1}{16}\sin\left(\frac{x}{2}\right)$$

Since the derivative of sine function follow a periodic cycle of 4, so

$$y_8 = \left(\frac{1}{2}\right)^8\sin\left(\frac{x}{2}\right)$$

$$\Rightarrow \quad y_{10} = -\left(\frac{1}{2}\right)^{10}\sin\left(\frac{x}{2}\right)$$

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$$y_{10} = -\frac{1}{1024}\sin\left(\frac{x}{2}\right)$$

and $$y_{11} = -\left(\frac{1}{2}\right)^{11}\sin\left(\frac{x}{2}\right)$$

$$= -\frac{1}{2048}\sin\left(\frac{x}{2}\right)$$

(ii) $y = \ell n(x^2 - a^2)$

Solution:

$$y = \ell n(x^2 - a^2)$$
$$y = \ell n(x-a)(x+a)$$
$$y = \ell n(x-a) + \ell n(x+a)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}\{\ell n(x-a) + \ell n(x+a)\}$$

$$y_1 = \frac{1}{x-a}(1) + \frac{1}{x+a}(1)$$

$$y_1 = (x-a)^{-1} + (x+a)^{-1}$$
$$y_2 = -1(x-a)^{-2} - (-1)(x+a)^{-2}$$
$$y_3 = (-1)(-2)(x-a)^{-3} + (-1)(-2)(x+a)^{-3}$$
$$= (-1)^2(1\cdot 2)(x-a)^{-3} + (-1)^2(1\cdot 2)(x+a)^{-3}$$
$$y_4 = (-1)^2(1\cdot 2)(-3)(x-a)^{-4} + (-1)^2(1\cdot 2)(-3)(x+a)^{-4}$$
$$= (-1)^3(3\cdot 2\cdot 1)\cdot(x-a)^{-4} + (-1)^3(3\cdot 2\cdot 1)(x+a)^{-4}$$
$$= \frac{(-1)^3\cdot 3!}{(x-a)^4} + \frac{(-1)^3\cdot 3!}{(x+a)^4}$$

$$y_4 = (-1)^3\cdot 3!\left\{\frac{1}{(x-a)^4} + \frac{1}{(x+a)^4}\right\}$$

According to above pattern, we have

$$y_{10} = (-1)^9\cdot 9!\left\{\frac{1}{(x-a)^{10}} + \frac{1}{(x+a)^{10}}\right\}$$

and $$y_{11} = (-1)^{10}\cdot 10!\left\{\frac{1}{(x-a)^{11}} + \frac{1}{(x+a)^{11}}\right\}$$

Q7. If $x^2 + y^2 = 25$, show that $y^3\frac{d^2y}{dx^2} + 25 = 0$.

Solution:

$$x^2 + y^2 = 25 \qquad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{2x}{2y}$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Differentiate w.r.t. 'x'

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x}{y}\right)$$
$$= -\frac{y\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(y)}{y^2}$$
$$\frac{d^2y}{dx^2} = -\frac{y\cdot 1 - x\frac{dy}{dx}}{y^2}$$
$$= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} \qquad \because \frac{dy}{dx} = -\frac{x}{y}$$
$$= -\frac{\frac{y^2 + x^2}{y}}{y^2} = -\frac{x^2 + y^2}{y^3}$$
$$\frac{d^2y}{dx^2} = -\frac{25}{y^3} \qquad \text{Using (1)}$$
$$y^3\frac{d^2y}{dx^2} = -25$$
$$y^3\frac{d^2y}{dx^2} + 25 = 0 \text{ (Proved)}$$

Q8. If $y = e^{ax}\cos bx$, show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$

Solution:

$$y = e^{ax}\cdot\cos bx \qquad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(e^{ax}\cdot\cos bx)$$
$$= e^{ax}\cdot\frac{d}{dx}(\cos bx) + \cos bx\cdot\frac{d}{dx}(e^{ax})$$
$$= e^{ax}(-\sin bx)b + \cos bx\cdot e^{ax}\cdot a$$
$$= -be^{ax}\sin bx + a\cdot e^{ax}\cos bx$$

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$$\frac{dy}{dx} = -be^{ax}\sin bx + ay \quad \text{...(2) Using (1)}$$

Differentiate w.r.t. 'x'

$$\frac{d^2y}{dx^2} = -b\frac{d}{dx}(e^{ax}\sin bx) + a\frac{dy}{dx}$$

$$= -b\left\{e^{ax}\cdot\frac{d}{dx}(\sin bx) + \sin bx\cdot\frac{d}{dx}(e^{ax})\right\} + a\frac{dy}{dx}$$

$$= -b\{e^{ax}\cos bx\cdot(b) + \sin bx\cdot e^{ax}\cdot(a)\} + a\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -b^2\cdot(e^{ax}\cos bx) + a(-be^{ax}\sin bx) + a\frac{dy}{dx}$$

From eq. (1) and eq. (2)

Put $y = e^{ax}\cos bx$ and $\frac{dy}{dx} - ay = -be^{ax}\sin bx$

$$\frac{d^2y}{dx^2} = -b^2y + a\left(\frac{dy}{dx} - ay\right) + a\frac{dy}{dx}$$

$$= -b^2y + a\frac{dy}{dx} - a^2y + a\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = 2a\frac{dy}{dx} - y(a^2 + b^2)$$

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0 \text{ (Proved)}$$

Q9. If $x = \sin t$ and $y = \cos^2 t$, then show that $\frac{d^2y}{dx^2} + 2 = 0$.

Solution:

$$x = \sin t$$

Differentiate w.r.t. 't'

$$\frac{dx}{dt} = \frac{d}{dt}(\sin t)$$

$$\frac{dx}{dt} = \cos t$$

$$y = \cos^2 t$$

Differentiate w.r.t. 't'

$$\frac{dy}{dt} = \frac{d}{dt}(\cos t)^2$$

$$= 2\cos t\cdot(-\sin t)$$

$$\frac{dy}{dt} = -2\sin t\cdot\cos t$$

Apply the Chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= (-2\sin t\cdot\cos t)\left(\frac{1}{\cos t}\right)$$

$$= -2\sin t$$

$$\frac{dy}{dx} = -2x \quad \because x = \sin t$$

Differentiate w.r.t. 'x'

$$\frac{d^2y}{dx^2} = -2\frac{d}{dx}(x)$$

$$\frac{d^2y}{dx^2} = -2(1)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + 2 = 0 \text{ (Proved)}$$

Q10. If $x = \ell n\, t$ and $y = \frac{1}{t}$, then show that $\frac{d^2y}{dx^2} - y = 0$.

Solution:

$$x = \ell n\, t$$

Differentiate w.r.t. 't'

$$\frac{dx}{dt} = \frac{d}{dt}(\ell n\, t)$$

$$\frac{dx}{dt} = \frac{1}{t}$$

$$y = \frac{1}{t}$$

Differentiate w.r.t. 't'

$$\frac{dy}{dt} = \frac{d}{dt}(t^{-1})$$

$$= -1t^{-2}$$

$$\frac{dy}{dt} = -\frac{1}{t^2}$$

Apply the Chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= -\frac{1}{t^2}\cdot t$$

$$\frac{dy}{dx} = -\frac{1}{t}$$

Differentiate w.r.t. 'x'

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}(t^{-1})$$

$$= -\left(-1t^{-2}\cdot\frac{dt}{dx}\right)$$

$$= \frac{1}{t^2}\cdot t \quad \because \frac{dt}{dx} = t$$

$$= \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = y \quad \because y = \frac{1}{t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - y = 0 \text{ (Proved)}$$

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Q11. If $y = a\cos(\ell n\, x) + b\sin(\ell n\, x)$, then show that $x^2 y_2 + xy_1 + y = 0$.

Solution:

$$y = a\cos(\ell n\, x) + b\sin(\ell n\, x) \quad \ldots(1)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}\{a\cos(\ell n\, x) + b\sin(\ell n\, x)\}$$

$$y_1 = a\frac{d}{dx}(\cos(\ell n\, x)) + b\frac{d}{dx}(\sin(\ell n\, x))$$

$$= a\left(-\sin(\ell n\, x)\cdot\frac{1}{x}\right) + b\left(\cos(\ell n\, x)\cdot\frac{1}{x}\right)$$

$$y_1 = \frac{1}{x}\{-a\sin(\ell n\, x) + b\cos(\ell n\, x)\}$$

$$\Rightarrow \quad xy_1 = -a\sin(\ell n\, x) + b\cos(\ell n\, x)$$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(xy_1) = -a\frac{d}{dx}(\sin(\ell n\, x)) + b\frac{d}{dx}(\cos(\ell n\, x))$$

$$x\frac{d}{dx}(y_1) + y_1\frac{d}{dx}(x) = -a\cos(\ell n\, x)\cdot\frac{1}{x} - b\sin(\ell n\, x)\cdot\frac{1}{x}$$

$$xy_2 + y_1(1) = -\frac{1}{x}\{a\cos(\ell n\, x) + b\sin(\ell n\, x)\}$$

$$\Rightarrow \quad x(xy_2 + y_1) = -y \qquad \text{Using (1)}$$

$$x^2 y_2 + xy_1 + y = 0 \text{ (Proved)}$$

Content Outline

- Increasing and Decreasing Functions
- Relative and Absolute Extrema

INCREASING AND DECREASING FUNCTIONS

Increasing Function	Decreasing Function
Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f is increasing if for any $x_1, x_2 \in I$. $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$	Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f is decreasing if for any $x_1, x_2 \in I$. $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

Recall!

$(a, b), [a, b], (a, b], [a, b), (-\infty, a), (-\infty, \infty)(-\infty, a], (a, \infty), [a, \infty)$ all are intervals.

Let f be a differentiable function defined on an interval I. Then

(i) f is increasing on I, if $f'(x) \ge 0$ for each $x \in I$.

(ii) f is decreasing on I, if $f'(x) \le 0$ for each $x \in I$.

Example 15: Prove that $f(x) = e^x$ defined on $R = (-\infty, \infty)$ is increasing.

Solution:

$f(x) = e^x$, where $x \in (-\infty, \infty)$

Differentiate w.r.t. 'x'

$f'(x) = e^x$

Since $f'(x) = e^x > 0$ for each $x \in (-\infty, \infty)$, so 'f' is an increasing function on $R = (-\infty, \infty)$

Example 16: Show that $f(x) = 2^{-x}$ defined on $R = (-\infty, \infty)$ is decreasing.

Solution:

$f(x) = 2^{-x}$, where $x \in (-\infty, \infty)$

Differentiate w.r.t. 'x'

$$f'(x) = 2^{-x}\cdot\ell n\, 2\frac{d}{dx}(-x) = 2^{-x}\ell n\, 2(-1)$$

$$f'(x) = -\frac{\ell n\, 2}{2^x}$$

Since $f'(x) = -\frac{\ell n\, 2}{2^x} < 0$ for each $x \in (-\infty, \infty)$, so 'f' is decreasing function on $R = (-\infty, \infty)$

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Stationary Point

A point on the graph where its derivative is zero is called a stationary point.

Example 17: Find the intervals on which the function $f(x) = |x|$, defined on $\mathbb{R} = (-\infty, \infty)$, is increasing or decreasing.

Solution:

$f(x) = |x|$, where $x \in (-\infty, \infty)$

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Differentiate w.r.t. 'x'

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Since $f'(x) = 1 > 0$ for each $x \in (0, \infty)$, so 'f' is an increasing function on $(0, \infty)$.

Since $f'(x) = -1 < 0$ for each $x \in (-\infty, 0)$, so f is decreasing function on $(-\infty, 0)$.

Now, we evaluate $f'(0)$ By definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) + f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h| - |0|}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{|h|}{h}$$

L.H.L $= \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$

R.H.L $= \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = -1$

As L.H.L $\neq$ R.H.L, so $\lim_{h \to 0} \frac{|h|}{h}$ does not exists. Hence $f'(0)$ is not defined.

$\Rightarrow$ The function has no stationary point, but a critical point at $x = 0$.

In the above example, the function has no stationary point, but it changes behavior around a point where its derivative is undefined.

Conclusion: From above discussion, we conclude that a function may change its:

1. Increasing/decreasing behavior around:
   - Stationary points where $f'(x) = 0$, or
   - Points where $f'(x)$ is undefined.

Critical Thinking

- Every stationary point is a critical point.
- Do you think every critical point is a stationary point?

Critical Point

For a function $f(x)$ a point $x = c, c \in \text{Dom} f$ is called a critical point if:

$$f'(c) = 0 \text{ or } f'(c) \text{ does not exist.}$$

Examples:

1. For $f(x) = |x|$, the derivative does not exist at $x = 0$, so $x = 0$ is a critical point.
2. For $f(x) = x^2$, the derivative $f'(x) = 2x$ is zero at $x = 0$, so $x = 0$ is a critical point.

Method to Find Intervals Where a Function is Increasing or Decreasing

To find the intervals where a function is increasing or decreasing, follow these steps:

Step 1: Differentiate the given function $f(x)$ to find $f'(x)$.

Step 2: Find critical points by solving $f'(x) = 0$ and identifying points where $f'(x)$ does not exist.

Step 3: If critical points exist, divide the domain into sub-intervals using the critical points.

Step 4: Select a test point from each sub-interval and substitute it into $f'(x)$:

- If $f'(x) > 0$ in a sub-interval, the function is increasing there.
- If $f'(x) < 0$ in a sub-interval, the function is decreasing there.

Point to Note

If $f'(x)$ is continuous then in each sub-interval of the domain determined by the critical points, the sign of the derivatives remains the same.

Example 18: Find the intervals in which the function $f(x) = 2x^3 - 3x^2 - 12x$ is increasing or decreasing.

Solution:

$f(x) = 2x^3 - 3x^2 - 12x$ defined for all $x \in R = (-\infty, \infty)$

Differentiate w.r.t. 'x'

$$f'(x) = 2(3x^2) - 3(2x) - 12(1)$$
$$= 6x^2 - 6x - 12$$

It is clear that $f'(x) = 6x^2 - 6x - 12$ is defined for each $x \in (-\infty, \infty)$.

