Exercise 4.1

Resolve into partial fractions.

1) $\dfrac{7x-9}{(x+1)(x-3)}$

Solution:

$\frac{7x-9}{(x+1)(x-3)}=\frac{A}{x+1}+\frac{B}{x-3}$

Multiplying both sides by $(x+1)(x-3)$, we get
$7x-9=A(x-3)+B(x+1)$ ....(1)

To find A, we put $x+1=0\Rightarrow x=-1$ in eq.(1), we get
$7(-1)-9=A(-1-3)+B(-1+1)$
$7-9=A(-4)+B(0)$
$-16=-4A$
$-4A=-16$

Dividing both sides by '-4', we get
$\mathrm{A}=4$

To find B, we put $\mathrm{x}-3=0\Rightarrow x=3$ in eq.(1), we get
$7(3)-9=A(3-3)+B(3+1)$
$21-9=A(0)+B(4)$
$12=4B$
$or\ 4B=12$

Dividing both sides by '4', we get
$B=3$

Thus required partial fraction are $\frac{4}{x+1}+\frac{3}{x-3}$

Hence, $\frac{7x-9}{(x+1)(x-3)}=\frac{4}{x+1}+\frac{3}{x-3}$

2) $\dfrac{x-11}{(x-4)(x+3)}$

Solution:

$\frac{x-11}{(x-4)(x+3)}=\frac{A}{x-4}+\frac{B}{x+3}$

Multiplying both sides by $(x-4)(x+3)$, we get
$x-11=A(x+3)+B(x-4)$ ....(1)

To find A, we put $x-4=0\Rightarrow x=4$ in eq.(1), we get
$4-11=A(4+3)+B(4-4)$
$-7=A(7)+B(0)$
$-7=7A$
$or\ 7A=-7$

Dividing both sides by '7', we get
$\mathrm{A}=-1$

To find B, we put $\mathrm{x}+3=0\Rightarrow x=-3$ in eq.(1), we get
$-3-11=A(-3+3)+B(-3-4)$
$-14=A(0)+B(-7)$
$-14=-7B$
$or\ -7B=-14$

Dividing both sides by '-7', we get
$B=2$

Thus required partial fraction are $\frac{-1}{x-4}+\frac{2}{x+3}$

Hence, $\frac{x-11}{(x-4)(x+3)}=-\frac{1}{x-4}+\frac{2}{x+3}$

3) $\dfrac{3x-1}{x^2-1}$

Solution:

$\frac{3x-1}{x^2-1}=\frac{3x-1}{(x-1)(x+1)}$
$Let\ \frac{3x-1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$

Multiplying both sides by $(x-1)(x+1)$, we get
$3x-1=A(x+1)+B(x-1)$ ....(1)

To find A, we put $x-1=0\Rightarrow x=1$ in eq.(1), we get
$3(1)-1=A(1+1)+B(1-1)$
$3-1=A(2)+B(0)$
$2=2A$
$or\ 2A=2$

Dividing both sides by '2', we get
$\mathrm{A}=1$

To find B, we put $\mathrm{x}+1=0\Rightarrow x=-1$ in eq.(1), we get
$3(-1)-1=A(-1+1)+B(-1-1)$

$-3-1=A(0)+B(-2)$
$-4=-2B$
$or\ -2B=-4$

Dividing both sides by '-2', we get
$B=2$

Thus required partial fraction are $\frac{1}{x-1}+\frac{2}{x+1}$

Hence, $\frac{3x-1}{x^2-1}=\frac{1}{x-1}+\frac{2}{x+1}$

4) $\dfrac{x-5}{x^2+2x-3}$

Solution:

$$\begin{aligned}\frac{x-5}{x^2+2x-3}&=\frac{x-5}{x^2+3x-x-3}\\&=\frac{x-5}{x(x+3)-1(x+3)}=\frac{x-5}{(x-1)(x+3)}\end{aligned}$$
$Let\ \frac{x-5}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}$

Multiplying both sides by $(x-1)(x+3)$, we get
$x-5=A(x+3)+B(x-1)$ ....(1)

To find A, we put $x-1=0\Rightarrow x=1$ in eq.(1), we get
$1-5=A(1+3)+B(1-1)$
$-4=A(1+3)+B(0)$
$-4=4A$
$or\ 4A=-4$

Dividing both sides by '4', we get
$\mathrm{A}=-1$

To find B, we put $\mathrm{x}+3=0\Rightarrow x=-3$ in eq.(1), we get
$-3-5=A(-3+3)+B(-3-1)$
$-8=A(0)+B(-4)$
$-8=-4B$
$or\ -4B=-8$

Dividing both sides by '-4', we get
$B=2$

Thus required partial fraction are $\frac{-1}{x-1}+\frac{2}{x+3}$

Hence, $\frac{x-5}{x^2+2x-3}=-\frac{1}{x-1}+\frac{2}{x+3}$

5) $\dfrac{3x+3}{(x-1)(x+2)}$

Solution:

$Let\ \frac{3x+3}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$

Multiplying both sides by $(x-1)(x+2)$, we get
$3x+3=A(x+2)+B(x-1)$ ....(1)

To find A, we put $x-1=0\Rightarrow x=1$ in eq.(1), we get
$3(1)+3=A(1+2)+B(1-1)$
$3+3=A(3)+B(0)$
$6=3A$

$or\ 3A=6$

Dividing both sides by '3', we get
$A=2$

To find B, we put $x+2=0 \Rightarrow x=-2$ in eq.(1), we get
$3(-2)+3=A(-2+2)+B(-2-1)$
$-6+3=A(0)+B(-3)$

$-3=-3B$
$or\ -3B=-3$

Dividing both sides by '-3', we get
$B=1$

Thus required partial fraction are $\frac{2}{x-1}+\frac{1}{x+2}$

Hence, $\frac{3x+3}{(x-1)(x+2)}=\frac{2}{x-1}+\frac{1}{x+2}$

6) $\frac{7x-25}{(x-4)(x-3)}$

Solution:

$Let\ \frac{7x-25}{(x-4)(x-3)}=\frac{A}{x-4}+\frac{B}{x-3}$

Multiplying both sides by $(x-4)(x-3)$, we get
$7x-25=A(x-3)+B(x-4)$ ....(1)

To find A, we put $x-4=0 \Rightarrow x=4$ in eq.(1), we get

$7(4)-25=A(4-3)+B(4-4)$
$28-25=A(1)+B(0)$
$3=A$
$or\ A=3$

To find B, we put $x-3=0 \Rightarrow x=3$ in eq.(1), we get
$7(3)-25=A(3-3)+B(3-4)$
$21-25=A(0)+B(-1)$
$-4=-B$
$or\ -B=-4$

Dividing both sides by '-1', we get
$B=4$

Thus required partial fraction are $\frac{3}{x-4}+\frac{4}{x-3}$

Hence, $\frac{7x-25}{(x-4)(x-3)}=\frac{3}{x-4}+\frac{4}{x-3}$

7) $\frac{x^2+2x+1}{(x-2)(x+3)}$

Solution:

$$\begin{aligned}\frac{x^2+2x+1}{(x-2)(x+3)}&=\frac{x^2+2x+1}{x^2+3x-2x-6}\\&=\frac{x^2+2x+1}{x^2+x-6}\end{aligned}$$

This is an improper fraction so by long division, we have

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2+x-6 & x^2+2x+1 \\ & \underline{\pm x^2 \pm x \mp 6} \\ & \quad x+7 \end{array}$$

$$\begin{aligned}\frac{x^2+2x+1}{(x-2)(x+3)}&=1+\frac{x+7}{x^2+x-6}\\&=1+\frac{x+7}{(x-2)(x+3)}\end{aligned}$$

$Let\ \frac{x+7}{(x-2)(x+3)}=\frac{A}{x-2}+\frac{B}{x+3}$

Multiplying both sides by $(x-2)(x+3)$, we get
$x+7=A(x+3)+B(x-2)$ ....(1)

To find A. we put $x-2=0 \Rightarrow x=2$ in eq.(1), we get
$2+7=A(2+3)+B(2-2)$
$9=A(5)+B(0)$
$9=5A$
$or\ 5A=9$

