بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ

(In the Name of Allah, the Most Merciful, the Most Compassionate.)

MATHEMATICS



PECTAA
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PUNJAB EDUCATION, CURRICULUM,
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Authors:

- **Mazhar Hussain, Associate Professor,**
  Govt. Islamia Graduate College, Railway Road Lahore.
- **Naveed Akhter, Associate professor**
  Govt. Graduate College of Science Wahdat Road, Lahore.
- **Javed Ali, Assistant Professor of Mathematics**
  Government Graduate College of Science, Wahdat Road, Lahore.
- **Ghulam Murtaza, SSS (Mathematics)**
  Govt. Pilot Higher Secondary School, Wahadat Road, Lahore.
- **Mr. Majid Hameed, Master Trainer (Mathematics)**
  Punjab Education Foundation, Lahore.
- **Dr. Taimoor Iqbal, Assistant Professor, UET, Lahore.**

External Review Committee

- **Sher Muhammad Khan, Assistant Professor of Mathematics**
  Govt. Associate College Shalimar Town Lahore.
- **Mr. Azmat Ali, Lecturer in Mathematics at Govt.**
  Government Shalimar Graduate College Lahore.
- **Mr. Abdul Quddos Akhter, Assistant Professor of Mathematics**
  Government Govt. College Township, Lahore.
- **Dr. Abdul Raheem, Assistant Professor of Mathematics**
  Government Graduate College Asghar Mall Rawalpindi.
- **Mr. Khalil Ahmad, Assistant Director.**
  National Book Foundation, Ministry of Federal Education and Professional Training.

Editors

- **Dr. Muhammad Danish Ikram, Assistant professor**
  Govt Shalimar Graduate College, Lahore.
- **Dr. Dr. Malik Anjum Javed, Principal,**
  Govt. Islamia Graduate College Railway Road, Lahore.
- **Dr. Irfan Ahmad Aslam,**
  Subject Specialist (Maths), PEF.

Supervised by: **Mr. Arshad Mahmood Awan,** SSS (Mathematics), PECTAA. **Dr. Muhammad Adnan Bashir** SS (Mathematics), PECTAA. **Mrs. Madeha Mahmood** SS(Statistics), PECTAA.

Director (Curriculum and Compliance): Aamir Raiz **Incharge Art Cell:** Aisha Sadiq

Deputy Director (Compliance Sciences): Syed Saghir-ul-Hassnain Tirmizi

Composed by: Kamran Afzal Butt, Atif Majeed **Designed and Illustrated by:** Kamran Afzal Butt, Atif Majeed

Unit 01 Graphical Representation of Functions

INTRODUCTION

This unit introduces students to the fundamental ideas of graphical representation of functions, enabling them to visualize and interpret mathematical relationships effectively. Students begin by learning how to draw graphs using factor form and how to predict equations from given graphs, particularly for quadratic functions. The unit further develops understanding by classifying functions into algebraic and transcendental types, followed by a detailed study of important transcendental functions such as trigonometric, inverse trigonometric, hyperbolic, logarithmic, and exponential functions. Emphasis is placed on understanding logarithms, their laws, and their applications in real-life contexts like growth, decay, and sound intensity. Students also explore exponential functions and their applications, including compound interest, and learn to solve related equations and inequalities. The graphs of all said functions will be discussed. In particular, graphical behaviors of exponential and logarithmic functions will be studied in details. In addition, the unit covers graph of inverse functions, and graphical transformations including shifts and scaling. Overall, this unit equips students with both analytical and graphical skills necessary for higher-level mathematics and practical problem solving.

1.1 Graph of Quadratic Function Using Factors

We have already sketched the graph of the quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ in Grade-XI. We know that its graph is a parabola, which has a vertex and opens either upward or downward depending on the value of a.

Now, we will learn how to sketch its graph when it is expressed in factorized form as a product of linear factors. This approach will also help you in higher classes to sketch the graphs of higher-degree polynomials that can be factorized into linear factors.

Method to Sketch the Graph

Suppose, after factorizing $y = f(x) = ax^2 + bx + c$, we obtain $y = a(x-h)(x-k)$, where $h \leq k$. To sketch the graph, follow the given steps:

Case-I: If $h < k$

Step-1: The graph will cut the x-axis at $x = h$ and $x = k$.

Step-2: Find value of y at $x = \dfrac{h+k}{2}$.

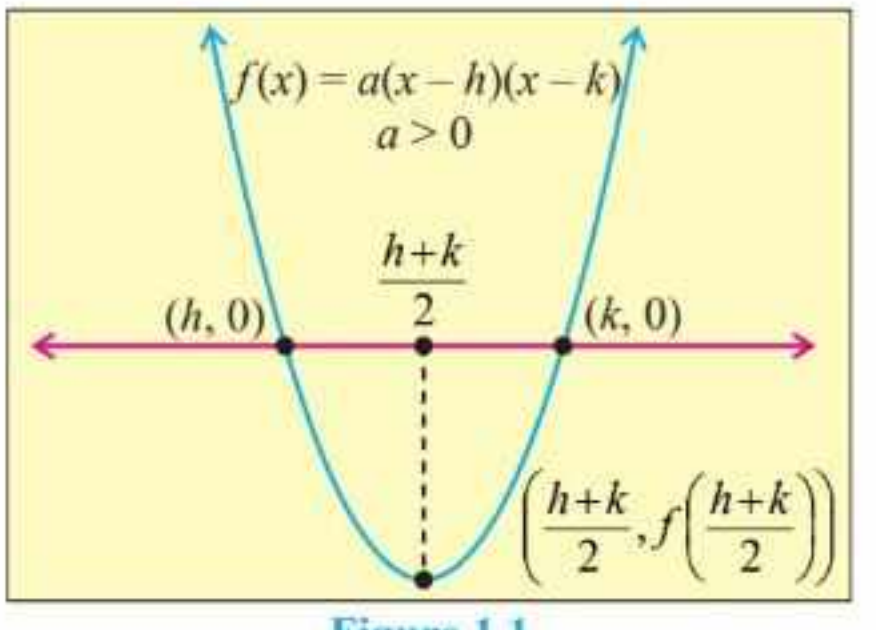


Figure 1.1

The point $\left(\frac{h+k}{2}, f\left(\frac{h+k}{2}\right)\right)$ represents the vertex of the parabola (see Figure 1.1).

Step-3: Determine the behaviour of y as $x \to \pm\infty$.

- If $a > 0$, then $y \to +\infty$.
- If $a < 0$, then $y \to -\infty$.

This statement means the graph behaves the **same way on both ends** (left and right).

Challenge!

If $y = a(x-h_1)(x-h_2)(x-h_3)$, then what will be the behaviour of y as $x \to -\infty$ and $x \to +\infty$

Step-4: Find the y-intercept:

Put $x = 0$ in the function $y = a(x-h)(x-k)$, the graph will cut the y-axis at $y = ahk$.

Case-II: If $h = k$, then the function becomes $y = a(x-h)^2$.

To sketch the graph, follow these steps:

Step-1: The graph **touches the x-axis** at $x = h$, therefore $(h, 0)$ is the vertex (see Figure 1.2).

Step-2: Then follow Steps 3 and 4 of Case-I to complete the sketch of the graph.

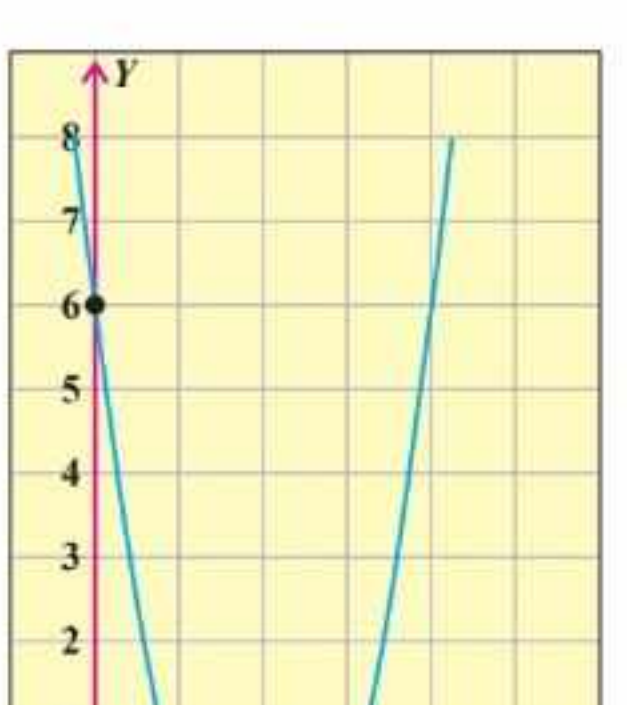


Figure 1.2

Note: Following Steps 1, 3, and 4, and by taking some points between the two x-intercepts, we can draw the graph of a polynomial of higher degree that can be factorized into linear factors.

Recall!

$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n$ is a polynomial of degree n, where n is a non-negative integer and $a_0 \neq 0$.

Example 1 Sketch the graph of $y = 2(x-1)(x-3)$.

Solution: To find the x-intercepts, set each factor equal to zero:

$$x - 1 = 0 \text{ and } x - 3 = 0$$
$$x = 1 \text{ and } x = 3$$

Hence, the graph meets the x-axis at the points (1, 0) and (3, 0).

To find y component of the vertex, put $x = \frac{1+3}{2} = 2$ into the equation:

$$y = 2(2-1)(2-3) = -2$$

So, the vertex of the graph is $(2, -2)$.

Since $a = 2 > 0$, so $y \to +\infty$ as $x \to -\infty$ or $x \to +\infty$.

The graph cuts the y-axis at $x = 0$ and $y = 2(-1)(-3) = 6$

So, the y-intercept is (0, 6).

Using the intercepts and the vertex, we can sketch the graph of the parabola as shown in Figure 1.3.

Figure 1.3

Example 2 Sketch the graph of $y = -4x^2 - 8x - 4$.

Solution: First, we factorize $y = -4x^2 - 8x - 4$.

$$y = -4(x^2 + 2x + 1)$$

$$y = -4(x + 1)^2$$To find the x-intercepts, set $x + 1$ equal to zero:

$$x + 1 = 0 \quad \Rightarrow \quad x = -1$$

Hence, the graph meets the x-axis at the point $(-1, 0)$.

Since the function is of the form $a(x-h)^2$, that is, $y = -4(x + 1)^2$, so $(-1, 0)$ is its vertex.

Since, $a = -4 < 0$, so $y \to -\infty$ as $x \to -\infty$ or $x \to +\infty$

The graph cuts the y-axis at $x = 0$:

$$y = -4(0 + 1)^2 = -4$$

So, the y-intercept is $(0, -4)$.

The sketch of the graph is shown in Figure 1.4.

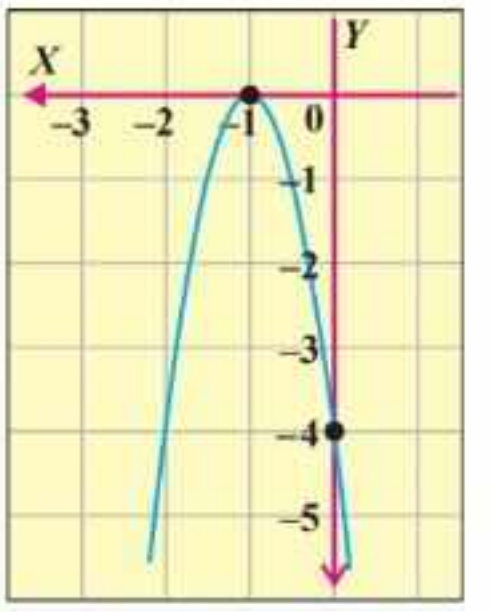


Figure 1.4

1.2 Predicting a Quadratic Function from its Graph

A quadratic function $f(x) = ax^2 + bx + c$ can be determined from its graph by using the **factor form** if the x-intercepts and one additional point on the curve are known.

If the graph crosses the x-axis at $x = h$ and $x = k$, then the function can be written as:

$$f(x) = a(x - h)(x - k)$$

Here, a is a constant that determines the shape and direction of the parabola. To find a, substitute the coordinates of a third known point into the equation.

Example 3 Find the function whose graph is shown in Figure 1.5.

Solution: From figure it is clear that x-intercepts of the graph are $h = -1$ and $k = 1$.

Therefore, the function can be written as:

$$y = a(x - (-1))(x - 1) = a(x + 1)(x - 1) \qquad \ldots(1)$$

To find the value of a, substitute the point $(0, 4)$ into equation (1):

$$4 = a(0 + 1)(0 - 1)$$

$$4 = -a \quad \Rightarrow \quad a = -4$$

Substituting the value of a into the equation (1):

$$y = -4(x + 1)(x - 1)$$

$$= -4(x^2 - 1)$$

$$y = -4x^2 + 4$$

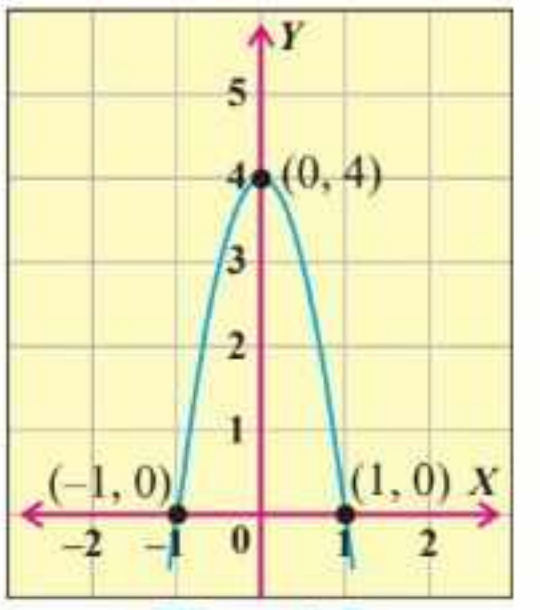


Figure 1.5

Example 4 Find an equation of the graph whose x-intercepts are $x = 1$ and $x = 2$ and passes through the point$(3, 4)$.