For other critical points, put $f'(x) = 0$

$$\Rightarrow \quad 6x^2 - 6x - 12 = 0$$
$$6(x^2 - x - 2) = 0$$

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$$x^2-2x+x-2=0$$
$$x(x-2)+1(x-2)=0$$
$$(x+1)(x+2)=0$$
$$\Rightarrow \quad x+1=0 \quad \text{and} \quad x-2=0$$
$$x=-1 \quad \text{and} \quad x=2$$

Real Numbers Line:

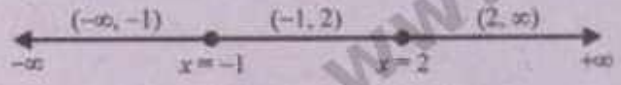

Table for sign of $f'(x)$ in each subinterval.

Interval	Test point	Sign of $f'(x)$	Conclusion
$(-\infty,-1)$	$x=-2$	$f'(-2)=6(-2)^2-6(-2)-12$ $=24>0$	Increasing
$(-1,2)$	$x=0$	$f'(0)=6(0)^2-6(0)-12$ $=12<0$	Decreasing
$(2,\infty)$	$x=3$	$f'(3)=6(3)^2-6(3)-12$ $=24>0$	Increasing

Hence the function $f(x)=2x^3-3x^2-12x$:

- Increases on $(-\infty,-1)$ and $(2,\infty)$.
- Decreases on $(-1,2)$.

Example 19: Find the intervals where the function $f(x)=x^{\frac{2}{3}}$ is increasing or decreasing on the domain $x\in(-\infty,\infty)$.

Solution:

$$f(x)=x^{\frac{2}{3}}, \text{ where } x\in(-\infty,\infty)$$

Differentiate w.r.t. 'x'

$$f'(x)=\frac{d}{dx}(x^{\frac{2}{3}})$$
$$=\frac{2}{3}x^{\frac{2}{3}-1}=\frac{2}{3}x^{\frac{-1}{3}}$$
$$f'(x)=\frac{2}{3x^{\frac{1}{3}}}$$

It is clear that $f'(x)=\frac{2}{3x^{\frac{1}{3}}}$ is not defined at $x=0 \Rightarrow x=0$ is a critical point.

For other critical points, put $f'(x)=0$

$$\Rightarrow \quad \frac{2}{3x^{\frac{1}{3}}}=0$$
$$2=0 \text{ (False)}$$

So, $f'(x)=0$ has no solution.

Real Numbers Line:

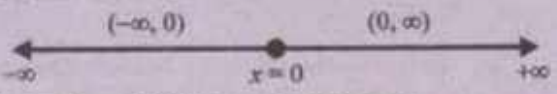

Table for sign of $f'(x)$ in each subinterval.

Interval	Test Point	Sign of $f'(x)$	Conclusion
$(-\infty,0)$	$x=-1$	$f'(-1)=\frac{2}{3(-1)^{1/3}}$ $=\frac{-2}{3}<0$	Decreasing
$(0,\infty)$	$x=1$	$f'(1)=\frac{2}{3(1)^{1/3}}$ $=\frac{2}{3}>0$	Increasing

Hence, the function $f(x)=x^{\frac{2}{3}}$ defined on $(-\infty,\infty)$:

- Decreases on $(-\infty,0)$
- Increases on $(0,\infty)$

RELATIVE AND ABSOLUTE EXTREMA

Global (Absolute) Maximum

Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f has a global (absolute) maximum at $x=c\in I$, if $f(c)\geq f(x)$ for all $x\in I$.

If $f(x)$ has a global (absolute) maximum at $x=c$, then $f(c)$ is called global (absolute) maximum value of the function.

Global (Absolute) Minimum

Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f has a global (absolute) minimum at $x=c\in I$, if $f(c)\leq f(x)$ for all $x\in I$.

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If $f(x)$ has a global (absolute) minimum at $x = c$, then $f(c)$ is called global (absolute) minimum value of the function.

Note: The global maximum and minimum of a function are collectively called its global extrema.

Examples:

(a) Let $f(x) = x^2$ be defined on the interval $[0, 2]$.

The function f has a global maximum at $x = 2$ and a global minimum at $x = 0$. The corresponding maximum and minimum values of the function are:

$f(2) = 4$ (global maximum value)

and $f(0) = 0$ (global minimum value) (See Figure 2.2).

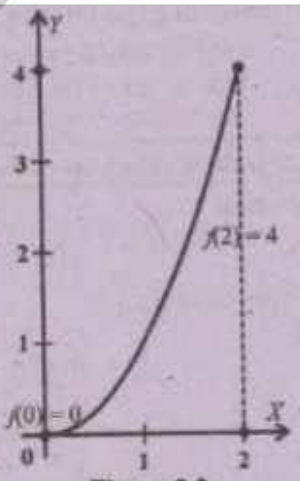

Figure 2.2

(b) The function $f(x) = x^2$ defined on $(-\infty, \infty)$ has an absolute minimum value at $x = 0$, where $f(0) = 0$. It has no absolute maximum value. (See Figure 2.3)

(c) The function $f(x) = -x^2 + 2$ defined on $(-\infty, \infty)$ has an absolute maximum value at $x = 0$, where $f(0) = 2$. It has no absolute minimum value. (See Fig. 2.4)

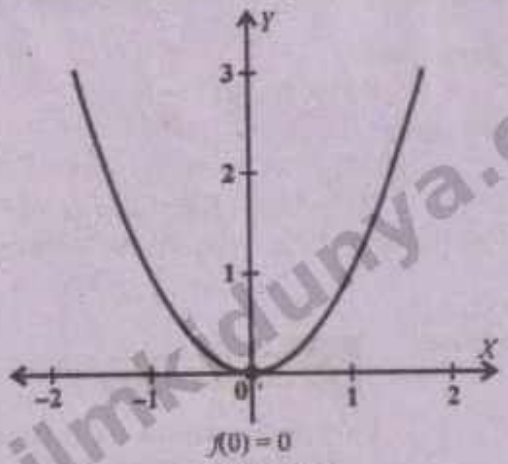

Figure 2.3

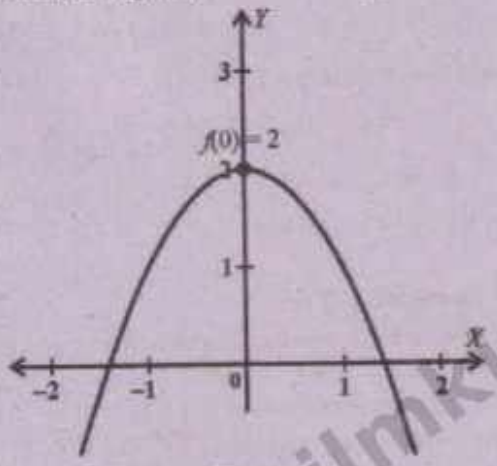

Figure 2.4

Local (Relative) Maxima	Local (Relative) Minima
Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f has a local (relative) maxima at $x = c \in I$, if there exists an open interval I_1 such that $c \in I_1$ and: $f(c) \geq f(x)$ for all $x \in I \cap I_1$ If $f(x)$ has a local (relative) maximum at $x = c$, then $f(c)$ is called local (relative) maximum value of the function. (See Figure 2.5)	Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f has a local (relative) minima at $x = e \in I$, if there exists an open interval I_1 such that $e \in I_1$ and: $f(e) \leq f(x)$ for all $x \in I \cap I_1$ If $f(x)$ has a local (relative) minimum at $x = e$, then $f(e)$ is called local (relative) minimum value of the function. (See Figure 2.5)

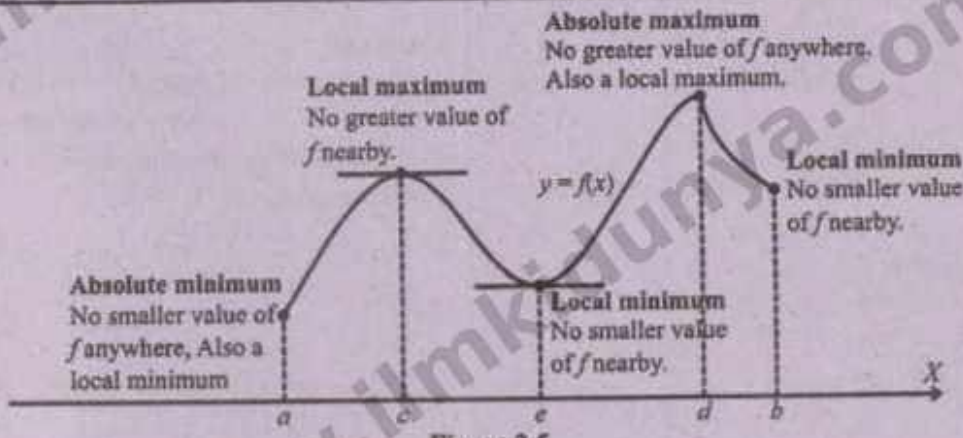

Figure 2.5

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Method to Find Global Extrema of a Continuous Functions Defined on a Closed Interval $[a, b]$

1. Find all critical points of the function within the interval $[a, b]$.
2. Find values of the function at critical points and at end points of the interval.
3. The largest of these values is the global maximum value, and the smallest is the global minimum value.

Example 20: Find the global extreme values of $f(x) = x^4 - 4x$ on $[-2, 3]$.

Solution:

$$f(x) = x^4 - 4x; x \in [-2, 3]$$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}(x^4 - 4x)$$
$$= 4x^3 - 4(1)$$
$$f'(x) = 4x^3 - 4$$

It is clear that $f'(x) = 4x^3 - 4$ is defined on $[-2, 3]$.

For other critical points, put $f'(x) = 0$

$$\Rightarrow \quad 4x^3 - 4 = 0$$
$$4(x^3 - 1^3) = 0$$
$$(x-1)(x^2 + x + 1) = 0$$

Either $x - 1 = 0$ or $x^2 + x + 1 = 0$

$x = 1$

Here $a = 1, b = 1, c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$
$$x = \frac{-1 \pm \sqrt{-3}}{2} \text{ (Complex roots)}$$

Critical point value:

$$f(1) = 1^4 - 4(1) = -3$$

End point values:

$$f(-2) = (-2)^4 - 4(-2) = 16 + 8 = 24$$
$$f(3) = (3)^4 - 4(3) = 81 - 12 = 69$$

- Global Maximum = 69
- Global Minimum = −3

Second Derivative Test

Let f be a twice differentiable function at $x = c$, where c lies in the domain of f. Suppose that $f'(c) = 0$. Then:

1. f has relative maximum at c if $f''(c) < 0$.
2. f has relative minimum at c, if $f''(c) > 0$.
3. If $f''(c) = 0$, then the test is inconclusive, and the nature of the critical point cannot be determined using the second derivative test alone.



Critical Thinking

If first derivative is not defined at a critical point, can we apply second derivative test at that critical point.

Example 21: Examine the function $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ for relative extreme values.

Solution:

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}\left(\frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1\right)$$
$$= \frac{1}{2}(4x^3) - \frac{4}{3}(3x^2) - 2x + 4(1) + 0$$
$$f'(x) = 2x^3 - 4x^2 - 2x + 4$$
$$f''(x) = 2(3x^2) - 4(2x) - 2(1) + 0$$
$$= 6x^2 - 8x - 2$$

For critical points, put $f'(x) = 0$

$$2x^3 - 4x^2 - 2x + 4 = 0$$
$$x^3 - 2x^2 - x + 2 = 0 \quad \text{(Dividing by '2')}$$
$$x^3 - x - 2x^2 + 2 = 0$$
$$x(x^2 - 1) - 2(x^2 - 1) = 0$$
$$(x-2)(x^2 - 1) = 0$$
$$(x-2)(x-1)(x+1) = 0$$
$$x - 2 = 0 \; ; \; x - 1 = 0 \; ; \; x + 1 = 0$$
$$x = 2 \; ; \; x = 1 \; ; \; x = -1$$

At $x = -1$:

$$f''(-1) = 6(-1)^2 - 8(-1) - 2$$
$$= 6 + 8 - 2 = 12 > 0$$

$\Rightarrow$ f has relative minimum at $x = -1$.

Minimum value $= f(-1)$

$$= \frac{1}{2}(-1)^4 - \frac{4}{3}(-1)^3 - (-1)^2 + 4(-1) + 1$$
$$= \frac{1}{2} + \frac{4}{3} - 1 - 4 + 1 = \frac{1}{2} + \frac{4}{3} - 4$$
$$= \frac{3 + 8 - 24}{6} = \frac{-13}{6}$$

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At $x = 1$:

$$f'(1) = 6(1)^2 - 8(1) - 2$$
$$= 6 - 8 - 2 = -4 < 0$$

$\Rightarrow$ f has relative maximum at $x = 1$.

Maximum value $= f(1)$

$$= \frac{1}{2}(1)^4 - \frac{4}{3}(1)^3 - (1)^2 + 4(1) + 1$$
$$= \frac{1}{2} - \frac{4}{3} - 1 + 4 + 1 = \frac{1}{2} - \frac{4}{3} + 4$$
$$= \frac{3 - 8 + 24}{6} = \frac{19}{6}$$

At $x = 2$:

$$f'(2) = 6(2)^2 - 8(2) - 2$$
$$= 24 - 16 - 2 = 6 > 0$$

$\Rightarrow$ f has relative minimum at $x = 2$.

Minimum value $= f(2)$

$$= \frac{1}{2}(2)^4 - \frac{4}{3}(2)^3 - (2)^2 + 4(2) + 1$$
$$= 8 - \frac{32}{3} - 4 + 8 + 1 = 13 - \frac{32}{3}$$
$$= \frac{39 - 32}{3} = \frac{7}{3}$$

EXERCISE 2.4

Q1. Determine the intervals on which f is increasing or decreasing for the domain mentioned in each case.

(i) $f(x) = \sin x; x \in (0, 2\pi)$

Solution:

$$f(x) = \sin x$$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}(\sin x)$$
$$f'(x) = \cos x$$

It is clear that $f'(x) = \cos x$ is defined for each $x \in (0, 2\pi)$.

