www.ilmkidunya.com

UNIT 01

Graphical Representation of Functions

INTRODUCTION

- This unit introduces students to the fundamental ideas of graphical representation of functions, enabling them to visualize and interpret mathematical relationships effectively.
- Students begin by learning how to draw graphs using factor form and how to predict equations from given graphs, particularly for quadratic functions.
- The unit further develops understanding by classifying functions into algebraic and transcendental types, followed by a detailed study of important transcendental functions such as trigonometric, inverse trigonometric, hyperbolic, logarithmic, and exponential functions.
- Emphasis is placed on understanding logarithms, their laws, and their applications in real-life contexts like growth, decay, and sound intensity.
- Students also explore exponential functions and their applications, including compound interest, and learn to solve related equations and inequalities. The graphs of all said functions will be discussed.
- In particular, graphical behaviors of exponential and logarithmic functions will be studied in detail.
- Additionally, the unit covers graphs of inverse functions, and graphical transformations including shifts and scaling.

Overall, this unit equips students with both analytical and graphical skills necessary for higher-level mathematics and practical problem solving.

Content Outline

- Graph of Quadratic Function Using Factors
- Predicting a Quadratic Function from its Graph

GRAPH OF QUADRATIC FUNCTION USING FACTORS

We have already sketched the graph of the quadratic function $y = ax^2 + bx + c (a \neq 0)$ in Grade-XI. We know that its graph is a parabola, which has a vertex and opens either upward or downward depending on the value of a.

Now, we will learn how to sketch its graph when it is expressed in factorized form as a product of linear factors. This approach will also help you in higher classes to sketch the graphs of higher-degree polynomials that can be factorized into linear factors.

Method to Sketch the Graph

Suppose, after factorizing we obtain $y = f(x) = ax^2 + bx + c$, we obtain $y = a(x-h)(x-k)$, where $h \leq k$ To sketch the graph, follow the given steps:

Case-I: If $h < k$

Step-1: The graph will cut the x-axis at $x = h$ and $x = k$.

Step-2: Find value of y at $x = \frac{h+k}{2}$.

The point $\left(\frac{h+k}{2}, f\left(\frac{h+k}{2}\right)\right)$ represents the vertex of the parabola (see Figure 1.1).

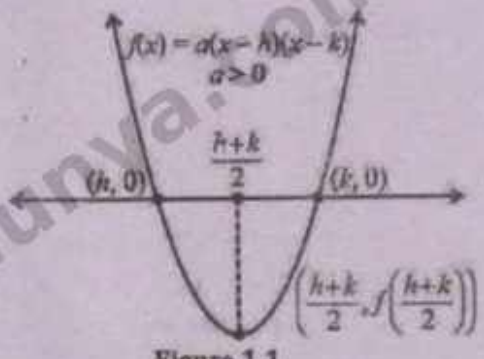


Figure 1.1

www.ilmkidunya.com

www.ilmkidunya.com

Step-3: Determine the behaviour of y as $x \to \pm\infty$.

- If $a > 0$, then $y \to +\infty$.
- If $a < 0$, then $y \to -\infty$.

This statement means the graph behaves the **same way on both ends** (left and right).

Step-4: Find the y-intercept:

Put $x = 0$ in the function $y = a(x-h)(x-k)$, the graph will cut the y-axis at $y = ahk$.

Case-II: If $h = k$, then the function becomes $y = a(x-h)^2$.

To sketch the graph, follow these steps:

Step-1: The graph **touches the x-axis** at $x = h$, therefore $(h, 0)$ is the vertex (see Figure 1.2).

Step-2: Then follow Steps 3 and 4 of Case-I to complete the sketch of the graph.

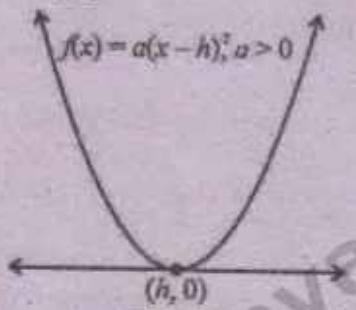


Figure 1.2

Challenge!

If $y = a(x-h_1)(x-h_2)(x-h_3)$, then what will be the behaviour of y as $x \to -\infty$ and $x \to +\infty$.

Solution:

$y = a(x-h_1)(x-h_2)(x-h_3)$

Since degree of y is 3, so

Case 1: If $a > 0$

- As $x \to +\infty$, then $y \to +\infty$
- As $x \to -\infty$, then $y \to -\infty$

Case 2: If $a < 0$

- As $x \to +\infty$, then $y \to -\infty$
- As $x \to -\infty$, then $y \to +\infty$

Recall!

$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$ is a polynomial of degree n, where n is a non-negative integer and $a_0 \neq 0$.

Note: Following Steps 1, 3, and 4, and by taking some points between the two x-intercepts, we can draw the graph of a polynomial of higher degree that can be factorized into linear factors.

Example 1: Sketch the graph of $y = 2(x-1)(x-3)$.

Solution:

Given Function: $y = 2(x-1)(x-3)$ ...(1)

For x-intercepts, put $y = 0$

$$2(x-1)(x-3) = 0$$

$$x - 1 = 0 \quad ; \quad x - 3 = 0$$

$$x = 1 \quad ; \quad x = 3$$

$\Rightarrow$ The graph meets the x-axis at $(1, 0)$ and $(3, 0)$.

For y-intercept, put $x = 0$

$$y = 2(0-1)(0-3)$$

$$y = 2(-1)(-3) = 6$$

$\Rightarrow$ The graph cuts the y-axis at $(0, 6)$.

Vertex:

$$x\text{-coordinate of vertex} = \frac{1+3}{2} = 2$$

Put $x = 2$ in eq. (1), we get

$$y = 2(2-1)(2-3)$$

$$= 2(1)(-1) = -2$$

$\Rightarrow$ Vertex $= (2, -2)$

As $a = 2 > 0$, therefore parabola opens upward.

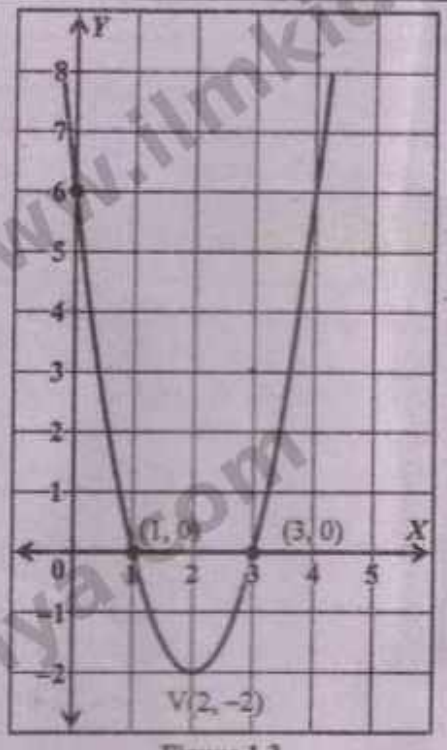


Figure 1.3

Example 2: Sketch the graph of $y = -4x^2 - 8x - 4$.

Solution:

Given Function: $y = -4x^2 - 8x - 4$

www.ilmkidunya.com

www.ilmkidunya.com

$$= -4(x^2 + 2x + 1)$$

$$y = -4(x+1)^2 \quad \ldots(1)$$

For x-intercepts, put $y = 0$

$$-4(x+1)^2 = 0$$

$$x + 1 = 0 \quad \Rightarrow \quad x = -1$$

$\Rightarrow$ The graph touches the x-axis at $(-1, 0)$.

Hence, vertex $= (-1, 0)$

For y-intercept, put $x = 0$ in eq. (1)

$$y = -4(0+1)^2 = -4$$

$\Rightarrow$ The graph cuts the y-axis at $(0, -4)$.

As $a = -4 < 0$, therefore parabola opens downward.

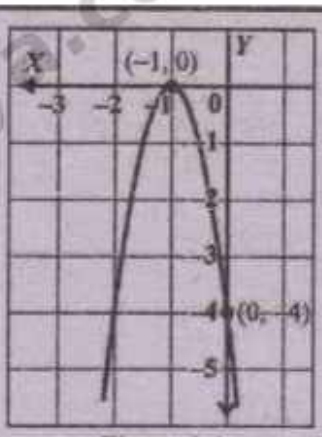


Figure 1.4

PREDICTING A QUADRATIC FUNCTION FROM ITS GRAPH

A quadratic function $f(x) = ax^2 + bx + c$ can be determined from its graph by using the **factor form** if the x-intercepts and one additional point on the curve are known.

If the graph crosses the x-axis at $x = h$ and $x = k$, then the function can be written as:

$$f(x) = a(x-h)(x-k)$$

Here, a is a constant that determines the shape and direction of the parabola. To find a, substitute the coordinates of a third known point into the equation.

Example 3: Find the function whose graph is shown in given figure.

Solution:

From figure, it is clear that x-intercepts of the graph are $(-1, 0)$ and $(1, 0)$

$\Rightarrow$ $h = -1$ and $k = 1$

Required function of graph is:

$$y = a(x-h)(x-k)$$

$$y = a(x+1)(x-1) \quad \ldots(1)$$

Since, the graph passes through $(0, 4)$, so

$$4 = a(0+1)(0-1)$$

$$4 = a(1)(-1) \quad \Rightarrow \quad a = -4$$

Put $a = -4$ in eq. (1), we get

$$y = -4(x+1)(x-1)$$

$$y = -4(x^2 - 1)$$

$$\Rightarrow \quad f(x) = -4x^2 + 4$$

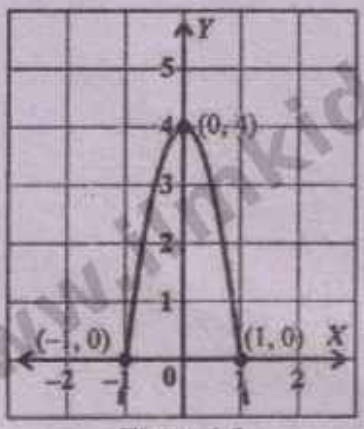


Figure 1.5

Example 4: Find an equation of the graph of a parabola whose x-intercepts are $x = 1$ and $x = 2$ and passes through the point $(3, 4)$.

Solution:

Since the x-intercepts of graph are $x = 1$ and $x = 2$, therefore $h = 1$ and $k = 2$

Required equation of graph is:

$$y = a(x-h)(x-k)$$

$$y = a(x-1)(x-2) \quad \ldots(1)$$

Since, the graph passes through $(3, 4)$, so

$$4 = a(3-1)(3-2)$$

$$4 = a(2)(1)$$

$$\Rightarrow \quad a = 2$$

Put $a = 2$ in eq. (1), we get

$$y = 2(x-1)(x-2)$$

$$y = 2(x^2 - 3x + 2)$$

$$y = 2x^2 - 6x + 4$$

www.ilmkidunya.com

www.ilmkidunya.com

EXERCISE 1.1

Q1. Sketch the graph of the following functions using the method of factorization.

(i) $y = 3(x+1)(x-1)$

Solution:

Given Function:

$$y = 3(x+1)(x-1) \quad \ldots(1)$$

For x-intercept, put $y = 0$

$$3(x+1)(x-1) = 0$$

$$x+1 = 0 \quad ; \quad x-1 = 0$$

$$x = -1 \quad ; \quad x = 1$$

$\Rightarrow$ The graph meets the x-axis at $(-1, 0)$ and $(1, 0)$.

For y-intercept, put $x = 0$

$$y = 3(0+1)(0-1)$$

$$y = 3(1)(-1) = -3$$

$\Rightarrow$ The graph cuts the y-axis at $(0, -3)$.

Vertex:

x-coordinate of vertex $= \dfrac{-1+1}{2} = 0$

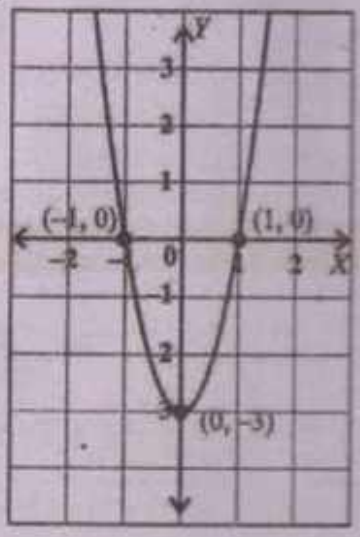
Put $x = 0$ in eq. (1), we get

$$y = 3(0+1)(0-1) = 3(1)(-1) = -3$$

$\Rightarrow$ Vertex $= (0, -3)$

As $a = 3 > 0$, therefore parabola opens upward.

Graph:



(ii) $f(x) = -2(x-1)(x-2)$

Solution:

Given Function:

$$y = -2(x-1)(x-2) \quad \ldots(1)$$

For x-intercept, put $y = 0$

$$-2(x-1)(x-2) = 0$$

$$x-1 = 0 \quad ; \quad x-2 = 0$$

$$x = 1 \quad ; \quad x = 2$$

$\Rightarrow$ The graph meets the x-axis at $(1, 0)$ and $(2, 0)$.

For y-intercept, put $x = 0$

$$y = -2(0-1)(0-2)$$

$$y = -2(-1)(-2) = -4$$

$\Rightarrow$ The graph cuts the y-axis at $(0, -4)$.

Vertex:

x-coordinate of vertex $= \dfrac{1+2}{2} = \dfrac{3}{2}$

Put $x = \dfrac{3}{2}$ in eq. (1), we get

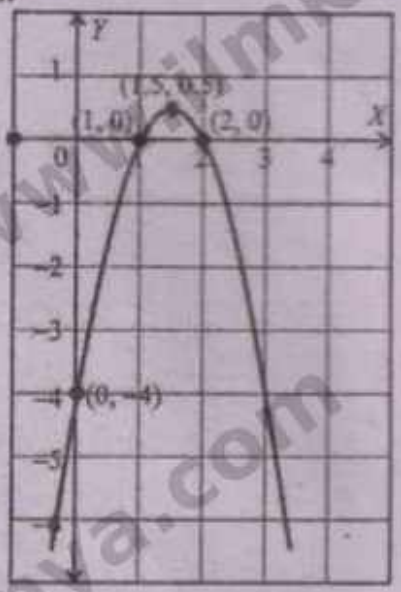
$$y = -2\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)$$

$$= -2\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right) = \frac{1}{2}$$

$\Rightarrow$ Vertex $= \left(\dfrac{3}{2}, \dfrac{1}{2}\right)$

As $a = -2 < 0$, therefore parabola opens downward.

Graph:



www.ilmkidunya.com

www.ilmkidunya.com

(iii) $y = (2x-1)(2x+1)$

Solution:

Given Function:

$$y = (2x-1)(2x+1)$$
$$= 2\left(x-\frac{1}{2}\right)\cdot 2\left(x+\frac{1}{2}\right)$$
$$y = 4\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right) \quad \ldots(1)$$

For x-intercept, put $y = 0$

$$4\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right) = 0$$
$$x-\frac{1}{2} = 0 \quad ; \quad x+\frac{1}{2} = 0$$
$$x = \frac{1}{2} \quad ; \quad x = -\frac{1}{2}$$

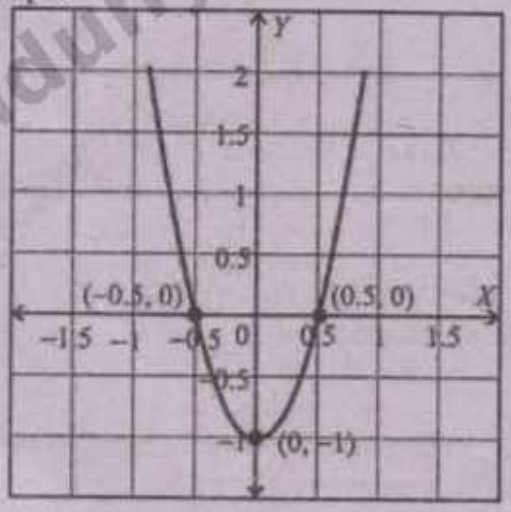
$\Rightarrow$ The graph meets the x-axis at $\left(-\frac{1}{2}, 0\right)$ and $\left(\frac{1}{2}, 0\right)$

For y-intercept, put $x = 0$

$$y = (2(0)-1)(2(0)+1)$$
$$y = (-1)(1) = -1$$

$\Rightarrow$ The graph cuts the y-axis at $(0, -1)$.

Graph:



Vertex:

x-coordinate of vertex $= \frac{-\frac{1}{2}+\frac{1}{2}}{2} = 0$

Put $x = 0$ in eq. (1), we get

$$y = (0-1)(0+1) = (-1)(1) = -1$$

$\Rightarrow$ Vertex $= (0, -1)$

As $a = 4 > 0$, therefore parabola opens upward.

(iv) $y = 2x^2 + x - 3$

Solution:

Given Function:

$$y = 2x^2 + x - 3$$
$$= 2x^2 + 3x - 2x - 3$$
$$= x(2x+3) - 1(2x+3)$$
$$= (x-1)(2x+3)$$
$$y = 2(x-1)\left(x+\frac{3}{2}\right) \quad \ldots(1)$$

For x-intercept, put $y = 0$

$$2(x-1)\left(x+\frac{3}{2}\right) = 0$$
$$x-1 = 0 \quad ; \quad x+\frac{3}{2} = 0$$
$$x = 1 \quad ; \quad x = -\frac{3}{2}$$

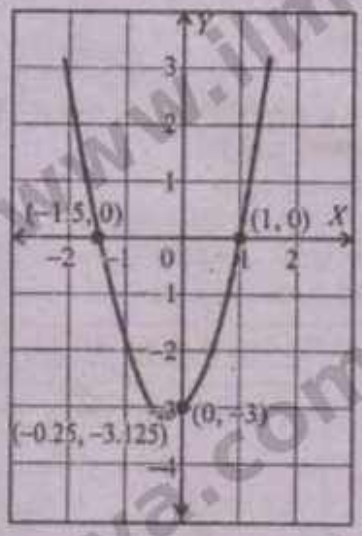
$\Rightarrow$ The graph meets the x-axis at $\left(-\frac{3}{2}, 0\right)$ and $(1, 0)$

For y-intercept, put $x = 0$

$$y = 2(0-1)\left(0+\frac{3}{2}\right)$$
$$y = 2(-1)\left(\frac{3}{2}\right) = -3$$

$\Rightarrow$ The graph cuts the y-axis at $(0, -3)$.

Graph:



Vertex:

x-coordinate of vertex $= \frac{\frac{-3}{2}+1}{2} = \frac{\frac{-1}{2}}{2} = \frac{-1}{4}$

Put $x = \frac{-1}{4}$ in eq. (1), we get

$$y = 2\left(\frac{-1}{4}-1\right)\left(\frac{-1}{4}+\frac{3}{2}\right)$$

www.ilmkidunya.com

www.ilmkidunya.com

$$y = 2\left(\frac{-5}{4}\right)\left(\frac{+5}{4}\right)$$
$$= \frac{-25}{8}$$

$\Rightarrow$ Vertex $= \left(\frac{-1}{4}, \frac{-25}{8}\right)$

As $a = 2 > 0$, therefore parabola opens upward.

(v) $f(x) = x^2 + 2x + 1$

Solution:

Given Function:

$$y = x^2 + 2x + 1$$
$$y = 1(x + 1)^2 \quad \ldots(1)$$

For x-intercept, put $y = 0$

$$(x + 1)^2 = 0$$
$$x + 1 = 0 \quad \Rightarrow \quad x = -1$$

$\Rightarrow$ The graph touches the x-axis at $(-1, 0)$

Hence, vertex $= (-1, 0)$

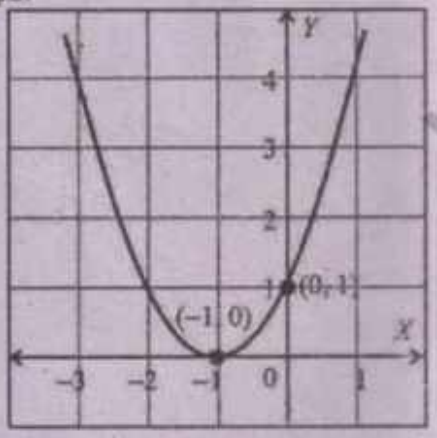
For y-intercept, put $x = 0$

$$y = (0 + 1)^2 = 1$$

$\Rightarrow$ The graph cuts the y-axis at $(0, 1)$.