Dividing both sides by '5', we get
$A=\frac{9}{5}$

To find B, we put $x+3=0 \Rightarrow x=-3$ in eq.(1), we get
$-3+7=A(-3+3)+B(-3-2)$
$4=A(0)+B(-5)$
$4=-5B$
$or\ -5B=4$
Dividing both sides by '-5', we get
$B=-\frac{4}{5}$

Thus required partial fraction are $\frac{9/5}{x-2}+\frac{-4/5}{x+3}$

Hence, $\frac{x^2+2x+1}{(x-2)(x+3)}=1+\frac{9}{5(x-2)}-\frac{4}{5(x+3)}$

8) $\frac{6x^3+5x^2-7}{3x^2-2x-1}$

Solution:

This an improper fraction so by long division, we have

$$\begin{array}{r|l} & \quad 2x+3 \\ \hline 3x^2-2x-1 & 6x^3+5x^2-7 \\ & \underline{\pm 6x^3 \mp 4x^2 \mp 2x} \\ & \quad 9x^2+2x-7 \\ & \quad \underline{\pm 9x^2 \mp 6x \mp 3} \\ & \qquad 8x-4 \end{array}$$

$$\begin{aligned}\frac{6x^3+5x^2-7}{3x^2-2x-1}&=2x+3+\frac{8x-4}{3x^2-2x-1}\\&=2x+3+\frac{8x-4}{3x^2-3x+x-1}\\&=2x+3+\frac{8x-4}{3x(x-1)+1(x-1)}\\&=2x+3+\frac{8x-4}{(3x+1)(x-1)}\end{aligned}$$

$Let\ \frac{8x-4}{(3x+1)(x-1)}=\frac{A}{3x+1}+\frac{B}{x-1}$

Multiplying both sides by $(3x+1)(x-1)$, we get
$8x-4=A(x-1)+B(3x+1)$ ....(1)

To find A, we put $3x+1=0 \Rightarrow 3x=-1 \Rightarrow x=-\frac{1}{3}$ in eq.(1), we get

$8\left(-\frac{1}{3}\right)-4=A\left(-\frac{1}{3}-1\right)+B\left[3\left(-\frac{1}{3}\right)+1\right]$

$-\frac{8}{3}-4=A\left(-\frac{4}{3}\right)+B(0)$

$-\frac{20}{3}=-\frac{4}{3}A$
$or -\frac{4}{3}A=-\frac{20}{3}$
$\frac{4}{3}A=\frac{20}{3}$
$A=\frac{20}{3}\times\frac{3}{4}$
$A=5$

To find B, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get
$8(1)-4=A(1-1)+B[3(1)+1]$
$8-4=A(0)+B(4)$
$4=4B$
$or\ 4B=4$

Dividing both sides by '4', we get
$B=1$

Thus required partial fraction are $\frac{5}{3x+1}+\frac{1}{x-1}$

Hence, $\frac{6x^3+5x^2-7}{3x^2-2x-1}=2x+3+\frac{5}{3x+1}+\frac{1}{x-1}$

Resolution of a fraction when D(x) consists of repeated linear factors:

Rule II:

If a linear factor (ax + b) occurs n times as a factor of D(x), then there are n partial fractions of the form.

$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+......+\frac{A_n}{(ax+b)^n}$ where A_1 ,A_2.......A_n are constants and $n \geq 2$ is a positive integer.

$$\frac{N(x)}{D(x)}=\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+......+\frac{A_n}{(ax+b)^n}$$

Exercise 4.2

Resolve into partial fraction.

$$(1)\ \frac{x^2-3x+1}{(x-1)^2(x-2)}$$

Solution:

Let $\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$

Multiplying both sides by $(x-1)^2(x-2)$, we get

$x^2-3x+1=A(x-1)(x-2)+B(x-2)+C(x-1)^2$ ....(1)

$x^2-3x+1=A(x^2-3x+2)+B(x-2)+C(x^2-2x+1)$

$x^2-3x+1=Ax^2-3Ax+2A+Bx-2B+Cx^2-2Cx+C$

$x^2-3x+1=Ax^2+Cx^2-3Ax+Bx-2Cx+2A-2B+C$ ....(2)

To find C, we put $x-2=0 \Rightarrow x=2$ in eq.(1), we get

$(2)^2-3(2)+1=A(2-1)(2-2)+B(2-2)+C(2-1)^2$

$4-6+1=A(1)(0)+B(0)+C(1)^2$

$5-6=A(0)+B(0)+C$

$-1=C$

or $C=-1$

To find B, we put $(x-1)^2=0 \Rightarrow x-1=0 \Rightarrow x=1$ in eq.(1), we get

$(1)^2-3(1)+1=A(1-1)(1-2)+B(1-2)+C(1-1)^2$

$1-3+1=A(0)(-1)+B(-1)+C(0)$

$2-3=A(0)+B(-1)+C(0)$

$-1=-B$

$\Rightarrow B=1$

To find A, equating coefficient of x^2 on both sides of (2), we get

$A+C=1$

$A+(-1)=1$

$A=1+1$

$A=2$

Thus required partial fractions are $\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{-1}{x-2}$

Hence, $\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{2}{x-1}+\frac{1}{(x-1)^2}-\frac{1}{x-2}$

$$(2)\ \frac{x^2+7x+11}{(x+2)^2(x+3)}$$

Solution:

Let $\frac{x^2+7x+11}{(x+2)^2(x+3)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x+3}$

Multiplying both sides by $(x+2)^2(x+3)$, we get

$x^2+7x+11=A(x+2)(x+3)+B(x+3)+C(x+2)^2$ ....(1)

$x^2+7x+11=A(x^2+5x+6)+B(x+3)+C(x^2+4x+4)$

$x^2+7x+11=Ax^2+5Ax+6A+Bx+3B+Cx^2+4Cx+4C$

$x^2+7x+11=Ax^2+Cx^2+5Ax+Bx+4Cx+6A+3B+4C$ ....(2)

To find C, we put $x+3=0 \Rightarrow x=-3$ in eq.(1), we get

$(-3)^2+7(-3)+11=A(-3+2)(-3+3)+B(-3+3)+C(-3+2)^2$

$9-21+11=A(-1)(0)+B(0)+C(-1)^2$

$20-21=A(0)+B(0)+C(1)$

$-1=C$

or $C=-1$

To find B, we put $(x+2)^2=0 \Rightarrow x+2=0 \Rightarrow x=-2$ in eq.(1), we get

$(-2)^2+7(-2)+11=A(-2+2)(-2+3)+B(-2+3)+C(-2+2)^2$

$4-14+11=A(0)(1)+B(1)+C(0)^2$

$15-14=A(0)+B(1)+C(0)$

$1=B$

$\Rightarrow B=1$

To find A, equating coefficient of x^2 on both sides of (2), we get

$A+C=1$

$A+(-1)=1$

$A=1+1$

$A=2$

Thus required partial fractions are $\frac{2}{x+2}+\frac{1}{(x+2)^2}+\frac{-1}{x+3}$

Hence, $\frac{x^2+7x+11}{(x+2)^2(x+3)}=\frac{2}{x+2}+\frac{1}{(x+2)^2}-\frac{1}{x+3}$

$$(3)\ \frac{9}{(x-1)(x+2)^2}$$

Solution:

Let $\frac{9}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$

Multiplying both sides by $(x-1)(x+2)^2$, we get

$9=A(x+2)^2+B(x-1)(x+2)+C(x-1)$ ....(1)

$9=A(x^2+4x+4)+B(x^2+x-2)+C(x-1)$

$9=Ax^2+4Ax+4A+Bx^2+Bx-2B+Cx-C$

$9=Ax^2+Bx^2+4Ax+Bx+Cx+4A-2B-C$ ....(2)

To find A, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$9=A(1+2)^2+B(1-1)(1+2)+C(1-1)$

$9=A(3)^2+B(0)(3)+C(0)$

$9=A(9)+B(0)+C(0)$

$9=9A$

or $9A=9$

Dividinge the both sides by '9', we get

$A=1$

To find C, we put $(x+2)^2=0 \Rightarrow x+2=0 \Rightarrow x=-2$ in eq.(1), we get

$9=A(-2+2)^2+B(-2-1)(-2+2)+C(-2-1)$

$9=A(0)^2+B(-3)(0)+C(-3)$

$9=A(0)+B(0)+C(-3)$

$9=-3C$

or $-3C=9$

Dividing both sides by '-3', we get

$C=-3$

To find B, equating coefficient of x^2 on both sides of (2), we get

$A+B=0$

$1+B=0$

$B=-1$

Thus required partial fractions are $\frac{1}{x-1}+\frac{-1}{x+2}+\frac{-3}{(x+2)^2}$