For other critical points, put $f'(x) = 0$

$$\cos x = 0$$

$\Rightarrow$ $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Real Numbers Line:

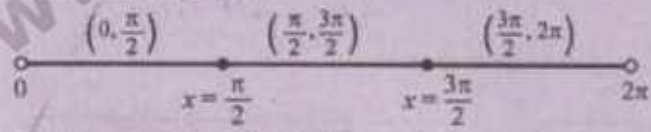

Table for sign of $f'(x)$ in each subinterval:

Interval	Test point	Sign of $f'(x)$	Conclusion
$\left(0, \frac{\pi}{2}\right)$	$x = \frac{\pi}{3}$	$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2} > 0$	Increasing
$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$	$x = \pi$	$f'(\pi) = \cos\pi = -1 < 0$	Decreasing
$\left(\frac{3\pi}{2}, 2\pi\right)$	$x = \frac{7\pi}{4}$	$f'\left(\frac{7\pi}{4}\right) = \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}} > 0$	Increasing

$\Rightarrow$ f is increasing on interval $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$ and decreasing on interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

(ii) $f(x) = 4 - x^2; x \in \mathbb{R} = (-\infty, \infty)$

Solution:

$$f(x) = 4 - x^2$$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}(4 - x^2)$$
$$f'(x) = 0 - 2x = -2x$$

It is clear that $f'(x) = -2x$ is defined for each $x \in \mathbb{R}$.

For other critical point, put $f'(x) = 0$

$\Rightarrow$ $-2x = 0$ $\Rightarrow$ $x = 0$

Real Numbers Line:

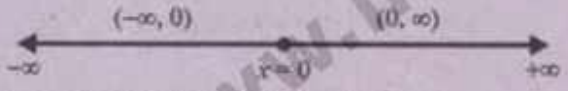

Table for sign of $f'(x)$ in each subinterval:

Interval	Test point	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	$x = -1$	$f'(-1) = -2(-1) = 2 > 0$	Increasing
$(0, \infty)$	$x = 1$	$f'(1) = -2(1) = -2 < 0$	Decreasing

$\Rightarrow$ f is increasing on interval $(-\infty, 0)$ and decreasing on interval $(0, \infty)$.

(iii) $f(x) = x^2 + 3x + 2; x \in [-4, 1]$

Solution:

$$f(x) = x^2 + 3x + 2$$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}(x^2 + 3x + 2)$$
$$= 2x + 3(1) + 0$$
$$f'(x) = 2x + 3$$

It is clear that $f'(x) = 2x + 3$ is defined for each $x \in [-4, 1]$.

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For other critical points, put $f'(x)=0$

$\Rightarrow \quad 2x+3=0 \Rightarrow \quad 2x=-3 \quad \Rightarrow \quad x=-\frac{3}{2}$

Real Numbers Line:

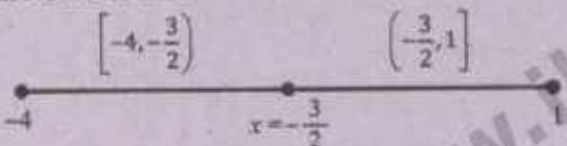

Table for sign of $f'(x)$ in each subinterval:

Interval	Test point	Sign of $f'(x)$	Conclusion
$\left[-4,-\frac{3}{2}\right)$	$x=-2$	$f'(-2)=2(-2)+3=-1<0$	Decreasing
$\left(-\frac{3}{2},1\right]$	$x=0$	$f'(0)=2(0)+3=3>0$	Increasing

$\Rightarrow$ f is decreasing on interval $\left[-4,-\frac{3}{2}\right)$ and increasing on interval $\left(-\frac{3}{2},1\right]$.

(iv) $f(x)=\sin x+\frac{1}{4}\cos 2x; x\in[0,2\pi]$

Solution:

$$f(x)=\sin x+\frac{1}{4}\cos 2x$$

Differentiate w.r.t. 'x'

$$f'(x)=\frac{d}{dx}\left(\sin x+\frac{1}{4}\cos 2x\right)$$
$$=\cos x+\frac{1}{4}(-\sin 2x\cdot 2)$$
$$=\cos x-\frac{1}{2}\sin 2x$$
$$=\cos x-\frac{1}{2}(2\sin x\cdot\cos x)$$
$$=\cos x-\sin x\cos x$$
$$f'(x)=\cos x(1-\sin x)$$

It is clear that $f'(x)=\cos x(1-\sin x)$ is defined for each $x\in[0,2\pi]$.

For other critical points, put $f'(x)=0$

$\Rightarrow \quad \cos x(1-\sin x)=0$

$\cos x=0$ and $1-\sin x=0$

$x=\frac{\pi}{2},\frac{3\pi}{2}$ and $\sin x=1 \Rightarrow x=\frac{\pi}{2}$

Real Numbers Line:

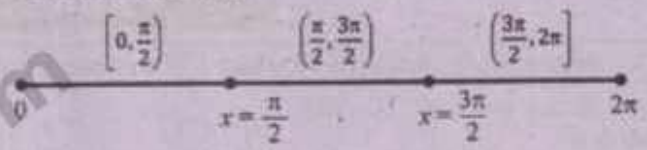

Table for sign of $f'(x)$ in each subinterval.

Interval	Test point	Sign of $f'(x)$	Conclusion
$\left[0,\frac{\pi}{2}\right)$	$x=0$	$f'(0)=\cos 0(1-\sin 0)$ $=1>0$	Increasing
$\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$	$x=\pi$	$f'(\pi)=\cos\pi(1-\sin\pi)$ $=-1<0$	Decreasing
$\left(\frac{3\pi}{2},2\pi\right]$	$x=2\pi$	$f'(2\pi)=\cos 2\pi(1-\sin 2\pi)$ $=1>0$	Increasing

$\Rightarrow$ 'f is increasing on interval $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right]$ and decreasing on interval $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$.

Q2. Using the Second Derivative Test, find local extreme values of the following functions:

(i) $f(x)=8-2x^2$

Solution:

$$f(x)=8-2x^2$$

Differentiate w.r.t. 'x'

$$f'(x)=\frac{d}{dx}(8-2x^2)$$
$$=0-2(2x)$$
$$=-4x$$
$$f''(x)=-4$$

For critical points, put $f'(x)=0$

$$-4x=0 \qquad \Rightarrow x=0$$

At $x=0$:

$$f''(0)=-4<0$$

$\Rightarrow$ f has local maximum at $x=0$

Maximum value $=f(0)$

$$=8-2(0)^2=8$$

(ii) $f(x)=x^2-x-2$

Solution:

$$f(x)=x^2-x-2$$

Differentiate w.r.t. 'x'

$$f'(x)=2x-1-0=2x-1$$
$$f''(x)=2(1)-0=2$$

For critical points, put $f'(x)=0$

$$2x-1=0 \qquad \Rightarrow x=\frac{1}{2}$$

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At $x=\frac{1}{2}$:

$$f''\left(\frac{1}{2}\right)=2>0$$

$\Rightarrow$ f has local minimum at $x=\frac{1}{2}$

Minimum value $=f\left(\frac{1}{2}\right)$

$$=\left(\frac{1}{2}\right)^2-\frac{1}{2}-2=\frac{1}{4}-\frac{1}{2}-2$$

$$=\frac{1-2-8}{4}=-\frac{9}{4}$$

(iii) $f(x)=x^4-8x^2+2$

Solution:

$$f(x)=x^4-8x^2+2$$

Differentiate w.r.t. 'x'

$$f'(x)=4x^3-8(2x)+0$$
$$=4x^3-16x$$
$$f''(x)=4(3x^2)-16(1)$$
$$=12x^2-16$$

For critical point, put $f'(x)=0$

$$4x^3-16x=0$$
$$4x(x^2-4)=0$$
$$x(x-2)(x+2)=0$$
$$x=0 \quad ; \quad x-2=0 \quad ; \quad x+2=0$$
$$x=2 \quad ; \quad x=-2$$

At $x=0$:

$$f''(0)=12(0)^2-16=-16<0$$

$\Rightarrow$ f has local maximum at $x=0$

Maximum value $=f(0)$

$$=(0)^4-8(0)^2+2$$
$$=2$$

At $x=2$:

$$f''(2)=12(2)^2-16=32>0$$

$\Rightarrow$ f has local minimum at $x=2$

Minimum value $=f(2)$

$$=(2)^4-8(2)^2+2$$
$$=16-32+2=-14$$

At $x=-2$:

$$f''(-2)=12(-2)^2-16=32>0$$

$\Rightarrow$ f has local minimum at $x=-2$

Minimum value $=f(-2)$

$$=(-2)^4-8(-2)^2+2$$
$$=16-32+2=-14$$

(iv) $f(x)=(x+2)^2(x-3)$

Solution:

$$f(x)=(x+2)^2(x-3)$$

Differentiate w.r.t. 'x'

$$f'(x)=\frac{d}{dx}\{(x+2)^2(x-3)\}$$
$$=(x+2)^2\frac{d}{dx}(x-3)+(x-3)\frac{d}{dx}(x+2)^2$$
$$=(x+2)^2\cdot(1)+(x-3)\cdot 2(x+2)(1)$$
$$=(x+2)(x+2+2x-6)$$
$$=(x+2)(3x-4)$$
$$f''(x)=\frac{d}{dx}\{(x+2)(3x-4)\}$$
$$=(x+2)(3)+(3x-4)(1)$$
$$=3x+6+3x-4$$
$$=6x+2$$

For critical points, put $f'(x)=0$

$$(x+2)(3x-4)=0$$
$$x+2=0 \quad ; \quad 3x-4=0$$
$$x=-2 \quad ; \quad x=\frac{4}{3}$$

At $x=-2$:

$$f''(-2)=6(-2)+2=-10<0$$

$\Rightarrow$ f has local maximum at $x=-2$

Maximum value $=f(-2)$

$$=(-2+2)^2(-2-3)$$
$$=(0)^2(-5)=0$$

At $x=\frac{4}{3}$:

$$f''\left(\frac{4}{3}\right)=6\left(\frac{4}{3}\right)+2=10>0$$

$\Rightarrow$ f has local minimum at $x=\frac{4}{3}$.

Minimum value $=f\left(\frac{4}{3}\right)$

$$=\left(\frac{4}{3}+2\right)^2\left(\frac{4}{3}-3\right)$$
$$=\left(\frac{10}{3}\right)^2\left(-\frac{5}{3}\right)$$
$$=\frac{100}{9}\times\frac{-5}{3}=-\frac{500}{27}$$

(v) $f(x)=(x^2-2x)e^x$

Solution:

$$f(x)=(x^2-2x)e^x$$

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Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}\{(x^2-2x)\cdot e^x\}$$
$$= (x^2-2x)\frac{d}{dx}(e^x)+e^x\cdot\frac{d}{dx}(x^2-2x)$$
$$= (x^2-2x)e^x\cdot 1+e^x(2x-2)$$
$$= e^x(x^2-2x+2x-2)$$
$$= e^x(x^2-2)$$
$$f''(x) = \frac{d}{dx}(e^x(x^2-2))$$
$$= e^x(2x-0)+(x^2-2)e^x\cdot 1$$
$$= e^x(2x+x^2-2)$$
$$= e^x(x^2+2x-2)$$

For critical point, put $f'(x)=0$

$$e^x(x^2-2) = 0$$

Since $e^x \neq 0$, so $x^2-2=0$

$$x^2 = 2 \quad \Rightarrow \quad x = \pm\sqrt{2}$$

At $x=\sqrt{2}$:

$$f''(\sqrt{2}) = e^{\sqrt{2}}((\sqrt{2})^2+2\sqrt{2}-2)$$
$$= e^{\sqrt{2}}(2+2\sqrt{2}-2)$$
$$= 2\sqrt{2}e^{\sqrt{2}} > 0$$

$\Rightarrow$ f has local minimum at $x=\sqrt{2}$.

Minimum value $= f(\sqrt{2})$
$$= ((\sqrt{2})^2-2\sqrt{2})e^{\sqrt{2}}$$
$$= (2-2\sqrt{2})e^{\sqrt{2}}$$

At $x=-\sqrt{2}$:

$$f''(-\sqrt{2}) = e^{-\sqrt{2}}((-\sqrt{2})^2+2(-\sqrt{2})-2)$$
$$= e^{-\sqrt{2}}(2-2\sqrt{2}-2) = \frac{-2\sqrt{2}}{e^{\sqrt{2}}} < 0$$

$\Rightarrow$ f has local maximum at $x=-\sqrt{2}$.

Maximum value $= f(-\sqrt{2})$
$$= ((-\sqrt{2})^2-2(-\sqrt{2}))e^{-\sqrt{2}}$$
$$= (2+2\sqrt{2})e^{-\sqrt{2}}$$

Q3. Discuss the global extrema of the following functions over specified domains.

(i) $f(x)=\sin x+\cos x$ on $[0,2\pi]$

Solution:

$$f(x) = \sin x+\cos x\,;\, x\in[0,2\pi]$$

Differentiate w.r.t. 'x'

$$f'(x) = \cos x-\sin x$$

It is clear that $f'(x)=\cos x-\sin x$ is defined on $[0,2\pi]$.

For other critical points, put $f'(x)=0$

$$\cos x-\sin x = 0$$
$$\cos x = \sin x \quad \Rightarrow \quad \frac{\sin x}{\cos x}=1$$
$$\tan x = 1$$
$$\Rightarrow \quad x = \tan^{-1}(1)$$
$$x = 45^\circ, 225^\circ = \frac{\pi}{4}, \frac{5\pi}{4} \in [0,2\pi]$$

Critical points value:

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4}+\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$
$$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4}+\cos\frac{5\pi}{4} = \frac{-1}{\sqrt{2}}-\frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

End points value:

$$f(0) = \sin 0+\cos 0 = 0+1 = 1$$
$$f(2\pi) = \sin 2\pi+\cos 2\pi = 0+1 = 1$$

$\Rightarrow$
- Global Maximum $=\sqrt{2}$ at $x=\frac{\pi}{4}$
- Global Minimum $=-\sqrt{2}$ at $x=\frac{5\pi}{4}$

(ii) $f(x)=x^{4/3}$ on $[-1,8]$

Solution:

$$f(x) = x^{\frac{4}{3}}; x\in[-1,8]$$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}(x^{\frac{4}{3}})$$
$$= \frac{4}{3}x^{\frac{4}{3}-1}\cdot 1 = \frac{4}{3}x^{\frac{1}{3}}$$

It is clear that $f'(x)=\frac{4}{3}x^{\frac{1}{3}}$ is defined on $[-1,8]$.