Solution: Since the x-intercepts are $x = 1$ and $x = 2$, the function can be written in factorized form as:

$$y \quad = a(x - 1)(x - 2) \qquad \ldots (1)$$

To find a, use the point (3, 4):

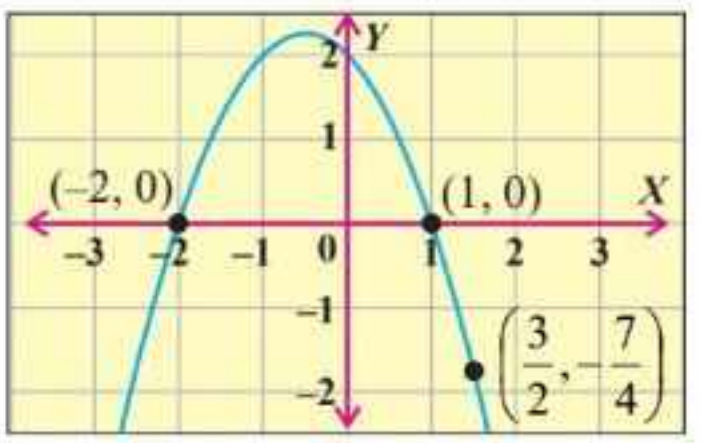
$$4 = a(3-1)(3-2) = 2a \quad \Rightarrow \quad a = \frac{4}{2} = 2$$

Substituting the value of a into the equation (1):

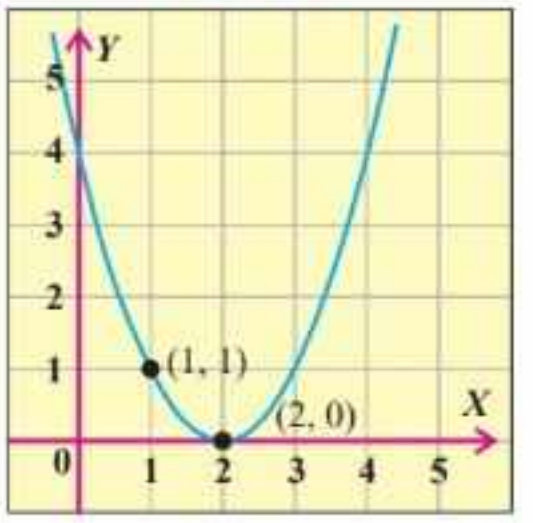
$$y = 2(x-1)(x-2)$$
$$y = 2(x^2 - 3x + 2) = 2x^2 - 6x + 4$$

EXERCISE 1.1

1. Sketch the graph of the following functions using the method of factorization.
   (i) $y = 3(x+1)(x-1)$ (ii) $f(x) = -2(x-1)(x-2)$
   (iii) $y = (2x-1)(2x+1)$ (iv) $y = 2x^2 + x - 3$
   (v) $f(x) = x^2 + 2x + 1$ (vi) $y = 3(x-1)^2$
2. Find the functions whose graphs are shown in the following figures.
   (i)



   (ii)



3. Find an equation of the graph whose x-intercepts are $x = -1$ and $x = -2$ and which passes through the point (–3, 2).
4. Find an equation of the graph which touches x-axis at $x = -1$ and cuts y-axis at $y = -2$.
5. Find an equation of the function whose graph cuts x-axis at $x = -1$ and $x = 1$, and whose vertex is at (0, –2).

1.3 Algebraic and Transcendental Functions

Algebraic and transcendental functions are two fundamental classes of functions studied in mathematics. Together, they play a vital role in modeling and understanding a wide range of real-world phenomena. In this section, we focus on the fundamental transcendental functions, namely exponential and logarithmic functions.

(i) Algebraic Functions

A function $y = f(x)$ is **algebraic** if it satisfies a polynomial equation of the form

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \cdots + P_1(x)y + P_0(x) = 0$$

where each $P_i(x)$ is a polynomial in x for $i = 0, 1, 2, \cdots, n$ and atleast one $P_i(x)$ is not identically zero for $i = 1, 2, 3, \ldots, n$.

In simple words, a function $y = f(x)$ is algebraic if it is obtained from x using a finite number of algebraic operations such as addition, subtraction, multiplication, division, and n^{th} roots.

$f(x) = 2x^2 - 3x$, $g(x) = \dfrac{x^2+1}{x-3}$, $x \neq 3$, $h(x) = (x+1)^{\frac{3}{2}}$ and $k(x) = \sqrt{x+1}$ are examples of algebraic functions.

(ii) Transcendental Functions

A function is called a **transcendental function** if it is not algebraic.

$y = e^x$, $y = 2^x$, $y = \ln x$, $y = \sin x$, $y = \sin^{-1} x$, $y = \sin x + \ln x$, $y = e^{x^2}$ are examples of transcendental functions.

1.3.1 Fundamental Transcendental Functions

Fundamental transcendental functions are the commonly used standard transcendental functions like $y = a^x$ $(a > 0, a \neq 1)$, $y = \ln x$, $y = \sin x$, $y = \sin^{-1} x$, which form the base for more complex transcendental functions.

1.3.2 Non-Fundamental Transcendental Functions

Functions formed by combining transcendental functions with algebraic functions, or by applying transformations and composition, are non-fundamental transcendental functions.

$y = \sin x + \ln x$, $y = e^{x^2}$, $y = x \sin x$, $y = \ln 2x$ are examples of functions which are not fundamental transcendental functions. On the other hand, $y = 10^{2x}$ is a standard transcental function, because, it can be written as $y = 100^x$.

Example 5 Given the following functions:

$$f(x) = \sqrt{x+1}, g(x) = e^x, h(x) = \frac{x+1}{x^2-2x+1}, q(x) = \ln 2x, r(x) = \sin x$$

Classify each function as algebraic or transcendental.

Solution: The following are algebraic functions:

$$f(x) = \sqrt{x+1},\ h(x) = \frac{x+1}{x^2-2x+1}$$

The following are transcendental functions:

$$g(x) = e^x,\ q(x) = \ln 2x,\ r(x) = \sin x$$

Example 6 Given the following functions:

$$f(x) = \cos 2x + 1,\ g(x) = 3^x,\ h(x) = \tan^{-1} x^2,\ q(x) = \ln x + e^x,\ r(x) = \tan x$$

Classify each function as fundamental or non-fundamental transcendental functions.

Solution: The following are fundamental transcendental functions:

$$g(x) = 3^x,\ r(x) = \tan x$$

The following are non-fundamental transcendental functions:

$f(x) = \cos 2x + 1$, $h(x) = \tan^{-1}(x^2)$, $q(x) = \ln x + e^x$

1.4 Exponential Functions

The function defined by $f(x) = a^x$ for all $x \in \mathbb{R}$, where $a > 0$ and $a \neq 1$ is called an **exponential function**. The number a is called the **base** of the exponential function.

The function $f(x) = e^x$ is called the **natural exponential function**, where e is a number

defined by $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ and its approximate value is $e \approx 2.71828$.

The exponential functions $f(x) = e^x$ and $g(x) = 2^x$ represent exponential growth, while the function $f(x) = \left(\frac{1}{2}\right)^x$ represents exponential decay.

1.4.1 Graph of Exponential Function

Before sketching the graph of an exponential function, we study its essential properties that determine its overall shape and behaviour. These include the domain and range, intercepts, injectivity, monotonicity, and asymptotic behaviour. Understanding these features help us to predict the graph without plotting many points. After this analysis, we use a few representative values to draw a smooth and accurate sketch of the function.

1.4.2 Properties of the Exponential Function

Consider the exponential function:

$$y = a^x \ (a > 0, a \neq 1) \qquad \ldots(1)$$

(i) Domain, Continuity, Positivity and Range of Exponential Function

Domain: The exponential function is defined for all real values of x. We can substitute any real number (positive, negative, or zero) in place of x, and the expression a^x is always meaningful. Hence, the domain of the exponential function is $]-\infty, \infty[$.

Continuity: Continuity at a point $c \in]-\infty, \infty[$ is shown using the equation:

$$\lim_{x \to c} a^x = a^c$$

This shows that the exponential function is continuous on the whole domain $]-\infty, \infty[$. Thus, graph of exponential function has no jump or hole.

Positivity: If $x = n$, a positive integer, then $a^n = a \cdot a \cdot \ldots \cdot a$ (n times).

Since, $a > 0$, product of positive numbers is positive. Thus, $a^n > 0$ also, $a^{-n} = \frac{1}{a^n}$.

Since, $a^n > 0$, its reciprocal is also positive. So, $a^{-n} > 0$.

Moreover, $a^0 = 1 > 0$.

If x is rational, then $x = \frac{p}{q}$, where p is an integer and q is a natural number. Thus:

$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

From integer case, $a^p > 0$. Since, the q^{th} root of a positive number is also positive, we have $a^{\frac{p}{q}} > 0$.

Thus, $a^x > 0$ for all rational numbers x. For irrational values of x, we accept that a^x is also positive. However, a rigorous proof of this fact is beyond the scope of this level.

Hence, $y = a^x > 0$ for all real numbers x. It follows that the graph of exponential function always lies above the x-axis.

Range: By observing the following facts:

1. $\lim_{x\to\infty} a^x = \infty$ and $\lim_{x\to-\infty} a^x = 0^+$ if $a > 1$.
2. $\lim_{x\to-\infty} a^x = \infty$ and $\lim_{x\to\infty} a^x = 0^+$ if $0 < a < 1$.
3. $y = a^x$ is continuous on $]-\infty, \infty[$.
4. $y = a^x > 0$ for all real numbers x.

We conclude that, the range of the exponential function (1) is $]0, \infty[$.

(ii) Intercepts of Exponential Function

***y*-intercept:** To find y-intercept, put $x = 0$ in equation (1):

$$y = a^0 = 1$$

So, the exponential function $y = a^x$ cuts the y-axis at the point (0, 1).

***x*-intercept:** To find x-intercept, put $y = 0$ in equation (1):

$$a^x = 0$$

But, $a^x > 0$ for all real x. It follows that there is no x-intercept.

Strictly Increasing and Strictly Decreasing Functions:

Let $y = f(x)$ be a real-valued function and let $x_1, x_2 \in D_f$ be any two points such that $x_1 < x_2$. The function f is called:

(i) **Strictly increasing** if $f(x_1) < f(x_2)$. In this case, the graph of the function rises as x moves from left to right along the x-axis.

(ii) **Strictly decreasing** if $f(x_1) > f(x_2)$. In this case, the graph of the function falls as x moves from left to right along the x-axis.

For Example: Consider the function $y = x^2$.

This function is strictly decreasing on $]-\infty, 0[$ and strictly increasing on $]0, \infty[$.

(iii) Exponential Function is Strictly Increasing or Strictly Decreasing

Let $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

Case-1: If $a > 1$

Let $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$. Then

$$a^{x_2} = a^{x_1} \cdot a^{x_2 - x_1}$$

Since $x_2 - x_1 > 0$ and $a > 1$, we have $a^{x_2 - x_1} > 1$. So:

$$a^{x_2} = a^{x_1} \cdot a^{x_2 - x_1} \Rightarrow a^{x_2} > a^{x_1}$$

Hence, $x_1 < x_2 \Rightarrow a^{x_1} < a^{x_2}$

Therefore, $y = a^x$ is **strictly increasing** when $a > 1$.

Case-2: If $0 < a < 1$

Let $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$. Then

$$a^{x_2} = a^{x_1} \cdot a^{x_2 - x_1}$$

Since $0 < a < 1$ and $x_2 - x_1 > 0$, we have $a^{x_2 - x_1} < 1$. So:

$$a^{x_2} = a^{x_1} \cdot a^{x_2 - x_1} \Rightarrow a^{x_2} < a^{x_1}$$

Hence, $x_1 < x_2 \Rightarrow a^{x_1} > a^{x_2}$

Therefore, $y = a^x$ is **strictly decreasing** when $0 < a < 1$.

(iv) Injectivity of Exponential Functions

Let $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

Further let, $f(x_1) = f(x_2)$ for some $x_1, x_2 \in]-\infty, \infty[$.

So: $a^{x_1} = a^{x_2}$

Case-1: If $x_1 < x_2$, then from the function of $f(x) = a^x$ either

$a^{x_1} < a^{x_2}$ (when $a > 1$) or $a^{x_1} > a^{x_2}$ (when $0 < a < 1$).

Which is impossible.

Case-2: If $x_1 > x_2$, then from the function of $f(x) = a^x$ either $a^{x_1} > a^{x_2}$ (when $a > 1$) or $a^{x_1} < a^{x_2}$ (when $0 < a < 1$). Which is again impossible.

Hence, we must have $x_1 = x_2$ and consequently, $f(x) = a^x$ is injective. It follows that any horizontal line meets the graph of $f(x) = a^x$ at most once.

Definition of Injectivity (One-to-One Function)

Recall that, a function $y = f(x)$ is called **injective (one-to-one)** if different inputs always give different outputs. Formally, a function is one-to-one if:

whenever for $x_1, x_2 \in$ Dom f such that $f(x_1) = f(x_2)$, we have $x_1 = x_2$.

A function is injective if every horizontal line meets its graph at most once; equivalently, its graph is strictly increasing or strictly decreasing.

(v) Horizontal Asymptote of Exponential Function

Note that if $a > 1$, then $a^x \to 0^+$ as $x \to -\infty$ and if $0 < a < 1$, then $a^x \to 0^+$ as $x \to \infty$. It follows that the horizontal line $y = 0$ represents the horizontal asymptote for the exponential function.

Note: (Horizontal Asymptote)

A **horizontal asymptote** is a horizontal line $y = c$ which a function approaches as x becomes very large (or very small), but the graph does not necessarily touch or cross it.

Formally the vertical line $y = c$ is called **horizontal asymptote** for the function $y = f(x)$ if $f(x) \to c$ as $x \to -\infty$ or $x \to \infty$.

Example 7 **Graph of $f(x) = e^x$**

Solution: Let $f(x) = e^x$.

- Domain of the function is $R = (-\infty, \infty)$.
- Range of $f(x) = e^x$ is $(0, \infty)$.
- The graph cuts y-axis at $(0, 1)$.
- $y = 0$ is the horizontal asymptote.
- f is continuous and strictly increasing, because $e > 1$.

Some approximate functional values are given in the following tables.

x	$f(x) = e^x$
1	$e = 2.718$
2	7.389
3	20.085
10	22026.465
100	2.688×10^{43}
$\to +\infty$	$\to +\infty$

x	$f(x) = e^x$
–1	0.367
–2	0.135
–3	0.0497
–10	0.0000454
–100	3.720×10^{-44}
$\to -\infty$	$\to 0$

With all above key points, the graph of $f(x) = e^x$ is shown in Figure 1.6.

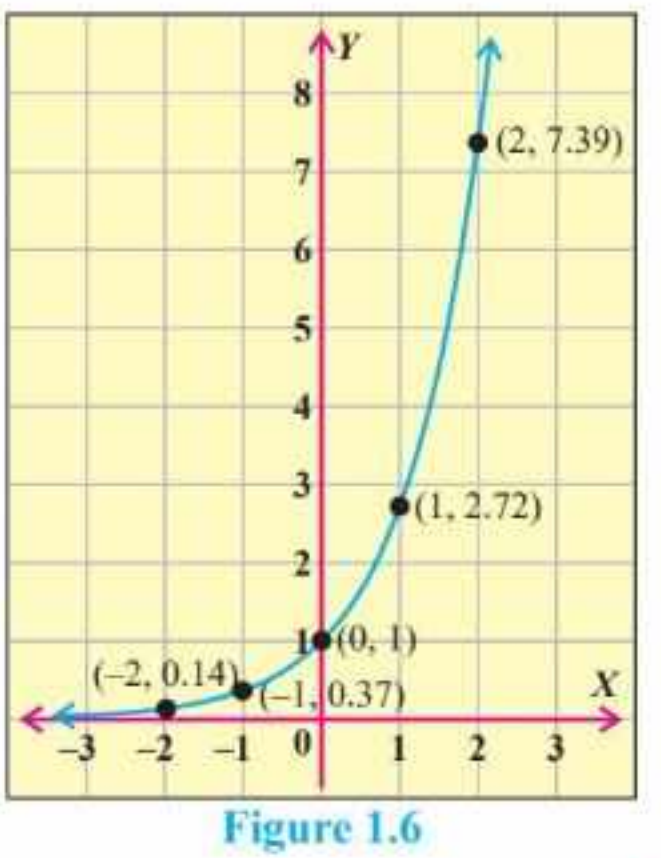


Figure 1.6

Note that the graph of all exponential functions $f(x) = a^x$, $a \in R$ and $a > 1$ is almost similar to the graph in Figure 1.6.