Hence, $\frac{9}{(x-1)(x+2)^2}=\frac{1}{x-1}-\frac{1}{x+2}-\frac{3}{(x+2)^2}$

$$(4)\ \frac{x^4+1}{x^2(x-1)}$$

Solution:

This is an improper fraction so by long division, we have

$$\begin{array}{rl} & \quad x+1 \\ x^3-x^2 & \overline{\big)\ x^4+1} \\ & \underline{\pm x^4 \mp x^3} \\ & \quad x^3+1 \\ & \underline{\pm x^3 \mp x^2} \\ & \quad x^2+1 \end{array}$$

$$\frac{x^4+1}{x^2(x-1)}=x+1+\frac{x^2+1}{x^2(x-1)}$$

Let $\frac{x^2+1}{x^2(x-1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$

Multiplying both sides by $x^2(x-1)$, we get

$$x^2+1=A(x)(x+1)+B(x-1)+Cx^2 \quad ....(1)$$
$$x^2+1=Ax^2-Ax+Bx-B+Cx^2$$
$$x^2+1=Ax^2+Cx^2-Ax+Bx-B \quad ....(2)$$

To find C, we put $x-1=0\Rightarrow x=1$ in eq.(1), we get

$$(1)^2+1=A(1)(1-1)+B(1-1)+C(1)^2$$
$$1+1=A(1)(0)+B(0)+C(1)$$
$$2=A(0)+B(0)+C(1)$$
$$2=C$$
$$or\ C=2$$

To find B, we put $x^2=0\Rightarrow x=0$ in eq.(1), we get

$$(0)^2+1=A(0)(0-1)+B(0-1)+C(0)^2$$
$$1=A(0)(-1)+B(-1)+C(0)$$
$$1=-B$$
$$or\ B=-1$$

To find A, equating coefficient of x^2 on both sides of (2), we get

$$A+C=1$$
$$A+2=1$$
$$A=1-2$$
$$A=-1$$

Thus required partial fractions are $\frac{-1}{x}+\frac{-1}{x^2}+\frac{2}{x-1}$

Hence, $\frac{x^4+1}{x^2(x-1)}=x+1-\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x-1}$

(5) $\frac{7x+4}{(3x+2)(x+1)^2}$

Solution:

Let $\frac{7x+4}{(3x+2)(x+1)^2}=\frac{A}{(3x+2)}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$

Multiplying both sides by $(3x+2)(x+1)^2$, we get

$$7x+4=A(x+1)^2+B(3x+2)(x+1)+C(3x+2) \quad ....(1)$$
$$7x+4=A(x^2+2x+1)+B(3x^2+5x+2)+C(3x+2)$$
$$7x+4=Ax^2+2Ax+A+3Bx^2+5Bx+2B+3Cx+2C$$
$$7x+4=Ax^2+3Bx^2+2Ax+5Bx+3Cx+A+2B+2C \quad ....(2)\backslash$$

To find A, we put $3x+2=0\Rightarrow 3x=-2\Rightarrow x=\frac{-2}{3}$ in eq.(1), we get

$$7\left(\frac{-2}{3}\right)+4=A\left(\frac{-2}{3}+1\right)^2+B\left(3\left(\frac{-2}{3}\right)+2\right)\left(\frac{-2}{3}+1\right)+C\left(3\left(\frac{-2}{3}\right)+2\right)$$
$$-\frac{14}{3}+4=A\left(\frac{1}{3}\right)^2+B(-2+2)\left(\frac{1}{3}\right)+C(-2+2)$$
$$-\frac{2}{3}=A\left(\frac{1}{9}\right)+B(0)+C(0)$$
$$-\frac{2}{3}=\frac{1}{9}A$$
$$or\ \frac{1}{9}A=-\frac{2}{3}$$
$$A=-\frac{2}{3}\times\frac{9}{1}$$
$$A=-6$$

To find C, we put $(x+1)^2=0\Rightarrow x+1=0\Rightarrow x=-1$ in eq.(1), we get

$$7(-1)+4=A(-1+1)^2+B(3(-1)+2)(-1+1)+C(3(-1)+2)$$
$$-7+4=A(0)^2+B(-3+2)(0)+C(-3+2)$$
$$-3=A(0)+B(0)+C(-1)$$
$$-3=-C$$
$$or-C=-3$$
$$\Rightarrow C=3$$

To find B, equating coefficient of x^2 on both sides of (2), we get

$$A+3B=0$$
$$-6+3B=0$$
$$3B=6$$
$$\Rightarrow B=2$$

Thus required partial fractions are $\frac{-6}{3x+2}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$

Hence, $\frac{7x+4}{(3x+2)(x+1)^2}=\frac{-6}{3x+2}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$

(6) $\frac{1}{(x-1)^2(x+1)}$

Solution:

Let $\frac{1}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$

Multiplying both sides by $(x-1)^2(x+1)$, we get

$$1=A(x-1)(x+1)+B(x+1)+C(x-1)^2 \quad ....(1)$$
$$1=A(x^2-1)+B(x+1)+C(x^2-2x+1)$$
$$1=Ax^2-A+Bx+B+Cx^2-2Cx+C$$
$$1=Ax^2+Cx^2+Bx-2Cx-A+B+C \quad ....(2)$$

To find C, we put $x+1=0\Rightarrow x=-1$ in eq.(1), we get

$$1=A(-1-1)(-1+1)+B(-1+1)+C(-1-1)^2$$
$$1=A(-2)(0)+B(0)+C(-2)^2$$
$$1=A(0)+B(0)+C(4)$$
$$1=4C$$
$$or\ 4C=1$$
$$\Rightarrow C=\frac{1}{4}$$

To find B, we put $(x-1)^2=0\Rightarrow x-1=0\Rightarrow x=1$ in eq.(1), we get

$$1=A(1-1)(1+1)+B(1+1)+C(1-1)^2$$
$$1=A(0)(2)+B(2)+C(0)^2$$
$$1=A(0)+B(2)+C(0)$$
$$1=2B$$
$$or\ 2B=1$$
$$\Rightarrow B=\frac{1}{2}$$

To find A, equating coefficient of x^2 on both sides of (2), we get

$$A+C=0$$
$$A+\frac{1}{4}=0$$
$$\Rightarrow A=-\frac{1}{4}$$

Thus required partial fractions are $\frac{-1/4}{x-1}+\frac{1/2}{(x-1)^2}+\frac{1/4}{x+1}$

Hence, $\frac{1}{(x-1)^2(x+1)}=\frac{-1}{4(x-1)}+\frac{1}{2(x-1)^2}+\frac{1}{4(x+1)}$

(7) $\frac{3x^2+15x+16}{(x+2)^2}$

Solution:

$$=\frac{3x^2+15x+16}{x^2+4x+4}$$

This is an improper fraction so by long division, we have

$$\begin{array}{r} 3 \\ x^2+4x+4\overline{\smash{)}3x^2+15x+16} \\ \underline{\pm3x^2\pm12x\pm12} \\ 3x+4 \end{array}$$

$$\frac{3x^2+15x+16}{x^2+4x+4}=3+\frac{3x+4}{(x+2)^2}$$

Let $\frac{3x+4}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$

Multiplying both sides by $(x+2)^2$, we get

$$3x+4=A(x+2)+B \quad ....(1)$$
$$3x+4=Ax+2A+B \quad ....(2)$$

To find B, we put $(x+2)^2=0\Rightarrow x+2=0\Rightarrow x=-2$ in eq.(1), we get

$$3(-2)+4=A(-2+2)+B$$
$$-6+4=A(0)+B$$
$$-2=B$$
$$or\ B=-2$$

To find A, equating coefficient of x on both sides of (2), we get

$$A=3$$

Thus required partial fractions are $\frac{3}{x+2}+\frac{-2}{(x+2)^2}$

Hence, $\frac{3x^2+15x+16}{(x+2)^2}=3+\frac{3}{x+2}-\frac{2}{(x+2)^2}$

(8) $\dfrac{1}{(x^2-1)(x+1)}$

Solution:

$$=\frac{1}{(x-1)(x+1)(x+1)}$$
$$=\frac{1}{(x-1)(x+1)^2}$$

Let $\dfrac{1}{(x-1)(x+1)^2}=\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$