For other critical points, put $f'(x)=0$

$$\frac{4}{3}x^{\frac{1}{3}} = 0$$
$$x^{\frac{1}{3}} = 0 \quad \Rightarrow x=0$$

Critical points value:

$$f(0) = (0)^{4/3} = 0$$

End points value:

$$f(-1) = (-1)^{4/3} = [(-1)^4]^{1/3} = (1)^{1/3} = 1$$
$$f(8) = (8)^{4/3} = (2^3)^{4/3} = 2^4 = 16$$

$\Rightarrow$
- Global Maximum $=16$ at $x=8$
- Global Minimum $=0$ at $x=0$

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Q4. Find the global and local extrema of the following functions over the specified domains. (Hint: Use the increasing and decreasing behavior of the functions including the end points).

(i) $f(x) = 4 - x^2$ on $[-1, 2]$

Solution:

$f(x) = 4 - x^2$ on $[-1, 2]$

Differentiate w.r.t 'x'

$f'(x) = -2x$

For critical points, put $f'(x) = 0$

$-2x = 0 \Rightarrow x = 0$

$\Rightarrow$ Subintervals are $[-1, 0)$ and $(0, 2]$.

If $x \in [-1, 0)$, then $f'(-0.5) = -2(-0.5) = 1 > 0$ (increasing)

If $x \in (0, 2]$, then $f'(1) = -2(1) = -2 < 0$ (decreasing)

$\Rightarrow$ f has local maximum at $x = 0$

Critical point value:

$f(0) = 4 - 0^2 = 4$

End points values:

$f(-1) = 4 - (1)^2 = 4 - 1 = 3$

$f(2) = 4 - 2^2 = 4 - 4 = 0$

Conclusion:

- Global Max = 4 at $x = 0$ (Also a local maximum)
- Global Min = 0 at $x = 2$
- Local Min = 3 at $x = -1$

(ii) $f(x) = \cos x - \sin x$ on $[0, 2\pi]$

Solution:

$f(x) = \cos x - \sin x$ on $[0, 2\pi]$

Differentiate w.r.t 'x'

$f'(x) = -\sin x - \cos x$

For critical points, put $f'(x) = 0$

$-\sin x - \cos x = 0 \Rightarrow \sin x = -\cos x \Rightarrow \dfrac{\sin x}{\cos x} = -1$

$\tan x = -1 < 0$ (2nd and 4th quadrants)

2nd–Quad	4th–Quad
$x = \pi - \dfrac{\pi}{4}$	$x = 2\pi - \dfrac{\pi}{4}$
$x = \dfrac{3\pi}{4}$	$x = \dfrac{7\pi}{4}$

$\Rightarrow$ Subintervals are $\left[0, \dfrac{3\pi}{4}\right), \left(\dfrac{3\pi}{4}, \dfrac{7\pi}{4}\right)$ and $\left(\dfrac{7\pi}{4}, 2\pi\right]$

If $x \in \left[0, \dfrac{3\pi}{4}\right)$, then $f'\left(\dfrac{\pi}{2}\right) = -\sin\dfrac{\pi}{2} - \cos\dfrac{\pi}{2} = -1 < 0$

(decreasing)

If $x \in \left(\dfrac{3\pi}{4}, \dfrac{7\pi}{4}\right)$, then $f'(\pi) = -\sin\pi - \cos\pi = 1 > 0$

(increasing)

If $x \in \left(\dfrac{7\pi}{4}, 2\pi\right]$, then $f'(2\pi) = -\sin 2\pi - \cos 2\pi = -1 < 0$

(decreasing)

$\Rightarrow$ f has local minimum at $x = \dfrac{3\pi}{4}$ and local maximum at $x = \dfrac{7\pi}{4}$.

Critical points value:

$$f\left(\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} - \sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$f\left(\frac{7\pi}{4}\right) = \cos\frac{7\pi}{4} - \sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

End points value:

$f(0) = \cos 0 - \sin 0 = 1 - 0 = 1$

$f(2\pi) = \cos 2\pi - \sin 2\pi = 1 - 0 = 1$

Conclusion:

- Global & Local Max = $\sqrt{2}$ at $x = \dfrac{7\pi}{4}$
- Global & Local Min = $-\sqrt{2}$ at $x = \dfrac{3\pi}{4}$
- Local Min = 1 at $x = 2\pi$
- Local Max = 1 at $x = 0$

Q5. Prove that $f(x) = \dfrac{\ell n(2x)}{x^2}$ has local maximum at $x = \dfrac{\sqrt{e}}{2}$.

Solution:

$$f(x) = \frac{\ell n(2x)}{x^2}$$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}\left(\frac{\ell n(2x)}{x^2}\right)$$

$$= \frac{x^2 \cdot \frac{d}{dx}(\ell n(2x)) - \ell n(2x) \cdot \frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2 \cdot \frac{1}{2x}(2) - \ell n(2x) \cdot (2x)}{x^4}$$

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$$f'(x)=\frac{x-2x\cdot \ell n(2x)}{x^4}$$

$$=\frac{x(1-2\ell n(2x))}{x^4}$$

$$f'(x)=\frac{1-2\ell n(2x)}{x^3}$$

Again, differentiate w.r.t. 'x'

$$f''(x)=\frac{d}{dx}\left(\frac{1-2\ell n(2x)}{x^3}\right)$$

$$=\frac{x^3\frac{d}{dx}(1-2\ell n(2x))-(1-2\ell n(2x))\cdot\frac{d}{dx}(x^3)}{(x^3)^2}$$

$$=\frac{x^3\left(0-2\cdot\frac{1}{2x}(2)\right)-(1-2\ell n(2x))\cdot(3x^2)}{x^6}$$

$$=\frac{x^3\left(\frac{-2}{x}\right)-3x^2(1-2\ell n(2x))}{x^6}$$

$$=\frac{-2x^2-3x^2+6x^2\ell n(2x)}{x^6}$$

$$=\frac{-5x^2+6x^2\ell n(2x)}{x^6}$$

$$=\frac{x^2(-5+6\ell n(2x))}{x^6}$$

$$f''(x)=\frac{-5+6\ell n(2x)}{x^4}$$

For critical points, put $f'(x)=0$

$$\frac{1-2\ell n(2x)}{x^4}=0 \Rightarrow 1-2\ell n(2x)=0$$

$$2\ell n(2x)=1 \Rightarrow \ell n(2x)=\frac{1}{2}$$

In Exponential Form:

$$2x=e^{1/2} \Rightarrow x=\frac{\sqrt{e}}{2}$$

At $x=\frac{\sqrt{e}}{2}$:

$$f''\left(\frac{\sqrt{e}}{2}\right)=\frac{-5+6\ell n\left(\frac{2\sqrt{e}}{2}\right)}{\left(\frac{\sqrt{e}}{2}\right)^4}$$

$$=\frac{-5+6\ell n(\sqrt{e})}{\frac{e^2}{16}}=\frac{-5+6\left(\frac{1}{2}\right)\ell n\, e}{\frac{e^2}{16}}$$

$$=\frac{-5+3(1)}{\frac{e^2}{16}}=\frac{-2}{\frac{e^2}{16}} \qquad \because \ell n\, e=1$$

$$f''\left(\frac{\sqrt{e}}{2}\right)=\frac{-32}{e^2}<0$$

Hence, $f(x)=\frac{\ell n(2x)}{x^2}$ has local maximum at $x=\frac{\sqrt{e}}{2}$.

ent Outline

RIZATION)

k with, so we use simpler functions to approximate them.
practical purposes.
ngent lines to approximate a function near a specific point.
ith a straight-line approximation, making calculations easier.
ring, and economics for quick, local estimates, especially

ng function
$(x-a)$

$=a$. The point $x=a$ is referred to as the center of

Critical points

$$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\sqrt{2}$$

$$f\left(\frac{5\pi}{4}\right)=\sin\frac{5\pi}{4}+\cos\frac{5\pi}{4}=-\sqrt{2}$$

End points value:

$$f(0)=\sin 0+\cos 0=0+1=1$$

$$f(2\pi)=\sin 2\pi+\cos 2\pi=0+1=1$$

$\Rightarrow$

- Global Maximum $=\sqrt{2}$ at $x=\frac{\pi}{4}$
- Global Minimum $=-\sqrt{2}$ at $x=\frac{5\pi}{4}$

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Recall!

A function of the form $y = mx + c$, where m is slope and c is the y-intercept, is called a linear function. The graph of a linear function is a straight line.

Let $(a, f(a))$ be a point on the graph of $y = f(x)$. If $f(x)$ is differentiable, then the equation of tangent line at $(a, f(a))$ is given by:

$$y = f(a) + f'(a)(x - a)$$

We can see in Fig. 2.6, the tangent line $L(x) = 2x - 1$ to the graph of the curve $f(x) = x^2$ at $(1, f(1))$ lies close to the curve near the point of tangency. For a small interval on either side of $x = 1$, y-values of the tangent line provide good approximations to the y-values of the curve $f(x) = x^2$. We observe this phenomenon by observing the table of values for the difference between the values of $L(x)$ and $f(x)$. The phenomenon is true not just for this curve; every differentiable curve behaves locally like its tangent line.

x	0.7	0.8	0.9	1	1.1	1.2	1.3
$f(x) = x^2$	0.49	0.64	0.81	1	1.21	1.44	1.69
$L(x) = 2x - 1$	0.4	0.6	0.8	1	1.2	1.4	1.6
$f(x) - L(x)$	0.09	0.04	0.01	0	0.01	0.04	0.09

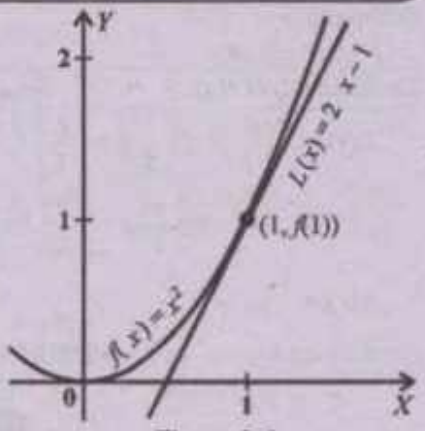

Figure 2.6

Example 22:

(a) Find the local linear approximation of $f(x) = \sqrt{2x-1}$ at $x = 5$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{9.2}$, and compare your approximation to the result produced directly by calculator.

Solution:

(a) $f(x) = \sqrt{2x-1}$ at $x = 5$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}(2x-1)^{\frac{1}{2}}$$

$$= \frac{1}{2}(2x-1)^{\frac{-1}{2}} \cdot (2)$$

$$f'(x) = \frac{1}{\sqrt{2x-1}}$$

Linearization of $f(x)$:

$$L(x) = f(5) + f'(5)(x-5)$$

$$= \sqrt{2(5)-1} + \frac{1}{\sqrt{2(5)-1}}(x-5)$$

$$= \sqrt{9} + \frac{1}{\sqrt{9}}(x-5)$$

$$= 3 + \frac{1}{3}(x-5)$$

$$= \frac{1}{3}(9 + x - 5)$$

$$L(x) = \frac{1}{3}(x+4)$$

Local linear approximation of $f(x)$:

$$f(x) \approx L(x)$$

$$\sqrt{2x-1} \approx \frac{1}{3}(x+4) \quad \ldots(1)$$

(b) To approximate $\sqrt{9.2}$, set

$$2x - 1 = 9.2 \quad \Rightarrow 2x = 10.2$$

$$x = \frac{10.2}{2} \quad \Rightarrow x = 5.1$$

Put $x = 5.1$ in eq. (1), we get

$$\sqrt{2(5.1)-1} \approx \frac{1}{3}(5.1+4)$$

$$\sqrt{9.2} \approx 3.0333 \quad \ldots(2)$$

Calculator value:

$$\sqrt{9.2} = 3.0331 \quad \ldots(3)$$

From (2) and (3), it is clear that the calculator value is very close to linear approximation.

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Example 23:

(a) Find the local linear approximation of $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

(b) Use the local linear approximation obtained in part (a) to approximate cos 91°, and compare your approximation to the result produced directly by a calculator.

Solution:

(a) $f(x) = \cos x$ at $x = \frac{\pi}{2}$

Differentiate w.r.t. 'x'

$f'(x) = -\sin x$

Linearization of $f(x)$:

$$L(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)$$

$$= \cos\frac{\pi}{2} - \sin\frac{\pi}{2}\left(x - \frac{\pi}{2}\right) = 0 - 1\left(x - \frac{\pi}{2}\right)$$

$$L(x) = -x + \frac{\pi}{2}$$

Local linear approximation of $f(x)$:

$$f(x) \approx L(x)$$

$$\cos x \approx -x + \frac{x}{2} \quad \ldots(1)$$

(b) To approximate cos 91°, put

$$x = 91° = \frac{91\pi}{180} \text{ rad in eq. (1)}$$

$$\cos 91° \approx \frac{-91\pi}{180} + \frac{\pi}{2}$$

$$\cos 91° \approx -1.58824962 + 1.57079633$$

$$\cos 91° \approx -0.01745329 \quad \ldots(2)$$

Calculator value:

$$\cos 91° = -0.01745240 \quad \ldots(3)$$

From (2) and (3), it is clear that the calculator value is very close to linear approximation.

EXERCISE 2.5

Q1. Find the linear approximation of $f(x) = \sqrt{x+1}$ at $x = 3$, and use it to approximate $\sqrt{3.1}$ and $\sqrt{2.9}$.