As $a = 1 > 0$, therefore parabola opens upward.

Graph:



(vi) $y = 3(x - 1)^2$

Solution:

Given Function:

$$y = 3(x - 1)^2 \quad \ldots(1)$$

For x-intercept, put $y = 0$

$$3(x - 1)^2 = 0$$
$$x - 1 = 0 \quad \Rightarrow \quad x = 1$$

$\Rightarrow$ The graph touches the x-axis at $(1, 0)$

Hence, vertex $= (1, 0)$

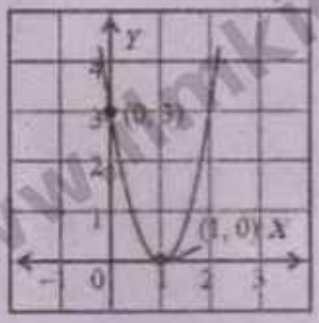
For y-intercept, put $x = 0$

$$y = 3(0 - 1)^2 = 3$$

$\Rightarrow$ The graph cuts the y-axis at $(0, 3)$.

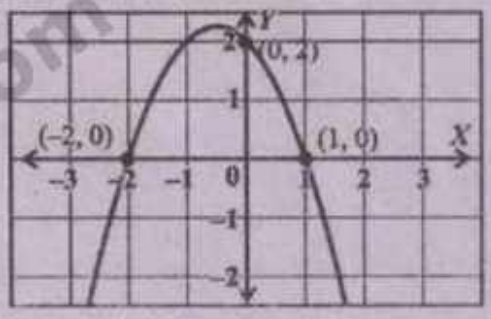
As $a = 3 > 0$, therefore parabola opens upward.

Graph:



Q2. Find the functions whose graphs are shown in the following figures.

(i)



Solution:

From figure, it is clear that x-intercepts of graph are $(-2, 0)$ and $(1, 0)$

$\Rightarrow$ $h = -2$ and $k = 1$

Required function of graph is:

$$y = a(x - h)(x - k)$$
$$y = a(x + 2)(x - 1) \quad \ldots(1)$$

Since the graph passes through $(0, 2)$, so

$$2 = a(0 + 2)(0 - 1)$$
$$2 = -2a$$
$\Rightarrow$
$$a = -1$$

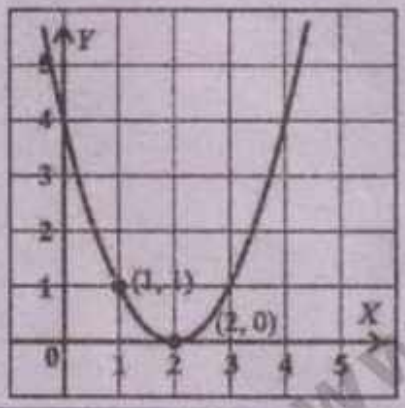
Put $a = -1$ in eq. (1), we get

$$y = -1(x + 2)(x - 1)$$
$$y = -(x + 2)(x - 1)$$

www.ilmkidunya.com

www.ilmkidunya.com

Q2.



Solution:

From figure, it is clear that the graph touches the x-axis at $(2, 0) \Rightarrow h = 2$

Required function of graph is:

$$y = a(x - h)^2$$
$$y = a(x - 2) \quad \ldots(1)$$

Since the graph passes through $(1, 1)$, so

$$1 = a(1 - 2)^2$$
$$1 = a(-1)^2 \Rightarrow a = 1$$

Put $a = 1$ in eq. (1), we get

$$y = 1(x - 2)^2$$

Q3. Find an equation of the graph of a parabola whose x-intercepts are $x = -1$ and $x = -2$ and which passes through the point $(-3, 2)$.

Solution:

Since, the x-intercepts of graph are $x = -2$ and $x = -1$ therefore $h = -2, k = -1$

Required equation of graph is:

$$y = a(x - h)(x - k)$$
$$y = a(x + 2)(x + 1) \quad \ldots(1)$$

Since, the graph passes through $(-3, 2)$, so

$$2 = a(-3 + 2)(-3 + 1)$$
$$2 = a(-1)(-2) \Rightarrow a = \frac{2}{2} = 1$$

Put $a = 1$ in eq. (1), we get

$$y = 1(x + 2)(x + 1)$$
$$y = (x + 1)(x + 2)$$

Q4. Find an equation of the graph of a parabola which touches x-axis at $x = -1$ and cuts y-axis at $y = -2$.

Solution:

Since, the graph touches the x-axis at $x = -1$, therefore $h = -1$.

Required equation of graph is:

$$y = a(x - h)^2$$
$$y = a(x + 1)^2 \quad \ldots(1)$$

Since, the graph cuts the y-axis at $y = -2$, so passes through $(0, -2)$

$$-2 = a(0 + 1)^2 \Rightarrow a = -2$$

Put $a = -2$ in eq. (1), we get

$$y = -2(x + 1)^2$$

Q5. Find an equation of a quadratic function whose graph cuts x-axis at $x = -1$ and $x = 1$, and whose vertex is at $(0, -2)$.

Solution:

Since, the graph cuts the x-axis at $x = -1$, therefore $x = 1$, therefore $h = -1, k = 1$

Required function of graph is:

$$y = a(x - h)(x - k)$$
$$y = a(x + 1)(x - 1) \quad \ldots(1)$$

Since, the graph passes through vertex $(0, -2)$, so

$$-2 = a(0 + 1)(0 - 1)$$
$$-2 = a(1)(-1) \Rightarrow a = 2$$

Put $a = 2$ in eq. (1), we get

$$y = 2(x + 1)(x - 1) = 2(x^2 - 1)$$
$$y = 2x^2 - 2$$

Content Outline

- Algebraic and Transcendental Functions
- Exponential Functions
- Introduction to Hyperbolic and Inverse Hyperbolic Functions

ALGEBRAIC AND TRANSCENDENTAL FUNCTIONS

Algebraic and transcendental functions are two fundamental classes of functions studied in mathematics. Together, they play a vital role in modeling and understanding a wide range of real-world phenomena. In this section, we distinguish between algebraic and transcendental functions.

www.ilmkidunya.com

www.ilmkidunya.com

(i) Algebraic Functions

A function $y = f(x)$ is algebraic if it satisfies a polynomial equation of the form

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \cdots + P_1(x)y + P_0(x) = 0$$

where each $P_i(x)$ is a polynomial in x for $i = 0,1,2,\cdots,n$ and atleast one $P_i(x)$ is not identically zero for $i = 1,2,3,\ldots,n$.

In simple words, a function $y = f(x)$ is algebraic if it is obtained from x using a finite number of algebraic operations such as addition, subtraction, multiplication, division, and n^{th} roots.

$f(x)=2x^2-3x, g(x)=\dfrac{x^2+1}{x-3}, x\neq 3, h(x)=(x+1)^{\frac{3}{2}}$ and $k(x)=\sqrt{x+1}$ are examples of algebraic functions.

(ii) Transcendental Functions

A function is called a transcendental function if it is not algebraic.

$y=e^x, y=2^x, y=\ln x, y=\sin x, y=\sin^{-1}x, y=\sin x+\ln x, y=e^{x^2}$ are examples of transcendental functions.

Example 5: Given the following functions:

$$f(x)=\sqrt{x+1}, g(x)=e^x, h(x)=\frac{x+1}{x^2-2x+1}, \quad q(x)=\ln 2x, r(x)=\sin x$$

Classify each function as algebraic or transcendental.

Solution:

Algebraic Functions:

$f(x)=\sqrt{x+1}, h(x)=\dfrac{x+1}{x^2-2x+1}$

Transcendental Functions:

$g(x) = e^x, q(x) = \ln 2x,$

$r(x) = \sin x$

Note: The sum, difference, product, quotient and n^{th} root of algebraic functions is again an algebraic function.

EXPONENTIAL FUNCTIONS

The function defined by $f(x) = a^x$ for all $x \in \mathbb{R}$, where $a > 0$ and $a \neq 1$ is called an exponential function. The number a is called the base of the exponential function.

Natural Exponential Function

The function $f(x) = e^x$ is called the natural exponential function, where e is a number defined by $e=\sum_{n=0}^{\infty}\frac{1}{n!}$ and its approximate value is $e \approx 2.71828$.

The exponential functions $f(x) = e^x$ and $g(x) = 2^x$ represent exponential growth, while the function $f(x)=\left(\frac{1}{2}\right)^x$ represents exponential decay.

1. Graph of Exponential Function

- Before sketching the graph of an exponential function, we study its essential properties that determine its overall shape and behaviour. These include the domain and range, intercepts, injectivity, monotonicity, and asymptotic behaviour.
- Understanding these features help us to predict the graph without plotting many points. After this analysis, we use a few representative values to draw a smooth and accurate sketch of the function.

2. Properties of the Exponential Function

Consider the exponential function:

$$y = a^x (a > 0, a \neq 1) \quad \ldots(1)$$

(i) Domain, Continuity, Positivity and Range of Exponential Function

Domain:

The exponential function is defined for all real values of x. We can substitute any real number (positive, negative, or zero) in place of x, and the expression a^x is always meaningful. Hence, the domain of the exponential function is $(-\infty, \infty)$.

Continuity:

Continuity at a point $c \in (-\infty, \infty)$ is shown using the equation:

$$\lim_{x \to c} a^x = a^c$$

www.ilmkidunya.com

www.ilmkidunya.com

This shows that the exponential function is continuous on the whole domain $(-\infty,\infty)$. Thus, graph of exponential function has no jump or hole.

Positivity:

If $x = n$, a positive integer, then $a^n = a \cdot a \cdot \ldots \cdot a$ (n times).

Since, $a > 0$, product of positive numbers is positive. Thus, $a^n > 0$ also, $a^{-n} = \frac{1}{a^n}$.

Since, $a^n > 0$, its reciprocal is also positive. So, $a^{-n} > 0$.

Moreover, $a^0 = 1 > 0$.

If x is rational, then $x = \frac{p}{q}$, where p is an integer and q is a natural number. Thus:

$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

From integer case, $a^p > 0$. Since, the qth root of a positive number is also positive, we have $a^{\frac{p}{q}} > 0$.

Thus, $a^x > 0$ for all rational numbers x. For irrational values of x, we accept that a^x is also positive. However, a rigorous proof of this fact is beyond the scope of this level. Hence, $y = a^x > 0$ for all real numbers x. It follows that the graph of exponential function always lies above the x-axis.

Range:

By observing the following facts:

1. $\lim_{x\to\infty} a^x = \infty$ and $\lim_{x\to-\infty} a^x = 0^+$ if $a > 1$.
2. $\lim_{x\to-\infty} a^x = \infty$ and $\lim_{x\to\infty} a^x = 0^+$ if $0 < a < 1$.
3. $y = a^x$ is continuous on $(-\infty,\infty)$.
4. $y = a^x > 0$ for all real numbers x.

We conclude that, the range of the exponential function (1) is $(0,\infty)$.

(ii) Intercepts of Exponential Function

y-intercept: To find y-intercept, put $x = 0$ in equation (1):

$$y = a^0 = 1$$

So, the exponential function $y = a^x$ cuts the y-axis at the point $(0,1)$.

x-intercept: To find x-intercept, put $y = 0$ in equation (1):

$$a^x = 0$$

But, $a^x > 0$ for all real x. It follows that there is no x-intercept.

DEFINITIONS: (Strictly Increasing and Strictly Decreasing Functions)

Let $y = f(x)$ be a real-valued function and let $x_1, x_2 \in D_f$ be any two points such that $x_1 < x_2$. The function f is called:

(i) **Strictly increasing** if $f(x_1) < f(x_2)$. In this case, the graph of the function rises as x moves from left to right along the x-axis.

(ii) **Strictly decreasing** if $f(x_1) > f(x_2)$. In this case, the graph of the function falls as x moves from left to right along the x-axis.

For Example: Consider the function $y = x^2$.

This function is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0,\infty)$.

(iii) Exponential Function is Strictly Increasing or Strictly Decreasing

Let $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

Case-1: If $a > 1$

First we show that $a^h > 1$ for every rational number $h > 0$. Put $h = \frac{m}{n}$ where $m, n \in \mathbb{Z}^+$. We can write $a^m = a \cdot a \cdot a \ldots a$ (m times). Since $a > 1$, we have $a^m > 1$, which implies that $(a^m)^{\frac{1}{n}} > (1)^{\frac{1}{n}}$ and so $a^h > 1$. For the irrational values of h, we skip the proof at this level and accept that $a^h > 1$.

Now, if $x_1 < x_2$ then $x_2 - x_1 > 0$ and so $a^{x_2 - x_1} > 1$ or $a^{x_2} > a^{x_1}$. Therefore, $f(x) = a^x$ is **strictly increasing** when $a > 1$.

Case-2: If $0 < a < 1$

First we show that $a^h < 1$ for every rational number $h > 0$. Put $h = \frac{m}{n}$ where $m, n \in \mathbb{Z}^+$. We can write $a^m = a \cdot a \cdot a \ldots a$ (m times). Since $a < 1$, we have $a^m < 1$, which implies that $(a^m)^{\frac{1}{n}} < (1)^{\frac{1}{n}}$ and so $a^h > 1$. For the irrational values of h, we skip the proof at this level and accept that $a^h < 1$.

Now, if $x_1 < x_2$ then $x_2 - x_1 > 0$ and so $a^{x_2 - x_1} < 1$ or $a^{x_2} < a^{x_1}$. Therefore, $f(x) = a^x$ is **strictly decreasing** when $0 < a < 1$.

www.ilmkidunya.com

www.ilmkidunya.com

(iv) Injectivity of Exponential Functions

Let $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

Further let, $f(x_1) = f(x_2)$ for some $x_1, x_2 \in (-\infty, \infty)$.

So: $a^{x_1} = a^{x_2}$

Case-1: If $x_1 < x_2$, then from the increasing or decreasing behaviour of the function of $f(x) = a^x$ either $a^{x_1} < a^{x_2}$ (when $a > 1$) or $a^{x_1} > a^{x_2}$ (when $0 < a < 1$). Which is impossible.

Case-2: If $x_1 > x_2$, then from the increasing or decreasing behaviour of the function of $f(x) = a^x$ either $a^{x_1} > a^{x_2}$ (when $a > 1$) or $a^{x_1} < a^{x_2}$ (when $0 < a < 1$). Which is again impossible.

Hence, we must have $x_1 = x_2$ and consequently, $f(x) = a^x$ is injective. It follows that any horizontal line meets the graph of $f(x) = a^x$ at most once.

DEFINITION: Injectivity/One-to-One Function

Recall that, a function $y = f(x)$ is called injective (one-to-one) if different inputs always give different outputs. Formally, a function is one-to-one if:

whenever for $x_1, x_2 \in \text{Dom} f$ such that

$f(x_1) = f(x_2)$, we have $x_1 = x_2$

A function is injective if every horizontal line meets its graph at most once.

(v) Horizontal Asymptote of Exponential Function

Note that if $a > 1$, then $a^x \to 0^+$ as $x \to -\infty$ and if $0 < a < 1$, then $a^x \to 0^+$ as $x \to \infty$. It follows that the horizontal line $y = 0$ represents the horizontal asymptote for the exponential function.

Note: (Horizontal Asymptote)

A **horizontal asymptote** is a horizontal line $y = c$ which a function approaches as x becomes very large (or very small), but the graph does not necessarily touch or cross it.

Formally the horizontal line $y = c$ is called **horizontal asymptote** for the function $y = f(x)$ if $f(x) \to c$ as $x \to -\infty$ or $x \to \infty$.

Example 6: Graph of $f(x) = e^x$

Solution:

Table of values for $f(x) = e^x$

x	$\to -\infty$	-3	-2	-1	0	1	2	3	$\to \infty$
$y = e^x$	$\to 0$	0.05	0.14	0.37	1	2.72	7.39	20.09	$\to \infty$

Graph:

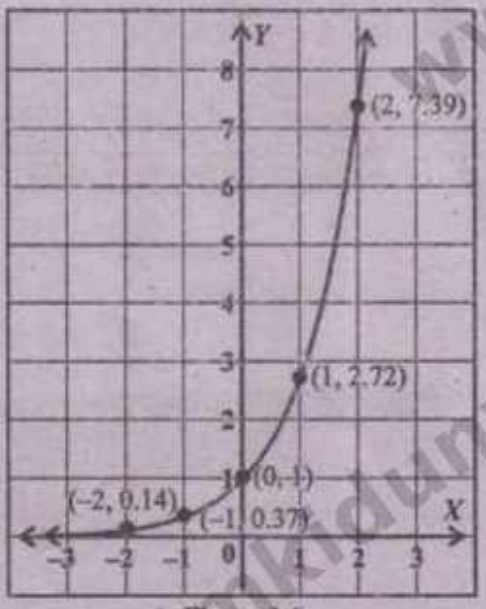


Figure 1.6

www.ilmkidunya.com

www.ilmkidunya.com

Example 7: Graphs of $f(x)=\left(\frac{1}{2}\right)^x=\frac{1}{2^x}$

Solution:

Table of values for $f(x)=\left(\frac{1}{2}\right)^x$

x	$\to-\infty$	-3	-2	-1	0	1	2	3	$\to\infty$
$y=\left(\frac{1}{2}\right)^x$	$\to\infty$	8	4	2	1	0.5	0.25	0.125	$\to 0$

Graph:

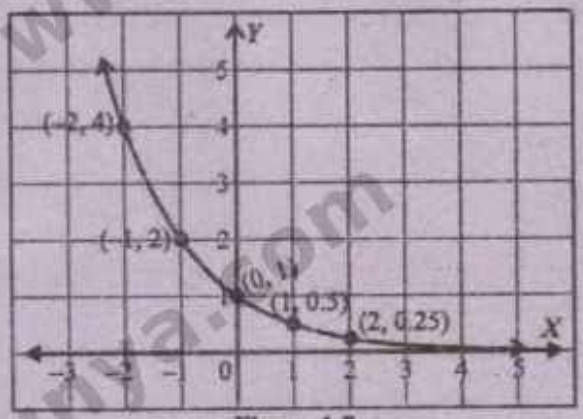


Figure 1.7

INTRODUCTION TO HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

- Hyperbolic functions are a class of transcendental functions closely related to exponential functions. They are defined using combinations of e^x and e^{-x}, and their graphs resemble trigonometric functions, although they arise from hyperbolas rather than circles. The basic hyperbolic functions include $\sinh x$, $\cosh x$ and $\tanh x$ which have important applications in mathematics, physics and engineering.
- Inverse hyperbolic functions are the inverse functions of hyperbolic functions, defined on suitable restricted domains. They are useful in solving equations involving hyperbolic functions and appear in various applications such as integration, differential equations, and modeling real-world phenomena.

Hyperbolic Functions:

The hyperbolic functions are defined as follows:

Function	Definition	Domain	Range
$\sinh x$	$\sinh x=\frac{e^x-e^{-x}}{2}$	$R=(-\infty,\infty)$	$R=(-\infty,\infty)$
$\cosh x$	$\cosh x=\frac{e^x+e^{-x}}{2}$	$R=(-\infty,\infty)$	$[1,\infty)$
$\tanh x=\frac{\sinh x}{\cosh x}$	$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$	$R=(-\infty,\infty)$	$(-1,1)$
$\coth x=\frac{\cosh x}{\sinh x}$	$\coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}$	$R\setminus\{0\}$	$(-\infty,-1)\cup(1,\infty)$
$\operatorname{sech} x=\frac{1}{\cosh x}$	$\operatorname{sech} x=\frac{2}{e^x+e^{-x}}$	$R=(-\infty,\infty)$	$(0,1]$
$\operatorname{csch} x=\frac{1}{\sinh x}$	$\operatorname{csch} x=\frac{2}{e^x-e^{-x}}$	$R\setminus\{0\}$	$(-\infty,0)\cup(0,\infty)$

Hyperbolic functions are so named because the parametric equations $x=\cosh t$ and $y=\sinh t$, satisfies the equation of the hyperbola $x^2-y^2=1$.

www.ilmkidunya.com

www.ilmkidunya.com

Example 8: Prove that $\cosh^2 x - \sinh^2 x = 1$.