Multiplying both sides by $(x-1)(x+1)^2$, we get

$1=A(x+1)^2+B(x-1)(x+1)+C(x-1)$ ....(1)

$1=A(x^2+2x+1)+B(x^2-1)+C(x-1)$

$1=Ax^2+2Ax+A+Bx^2-B+Cx-C$

$1=Ax^2+Bx^2+2Ax+Cx+A-B-C$ ....(2)

To find A, we put $x-1=0\Rightarrow x=1$ in eq.(1), we get

$1=A(1+1)^2+B(1-1)(1+1)+C(1-1)$

$1=A(2)^2+B(0)(2)+C(0)$

$1=A(4)+B(0)+C(0)$

$1=4A$

$or\ 4A=1$

$\Rightarrow A=\dfrac{1}{4}$

To find C, we put $(x+1)^2=0\Rightarrow x+1=0\Rightarrow x=-1$ in eq.(1), we get

$1=A(-1+1)^2+B(-1-1)(-1+1)+C(-1-1)$

$1=A(0)^2+B(-2)(0)+C(-2)$

$1=A(0)+B(0)+C(-2)$

$1=-2C$

$or\ -2C=1$

$\Rightarrow C=-\dfrac{1}{2}$

To find B, equating coefficient of x^2 on both sides of (2), we get

$A+B=0$

$\dfrac{1}{4}+B=0$

$\Rightarrow B=-\dfrac{1}{4}$

Thus required partial fractions are $\dfrac{1/4}{x-1}+\dfrac{-1/4}{x+1}+\dfrac{-1/2}{(x+1)^2}$

Hence, $\dfrac{1}{(x^2-1)(x+1)}=\dfrac{1}{4(x-1)}-\dfrac{1}{4(x+1)}-\dfrac{1}{2(x+1)^2}$

Resolution of fraction when D (x) consists of non-repeated irreducible quadratic factors.

Rule III:

If a quadratic factor ($ax^2 + bx + c$) with $a \neq 0$ occur once as a factor of D(x), the partial fraction is of the form $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be found.

Exercise 4.3

Resolve into partial fractions.

(1) $\frac{3x-11}{(x+3)(x^2+1)}$

Solution:

$$\frac{3x-11}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(x+3)(x^2+1)$, we get

$3x-11 = A(x^2+1)+(Bx+C)(x+3)$ ....(1)

$3x-11 = Ax^2+A+Bx^2+3Bx+Cx+3C$

$3x-11 = Ax^2+Bx^2+3Bx+Cx+A+3C$ ....(2)

To find A, we put $x+3=0 \Rightarrow x=-3$ in eq.(1), we get

$3(-3)-11 = A((-3)^2+1)+(B(-3)+C)(-3+3)$

$-9-11 = A(9+1)+(-3B+C)(0)$

$-20 = 10A$

$or\ 10A = -20$

Dividing both sides by '10', we get

$\Rightarrow A = -2$

To find B and C, equating coefficient of x^2 and constant on both sides of eq.(2), we get

$A+B=0$

$-2+B=0$

$\Rightarrow B=2$

And $A+3C=-11$

$-2+3C=-11$

$3C=-11+2$

$3C=-9$

Dividing both sides by '3', we get

$C=-3$

Thus required partial fractions are $\frac{-2}{x+3}+\frac{2x+(-3)}{x^2+1}$

Hence, $\frac{3x-11}{(x+3)(x^2+1)} = \frac{-2}{x+3}+\frac{2x-3}{x^2+1}$

(2) $\frac{3x+7}{(x^2+1)(x+3)}$

Solution:

$$\frac{3x+7}{(x^2+1)(x+3)} = \frac{Ax+B}{x^2+1}+\frac{C}{x+3}$$

Multiplying both sides by $(x^2+1)(x+3)$, we get

$3x+7 = (Ax+B)(x+3)+C(x^2+1)$ ....(1)

$3x+7 = Ax^2+3Ax+Bx+3B+Cx^2+C$

$3x+7 = Ax^2+Cx^2+3Ax+Bx+3B+C$ ....(2)

To find C, we put $x+3=0 \Rightarrow x=-3$ in eq.(1), we get

$3(-3)+7 = (A(-3)+B)(-3+3)+C((-3)^2+1)$

$-9+7 = (-3A+B)(0)+C(9+1)$

$-2 = 10C$

$or\ 10C = -2$

Dividing both sides by '10', we get

$\Rightarrow C = -\frac{1}{5}$

To find A and B, equating coefficient of x^2 and constant on both sides of (2), we get

$A+C=0$

$A+\left(-\frac{1}{5}\right)=0$

$\Rightarrow A=\frac{1}{5}$

And $3B+C=7$

$3B+\left(-\frac{1}{5}\right)=7$

$3B=7+\frac{1}{5}$

$3B=\frac{36}{5}$

$B=\frac{36}{5}\times\frac{1}{3}$

$B=\frac{12}{5}$

Thus required partial fractions are $\frac{\frac{1}{5}x+\frac{12}{5}}{x^2+1}+\frac{\frac{-1}{5}}{x+3}$

Hence, $\frac{3x+7}{(x^2+1)(x+3)} = \frac{x+12}{5(x^2+1)}-\frac{1}{5(x+3)}$

(3) $\frac{1}{(x+1)(x^2+1)}$

Solution:

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(x+1)(x^2+1)$, we get

$1 = A(x^2+1)+(Bx+C)(x+1)$ ....(1)

$1 = Ax^2+A+Bx^2+Bx+Cx+C$

$1 = Ax^2+Bx^2+Bx+Cx+A+C$ ....(2)

To find A, we put $x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$1 = A((-1)^2+1)+(B(-1)+C)(-1+1)$

$1 = A(1+1)+(-B+C)(0)$

$1 = A(2)$

$or\ 2A=1$

Dividing both sides by '2', we get

$\Rightarrow A=\frac{1}{2}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$A+B=0$

$\frac{1}{2}+B=0 \quad \because A=\frac{1}{2}$

$\Rightarrow B=-\frac{1}{2}$

And $A+C=1$

$\frac{1}{2}+C=1 \quad \because A=\frac{1}{2}$

$C=1-\frac{1}{2}$

$C=\frac{1}{2}$

Thus required partial fractions are $\frac{\frac{1}{2}}{x+1}+\frac{\frac{-1}{2}x+\frac{1}{2}}{x^2+1}$

Hence, $\frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)}-\frac{x-1}{2(x^2+1)}$

(4) $\frac{9x-7}{(x+3)(x^2+1)}$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3}+\frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(x+3)(x^2+1)$, we get

$9x-7 = A(x^2+1)+(Bx+C)(x+3)$ ....(1)

$9x-7 = Ax^2+A+Bx^2+3Bx+Cx+3C$

$9x-7 = Ax^2+Bx^2+3Bx+Cx+A+3C$ ....(2)

To find A, we put $x+3=0 \Rightarrow x=-3$ in eq.(1), we get

$9(-3)-7 = A((-3)^2+1)+(B(-3)+C)(-3+3)$

$-27-7 = A(9+1)+(-3B+C)(0)$

$-34 = 10A$

$or\ 10A = -34$

Dividing both sides by '10', we get

$\Rightarrow A=\frac{-34}{10}=-\frac{17}{5}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$$A+B=0$$
$$-\frac{17}{5}+B=0 \quad \because A=-\frac{17}{5}$$
$$\Rightarrow B=\frac{17}{5}$$

And $A+3C=-7$

$$-\frac{17}{5}+3C=-7 \qquad \because A=-\frac{17}{5}$$
$$3C=-7+\frac{17}{5}$$
$$3C=-\frac{18}{5}$$
$$C=-\frac{18}{5}\times\frac{1}{3}$$
$$C=-\frac{6}{5}$$

Thus required partial fractions are $\frac{\frac{-17}{5}}{x+1}+\frac{\frac{17}{5}x-\frac{6}{5}}{x^2+1}$

Hence, $\frac{9x-7}{(x+3)(x^2+1)}=\frac{-17}{5(x+1)}+\frac{17x-6}{5(x^2+1)}$

(5) $\frac{3x+7}{(x+3)(x^2+4)}$

Solution:

Let $\frac{3x+7}{(x+3)(x^2+4)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+4}$