Example 8: Graphs of $f(x) = \left(\frac{1}{2}\right)^x = \frac{1}{2^x}$

Solution: The graph of the exponential function $f(x) = \left(\frac{1}{2}\right)^x$ has following important properties:

- Domain of the function is $R = (-\infty, \infty)$.
- Range of $f(x) = \left(\frac{1}{2}\right)^x$ is $(0, \infty)$.
- The graph cuts y-axis at $(0, 1)$.
- $y = 0$ is the horizontal asymptote.
- f is continuous and strictly decreasing because $a = \frac{1}{2} < 1$.

Some approximate functional values are given in the following tables.

x	$f(x) = \left(\frac{1}{2}\right)^x$
1	0.5
2	0.25
3	0.125
10	0.0009766
100	7.89×10^{-31}
$\to +\infty$	$\to 0$

x	$f(x) = \left(\frac{1}{2}\right)^x$
–1	2
–2	4
–3	8
–10	1024
–100	1.267×10^{30}
$\to -\infty$	$\to \infty$

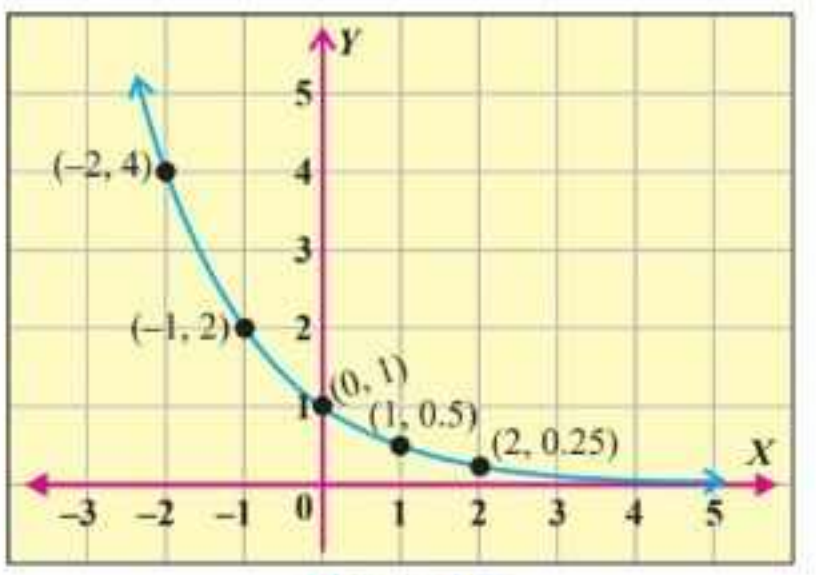


Figure 1.7

With all the above key points, the graph of $f(x) = \left(\frac{1}{2}\right)^x$ is shown in Figure 1.7.

Note that the graph of all exponential functions $f(x) = a^x$, $a \in R$ and $0 < a < 1$ is almost similar to the graph in Figure 1.7.

1.5 Introduction to Hyperbolic and Inverse Hyperbolic Functions

Hyperbolic functions are a class of transcendental functions closely related to exponential functions. They are defined using combinations of e^x and e^{-x}, and their graphs resemble trigonometric functions, although they arise from hyperbolas rather than circles. The basic hyperbolic functions include sinh x, cosh x and tanh x which have important applications in mathematics, physics and engineering.

Inverse hyperbolic functions are the inverse functions of hyperbolic functions, defined

on suitable restricted domains. They are useful in solving equations involving hyperbolic functions and appear in various applications such as integration, differential equations, and modeling real-world phenomena.

1.5.1 Hyperbolic Functions

The hyperboli c functions are defined as follows:

Function	Definition	Domain	Range
$\sinh x$	$\sinh x = \frac{e^x - e^{-x}}{2}$	$R = (-\infty, \infty)$	$R = (-\infty, \infty)$
$\cosh x$	$\cosh x = \frac{e^x + e^{-x}}{2}$	$R = (-\infty, \infty)$	$[1, \infty)$
$\tanh x = \frac{\sinh x}{\cosh x}$	$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	$R = (-\infty, \infty)$	$(-1, 1)$
$\coth x = \frac{\cosh x}{\sinh x}$	$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$	$R\backslash\{0\}$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech} x = \frac{1}{\cosh x}$	$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$	$R = (-\infty, \infty)$	$(0, 1]$
$\operatorname{csch} x = \frac{1}{\sinh x}$	$\operatorname{csch} x = \frac{2}{e^x - e^{-x}}$	$R\backslash\{0\}$	$(-\infty, 0) \cup (0, \infty)$

Hyperbolic functions are so named because the parametric equations $x = \cosh t$ and $y = \sinh t$, satisfies the equation of the hyperbola $x^2 - y^2 = 1$.

Example 9 Prove that $\cosh^2 x - \sinh^2 x = 1$.

Solution: We know $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$

Therefore, $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \left(\frac{e^{2x} + e^{-2x} + 2}{4}\right) - \left(\frac{e^{2x} + e^{-2x} - 2}{4}\right)$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4} = \frac{4}{4} = 1$$

Hence, $\cosh^2 x - \sinh^2 x = 1$

1.5.2 Graphs of Hyperbolic Functions

(i) Graph of $\sinh x$

We know that $\sinh x = \frac{e^x - e^{-x}}{2}$

This can be written as $\sinh x = \dfrac{e^x - \dfrac{1}{e^x}}{2} = \dfrac{e^x - \left(\dfrac{1}{e}\right)^x}{2}$

Since, domain of e^x and $\left(\dfrac{1}{e}\right)^x$ is R, therefore, the domain of $\sinh x$ is $R = (-\infty, \infty)$.

As $x \to \infty$, $e^x \to \infty$ and $\left(\dfrac{1}{e}\right)^x \to 0$.

Hence, $\sinh x = \dfrac{e^x - \left(\dfrac{1}{e}\right)^x}{2} \to \infty$

As $x \to -\infty$, $e^x \to 0$ and $\left(\dfrac{1}{e}\right)^x \to \infty$.

Hence, $\sinh x = \dfrac{e^x - \left(\dfrac{1}{e}\right)^x}{2} \to -\infty$

Therefore, the range of $\sinh x$ is $R = (-\infty, \infty)$.

At $x = 0$, $\sinh 0 = \dfrac{e^0 - e^{-0}}{2} = 0$, so the graph of $\sinh x$ passes through the point $(0, 0)$.

x	$-\infty \leftarrow$	-3	-2	-1	0	1	2	3	$\rightarrow \infty$
$y = \sinh x$	$-\infty \leftarrow$	-10.03	-3.63	-1.18	0	1.18	3.63	10.03	$\rightarrow \infty$

The graph of $\sinh x$ is shown in Figure 1.8.

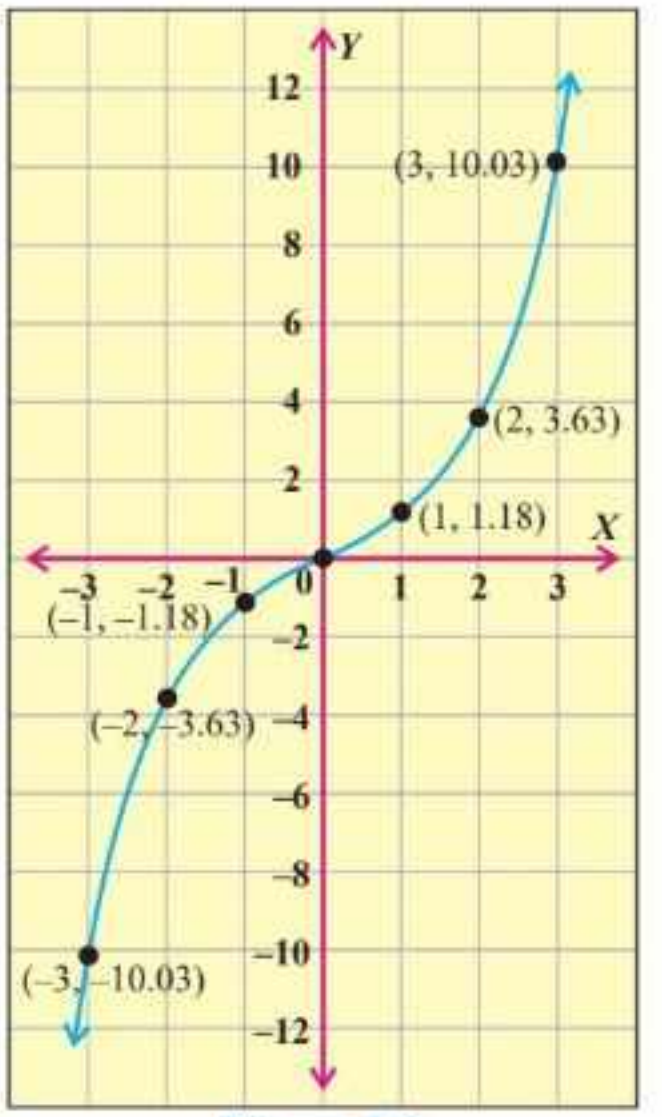


Figure 1.8

(ii) Graph of cosh x

We know that $\cosh x = \dfrac{e^x + e^{-x}}{2}$

This can be written as $\cosh x = \dfrac{e^x + \dfrac{1}{e^x}}{2} = \dfrac{e^x + \left(\dfrac{1}{e}\right)^x}{2}$.

Since domain of e^x and $\left(\dfrac{1}{e}\right)^x$ is R, therefore, the domain of $\cosh x$ is $R = (-\infty, \infty)$.

As $x \to \infty$, $e^x \to \infty$ and $\left(\dfrac{1}{e}\right)^x \to 0$.

Hence, $\cosh x = \dfrac{e^x + \left(\dfrac{1}{e}\right)^x}{2} \to \infty$

As $x \to -\infty$, $e^x \to 0$ and $\left(\dfrac{1}{e}\right)^x \to \infty$.

Hence, $\cosh x = \dfrac{e^x + \left(\dfrac{1}{e}\right)^x}{2} \to \infty$

At $x = 0$, $\cosh 0 = \dfrac{e^0 + e^{-0}}{2} = 1$, so the graph of $\cosh x$ passes through the point $(0, 1)$ and $y = 1$ is minimum value of the graph.

Therefore, the range of $\cosh x$ is $[1, \infty)$. To sketch the graph, we take a table of values:

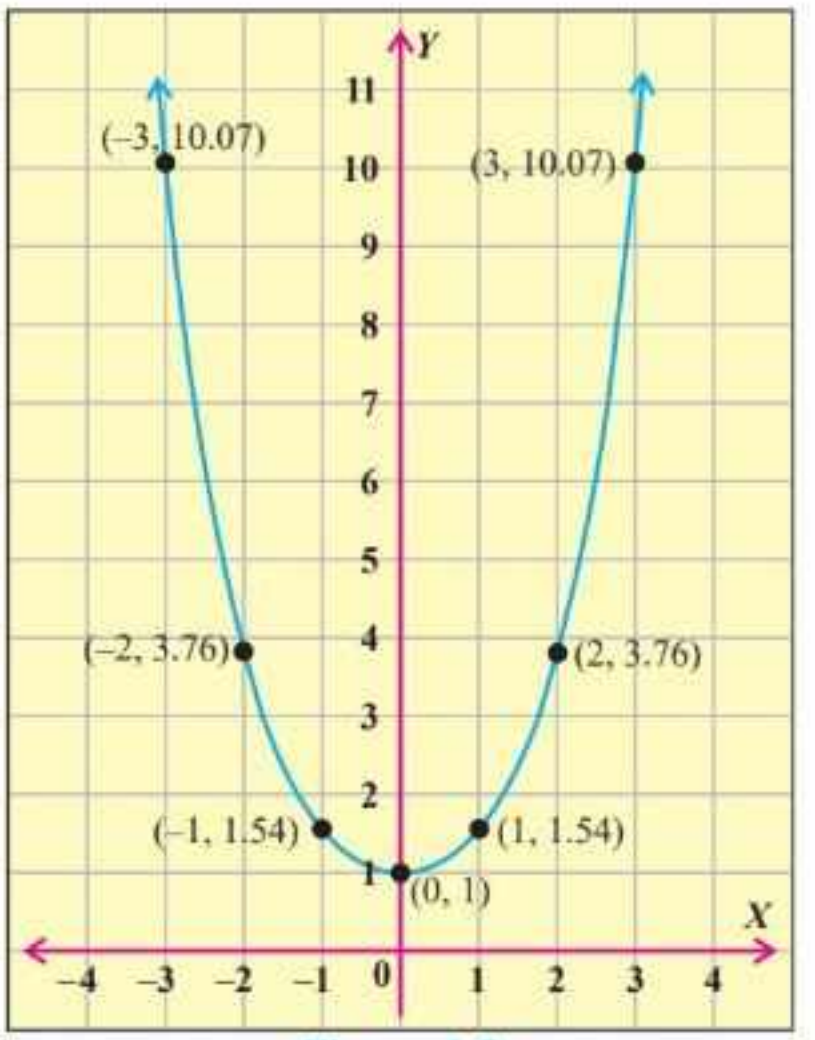


Figure 1.9

x	$-\infty \leftarrow$	–3	–2	–1	0	1	2	3	$\rightarrow \infty$
$y = \cosh x$	$+\infty \leftarrow$	10.07	3.76	1.54	1	1.54	3.76	10.07	$\rightarrow \infty$

The graph of $\cosh x$ is shown in Figure 1.9.

1.5.3 Inverse Hyperbolic Functions

The formulae, domains and ranges of all inverse hyperbolic functions are given in the following table.

Function	Formula	Domain	Range
$\sinh^{-1} x$	$\ln(x + \sqrt{x^2 + 1})$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\cosh^{-1} x$	$\ln(x + \sqrt{x^2 - 1})$	$[1, \infty)$	$[0, \infty)$
$\tanh^{-1} x$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$(-1, 1)$	$(-\infty, \infty)$
$\coth^{-1} x$	$\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\operatorname{sech}^{-1} x$	$\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$(0, 1]$	$(0, \infty[$
$\operatorname{csch}^{-1} x$	$\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right)$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

1.5.2 Inverse Hyperbolic Functions

(i) Inverse Hyperbolic sine Functions

From the graph of $y = \sinh x$ (see Figure 1.8), it is clear from the horizontal line test that the function is bijective. Hence, its inverse function exists, and we define it as follows:

$$x = \sinh^{-1} y \text{ if and only if } y = \sinh x$$

Example 10 Prove that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$.

Solution: Let $y = \sinh^{-1} x$, where $x, y \in]-\infty, \infty[$.

$$x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2} \quad \Rightarrow \quad 2x = e^y - \frac{1}{e^y}$$

$$\Rightarrow \quad 2x = \frac{e^{2y}-1}{e^y} \quad \Rightarrow \quad 2xe^y = e^{2y} - 1$$

$$\Rightarrow \quad (e^y)^2 - 2xe^y - 1 = 0$$

Using quadratic formula:

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Note that, $x - \sqrt{x^2+1} < 0 \ \forall \ x \in]-\infty, \infty[$. Since $e^y > 0$, we take the positive sign:

$$e^y = x + \sqrt{x^2+1} \quad \Rightarrow \quad y = \ln(x + \sqrt{x^2+1})$$

Hence, $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

Note

If $y = f(x)$ is a bijective function, then $x = f^{-1}(y)$ is its inverse.