Solution:

$f(x) = \sqrt{x+1}$ at $x = 3$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}(x+1)^{\frac{1}{2}}$$

$$= \frac{1}{2}(x+1)^{\frac{-1}{2}} \cdot 1$$

$$f'(x) = \frac{1}{2\sqrt{x+1}}$$

Linearization of $f(x)$ at $x = 3$:

$$L(x) = f(3) + f'(3)(x-3)$$

$$= \sqrt{3+1} + \frac{1}{2\sqrt{3+1}}(x-3)$$

$$= \sqrt{4} + \frac{1}{2\sqrt{4}}(x-3)$$

$$L(x) = 2 + \frac{1}{4}(x-3) \quad \ldots(1)$$

Local linear approximation of $f(x)$:

$$f(x) \approx L(x)$$

$$\sqrt{x+1} \approx 2 + \frac{1}{4}(x-3) \quad \ldots(1)$$

(a) To approximate $\sqrt{3.1}$, set

$$x + 1 = 3.1 \Rightarrow x = 3.1 - 1 \Rightarrow x = 2.1$$

Put $x = 2.1$ in (1), we get

$$\sqrt{2.1+1} \approx 2 + \frac{1}{4}(2.1-3)$$

$$\sqrt{3.1} \approx 2 - 0.225 = 1.775$$

(b) To approximate $\sqrt{2.9}$, set

$$x + 1 = 2.9 \Rightarrow x = 2.9 - 1 \Rightarrow x = 1.9$$

Put $x = 1.9$ in (1), we get

$$\sqrt{1.9+1} \approx 2 + \frac{1}{4}(1.9-3)$$

$$\sqrt{2.9} \approx 2 - 0.275 = 1.725$$

Q2. (a) Find the local linear approximation of $f(x) = \sin x$ at $x = 0$.

Solution:

$f(x) = \sin x$ at $x = 0$

Differentiate w.r.t. 'x'

$f'(x) = \cos x$

Linearization of $f(x)$ at $x = 0$:

$$L(x) = f(0) + f'(0)(x-0)$$

$$= \sin 0 + \cos 0(x-0) = 0 + 1(x)$$

$$L(x) = x$$

Local linear approximation of $f(x)$:

$$f(x) \approx L(x)$$

$$\sin x \approx x \quad \ldots(1)$$

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(b) Use the local linear approximation obtained in part (a) to approximate sin 2°, and compare your approximation to the result produced directly by a calculator.

Solution:

To approximate sin 2°, put

$$x = 2° = 2\times\frac{\pi}{180}\text{rad} = \frac{\pi}{90} \text{ in (1)}$$

$$\sin 2° \approx \frac{\pi}{90}$$

$$\sin 2° \approx 0.034907 \quad \ldots(2)$$

Calculator Value:

$$\sin 2° = 0.034899 \quad \ldots(3)$$

From (2) and (3), it is clear that the calculator value is very close to linear approximation.

Q3. Find the linear approximation of the following functions at the given points:

(i) $f(x)=\sqrt{x^2+16}$ at $x=3$

Solution:

$$f(x) = \sqrt{x^2+16} \text{ at } x = 3$$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}(x^2+16)^{\frac{1}{2}}$$

$$= \frac{1}{2}(x^2+16)^{-\frac{1}{2}}\cdot(2x)$$

$$f'(x) = \frac{x}{\sqrt{x^2+16}}$$

Linear approximation of $f(x)$ at $x=3$:

$$L(x) = f(3)+f'(3)(x-3)$$

$$= \sqrt{3^2+16}+\frac{3}{\sqrt{3^2+16}}(x-3)$$

$$= \sqrt{25}+\frac{3}{\sqrt{25}}(x-3)$$

$$L(x) = 5+\frac{3}{5}(x-3)$$

(ii) $f(x) = \tan x$ at $x=\pi$

Solution:

$$f(x) = \tan x \text{ at } x = \pi$$

Differentiate w.r.t. 'x'

$$f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$$

Linear approximation of $f(x)$ at $x=\pi$:

$$L(x) = f(x)+f'(\pi)(x-\pi)$$

$$= \tan\pi+\frac{1}{\cos^2\pi}(x-\pi) = 0+\frac{1}{(-1)^2}(x-\pi)$$

$$L(x) = x-\pi$$

(iii) $f(x) = \cos x$ at $x=0$

Solution:

$$f(x) = \cos x \text{ at } x = 0$$

Differentiate w.r.t. 'x'

$$f'(x) = -\sin x$$

Linear approximation of $f(x)$ at $x=0$:

$$L(x) = f(0)+f'(0)(x-0)$$

$$= \cos 0-\sin 0(x-0) = 1-0(x)$$

$$L(x) = 1$$

Q4. Use a local linear approximation to estimate the following:

(i) $(82)^{\frac{1}{3}}$

Solution:

Let $f(x) = x^{\frac{1}{3}}$ at $x=64$

Differentiate w.r.t. 'x'

$$f'(x) = \frac{d}{dx}(x^{\frac{1}{3}})$$

$$= \frac{1}{3}x^{\frac{1}{3}-1}$$

$$= \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$$

Linearization of $f(x)$ at $x=64$:

$$L(x) = f(64)+f'(64)(x-64)$$

$$L(x) = (64)^{\frac{1}{3}}+\frac{1}{3(64)^{\frac{2}{3}}}(x-64)$$

$$= (4^3)^{\frac{1}{3}}+\frac{1}{3(4^3)^{\frac{2}{3}}}(x-64)$$

$$= 4+\frac{1}{48}(x-64) = \frac{1}{48}(192+x-64)$$

$$L(x) = \frac{1}{48}(x+128)$$

Local linear approximation of $f(x)$:

$$f(x) \approx L(x)$$

$$x^{\frac{1}{3}} \approx \frac{1}{48}(x+128)$$

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Put $x = 82$, we get

$$(82)^{\frac{1}{3}} \approx \frac{1}{48}(82+128)$$

$$(82)^{\frac{1}{3}} \approx 4.375$$

(ii) sin 29°

Solution:

Let $f(x) = \sin x$ at $x = 30°$

Differentiate w.r.t. 'x'

$$f'(x) = \cos x$$

Linearization of $f(x)$ at $x = 30°$:

$$L(x) = f(30°) + f'(30°)(x - 30°)$$
$$= \sin 30° + \cos 30°(x - 30°)$$
$$L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - 30°)$$

Local linear approximation of $f(x)$:

$$f(x) \approx L(x)$$
$$\sin x \approx \frac{1}{2} + \frac{\sqrt{3}}{2}(x - 30°)$$

Put $x = 29°$, we get

$$\sin 29° \approx \frac{1}{2} + \frac{\sqrt{3}}{2}(29° - 30°)$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2}(-1°)$$
$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right)$$
$$= 0.5 - 0.01511$$
$$\Rightarrow \quad \sin 29° \approx 0.4849$$

Content Outline

- Differentials
- Error in Differential Approximations

DIFFERENTIALS

Assume that f is differentiable at a point x, if $dx \neq 0$ is an independent variable that can take values in the neighborhood of 0 and not 0 , then we define dependent variable dy, by the formula:

$$dy = f'(x)dx \qquad \ldots(1)$$

Since $dx \neq 0$, we can divide both sides of (1) by dx to obtain:

$$\frac{dy}{dx} = f'(x)$$

This achieves our goal of defining dy and dx so that their ratio is equal to the derivative $f'(x)$.

Example 24:

(a) Find dy if $y = x^4 + 7x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.1$.

Solution:

(a) $y = x^4 + 7x$

Taking differentials on both sides

$$dy = d(x^4 + 7x)$$
$$= d(x^4) + d(7x) = 4x^3dx + 7(1)dx$$
$$dy = (4x^3 + 7)dx \qquad \ldots(1)$$

(b) Put $x = 1$ and $dx = 0.1$ in eq. (1)

$$dy = (4(1)^3 + 7)(0.1)$$
$$= (11)(0.1)$$
$$dy = 1.1$$

Using dy to Approximate Δy

When a function $y = f(x)$ is differentiable at a point $P(x, y)$, we can use the differential dy to estimate the actual change Δy in y, for small change Δx in x.

The actual change Δy in y is given by:

$$\Delta y = f(x + \Delta x) - f(x) \qquad \ldots(1)$$

But calculating Δy exactly often requires complicated formulas. Instead, we use the differential dy, defined as:

$$dy = f'(x)dx$$

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If we take $dx = \Delta x$, then: $dy = f'(x)\Delta x$ ...(2)

Let $Q(x + \Delta x, f(x + \Delta x))$ be a neighboring point of $P(x, y)$ on $y = f(x)$.

- The slope of the tangent line at P is $f'(x)$
- The slope of the secant line connecting P and Q is $\frac{\Delta y}{\Delta x}$

As $\Delta x \to 0$, the point Q approaches P, and secant line PQ approaches the tangent line at P. (See Fig. 2.7)

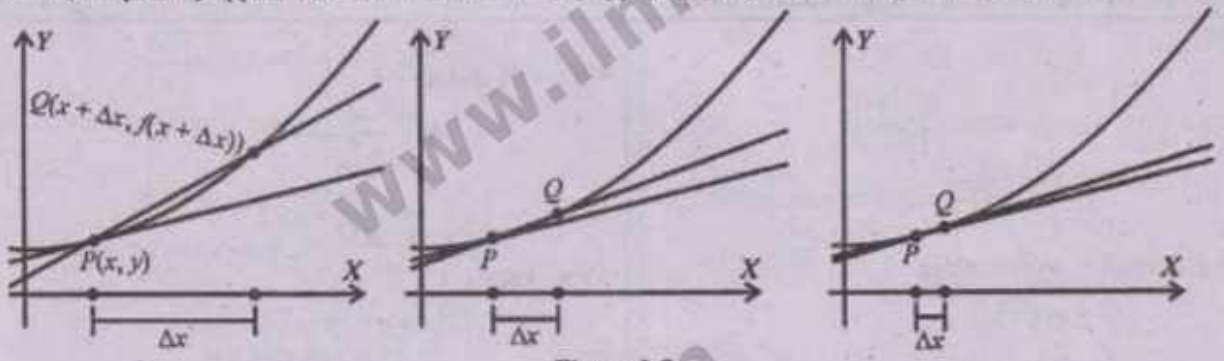

Figure 2.7

Therefore, for small Δx:

Slope of tangent line at $P \approx$ Slope of secant line PQ

$$\Rightarrow \quad f'(x) \approx \frac{\Delta y}{\Delta x}$$

Substitute $f'(x) \approx \frac{\Delta y}{\Delta x}$ into equation (2):

$$dy \approx \frac{\Delta y}{\Delta x} \times \Delta x$$

$$dy \approx \Delta y \text{ (for } \Delta x \text{ is small)}$$

Thus, for small change Δx, the actual change Δy is approximately equal to the differential dy, that is:

$$\Delta y \approx dy = f'(x)dx$$

Using Differentials to Approximate

Since $\Delta y = f(x + \Delta x) - f(x)$ and for small change Δx

$dx = \Delta x$ and $\Delta y \approx dy$

$\Rightarrow dy \approx f(x + \Delta x) - f(x)$

and hence: $(x + \Delta x) \approx f(x) + dy$

ERROR IN DIFFERENTIAL APPROXIMATIONS

Suppose f is a differentiable function at a point x, and there is a small change Δx in x. If:

- Δy is actual corresponding change in y, given by
$$\Delta y = f(x + \Delta x) - f(x)$$
- dy is the approximation change in y obtained using differentials, given by
$$dy = f'(x)dx \text{ where } dx = \Delta x$$

Then:

Error in Δy is: Error $= \Delta y - dy$

Relative error in Δy is: Relative Error $= \frac{\Delta y - dy}{\Delta y}$, when $\Delta y \neq 0$

Percentage error Δy is: Percentage Error $= \left|\frac{\Delta y - dy}{\Delta y}\right| \times 100$ when $\Delta y \neq 0$

Example 25: Find dy and Δy for the function $f(x) = x^3$ when $x = 1$ and $dx = 0.1$.

Solution:

$f(x) = x^3$

$\Rightarrow \quad y = x^3$ $\quad \because f(x) = y$

Taking differentials on both sides

$dy = d(x^3)$

$dy = 3x^2dx$

Put $x = 1$ and $dx = 0.1$

$dy = 3(1)^2(0.1)$

$\boxed{dy = 0.3}$

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As $y = x^3$

$\Rightarrow y + \Delta y = (x + \Delta x)^3$

$\Delta y = (x + \Delta x)^3 - y$

$= (x + \Delta x)^3 - x^3$

Put $x = 1$ and $\Delta x = dx = 0.1$

$\Delta y = (1 + 0.1)^3 - (1)^3$

$= (1.1)^3 - 1 = 1.331 - 1$

$\boxed{\Delta y = 0.331}$

Example 26: Use differentials to approximate $\sqrt{24}$.

Solution:

Let $y = \sqrt{x}$

We take $x = 25$, $dx = \Delta x = 24 - 25 = -1$

$\Rightarrow y = \sqrt{25} = 5$

Now $y = \sqrt{x}$

Taking differentials on both sides

$dy = d(\sqrt{x})$

$= \frac{1}{2\sqrt{x}} dx$

Put $x = 25$ and $dx = 1$

$dy = \frac{1}{2\sqrt{25}}(-1)$

$= \frac{-1}{2(5)} = -0.1$

As we know

$\sqrt{24} \approx y + dy$

$= 5 + (-0.1) = 4.9$

Hence, $\sqrt{24} \approx 4.9$

Example 27: Using differentials approximate the value of cos 61°.

Solution:

Let $y = \cos x$

We take $x = 60°$, $dx = \Delta x = 61° - 60° = 1°$

$\Rightarrow y = \cos 60°$

$y = \frac{1}{2} = 0.5$

Now, $y = \cos x$

Taking differentials on both sides

$dy = d(\cos x)$

$= -\sin x \cdot dx$

Put $x = 60°$ and $dx = 1° = \frac{\pi}{180}$ rad

$dy = -\sin 60° \left(\frac{\pi}{180}\right)$

$= \frac{-\sqrt{3}}{2} \times \frac{\pi}{180}$

$dy = -0.0151$

As we know

$\cos 61° \approx y + dy$

$= 0.5 + (-0.0151)$

$= 0.4849$

Hence, $\cos 61° \approx 0.4849$

Example 28: The side of a cube is measured to be 25 cm with a possible error of ±1 cm.

(a) Use differentials to estimate the error in the calculated volume.

(b) Estimate the percentage errors in the side and volume.

Solution:

Let length of each side of cube = x cm

Volume of cube = $x \cdot x \cdot x$

$V = x^3$

(a) Taking differentials on both sides

$dV = d(x^3)$

$dV = 3x^2 \cdot dx$

Put $x = 25$ and $dx = \pm 1$

$dV = 3(25)^2(\pm 1)$

$= \pm 1875$ cm³ (Error)

(b) Percentage error in side $= \left|\frac{dx}{x}\right| \times 100$

$= \left|\frac{\pm 1}{25}\right| \times 100 = \frac{1}{25} \times 100 = 4\%$

Percentage error in volume $= \left|\frac{dV}{V}\right| \times 100$

$= \left|\frac{\pm 1875}{25^3}\right| \times 100$

$= \frac{1875}{15625} \times 100 = 12\%$

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Example 29: The radius r of a circle is measured with an error of at most 2%. What is the maximum corresponding percentage error in computing the circle's
(a) circumference? (b) area?