Multiplying both sides by $(x+3)(x^2+4)$, we get

$3x+7=A(x^2+4)+(Bx+C)(x+3) \quad ....(1)$

$3x+7=Ax^2+4A+Bx^2+3Bx+Cx+3C$

$3x+7=Ax^2+Bx^2+3Bx+Cx+4A+3C \quad ....(2)$

To find A, we put $x+3=0\Rightarrow x=-3$ in eq.(1), we get

$3(-3)+7=A((-3)^2+4)+(B(-3)+C)(-3+3)$

$-9+7=A(9+4)+(-3B+C)(0)$

$-2=13A$

or $13A=-2$

Dividing both sides by '13', we get

$\Rightarrow A=-\frac{2}{13}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$$A+B=0$$
$$-\frac{2}{13}+B=0 \quad \because A=-\frac{2}{13}$$
$$\Rightarrow B=\frac{2}{13}$$

And $4A+3C=7$

$$4\left(-\frac{2}{13}\right)+3C=7 \qquad \because A=-\frac{2}{13}$$
$$-\frac{8}{13}+3C=7$$
$$3C=7+\frac{8}{13}$$
$$3C=\frac{99}{13}$$

$$C=\frac{99}{13}\times\frac{1}{3}$$
$$C=\frac{33}{13}$$

Thus required partial fractions are $\frac{\frac{-2}{13}}{x+3}+\frac{\frac{2}{13}x+\frac{33}{13}}{x^2+4}$

Hence, $\frac{3x+7}{(x+3)(x^2+4)}=\frac{-2}{13(x+3)}+\frac{2x+33}{13(x^2+4)}$

(6) $\frac{x^2}{(x+2)(x^2+4)}$

Solution:

Let $\frac{x^2}{(x+2)(x^2+4)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+4}$

Multiplying both sides by $(x+2)(x^2+4)$, we get

$x^2=A(x^2+4)+(Bx+C)(x+2) \quad ....(1)$

$x^2=Ax^2+4A+Bx^2+2Bx+Cx+2C$

$x^2=Ax^2+Bx^2+2Bx+Cx+4A+2C \quad ....(2)$

To find A, we put $x+2=0\Rightarrow x=-2$ in eq.(1), we get

$(-2)^2=A((-2)^2+4)+(B(-2)+C)(-2+2)$

$4=A(4+4)+(-2B+C)(0)$

$4=8A$

or $8A=4$

Dividing both sides by '8', we get

$\Rightarrow A=\frac{4}{8}=\frac{1}{2}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$$A+B=1$$
$$\frac{1}{2}+B=1 \quad \because A=\frac{1}{2}$$
$$B=1-\frac{1}{2}$$
$$\Rightarrow B=\frac{1}{2}$$

And $4A+2C=0$

$$4\left(\frac{1}{2}\right)+2C=0 \qquad \because A=\frac{1}{2}$$
$$2+2C=0$$
$$2C=-2$$
$$\Rightarrow \quad C=-1$$

Thus required partial fractions are $\frac{\frac{1}{2}}{x+2}+\frac{\frac{1}{2}x-1}{x^2+4}$

Hence, $\frac{x^2}{(x+2)(x^2+4)}=\frac{1}{2(x+2)}+\frac{x-2}{2(x^2+4)}$

(7) $\frac{1}{x^3+1}$

Solution:

$$\frac{1}{x^3+1}=\frac{1}{(x)^3+(1)^3}=\frac{1}{(x+1)(x^2-x+1)}$$

Let $\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$

Multiplying both sides by $(x+1)(x^2-x+1)$, we get

$1=A(x^2-x+1)+(Bx+C)(x+1) \quad ....(1)$

$1=Ax^2-Ax+A+Bx^2+Bx+Cx+C$

$1=Ax^2+Bx^2-Ax+Bx+Cx+A+C \quad ....(2)$

To find A, we put $x+1=0\Rightarrow x=-1$ in eq.(1), we get

$1=A((-1)^2-(-1)+1)+(B(-1)+C)(-1+1)$

$1=A(1+1+1)+(-B+C)(0)$

$1=A(3)+(-B+C)(0)$

or $3A=1$

Dividing both sides by '3', we get

$\Rightarrow A=\frac{1}{3}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$$A+B=0$$
$$\frac{1}{3}+B=0 \quad \because A=\frac{1}{3}$$
$$B=-\frac{1}{3}$$

And $A+C=1$

$$\frac{1}{3}+C=1 \qquad \because A=\frac{1}{2}$$
$$C=1-\frac{1}{3}$$
$$C=\frac{2}{3}$$

Thus required partial fractions are $\frac{\frac{1}{3}}{x+1}+\frac{\frac{-1}{3}x+\frac{2}{3}}{x^2-x+1}$

Hence, $\frac{1}{x^3+1}=\frac{1}{3(x+1)}-\frac{x-2}{3(x^2-x+1)}$

$$(8)\ \frac{x^2+1}{x^3+1}$$

Solution:

$$\frac{x^2+1}{x^3+1}=\frac{x^2+1}{(x)^3+(1)^3}=\frac{x^2+1}{(x+1)(x^2-x+1)}$$

Let $$\frac{x^2+1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

Multiplying both sides by $(x+1)(x^2-x+1)$, we get

$x^2+1=A(x^2-x+1)+(Bx+C)(x+1)$ ....(1)

$x^2+1=Ax^2-Ax+A+Bx^2+Bx+Cx+C$

$x^2+1=Ax^2+Bx^2-Ax+Bx+Cx+A+C$ ....(2)

To find A, we put $x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$(-1)^2+1=A[(-1)^2-(-1)+1]+(B(-1)+C)(-1+1)$

$1+1=A(1+1+1)+(-B+C)(0)$

$2=A(3)+(-B+C)(0)$

$2=A(3)$

$or\ 3A=2$

$\Rightarrow A=\frac{2}{3}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$A+B=1$

$\frac{2}{3}+B=1 \quad \because A=\frac{2}{3}$

$B=1-\frac{2}{3}$

$B=\frac{1}{3}$

And $A+C=1$

$\frac{2}{3}+C=1 \quad \because A=\frac{2}{3}$

$C=1-\frac{2}{3}$

$C=\frac{1}{3}$

Thus required partial fractions are $\dfrac{\frac{2}{3}}{x+1}+\dfrac{\frac{1}{3}x+\frac{1}{3}}{x^2-x+1}$

Hence, $\dfrac{x^2+1}{x^3+1}=\dfrac{2}{3(x+1)}+\dfrac{x+1}{3(x^2-x+1)}$

Resolution of a fraction when D (x) has repeated irreducible quadratic factors.

Rule IV:

If a quadratic factor ($ax^2 + bx + c$) with $a \neq 0$, occurs twice in the denominator, the corresponding partial fractions are

$$\frac{Ax+B}{\left(ax^2+bx+c\right)}+\frac{Cx+D}{\left(ax^2+bx+c\right)^2}$$

The constants A, B, C and D are found in the usual way.

Exercise 4.4

Resolve into partial fraction.

(1) $\frac{x^3}{\left(x^2+4\right)^2}$

Solution:

Let $\frac{x^3}{\left(x^2+4\right)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{\left(x^2+4\right)^2}$

Multiplying both sides by $\left(x^2+4\right)^2$, we get

$x^3=\left(Ax+B\right)\left(x^2+4\right)+Cx+D$ ....(1)

$x^3=Ax^3+4Ax+Bx^2+4B+Cx+D$ ....(2)

To find A,B,C and D, equating coefficient of x^3, x^2, x and constant on both sides of eq.(2)

We get

Coefficient of x^3 : $A=1$

Coefficient of x^2 : $B=0$

Coefficient of x : $4A+C=0$ ....(3)

Constant: $4B+D=0$ ....(4)

Put $A=1$ in eq.(3), we get

$4(1)+C=0$

$4+C=0$

$C=-4$

Put $B=0$ in eq.(4), we get

$4(0)+D=0$

$D=0$

Thus required partial fractions are $\frac{(1)x+0}{4}+\frac{(-4)x+0}{\left(x^2+4\right)^2}$

Hence, $\frac{x^3}{\left(x^2+4\right)^2}=\frac{x}{x^2+4}-\frac{4x}{\left(x^2+4\right)^2}$

(2) $\frac{x^4+3x^2+x+1}{\left(x+1\right)\left(x^2+1\right)^2}$

Solution:

Let $\frac{x^4+3x^2+x+1}{\left(x+1\right)\left(x^2+1\right)^2}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$

Multiplying both sides by $\left(x+1\right)\left(x^2+1\right)^2$, we get

$x^4+3x^2+x+1=A\left(x^2+1\right)^2+\left(Bx+C\right)\left(x+1\right)\left(x^2+1\right)+\left(Dx+E\right)\left(x+1\right)$ ....(1)

$x^4+3x^2+x+1=A\left(x^4+2x^2+1\right)+\left(Bx+C\right)\left(x^3+x^2+x+1\right)+Dx^2+Dx+Ex+E$

$x^4+3x^2+x+1=Ax^4+2Ax^2+A+Bx^4+Bx^3+Bx^2+Bx+Cx^3+Cx^2$
$\quad +Cx+C+Dx^2+Dx+Ex+E$

$x^4+3x^2+x+1=Ax^4+Bx^4+Bx^3+Cx^3+2Ax^2+Bx^2+Cx^2$
$\quad +Dx^2+Bx+Cx+Dx+Ex+A+C+E$ ....(2)

To find A, we put $x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$(-1)^4+3(-1)^2+(-1)+1=A\left((-1)^2+1\right)^2$

$1+3-1+1=A(1+1)^2$

$4=A(2)^2$

$4=4A$

$4A=4$

$\Rightarrow A=1$

To find B,C,D and E, equating coefficient of x^4, x^3, x^2 and x on both sides of eq.(2)

We get

Coefficient of x^4 : $4+B=1$ ....(3)

Coefficient of x^3 : $B+C=0$ ....(4)

Coefficient of x^2 : $2A+B+C+D=3$ ....(5)

Coefficient of x : $B+C+D+E=1$ ....(6)

Put $A=1$ in eq.(3), we get

$1+B=1$

$B=1-1$

$B=0$

Put $B=0$ in eq.(4), we get

$0+C=0$

$C=0$

Put A=1, B = 0, C=0 in eq.(5), we get

$2(1)+0+0+D=3$

$2+D=3$

$D=3-2$

$D=1$

Put $B=0, C=0, D=1$ in eq.(6), we get

$0+0+1+E=1$

$1+E=1$

$E=1-1$

$E=0$

Thus required partial fractions are $\frac{1}{x+1}+\frac{(0)x+(0)}{x^2+1}+\frac{(1)x+(0)}{\left(x^2+1\right)^2}$

Hence, $\frac{x^4+3x^2+x+1}{\left(x+1\right)\left(x^2+1\right)^2}=\frac{1}{x+1}+\frac{x}{\left(x^2+1\right)^2}$

(3) $\frac{x^2}{\left(x+1\right)\left(x^2+1\right)^2}$

Solution:

Let $\frac{x^2}{\left(x+1\right)\left(x^2+1\right)^2}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$

Multiplying both sides by $\left(x+1\right)\left(x^2+1\right)^2$, we get

$x^2=A\left(x^2+1\right)^2+\left(Bx+C\right)\left(x^2+1\right)\left(x+1\right)+\left(Dx+E\right)\left(x+1\right)$ ....(1)

$x^2=A\left(x^4+2x^2+1\right)+\left(Bx+C\right)\left(x^3+x^2+x+1\right)$
$\quad +Dx^2+Dx+Ex+E$

$x^2=Ax^4+2Ax^2+A+Bx^4+Bx^3+Bx^2+Bx+Cx^3$
$\quad +Cx^2+Cx+C+Dx^2+Dx+Ex+E$

$x^2=Ax^4+Bx^4+Bx^3+Cx^3+2Ax^2+Bx^2+Cx^2+Dx^2$
$\quad +Bx+Cx+Dx+Ex+A+C+E$ ....(2)

To find A, we put $x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$(-1)^2=A\left((-1)^2+1\right)^2$

$1=A(1+1)^2$

$1=A(2)^2$

$1=4A$

or $4A=1$

$\Rightarrow A=\frac{1}{4}$

To find B,C,D and E, equating coefficient of x^4, x^3, x^2 and x on both sides of eq.(2)

We get

Coefficient of x^4 : $A+B=0$ ....(3)

Coefficient of x^3 : $B+C=0$ ....(4)

Coefficient of x^2 : $2A+B+C+D=1$ ....(5)

Coefficient of x : $B+C+D+E=0$ ....(6)

Put $A=\frac{1}{4}$ in eq.(4), we get

$\frac{1}{4}+B=0$

$B=-\frac{1}{4}$

Put $B=-\frac{1}{4}$ in eq.(4), we get

$-\frac{1}{4}+C=0$

$C=\frac{1}{4}$

Put $A=\frac{1}{4}, B=-\frac{1}{4}, C=\frac{1}{4}$ in eq.(5), we get

$2\left(\frac{1}{4}\right)+\left(-\frac{1}{4}\right)+\left(\frac{1}{4}\right)+D=1$

$$\frac{2}{4}-\frac{1}{4}+\frac{1}{4}+D=1$$
$$\frac{1}{2}+D=1$$
$$D=1-\frac{1}{2}$$
$$D=\frac{1}{2}$$

Put $B=-\frac{1}{4}, C=\frac{1}{4}, D=\frac{1}{2}$ in eq.(6), we get

$$\left(-\frac{1}{4}\right)+\left(\frac{1}{4}\right)+\left(\frac{1}{2}\right)+E=0$$
$$-\frac{1}{4}+\frac{1}{4}+\frac{1}{2}+E=0$$
$$E=-\frac{1}{2}$$

Thus required partial fractions are $\frac{\frac{1}{4}}{x+1}+\frac{\left(\frac{-1}{4}\right)x+\left(\frac{1}{4}\right)}{x^2+1}+\frac{\left(\frac{1}{2}\right)x+\left(\frac{-1}{2}\right)}{\left(x^2+1\right)^2}$

Hence, $\frac{x^2}{(x+1)\left(x^2+1\right)^2}=\frac{1}{4(x+1)}-\frac{x-1}{4\left(x^2+1\right)}+\frac{x-1}{2\left(x^2+1\right)^2}$

(4) $\frac{x^2}{(x-1)\left(x^2+1\right)^2}$

Solution:

Let $\frac{x^2}{(x-1)\left(x^2+1\right)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$

Multiplying both sides by $(x-1)\left(x^2+1\right)^2$, we get

$$x^2=A\left(x^2+1\right)^2+(Bx+C)\left(x^2+1\right)(x-1)+(Dx+E)(x-1) \quad ....(1)$$
$$x^2=A\left(x^4+2x^2+1\right)+(Bx+C)\left(x^3-x^2+x-1\right)$$
$$+Dx^2-Dx+Ex-E$$
$$x^2=Ax^4+2Ax^2+A+Bx^4-Bx^3+Bx^2-Bx+Cx^3$$
$$-Cx^2+Cx-C+Dx^2-Dx+Ex-E$$
$$x^2=Ax^4+Bx^4-Bx^3+Cx^3+2Ax^2+Bx^2-Cx^2+Dx^2$$
$$-Bx+Cx-Dx+Ex+A-C-E \quad ....(2)$$

To find A, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$$(1)^2=A\left((1)^2+1\right)^2$$
$$1=A(1+1)^2$$
$$1=A(2)^2$$
$$1=4A$$
$$or\ 4A=1$$
$$\Rightarrow A=\frac{1}{4}$$

To find B,C,D and E, equating coefficient of x^4, x^3, x^2 and x on both sides of eq.(2)

We get

Coefficient of x^4: $A+B=0$ ....(3)
Coefficient of x^3: $-B+C=0$ ....(4)

Coefficient of x^2: $2A+B-C+D=1$ ....(5)
Coefficient of x: $-B+C-D+E=0$ ....(6)

Put $A=\frac{1}{4}$ in eq.(4), we get

$$\frac{1}{4}+B=0$$
$$B=-\frac{1}{4}$$

Put $B=-\frac{1}{4}$ in eq.(4), we get

$$-\left(-\frac{1}{4}\right)+C=0$$
$$\frac{1}{4}+C=0$$
$$C=-\frac{1}{4}$$

Put $A=\frac{1}{4}, B=-\frac{1}{4}, C=-\frac{1}{4}$ in eq.(5), we get

$$2\left(\frac{1}{4}\right)+\left(-\frac{1}{4}\right)-\left(-\frac{1}{4}\right)+D=1$$
$$\frac{2}{4}-\frac{1}{4}+\frac{1}{4}+D=1$$
$$\frac{1}{2}+D=1$$
$$D=1-\frac{1}{2}$$
$$D=\frac{1}{2}$$