(ii) Inverse Hyperbolic cosine Function

From the graph of $y = \cosh x$ (see Figure 1.9), it is clear from the horizontal line test that the function is not one-to-one on the domain $(-\infty, \infty)$. However, if we restrict the domain to $[0, \infty)$, the range set is not disturbed. It follows that $y = \cosh x$ becomes one-to-one for the restricted domain $[0, \infty)$ and thus bijective. Therefore, on the domain $[0, \infty)$, the inverse hyperbolic cosine function exists and we define it as follows:

$$x = \cosh^{-1} y \text{ if and only if } y = \cosh x$$

Example 11 Prove that $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ for all $x \in [1, \infty)$.

Solution: For $y \in [0, \infty)$ = Restricted Dom of $\cosh x$ and $x \in [1, \infty)$ = Range of $\cosh x$.

Let $y = \cosh^{-1} x$

$$x = \cosh y$$

$$x = \frac{e^y + e^{-y}}{2} \quad \Rightarrow \quad 2x = e^y + \frac{1}{e^y}$$

$$\Rightarrow \quad 2x = \frac{e^{2y}+1}{e^y} \quad \Rightarrow \quad 2xe^y = e^{2y} + 1$$

$$\Rightarrow \quad (e^y)^2 - 2xe^y + 1 = 0$$

Remember!

Dom f = Range f^{-1}

Range f = Dom f^{-1}

Using quadratic formula:

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$e^y = x \pm \sqrt{x^2 - 1}, \text{ where } x \geq 1,\ y \geq 0 \text{ and } e^y > 0.$$

$$y = \ln(x \pm \sqrt{x^2 - 1})$$

It can easily be observed that $0 < x - \sqrt{x^2 - 1} < 1$ for all $x \geq 1$. Thus, $y = \ln\left(x - \sqrt{x^2 - 1}\right) < 0$. But, $y \geq 0$, which is possible only if the positive sign is taken. Hence,

$$y = \ln(x + \sqrt{x^2 - 1})$$

$$\Rightarrow \quad \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

EXERCISE 1.2

1. Which of the following functions are algebraic?

$f(x) = \sec x,\ g(x) = \dfrac{2x-3}{x+1},\ h(x) = \sqrt{x^2 + 2x},\ k(x) = \dfrac{e^x - e^{-x}}{2},\ r(x) = (x-1)^3$

2. Which of the following functions are transcendental?

$f(x) = \dfrac{1}{x}, g(x) = \cot x, h(x) = e^{2x} + \sqrt{x^2 + 2x}, k(x) = \dfrac{e^x + e^{-x}}{2}, r(x) = (x-1)^{-\frac{3}{2}}$

3. Which of the following functions are fundamental transcendental functions?

$f(x) = e^x + \cos x, g(x) = \sin x, h(x) = e^{2x},\ k(x) = \dfrac{e^x - e^{-x}}{2}, r(x) = 15^x$

4. Which of the following functions are non-fundamental transcendental functions?

$f(x) = e^{2x},\ g(x) = \cos x,\ h(x) = 2^x,\ k(x) = \cos x^2,\ r(x) = 2^{-x}$

5. Discuss domain, range, and graph of the function $f(x) = 3^x$.

6. Discuss domain, range, and graph of the function $f(x) = \left(\dfrac{1}{3}\right)^x$.

7. Prove the following hyperbolic identities:

(i) $1 - \tanh^2 x = \operatorname{sech}^2 x$ (ii) $\operatorname{csch}^2 x = \coth^2 x - 1$

(iii) $\sinh 2x = 2 \sinh x \cosh x$ (iv) $\cosh 2x = \cosh^2 x + \sinh^2 x$

8. Prove the following inverse hyperbolic formulae.

(i) $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), x \neq 1$ (ii) $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), x \neq 1$

(iii) $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right), x \neq 0$ (iv) $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x}+\sqrt{\dfrac{1}{x^2}+1}\right), x \neq 0$

1.6 Logarithmic Function

A logarithmic function is defined by $y = \log_a x$, $a > 0$, $a \neq 1$, $x > 0$. It assigns to each positive real number x a unique real number y, such that $a^y = x$.

Logarithmic Function as Inverse of Exponential Function

We know that the exponential function

$$y = a^x, (a > 0, a \neq 1)$$

is a one-to-one function with domain $]-\infty, \infty[$ and range $]0, \infty[$. Thus, it is invertible. Interchanging x and y, we obtain:

$$x = a^y \quad \Rightarrow \quad y = \log_a x$$

Hence, the logarithmic function is the inverse of the exponential function with domain $]0, \infty[$ and range $]-\infty, \infty[$.

A logarithm with base 10 is called a common logarithm and a logarithm with base e is called a natural logarithm and we write:

$$\log_{10} b = \log b, \log_e b = \ln b$$

Basic Logarithmic Identities

For $a > 0$, $a \neq 1$, $x > 0$, we have: $\log_a 1 = 0$, $\log_a a = 1$

$\log_a(a^x) = x$, $\log_a x = x$

Laws of Logarithms and Change of Base Formula

For $a > 0$, $a \neq 1$, $x > 0$, $y > 0$, $t \in \mathbb{R}$, we have studied the following laws of logarithms and change of base formula:

1. Product Law: $\log_a(xy) = \log_a x + \log_a y$
2. Quotient Law: $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
3. Power Law: $\log_a(x^t) = t \log_a x$
4. Change of Base Formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$ for every $b > 0$, $b \neq 1$. In particular:

$$\log_a x = \frac{\ln x}{\ln a} \quad \text{or} \quad \log_a x = \frac{\log x}{\log a}$$

1.6.1 Graphical Behaviour of Logarithmic Functions

In this section, we study the graphical behaviour of logarithmic functions in a structured manner. For a logarithmic function $y = \log_a x$ with domain $]0, \infty[$ and range

$]-\infty, \infty[$, we analyze its vertical asymptotic behaviour, and its overall shape for different values of the base a. We also examine important graphical features such as intercepts, continuity, injectivity and whether the function is increasing or decreasing. These properties together help us to understand how the logarithmic function behaves as a curve and provide the necessary foundation for sketching and interpreting its graph accurately.

(i) Injectivity of Logarithmic Functions

Let $f(x) = \log_a x$, where $a > 0$ and $a \neq 1$, $x \in]0, \infty[$. Let $s, t \in]0, \infty[$ be arbitrary numbers such that, $f(s) = f(t)$

$$\Rightarrow \quad \log_a s = \log_a t \quad \Rightarrow \quad a^{\log_a s} = a^{\log_a t} \quad \Rightarrow \quad s = t$$

Hence, the logarithmic function is one to one.

(ii) Vertical Asymptote of Logarithmic Functions

Let $f(x) = \log_a x$, where $a > 0, a \neq 1$ and $x > 0$. Then $x = a^y$. We consider the following cases:

Case-1: $a > 1$

Note that: $x \to 0^+$

$\Rightarrow a^y \to 0^+ \Rightarrow y \to -\infty$

$\Rightarrow \log_a x \to -\infty$.

Hence, $\lim\limits_{x \to 0^+} f(x) = -\infty$.

Case-2: $0 < a < 1$

Note that: $x \to 0^+$

$\Rightarrow a^y \to 0^+ \Rightarrow y \to \infty$

$\Rightarrow \log_a x \to \infty$.

Hence $\lim\limits_{x \to 0^+} f(x) = \infty$.

In both cases, we have: $\lim\limits_{x \to 0^+} f(x) = \pm\infty$.

Vertical Asymptote of a Function

A **vertical asymptote** is a vertical line $x = c$ such that the function $f(x)$ increases or decreases without bound as $x \to c$. In limit form, $x = c$ is a vertical asymptote of the function $y = f(x)$ if:

$\lim\limits_{x \to c^+} f(x) = \pm\infty$ or $\lim\limits_{x \to c^-} f(x) = \pm\infty$

Hence, the line $x = 0$ is the vertical asymptote of $f(x) = \log_a x$. Since, $y = \log_a x$ is defined for all positive values of x, there is no other asymptote except $x = 0$.

(iii) Logarithmic Functions are Strictly Increasing or Strictly Decreasing

Consider the logarithmic function:

$$y(x) = f(x) = \log_a x \quad (a > 0, a \neq 1, x > 0)$$

Let $x_1, x_2 \in]0, \infty[$ such that $x_1 < x_2$. Then, $y_1 = f(x_1) = \log_a x_1$ and $y_2 = f(x_2) = \log_a x_2$.

So that: $x_1 = a^{y_1}$ and $x_2 = a^{y_2}$

Since, $x_1 < x_2$, we have $a^{y_1} < a^{y_2}$. We consider the following cases:

Case-1:

If $a > 1$, then $a^{y_1} < a^{y_2} \Rightarrow y_1 < y_2$.

It follows that:

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Hence, $y = \log_a x$ is strictly increasing when $a > 1$.

Case-2:

If $0 < a < 1$, then $a^{y_1} < a^{y_2} \Rightarrow y_1 > y_2$.

It follows that:

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

Hence, $y = \log_a x$ is strictly decreasing when $0 < a < 1$.

(iv) Intercepts and Continuity of Logarithmic Functions

For the logarithmic function: $y = \log_a x \ \ (a > 0, a \neq 1, x > 0)$

Intercepts:

(i) The x-intercept is obtained by substituting $y = 0$:

$$\log_a x = 0 \quad \Rightarrow \quad x = a^0 = 1$$

Hence, the graph of $y = \log_a x$ intersects x-axis at the point $(1, 0)$.

(ii) Since, the function $y = \log_a x$ is undefined at $x = 0$, it has no y-intercept.

Continuity:

The function $y = \log_a x$ is continuous for all $x > 0$. This can be seen from the fact that:

$\lim\limits_{x \to c} \log_a x = \log_a c$ for all $c > 0$.

Hence, the function has no breaks or jumps in its domain.

1.6.2 Graph of Logarithmic Function

The graph of the logarithmic function $y = \log_a x$ is constructed using the properties discussed in the above subsections. In particular, the domain $x > 0$, the vertical asymptote $x = 0$, the intercept $(1, 0)$, the continuity, the injectivity and the monotonic behaviour depending on the value of a together determine the shape and position of the curve. Using these features, the graph can be drawn accurately for different values of the base a.

Figure 1.10 shows the graphs of logarithmic functions with bases greater than 1, whereas Figure 1.11 represents the graphs of logarithmic functions with bases between 0 and 1.

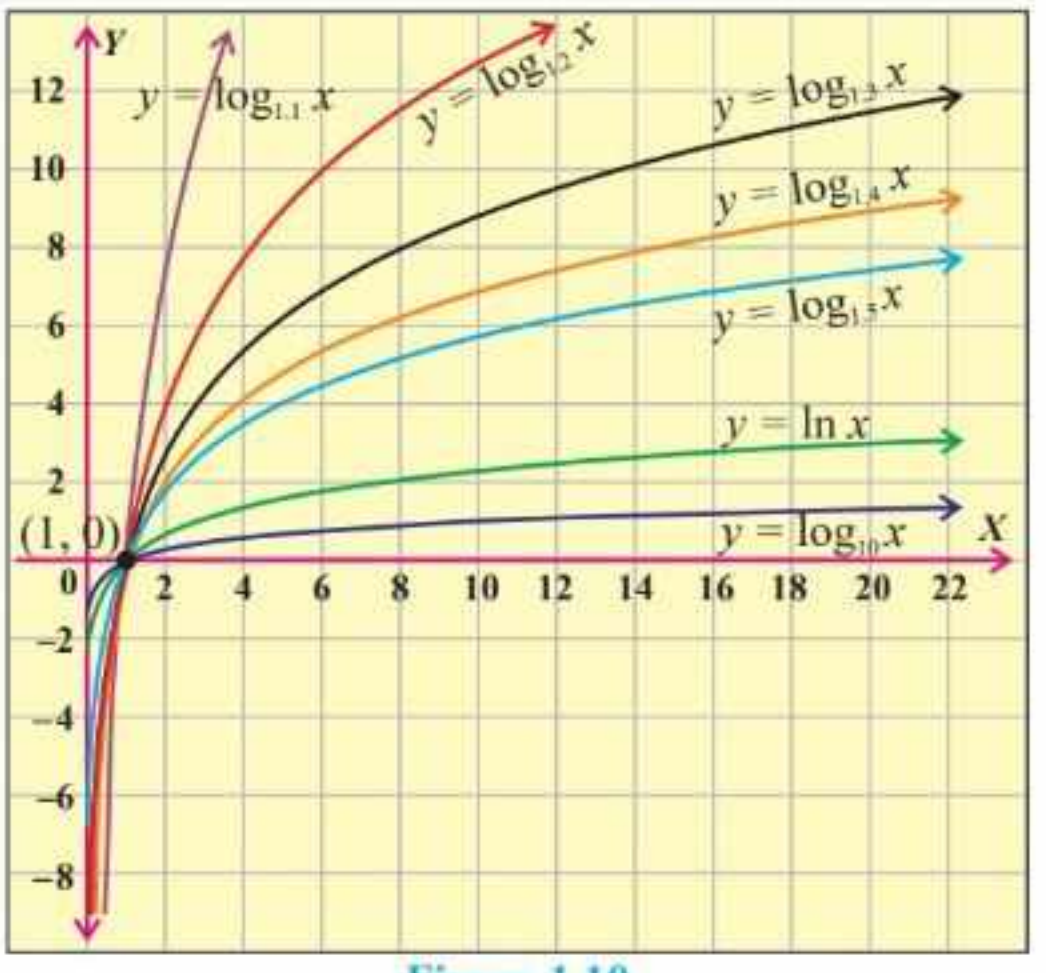


Figure 1.10

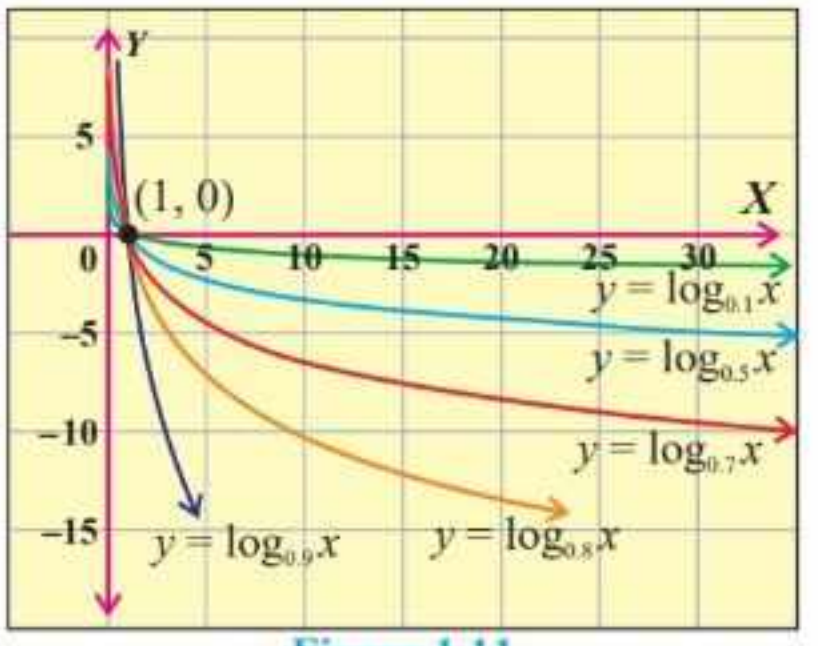


Figure 1.11

1.7 Modulus Function and Its Graph

The **modulus function** or **absolute value function** is denoted and defined by:

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}$$

For example, if $x = 2 > 0$ then $|x| = |2| = 2$ and if $x = -2 < 0$ then $|x| = |-2| = -(-2) = 2$. Similarly, $x = 0$ then $|x| = |0| = 0$. Thus, the modulus of a number is always non-negative.