Solution:

Radius of circle = r

Percentage error in radius = 2%

$$\left|\frac{dr}{r}\right| \times 100 = 2\% \quad \ldots(1)$$

(a) Circumference of circle = $2\pi r$

$$C = 2\pi r$$

$$\Rightarrow \quad dC = d(2\pi r) = 2\pi dr$$

Percentage error in circumference $= \left|\frac{dC}{C}\right| \times 100$

$$= \left|\frac{2\pi dr}{2\pi r}\right| \times 100$$

$$= \left|\frac{dr}{r}\right| \times 100$$

$= 2\%$ Using eq. (1)

(b) Area of circle = πr^2

$$A = \pi r^2$$

$$\Rightarrow \quad dA = d(\pi r^2)$$

$$= \pi(2r dr)$$

$$= 2\pi r dr$$

Percentage error in area $= \left|\frac{dA}{A}\right| \times 100$

$$= \left|\frac{2\pi r dr}{\pi r^2}\right| \times 100$$

$$= 2\left|\frac{dr}{r}\right| \times 100$$

$= 2(2\%)$

$= 4\%$ Using eq. (1)

EXERCISE 2.6

Q1. Find the differential dy in the following cases:

(i) $y = x^3 - 3\sqrt{x}$

Solution:

$$y = x^3 - 3\sqrt{x}$$

Taking differentials on both sides

$$dy = d(x^3 - 3\sqrt{x})$$

$$= d(x^3) - 3d(\sqrt{x}) = 3x^2 dx - 3 \cdot \frac{1}{2\sqrt{x}} dx$$

$$dy = \left(3x^2 - \frac{3}{2\sqrt{x}}\right) dx$$

(ii) $y = \cos x^2$

Solution:

$$y = \cos x^2$$

Taking differentials on both sides

$$dy = d(\cos x^2)$$

$$= -\sin x^2 \cdot d(x^2)$$

$$= -\sin x^2 \cdot (2x dx)$$

$$dy = -2x \sin x^2 \, dx$$

Q2. Use dy to approximate Δy for the following functions:

(i) $y = x^3 - x$ when x changes from 3 to 3.02.

Solution:

$$y = x^3 - x$$

Taking differentials on both sides

$$dy = d(x^3 - x)$$

$$= d(x^3) - d(x)$$

$$= 3x^2 dx - dx$$

$$dy = (3x^2 - 1) dx$$

As x changes from 3 to 3.02, so

$$x = 3 \text{ and } dx = \Delta x = 3.02 - 3 = 0.02$$

$$\Rightarrow \quad dy = (3(3)^2 - 1)(0.02)$$

$$= 26(0.02) = 0.52$$

Thus, $\Delta y \approx dy = 0.52$

(ii) $y = \sin x + \cos x$ when x changes from 30° to 29°.

Solution:

$$y = \sin x + \cos x$$

Taking differentials on both sides

$$dy = d(\sin x + \cos x)$$

$$= d(\sin x) + d(\cos x)$$

$$= \cos x \, dx - \sin x \, dx$$

$$dy = (\cos x - \sin x) dx$$

As x changes from 30° to 29°, so

$x = 30°$ and $dx = \Delta x = 29° - 30°$

$$dx = -1° = \frac{-\pi}{180} \text{rad}$$

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$$\Rightarrow \quad dy = (\cos 30^\circ - \sin 30^\circ)\left(\frac{-\pi}{180}\right)$$
$$= (0.8660 - 0.5)(-0.0175)$$
$$= (0.366)(-0.0175)$$
$$\approx -0.0064$$

Thus, $\Delta y \approx dy = -0.0064$

(iii) $y = \frac{1}{x}$ when $x = 0.5$ and $dx = -0.1$

Solution:

$$y = \frac{1}{x}$$

Taking differentials on both sides

$$dy = d(x^{-1})$$
$$= -1x^{-2}dx$$
$$dy = -\frac{1}{x^2}dx$$

Put $x = 0.5$ and $dx = -0.1$

$$dy = -\frac{1}{(0.5)^2}(-0.1)$$
$$= +\frac{0.1}{0.25} = 0.4$$

Thus, $\Delta y \approx \Delta y = 0.4$

Q3. Use differentials to approximate the value of:

(i) $\sqrt{26}$

Solution:

Let $y = \sqrt{x}$

We take $x = 25$, $dx = \Delta x = 26 - 25 = 1$

$\Rightarrow \quad y = \sqrt{25} = 5$

Now, $y = \sqrt{x}$

Taking differentials on both sides

$$dy = d(\sqrt{x})$$
$$= \frac{1}{2\sqrt{x}}dx$$

Put $x = 25$ and $dx = 1$

$$dy = \frac{1}{2\sqrt{25}}(1)$$
$$= \frac{1}{2(5)}$$
$$dy = 0.1$$

As we know

$$\sqrt{26} \approx y + dy$$
$$= 5 + 0.1$$
$$= 5.1$$

Hence, $\sqrt{28} \approx 5.1$

(ii) $(33)^{1/5}$

Solution:

Let $y = x^{1/5}$

We take $x = 32$, $dx = \Delta x = 33 - 32 = 1$

$\Rightarrow \quad y = (32)^{1/5}$
$= (2^5)^{1/5} = 2$

Now, $y = x^{1/5}$

Taking differentials on both sides

$$dy = d(x^{1/5})$$
$$= \frac{1}{5}x^{\frac{1}{5}-1} \cdot dx$$
$$= \frac{1}{5}x^{\frac{-4}{5}}dx$$
$$dy = \frac{1}{5x^{4/5}}dx$$

Put $x = 32$ and $dx = 1$

$$dy = \frac{1}{5(32)^{4/5}}(1) = \frac{1}{5(2^5)^{4/5}}$$
$$= \frac{1}{5(2^4)} = \frac{1}{80} = 0.0125$$

As we know

$$(33)^{1/5} \approx y + dy$$
$$= 2 + 0.0125$$
$$= 2.0125$$

Thus, $(33)^{1/5} \approx 2.0125$

(iii) $\tan 46^\circ$

Solution:

Let $y = \tan x$

We take $x = 45^\circ$, $dx = \Delta x = 46^\circ - 45^\circ$, $dx = 1^\circ = \frac{\pi}{180}$ rad

$\Rightarrow \quad y = \tan 45^\circ = 1$

Now $y = \tan x$

Taking differentials on both sides

$$dy = d(\tan x)$$
$$= \sec^2 x\, dx$$
$$dy = \frac{1}{\cos^2 x}dx$$

Put $x = 45^\circ$ and $dx = \frac{\pi}{180}$

$$dy = \frac{1}{\cos^2 45^\circ}\left(\frac{\pi}{180}\right)$$

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$$dy = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}(0.01745)$$

$$dy = 2(0.01745) \approx 0.0349$$

As we know

$$\tan 46^\circ \approx y + dy$$
$$= 1 + 0.0349$$
$$= 1.0349$$

Thus, $\tan 46^\circ \approx 1.0349$

Q4. The side of a cube is measured to be 30 cm with a maximum error of 0.14 cm in its measurement. Find the maximum error in the calculated volume of the cube.

Solution:

Let the length of each side of cube = x cm

Volume of cube = $x \cdot x \cdot x$

$$V = x^3$$

Taking differentials on both sides

$$dV = d(x^3)$$

$$dV = 3x^2 \cdot dx$$

Put $x = 30$ and $dx = 0.14$

$$dV = 3(30)^2(0.14)$$
$$= 3(900)(0.14)$$
$$dV = 378 \text{ cm}^3 \text{ (Error)}$$

Q5. The radius of a sphere is measured as 100 ± 0.5 cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

Solution:

Let radius of sphere = r cm

Volume of sphere = $\frac{4}{3}\pi r^3$

$$V = \frac{4}{3}\pi r^3$$

Taking differentials on both sides

$$dV = \frac{4}{3}\pi d(r^3)$$
$$= \frac{4}{3}\pi(3r^2 dr)$$
$$= 4\pi r^2 dr$$

Put $r = 100$ and $dr = \pm 0.5$

$$dV = 4\pi(100)^2(\pm 0.5)$$
$$= 4000\pi(\pm 0.5)$$
$$= \pm 2000\pi$$
$$\boxed{dV = 6283.19 \text{ cm}^3} \text{ Error}$$

Percentage error in volume $= \left|\frac{dV}{V}\right| \times 100$

$$= \left|\frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3}\right| \times 100$$

$$= \left|\frac{3dr}{r}\right| \times 100$$

$$= 3\left|\frac{\pm 0.5}{100}\right| \times 100$$

$$= 3\left(\frac{0.5}{100}\right) \times 100$$

$$= 1.5\%$$

Content Outline

- Applying Extrema to Real-World Problems

APPLYING EXTREMA TO REAL-WORLD PROBLEMS

To solve extrema problems in real-life scenarios, follow these steps:

1. **Understand the Problem** - Identify what needs to be optimized.
2. **Define Variables** - Assign variables to unknown quantities.
3. **Formulate the Function** - Write an equation representing the quantity to optimize.
4. **Find Constraints** - If applicable, express limitations.
5. **Find Critical Points** - Take the derivative and find critical points.
6. **Verify Extrema** - Use the second derivative test to confirm maxima/minima.
7. **Interpret the Solution** - Relate the mathematical answer back to the real-world context.

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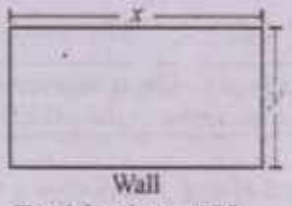
Example 30: A rectangular garden is to be fenced on three sides with 100 meters of fencing. The fourth side is along a wall and requires no fence. Find the dimensions that maximize the area and the maximum area.

Solution:

Let side opposite to wall = x m

Each side perpendicular to wall = y m

Area of garden = $x \cdot y$



Let $f(x) = x\left(\frac{100-x}{2}\right)$ Using (1)

$= \frac{1}{2}(100x - x^2)$

Differentiate w.r.t. 'x'

$f'(x) = \frac{1}{2}(100(1) - 2x)$

$= 50 - x$

Again, differentiate w.r.t. 'x'

$f''(x) = -1$

For critical points, put $f'(x) = 0$

$50 - x = 0 \Rightarrow x = 50$

At $x = 50$:

$f''(50) = -1 < 0$

Total fencing = 100

$x + 2y = 100$

$2y = 100 - x$

$y = \frac{100-x}{2}$ ...(1)

$\Rightarrow$ The area is maximum at $x = 50\ m$

- Side opposite to wall = $x = 50\ m$
- Each side perpendicular to wall: $y = \frac{100-50}{2} = 25m$
- Maximum fenced area of garden: $A = xy = (50)(25) = 1250\ m^2$

Example 31: Find the dimensions of a rectangle of maximum area that can be inscribed under the curve $y = 4 - x^2$ and above the x-axis.

Solution:

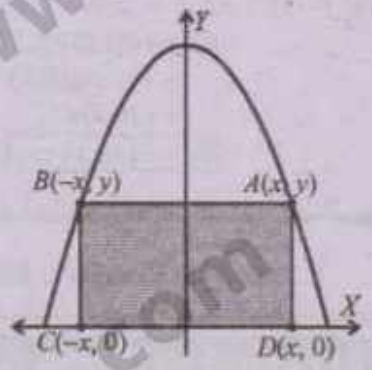
Curve: $y = 4 - x^2$

Let $A(x, y)$ and $B(-x, y)$ be the two vertices of rectangle on parabola in 1st and 2nd quadrants respectively and $x > 0$.

Height of rectangle = y

Width of rectangle = $x + x = 2x$

Area of rectangle = $(2x)(y)$



Let $f(x) = 2x(4 - x^2)$ $\because$ Vertex $A(x, y)$ lies on the curve.

$f(x) = 8x - 2x^3$

Differentiate w.r.t. 'x'

$f'(x) = 8 - 6x^2$

Again, differentiate w.r.t. 'x'

$f''(x) = -12x$

For critical points, put $f'(x) = 0$

$8 - 6x^2 = 0 \Rightarrow 6x^2 = 8$

$x^2 = \frac{8}{6} = \frac{4}{3} \Rightarrow x = \frac{2}{\sqrt{3}}$ $\because x > 0$

At $x = \frac{2}{\sqrt{3}}$:

$f''\left(\frac{2}{\sqrt{3}}\right) = -12\left(\frac{2}{\sqrt{3}}\right)$

$= \frac{-24}{\sqrt{3}} < 0$

$\Rightarrow$ The area is maximum at $x = \frac{2}{\sqrt{3}}$.

Dimensions of rectangle are:

- Height = $y = 4 - x^2 = 4 - \left(\frac{2}{\sqrt{3}}\right)^2 = 4 - \frac{4}{3} = \frac{8}{3}$ units
- Width = $2x = 2\left(\frac{2}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}}$ units

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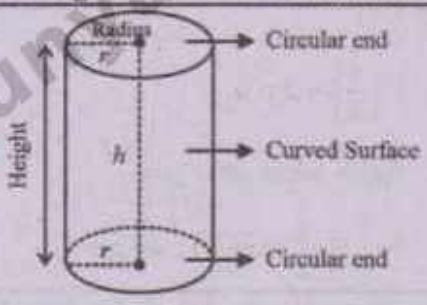
Example 32: A cylindrical can must hold 500 cm³ of liquid. Find the radius and height that minimizes the material cost.