Put $B=-\frac{1}{4}, C=-\frac{1}{4}, D=\frac{1}{2}$ in eq.(6), we get

$$-\left(-\frac{1}{4}\right)+\left(-\frac{1}{4}\right)-\frac{1}{2}+E=0$$
$$\frac{1}{4}-\frac{1}{4}-\frac{1}{2}+E=0$$
$$E=\frac{1}{2}$$

Thus required partial fractions are $\frac{\frac{1}{4}}{x-1}+\frac{\left(\frac{-1}{4}\right)x+\left(\frac{-1}{4}\right)}{x^2+1}+\frac{\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)}{\left(x^2+1\right)^2}$

Hence, $\frac{x^2}{(x-1)\left(x^2+1\right)^2}=\frac{1}{4(x-1)}-\frac{x+1}{4\left(x^2+1\right)}+\frac{x+1}{2\left(x^2+1\right)^2}$

(5) $\frac{x^4}{\left(x^2+2\right)^2}$

Solution:

$$\frac{x^4}{\left(x^2+2\right)^2}=\frac{x^4}{x^4+4x^2+4}$$

This is an improper fraction so by long division, we have

$$\begin{array}{r|l} & \quad 1 \\ \hline x^4+4x^2+4 & x^4 \\ & \underline{\pm x^4 \pm 4x^2 \pm 4} \\ & -4x^2-4 \\ & -\left(4x^2+4\right) \end{array}$$

$$\frac{x^4}{\left(x^2+2\right)^2}=1-\frac{4x^2+4}{\left(x^2+2\right)^2}$$

Let $\frac{4x^2+4}{\left(x^2+2\right)^2}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{\left(x^2+2\right)^2}$

Multiplying both sides by $\left(x^2+2\right)^2$, we get

$$4x^2+4=(Ax+B)\left(x^2+2\right)+(Cx+D)$$
$$4x^2+4=Ax^3+2Ax+Bx^2+2B+Cx+D$$
$$4x^2+4=Ax^3+Bx^2+2Ax+Cx+2B+D \quad ....(1)$$

To find A,B,C and D, equating coefficient of x^3, x^2, x and constant on both sides of eq.(1), We get

Coefficient of x^3: $A=0$
Coefficient of x^2: $B=4$
Coefficient of x: $2A+C=0$ ....(2)
Constant: $2B+D=4$ ....(3)

Put $A=0$ in eq.(2), we get

$$2(0)+C=0$$

$$C=0$$

Put $B=4$ in eq.(3), we get

$$2(4)+D=4$$
$$8+D=4$$
$$D=4-8$$
$$D=-4$$

Thus required partial fractions are $\frac{(0)x+(4)}{x^2+2}+\frac{(0)x+(-4)}{\left(x^2+2\right)^2}$

Hence, $\frac{x^4}{\left(x^2+2\right)^2}=1-\left[\frac{4}{x^2+2}-\frac{4}{\left(x^2+2\right)^2}\right]$

$$=1-\frac{4}{x^2+2}+\frac{4}{\left(x^2+2\right)^2}$$

(6) $\frac{x^5}{\left(x^2+1\right)^2}$

Solution:

$$\frac{x^5}{\left(x^2+1\right)^2}=\frac{x^5}{x^4+2x^2+1}$$

This is an improper fraction so by long division, we have

$$\begin{array}{r} x \\ x^4+2x^2+1 \overline{\big)\, x^5 \qquad} \\ \underline{\pm x^5 \pm 2x^3 \pm x} \\ -2x^3-x \\ -\left(2x^3+x\right) \end{array}$$

$$\frac{x^5}{\left(x^2+1\right)^2}=x-\frac{2x^3+x}{\left(x^2+1\right)^2}$$

Let $\dfrac{2x^3+x}{\left(x^2+1\right)^2}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{\left(x^2+1\right)^2}$

Multiplying both sides by $\left(x^2+1\right)^2$, we get

$2x^3+x=(Ax+B)\left(x^2+1\right)+(Cx+D)$

$2x^3+x=Ax^3+Ax+Bx^2+B+Cx+D$

$2x^3+x=Ax^3+Bx^2+Ax+Cx+B+D$ ....(1)

To find A,B,C and D, equating coefficient of x^3, x^2, x and constant on both sides of eq.(1), We get

Coefficient of x^3 : $A=2$

Coefficient of x^2 : $B=0$

Coefficient of x : $A+C=1$ ....(2)

Constant: $B+D=0$ ....(3)

Put $A=2$ in eq.(2), we get

$A+C=1$

$2+C=1$

$C=1-2$

$C=-1$

Put B $=0$ in eq.(3), we get

$0+D=0$

$D=0$

Thus required partial fractions are $\dfrac{2x+0}{x^2+1}+\dfrac{(-1)x+0}{\left(x^2+1\right)^2}$

Hence, $\dfrac{x^5}{\left(x^2+1\right)^2}=x-\left[\dfrac{2x}{x^2+1}+\dfrac{-x}{\left(x^2+1\right)^2}\right]$

Miscellaneous Exercise 4

Q1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) **The identity $(5x + 4)^2 = 25x^2 + 40x + 16$ is true for**

(a) one value of x **(b)** two values of x
(c) all values of x **(d)** none of these

(ii) **A function of the form $f(x) = \frac{N(x)}{D(x)}$, with $D(x) \neq 0$, where N(x) and D(x) are polynomials in x is called**

(a) an identity **(b)** an equation
(c) a fraction **(d)** none of these

(iii) **A fraction in which the degree of the numerator is greater or equal the degree of denominator is called:**

(a) a proper fraction **(b)** an improper fraction
(c) an equation **(d)** algebraic relation

(iv) **A fraction in which the degree of numerator is less than the degree of the denominator is called**

(a) an equation **(b)** an improper fraction
(c) an identity **(d)** a proper fraction

(v) $\frac{2x+1}{(x+1)(x-1)}$ is

(a) an improper fraction **(b)** an equation
(c) a proper fraction **(d)** none of these

(vi) $(x + 3)^2 = x^2 + 6x + 9$ is:

(a) a linear equation **(b)** an equation
(c) an identity **(d)** none of these

(vii) $\frac{x^3+1}{(x-1)(x+2)}$ is

(a) a proper fraction **(b)** an improper fraction
(c) an identity **(d)** a constant term

(viii) Partial fractions of $\frac{x-2}{(x-1)(x+2)}$ are of the form

(a) $\frac{A}{x-1}+\frac{B}{x+2}$ **(b)** $\frac{Ax}{x-1}+\frac{B}{x+2}$
(c) $\frac{Ax}{x-1}+\frac{Bx+C}{x+2}$ **(d)** $\frac{Ax+B}{x-1}+\frac{C}{x+2}$

(ix) Partial fractions of $\frac{x+2}{(x+1)(x^2+2)}$ are of the form

(a) $\frac{A}{x+1}+\frac{B}{x^2+2}$ **(b)** $\frac{A}{x+1}+\frac{Bx+C}{x^2+2}$
(c) $\frac{Ax+B}{x+1}+\frac{C}{x^2+2}$ **(d)** $\frac{A}{x+1}+\frac{Bx}{x^2+2}$

(x) Partial fractions of $\frac{x^2+1}{(x+1)(x-1)}$ are of the form

(a) $\frac{A}{x+1}+\frac{B}{x-1}$ **(b)** $1+\frac{A}{x+1}+\frac{Bx+C}{x-1}$
(c) $1+\frac{A}{x+1}+\frac{B}{x-1}$ **(d)** $\frac{Ax+B}{(x+1)}+\frac{C}{x-1}$

Answers:

(i)	c	**(ii)**	c	**(iii)**	b	**(iv)**	d	**(v)**	c
(vi)	c	**(vii)**	b	**(viii)**	a	**(ix)**	b	**(x)**	c

Q2. Write short answers of the following questions.

(i) Define a rational fraction.

Ans: Rational Fraction

An expression of the form $\frac{N(x)}{D(x)}$ with $D(x) \neq 0$ is called a rational fraction.