The graph of the modulus function is V-shaped. It consists of two straight lines meeting at the origin. For $x \geq 0$, the graph is the line $y = x$, while for $x < 0$, the graph is the line $y = -x$. Thus, the graph of modulus function can be sketched using the points (0, 0), (2, 2) and (–2, 2) as shown in Figure 1.12.

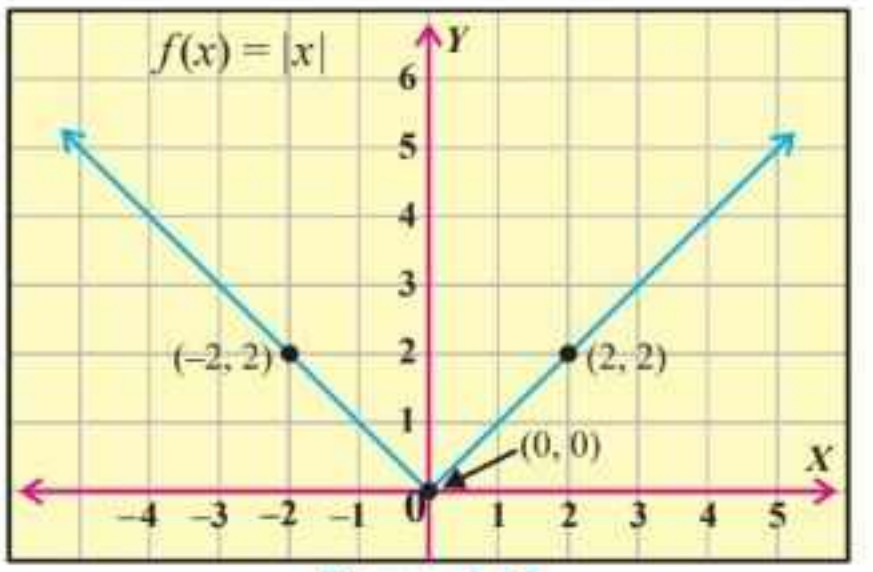


Figure 1.12

1.8 Interpreting the Relation Between a One-to-One Function and Its Inverse through Graph

A function has an inverse only if it is bijective, that is, both one-to-one and onto. However, if the co-domain of a function is taken equal to its range, then every value in the co-domain is already attained by the function. Therefore, the function automatically becomes onto, and only the one-to-one condition is required for the existence of its inverse.

The graph of a function and its inverse are related through a geometric symmetry about the line $y = x$. This line is used because any point (x, y) on a function represents an input-output pair, and for its inverse, the roles of input and output are interchanged. The line $y = x$ is exactly the set of all points where input and output values are equal, making it the natural axis of reflection for this interchange.

1.8.1 Method of Constructing the Graph of an Inverse Function

The graph of an inverse function is obtained geometrically as follows:

(i) Draw the graph of the given one-to-one function.

(ii) Consider the line $y = x$ on the same coordinate plane.

(iii) Reflect the graph of the function about the line $y = x$.

(iv) The resulting reflected curve represents the graph of the inverse function.

The reflection ensures that each point on the original graph is mapped to a corresponding point on the inverse graph by maintaining equal perpendicular distance from the line $y = x$, preserving the geometric symmetry between the two graphs.

Example 12 **(Logarithmic Graph from Exponential Graph)**

Show graphically that the function $y = \ln x$ is the inverse of the function $y = e^x$.

Solution:

Consider the exponential function $y = e^x$

Note that the function $y = e^x$ is one to one on $]-\infty, \infty[$ and its range is $]0, \infty[$. Hence, it is invertible and its inverse is defined on $]0, \infty[$. We draw the graph of $y = e^x$ along with the line $y = x$ (see Figure 1.13). To obtain the graph of the inverse function, points on the curve $y = e^x$ are reflected about the line $y = x$. The curve passing through these reflected points is then drawn (see Figure 1.13). The resulting curve represents the function:

$$y = \ln x$$

which is clearly defined on the interval $]0, \infty[$.

Hence, the function $y = \ln x$ is the inverse of $y = e^x$.

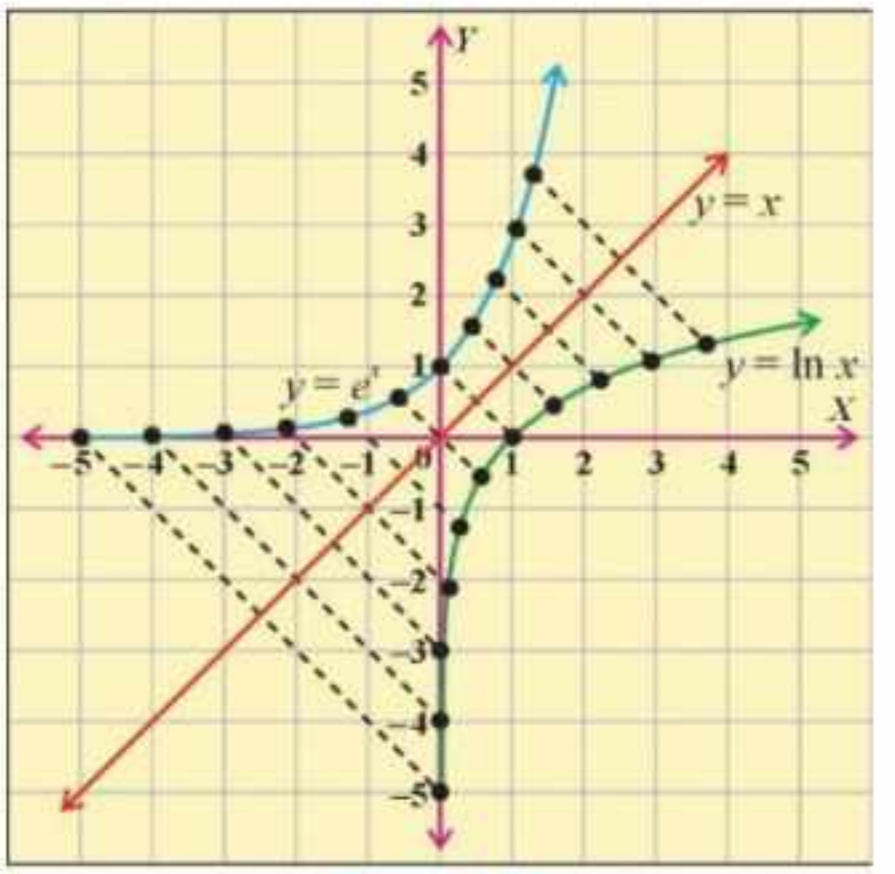


Figure 1.13

1.9 Transformation of Graphs of Functions

Graphs of functions can be transformed in various ways to obtain new graphs from basic functions. These transformations include reflection, rotation, stretching, compression and shifting, and are applied to represent changes in functions in algebraic, graphical and real-life contexts. One of the most important applications of these transformations is in computer graphics, where they are used to model, manipulate and animate images and shapes. However, in this section, we shall focus only on three fundamental transformations: horizontal shift, vertical shift and scaling, which are most commonly used in the graphical analysis of functions.

(i) Shifting of Graphs

Shifting of graphs refers to the movement of a graph from its original position without changing its shape or size. In shifting, the graph is translated either horizontally or vertically.

In mathematical modeling, shifting is used to represent changes in a function due to variations in input or output values. It allows the same basic function to describe different situations by adjusting its position on the coordinate plane.

Horizontal Shift

A graph of the function $y = f(x)$ is **shifted horizontally** when the input variable x is changed to $x - h$. The resulting graph is represented by the function:

$$y = f(x - h)$$

- If $h > 0$: shift h units to the right
- If $h < 0$: shift $|h|$ units to the left

Vertical Shift

A graph of the function $y = f(x)$ is **shifted vertically** when the output variable y is changed to $y - k$. The resulting graph is represented by the function:

$$y = f(x) + k$$

- If $k > 0$: shift k units upward
- If $k < 0$: shift $|k|$ units downward

For better understanding of horizontal and vertical shifts, we consider the function:

$$y = x^2$$

This equation represents a parabola with vertex at (0, 0), as shown in Figure 1.14.

Next, we consider the function $y = (x - 2)^2$

This also represents a parabola with vertex at (2, 0). Its graph can be seen in Figure 1.15.

Note that the graph in Figure 1.15 can be obtained directly from the graph in Figure 1.14 by shifting it 2 units horizontally to the right.

Now, consider another function $y = x^2 - 2$

This equation also represents a parabola with vertex at (0, –2). Its graph is shown in Figure 1.16.

Note that this graph can be obtained directly from the graph in Figure 1.14 by shifting it 2 units downward.

Challenge!

Draw the graph of the function

$$y = (x - 2)^2 - 2$$

and observe how it can be obtained from the graph of $y = x^2$

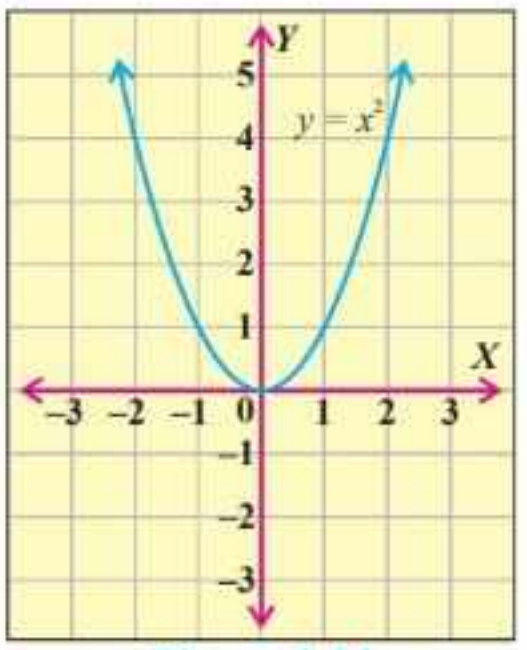


Figure 1.14

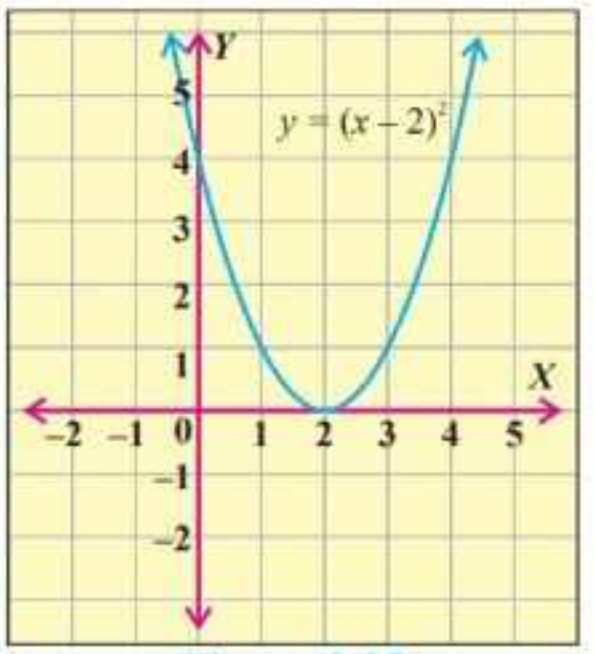


Figure 1.15

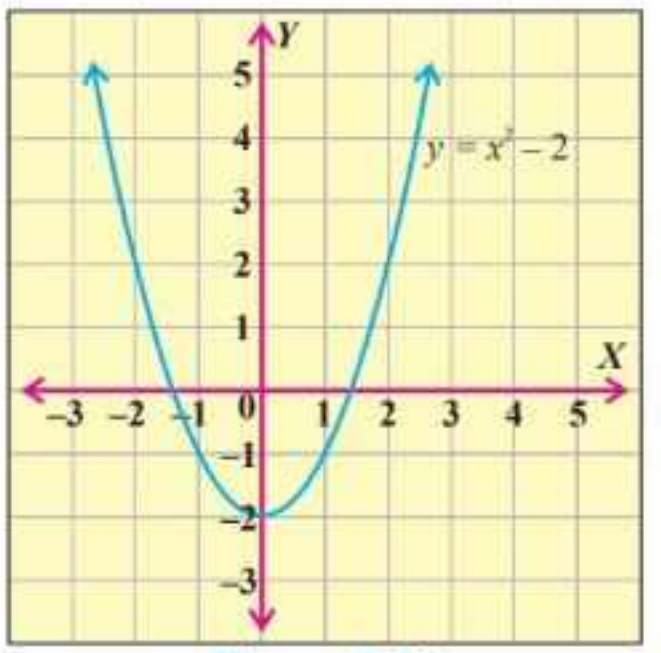


Figure 1.16

General Formula for Vertical and Horizontal Shift

If both horizontal and vertical shifts are applied together to the function $y = f(x)$, then the general form of the transformed function is given by:

$$y = f(x - h) + k$$

(ii) Scaling of Graphs

Scaling of graphs refers to the stretching or compression of a graph without changing its general shape. In scaling, the graph is enlarged or reduced either in the vertical direction or in the horizontal direction.

Scaling helps in understanding how a function changes when its input or output values are multiplied by a positive constant factor.

Vertical Scaling

If a function $y = f(x)$ is multiplied by a positive constant a, then the new function:

$$y = af(x), \quad a > 0$$

represents a **vertical scaling** of the graph of $y = f(x)$.

- If $a > 1$, then the graph is **stretched vertically** by factor a.
- If $0 < a < 1$, then the graph is **compressed vertically** by factor $\frac{1}{a} > 1$.
- Only the y-values of the graph change, while x-values remain unchanged.

Horizontal Scaling

If a function $y = f(x)$ is transformed to:

$$y = f(ax), \quad a > 0$$

then, it represents a **horizontal scaling** of the graph of $y = f(x)$.

- If $a > 1$, then the graph is **compressed horizontally** by a factor a
- If $0 < a < 1$, then the graph is **stretched horizontally** by a factor $\frac{1}{a} > 1$.
- Only the x-values of the graph change, while y-values remain unchanged

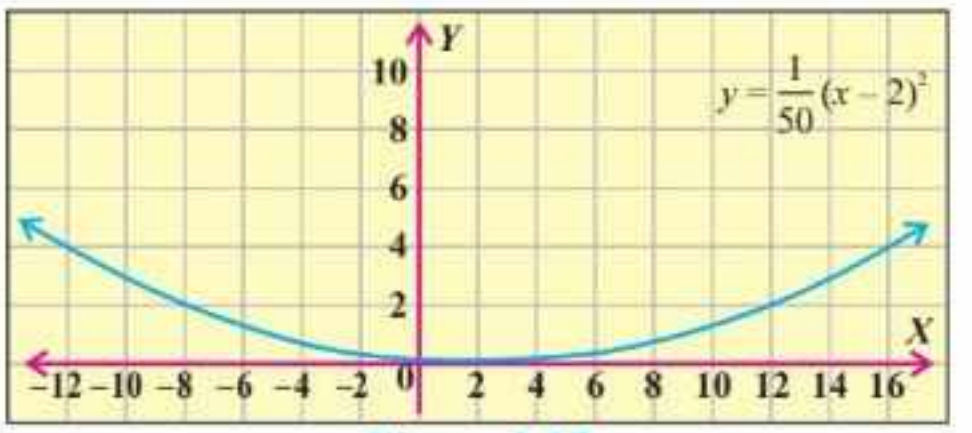


Figure 1.17

Important Note:

If $a < 0$, scaling is not purely stretching or compression. It results in:

- reflection along the x-axis in $y = af(x)$
- reflection along the y-axis in $y = f(ax)$

Therefore, in this section we consider only $a > 0$ to focus on scaling alone.