Solution:

$$\text{L.H.S} = \cosh^2 x - \sinh^2 x$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x} + 2e^x \cdot e^{-x}}{4} - \frac{e^{2x} + e^{-2x} - 2e^x \cdot e^{-x}}{4} = \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4} = \frac{4}{4}$$

$$= 1 = \text{R.H.S (Proved)}$$

Graphs of Hyperbolic Functions

(i) Draw the graph of $\sinh x$

Solution:

Let $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2} = \dfrac{e^x - \left(\frac{1}{e}\right)^x}{2}$

Table of values:

x	$\to -\infty$	-3	-2	-1	0	1	2	3	$\to \infty$
$y = \sinh x$	$\to -\infty$	-10.03	-3.63	-1.18	0	1.18	3.63	10.03	$\to \infty$

Graph:

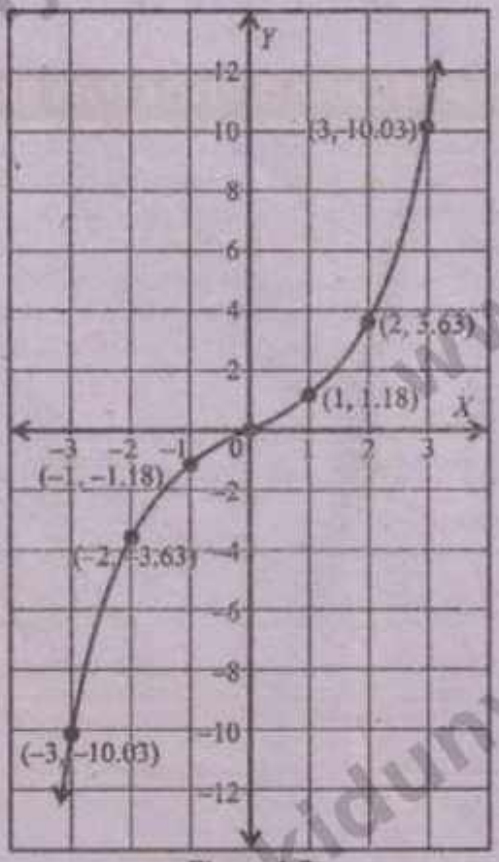


Figure 1.8

www.ilmkidunya.com

www.ilmkidunya.com

(iii) **Draw the graph of cosh x**

Solution:

Let $f(x)=\cosh x=\dfrac{e^x+e^{-x}}{2}=\dfrac{e^x+\left(\frac{1}{e}\right)^x}{2}$

Table of values:

x	$\to-\infty$	-3	-2	-1	0	1	2	3	$\to\infty$
$y=\cosh x$	$\to+\infty$	10.07	3.76	1.54	1	1.54	3.76	10.07	$\to\infty$

Graph:

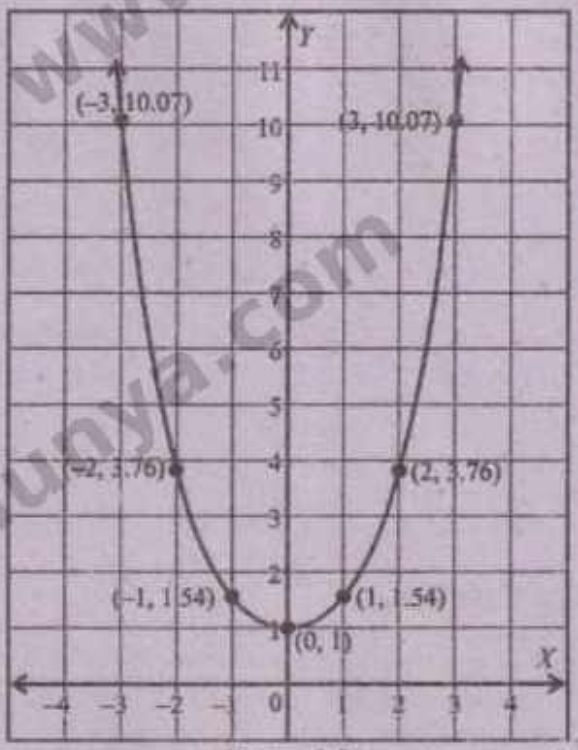


Figure 1.9

Inverse Hyperbolic Functions:

The formulae, domains and ranges of all inverse hyperbolic functions are given in the following table.

Function	Formula	Domain	Range
$\sinh^{-1}x$	$\ln(x+\sqrt{x^2+1})$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\cosh^{-1}x$	$\ln(x+\sqrt{x^2-1})$	$[1,\infty)$	$[0,\infty)$
$\tanh^{-1}x$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$(-1,1)$	$(-\infty,\infty)$
$\coth^{-1}x$	$\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$
$\operatorname{sech}^{-1}x$	$\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$(0,1]$	$[0,\infty)$
$\operatorname{csch}^{-1}x$	$\ln\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}+1}\right)$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$

www.ilmkidunya.com

www.ilmkidunya.com

(i) Inverse Hyperbolic sine Functions

From the graph of $y = \sinh x$ (see Figure 1.8), it is clear from the horizontal line test that the function is bijective. Hence, its inverse function exists, and we define it as follows:

$$x = \sinh^{-1} y \text{ if and only if } y = \sinh x$$

Example 9: Prove that $\sinh^{-1} x = \ln(x+\sqrt{x^2+1})$.

Solution:

Let $y = \sinh^{-1} x$, where $x, y \in \mathbb{R}$

$$x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - \frac{1}{e^y}$$

Multiply both sides by e^y

$$2xe^y = (e^y)^2 - 1$$

$$1(e^y)^2 - 2x(e^y) - 1 = 0$$

Here $a = 1, b = -2x, c = -1$

By using quadratic formula

$$e^y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2x \pm \sqrt{4(x^2+1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2+1}}{2} = \frac{2(x \pm \sqrt{x^2+1})}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

Since e^y must always be positive and $\sqrt{x^2+1} > x$

$\Rightarrow$ $x - \sqrt{x^2+1} < 0$, so we take

$$e^y = x + \sqrt{x^2+1}$$

Taking $\ln$ on both sides:

$$\ln e^y = \ln(x+\sqrt{x^2+1})$$

$$y \ln e = \ln(x+\sqrt{x^2+1}) \quad \because \ln e = 1$$

$$y = \ln(x+\sqrt{x^2+1})$$

$$\Rightarrow \sinh^{-1} x = \ln(x+\sqrt{x^2+1})$$

Note: If $y = f(x)$ is a bijective function, then $x = f^{-1}(y)$ is its inverse.

(ii) Inverse Hyperbolic cosine Function

From the graph of $y = \cosh x$ (see Figure 1.9), it is clear from the horizontal line test that the function is not one-to-one on the domain $(-\infty, \infty)$. However, if we restrict the domain to $[0, \infty)$, the range set is not disturbed. It follows that $y = \cosh x$ becomes one-to-one for the restricted domain $[0, \infty)$ and thus bijective. Therefore, on the domain $[0, \infty)$, the inverse hyperbolic cosine function exists and we define it as follows:

$$x = \cosh^{-1} y \text{ if and only if } y = \cosh x$$

Example 10: Prove that $\cosh^{-1} x = \ln(x+\sqrt{x^2-1})$ for all $x \in [1, \infty)$.

Solution:

Let $y = \cosh^{-1} x$, where $x \geq 1$ and $y \geq 0$

$$x = \cosh y$$

$$x = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + \frac{1}{e^y}$$

Remember!
Dom$f$ = Range f^{-1}
Range f = Domf^{-1}

Multiply both sides by e^y

$$2xe^y = (e^y)^2 + 1$$

$$1(e^y) - 2x(e^y) + 1 = 0$$

Here $a = 1, b = -2x, c = 1$

By using quadratic formula:

$$e^y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$= \frac{2x \pm \sqrt{4(x^2-1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2-1}}{2}$$

www.ilmkidunya.com

www.ilmkidunya.com

$$= \frac{2(x \pm \sqrt{x^2-1})}{2}$$

$$e^y = x \pm \sqrt{x^2-1}$$

Since $y \geq 0 \Rightarrow e^y \geq e^0 = 1$, so we take

$$e^y = x + \sqrt{x^2-1} \quad \because e^y = x - \sqrt{x^2-1} \leq 1 \text{ for any } x \geq 1$$

Taking $\ln$ on both sides:

$$\ln e^y = \ln(x+\sqrt{x^2-1})$$

$$y \ln e = \ln(x+\sqrt{x^2-1})$$

$$y = \ln(x+\sqrt{x^2-1}) \quad \because \ln e = 1$$

$$\Rightarrow \quad \cosh^{-1} x = \ln(x+\sqrt{x^2-1})$$

EXERCISE 1.2

Q1. Which of the following functions are algebraic?

$$f(x)=\sec x, g(x)=\frac{2x-3}{x+1}, h(x)=\sqrt{x^2+2x}, k(x)=\frac{e^x-e^{-x}}{2}, r(x)=(x-1)^3$$

Solution:

Algebraic Functions:

$$g(x)=\frac{2x-3}{x+1}, h(x)=\sqrt{x^2+2x}, r(x)=(x-1)^3$$

Transcendental Functions:

$$f(x)=\sec x, k(x)=\frac{e^x-e^{-x}}{2}$$

Q2. Which of the following functions are transcendental?

$$f(x)=\frac{1}{x}, g(x)=\cot x, h(x)=e^{2x}+\sqrt{x^2+2x}, k(x)=\frac{e^x+e^{-x}}{2}, r(x)=(x-1)^{\frac{3}{2}}$$

Solution:

Transcendental Functions:

$$g(x)=\cot x, h(x)=e^{2x}+\sqrt{x^2+2x}, k(x)=\frac{e^x+e^{-x}}{2}$$

Algebraic Functions:

$$f(x)=\frac{1}{x}, r(x)=(x-1)^{\frac{3}{2}}$$

Q3. Discuss domain, range, and graph of the function $f(x)=3^x$.

Solution:

Given Function: $f(x) = 3^x$

Domain:

Since $f(x) = 3^x$ is defined for all real values of x, so

$$D_f =]-\infty, \infty[$$

Range:

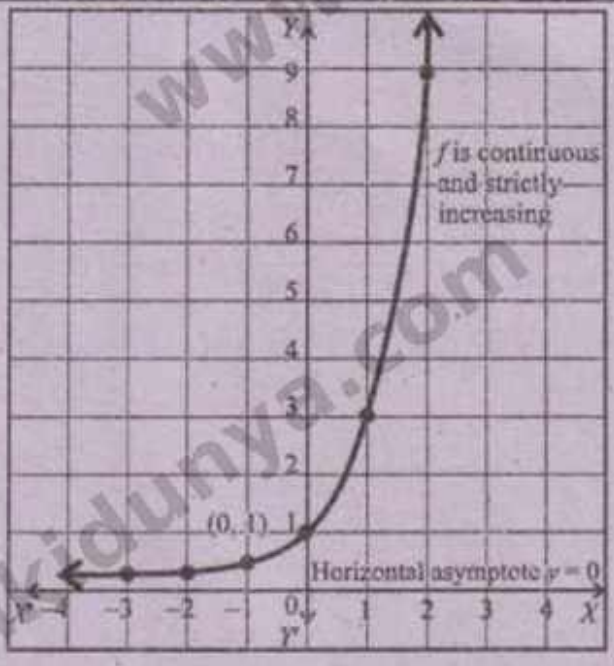
If $x \to \infty$, then $3^x \to \infty$

If $x \to -\infty$, then $3^x \to 0$

$$R_f =]0, \infty[$$

Table of values for $f(x) = 3^x$

x	$\to -\infty$	-3	-2	-1	0	1	2	3	$\to \infty$
$y = 3^x$	$\to \infty$	0.04	0.11	0.33	1	3	9	27	$\to \infty$



www.ilmkidunya.com

www.ilmkidunya.com

Q4. Discuss domain, range, and graph of the function $f(x)=\left(\frac{1}{3}\right)^x$.

Solution:

Given Function: $f(x)=\left(\frac{1}{3}\right)^x$

Domain: Since $f(x)=\left(\frac{1}{3}\right)^x$ is defined for all real values of x, so

$$D_f=]-\infty,\infty[$$

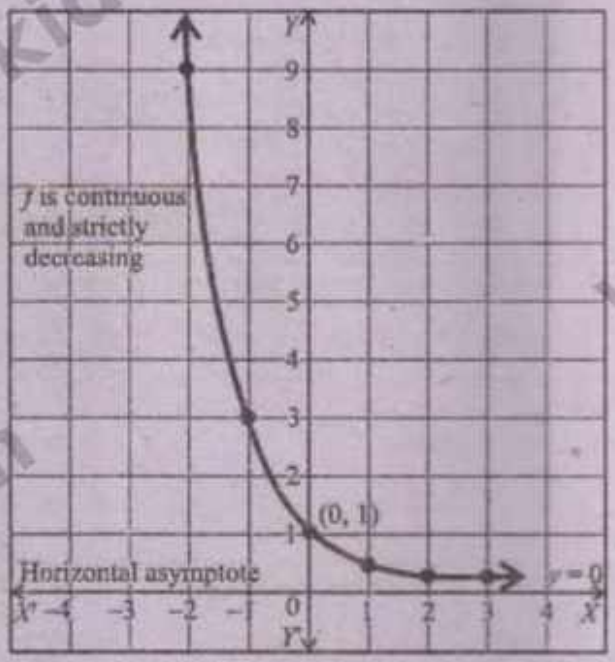
Range: If $x\to\infty$, then $\left(\frac{1}{3}\right)^x\to 0$

If $x\to-\infty$, then $\left(\frac{1}{3}\right)^x\to\infty$

$$R_f=]0,\infty[$$

Table of values for $f(x)=\left(\frac{1}{3}\right)^x$

x	$\to-\infty$	-3	-2	-1	0	1	2	3	$\to\infty$
$y=\left(\frac{1}{3}\right)^x$	$\to\infty$	27	9	3	1	0.33	0.11	0.04	$\to 0$



Q5. Prove the following hyperbolic identities:

(i) $1-\tanh^2 x=\operatorname{sech}^2 x$

Solution:

L.H.S $=1-\tanh^2 x$

$$=1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2$$

$$=1-\frac{(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$$

$$=\frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$$

$$=\frac{(e^{2x}+e^{-2x}+2e^xe^{-x})-(e^{2x}+e^{-2x}-2e^xe^{-x})}{(e^x+e^{-x})^2}$$

$$=\frac{e^{2x}+e^{-2x}+2-e^{2x}-e^{-2x}+2}{(e^x+e^{-x})^2}$$

$$=\frac{4}{(e^x+e^{-x})^2}$$

$$=\left(\frac{2}{e^x+e^{-x}}\right)^2$$

$=\operatorname{sech}^2 x$

= R.H.S (Proved)

(ii) $\operatorname{csch}^2 x=\coth^2 x-1$

Solution:

R.H.S $=\coth^2 x-1$

$$=\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)-1$$

$$=\frac{(e^x+e^{-x})^2}{(e^x-e^{-x})^2}-1$$

$$=\frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x-e^{-x})^2}$$

$$=\frac{(e^{2x}+e^{-2x}+2e^xe^{-x})-(e^{2x}+e^{-2x}-2e^xe^{-x})}{(e^x-e^{-x})^2}$$

$$=\frac{e^{2x}+e^{-2x}+2-e^{2x}-e^{-2x}+2}{(e^x-e^{-x})^2}$$

$$=\frac{4}{(e^x-e^{-x})^2}$$

$$=\left(\frac{2}{e^x-e^{-x}}\right)^2$$

$=\operatorname{csch}^2 x$

= L.H.S (Proved)

www.ilmkidunya.com

www.ilmkidunya.com

(iii) $\sinh 2x = 2\sinh x \cosh x$

Solution:

R.H.S $= 2\sinh x \cosh x$

$$= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{(e^x)^2 - (e^{-x})^2}{2} \qquad \because (a+b)(a-b) = a^2 - b^2$$

$$= \frac{e^{2x} - e^{-2x}}{2}$$

$= \sinh 2x$

= L.H.S (Proved)

(iv) $\cosh 2x = \cosh^2 x + \sinh^2 x$

Solution:

R.H.S $= \cosh^2 x + \sinh^2 x$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x}}{4} + \frac{e^{2x} + e^{-2x} - 2e^x e^{-x}}{4}$$

$$= \frac{e^{2x} + e^{-2x} + 2 + e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{2(e^{2x} + e^{-2x})}{4} = \frac{e^{2x} + e^{-2x}}{2}$$

$= \cosh 2x$ = L.H.S (Proved)

Q6. Prove the following inverse hyperbolic formulae.

(i) $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$

Solution:

Let $y = \tanh^{-1} x$

$x = \tanh y$

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$x = \frac{e^y - \frac{1}{e^y}}{e^y + \frac{1}{e^y}}$$

$$x = \frac{\frac{e^{2y} - 1}{e^y}}{\frac{e^{2y} + 1}{e^y}}$$

$$x(e^{2y} + 1) = e^{2y} - 1$$

$$xe^{2y} + x = e^{2y} - 1$$

$$1 + x = e^{2y} - xe^{2y}$$

$$e^{2y}(1 - x) = 1 + x$$

$$e^{2y} = \frac{1+x}{1-x}$$

Taking $\ln$ on both sides:

$$\ln e^{2y} = \ln\left(\frac{1+x}{1-x}\right)$$

$$2y \cdot \ln e = \ln\left(\frac{1+x}{1-x}\right)$$

$$2y = \ln\left(\frac{1+x}{1-x}\right) \qquad \because \ln e = 1$$

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow \quad \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

(ii) $\coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$

Solution:

Let $y = \coth^{-1} x$

$x = \coth y$

$$x = \frac{e^y + e^{-y}}{e^y - e^{-y}}$$

$$x = \frac{e^y + \frac{1}{e^y}}{e^y - \frac{1}{e^y}}$$

$$x = \frac{\frac{e^{2y} + 1}{e^y}}{\frac{e^{2y} - 1}{e^y}}$$

$$x(e^{2y} - 1) = e^{2y} + 1$$

$$xe^{2y} - x = e^{2y} + 1$$

$$xe^{2y} - e^{2y} = x + 1$$

$$e^{2y}(x - 1) = x + 1$$

$$e^{2y} = \frac{x+1}{x-1}$$

Taking $\ln$ on both sides:

$$\ln e^{2y} = \ln\left(\frac{x+1}{x-1}\right)$$

$$2y \cdot \ln e = \ln\left(\frac{x+1}{x-1}\right)$$

$$2y = \ln\left(\frac{x+1}{x-1}\right) \qquad \because \ln e = 1$$

$$y = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\Rightarrow \quad \coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

www.ilmkidunya.com

www.ilmkidunya.com

(iii) $\operatorname{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

Solution:

Let $y = \operatorname{sech}^{-1}x$, where $x \in (0,1]$ and $y \in [0,\infty)$

$x = \operatorname{sech} y$

$$x = \frac{2}{e^y + e^{-y}}$$

$$x(e^y + e^{-y}) = 2$$

$$x\left(e^y + \frac{1}{e^y}\right) = 2$$

$$x\left(\frac{e^{2y}+1}{e^y}\right) = 2$$

$$xe^{2y} + x = 2e^y$$

$$x(e^y)^2 - 2(e^y) + x = 0$$

Here $a = x, b = -2, c = x$

By using quadratic formula

$$e^y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{(-2)^2 - 4(x)(x)}}{2(x)}$$

$$= \frac{2 \pm \sqrt{4-4x^2}}{2x} = \frac{2 \pm \sqrt{4(1-x^2)}}{2x}$$

$$= \frac{2 \pm 2\sqrt{1-x^2}}{2x} = \frac{2(1 \pm \sqrt{1-x^2})}{2x}$$

$$e^y = \frac{1 \pm \sqrt{1-x^2}}{x}$$

Since $y \geq 0 \Rightarrow e^y \geq e^0 = 1$, so we take

$$e^y = \frac{1+\sqrt{1-x^2}}{x} \quad \because e^y = \frac{1-\sqrt{1-x^2}}{x} \leq 1, \forall x \in (0,1]$$

Taking $\ln$ on both sides:

$$\ln e^y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

$$y \ln e = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

$$y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \because \ln e = 1$$

$$\Rightarrow \quad \operatorname{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

(iv) $\operatorname{csch}^{-1}x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right)$

Solution:

Let $y = \operatorname{csch}^{-1}x$, where $x \in R, x \neq 0$

$x = \operatorname{csch} y$

$$x = \frac{2}{e^y - e^{-y}}$$

$$x(e^y - e^{-y}) = 2$$

$$x\left(e^y - \frac{1}{e^y}\right) = 2$$

$$x\left(\frac{e^{2y}-1}{e^y}\right) = 2$$

$$xe^{2y} - x = 2e^y$$

$$x(e^y)^2 - 2(e^y) - x = 0$$

By using quadratic formula

$$e^y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{(-2)^2 - 4(x)(-x)}}{2(x)} = \frac{2 \pm \sqrt{4+4x^2}}{2x}$$

$$= \frac{2 \pm \sqrt{4(1+x^2)}}{2x} = \frac{2 \pm 2\sqrt{1+x^2}}{2x}$$

$$= \frac{2(1 \pm \sqrt{1+x^2})}{2x} = \frac{1 \pm \sqrt{1+x^2}}{x}$$

$$= \frac{1}{x} \pm \frac{\sqrt{1+x^2}}{x} = \frac{1}{x} \pm \frac{\sqrt{1+x^2}}{\sqrt{x^2}}$$

$$e^y = \frac{1}{x} \pm \sqrt{\frac{1}{x^2}+1}$$

Since e^y must always be positive and $\sqrt{\frac{1}{x^2}+1} > \frac{1}{x}$

$\Rightarrow \quad \frac{1}{x} - \sqrt{\frac{1}{x^2}+1} < 0$ for all $x \in R, x \neq 0$, so we take

$$e^y = \frac{1}{x} + \sqrt{\frac{1}{x^2}+1}$$

Taking $\ln$ on both sides:

$$\ln e^y = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right)$$

$$y \ln e = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right)$$

$$y = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right) \qquad \because \ln e = 1$$

$$\Rightarrow \quad \operatorname{csch}^{-1}x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right)$$

www.ilmkidunya.com

www.ilmkidunya.com

Content Outline

- Logarithmic Function
- Modulus Function and Its Graph
- Interpreting the Relation Between a One-to-One Function and Its Inverse through Graph
- Transformation of Graphs of Functions

LOGARITHMIC FUNCTION

A logarithmic function is defined by $y = \log_a x, a > 0, a \neq 1, x > 0$. It assigns to each positive real number x a unique real number y, such that $a^y = x$.