(ii) What is a proper fraction?

Ans: Proper Fraction

A rational fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ is called a proper fraction if degree of the polynomial N(x) < degree of the polynomial D (x).

(iii) What is an improper fraction?

Ans: Improper Fraction

A rational fraction $\frac{N(x)}{D(x)}$ · with $D(x) \neq 0$ is called an improper fraction if degree of the polynomial N(x) is greater or equal to the degree of D (x).

(iv) What are partial fractions?

Ans: Partial Fraction

A single fraction written in the forms of its components is said to be resolved into partial fraction.

(v) How can we make partial fractions of $\frac{x-2}{(x+2)(x+3)}$?

Ans: It is written as: $\frac{x-2}{(x+2)(x+3)}=\frac{A}{x+2}+\frac{B}{x+3}$, then values of A and B are found as below.

$$\text{Let}\quad \frac{x-2}{(x+2)(x+3)}=\frac{A}{x+2}+\frac{B}{x+3}\;\ldots\ldots(i)$$

Multiplying both sides by $(x+2)(x+3)$, we get

$$x-2=A(x+3)+B(x+2)\qquad\ldots\ldots(ii)$$

$$\begin{aligned}&\text{Put } x+2=0 \Rightarrow x=-2 \text{ in } (ii)\\&-2-2=A(-2+3)+B(-2+2)\\&-4=A(1)\\&\boxed{A=-4}\end{aligned}$$

$$\begin{aligned}&\text{Put } x+3=0 \Rightarrow x=-3 \text{ in eq.}(ii)\\&-3-2=A(-3+3)+B(-3+2)\\&-5=B(-1)\\&\boxed{B=5}\end{aligned}$$

Putting values of A,B in (i)

$$\frac{x-2}{(x+2)(x+3)}=\frac{-4}{x+2}+\frac{5}{x+3}$$

(vi) Resolve $\frac{1}{x^2-1}$ into partial fractions.

Ans:

$$\frac{1}{x^2-1}$$

$$=\frac{1}{(x+1)(x-1)}$$

$$\text{Let}\quad \frac{1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}\;\ldots\ldots(i)$$

Multiplying both sides by $(x+1)(x-1)$, we get

$$1=A(x-1)+B(x+1) \qquad \text{........}(ii)$$

$$\begin{aligned}&Put\ x+1=0 \Rightarrow x=-1 \text{ in } (ii)\\&1=A(-1-1)+B(-1+1)\\&1=A(-2)+B(0)\\&1=-2A\\&\boxed{A=-\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}&\text{Put } x-1=0 \Rightarrow x=1 \text{ in eq.}(ii)\\&1=A(1-1)+B(1+1)\\&1=A(0)+B(2)\\&1=2B\\&\boxed{B=\frac{1}{2}}\end{aligned}$$

Putting values of A,B in (i)

$$\begin{aligned}\frac{1}{(x+1)(x-1)}&=\frac{-1}{2(x+1)}+\frac{1}{2(x-1)}\\&=\frac{1}{2(x-1)}-\frac{1}{2(x+1)}(\text{Partial Fractions})\end{aligned}$$

(vii) Find partial fractions of $\frac{3}{(x+1)(x-1)}$.

Ans:

$$Let \quad \frac{3}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1} \quad \text{.......}(i)$$

Multiplying both sides by $(x+1)(x-1)$, we get

$$3=A(x-1)+B(x+1) \qquad \text{........}(ii)$$

$$\begin{aligned}&Put\ x-1=0 \Rightarrow x=1 \text{ in } (ii)\\&3=A(1-1)+B(1+1)\\&3=2B\\&2B=3\\&\boxed{B=\frac{3}{2}}\end{aligned}$$

$$\begin{aligned}&\text{Put } x+1=0 \Rightarrow x=-1 \text{ in eq.}(ii)\\&3=A(-1-1)+B(-1+1)\\&3=-2A\\&-2A=3\\&\boxed{A=-\frac{3}{2}}\end{aligned}$$

Putting values of A,B in (i)

$$\begin{aligned}\frac{3}{(x+1)(x-1)}&=\frac{-3}{2(x+1)}+\frac{3}{2(x-1)}\\&=\frac{3}{2(x-1)}-\frac{3}{2(x+1)}\\&=\frac{3}{2}\left[\frac{1}{x-1}-\frac{1}{x+1}\right]\end{aligned}$$

(viii) Resolve $\frac{x}{(x-3)^2}$ into partial fractions.

Ans:

$$Let \quad \frac{x}{(x-3)^2}=\frac{A}{x-3}+\frac{B}{(x-3)^2} \quad \text{.......}(i)$$

Multiplying both sides by $(x-3)^2$, we get

$$x=A(x-3)+B \qquad \text{........}(ii)$$

$$\begin{aligned}&Put\ x-3=0 \Rightarrow x=3 \text{ in } (ii)\\&3=A(3-3)+B\\&3=B\\&B=3\end{aligned}$$

Comparing coefficients of x

$$\begin{aligned}&1=A\\&A=1\end{aligned}$$

Putting values of A,B in (i)

$$\frac{x}{(x-3)^2}=\frac{1}{x-3}+\frac{3}{(x-3)^2}$$

(ix) How we can make the partial fractions of $\frac{x}{(x+a)(x-a)}$?

Ans:

$$Let \quad \frac{x}{(x+a)(x-a)}=\frac{A}{(x+a)}+\frac{B}{(x-a)} \quad \text{.......}(i)$$

Multiplying by $(x+a)(x-a)$, we get

$$x=A(x-a)+B(x+a) \qquad \text{........}(ii)$$

$$\begin{aligned}&Put\ x-a=0 \Rightarrow x=a \text{ in } (ii)\\&a=A(a-a)+B(a+a)\\&a=B(2a)\end{aligned}$$

$$\begin{aligned}&A=\frac{a}{2a}\\&\boxed{B=\frac{1}{2}}\end{aligned}$$

$$Put\ x+a=0 \Rightarrow x=-a \text{ in } (ii)$$

$$\begin{aligned}&-a=A(-a-a)+B(-a+a)\\&-a=A(-2a)\\&A=\frac{-a}{-2a}\\&\boxed{A=\frac{1}{2}}\end{aligned}$$

Putting values of A,B in (i)

$$\begin{aligned}\frac{x}{(x+a)(x-a)}&=\frac{1}{2(x+a)}+\frac{1}{2(x-a)}\\&=\frac{1}{2}\left[\frac{1}{x+a}+\frac{1}{x-a}\right](\text{Partial Fraction})\end{aligned}$$

(x) Whether $(x+3)^2=x^2+6x+9$ is an identity?

Ans:

An identity is an equation which is satisfied by all the values of the variables involved.

$$(x+3)^2=x^2+6x+9 \quad \text{.......}(i)$$

$$\begin{aligned}&Put\ x=7 \text{ in } (i)\\&(7+3)^2=(7)^2+6(7)+9\\&100=100\end{aligned}$$

$$\begin{aligned}&Put\ x=-7 \text{ in } (i)\\&(-7+3)^2=(-7)^2+6(-7)+9\\&(-4)^2=49-42+9\\&16=58-42\\&16=16\end{aligned}$$

Yes, this is an identity.
It is true for every value of x.

Summary

✓ A **fraction** is an indicated quotient of two numbers or algebraic expressions.

✓ An expression of the form $\frac{N(x)}{D(x)}$, with D(x) ≠ 0 and N(x) and D(x) are polynomials in x with real coefficients, is called a **rational fraction**. Every fractional expression can be expressed as a quotient of two polynomials.

✓ A rational fraction $\frac{N(x)}{D(x)}$, with D (x) ≠ 0 is called a **proper fraction** if degree of the polynomial N(x) in the numerator is less than the degree of the polynomial D(x), in the denominator.

✓ A rational fraction $\frac{N(x)}{D(x)}$, with D (x) ≠ 0 is called an **improper fraction** if degree of the polynomial N(x) is greater or equal to the degree of the polynomial D(x).

✓ **Partial fractions:** Decomposition of resultant fraction $\frac{N(x)}{D(x)}$, with D (x) ≠ 0, when

(a) D(x) consists of non-repeated linear factors.
(b) D(x) consists of repeated linear factors.
(c) D(x) consists of non-repeated irreducible quadratic factors.
(d) D(x) consists of repeated irreducible quadratic factors.