For better understanding of scaling, we consider the following functions:

$$y = \frac{1}{50}(x-2)^2 \quad \text{and} \quad y = (3x-2)^2$$

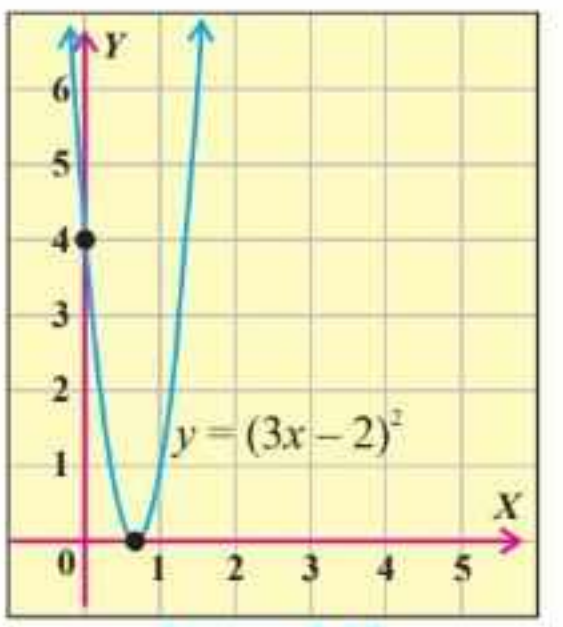


Figure 1.18

The graph of $y = \frac{1}{50}(x-2)^2$ is shown in Figure 1.17 which can be obtained from the graph of $y = (x-2)^2$ (Figure 1.15) using a vertical compression by factor of 50. Similarly, the graph of $y = (3x-2)^2$ can be obtained from the graph of $y = (x-2)^2$ by a horizontal compress, that is, by a factor of 3 (see Figure 1.18).

General Formula for Scaling of Graphs

If a function $y = f(x)$ is transformed to

$$y = af(bx), \quad a > 0, b > 0$$

then it represents both vertical and horizontal scaling of the graph of $y = f(x)$.

If $a > 1$, the graph is stretched vertically by factor a. If $0 < a < 1$, the graph is compressed vertically by factor $\frac{1}{a}$. On the other hand, if $b > 1$, the graph is compressed horizontally by factor b and if $0 < b < 1$, the graph is stretched horizontally by factor $\frac{1}{b}$.

Example 13 Let $f(x) = \ln(x^2 + 1)$. Find the function obtained after:

(i) a vertical stretch by factor 2 (ii) then a shift 1 unit to the left.

Solution: We have: $f(x) = \ln(x^2 + 1)$.

(i) Formula for the vertical scaling is $y = af(x)$

Since, there is a vertical stretch by factor 2, we have $a = 2$. Thus, the resulting function becomes:

$$y = 2\ln(x^2 + 1)$$

(ii) Now we have, $g(x) = 2\ln(x^2 + 1)$.

Formula for the horizontal translation is $y = g(x - h)$.

Since, there is a shift of 1 unit to the left, we have $h = -1$. Thus, the resulting function becomes:

$$y = g(x - (-1)) = g(x + 1)$$

$$= 2\ln[(x + 1)^2 + 1] = 2\ln(x^2 + 2x + 1 + 1)$$

Hence, the transformed function is $y = 2\ln(x^2 + 2x + 2)$

Example 14 Let $f(x) = \frac{x}{x+1}$. Find the function obtained after applying all the following transformations with the given order:

(i) vertical stretch by 2 (ii) horizontal stretch by 3

(iii) shift 2 units to the right (iv) shift 1 unit downward

Solution: Given function: $f(x) = \frac{x}{x+1}$.

(i) Vertical stretch by 2.

The resulting function is $y = af(x)$ where $a = 2$. Thus, we get:

$$y = 2\frac{x}{x+1}$$

$$= \frac{2x}{x+1}$$

We set, $g(x) = \frac{2x}{x+1}$.

(ii) Horizontal stretch by 3.

The resulting function is $y = g(bx)$, where $b = \frac{1}{3}$. So, we have:

$$y = g\left(\frac{x}{3}\right) = \frac{2\frac{x}{3}}{\frac{x}{3}+1} = \frac{2x}{x+3}$$

Let $s(x) = \frac{2x}{x+3}$

(iii) Shift 2 units to the right

The resulting function is $y=(x-h)$, where $h=2$. So, we get:

$$y=s(x-2)=\frac{2(x-2)}{(x-2)+3}=\frac{2x-4}{x+1}$$

Let $t(x)=\dfrac{2x-4}{x+1}$

(iv) Shift 1 unit downward

The resulting function is $y=t(x)+k$, where $k=-1$. Thus, we have:

$$y=t(x)-1=\frac{2x-4}{x+1}-1=\frac{x-5}{x+1}$$

Hence, transformed function is $y=\dfrac{x-5}{x+1}$.

EXERCISE 1.3

1. Let $f(x)=\log_4(x-2)$
   (i) Show that $f(x)$ is a one-to-one function using the horizontal line test.
   (ii) Find the inverse function $f^{-1}(x)$ algebraically.
   (iii) State the domain and range of $f(x)$ and $f^{-1}(x)$.
   (iv) Verify graphically that the graphs of $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y=x$.
2. Let $f(x)=\sin^{-1}x$
   (i) State the domain and range of $f(x)$ and $f^{-1}(x)$.
   (ii) Sketch the graph of $f(x)$ and show that it is one-to-one using the horizontal line test.
   (iii) Draw the graph of $f^{-1}(x)$ and verify that it is the reflection of $f(x)$ in the line $y=x$.
3. A one-to-one function is defined by $f(x)=(x-1)^3, \quad x\in[0,3]$
   Using the concept of reflection in the line $y=x$, sketch the graph of the inverse function $f^{-1}(x)$.
4. Given the function $f(x)=\sqrt{x}$
   Find the equation of the function after:
   (i) vertical stretch by factor 2 (ii) horizontal shift 3 units to the right
   (iii) applying both transformations (i) and (ii).
5. Given the function $f(x)=\dfrac{1}{x}, x\neq 0$. Find the equation of the transformed function after applying the following transformations in order.
   Horizontal shift 2 units to the left, horizontal compression by factor 2, vertical stretching by factor 3, and vertical shift 1 unit upward.
6. Given the function $f(x)=\ln x$, find the equation of the transformed function obtained after shifting the graph 1 unit to the right, vertically compressing it by a factor of 2, and then shifting it 3 units upward.

7. Given the function $f(x)=\sqrt{\frac{2x-1}{x+3}}$, $x \neq -3$. Find the equation of the transformed function after applying in order, the horizontal shift 2 units to the left, horizontal compression by factor 3, vertical compression by factor 2, and vertical shift 1 unit downward.
8. Given the function $f(x)=3^x$ and its transformed function $g(x)=3^{x-2}+1$, draw the graphs of both functions on the same coordinate plane and compare their shapes.
9. Given the function $f(x)=\ln x$. Find the equation of the transformed function after applying in order, the vertical stretch by factor 4, shift 6 units to right, horizontal compress by factor 3, and shift 1 unit upward. Draw the graphs of both the original and transformed functions on the same coordinate plane.
10. Draw the graph of $f(x)=|x|$ after applying in order, shift of 7 units to right, horizontal compress by factor 2, vertical shift 6 units downward, and vertical compress by factor 2.

1.10 Solving Problems Involving Exponential and Logarithmic Equations and Inequalities

In this section, we solve exponential and logarithmic equations and inequalities in which the variable appears in the exponent or inside a logarithmic expression. Before solving, the domain must be considered wherever the expression is not defined for all real values of the variable. The methods of solving equations and inequalities are illustrated in the following examples.

Examples of Exponential and Logarithmic Equations

Example 15 Solve the equation: $4^{x-1}=8$.

Solution: The given equation is: $4^{x-1}=8$.

Domain: Since the expression 4^{x-1} is defined for every real number

$$\text{Domain} =]-\infty, \infty[.$$

Since $4=2^2$ and $8=2^3$, the above equation can be written as:

$$(2^2)^{x-1}=2^3 \text{ or } 2^{2x-2}=2^3$$

Since the exponential function is one to one, we must have:

$$2x-2=3 \quad \Rightarrow \quad x=\frac{5}{2}$$

Solution Set: $\left\{\frac{5}{2}\right\}$

Example 16

Find the solution of $2^{2x} - 3^{5x+3} = 0$.

Solution: The given equation can be written as:

$$2^{2x} = 3^{5x+3} \quad \ldots(1)$$

Domain: Since the expressions 2^{2x} and 3^{5x+3} are defined for every real number.

So, Domain $=]-\infty, \infty[$.

Applying natural logarithms on both sides of equation (1):

$$\ln 2^{2x} = \ln 3^{5x+3}$$

Applying power law of logarithms:

$$2x \ln 2 = (5x+3) \ln 3$$

$$\Rightarrow \quad 2x \ln 2 = 5x \ln 3 + 3 \ln 3$$

$$\Rightarrow \quad (2 \ln 2 - 5 \ln 3)x = 3 \ln 3$$

Applying power law of logarithms:

$$(\ln 2^2 - \ln 3^5)x = \ln 3^3$$

$$\Rightarrow \quad (\ln 4 - \ln 243)x = \ln 27$$

Applying quotient law of logarithms:

$$x \ln\left(\frac{4}{243}\right) = \ln 27$$

$$\Rightarrow \quad x = \ln 27 / \ln\left(\frac{4}{243}\right) = \log_{\frac{4}{243}}(27)$$

Solution Set: $\left\{\log_{\frac{4}{243}}(27)\right\}$

Example 16 Solve the equation:

$$\log x + \log(x-3) = 1$$

Solution:

The given equation is:

$$\log x + \log(x-3) = 1 \quad \ldots(1)$$

Domain: Note that the expressions $\log x$ and $\log(x-3)$ are defined only when:

$$x > 0 \text{ and } x - 3 > 0 \quad \Rightarrow \quad x > 3$$

Hence, Domain $=]3, \infty[$.

Applying product law of logarithms on the L.H.S of the given equation (1):

$$\log[x(x-3)] = 1$$

Writing in exponential form:

$$x(x-3) = 10^1 = 10$$

Factorizing: $x^2 - 3x - 10 = 0$

$$(x-5)(x+2) = 0$$

$$\Rightarrow \quad x = 5 \text{ or } x = -2$$

Since $5 \in]3, \infty[$ whereas, $-2 \notin]3, \infty[$, the solution set is $\{5\}$.

Example 17 Solve the equation: $\log_x(x^2 - 3x + 2) = 2$.

Solution: Domain:

- Base condition: $x > 0, x \neq 1$
- Argument condition: $x^2 - 3x + 2 > 0$

The given equation is: $\log_x(x^2 - 3x + 2) = 2$.

Convert to exponential form: $x^2 = x^2 - 3x + 2$

$$\Rightarrow \quad 3x - 2 = 0 \quad \Rightarrow \quad 3x - 2 = 0 \quad \Rightarrow \quad x = \frac{2}{3}$$

Check domain:

- Base condition: $x = \frac{2}{3} > 0$ and $x \neq 1$.
- Argument condition:

$$x^2 - 3x + 2 = \left(\frac{2}{3}\right)^2 - 3\left(\frac{2}{3}\right) + 2 = \frac{4}{9} - 2 + 2 = \frac{4}{9} > 0$$

Hence, $x = \frac{2}{3}$ is a solution of the given equation.

Solution set: $\left\{\frac{2}{3}\right\}$

Examples of Exponential and Logarithmic Inequations

Example 19 Solve the inequality: $2^{x+1} > 5^x$.

Solution:

Domain: Note that, the expression 2^{x+1} and 5^x are defined for every real number.

The given inequality can be written as: $2 \cdot 2^x > 5^x$

Since, $2^x > 0$ for every real number, we have:

$$2 > \frac{5^x}{2^x} \quad \Leftrightarrow \quad 2 > \left(\frac{5}{2}\right)^x$$

Since, the logarithmic function with base 10 is strictly increasing, we have:

$$2 > \frac{5^x}{2^x} \quad \Leftrightarrow \quad \log 2 > \log\left(\frac{5}{2}\right)^x$$

Applying power law of logarithms: $\log 2 > x \ \log\left(\frac{5}{2}\right)$

Since, $\frac{5}{2} > 0 \ \Rightarrow \ \log\left(\frac{5}{2}\right) > 0$, we have:

$$x < \frac{\log 2}{\log\left(\frac{5}{2}\right)} = \log_{\frac{5}{2}}(2) \approx 0.7565$$

Hence, $2^{x+1} > 5^x \ \Rightarrow \ x < \log_{\frac{5}{2}}(2)$

Solution Set: $]-\infty, \log_{\frac{5}{2}}(2)[$.

Example 20 Solve the inequality: $\log(2x-1) \geq \log(5-3x)$

Solution:

Domain: Note that $\log(2x-1)$ and $\log(5-3x)$ are defined when:

$$2x-1 > 0 \ \Rightarrow \ x > \frac{1}{2} \quad \text{and} \quad 5-3x > 0 \ \Rightarrow \ x < \frac{5}{3}$$

So, we have: $\frac{1}{2} < x < \frac{5}{3}$

The given inequality is: $\log(2x-1) \geq \log(5-3x)$

Since, the logarithmic function with base 10 is strictly increasing, we must have:

$$\log(2x-1) \geq \log(5-3x) \quad \Leftrightarrow \quad 2x-1 \geq 5-3x$$

Solve inequality: $2x+3x \geq 5+1 \quad \Leftrightarrow \quad 5x \geq 6 \quad \Leftrightarrow \quad x \geq \frac{6}{5}$

Compatibility with domain: $\frac{6}{5} \leq x < \frac{5}{3}$

It follows that: $\log(2x-1) \geq \log(5-3x)$ iff $\frac{6}{5} \leq x < \frac{5}{3}$

Solution Set: $\left[\frac{6}{5}, \frac{5}{3}\right[$

Example 21 Solve the inequality: $\log_{\frac{7}{9}}(x^2-3x+2) \geq \log_{\frac{7}{9}}(2x-1)$

Solution:

Domain: For the expression $\log_{\frac{7}{9}}(x^2-3x+2)$ to be defined, we have:

$x^2-3x+2>0 \quad \Rightarrow \quad (x-1)(x-2)>0 \quad \Rightarrow \quad$ either $x<1$ and $x<2$ (which gives $x<1$) or $x>1$ and $x>2$ (which gives $x>2$). Hence $x<1$ or $x>2$.