Logarithmic Function as Inverse of Exponential Function

We know that the exponential function

$$y = a^x, (a > 0, a \neq 1)$$

is a one-to-one function with domain $(-\infty, \infty)$ and range $(0, \infty)$. Thus, it is inverse exists.
Interchanging x and y, we obtain:

$$x = a^y \Rightarrow y = \log_a x$$

Hence, the logarithmic function is the inverse of the exponential function with domain $(0, \infty)$ and range $(-\infty, \infty)$.

- A logarithm with base 10 is called a **common logarithm** and we write: $\log_{10} b = \log b$
- A logarithm with base e is called a **natural logarithm** and we write: $\log_e b = \ln b$

Basic Logarithmic Identities

For $a > 0, a \neq 1, x > 0$, we have:

$\log_a 1 = 0$	$\log_a a = 1$	$\log a(a^x) = x$	$\log^{\log_a x} = x$

Laws of Logarithms and Change of Base Formula

For $a > 0, a \neq 1, x > 0, y > 0, t \in \mathbb{R}$, we have studied the following laws of logarithms and change of base formula:

1. Product Law: $\log_a(xy) = \log_a x + \log_a y$
2. Quotient Law: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. Power Law: $\log_a(x^t) = t\log_a x$
4. Change of Base Formula: $\log_a x = \frac{\log_b x}{\log_b a}$ for every $b > 0, b \neq 1$.

In particular: for $b = e \Rightarrow \log_a x = \frac{\ln x}{\ln a}$ and for $b = 10 \Rightarrow \log_a x = \frac{\log x}{\log a}$

Graphical Behaviour of Logarithmic Functions

In this section, we study the graphical behaviour of logarithmic functions in a structured manner.

(i) Injectivity of Logarithmic Functions

Let $f(x) = \log_a x$, where $a > 0$ and $a \neq 1, x \in (0, \infty)$. Let $s, t \in (0, \infty)$ be arbitrary numbers such that,

$$f(s) = f(t) \Rightarrow \log_a s = \log_a t \Rightarrow a^{\log_a s} = a^{\log_a t} \Rightarrow s = t$$

Hence, the logarithmic function is one to one.

(ii) Vertical Asymptote of Logarithmic Functions

Let $f(x) = \log_a x$, where $a > 0, a \neq 1$ and $x > 0$. Then $x = a^y$.
We consider the following cases:

Case-1: $a > 1$
Note that: $x \to 0^+$
$\Rightarrow$ $a^y \to 0^+ \Rightarrow y \to -\infty$
$\Rightarrow$ $\log_a x \to -\infty$.
Hence, $\lim\limits_{x \to 0^+} f(x) = -\infty$.

Case-2: $0 < a < 1$
Note that: $x \to 0^+$
$\Rightarrow$ $a^y \to 0^+ \Rightarrow y \to \infty$
$\Rightarrow$ $\log_a x \to \infty$.
Hence $\lim\limits_{x \to 0^+} f(x) = \infty$

In both cases, we have: $\lim\limits_{x \to 0^+} f(x) = \pm\infty$.

Hence, the line $x = 0$ is the vertical asymptote of $f(x) = \log_a x$. Since, $y = \log_a x$ is defined for all positive values of x, there is no other asymptote except $x = 0$.

Vertical Asymptote of a Function

A **vertical asymptote** is a vertical line $x = c$ such that the function $f(x)$ increases or decreases without bound as $x \to c$. In limit form, $x = c$ is a vertical asymptote of the function $y = f(x)$ if:

$$\lim_{x \to c^+} f(x) = \pm\infty \text{ or } \lim_{x \to c^-} f(x) = \pm\infty$$

www.ilmkidunya.com

www.ilmkidunya.com

(iii) Logarithmic Functions are Strictly Increasing or Strictly Decreasing

Consider the logarithmic function:

$$y(x) = f(x) = \log_a x (a > 0, a \neq 1, x > 0)$$

Let $x_1, x_2 \in (0, \infty)$ such that $x_1 < x_2$.

Further let, $y_1 = f(x_1) = \log_a x_1$ and $y_2 = f(x_2) = \log_a x_2$. Then $x_1 = a^{y_1}$ and $x_2 = a^{y_2}$

Since, $x_1 < x_2$, we have $a^{y_1} < a^{y_2}$. We consider the following cases:

Case-1:

If $a > 1$, then $a^{y_1} < a^{y_2} \Rightarrow y_1 < y_2$.

It follows that:

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Hence, $y = \log_a x$ is strictly increasing when $a > 1$.

Case-2:

If $0 < a < 1$, then $a^{y_1} < a^{y_2} \Rightarrow y_1 > y_2$.

It follows that:

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

Hence, $y = \log_a x$ is strictly decreasing when $0 < a < 1$.

(iv) Intercepts and Continuity of Logarithmic Functions

For the logarithmic function: $y = \log_a x (a > 0, a \neq 1, x > 0)$

Intercepts:

(i) The x-intercept is obtained by substituting $y = 0$:

$$\log_a x = 0 \Rightarrow x = a^0 = 1$$

Hence, the graph of $y = \log_a x$ intersects x-axis at the point $(1,0)$.

(ii) Since, the function $y = \log_a x$ is undefined at $x = 0$, it has no y-intercept.

Continuity:

The function $y = \log_a x$ is continuous for all $x > 0$. This can be seen from the fact that: $\lim_{x \to c} \log_a x = \log_a c$ for all $c > 0$.

Hence, the function has no breaks or jumps in its domain.

Graph of Logarithmic Function

The graph of the logarithmic function $y = \log_a x$ is constructed using the properties discussed in the above subsections. In particular, the domain $x > 0$, the vertical asymptote $x = 0$, the intercept $(1, 0)$, the continuity, the injectivity and the monotonic behaviour depending on the value of a together determine the shape and position of the curve. Using these features, the graph can be drawn accurately for different values of the base a.

- Figure 1.10 shows the graphs of logarithmic functions with bases greater than 1.
- Figure 1.11 represents the graphs of logarithmic functions with bases between 0 and 1.

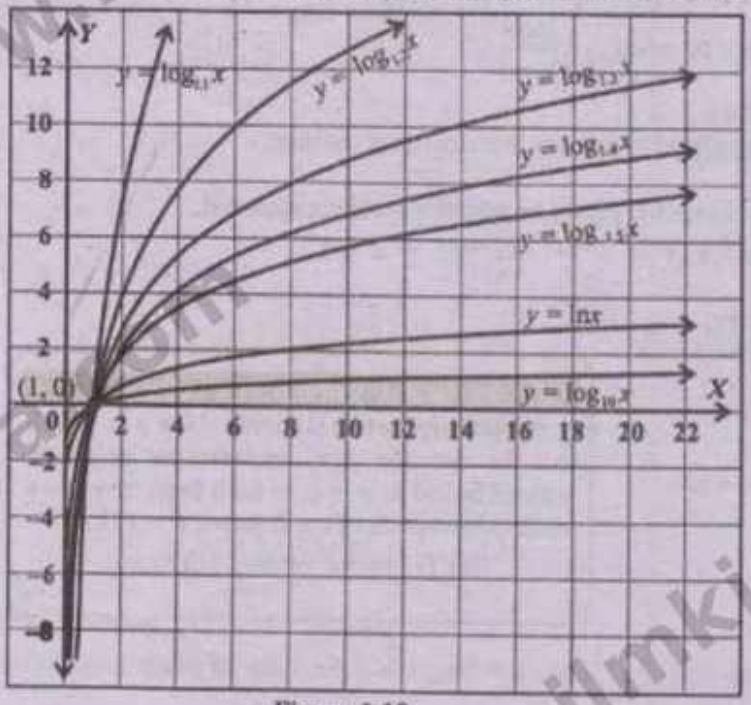


Figure 1.10

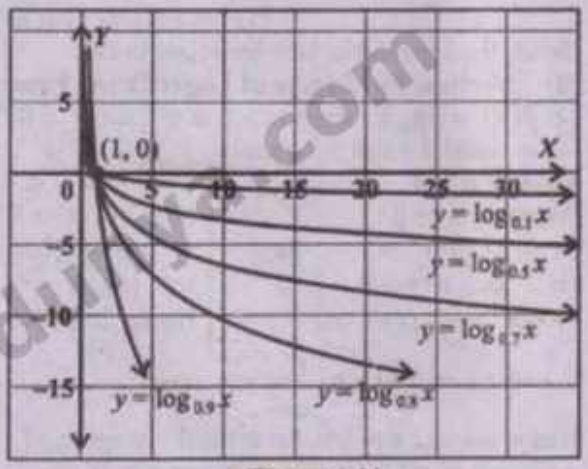


Figure 1.11

www.ilmkidunya.com

www.ilmkidunya.com

MODULUS FUNCTION AND ITS GRAPH

The modulus function or absolute value function is denoted and defined by:

$$f(x)=|x|=\begin{cases} x, & x\geq 0 \\ -x & x<0 \end{cases}$$

For example: If $x = 2 > 0$ then $|x| = |2| = 2$ and if $x = -2 < 0$ then $|x| = |-2| = -(-2) = 2$.
Similarly, if $x = 0$ then $|x| = |0| = 0$. Thus, the modulus of a number is always non-negative.

Graph:

- The graph of the modulus function is V-shaped. It consists of two rays starting from origin.
- For $x \geq 0$, the graph is the line $y = x$, while for $x < 0$, the graph is the line $y = -x$.

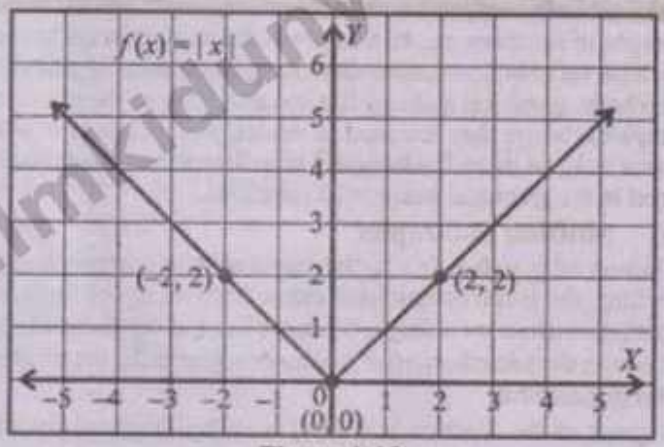


Figure 1.12

Thus, the graph of modulus function can be sketched using the points (0, 0), (2, 2) and (−2, 2) as shown in Figure 1.12.

INTERPRETING THE RELATION BETWEEN A ONE-TO-ONE FUNCTION AND ITS INVERSE THROUGH GRAPH

- A function has an inverse only if it is bijective, that is, both one-to-one and onto. However, if the co-domain of a function is taken equal to its range, then every value in the co-domain is already attained by the function. Therefore, the function automatically becomes onto, and only the one-to-one condition is required for the existence of its inverse.
- The graph of a function and its inverse are related through a geometric symmetry about the line $y = x$. This line is used because any point (x, y) on a function represents an input-output pair, and for its inverse, the roles of input and output are interchanged. The line $y = x$ is exactly the set of all points where input and output values are equal, making it the natural axis of reflection for this interchange.

Method of Constructing the Graph of an Inverse Function

The graph of an inverse function is obtained geometrically as follows:

(i) Draw the graph of the given one-to-one function.
(ii) Consider the line $y = x$ on the same coordinate plane.
(iii) Reflect the graph of the function about the line $y = x$.
(iv) The resulting reflected curve represents the graph of the inverse function.

The reflection ensures that each point on the original graph is mapped to a corresponding point on the inverse graph by maintaining equal perpendicular distance from the line $y = x$, preserving the geometric symmetry between the two graphs.

Example 11: (Logarithmic Graph from Exponential Graph)
Show graphically that the function $y = \ln x$ is the inverse of the function $y = e^x$.

Solution:

Consider $y = e^x$ (one-to-one)
Domain $= (-\infty, \infty)$
Range $= (0, \infty)$

Table of coordinates for $y = e^x$:

x	−1	0	1	1.5
$y = e^x$	0.37	1	2.72	4.48
(x, y)	(−1, 0.37)	(0, 1)	(1, 2.72)	(1.5, 4.48)

Table of reflected coordinates for $y = \ln x$:

x	0.37	1	2.72	4.48
$y = \ln x$	−1	0	1	1.5
(x, y)	(0.37, −1)	(1, 0)	(2.72, 1)	(4.48, 1.5)

As every point, asymptote and boundary swaps places across the line $y = x$, so the graphs are geometric reflections. Hence, they are inverses.

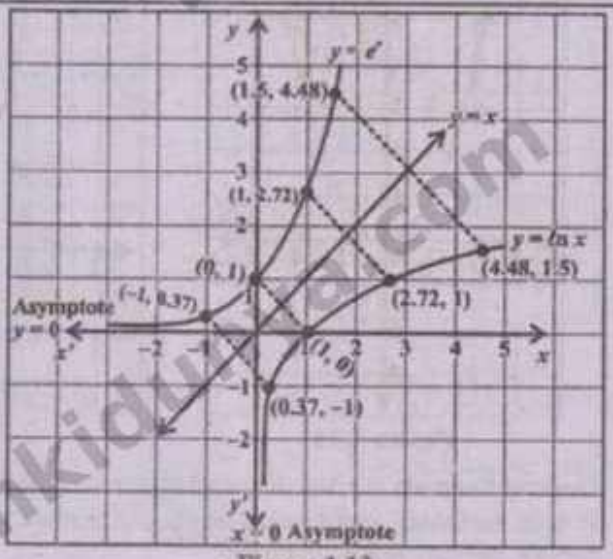


Figure 1.13

www.ilmkidunya.com

www.ilmkidunya.com

TRANSFORMATION OF GRAPHS OF FUNCTIONS

Graphs of functions can be transformed in various ways to obtain new graphs from basic functions. These transformations include reflection, rotation, stretching, compression and shifting, and are applied to represent changes in functions in algebraic, graphical and real-life contexts. One of the most important applications of these transformations is in computer graphics, where they are used to model, manipulate and animate images and shapes. However, in this section, we shall focus only on three fundamental transformations: horizontal shift, vertical shift and scaling, which are most commonly used in the graphical analysis of functions.

(i) Shifting of Graphs

Shifting of graphs refers to the movement of a graph from its original position without changing its shape or size. In shifting, the graph is translated either horizontally or vertically.

In mathematical modeling, shifting is used to represent changes in a function due to variations in input or output values. It allows the same basic function to describe different situations by adjusting its position on the coordinate plane.

Horizontal Shift

A graph of the function $y = f(x)$ is shifted horizontally when the input variable x is changed to $x - h$. The resulting graph is represented by the function:

$$y = f(x - h)$$

- If $h > 0$: shift h units to the right
- If $h < 0$: shift $|h|$ units to the left

Vertical Shift

A graph of the function $y = f(x)$ is shifted vertically when the output variable y is changed to $y - k$. The resulting graph is represented by the function:

$$y = f(x) + k$$

- If $k > 0$: shift k units upward
- If $k < 0$: shift $|k|$ units downward

Challenge!

Draw the graph of the function:

$$y = (x - 2)^2 - 2$$

and observe how it can be obtained from the graph of $y = x^2$

Example:

For better understanding of horizontal and vertical shifts, we consider the function:

$$y = x^2$$

This equation represents a parabola with vertex at $(0, 0)$, as shown in Figure 1.14.

(i) Next, we consider the function $y = (x - 2)^2$

This also represents a parabola with vertex at $(2, 0)$. Its graph can be seen in Figure 1.15.

Note that the graph in Figure 1.15 can be obtained directly from the graph in Figure 1.14 by shifting it 2 units horizontally to the right.

(ii) Now, consider another function $y = x^2 - 2$

This equation also represents a parabola with vertex at $(0, -2)$. Its graph is shown in Figure 1.16.

Note that this graph can be obtained directly from the graph in Figure 1.14 by shifting it 2 units downward.

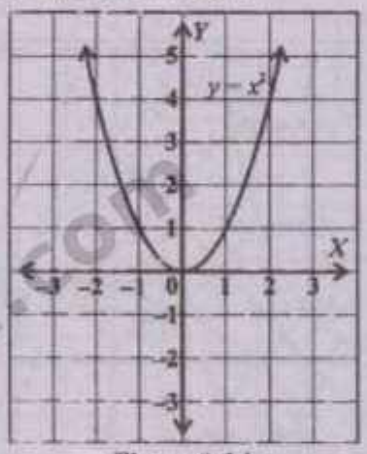


Figure 1.14

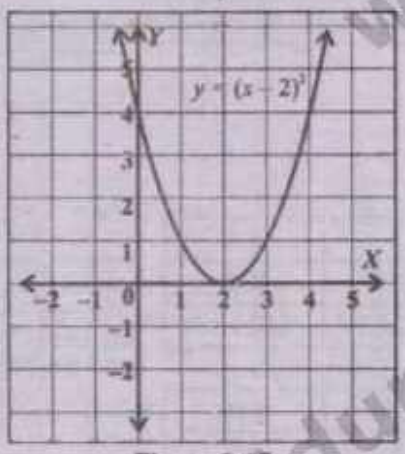


Figure 1.15

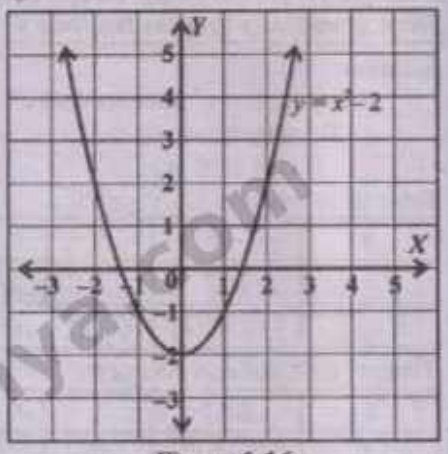


Figure 1.16

General Formula for Vertical and Horizontal Shift

If both horizontal and vertical shifts are applied together to the function $y = f(x)$, then the general form of the transformed function is given by:

$$y = f(x - h) + k$$

www.ilmkidunya.com

www.ilmkidunya.com

(ii) Scaling of Graphs

Scaling of graphs refers to the stretching or compression of a graph without changing its general shape. In scaling, the graph is enlarged or reduced either in the vertical direction or in the horizontal direction.

Scaling helps in understanding how a function changes when its input or output values are multiplied by a positive constant factor.