Solution:



Let Radius of base = r cm

Height of cylinder = h cm

Surface area of two circular ends = $\pi r^2 + \pi r^2 = 2\pi r^2$

Curved surface area = $2\pi rh$

Total surface area of cylinder = $2\pi r^2 + 2\pi rh$

Let $f(r) = 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$ Using (1)

$f(r) = 2\pi r^2 + 1000r^{-1}$

Differentiate w.r.t 'r'

$f'(r) = 2\pi(2r) - 1000r^{-2} = 4\pi r - 1000r^{-2}$

Again, differentiate w.r.t 'x'

$f''(r) = 4\pi(1) - 1000(-2r^{-3}\cdot 1) = 4\pi + \frac{2000}{r^3}$

For critical points, put $f'(r) = 0$

$4\pi r - \frac{1000}{r^2} = 0$

$4\pi r = \frac{1000}{r^2} \Rightarrow r^3 = \frac{1000}{4\pi}$

$r^3 = \frac{250}{\pi} \Rightarrow r = \left(\frac{250}{\pi}\right)^{\frac{1}{3}}$

At $r = \left(\frac{250}{\pi}\right)^{\frac{1}{3}}$:

$f''\left(\left(\frac{250}{\pi}\right)^{1/3}\right) = 4\pi + \frac{2000}{\left[\left(\frac{250}{\pi}\right)^{1/3}\right]^3} = 4\pi + 8\pi = 12\pi > 0$

$V = 500$

$\pi r^2 h = 500$

$h = \frac{500}{\pi r^2}$ ...(1)

$\Rightarrow$ The surface area is minimum at $r = \left(\frac{250}{\pi}\right)^{1/3}$

Radius: $r = \left(\frac{250}{\pi}\right)^{1/3}$ cm

Height: $h = \frac{500}{\pi r^2}$

$h = \frac{500}{\pi\left(\left(\frac{250}{\pi}\right)^{1/3}\right)^2} = \frac{500}{\frac{\pi(250)^{2/3}}{\pi^{2/3}}}$

$h = \frac{500}{(250)^{2/3}\cdot\pi^{1-\frac{2}{3}}}$

$h = \frac{500}{(250)^{2/3}\pi^{1/3}}$ cm

Example 33: A car moves along the parabolic road $y = 4 - x^2$. A mobile tower is at the origin. Find the position of the car where the signal strength is strongest.

Solution:

Parabola: $y = 4 - x^2$ ...(1)

Let $P(x, y)$ be the position of car on the parabola.

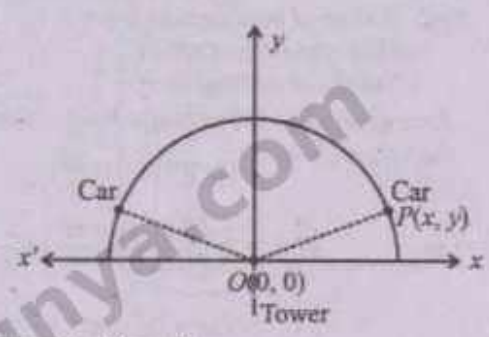
By distance formula

$|\overline{OP}| = \sqrt{(x-0)^2 + (y-0)^2}$

$= \sqrt{x^2 + y^2}$

$= \sqrt{4 - y + y^2}$ $\because y = 4 - x^2 \Rightarrow x^2 = 4 - y$

$\Rightarrow |\overline{OP}|^2 = y^2 - y + 4$



Let $f(y) = y^2 - y + 4$

Differentiate w.r.t 'y'

$f'(y) = 2y - 1$

Again, differentiate w.r.t 'x'

$f''(y) = 2$

For critical points, put $f'(y) = 0$

$2y - 1 = 0$

$y = \frac{1}{2}$

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At $y=\frac{1}{2}$:

$$f''\left(\frac{1}{2}\right) = 2 > 0$$

$\Rightarrow$ $f(x)$ is minimum at $y=\frac{1}{2}$

Put $y=\frac{1}{2}$ in eq. (1), we get

$$\frac{1}{2} = 4 - x^2 \Rightarrow x^2 = 4 - \frac{1}{2} = \frac{7}{2}$$

$$\sqrt{x^2} = \sqrt{\frac{7}{2}} \Rightarrow x = \pm\sqrt{\frac{7}{2}}$$

The positions of the car where signal strength is maximum are $\left(\sqrt{\frac{7}{2}}, \frac{1}{2}\right)$ and $\left(-\sqrt{\frac{7}{2}}, \frac{1}{2}\right)$

Example 34: A software company is designing a data processing algorithm for analyzing batches of customer transactions. The runtime $T(n)$ in milliseconds depends on the size of the batch n (the number of transactions processed together) and is modeled by the equation:

$$T(n) = n^3 - 12n^2 + 36n$$

- **The term n^3 reflects the time complexity of a sorting step in the algorithm.**
- **The term $-12n^2$ accounts for optimizations that reduce processing time for moderate batch sizes.**
- **The term $36n$ represents overhead that scales linearly with batch size.**

The company wants to find the optimal batch size n that minimizes the algorithm's total runtime.

Solution:

$$T(n) = n^3 - 12n^2 + 36n$$

Differentiate w.r.t. 'n'

$$T'(n) = 3n^2 - 24n + 36$$

Again, differentiate w.r.t. 'x'

$$T''(n) = 6n - 24$$

For critical points, put $T'(n) = 0$

$$3n^2 - 24n + 36 = 0$$

$$n^2 - 8n + 12 = 0 \qquad \text{Divide by '3'}$$

$$n^2 - 6n - 2n + 12 = 0$$

$$n(n-6) - 2(n-6) = 0$$

$$(n-2)(n-6) = 0$$

$\Rightarrow$ $n-2=0$; $n-6=0$

$n=2$; $n=6$

At $n=2$:

$$T''(2) = 6(2) - 24 = 12 - 24 = -12 < 0$$

$\Rightarrow$ $T(n)$ has a local maximum at $n=2$.

At $n=6$:

$$T''(6) = 6(6) - 24 = 36 - 12 = 12 > 0$$

$\Rightarrow$ $T(n)$ has a local minimum at $n=6$

Hence, the runtime is minimum when the input size is $n=6$.

EXERCISE 2.7

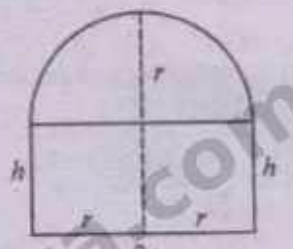
Q1. A window consists of a rectangle surmounted by a semicircular opening. If the total perimeter is 21 meter, find the dimensions that maximize the area.

Solution:

Let Radius of semicircular top $= r$

Width of rectangle $= 2r$

Height of rectangle $= h$



Area of window = Rectangle Area + Semicircle Area

$$= (2r)(h) + \frac{1}{2}(\pi r^2)$$

Let $f(x) = 2r\left(\frac{21-2r-\pi r}{2}\right) + \frac{1}{2}\pi r^2$ Using (1)

$$= 21r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2$$

$$= -2r^2 + 21r + \frac{-2\pi r^2 + \pi r^2}{2}$$

$$f(r) = -2r^2 + 21r - \frac{1}{2}\pi r^2$$

Perimeter $= 21$ m

$$2r + 2h + \pi r = 21$$

$$2h = 21 - 2r - \pi r$$

$$h = \frac{21 - r(2+\pi)}{2} \qquad \ldots(1)$$

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Differentiate w.r.t. 'r'

$$f'(r) = -2(2r) + 21(1) - \frac{1}{2}\pi(2r)$$
$$= -4r + 21 - \pi r$$

Again, differentiate w.r.t. 'x'

$$f''(r) = -4 + 0 - \pi = -(4 + \pi)$$

For critical points, put $f'(r) = 0$

$$-4r + 21 - \pi r = 0$$
$$21 = \pi r + 4r \Rightarrow r(\pi + 4) = 21 \Rightarrow r = \frac{21}{\pi + 4}$$

At $r = \frac{21}{\pi+4}$:

$$f''\left(\frac{21}{\pi+4}\right) = -(\pi + 4) < 0$$

$\Rightarrow$ The area of window is maximum at $r = \frac{21}{\pi + r}$

Dimensions:

- Radius $= r = \frac{21}{4+\pi}$ m
- Width $= 2r = 2\left(\frac{21}{4+\pi}\right) = \frac{42}{4+\pi}$ m
- Height $= h = \frac{21 - \frac{21}{\pi+4}(2+\pi)}{2}$

$$= \frac{1}{2} \cdot 21\left(1 - \frac{2+\pi}{\pi+4}\right)$$
$$= \frac{21}{2}\left(\frac{\pi + 4 - 2 - \pi}{\pi + 4}\right)$$
$$= \frac{21}{\pi+4} \text{ m}$$

Q2. A rectangular sheet of metal with dimensions 40 cm by 25 cm is to be made into an open box by cutting equal squares from the corners and folding up the sides. Find the size of the squares that maximize the volume.

Solution:

Let side length of corner square = x cm
Height of box = x
Length of box = $40 - 2x$
Width of box = $25 - 2x$
Volume of box = $x(40 - 2x)(25 - 2x)$

Let $f(x) = x(1000 - 80x - 50x + 4x^2) = x(4x^2 - 130x + 1000)$

$$f(x) = 4x^3 - 130x^2 + 1000x$$

Differentiate w.r.t. 'x'

$$f'(x) = 12x^2 - 260x + 1000$$

Again, differentiate w.r.t. 'x'

$$f''(x) = 24x - 260$$

For critical points, put $f'(x) = 0$

$$12x^2 - 260x + 1000 = 0$$
$$3x^2 - 65x + 250 = 0 \quad \text{Divide by '4'}$$
$$3x^2 - 50x - 15x + 250 = 0$$
$$x(3x - 50) - 5(3x - 50) = 0$$
$$(x - 5)(3x - 50) = 0$$
$$x - 5 = 0 \quad ; \quad 3x - 50 = 0$$
$$x = 5 \quad ; \quad x = \frac{50}{3} = 16.67 \text{ (Invalid)}$$

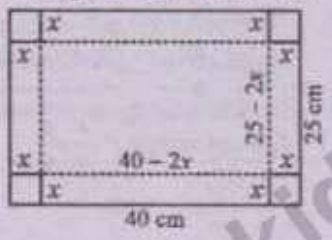
At $x = 5$:

$$f''(5) = 24(5) - 260 = -140 < 0$$

$\Rightarrow$ The volume of box is maximum at $x = 5$

Hence, size of square cut = $x = 5$ cm

Rectangular Sheet of Metal



Q3. A closed cylindrical tank is to be constructed to hold 54π liters of water. Find the dimensions that minimize the construction cost of the tank.

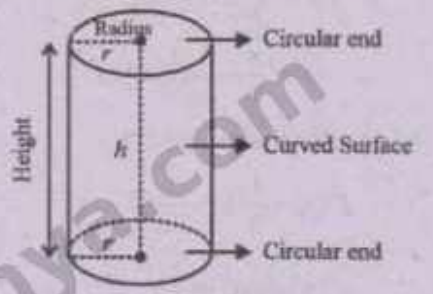
Solution:

Let Radius of base = r dm
Height of cylinder = h dm

Surface area of two circular ends = $\pi r^2 + \pi r^2 = 2\pi r^2$
Curved surface area = $2\pi rh$
Total surface area of cylinder = $2\pi r^2 + 2\pi rh$

Let $f(r) = 2\pi r^2 + 2\pi r\left(\frac{54}{r^2}\right)$ Using (1)

$$f(r) = 2\pi r^2 + \frac{108\pi}{r}$$
$$f(r) = 2\pi r^2 + 108\pi r^{-1}$$



$$V = 54\pi$$
$$\pi r^2 h = 54\pi$$
$$h = \frac{54\pi}{\pi r^2}$$
$$h = \frac{54}{r^2} \quad \ldots(1)$$

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Differentiate w.r.t. 'x'

$$f'(r) = 2\pi(2r) + 108\pi(-r^{-2})$$
$$= 4\pi r - 108\pi r^{-2}$$

Again, differentiate w.r.t. 'x'

$$f''(r) = 4\pi(1) - 108\pi(-2r^{-3})$$
$$= 4\pi + \frac{216\pi}{r^3}$$

For critical points, put $f'(r) = 0$

$$4\pi r - \frac{108\pi}{r^2} = 0 \Rightarrow 4\pi r = \frac{108\pi}{r^2}$$

$$r^3 = \frac{108\pi}{4\pi} \Rightarrow r^3 = 27 \Rightarrow r^3 = 3^3 \Rightarrow r = 3$$

At $r = 3$:

$$f''(3) = 4\pi + \frac{216\pi}{3^3} = 4\pi + 8\pi = 12\pi > 0$$

$\Rightarrow$ The surface area is minimum at $r = 3$

Dimensions of tank:

Radius: $r = 3$ dm $= 3(10)$ cm $= 30$ cm

Height: $h = \frac{54}{r^2} = \frac{54}{3^2} = 6\,dm = 60$ cm

Q4. A wire of 20 meters is cut into two pieces. One piece is bent into a square and the other into a circle. How should the wire be cut to maximize the total area enclosed?