For the expression $\log_{\frac{7}{9}}(2x-1)$ to be defined, we have:

$$2x-1>0 \Rightarrow x>\frac{1}{2}$$

Combined domain: $\left]\frac{1}{2},1\right[\cup]2,\infty[$

The given inequality is: $\log_7(x^2-3x+2) \geq \log_7(2x-1)$

Since, $0<\frac{7}{9}<1$, the logarithmic function with base $\frac{7}{9}$ is strictly decreasing. Thus, we have:

$$\log_{\frac{7}{9}}(x^2-3x+2) \geq \log_{\frac{7}{9}}(2x-1) \quad \Leftrightarrow \quad x^2-3x+2 \leq 2x-1$$

$$\Leftrightarrow \quad x^2-3x+2-2x+1 \leq 0 \quad \Leftrightarrow \quad x^2-5x+3 \leq 0$$

Solving the quadratic equation $x^2-5x+3=0$:

$$x=\frac{5\pm\sqrt{25-12}}{2}=\frac{5\pm\sqrt{13}}{2}$$

Since, $x^2 - 5x + 3 \leq 0$, either $x \leq \frac{5-\sqrt{13}}{2}$ and $x \geq \frac{5+\sqrt{13}}{2}$ (which is impossible) or

$x \geq \frac{5-\sqrt{13}}{2}$ and $x \leq \frac{5+\sqrt{13}}{2}$ $\left(\text{which gives } \frac{5-\sqrt{13}}{2} \leq x \leq \frac{5+\sqrt{13}}{2}\right)$.

Compatibility with the domain: $\left]\frac{1}{2}, 1\right[\cup]2, \infty[$.

If $x \in \left]\frac{1}{2}, 1\right[\cup]2, \infty[$ and $\frac{5-\sqrt{13}}{2} \leq x \leq \frac{5+\sqrt{13}}{2}$ then, we have:

$$x \in \left[\frac{5-\sqrt{13}}{2}, 1\right[\cup \left]2, \frac{5+\sqrt{13}}{2}\right]$$

Hence, $\log_{\frac{7}{9}}(x^2 - 3x + 2) \geq \log_{\frac{7}{9}}(2x - 1)$ iff $x \in \left[\frac{5-\sqrt{13}}{2}, 1\right[\cup \left]2, \frac{5+\sqrt{13}}{2}\right]$.

Solution Set: $\left[\frac{5-\sqrt{13}}{2}, 1\right[\cup \left]2, \frac{5+\sqrt{13}}{2}\right]$

EXERCISE 1.4

1. Solve the following equations:

   (i) $\log(3x-2) = \log(x^2 - 5x + 6)$ (ii) $\ln(x^2 - 1) = \ln(3x - 2)$

   (iii) $\log(x) + \log(x^2 - 5x + 7) = \log(3)$ (iv) $3^{(x+1)} - 2 \cdot 9^x + 9 = 0$

   (v) $x \log_{\frac{1}{e}} 4 = \log_{\frac{1}{e}}(16^3 - 3)$ (vi) $2^x + 2^{-x} = 5$

2. Solve the following inequalities (if its solution exists).

   (i) $2^x + 2^{1-x} \leq 3$ (ii) $2^x \geq \log_2(16)$

   (iii) $\log_2(x-1) \geq \log_2(5-x)$ (iv) $\log_{\frac{1}{3}}(x^2 - 4x + 3) > \log_{\frac{1}{3}}(2x - 1)$

   (v) $\log_2(x-2) + \log_2(x-3) \geq 3$ (vi) $\log(x) + \log(x-3) \geq \log(2x) + \log(x-1)$

1.11 Applications of Exponential and Logarithmic Functions in Real-Life Situations

Although there are many real-life situations involving exponential and logarithmic functions, in this section we focus on selected applications such as growth and decay processes, the measurement of sound intensity in decibels, and compound interest.

Exponential models are used to describe quantities that change over time, while logarithmic functions help in determining unknown quantities in such models. The formula for growth and decay will be used as a given model and will be derived later in the chapter on differential equations, while the formulas for sound intensity and compound interest will be taken as standard results and will not be derived here.

(i) Growth and Decay

Many natural and physical processes such as population growth and radioactive decay change in such a way that the rate of change is proportional to the current amount. These processes are known as exponential growth or decay.

The general mathematical model is:

$$N = N_0\ e^{kt}$$

Where N_0 is the initial amount, N is the amount at time t, and k is a constant which is positive for growth and negative for decay.

In growth, the quantity increases with time, while in decay it decreases with time. Logarithms are used to determine unknown quantities such as time or rate in these models.

Example 22 (Growth)

A population is given by: $N = 200e^{0.04t}$

Find the time t when the population becomes 400, where t is measured in hours.

Solution: We are given that:

$$N = 200e^{0.04t}$$

Substituting $N = 400$:

$$400 = 200e^{0.04t}$$

$$2 = e^{0.04t}$$

Take natural logarithm on both sides:

$$\ln 2 = 0.04t$$

$$t = \frac{\ln 2}{0.04} \approx 17.33 \text{ hours}$$

Hence the populations become 400 in 17.33 hours approximately.

Example 23 (Decay)

A radioactive substance decays according to: $N = 500e^{kt}$

After 20 hours, the amount becomes 300.

Find the value of k, where t is in hours.

Solution: We have:

$$N = 500e^{kt}$$

Substituting $t = 20$ and $N = 300$:

$$300 = 500e^{20k}$$

$$\frac{3}{5} = e^{20k}$$

$$\ln\left(\frac{3}{5}\right) = 20k$$

$$k = \frac{1}{20}\ln\left(\frac{3}{5}\right) \approx -0.0256$$

Hence, $k \approx -0.0256$

(ii) Sound Intensity (Decibels)

The intensity of sound is the amount of sound energy passing through a unit area per unit time. Since sound intensity varies over a very wide range, it is measured on a logarithmic scale called the decibel (dB) scale.

The sound level L is given by:

$$L = 10 \log \left(\frac{I}{I_0} \right)$$

where I is the intensity of sound and I_0 is the **threshold of hearing**, which is the minimum sound intensity that a normal human ear can detect. Its standard value is given by:

$$I_0 = 10^{-12} \text{ W/m}^2$$

where, W/m^2 (watts per square meter) is the unit of sound intensity. It represents the amount of sound power (energy per second) passing through a unit area.

Because of the logarithmic scale, even a small increase in decibels represents a large change in actual sound intensity, which makes logarithmic laws very useful in comparing sound levels in real-life situations.

Example 24 **(Finding sound level)**

A sound has intensity $I = 10^{-4}$ and reference intensity $I_0 = 10^{-12}$. Find the sound level.

Solution: We have: $L = 10 \log \left(\frac{I}{I_0} \right)$

Substituting, $I = 10^{-4}$ and $I = 10^{-12}$:

$$L = 10 \log \left(\frac{10^{-4}}{10^{-12}} \right)$$

$$= 10 \log(10^8)$$

$$= 10 \times 8$$

$$= 80 \text{ dB}$$

Example 25 A sound has level $L = 60\text{dB}$ and $I_0 = 10^{-12}$. Find the intensity I.

Solution: We have: $L = 10 \log \left(\frac{I}{I_0} \right)$

Substituting $L = 60\text{dB}$ and $I_0 = 10^{-12}$:

$$60 = 10 \log \left(\frac{I}{10^{-12}} \right)$$

$$6 = \log \left(\frac{I}{10^{-12}} \right)$$

Convert to exponential form:

$$10^6 = \frac{I}{10^{-12}}$$

$$I = 10^6 \times 10^{-12} = 10^{-6} \text{ W/m}^2$$

Example 26 The sound levels at two points are $L_1 = 90$ dB and $L_2 = 60$ dB. Find the ratio of their intensities $\frac{I_1}{I_2}$, where $I_0 = 10^{-12}$ W/m^2.

Solution: Given: $L_1 = 90$ dB, $L_2 = 60$ dB

Formula: $L = 10 \log \left(\frac{I}{I_0} \right)$

For the given values of L_1 and L_2, we get:

$$90 = 10 \log\left(\frac{I_1}{I_0}\right) \quad \text{or} \quad 9 = \log\left(\frac{I_1}{I_0}\right)$$

$$60 = 10 \log\left(\frac{I_2}{I_0}\right) \quad \text{or} \quad 6 = \log\left(\frac{I_2}{I_0}\right)$$

Converting in exponential form:

$$\frac{I_1}{I_0} = 10^9 \quad \text{and} \quad \frac{I_2}{I_0} = 10^6$$

Dividing the above equations:

$$\frac{I_1}{I_2} = \frac{10^9}{10^6} \quad \text{and} \quad \frac{I_1}{I_2} = 10^3 = 1000$$

Which is the required ratio of intensities of the sound levels.

(iii) Compound Interest

Compound interest is a method of calculating interest in which the interest earned in each period is added to the **principal** (initial amount), so that interest is earned on both the principal and the accumulated interest. This results in exponential growth of money over time and is widely used in banking and financial systems to model investment growth. Compound interest is generally classified into two types: discrete compounding and continuous compounding.

Discrete Compound Interest

Discrete compound interest is a method of calculating interest in which the interest is added to the principal at fixed time intervals such as yearly, half-yearly, quarterly, or monthly. In this method, interest is not added continuously, but only at specific intervals, and the future amount is calculated using a fixed compounding frequency. This type of compounding leads to exponential growth of money over time and is commonly used in banking systems.

The discrete compound interest is given by the formula:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

where, A = Final amount

P = Principal (initial amount)

r = Interest rate per unit time (in decimal form)

n = Number of times interest is compounded per unit time

t = Time (measured in the same unit as the compounding period)

Continuous Compound Interest

Continuous compound interest is a method of calculating interest in which the interest is assumed to be added to the principal continuously at every instant of time. In this case, the compounding occurs without any fixed intervals, and the amount increases smoothly over time. This type of compounding represents the limiting case of discrete compounding when the number of compounding periods becomes infinitely large.

The formula for continuous compounding is:

$$A = Pe^{rt}$$

where,

A = Final amount after time t (principal + interest)

P = Principal (initial amount invested or borrowed)

r = Rate of interest per unit time (in decimal form)

t = Time (measured in the same unit as the rate is based on, such as years or months)

Example 27: A sum of Rs. 1000 is invested at a rate of 5% per year compounded continuously for 3 years. Find the final amount.

Solution: Given: $P = 1000$,

$r = 5\% = 0.05$

$t = 3$

Formula for the continuous compound interest:

$$A = Pe^{rt}$$

Substituting the values:

$$A = 1000e^{0.05 \times 3}$$

$$A = 1000e^{0.15}$$

$$A \approx 1000 \times 1.1618$$

$$= 1161.8$$

Hence, the final amount after 3 years is

$A \approx$ Rs. 1161.8.

Example 28: A sum of Rs. 5000 is invested at a rate of 8% per annum, compounded quarterly. Find the amount after 3 years.

Solution: We have: $P = 5000$

$r = 0.08$ (8% per annum)

Quarterly compounding $n = 4$

$t = 3$ years

Formula for the discrete compound interest:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Substituting the values:

$$A = 5000\left(1+\frac{0.08}{4}\right)^{4\times3}$$

$$A = (5000)(1.02)^{12}$$

$$\approx 6341.2$$

Hence, the amount after 3 years is

$A \approx$ Rs. 6341.2.

EXERCISE 1.5

1. A population is given by $N = 300e^{0.02t}$, where t is measured in hours. Find the time when the population becomes 600.
2. A culture grows according to $N = 150e^{kt}$, where time t is measured in hours. After 10 hours, the population becomes 250. Find the growth constant k.
3. A radioactive substance decays as $N = 800e^{-0.05t}$, where t is in hours. Find the time when the amount reduces to 200.
4. A substance decays according to $N = 600e^{kt}$, where time t is measured in hours. After 8 hours, the amount becomes 400. Find the decay constant k.
5. A sound has intensity $I = 10^{-6}$ W/m^2 and reference intensity $I_0 = 10^{-12}$ W/m^2. Find the sound level in decibels.
6. A sound has level $L = 80$ dB and $I_0 = 10^{-12}$ W/m^2. Find the intensity I.
7. A quantity changes according to $N = 500e^{kt}$. If $N = 250$ when $t = 5$, find:
   (i) the value of constant k.
   (ii) the quantity N when $t = 10$.
8. Two sounds have intensities I_1 and I_2 such that $L_1 = 85$ dB and $L_2 = 65$ dB.
   Find the ratio $\frac{I_1}{I_2}$, where $I_0 = 10^{-12}$ W/m^2.
9. A sum of Rs. 4000 is invested at 6% per annum, compounded annually for 3 years. Find the final amount.
10. A sum of Rs. 5000 becomes 6324.48 when invested at 8% per annum, compounded annually. Find the time t.
11. A sum of Rs. 3000 is invested at 5% per annum, compounded continuously for 4 years. Find the final amount.
12. An amount of Rs. 2000 becomes 3320.12 under continuous compounding at 6% per annum. Find the time t.

Unit 02 Further Differentiation

INTRODUCTION

Differentiation, a fundamental concept in calculus introduced by Newton and Leibniz, allows us to study how quantities change and evolve. In our previous class, we studied the definition of differentiation along with sum, difference, product, and quotient rules, and applied them to find derivatives of polynomials and rational functions. Building on that foundation, we will now explore how to find the derivatives of trigonometric, inverse trigonometric, exponential, and logarithmic functions. We will apply differentiation to identify increasing and decreasing functions and to find equations of tangents and normals to curves at specific points. Higher-order derivatives of algebraic, implicit, parametric, and transcendental functions will also be studied. We will use linear approximations and differentials to estimate function values, changes, and calculate relative and percentage errors in real-life contexts like volumes of geometric shapes. The unit also delves into finding local and global extrema using critical points

2.1 Review of Differentiation Basics

Definition: The derivative of the function $y = f(x)$ is denoted by $\frac{dy}{dx}$ or $f'(x)$ and is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ if exists or } \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \text{ if exists.}$$

The derivative gives the instantaneous rate of change of the function at a point.

Recall!

- Power Rule: $\frac{d}{dx}\left(x^n\right) = n.x^{n-1}$ where $n \in \mathbb{R}$.
- Constant Rule: $\frac{d}{dx}(c) = 0$, for any $c \in \mathbb{R}$.

In the following rules $u = u(x), v = v(x)$ are differentiable functions and c is any arbitrary constant.

- Constant Multiple Rule: $\frac{d}{dx}(cu) = c\frac{du}{dx} = cu'$.

- Sum and Difference Rule: $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx} = u' \pm v'$
- Product Rule: This rule helps to differentiate the product of two functions.

$$\frac{d}{dx}(uv) = u'v + uv'$$

- Quotient Rule: This rule is used to differentiate the division of two functions.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

where $v(x) \neq 0$ for all x in the domain.

2.2 Derivatives of Trigonometric Functions

Derivative of the Sine Function

While finding derivatives of trigonometric functions, we assume that the angle is measured in radians. To calculate the derivative of $f(x) = \sin x$.

Here, $f(x+h) = \sin(x+h)$, and

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$ (By the definition of derivative)

$$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$ (By substituting the values)

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h} \quad \because \sin p - \sin q = 2\cos\frac{p+q}{2}\sin\frac{p-q}{2}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\cdot\lim_{h\to 0}\frac{\sin(h/2)}{h/2}$$ applying limit $h \to 0$

$$= \cos x(1) = \cos x \qquad \left(\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1,\ \theta \text{ is measured in radians}\right)$$

Thus $\frac{d}{dx}(\sin x) = \cos x$

The derivative of the sine function is the cosine function

$$\frac{d}{dx}(\sin x) = \cos x$$

Try yourself!