Vertical Scaling

If a function $y = f(x)$ is multiplied by a positive constant a, then the new function:

$$y = af(x), \quad a > 0$$

represents a **vertical scaling** of the graph of $y = f(x)$.

- If $a > 1$, then the graph is **stretched vertically** by factor a.
- If $0 < a < 1$, then the graph is **compressed vertically** by factor $\frac{1}{a} > 1$.
- Only the y-values of the graph change, while x-values remain unchanged.

Horizontal Scaling

If a function $y = f(x)$ is transformed to:

$$y = f(ax), a > 0$$

then, it represents a **horizontal scaling** of the graph of $y = f(x)$.

- If $a > 1$, then the graph is **compressed horizontally** by a factor a
- If $0 < a < 1$, then the graph is **stretched horizontally** by a factor $\frac{1}{a} > 1$.
- Only the x-values of the graph change, while y-values remain unchanged.

Important Note:

- If $a < 0$, scaling is not purely stretching or compression. It also results in:
  - reflection along the x-axis in $y = af(x)$
  - reflection along the y-axis in $y = f(ax)$

Therefore, in this section we consider only $a > 0$ to focus on scaling alone.

Example:

For better understanding of scaling, we consider the following functions:

$$y = \frac{1}{50}(x-2)^2 \text{ and } y = (3x-2)^2$$

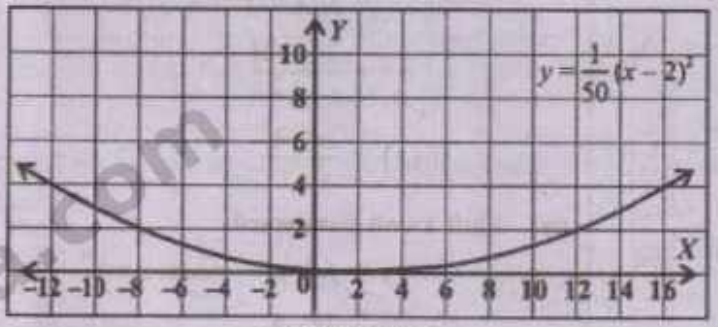


Figure 1.17

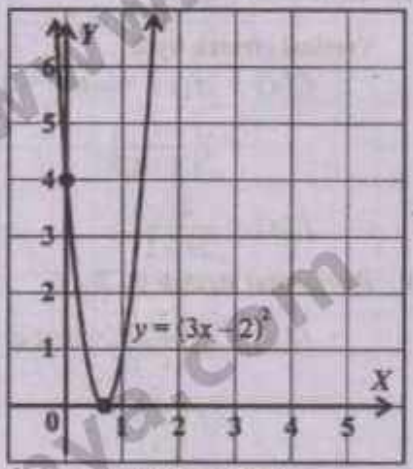


Figure 1.18

The graph of $y = \frac{1}{50}(x-2)^2$ is shown in Figure 1.17 which can be obtained from the graph of $y = (x-2)^2$ (Figure 1.15) using a vertical compression by factor of 50. Similarly, the graph of $y = (3x-2)^2$ can be obtained from the graph of $y = (x-2)^2$ by a horizontal compress, that is, by a factor of 3 (see Figure 1.18).

www.ilmkidunya.com

www.ilmkidunya.com

General Formula for Scaling of Graphs

If a function $y = f(x)$ is transformed to

$$y = af(bx), a > 0, b > 0$$

then it represents both vertical and horizontal scaling of the graph of $y = f(x)$.

- If $a > 1$, the graph is stretched vertically by factor a.
- If $0 < a < 1$, the graph is compressed vertically by factor $\frac{1}{a}$.

On the other hand,

- If $b > 1$, the graph is compressed horizontally by factor b.
- If $0 < b < 1$ the graph is stretched horizontally by factor $\frac{1}{b}$.

Example 12: Let $f(x) = \ell n(x^2+1)$. Find the function obtained after:

(i) a vertical stretch by factor 2 **(ii) then a shift 1 unit to the left.**

Solution:

Given function: $f(x) = \ell n(x^2+1)$

(i) Vertical stretch by factor 2

$f_1(x) = cf(x)$, where $c = 2$

$f_1(x) = 2\ell n(x^2+1)$

(ii) Shift 1 unit to the left

$f_2(x) = f_1(x+c)$ where $c = 1$

$= f_1(x+1)$

$= 2\ell n((x+1)^2+1)$

$= 2\ell n(x^2+2x+1+1)$

$f_2(x) = 2\ell n(x^2+2x+2)$ (Transformed function)

Example 13: Let $f(x) = \frac{x}{x+1}$. Find the function obtained after applying all the following transformations with the given order:

(i) vertical stretch by 2 **(ii) horizontal stretch by 3** **(iii) shift 2 units to the right** **(iv) shift 1 unit downward**

Solution:

Given function: $f(x) = \frac{x}{x+1}$.

(i) Vertical stretch by 2

$f_1(x) = cf(x)$, where $c = 2$

$= 2\left(\frac{x}{x+1}\right)$

$f_1(x) = \frac{2x}{2x+1}$

(ii) Horizontal stretch by 3

$f_2(x) = f_1\left(\frac{1}{c}x\right)$, where $c = 3$

$= f_1\left(\frac{x}{3}\right)$

$= \frac{2\left(\frac{x}{3}\right)}{\frac{x}{3}+1} = \frac{\frac{2x}{3}}{\frac{x+3}{3}}$

$f_2(x) = \frac{2x}{x+3}$

(iii) Shift 2 units to the right

$f_3(x) = f_2(x-c)$, where $c = 2$

$= f_2(x-2)$

$= \frac{2(x-2)}{x-2+3}$

$f_3(x) = \frac{2x-4}{x+1}$

(iv) Shift 1 unit downward

$y = f_3(x) - c$, where $c = 1$

$= \frac{2x-4}{x+1} - 1$

$= \frac{2x-4-x-1}{x+1}$

$y = \frac{x-5}{x+1}$ (Transformed function)

www.ilmkidunya.com

www.ilmkidunya.com

EXERCISE 1.3

Q1. Let $f(x) = \log_4(x-2)$

(i) Show that $f(x)$ is a one-to-one function using the horizontal line test.

Solution:

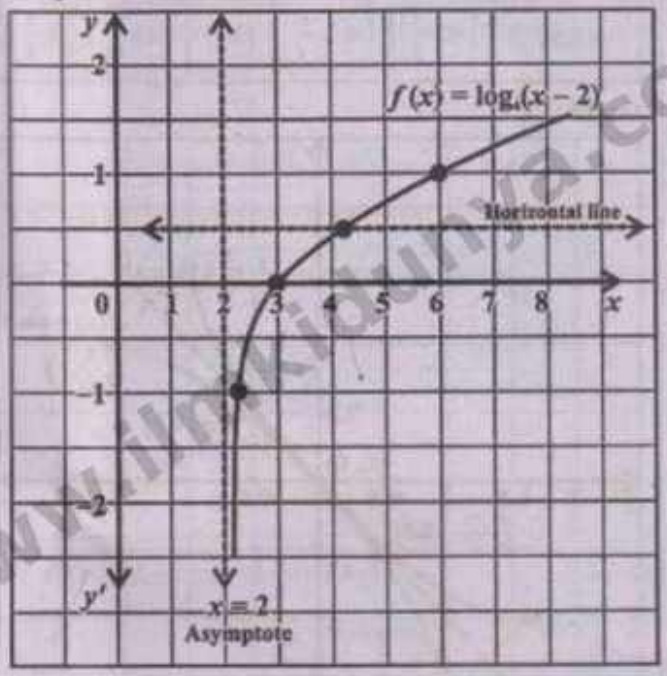
$f(x) = \log_4(x-2)$

Here base $= 4 > 1 \Rightarrow f$ is strictly increasing

Table of coordinates for $f(x) = \log_4(x-2)$:

x	2.25	3	6	$\to \infty$
$y = \log_4(x-2)$	-1	0	1	$\to \infty$
(x, y)	$(2.25, -1)$	$(3, 0)$	$(6, 1)$	

Graph:



From graph, it is clear that any horizontal line crosses the curve of f only once. Which indicates that f is one-to-one.

(ii) Find the inverse function $f^{-1}(x)$ algebraically.

Solution:

$$f(x) = \log_4(x-2)$$
$$\Rightarrow \quad y = \log_4(x-2) \qquad \because f(x) = y$$

Interchanging x and y

$$x = \log_4(y-2)$$

In exponential form

$$4^x = y - 2$$
$$y = 4^x + 2$$
$$\Rightarrow \quad f^{-1}(x) = 4^x + 2$$

(iii) State the domain and range of $f(x)$ and $f^{-1}(x)$.

Solution:

$$f(x) = \log_4(x-2)$$

For Domain: $x - 2 > 0$

$$\Rightarrow \quad x > 2$$
$$D_f = (2, \infty)$$

Range: $R_f = (-\infty, \infty)$

By definition of inverse function

$$D_{f^{-1}} = (-\infty, \infty)$$
$$R_{f^{-1}} = (2, \infty)$$

(iv) Verify graphically that the graphs of $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = x$.

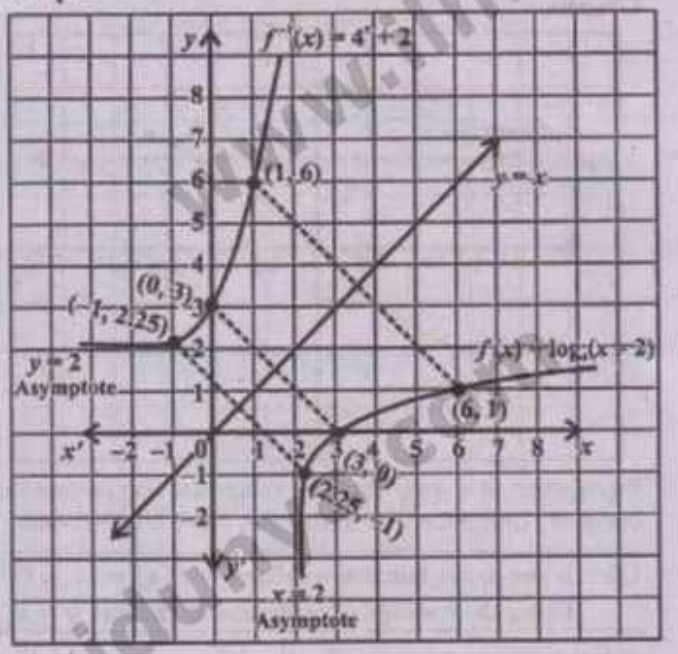
Solution:

$f(x) = \log_4(x-2)$; $f^{-1}(x) = 4^x + 2$

Table of reflected coordinates for $f^{-1}(x) = 4^x + 2$:

x	-1	0	1
$y = 4^x + 2$	2.25	3	6
(x, y)	$(-1, 2.25)$	$(0, 3)$	$(1, 6)$

Graph:



The graphs are perfectly symmetric about $y = x$ because reflecting any point of $f(x)$ across this line gives a corresponding mirror point on $f^{-1}(x)$.

www.ilmkidunya.com

www.ilmkidunya.com

Q2. Let $f(x)=\sin x$ for all $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

(i) State the domain and range of $f(x)$ and $f^{-1}(x)$.

Solution:

$$f(x)=\sin x \text{ for all } x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$D_f=\left[-\frac{\pi}{2},\frac{\pi}{2}\right];\ R_f=[-1,1]$$

By definition of inverse function

$$D_{f^{-1}}=[-1,1];\ R_{f^{-1}}=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

(ii) Sketch the graph of $f(x)$ and show that it is one-to-one using the horizontal line test.

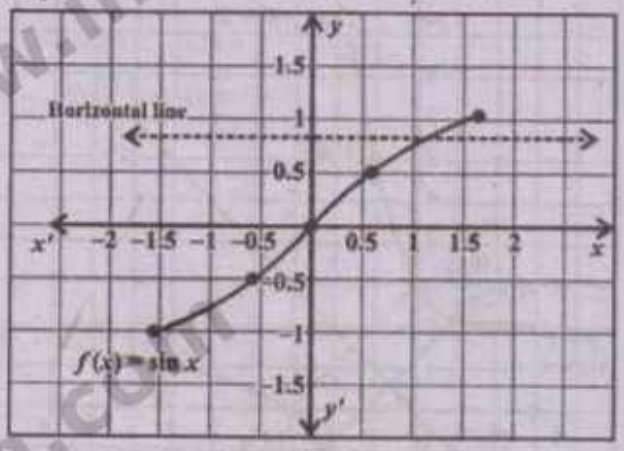
Solution:

$$f(x)=\sin x \text{ for all } x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

Table of coordinates for $f(x)=\sin x$

x	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
	−1.57	−0.52	0	0.52	1.57
$y=\sin x$	−1	−0.5	0	0.5	1
(x,y)	$\left(-\frac{\pi}{2},-1\right)$	$\left(-\frac{\pi}{6},-0.5\right)$	(0,0)	$\left(\frac{\pi}{6},0.5\right)$	$\left(\frac{\pi}{2},1\right)$

Graph:



From graph, it is clear that any horizontal line crosses the curve of f only once which indicates that f is one-to-one.

(iii) Draw the graph of $f^{-1}(x)$ and verify that it is the reflection of $f(x)$ in the line $y=x$.

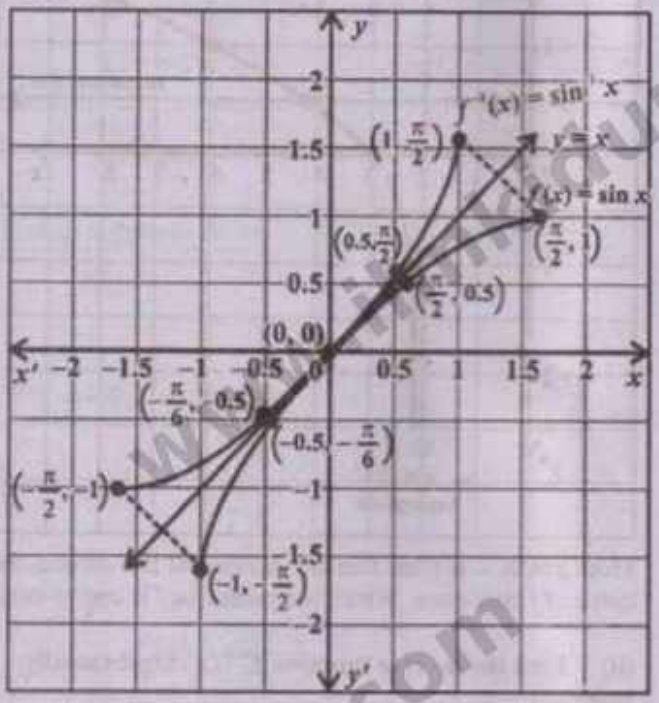
Solution:

$$f(x)=\sin x;\ f^{-1}(x)=\sin^{-1}x$$

Table of reflected coordinates for $f^{-1}(x)=\sin^{-1}x$:

x	−1	−0.5	0	0.5	1
$y=\sin^{-1}x$	−1.57	0.52	0	0.52	1.57
	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
(x,y)	$\left(-1,-\frac{\pi}{2}\right)$	$\left(-0.5,-\frac{\pi}{6}\right)$	(0,0)	$\left(0.5,\frac{\pi}{6}\right)$	$\left(1,\frac{\pi}{2}\right)$

Graph:



The graphs are perfectly symmetric about $y=x$ because reflecting any point of $f(x)$ across this line gives a corresponding mirror point on $f^{-1}(x)$.

Q3. A one-to-one function is defined by $f(x)=(x-1)^3, x\in[0,3]$
Using the concept of reflection in the line $y=x$, sketch the graph of the inverse function $f^{-1}(x)$.

Solution:

Given function: $f(x)=(x-1)^3, x\in[0,3]$

Since f is one-to-one, therefore f^{-1} exists.

www.ilmkidunya.com

www.ilmkidunya.com

Table of coordinates for $f(x)=(x-1)^3$:

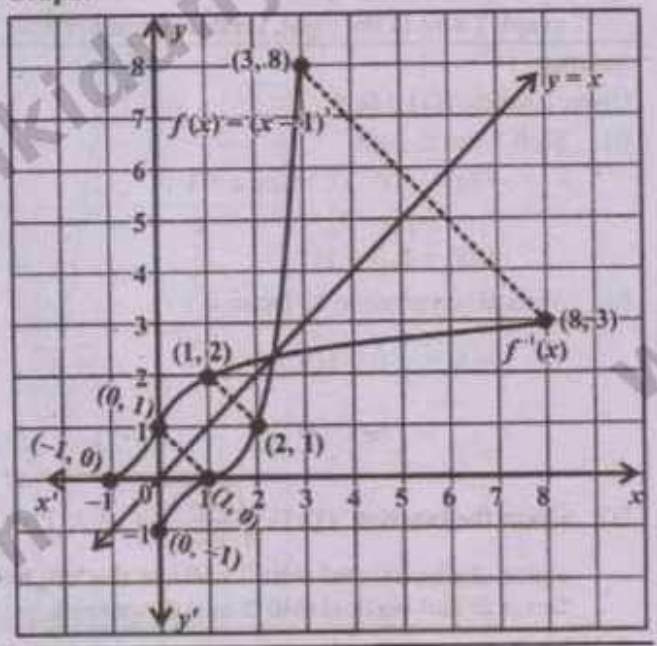
x	0	1	2	3
$y=(x-1)^3$	-1	0	1	8
(x,y)	$(0,-1)$	$(1,0)$	$(2,1)$	$(3,8)$

Table of reflected coordinates for $f^{-1}(x)$:

x	-1	0	1	8
$y=f^{-1}(x)$	0	1	2	3
(x,y)	$(-1,0)$	$(0,1)$	$(1,2)$	$(8,3)$

The graphs are perfectly symmetric about $y = x$ because reflecting any point of $f(x)$ across this line gives a corresponding mirror point on $f^{-1}(x)$.

Graph:



Q4. Given the function $f(x)=\sqrt{x}$. Find the equation of the function after:

(i) vertical stretch by factor 2

(ii) horizontal shift 3 units to the right

(iii) applying both transformations (i) and (ii).

Solution:

Given function: $f(x)=\sqrt{x}$

(i) Vertical stretch by factor 2

$f_1(x) = cf(x)$, where $c = 2$

$= 2\sqrt{x}$

(ii) Horizontal shift 3 units to right

$f_2(x) = f(x-c)$, where $c = 3$

$= f(x-3) = \sqrt{x-3}$

(iii) Apply both transformations (i) and (ii)

$y = 2f(x-3)$

$y = 2\sqrt{x-3}$ (Transformed function)

Q5. Given the function $f(x)=\frac{1}{x}, x\neq 0$. Find the equation of the transformed function after applying the following transformations in order.

Horizontal shift 2 units to the left, horizontal compression by factor 2, vertical stretching by factor 3, and vertical shift 1 unit upward.

Solution:

Given function: $f(x)=\frac{1}{x}, x\neq 0$

(i) Horizontal shift 2 unit to left

$f_1(x) = f(x+c)$, where $c = 2$

$= f(x+2)$

$f_1(x) = \frac{1}{x+2}$

(ii) Horizontal compression by factor 2

$f_2(x) = f_1(cx)$, where $c = 2$

$= f_1(2x)$

$f_2(x) = \frac{1}{2x+2}$

(iii) Vertical stretch by factor 3

$f_3(x) = cf_2(x)$, where $c = 3$

$f_3(x) = \frac{3}{2x+2}$

(iv) Vertical shift 1 unit upward

$y = f_3(x) + c$, where $c = 1$

$= \frac{3}{2x+2} + 1$

$= \frac{3+2x+2}{2x+2}$

$y = \frac{2x+5}{2x+2}$ (Transformed function)

www.ilmkidunya.com

www.ilmkidunya.com

Q6. Given the function $f(x) = \ln x$, find the equation of the transformed function obtained after shifting the graph 1 unit to the right, vertically compressing it by a factor of 2, and then shifting it 3 units upward.