Solution:

Let Length of wire for square $= x$ m

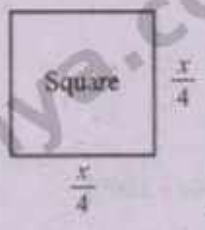
Length of wire for circle $= (20 - x)$m

For the square:

Perimeter $= x$

$\Rightarrow$ $4(\text{Side}) = x$

$\text{Side} = \frac{x}{4}$



Area of square $= \frac{x}{4}\cdot\frac{x}{4}$

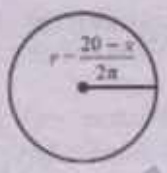
$$A_1 = \frac{x^2}{16}$$

For the circle:

Circumference $= 20 - x$

$\Rightarrow$ $2\pi r = 20 - x$

$$r = \frac{20-x}{2\pi}$$



Area of circle $= \pi r^2$

$$A_2 = \pi\left(\frac{20-x}{2\pi}\right)^2 = \frac{\pi(20-x)^2}{4\pi^2} = \frac{(20-x)^2}{4\pi}$$

Total Area $= A_1 + A_2$

Let $f(x) = \frac{x^2}{16} + \frac{(20-x)^2}{4\pi}$

Differentiate w.r.t 'x'

$$f'(x) = \frac{1}{16}(2x) + \frac{1}{4\pi}\cdot 2(20-x)^1\cdot(-1)$$
$$= \frac{x}{8} - \frac{1}{2\pi}(20-x)$$

Again, differentiate w.r.t 'x'

$$f''(x) = \frac{1}{8}(1) - \frac{1}{2\pi}(0-1)$$
$$f''(x) = \frac{1}{8} + \frac{1}{2\pi}$$

For critical points, put $f'(x) = 0$

$$\frac{x}{8} - \frac{20-x}{2\pi} = 0 \Rightarrow \frac{x}{8} = \frac{20-x}{2\pi}$$
$$2\pi x = 160 - 8x \Rightarrow 2\pi x + 8x = 160$$
$$2x(\pi+4) = 160 \Rightarrow x = \frac{160}{2(\pi+4)} \Rightarrow x = \frac{80}{\pi+4}$$

At $x = \frac{80}{\pi+4}$:

$$f''\left(\frac{80}{\pi+4}\right) = \frac{1}{8} + \frac{1}{2\pi} > 0$$

$\Rightarrow$ The area is minimum at $x = \frac{80}{\pi+4}$

Wire for square $= x = \frac{80}{\pi+4} \approx 11.2$ m

Wire for circle $= 20 - \frac{80}{\pi+4}$

$$= \frac{20\pi + 80 - 80}{\pi+4}$$
$$= \frac{20\pi}{x+4} \approx 8.8 \text{ m}$$

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Q5. **A team of environmental scientists is monitoring the oxygen level in a lake over time to ensure the water quality remains suitable for aquatic life. The oxygen level in the lake, measured in milligrams per liter (mg/L), is modeled by the equation:**

$$O(t) = t^3 - 18t^2 + 81t + 100$$

Where $O(t)$ is the oxygen level at time t (in days after monitoring began).

Scientists want to determine when during the monitoring period the oxygen level is at its lowest, as low oxygen levels can be harmful to fish and other aquatic organisms.

Find the time t when the oxygen level is at its minimum.

Solution:

$$O(t) = t^3 - 18t^2 + 81t + 100$$

Differentiate w.r.t. 't'

$$O'(t) = 3t^2 - 36t + 81$$

Again, differentiate w.r.t. 't'

$$O''(t) = 6t - 36$$

For critical points, put $O'(t) = 0$

$$3t^2 - 36t + 81 = 0$$

$$t^2 - 12t + 27 = 0 \qquad \text{Divide by '3'}$$

$$t^2 - 9t - 3t + 27 = 0$$

$$t(t-9) - 3(t-9) = 0$$

$$(t-3)(t-9) = 0$$

$$t - 3 = 0 \quad ; \quad t - 9 = 0$$

$$t = 3 \quad ; \quad t = 9$$

At $t = 3$:

$$O''(3) = 6(3) - 36 = 18 - 36 = -18 < 0$$

$\Rightarrow$ $O(t)$ is maximum at $t = 3$.

At $t = 9$:

$$O''(9) = 6(9) - 36 = 54 - 36 = 18$$

$\Rightarrow$ $O(t)$ is minimum at $t = 9$

Hence, the oxygen level in the lake reaches its minimum value 9 days after the monitoring period began.

Q6. **The perimeter of a triangle is 16 cm . If one side is of length 6 cm , what are the lengths of the other sides for maximum area of the triangle.**

Solution:

Let Length of 1st side = 6 cm

Length of 2nd side = x cm

Length of 3rd side = $16 - 6 - x = (10 - x)$ cm



By Hero's formula

$$\text{Area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Delta = \sqrt{8(8-6)(8-x)(8-10+x)}$$

$$= \sqrt{8(2)(8-x)(-2+x)}$$

$$\Delta = \sqrt{16(-16+10x-x^2)}$$

Taking square on both sides

$$\Delta^2 = 16(-x^2 + 10x - 16)$$

Let $f(x) = 16(-x^2 + 10x - 16)$

Differentiate w.r.t. 'x'

$$f'(x) = 16(-2x + 10)$$

Again, differentiate w.r.t. 'x'

$$f''(x) = 16(-2 + 0) = -32$$

For critical points, put $f'(x) = 0$

$$16(-2x + 10) = 0$$

$\Rightarrow$ $-2x + 10 = 0 \Rightarrow 2x = 10 \Rightarrow x = 5$

As we know

$$s = \frac{a+b+c}{2}$$

$$s = \frac{16}{2} = 8 \qquad \ldots(1)$$

At $x = 5$:

$$f''(5) = -32 < 0$$

$\Rightarrow$ The area of triangle is maximum at $x = 5$

Dimensions of Triangle:

- Length of 1st side = 6 cm
- Length of 2nd side = $x = 5$ cm
- Length of 3rd side = $10 - x = 10 - 5 = 5$ cm

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Q7. A poster must have 200 cm² printed area with margins of 2 cm on top/bottom and 3 cm on sides. Find poster dimensions to minimize paper used.

Solution:

Let Length of printed area = x cm

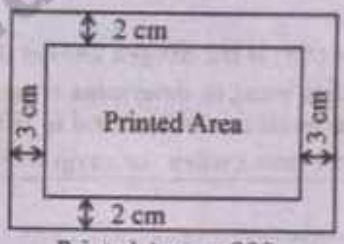
Width of printed area = y cm

According to the statement, we have

Total length = $(x+6)$ cm

Total width = $(y+4)$ cm

Area of paper = $(x+6)\cdot(y+4)$



Printed Area = 200

$x \cdot y = 200$

$y = \dfrac{200}{x}$ ...(1)

Let $f(x) = (x+6)\left(\dfrac{200}{x}+4\right)$

$= 200 + 4x + \dfrac{1200}{x} + 24$

$f(x) = 4x + 1200x^{-1} + 24$

Differentiate w.r.t. 'x'

$f'(x) = 4(1) + 1200(-1x^{-2}) = 4 - 1200x^{-2}$

Again, differentiate w.r.t. 'x'

$f''(x) = 0 - 1200(-2x^{-3}) = \dfrac{2400}{x^3}$

For critical points, put $f'(x) = 0$

$4 - \dfrac{1200}{x^2} = 0 \Rightarrow \dfrac{1200}{x^2} = 4 \Rightarrow x^2 = \dfrac{1200}{4}$

$x^2 = 300 \Rightarrow x = \sqrt{300} = 10\sqrt{3}$ ∵ length is always +ve

At $x = 10\sqrt{3}$:

$f''(10\sqrt{3}) = \dfrac{1200}{(10\sqrt{3})^3} > 0$

$\Rightarrow$ The paper area is minimum at $x = 10\sqrt{3}$

Dimensions of poster:

- Length of paper $= x + 6 = (6 + 10\sqrt{3})$ cm
- Width of paper $= y + 4 = \dfrac{200}{x} + 4$

$= \dfrac{200}{10\sqrt{3}} + 4 = \left(4 + \dfrac{20}{\sqrt{3}}\right)$ cm

Q8. Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.

Solution:

Let 1st +ve integer = x

2nd +ve integer = $12 - x$

Product of one with square of other = $(12-x)\cdot x^2$

Let $f(x) = 12x^2 - x^3$

Differentiate w.r.t 'x'

$f'(x) = 24x - 3x^2$

Again, differentiate w.r.t 'x'

$f''(x) = 24 - 6x$

For critical points, put $f'(x) = 0$

$24x - 3x^2 = 0 \Rightarrow 3x(8-x) = 0$

Either $8 - x = 0$ or $3x = 0$

$x = 8$ or $x = 0$ (Not possible)

At $x = 8$:

$f''(8) = 24 - 6(8) = 24 - 48 = -24 < 0$

$\Rightarrow$ $f(x)$ is maximum at $x = 8$

1st +ve integer = $x = 8$

2nd +ve integer = $12 - x = 12 - 8 = 4$

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Multiple Choice Questions (MCQs)

1. **The derivative of $f(x)$ at a point is defined by:**

A) $\lim_{h\to 0}\frac{f(x+h)+f(x)}{h}$ B) $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ C) $\lim_{h\to\infty}\frac{f(x+h)-f(x)}{h}$ D) $\lim_{h\to 0}\frac{f(x+h)-f(x)}{x}$

2. **The derivative of x^n, $n \in R$, is:**

A) $n\,x^{n-1}$ B) x^{n-1} C) $n\,x^n$ D) $\frac{x^n}{n}$

3. **The Product Rule of derivative is:**

A) $(uv)' = u'v + uv'$ B) $(uv)' = u'v'$ C) $(uv)' = u' + v'$ D) $(uv)' = \frac{u}{v}$

4. $\frac{d}{dx}(\tan x)$ **equals:**

A) $\sec^2 x$ B) $\csc^2 x$ C) $-\sec^2 x$ D) $\cot x$

5. $\frac{d}{dx}(\sec x)$ **equals:**

A) $\sec x \tan x$ B) $-\sec x \tan x$ C) $\csc x \cot x$ D) $\sec^2 x$

6. **If $y = f(u)$ and $u = g(x)$, the chain rule gives:**

A) $\frac{dy}{dx} = \frac{dy}{du} + \frac{du}{dx}$ B) $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$ C) $\frac{dy}{dx} = \frac{du}{dy}$ D) $\frac{dy}{dx} = f(u)\cdot g(x)$

7. **For $y = \sin(x^2 - x)$, $\frac{dy}{dx}$ is:**

A) $\cos(x^2 - x)$ B) $(2x-1)\cos(x^2 - x)$ C) $(2x-1)\sin(x^2 - x)$ D) $2x\cos x$

8. **For $x = f(t)$, $y = g(t)$, $\frac{dy}{dx}$ is:**

A) $\frac{\frac{dx}{dt}}{\frac{dy}{dt}}$ B) $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ C) $\frac{dx}{dy}$ D) $\frac{dy}{dt} + \frac{dx}{dt}$

9. **For $x = a\cos t$ and $y = a\sin t$, $\frac{dy}{dx}$ is:**

A) $\tan t$ B) $-\tan t$ C) $\cot t$ D) $-\cot t$

10. $\frac{d}{dx}(\sec^{-1} x)$ **(sec^{-1}x) is:**

A) $\frac{1}{x\sqrt{x^2-1}}$ B) $\frac{1}{|x|\sqrt{x^2-1}}$ C) $-\frac{1}{|x|\sqrt{x^2-1}}$ D) $\frac{1}{1+x^2}$

11. $\frac{d}{dx}(\log_a x)$ **is:**

A) $\frac{1}{x\ln a}$ B) $\frac{\ln a}{x}$ C) $\frac{a}{x}$ D) $\frac{1}{ax}$

12. **For $y = \ln(\cos e^x)$, $\frac{dy}{dx}$ is:**

A) $-e^x \tan(e^x)$ B) $e^x \tan(e^x)$ C) $\tan(e^x)$ D) $-\tan x$

13. **If tangent slope is $m \neq 0$, normal slope is:**

A) m B) $-m$ C) $\frac{1}{m}$ D) $\frac{-1}{m}$

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14. A critical point occurs at $x = c$, $c \in \text{Dom } f$ where:
A) $f'(c) = 1$ only
B) $f'(c) = 0$ or $f'(c)$ does not exist
C) $f(c) = 0$ only
D) $f''(c) = 0$ only

15. For $f(x) = x^2 - 4x$, the critical point is:
A) $x = 1$ B) $x = 2$ C) $x = 3$ D) $x = 4$

16. If $f'(x) > 0$ throughout an interval, f is:
A) decreasing B) increasing C) constant D) undefined

17. If $f'(x) < 0$ throughout an interval, f is:
A) increasing B) decreasing C) constant D) periodic

18. If $f'(c) = 0$ and $f''(c) < 0$, f has _____ at $x = c$.
A) local minimum B) local maximum C) no critical point D) vertical tangent

19. The linearization of f at $x = a$ is:
A) $L(x) = f(a) + f'(a)(x - a)$ B) $L(x) = f'(a) + f(a)(x - a)$ C) $L(x) = f(a)x$ D) $L(x) = f'(x)$

20. If $y = f(x)$, the differential dy is:
A) $F'(x)dx$ B) $f(x)dx$ C) $\frac{dx}{f'(x)}$ D) $f'(x)$

21. If $f(x) = e^x - e^{-x}$, $f'(x)$ is:
A) $e^x - e^{-x}$ B) $e^x + e^{-x}$ C) $-e^x - e^{-x}$ D) e^x

22. For $y = \ln(1 - \cos^2 x)$, the derivative of the inner expression is:
A) $-2\sin x \cos x$ B) $2\sin x \cos x$ C) $\cos x$ D) $-\sin x$

23. For $y = \sin^{-1}\left(\frac{x}{2}\right) + \cos x$ at $x = \frac{\pi}{2}$, the tangent slope is:
A) $\frac{1}{\sqrt{\frac{4-\pi^2}{4}}} - 1$ B) $\frac{1}{\sqrt{\frac{4-\pi^2}{4}}} + 1$ C) -1 D) 0

24. Function $f(x) = 4 - x^2$ is increasing when:
A) $x < 0$ B) $x > 0$ C) $x = 0$ only D) all real x

25. Using $L(x) = \frac{x+4}{3}$, an approximation to $\sqrt{9.2}$ is:
A) 3.0333 B) 3.2 C) 2.9667 D) 4.0333

26. For $f(x) = \cos x$ at $x = \frac{\pi}{2}$, its linearization is:
A) $x - \frac{\pi}{2}$ B) $-x + \frac{\pi}{2}$ C) 1 D) $\cos x$

27. Local linear approximation is based on the:
A) secant line B) tangent line C) normal line only D) quadratic curve

28. For $f(x) = x^3 - x$, $x = 3$ and $dx = 0.02$, dy is:
A) 0.02 B) 0.26 C) 0.52 D) 1.04

29. For $y = \sin x + \cos x$, dy is:
A) $(\cos x - \sin x)dx$ B) $(\sin x + \cos x)dx$ C) $\sin x\, dx$ D) $\cos x\, dx$

30. If side measurement has 4% error, approximate percentage error in volume $V = x^3$ is:
A) 4% B) 8% C) 12% D) 16%

ANSWER KEY

1.	B	2.	A	3.	A	4.	A	5.	A	6.	B	7.	B	8.	B	9.	B	10.	B
11.	A	12.	A	13.	D	14.	B	15.	B	16.	B	17.	B	18.	B	19.	A	20.	A
21.	B	22.	B	23.	A	24.	A	25.	A	26.	B	27.	B	28.	C	29.	A	30.	C

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