Solution:

Given function: $f(x) = \ln x$

(i) Shift 1 unit to right

$$f_1(x) = f(x-c), \text{ where } c = 1$$
$$= f(x-1)$$
$$f_1(x) = \ln(x-1)$$

(ii) Vertical compression by factor 2

$$f_2(x) = \frac{1}{c} f_1(x), \text{ where } c = 2$$
$$= \frac{1}{2}\ln(x-1)$$

(iii) Shift 3 units upward

$$y = f_2(x) + c, \text{ where } c = 3$$
$$= \frac{1}{2}\ln(x-1) + 3$$
$$= \ln(x-1)^{\frac{1}{2}} + 3$$
$$y = \ln\sqrt{x-1} + 3 \quad \text{(Transformed function)}$$

Q7. Given the function $f(x) = \sqrt{\frac{2x-1}{x+3}}, x \neq -3$. Find the equation of the transformed function after applying in order, the horizontal shift 2 units to the left, horizontal compression by factor 3, vertical compression by factor 2, and vertical shift 1 unit downward.

Solution:

Given function: $f(x) = \sqrt{\frac{2x-1}{x+3}}, x \neq -3$

(i) Horizontal shift 2 units to left

$$f_1(x) = f(x+c), \text{ where } c = 2$$
$$= f(x+2)$$
$$= \sqrt{\frac{2(x+2)-1}{x+2+3}}$$
$$f_1(x) = \sqrt{\frac{2x+3}{x+5}}$$

(ii) Horizontal compression by factor 3

$$f_2(x) = f_1(cx), \text{ where } c = 3$$
$$= f_1(3x)$$
$$= \sqrt{\frac{2(3x)+3}{3x+5}}$$
$$f_2(x) = \sqrt{\frac{6x+3}{3x+5}}$$

(iii) Vertical compression by factor 2

$$f_3(x) = \frac{1}{c} f_2(x), \text{ where } c = 2$$
$$= \frac{1}{2}\sqrt{\frac{6x+3}{3x+5}}$$

(iv) Vertical shift 1 unit downward

$$y = f_3(x) - c, \text{ where } c = 1$$
$$y = \frac{1}{2}\sqrt{\frac{6x+3}{3x+5}} - 1 \quad \text{(Transformed function)}$$

Q8. Given the function $f(x) = 3^x$ and its transformed function $g(x) = 3^{x-2} + 1$, draw the graphs of both functions on the same coordinate plane and compare their shapes.

Solution:

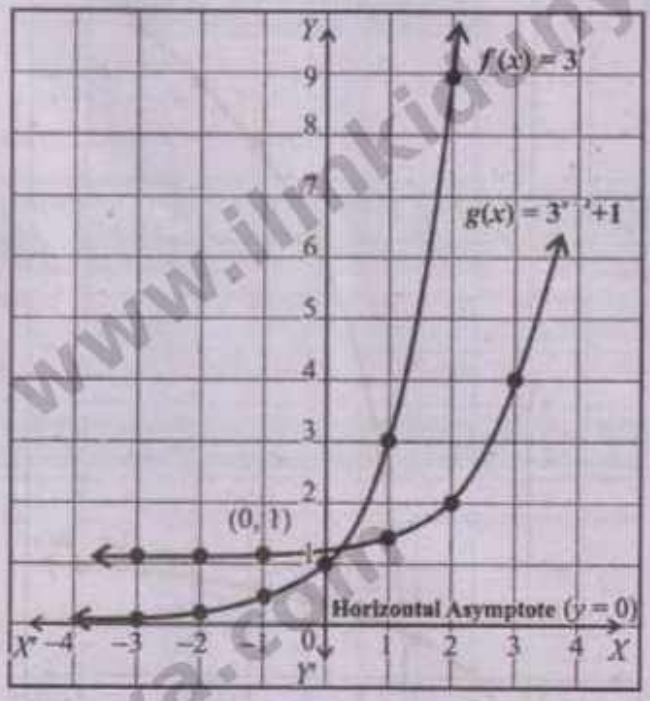
Table of values for $f(x) = 3^x$ and $g(x) = 3^{x-2} + 1$

x	$\to -\infty$	-3	-2	-1	0	1	2	3	$\to \infty$
$f(x) = 3^x$	$\to 0$	0.04	0.11	0.33	1	3	9	27	$\to \infty$
$g(x) = 3^{x-2} + 1$	$\to 1$	1.004	1.01	1.04	1.11	1.33	2	4	$\to \infty$

www.ilmkidunya.com

www.ilmkidunya.com

Graph:



Comparison: From the figure, it is clear that the graph of $g(x)$ is obtain from $f(x)$ by shifting it 2 units to the right and 1 unit upward. Therefore, both the graphs are identical in shape but differ in their position on the coordinate plane.

Q9. Given the function $f(x) = \ln x$. Find the equation of the transformed function after applying in order, the vertical stretch by factor 4, shift 6 units to right, horizontal compress by factor 3, and shift 1 unit upward. Draw the graphs of both the original and transformed functions on the same coordinate plane.

Solution:

Given function: $f(x) = \ln x$

(i) Vertical stretch by factor 4

$f_1(x) = cf(x)$, where $c = 4$

$= 4\ln x$

(ii) Shift 6 units to right

$f_2(x) = f_1(x - c)$, where $c = 6$

$= f_1(x - 6)$

$f_2(x) = 4\ln(x - 6)$

(iii) Horizontal compress by factor 3

$f_3(x) = f_2(cx)$, where $c = 3$

$= f_2(3x)$

$f_3(x) = 4\ln(3x - 6)$

(iv) Vertical shift 1 unit upward

$f_4(x) = f_3(x) + c$, where $c = 1$

$f_4(x) = 4\ln(3x - 6) + 1$

Let $g(x) = 4\ln(3x - 6) + 1$ (Transformed function)

Table of values for $f(x) = \ln x$, $x > 0$

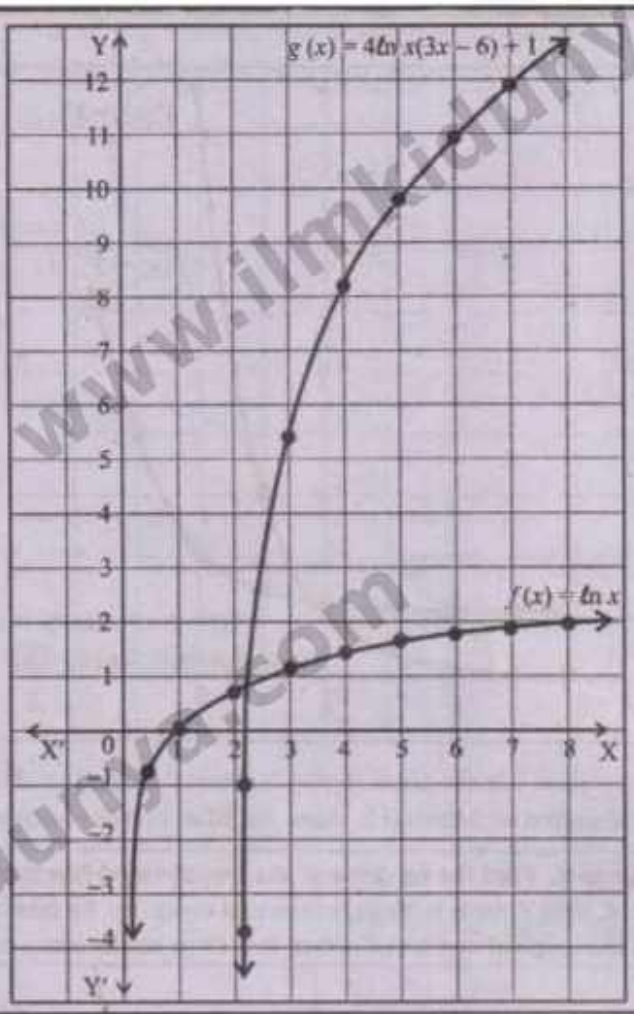
x	$\to 0$	0.5	1	2	3	4	5	6	7	8	$\to \infty$
$f(x) = \ln x$	$\to -\infty$	−0.69	0	0.69	1.1	1.39	1.61	1.79	1.95	2.08	$\to \infty$

Table of values for $g(x) = 4\ln(3x - 6) + 1$, $x > 2$

x	$\to 2$	2.1	2.2	3	4	5	6	7	$\to \infty$
$g(x) = 4\ln(3x - 6) + 1$	$\to -\infty$	−3.82	−1.04	5.39	8.17	9.79	10.94	11.83	$\to \infty$

www.ilmkidunya.com

www.ilmkidunya.com



Q10. Draw the graph of $f(x) = |x|$ after applying in order, shift of 7 units to right, horizontal compress by factor 2, vertical shift 6 units downward, and vertical compress by factor 2.

Solution:

Given function: $f(x) = |x|$

(i) Shift of 7 units to right

$f_1(x) = f(x - c)$, where $c = 7$

$= f(x - 7)$

$= |x - 7|$

(ii) Horizontal compress by factor 2

$f_2(x) = f_1(cx)$, where $c = 2$

$= f_1(2x)$

$f_2(x) = |2x - 7|$

(iii) Vertical shift 6 units downward

$f_3(x) = f_2(x) - c$, where $c = 6$

$f_3(x) = |2x - 7| - 6$

(iv) Vertical compress by factor 2

$g(x) = \frac{1}{c} f_3(x)$, where $c = 2$

$= \frac{1}{2}(|2x - 7| - 6)$

$g(x) = \frac{1}{2}|2x - 7| - 3$

(Transformed function)

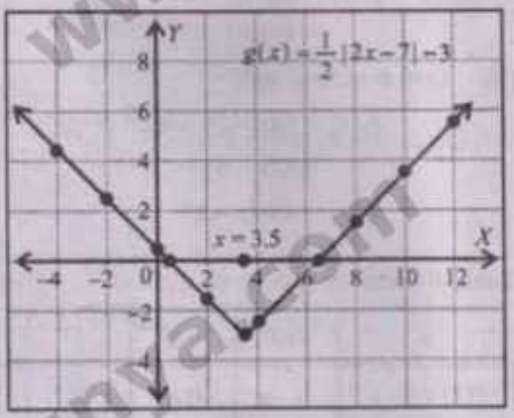


Table of values for $g(x) = \frac{1}{2}|2x - 7| - 3$

x	$\to -\infty$	−4	−2	0	2	4	6	8	10	12	$\to \infty$
$g(x) = \frac{1}{2}\|2x - 7\| - 3$	$\to \infty$	4.5	2.5	0.5	−1.5	−2.5	−0.5	1.5	3.5	5.5	$\to \infty$

www.ilmkidunya.com

www.ilmkidunya.com

Content Outline

- Solving Problems Involving Exponential and Logarithmic Equations and Inequalities
- Examples of Exponential and Logarithmic Equations
- Examples of Exponential and Logarithmic Inequalities

SOLVING PROBLEMS INVOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS AND INEQUALITIES

In this section, we solve exponential and logarithmic equations and inequalities in which the variable appears in the exponent or inside a logarithmic expression. Before solving, the domain must be considered wherever the expression is not defined for all real values of the variable. The methods of solving equations and inequalities are illustrated in the following examples:

EXAMPLES OF EXPONENTIAL AND LOGARITHMIC EQUATIONS

Example 14: Solve the equation: $4^{x-1} = 8$.

Solution:

$$4^{x-1} = 8$$

Domain: Since the expression 4^{x-1} is defined for every real number, so

$$\text{Domain} =]-\infty, \infty[$$

Now,
$$4^{x-1} = 8$$
$$(2^2)^{x-1} = 2^3$$
$$2^{2x-2} = 2^3$$

Equating the exponents, we get

$$2x - 2 = 3$$
$$2x = 3 + 2$$
$$x = \frac{5}{2} \in]-\infty, \infty[$$

Solution set $= \left\{\frac{5}{2}\right\}$

Example 15: Find the solution of $2^{2x} - 3^{5x+3} = 0$.

Solution:

$$2^{2x} - 3^{5x+3} = 0$$

Domain: Since 2^{2x} and 3^{5x+3} are defined for every real number, so

$$\text{Domain} =]-\infty, \infty[$$

Now,
$$2^{2x} - 3^{5x+3} = 0$$
$$2^{2x} = 3^{5x+3}$$

Taking $\ln$ of both sides

$$\ln(2^{2x}) = \ln(3^{5x+3})$$
$$2x \cdot \ln 2 = (5x+3) \cdot \ln 3 \quad \because \ln(a^b) = b \ln a$$
$$2x \cdot \ln 2 = 5x \cdot \ln 3 + 3 \ln 3$$
$$2x \ln 2 - 5x \ln 3 = 3 \ln 3$$
$$x(2 \ln 2 - 5 \ln 3) = 3 \ln 3$$
$$x(\ln 2^2 - \ln 3^5) = \ln 3^3$$
$$x(\ln 4 - \ln 243) = \ln 27 \quad \because \ln(a^b) = b \ln a$$
$$x \cdot \ln\left(\frac{4}{243}\right) = \ln 27 \quad \because \ln a - \ln b = \ln\frac{a}{b}$$
$$x = \frac{\ln 27}{\ln\left(\frac{4}{243}\right)}$$
$$x = \log_{\frac{4}{243}}(27) = -0.8025 \in]-\infty, \infty[$$

Solution set $= \left\{\log_{\frac{4}{243}}(27)\right\}$

Example 16: Solve the equation: $\log x + \log(x-3) = 1$

Solution:

$$\log x + \log(x-3) = 1$$

Domain:

$\log x$ is defined if $x > 0$
$\Rightarrow \quad]0, \infty[$

$\log(x+3)$ is defined if $x - 3 > 0 \Rightarrow x > 3$
$\Rightarrow \quad]3, \infty[$

Domain $=]0, \infty[\cap]3, \infty[=]3, \infty[$

Now, $\log x + \log(x-3) = 1$
$\log x(x-3) = 1 \quad \because \log a + \log b = \log(a \cdot b)$
$\Rightarrow \log_{10}(x^2 - 3x) = 1 \quad \because \log = \log_{10}$

In exponential form:

$$x^2 - 3x = 10^1$$
$$x^2 - 3x - 10 = 0$$

www.ilmkidunya.com

www.ilmkidunya.com

$x^2 - 5x + 2x - 10 = 0$
$x(x-5) + 2(x-5) = 0$
$(x-5)(x+2) = 0$

Either $x - 5 = 0$ or $x + 2 = 0$
$x = 5$ or $x = -2 \notin]3, \infty[$
Solution set = {5}

Example 17: Solve the equation: $\log_x(x^2 - 3x + 2) = 2$.

Solution:

$\log_x(x^2 - 3x + 2) = 2$

Domain:

- Base condition: $x > 0, x \neq 1$
- Argument condition: $x^2 - 3x + 2 > 0$

Now, $\log_x(x^2 - 3x + 2) = 2$

In exponential form:

$x^2 - 3x + 2 = x^2$
$x^2 - 3x + 2 - x^2 = 0$
$\Rightarrow \quad -3x + 2 = 0$
$3x = 2 \quad \Rightarrow \quad x = \frac{2}{3}$

Check Domain:

- Base condition: $x = \frac{2}{3} > 0$ and $\neq 1$
- Argument condition:

$$x^2 - 3x + 2 = \left(\frac{2}{3}\right)^2 - 3\left(\frac{2}{3}\right) + 2$$
$$= \frac{4}{9} - 2 + 2 = \frac{4}{9} > 0$$

$\Rightarrow$ Both the conditions are satisfied.

Solution set = $\left\{\frac{2}{3}\right\}$

EXAMPLES OF EXPONENTIAL AND LOGARITHMIC INEQUALITIES

Example 18: Solve the inequality: $2^{x+1} > 5^x$.

Solution:

$2^{x+1} > 5^x$

Domain: Since 2^{x+1} and 5^x are defined for all real number, so

Domain = $]-\infty, \infty[$

Now, $2^{x+1} > 5^x$

$2^x \cdot 2^1 > 5^x$

$2 > \frac{5^x}{2^x}$

$2 > \left(\frac{5}{2}\right)^x$

Taking $\ln$ on both sides:

$$\ln(2) > \ln\left(\frac{5}{2}\right)^x \quad \because 2 > 0, \left(\frac{5}{2}\right)^x > 0 \ \forall x \in R$$

$$\ln(2) > x \cdot \ln\left(\frac{5}{2}\right) \quad \because \ln(a^b) = b \cdot \ln a$$

$$\Rightarrow \quad x < \frac{\ln 2}{\ln \frac{5}{2}} = \log_{\frac{5}{2}}(2) \approx 0.7565$$

Solution set = $]-\infty, \log_{\frac{5}{2}}(2)[$

Example 19: Solve the inequality: $\log(2x-1) \geq \log(5-3x)$

Solution:

$\log(2x-1) \geq \log(5-3x)$

Domain:

$\log(2x-1)$ is defined if $2x - 1 > 0$
$2x > 1$
$x > \frac{1}{2}$

$\log(5-3x)$ is defined if $5 - 3x > 0$
$5 > 3x$
$x < \frac{5}{3}$

Domain = $\left]\frac{1}{2}, \infty\right[\cap]-\infty, \frac{5}{3}[$
$= \left]\frac{1}{2}, \frac{5}{3}\right[=]0.5, 1.67[$

Now, $\log(2x-1) \geq \log(5-3x)$
$\Rightarrow \quad 2x - 1 \geq 5 - 3x \quad \because \log \text{ base: } 10 > 1$
$2x + 3x \geq 5 + 1$
$5x \geq 6$
$x \geq \frac{6}{5} = \left[\frac{6}{5}, \infty\right[$

Intersection of $\left[\frac{6}{5}, \infty\right[$ and domain $\left]\frac{1}{2}, \frac{5}{3}\right[$, gives

Solution set = $\left[\frac{6}{5}, \frac{5}{3}\right[$

www.ilmkidunya.com

www.ilmkidunya.com

Example 20: Solve the inequality: $\log_{\frac{7}{9}}(x^2-3x+2) \geq \log_{\frac{7}{9}}(2x-1)$

Solution:

$$\log_{\frac{7}{9}}(x^2-3x+2) \geq \log_{\frac{7}{9}}(2x-1)$$

Domain:

$\log_{\frac{7}{9}}(2x-1)$ is defined if

$2x-1>0$

$2x>1$

$x>\frac{1}{2}$

$\log_{\frac{7}{9}}(x^2-3x+2)$ is defined if

$x^2-3x+2>0$

$(x-1)(x-2)>0$

$x<1$ or $x>2$

Domain $= \left]\frac{1}{2},\infty\right[\cap \{]-\infty,1[\text{ or }]2,\infty[\}$

$= \left]\frac{1}{2},1\right[\cup]2,\infty[$

Now, $\log_{\frac{7}{9}}(x^2-3x+2) \geq \log_{\frac{7}{9}}(2x-1)$

$\Rightarrow \quad x^2-3x+2 \leq 2x-1 \qquad \because \text{log base: } 0<\frac{7}{9}<1$

$x^2-3x+2-2x+1 \leq 0$

$x^2-5x+3 \leq 0$

Associated equation is

$x^2-5x+3=0$

Here $a=1, b=-5, c=3$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{5\pm\sqrt{(-5)^2-4(1)(3)}}{2(1)}$$

$$=\frac{5\pm\sqrt{25-12}}{2}$$

$$x=\frac{5\pm\sqrt{13}}{2}$$

$x=\frac{5-\sqrt{13}}{2}$ | $x=\frac{5+\sqrt{13}}{2}$

$x-\frac{5-\sqrt{13}}{2}=0$ | $x-\frac{5+\sqrt{13}}{2}=0$

$$\Rightarrow \left(x-\frac{5-\sqrt{13}}{2}\right)\left(x-\frac{5+\sqrt{13}}{2}\right) \leq 0$$

$$\Rightarrow \quad \frac{5-\sqrt{13}}{2} \leq x \leq \frac{5+\sqrt{13}}{2} = \left[\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}\right]$$

Intersection of $\left[\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}\right]$ and domain $\left]\frac{1}{2},1\right[\cup]2,\infty[$, gives

Solution set $= \left[\frac{5-\sqrt{13}}{2},1\right[\cup \left]2,\frac{5+\sqrt{13}}{2}\right]$

EXERCISE 1.4

Q1. Solve the following equations:

(i) $\log(3x-2) = \log(x^2-5x+6)$

Solution:

Domain:

$\log(3x-2)$ is defined if

$3x-2>0$

$3x>2$

$x>\frac{2}{3}$

$\log(5-3x)$ is defined if

$x^2-5x+6>0$

$(x-2)(x-3)>0$

$\Rightarrow \quad x<2$ or $x>3$

Domain $= \left]\frac{2}{3},\infty\right[\cap \{]-\infty,2[\text{ or }]3,\infty[\}$

$= \left]\frac{2}{3},2\right[\cup]3,\infty[$

Now, $\log(3x-2) = \log(x^2-5x+6)$

$\Rightarrow \quad 3x-2 = x^2-5x+6$

$x^2-5x+6-3x+2=0$

$x^2-8x+8=0$

Here $a=1, b=-8, c=8$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{8\pm\sqrt{(-8)^2-4(1)(8)}}{2(1)}$$

$$=\frac{8\pm\sqrt{64-32}}{2} = \frac{8\pm\sqrt{32}}{2}$$

$$=\frac{8\pm4\sqrt{2}}{2} = \frac{2(4\pm2\sqrt{2})}{2}$$

$x=4\pm2\sqrt{2}$

$\Rightarrow \quad x=4+2\sqrt{2}$ | $x=4-2\sqrt{2}$

$=6.83 \in]3,\infty[$ | $=1.17 \in \left]\frac{2}{3},2\right[$

Solution set $= (4\pm2\sqrt{2})$

www.ilmkidunya.com

www.ilmkidunya.com

(ii) $\ln(x^2-1)=\ln(3x-2)$

Solution:

Domain:

$\ln(3x-2)$ is defined if $3x-2>0$, $3x>2$, $x>\frac{2}{3}$

$\ln(x^2-1)$ is defined if $x^2-1>0$, $(x+1)(x-1)>0$, $(x-(-1))(x-1)>0$, $\Rightarrow x<-1$ or $x>1$

$$\text{Domain} = \left]\frac{2}{3},\infty\right[\cap \{]-\infty,-1[\text{ or }]1,\infty[\}$$

$$=]1,\infty[$$

Now, $\ln(x^2-1)=\ln(3x-2)$

$\Rightarrow x^2-1=3x-2$

$x^2-1-3x+2=0$

$x^2-3x+1=0$

Here $a=1, b=-3, c=1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{3\pm\sqrt{(-3)^2-4(1)(1)}}{2(1)}$$

$$=\frac{3\pm\sqrt{9-4}}{2}$$

$$x=\frac{3\pm\sqrt{5}}{2}$$

$x=\frac{3+\sqrt{5}}{2}$, $\approx 2.62 \in]1,\infty[$

$x=\frac{3-\sqrt{5}}{2}$, $\approx 0.38 \notin]1,\infty[$

Solution set $=\left\{\frac{3+\sqrt{5}}{2}\right\}$

(iii) $\log(x)+\log(x^2-5x+7)=\log(3)$

Solution:

Domain:

$\log x$ is defined if $x>0$

$\log(x^2-5x+7)$ is defined if $x^2-5x+7>0\ \forall x\in R$

Domain $=]0,\infty[\cap]-\infty,\infty[=]0,\infty[$

Now, $\log x+\log(x^2-5x+7)=\log 3$

$\log x(x^2-5x+7)=\log 3 \quad \because \log a+\log b=\log(ab)$

$\Rightarrow x^3-5x^2+7x=3$

$x^3-5x^2+7x-3=0$

At $x=1$:

$(1)^3-5(1)^2+7(1)-3=1-5+7-3=0$

$\Rightarrow (x-1)$ is a factor.

By Synthetic Division

1	1	−5	7	−3
	↓	1	−4	3
	1	−4	3	0

Quotient $= x^2-4x+3$

$\Rightarrow (x-1)(x^2-4x+3)=0$

$(x-1)(x^2-3x-x+3)=0$

$(x-1)(x(x-3)-1(x-3))=0$

$(x-1)(x-1)(x-3)=0$

$\Rightarrow x-1=0$, $x=1\in]0,\infty[$ | $x-3=0$, $x=3\in]0,\infty[$

Solution set $=\{1,3\}$

(iv) $3^{(x+1)}-2\cdot 9^x+9=0$

Solution:

Domain: Since 3^{x+1} and 9^x are defined for every real number, so

$$\text{Domain} =]-\infty,\infty[$$

Now, $3^{x+1}-2\cdot 9^x+9=0$

$3^x\cdot 3-2(3^x)^2+9=0$

Put $3^x=y$, where $y>0$ ...(1)

$3y-2y^2+9=0$

$\Rightarrow 2y^2-3y-9=0$

$2y^2-6y+3y-9=0$

$2y(y-3)+3(y-3)=0$

$(y-3)(2y+3)=0$

$\Rightarrow y-3=0$, $y=3$ | or $2y+3=0$, $y=\frac{-3}{2}$ (Not possible)

Put in eq. (1)

$3^x=3^1$

$\Rightarrow x=1\in]-\infty,\infty[$

Solution set $=\{1\}$

(v) $x\log_{\frac{1}{e}}4=\log_{\frac{1}{e}}(16^x-3)$

Solution:

$$x\log_{\frac{1}{e}}4=\log_{\frac{1}{e}}(16^x-3)$$

Domain: $\log_{\frac{1}{e}}(16^x-3)$ is defined if

$16^x-3>0$

$16^x>3$

$x>\log_{16}(3)$ Exponential form

$x>0.40$

Domain $=]0.4,\infty[$

www.ilmkidunya.com

www.ilmkidunya.com

Now, $x\log_{\frac{1}{e}} 4 = \log_{\frac{1}{e}}(16^x - 3)$

$\log_{\frac{1}{e}}(4^x) = \log_{\frac{1}{e}}(16^x - 3) \quad \because b\log a = \log a^b$

$\Rightarrow \quad 4^x = 16^x - 3$

$4^x = 4^{2x} - 3$

$(4^x)^2 - 4^x - 3 = 0$

Put $4^x = y$, where $y > 0$ ...(1)

$y^2 - y - 3 = 0$

Here $a = 1, b = -1, c = -3$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+12}}{2}$$

$$y = \frac{1 \pm \sqrt{13}}{2}$$

$\Rightarrow y = \frac{1+\sqrt{13}}{2}$ | $y = \frac{1-\sqrt{13}}{2}$

$y \approx -1.30$ (Not possible)

Put in eq. (1)

$$4^x = \frac{1+\sqrt{13}}{2}$$

$$2^{2x} = \frac{1+\sqrt{13}}{2}$$

$\Rightarrow$

$$2x = \log_2\left(\frac{1+\sqrt{13}}{2}\right)$$

$$x = \frac{1}{2}\log_2\left(\frac{1+\sqrt{13}}{2}\right) \in]0.4, \infty[$$

Solution set $= \left\{\frac{1}{2}\log_2\left(\frac{1+\sqrt{13}}{2}\right)\right\}$

(vi) $2^x + 2^{-x} = 5$

Solution:

Domain: Since 2^x and 2^{-x} are defined for every real number, so

Domain $=]-\infty, \infty[$

Now, $2x + 2^{-x} = 5$

$$2^x + \frac{1}{2^x} = 5$$

Multipl both sides by 2^x

$(2^x)^2 + 1 = 5(2^x)$

$(2^x)^2 - 5(2^x) + 1 = 0$

Put $2^x = y$, where $y > 0$ ...(1)

$y^2 - 5y + 1 = 0$

Here $a = 1, b = -5, c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{(-5)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25-4}}{2}$$

$$x = \frac{5 \pm \sqrt{21}}{2}$$

$x = \frac{5+\sqrt{21}}{2}$ | $x = \frac{5-\sqrt{21}}{2}$

Put in eq. (1) | Put in eq. (1)

$2^x = \frac{5+\sqrt{21}}{2}$ | $2^x = \frac{5-\sqrt{21}}{2}$

$\Rightarrow x = \log_2\left(\frac{5+\sqrt{21}}{2}\right)$ | $\Rightarrow x = \log_2\left(\frac{5-\sqrt{21}}{2}\right)$

Solution set $= \left\{\log_2\left(\frac{5 \pm \sqrt{21}}{2}\right)\right\}$

Q2. Solve the following inequalities (if its solution exists).

(i) $2^x + 2^{1-x} \leq 3$

Solution:

Domain: Since 2^x and 2^{1-x} are defined for all real number, so

Domain $=]-\infty, \infty[$

Now, $2^x + 2^{1-x} \leq 3$

$2^x + 2 \cdot 2^{-x} \leq 3$

$$2^x + \frac{2}{2^x} \leq 3$$

Multiply both sides by 2^x

$(2^x)^2 + 2 \leq 3 \cdot 2^x$

Put $2^x = y$ ...(1)

$y^2 + 2 \leq 3y$

$y^2 - 3y + 2 \leq 0$

$y^2 - 2y - y + 2 \leq 0$

$y(y-2) - 1(y-2) \leq 0$

$(y-1)(y-2) \leq 0$

$\Rightarrow \quad 1 \leq y \leq 2$

$1 \leq 2^x \leq 2$ Using (1)

$2^0 \leq 2^x \leq 2^1$

$\Rightarrow \quad 0 \leq x \leq 1$

Solution set $= [0, 1]$

www.ilmkidunya.com

www.ilmkidunya.com

(ii) $2^x \geq \log_2(16)$

Solution:

Domain: Since 2^x is defined for all real number, so

Domain $=]-\infty, \infty[$

Now, $2^x \geq \log_2(16)$

$2^x \geq \log_2(2^4)$

$2^x \geq 4 \cdot \log_2 2$ $\quad \because \log(a^b) = b\log(a)$

$2^x \geq 4 \cdot (1)$ $\quad \because \log_2(2) = 1$

$2^x \geq 2^2$

$\Rightarrow \quad x \geq 2$

Solution set $= [2, \infty[$ or $[2, \infty)$

(iii) $\log_2(x-1) \geq \log_2(5-x)$

Solution:

Domain:

$\log_2(x-1)$ is defined	$\log_2(5-x)$ is defined
if $x-1>0$	if $5-x>0$
$x>1$	$x<5$

Domain $=]1, \infty[\cap]-\infty, 5[=]1, 5[$

Now, $\log_2(x-1) \geq \log_2(5-x)$

$\Rightarrow \quad x-1 \geq 5-x$ $\quad$ log base: $2>1$

$x+x \geq 5+1$

$2x \geq 6$

$x \geq \frac{6}{2}$

$x \geq 3 \quad \Rightarrow \quad x \in [3, \infty[$

Intersection of $[3, \infty[$ with domain $]1, 5[$ gives

Solution set $= [3, 5[$ or $[3, 5)$

(iv) $\log_{\frac{1}{3}}(x^2-4x+3) > \log_{\frac{1}{3}}(2x-1)$

Solution:

Domain:

$\log_{\frac{1}{3}}(2x-1)$ is defined	$\log_{\frac{1}{3}}(x^2-4x+3)$ is defined
if $2x-1>0$	if $x^2-4x+3>0$
$2x>1$	$(x-1)(x-3)>0$
$x>\frac{1}{2}$	$\Rightarrow \quad x<1$ or $x>3$

Domain $= \left]\frac{1}{2}, \infty\right[\cap \{]-\infty, 1[\text{ or }]3, \infty[\}$

$= \left]\frac{1}{2}, 1\right[\cup]3, \infty[$

Now, $\log_{\frac{1}{3}}(x^2-4x+3) > \log_{\frac{1}{3}}(2x-1)$

$\Rightarrow \quad x^2-4x+3 < 2x-1$ $\quad \because$ log base: $0<\frac{1}{3}<1$

$x^2-4x+3-2x+1<0$

$x^2-6x+4<0$

Associated equation is

$x^2-6x+4=0$

Here $a=1, b=-6, c=4$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{(-6)^2-4(1)(4)}}{2(1)} = \frac{6 \pm \sqrt{36-16}}{2}$$

$$= \frac{6 \pm \sqrt{20}}{2} = \frac{6 \pm 2\sqrt{5}}{2} = 2\left(\frac{3 \pm \sqrt{5}}{2}\right)$$

$x = 3 \pm \sqrt{5}$

$x = 3+\sqrt{5}$	$x = 3-\sqrt{5}$
$x-(3+\sqrt{5}) = 0$	$x-(3-\sqrt{5}) = 0$

$\Rightarrow \quad (x-(3-\sqrt{5}))(x-(3+\sqrt{5})) < 0$

$\Rightarrow \quad 3-\sqrt{5} < x < 3+\sqrt{5} =]3-\sqrt{5}, 3+\sqrt{5}[$

Intersection of $]3-\sqrt{5}, 3+\sqrt{5}[$ and domain $\left]\frac{1}{2}, 1\right[\cup]3, \infty[$ gives

Solution set $=]3-\sqrt{5}, 1[\cup]3, 3+\sqrt{5}[$

(v) $\log_2(x-2) + \log_2(x-3) \geq 3$

Solution:

Domain:

$\log_2(x-2)$ is defined	$\log_2(x-3)$ is defined
if $x-2>0$	if $x-3>0$
$x>2$	$x>3$

Domain $=]2, \infty[\cap]3, \infty[=]3, \infty[$

Now,

$\log_2(x-2) + \log_2(x-3) \geq 3$

$\log_2(x-2)(x-3) \geq 3$ $\quad \because \log a + \log b = \log(ab)$

$\Rightarrow \quad (x-2)(x-3) \geq 2^3$ $\quad$ Exponential form

$x^2-5x+6-8 \geq 0$

$x^2-5x-2 \geq 0$

Associated equation is

$x^2-5x-2=0$

Here $a=1, b=-5, c=-2$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{(-5)^2-4(1)(-2)}}{2(1)}$$

www.ilmkidunya.com

www.ilmkidunya.com

$$x = \frac{5 \pm \sqrt{25+8}}{2}$$

$$= \frac{5 \pm \sqrt{33}}{2}$$

$$x = \frac{5+\sqrt{33}}{2} \qquad x = \frac{5-\sqrt{33}}{2}$$

$$x - \frac{5+\sqrt{33}}{2} = 0 \qquad x - \frac{5-\sqrt{33}}{2} = 0$$

$$\Rightarrow \left(x - \frac{5-\sqrt{33}}{2}\right)\left(x - \frac{5+\sqrt{33}}{2}\right) \geq 0$$

$$\Rightarrow x \leq \frac{5-\sqrt{33}}{2} \quad \text{or} \quad x \geq \frac{5+\sqrt{33}}{2}$$

$$\left]-\infty, \frac{5-\sqrt{33}}{2}\right] \quad \text{or} \quad \left[\frac{5+\sqrt{33}}{2}, \infty\right[$$

Intersection of $\left]-\infty, \frac{5-\sqrt{33}}{2}\right]$ or $\left[\frac{5+\sqrt{33}}{2}, \infty\right[$ and domain $]3, \infty[$ gives

Solution set $= \left[\frac{5+\sqrt{33}}{2}, \infty\right[$

(vi) $\log(x) + \log(x-3) \geq \log(2x) + \log(x-1)$

Solution:

Domain: $\log x$ and $\log(2x)$ are defined, if $x > 0$

$\log(x-3)$ is defined	$\log(x-1)$ is defined
if $x - 3 > 0$	if $x - 1 > 0$
$x > 3$	$x > 1$

Domain $=]0, \infty[\cap]3, \infty[\cap]1, \infty[=]3, \infty[$

Now,

$$\log x + \log(x-3) \geq \log(2x) + \log(x-1)$$

$$\log x(x-3) \geq \log(2x)(x-1) \quad \because \log a + \log b = \log(ab)$$

$$\Rightarrow x(x-3) \geq 2x(x-1) \quad \because \text{log base: } 10 > 1$$

$$x^2 - 3x \geq 2x^2 - 2x$$

$$0 \geq 2x^2 - 2x - x^2 + 3x$$

$$x^2 + x \leq 0$$

$$x(x+1) \leq 0$$

$$(x-(-1))(x-0) \leq 0$$

$$\Rightarrow -1 \leq x \leq 0 = [-1, 0]$$

Intersection of $[-1, 0]$ and domain $]3, \infty[$ gives

Solution set $= \{\} = \phi$

Content Outline

- Applications of Exponential and Logarithmic Functions in Real-Life Situations

APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS IN REAL-LIFE SITUATIONS

Although there are many real-life situations involving exponential and logarithmic functions, in this section we focus on selected applications such as growth and decay processes, the measurement of sound intensity in decibels, and compound interest. Exponential models are used to describe quantities that change over time, while logarithmic functions help in determining unknown quantities in such models. The formula for growth and decay will be used as a given model and will be derived later in the chapter on differential equations, while the formulas for sound intensity and compound interest will be taken as standard results and will not be derived here.

(i) Growth and Decay

Many natural and physical processes such as population growth and radioactive decay change in such a way that the rate of change is proportional to the current amount. These processes are known as exponential growth or decay.
The general mathematical model is:

$$N = N_0 e^{kt}$$

Where N_0 is the initial amount, N is the amount at time t, and k is a constant which is positive for growth and negative for decay. In growth, the quantity increases with time, while in decay it decreases with time. Logarithms are used to determine unknown quantities such as time or rate in these models.

www.ilmkidunya.com

www.ilmkidunya.com

Example 21 (Growth): A population is given by: $N = 200e^{0.04t}$ Find the time t when the population becomes 400, where t is measured in hours.

Solution:

Given that: $N = 200e^{0.04t}$

Put $N = 400$, we get

$$400 = 200e^{0.04t}$$

$$\frac{400}{200} = e^{0.04t}$$

$$2 = e^{0.04t}$$

Take $\ln$ on both sides:

$$\ln 2 = \ln(e^{0.04t})$$

$$0.6931 = (0.04t)\ln e \quad \because \ln(a^b) = b\ln(a)$$

$$0.6931 = 0.04t \quad \because \ln e = 1$$

$$t = \frac{0.6931}{0.04} \approx 17.33 \text{ hours}$$

Hence, $t \approx 17.33$

Example 22 (Decay): A radioactive substance decays according to: $N = 500e^{kt}$. After 20 hours, the amount becomes 300. Find the value of k, where t is in hours.

Solution:

Given that: $N = 500e^{kt}$

Put $t = 20$ and $N = 300$, we get

$$300 = 500e^{20k}$$

$$\frac{300}{500} = e^{20k}$$

$$\frac{3}{5} = e^{20k}$$

Take $\ln$ on both sides:

$$\ln\left(\frac{3}{5}\right) = \ln(e^{20k})$$

$$-0.5108 = (20k)\ln e \quad \because \ln(a^b) = b\ln(a)$$

$$-0.5108 = 20k \quad \because \ln e = 1$$

$$k = \frac{-0.5108}{20} \approx -0.0255$$

Hence, $k \approx -0.0255$

(ii) Sound Intensity (Decibels)

The intensity of sound is the amount of sound energy passing through a unit area per unit time. Since sound intensity varies over a very wide range, it is measured on a logarithmic scale called the decibel (dB) scale.

The sound level L is given by:

$$L = 10\log\left(\frac{I}{I_0}\right)$$

where I is the intensity of sound and I_0 is the **threshold of hearing**, which is the minimum sound intensity that a normal human ear can detect. Its standard value is given by:

$$I_0 = 10^{-12} \text{ W/m}^2$$

where, W/m^2 (watts per square meter) is the unit of sound intensity. It represents the amount of sound power (energy per second) passing through a unit area.

Because of the logarithmic scale, even a small increase in decibels represents a large change in actual sound intensity, which makes logarithmic laws very useful in comparing sound levels in real-life situations.

Example 23 (Finding sound level): A sound has intensity $I = 10^{-4}$ and reference intensity $I_0 = 10^{-12}$. Find the sound level.

Solution:

As we know

$$L = 10\log\left(\frac{I}{I_0}\right)$$

Put $I = 10^{-4}$ and $I_0 = 10^{-12}$, we get

$$L = 10\log\left(\frac{10^{-4}}{10^{-12}}\right)$$

$$L = 10\log(10^{-4+12})$$

$$= 10\log(10^8)$$

$$= 10 \times 8\log(10) \quad \because \log(a^b) = b\cdot\log(a)$$

$$= 80(1) \quad \because \log_{10}(10) = 1$$

$$L = 80 \text{ dB}$$

www.ilmkidunya.com

www.ilmkidunya.com

Example 24: A sound has level $L = 60$ dB and $I_0 = 10^{-12}$. Find the intensity I.

Solution:

As we know

$$L = 10\log\left(\frac{I}{I_0}\right)$$

Put $L = 60$ dB and $I_0 = 10^{-12}$, we get

$$60 = 10\log\left(\frac{I}{10^{-12}}\right)$$

$$\log\left(\frac{I}{10^{-12}}\right) = \frac{60}{10}$$

$$\log\left(\frac{I}{10^{-12}}\right) = 6$$

In exponential form

$$\frac{I}{10^{-12}} = 10^6$$

$$I = 10^6 \times 10^{-12}$$

$$I = 10^{6-12}$$

$$I = 10^{-6}\ \text{W/m}^2$$

Example 25: The sound levels at two points are $L_1 = 90$ dB and $L_2 = 60$ dB. Find the ratio of their intensities $\frac{I_1}{I_2}$, where $I_0 = 10^{-12}\ \text{W/m}^2$.

Solution:

Here $L_1 = 90$ dB, $L_2 = 60$ dB, $I_0 = 10^{-12}\ \text{W/m}^2$, $\frac{I_1}{I_2} = ?$

As we know

$$L = 10\log\left(\frac{I}{I_0}\right)$$

$$\Rightarrow \quad L_1 = 10\log\left(\frac{I_1}{I_0}\right)$$

Put $L_1 = 90$

$$90 = 10\log\left(\frac{I_1}{I_0}\right)$$

$$\frac{90}{10} = \log\left(\frac{I_1}{I_0}\right)$$

$$\log\left(\frac{I_1}{I_0}\right) = 9$$

In exponential form:

$$\frac{I_1}{I_0} = 10^9$$

$$L_2 = 10\log\left(\frac{I_2}{I_0}\right)$$

Put $L_2 = 60$

$$60 = 10\log\left(\frac{I_2}{I_0}\right)$$

$$\frac{60}{10} = \log\left(\frac{I_2}{I_0}\right)$$

$$\log\left(\frac{I_2}{I_0}\right) = 6$$

In exponential form:

$$\frac{I_2}{I_0} = 10^6$$

Dividing both the equations:

$$\frac{\frac{I_1}{I_0}}{\frac{I_2}{I_0}} = \frac{10^9}{10^6}$$

$$\frac{I_1}{I_2} = 10^{9-6} = 10^3$$

$$I_1 : I_2 = 1000$$

(iii) Compound Interest

Compound interest is a type of interest in which the interest earned in each period is added to the principal (initial amount), so that interest is earned on both the principal and the accumulated interests. This results in exponential growth of money over time and is widely used in banking and financial systems to model investment growth. Compound interest is generally classified into two types: discrete compounding and continuous compounding.

Discrete Compound Interest

Discrete compound interest is a type of interest in which the interests is added to the principal at fixed time intervals such as yearly, half-yearly, quarterly, or monthly. In this method, interest is not added continuously, but only at specific intervals, and the future amount is calculated using a fixed compounding frequency. This type of compounding leads to exponential growth of money over time and is commonly used in banking systems.

The discrete compound interest is given by the formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where,

A = Final amount

P = Principal (initial amount)

r = Interest rate per unit time (in decimal form)

n = Number of times interest is compounded per unit time

t = Time (measured in the same unit as the compounding period)

www.ilmkidunya.com

www.ilmkidunya.com

Continuous Compound Interest

Continuous compound interest is a type of interest in which the interest is assumed to be added to the principal continuously at every instant of time. In this case, the compounding occurs without any fixed intervals, and the amount increases smoothly over time. This type of compounding represents the limiting case of discrete compounding when the number of compounding periods becomes infinitely large. The formula for continuous compounding is:

$$A = Pe^{rt}$$

where, A = Final amount after time t (principal + interest)
P = Principal (initial amount invested or borrowed)
r = Rate of interest per unit time (in decimal form)
t = Time (measured in the same unit as the rate is based on, such as years or months)

Example 26: A sum of Rs. 1000 is invested at a rate of 5% per year compounded continuously for 3 years. Find the final amount.

Solution:

Here $P = 1000$, $t = 3$,

$$r = 5\% = \frac{5}{100} = 0.05, A = ?$$

As we know

$$A = Pe^{rt}$$

$$\Rightarrow \quad A = 1000e^{(0.05)(3)}$$
$$= 1000e^{0.15}$$
$$\approx 1000 \times 1.1618$$
$$= 1161.8$$

Hence, $A \approx$ Rs. 1161.8

Example 27: A sum of Rs. 5000 is invested at a rate of 8% per annum, compounded quarterly. Find the amount after 3 years.

Solution:

Here $P = 5000$, $n = 4$ (∵ Compounded quarterly)

$$r = 8\% = \frac{8}{100} = 0.08, t = 3, A = ?$$

As we know

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 5000\left(1 + \frac{0.08}{4}\right)^{4(3)}$$
$$= 5000(1 + 0.02)^{12} = 5000(1.02)^{12}$$
$$\approx 6341.2$$

Hence, $A \approx$ Rs. 6341.2

EXERCISE 1.5

Q1. A population is given by $N = 300e^{0.02t}$, where t is measured in hours. Find the time when the population becomes 600.

Solution:

Given that: $N = 300e^{0.02t}$

Put $N = 600$, we get

$$600 = 300e^{0.02t}$$
$$\frac{600}{300} = e^{0.02t}$$
$$2 = e^{0.02t}$$

Take $\ln$ on both sides:

$$\ln 2 = \ln(e^{0.02t})$$
$$0.6931 = (0.02t)\ln e$$
$$0.6931 = 0.02t \qquad \because \ln e = 1$$
$$t = \frac{0.6931}{0.02} \approx 34.66$$

Hence, $t \approx 34.65$ hours

Q2. A culture grows according to $N = 150e^{kt}$, where time t is measured in hours. After 10 hours, the population becomes 250. Find the growth constant k.

Solution:

Given that: $N = 150e^{kt}$

Put $t = 10$, $N = 250$, we get

$$250 = 150e^{10k}$$
$$\frac{250}{150} = e^{10k}$$
$$\frac{5}{3} = e^{10k}$$

www.ilmkidunya.com

www.ilmkidunya.com

Take $\ln$ on both sides:

$$\ln\left(\frac{5}{3}\right) = \ln(e^{10k})$$

$$0.5108 = (10k)\ln e \qquad \because \ln(a^b) = b\ln(a)$$

$$0.5108 = 10k \qquad \because \ln e = 1$$

$$k = \frac{0.5108}{10} \approx 0.0511$$

Hence, $k \approx 0.0511$

Q3. A radioactive substance decays as $N = 800e^{-0.05t}$, where t is in hours. Find the time when the amount reduces to 200.

Solution:

Given that: $N = 800e^{-0.05t}$

Put $N = 200$, we get

$$200 = 800e^{-0.05t}$$

$$\frac{200}{800} = e^{-0.05t}$$

$$\frac{1}{4} = e^{-0.05t}$$

Take $\ln$ on both sides:

$$\ln\left(\frac{1}{4}\right) = \ln(e^{-0.05t})$$

$$-1.3863 = (-0.05t)\ln e \qquad \because \ln(a^b) = b\ln(a)$$

$$1.3863 = 0.05t \qquad \because \ln e = 1$$

$$t = \frac{1.3863}{0.05} \approx 27.73$$

Hence, $t \approx 27.73$ hours

Q4. A substance decays according to $N = 600e^{kt}$, where time t is measured in hours. After 8 hours, the amount becomes 400. Find the decay constant k.

Solution:

Given that: $N = 600e^{kt}$

Put $t = 8$, $N = 400$, we get

$$400 = 600e^{8k}$$

$$\frac{400}{600} = e^{8k}$$

$$\frac{2}{3} = e^{8k}$$

Take $\ln$ on both sides:

$$\ln\left(\frac{2}{3}\right) = \ln e^{8k}$$

$$-0.4055 = (8k)\ln e \qquad \because \ln(a^b) = b\ln(a)$$

$$-0.4055 = 8k$$

$$k = \frac{-0.4055}{8} \approx -0.0507$$

Hence, $k \approx -0.0507$

Q5. A sound has intensity $I = 10^{-6}$ W/m^2 and reference intensity $I_0 = 10^{-12}$ W/m^2. Find the sound level in decibels.

Solution:

As we know

$$L = 10\log\left(\frac{I}{I_0}\right)$$

Put $I = 10^{-6}$, $I_0 = 10^{-12}$, we get

$$L = 10\log\left(\frac{10^{-6}}{10^{-12}}\right)$$

$$L = 10\log(10^{-6+12})$$

$$= 10\log(10^6)$$

$$= 10 \times 6\log(10) \qquad \because \log(a^b) = b\log(a)$$

$$= 60(1) \qquad \because \log_{10}(10) = 1$$

$$L = 60 \text{ dB}$$

Q6. A sound has level $L = 80$ dB and $I_0 = 10^{-12}$ W/m^2. Find the intensity I.

Solution:

As we know

$$L = 10\log\left(\frac{I}{I_0}\right)$$

Put $L = 80$ dB and $I_0 = 10^{-12}$, we get

$$80 = 10\log\left(\frac{I}{10^{-12}}\right)$$

$$\frac{80}{10} = \log\left(\frac{I}{10^{-12}}\right)$$

$$\log\left(\frac{I}{10^{-12}}\right) = 8$$

In exponential form

$$\frac{I}{10^{-12}} = 10^8$$

$$I = 10^8 \times 10^{-12}$$

$$I = 10^{8-12}$$

$$I = 10^{-4} \text{ W/m}^2$$

www.ilmkidunya.com

www.ilmkidunya.com

Q7. A quantity changes according to $N = 500e^{kt}$. If $N = 250$ when $t = 5$, find:
(i) the value of constant k.
(ii) the quantity N when $t = 10$.

Solution:

Given that: $N = 500e^{kt}$ ...(1)

(i) $k = ?$, when $N = 250$, $t = 5$

Put $N = 250$, $t = 5$ in eq. (1)

$$250 = 500e^{5k}$$

$$\frac{250}{500} = e^{5k}$$

$$\frac{1}{2} = e^{5k}$$

Take $\ln$ on both sides:

$$\ln\left(\frac{1}{2}\right) = \ln(e^{5k})$$

$$-0.6931 = (5k)\ln e \qquad \because \ln(a^b) = b\ln(a)$$

$$-0.6931 = 5k \qquad \because \ln e = 1$$

$$k = \frac{-0.6931}{5} \approx 0.1386$$

Hence, $k \approx -0.1386$

(ii) $N = ?$, when $t = 10$, $k = -0.1386$

Put $t = 10$ and $k = -0.1386$ in eq. (1)

$$N = 500e^{-0.1386(10)}$$

$$= 500e^{-1.386}$$

$$= 500 \times 0.25$$

$$N = 125$$

Q8. Two sounds have intensities I_1 and I_2 such that $L_1 = 85$ dB and $L_2 = 65$ dB. Find the ratio $\frac{I_1}{I_2}$, where $I_0 = 10^{-12}$ W/m^2.

Solution:

Here $L_1 = 85$ db, $L_2 = 65$ dB, $I_0 = 10^{-12}$ W/m^2

$$\frac{I_1}{I_2} = ?$$

As we know

$$L = 10\log\left(\frac{I}{I_0}\right)$$

$$\Rightarrow \quad L_1 = 10\log\left(\frac{I_1}{I_0}\right) \qquad \Rightarrow \quad L_2 = 10\log\left(\frac{I_2}{I_0}\right)$$

Put $L_1 = 85$ | Put $L_2 = 65$

$$85 = 10\log\left(\frac{I_1}{I_0}\right) \qquad 65 = 10\log\left(\frac{I_2}{I_0}\right)$$

$$\frac{85}{10} = \log\left(\frac{I_1}{I_0}\right) \qquad \frac{65}{10} = \log\left(\frac{I_2}{I_0}\right)$$

$$\log\left(\frac{I_1}{I_0}\right) = 8.5 \qquad \log\left(\frac{I_2}{I_0}\right) = 6.5$$

In exponential form | In exponential form

$$\frac{I_1}{I_0} = 10^{8.5} \qquad \frac{I_2}{I_0} = 10^{6.5}$$

Dividing both the equations

$$\frac{\frac{I_1}{I_0}}{\frac{I_2}{I_0}} = \frac{10^{8.5}}{10^{6.5}}$$

$$\frac{I_1}{I_2} = 10^{8.5-6.5}$$

$$\frac{I_1}{I_2} = 10^2$$

$$\Rightarrow \quad I_1 : I_2 = 100$$

Q9. A sum of Rs. 4000 is invested at 6% per annum, compounded annually for 3 years. Find the final amount.

Solution:

Here $P = 4000$, $n = 1$ ($\because$ Compounded annually)

$r = 6\% = \frac{6}{100} = 0.06$, $t = 3$, $A = ?$

As we know

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 4000\left(1 + \frac{0.06}{1}\right)^{1(3)}$$

$$= 4000(1.06)^3$$

$$= 4000(1.191016)$$

Hence, $A = 4764.064$

www.ilmkidunya.com

www.ilmkidunya.com

Q10. A sum of Rs. 5000 becomes 6324.48 when invested at 8% per annum, compounded annually. Find the time t.

Solution:

Here $P = 5000$, $A = 6324.48$, $n = 1$ ($\because$ Compounded annually)
$r = 8\% = 0.08$, $t = ?$
As we know

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$6324.48 = 5000\left(1+\frac{0.08}{1}\right)^{1t}$$

$$\frac{6324.48}{5000} = (1.08)^t$$

$$(1.08)^t = 1.264896$$

Take $\ln$ on both sides:

$$\ln(1.08)^t = \ln(1.264896)$$

$$t\cdot\ln(1.08) = \ln(1.264896) \qquad \because \log(a^b) = b\log a$$

$$t = \frac{\ln(1.264896)}{\ln(1.08)} \approx 3.05$$

Hence, $t \approx 3.05$ years

Q11. A sum of Rs. 3000 is invested at 5% per annum, compounded continuously for 4 years. Find the final amount.

Solution:

Here $P = 3000$, $t = 4$, $r = 5\% = 0.05$, $A = ?$
As we know

$$A = Pe^{rt}$$

$$A = 3000e^{(0.05)(4)}$$

$$= 3000e^{0.2}$$

$$\approx 3000(1.2214)$$

$$A = 3664.2$$

Hence, $A \approx$ Rs. 3664.2

Q12. An amount of Rs. 2000 becomes 3320.12 under continuous compounding at 6% per annum. Find the time t.

Solution:

Here $P = 2000$, $A = 3320.12$, $r = 6\% = 0.06$, $t = ?$
As we know

$$A = Pe^{rt}$$

$$3320.12 = 2000e^{0.06t}$$

$$\frac{3320.12}{2000} = e^{0.06t}$$

$$1.66006 = e^{0.06t}$$

Take $\ln$ on both sides:

$$\ln(1.66006) = \ln e^{0.06t}$$

$$0.5069 \approx (0.06t)\ln e$$

$$0.5069 = 0.06t \qquad \because \ln e = 1$$

$$t = \frac{0.5069}{0.06} \approx 8.45$$

Hence, $t \approx 8.45$ hours

www.ilmkidunya.com

www.ilmkidunya.com

Multiple Choice Questions (MCQs)

1. If $f(x)=a(x-h)(x-k)$, where $a \neq 0$, then the graph cuts the x-axis at:
   A) $x=a$ and $x=h$ B) $x=h$ and $x=k$
   C) $x=a$ and $x=k$ D) $x=-h$ and $x=-k$
2. The vertex of the parabola $f(x)=a(x-h)(x-k)$ has x-coordinate:
   A) $h+k$ B) $h-k$ C) $\frac{h+k}{2}$ D) $\frac{hk}{2}$
3. The vertex of the parabola $y=-4(x+1)^2$ is:
   A) $(1, 0)$ B) $(-1, 0)$ C) $(0, -1)$ D) $(0, 1)$
4. A quadratic function whose x-intercepts are $x=h$ and $x=k$ can be written in the form:
   A) $y=a(x+h)(x+k)$ B) $y=a(x-h)(x-k)$ C) $y=a(x+h)(x-k)$ D) $y=(x-h)+(x-k)$
5. The function $f(x)=e^x$ is classified as:
   A) algebraic B) transcendental C) polynomial D) rational
6. For the exponential function $f(x)=a^x$, the required conditions are:
   A) $a<0$ and $a \neq 1$ B) $a>0$ and $a \neq 1$ C) $a=0$ only D) $a=1$ only
7. The domain of $f(x)=a^x$, where $a>0, a \neq 1$, is:
   A) $(0, \infty)$ B) $[0, \infty)$ C) R D) $(-\infty, 0)$
8. The range of the exponential function $f(x)=a^x$ is:
   A) R B) $(0, \infty)$ C) $[0, \infty)$ D) $(-\infty, 0)$
9. The graph of an exponential function $y=a^x$ has the horizontal asymptote:
   A) $y=1$ B) $x=0$ C) $y=0$ D) $x=1$
10. The y-intercept of $y=a^x$ is:
    A) $(0, 0)$ B) $(1, 0)$ C) $(0, 1)$ D) $(1, 1)$
11. The exponential function $f(x)=a^x$ is strictly increasing when:
    A) $0<a<1$ B) $a=0$ C) $a>1$ D) $a<0$
12. The function $f(x)=\left(\frac{1}{2}\right)^x$ is:
    A) strictly increasing B) strictly decreasing
    C) constant D) neither increasing nor decreasing
13. The function $f(x)=a^x$ is one-to-one because:
    A) it is discontinuous B) it is strictly increasing or strictly decreasing
    C) its range is R D) its domain is $(0, \infty)$
14. The function sinh x is defined as:
    A) $\frac{e^x+e^{-x}}{2}$ B) $\frac{e^x-e^{-x}}{2}$ C) e^x-e^{-x} D) e^x+e^{-x}
15. The range of cosh x is:
    A) $(-\infty, \infty)$ B) $(-1, 1)$ C) $[1, \infty)$ D) $(0, 1)$
16. The identity $\cosh^2 x-\sinh^2 x=$
    A) cosh $2x$ B) sinh $2x$ C) 1 D) 2
17. $\sinh^{-1} x=$:
    A) $\ln(x+\sqrt{x^2+1})$ B) $\ln(x+\sqrt{x^2-1})$ C) $\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ D) $\ln(x^2+1)$

www.ilmkidunya.com

www.ilmkidunya.com

18. **A logarithmic function is the inverse a/an _____ function.**
A) polynomial B) exponential C) modulus D) hyperbolic

19. **A logarithm with base e is called:**
A) common logarithm B) natural logarithm
C) decimal logarithm D) binary logarithm

20. **The value of $\log_a 1$ is _____ for $a > 0, a \neq 1$.**
A) 0 B) 1 C) a D) -1

21. **For $a > 0, a \neq 1$, the value of $\log_a a$ is:**
A) 0 B) a C) 1 D) -1

22. **The change-of-base formula is:**
A) $\log_a x = \frac{\log_a x}{\log_b x}$ B) $\log_a x = \frac{\log_b x}{\log_a b}$
C) $\log_a x = \frac{\log_b x}{\log_b a}$ D) $\log_a x = \log_b x + \log_b a$

23. **The vertical asymptote of $y = \log_a x$ is:**
A) $y = 0$ B) $x = 0$ C) $x = 1$ D) $y = 1$

24. **The x-intercept of $y = \log_a x$ is:**
A) (0, 1) B) (1, 0) C) (0, 0) D) (1, 1)

25. **If $0 < a < 1$, the logarithmic function $y = \log_a x$ is:**
A) strictly increasing B) strictly decreasing C) constant D) linear

26. **The graph of $y = |x|$ consists of:**
A) a circle B) two rays forming a V-shape
C) a parabola D) a hyperbola

27. **The graph of a function and its inverse function is symmetric about:**
A) x-axis B) y-axis C) $y = x$ D) $y = -x$

28. **A function must be _____ for its inverse function to exist.**
A) onto only B) one-to-one only C) bijective D) constant

29. **If $y = f(x)$ is shifted horizontally to the right by h units, the resulting function is:**
A) $y = f(x + h)$ B) $y = f(x - h)$
C) $y = f(x) + h$ D) $y = f(x) - h$

30. **If $y = f(x)$ is shifted vertically upward by k units, the resulting function is:**
A) $y = f(x - k)$ B) $y = f(x + k)$
C) $y = f(x) + k$ D) $y = f(x) - k$

ANSWER KEY

1.	B	2.	C	3.	B	4.	B	5.	B	6.	B	7.	C	8.	B	9.	C	10.	C
11.	C	12.	B	13.	B	14.	B	15.	C	16.	C	17.	A	18.	B	19.	B	20.	A
21.	C	22.	C	23.	B	24.	B	25.	B	26.	B	27.	C	28.	C	29.	B	30.	C

www.ilmkidunya.com