Prove by definition

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Derivative of the Other Trigonometric Functions

After finding derivatives of $\sin x$ and $\cos x$ by definition, the derivatives of the remaining trigonometric functions can be found using product rule and quotient rule of differentiation.

Example 1 Find $f'(x)$ if $f(x)=\tan x$ (By using definition)

Solution: $f(x)=\tan x=\dfrac{\sin x}{\cos x}$

Differentiating w.r.t. 'x', we have

$$f'(x)=\frac{(\sin x)'\cos x-\sin x(\cos x)'}{\cos^2 x} \qquad \because \left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$$

$$=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x} \qquad \because \ (\sin x)'=\cos x,\ (\cos x)'=-\sin x$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$

$$=\frac{1}{\cos^2 x}=\sec^2 x \qquad \because \ \sin^2 x+\cos^2 x=1$$

Thus, $\dfrac{d}{dx}(\tan x)=\sec^2 x$.

The Derivatives of Six Trigonometric Functions

(i) $\dfrac{d}{dx}(\sin x)=\cos x$ (ii) $\dfrac{d}{dx}(\cos x)=-\sin x$

(iii) $\dfrac{d}{dx}(\tan x)=\sec^2 x$ (iv) $\dfrac{d}{dx}(\cot x)=-\csc^2 x$

(v) $\dfrac{d}{dx}(\sec x)=\sec x\tan x$ (vi) $\dfrac{d}{dx}(\csc x)=\csc x\cot x$

Example 2 Find $\dfrac{dy}{dx}$ if $y=x^2\csc x$

Solution: $y=x^2\csc x$

Differentiating w.r.t. 'x'

$$\frac{dy}{dx}=(x^2)'\csc x+x^2(\csc x)'=2x\csc x+x^2(-\csc x.\cot x) \quad =x\csc x(2-x\cot x)$$

2.3 The Chain Rule

2.3.1 Composition of Functions

Let $g: X\to Y$, and $f: Y\to Z$, then their composition $f\circ g: X\to Y$ is defined as:

$$f\circ g(x)=f(g(x))$$

For example, consider two real valued functions $g(x)=x^2-x$ and $f(x)=\sin x$, then

$$f\circ g(x)=f\left(g(x)\right)=f\left(x^2-x\right)=\sin\left(x^2-x\right)$$

Theorem 1: (The Chain Rule)

If $f(u)$ is differentiable at the point $u=g(x)$ and $g(x)$ is differentiable at x, then the composite function $f\circ g(x)=f(g(x))$ is differentiable at x, an $f\circ g'(x)=f'(g(x))\cdot g'(x)$

In Leibniz's notation, if $y=f(u)$ and $u=g(x)$, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

Critical Thinking

Suppose $f(x)=x^2$ and $g(x)=|x|$. Then the composition $(f\circ g)(x)=x^2$ is differentiable at $x=0$, although $g(x)$ is not differentiable at $x=0$. Does this contradict the Chain Rule?

Example 3 Differentiate $y=\sin(x^2-x)$ w.r.t 'x'.

Solution: Let $u=x^2-x$ ...(1)

Substituting $u=x^2-x$ into $y=\sin(x^2-x)$, we have $y=\sin u$.

Now, we have $y=\sin u$ and $u=x^2-x$

Differentiate with respect to u.

$$\frac{dy}{du}=\cos u$$

Differentiate with respect to x.

$$\frac{du}{dx}=2x-1$$

By Chain Rule $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\cos u\cdot(2x-1)$

$$\boxed{\frac{dy}{dx}=\cos(x^2-x)(2x-1)}$$ (By using equation 1)

2.3.2 Derivative of a Function Given in the Form of Parametric Equations

Let a function be represented by parametric equations

$$x=f(t) \quad \text{and} \quad y=g(t)$$

From $x=f(t)$, we have

$$t=f^{-1}(x)$$ (Provided f is bijective)

Now, by Chain Rule applying to $y=g(t)$ and $t=f^{-1}(x)$, we get:

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \left(\because \frac{dt}{dx}=\frac{1}{\frac{dx}{dt}}\right)$$

Example 4 Find $\frac{dy}{dx}$ if $y = a\cos t$ and $x = a\sin t$.

Solution: Differentiate $y = a\cos t$ and $x = a\sin t$ w.r.t 't'

$$\frac{dy}{dt} = -a\sin t \quad \text{and} \quad \frac{dx}{dt} = a\cos t$$

Apply the Chain Rule $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{a\sin t}{a\cos t} = -\tan t$$

2.3.3 Implicit Differentiation

When an implicit relation cannot be easily solved for y (or when multiple explicit functions are involved), we can still compute $\frac{dy}{dx}$ using implicit differentiation. The steps are as follows:

1. Differentiate both sides of $F(x, y) = 0$ with respect to x, treating y as an implicit function of x.
2. Apply the Chain Rule to terms containing y.
3. Solve the resulting equation for $\frac{dy}{dx}$.

Example 5 Find $\frac{dy}{dx}$ if $x + y^2 + \sin(xy^2) = 0$

Solution: Differentiate $x + y^2 + \sin(xy^2) = 0$ with respect to x to obtain:

$$\frac{d}{dx}(x) + \frac{d}{dx}(y^2) + \frac{d}{dx}\left(\sin(xy^2)\right) = 0$$

$$1 + 2y\frac{dy}{dx} + \cos(xy^2)\cdot\frac{d}{dx}(xy^2) = 0$$

$$2y\frac{dy}{dx} + \cos\left(xy^2\right)\left\{y^2 + x\left(2y\frac{dy}{dx}\right)\right\} = -1$$

$$2y\frac{dy}{dx} + y^2\cos(xy^2) + 2xy\frac{dy}{dx}\cos(xy^2) = -1$$

$$2y\frac{dy}{dx}\left\{1 + x\cos\left(xy^2\right)\right\} = -1 - y^2\cos\left(xy^2\right)$$

$$\boxed{\frac{dy}{dx} = -\frac{1 + y^2\cos(xy^2)}{2y\left\{1 + x\cos(xy^2)\right\}}}$$

2.4 Derivatives of Inverse Trigonometric Functions

Here, we will prove that:

(i) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$

(ii) $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$

(iii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$, for $x \in \mathbb{R}$

(iv) $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$, for $x \in \mathbb{R}$

(v) $\dfrac{d}{dx}(\sec^{-1}x)=\dfrac{1}{|x|\sqrt{x^2-1}}$, for $x\in(-\infty,-1)\cup(1,\infty)$

(vi) $\dfrac{d}{dx}(\csc^{-1}x)=-\dfrac{1}{|x|\sqrt{x^2-1}}$, for $x\in(-\infty,-1)\cup(1,\infty)$

Proof (1): Let $y=\sin^{-1}(x)$, where $x\in[-1,1]$ and $y\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$.

$$\sin y=x \quad \ldots(1)$$

Differentiate equation (1) w.r.t. x, we have

$$\cos y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{\cos^2 y}}=\frac{1}{\sqrt{1-\sin^2 y}}$$

$$\boxed{\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}},\quad \text{for } x\in(-1,1)\quad (\because \sin y=x)}$$

Proof (2): Let $y=\cot^{-1}(x)$, where $y\in(0,\pi)$ and $x\in\mathbb{R}$.

$$\cot y=x \qquad \ldots(2)$$

We differentiate equation (1) w.r.t. x,

$$-\csc^2 y\frac{dy}{dx}=1 \qquad \left((\cot y)'=-\csc^2 y\right)$$

$$\frac{dy}{dx}=-\frac{1}{\csc^2 y} \qquad \left(\csc y\neq 0,\ \text{for } y\in(0,\pi)\right)$$

$$=-\frac{1}{1+\cot^2 y} \qquad (\because\ 1+\cot^2 y=\csc^2 y)$$

$$\boxed{\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}}\ \text{for } x\in R \qquad (\because \cot y=x)$$

Proof (3): Let $y=\sec^{-1}x$, where $x\in(-\infty,-1]\cup[1,\infty)$ and $y\in[0,\pi]-\left\{\dfrac{\pi}{2}\right\}$

$$\sec y=x \qquad \ldots(1) \quad , \quad |x|\geq 1$$

We differentiate equation (1) w.r.t. x:

$$\sec y\tan y\frac{dy}{dx}=1 \qquad \left(\because (\sec y)'=\sec y\tan y\right)$$

$$\frac{dy}{dx}=\frac{1}{\sec y\tan y} \qquad \text{where, } \sec y\neq 0 \text{ and } \tan y\neq 0$$

$\tan y \neq 0 \Rightarrow y \neq 0$ and $y \neq \pi \Rightarrow x \neq -1, 1$ and from equation (1).

$$\frac{dy}{dx} = \frac{1}{x \tan y} \qquad ...(2) \qquad (\because \sec y = x)$$

$$\tan y = \begin{cases} \sqrt{\tan^2 y}, & y \in \left(0, \frac{\pi}{2}\right) \\ -\sqrt{\tan^2 y}, & y \in \left(\frac{\pi}{2}, \pi\right) \end{cases} = \begin{cases} \sqrt{\sec^2 y - 1}, & y \in \left(0, \frac{\pi}{2}\right) \\ -\sqrt{\sec^2 y - 1}, & y \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

$$\tan y = \begin{cases} \sqrt{x^2 - 1}, & x \in (1, \infty) \\ -\sqrt{x^2 - 1}, & x \in (-\infty, -1) \end{cases}$$

Challenge!

Prove formulae (2), (3) and (5).

$$\tan y = \begin{cases} \sqrt{x^2 - 1} & x \in (1, -1) \\ -\sqrt{x^2 - 1}, & x \in (-\infty, 1) \end{cases} \left(\because \sec y = x, \ and \ x \in (1, \infty) \text{ when } y \in \left(0, \frac{\pi}{2}\right), \text{ while } x \in (-\infty, -1) \text{ when } y \in \left(\frac{\pi}{2}, \pi\right) \right)$$

Substitute the value of $\tan y$ in equation (2)

$$\frac{dy}{dx} = \begin{cases} \frac{1}{x\sqrt{x^2 - 1}}, & x \in (1, \infty) \\ \frac{1}{-x\sqrt{x^2 - 1}}, & x \in (-\infty, -1) \end{cases}$$

Recall!

$$\sqrt{x^2} = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Hence, $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$ $\left(\because |x| = \begin{cases} x & if \ x \geq 0 \\ -x & if \ x < 0 \end{cases} \right)$

Example 6 Find $\frac{dy}{dx}$ if $y = x \cos^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2}$.

Solution: Differentiate $y = x \cos^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2}$ w.r.t. x to obtain:

$$\frac{dy}{dx} = \cos^{-1}\left(\frac{x}{a}\right) + x\left(-\frac{1}{\sqrt{1-(x/a)^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)\right) - \frac{1}{2}(a^2 - x^2)^{-1/2} \cdot \frac{d}{dx}(a^2 - x^2)$$

$$= \cos^{-1}\left(\frac{x}{a}\right) - \frac{x}{\sqrt{\frac{a^2 - x^2}{a^2}}} \cdot \frac{1}{a} - \frac{1}{2} \cdot \frac{1}{\sqrt{a^2 - x^2}}(-2x)$$

$$\frac{dy}{dx} = \cos^{-1}\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right)$$

EXERCISE 2.1

1. Derive the formula for the derivative of the following functions w.r.t. x:
   (a) $\sec x$ (b) $\csc x$ (c) $\cot x$

In 2 - 6, differentiate with respect to the variable involved.

2. $y = x^2 \cos x$
3. $y = x\sec x + \dfrac{1}{x}$
4. $y = \dfrac{\sin t}{1-\cos t}$
5. $y = (1+\csc t)\sin t$
6. $y = \sec x + 4\sqrt{x} - 10$

In 7 - 9, find $\dfrac{dy}{dx}$ by making suitable substitutions and applying the Chain Rule.

7. $y = \sqrt{\dfrac{1+x}{1-x}}$
8. $y = (\sin x + \cos x)^{3/2}$

In 9 - 12, differentiate w.r.t the variable involved.

9. $y = \sec(\tan x)$
10. $y = \sqrt{\sin x} + \cos\sqrt{x}$
11. $y = \tan^2 x + \cot(x^2)$
12. $y = \sin^3(\cos 2v)$
13. Find $\dfrac{dy}{dx}$ for the following parametric equations:
    (i) $x = \sin^2 t$ and $y = \cos 2t$ (ii) $x = \sqrt{1+\sec t}$, and $y = \sqrt{\tan t}$
14. If $x = a\sec t$ and $y = b\tan t$, show that $\dfrac{dy}{dx} = \dfrac{-b^2 x}{a^2 y}$.
15. Use implicit differentiation to find $\dfrac{dy}{dx}$ if:
    (i) $ax^2 + by^2 + 2gx + 2fy + c = 0$ (ii) $x + \tan(xy) = 0$
16. Differentiate w.r.t. the variables involved.
    (i) $\sin^{-1}\left(\dfrac{x}{a}\right)$ (ii) $\csc^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)$
    (iii) $\cot^{-1}\left(\sqrt{x}\right) + \sqrt{\cot^{-1} x}$
    (iv) $(\tan t)^{-1} + \dfrac{1}{\tan^{-1} t}$

Point to Note

$\left|\dfrac{x^2+1}{x^2-1}\right| = \dfrac{x^2+1}{x^2-1} > 1$ for $x \in (-\infty,-1) \cup (1,\infty)$, and

$\dfrac{x^2+1}{x^2-1} < -1$ has no solution.

2.5 Derivative of Exponential Functions

Recall!

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a$$

A function defined by

$$f(x) = a^x, \quad \text{where } a > 0 \text{ and } a \neq 1$$

is called an exponential function.

If $a = e \approx 2.71828$ then $f(x) = e^x$, is called natural exponential function.

Now, we shall find the derivative of $f(x) = a^x$, using the definition.

Let $f(x) = a^x$ ... (1)

From equation (1) we have $f(x+h) = a^{x+h}$ ...(2)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$ (definition of derivative)

After substituting the expression for $f(x)$ and $f(x+h)$ from equations (1) and (2), we get:

$$= \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h-1)}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h-1}{h} = a^x \ln a$$

Hence, we have proved that $\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$.

Challenge!

- Prove that $\frac{d}{dx}(e^x) = e^x$
- Do you know $\ln e = 1$?

Example 7 Find $\frac{dy}{dx}$ if (i) $y = e^{\sin^2 x}$ (ii) a^{x^3}

Solution: **(i)** Differentiate $y = e^{\sin^2 x}$ w.r.t. x and apply the Chain Rule, to obtain:

$$\frac{dy}{dx} = e^{\sin^2 x} \frac{d}{dx}(\sin^2 x) \qquad \left(\because \frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}\right)$$

$$\frac{dy}{dx} = e^{\sin^2 x} 2\sin x \cdot \frac{d}{dx}(\sin x) = 2\sin x\, e^{\sin^2 x} \cos x$$

$$= 2\sin x \cos x\, e^{\sin^2 x}$$

(ii) Differentiate $y = a^{x^3}$ w.r.t. x and apply the Chain Rule, to obtain:

$$\frac{dy}{dx} = a^{x^3} \ln a \cdot \frac{d}{dx}(x^3) \qquad \left(\because (a^u)' = a^u \ln a \cdot \frac{du}{dx}\right)$$

$$= a^{x^3} \ln a (3x^2)$$

$$= 3x^2 a^{x^3} \ln a$$

2.6 Derivative of Logarithmic Functions

If $a^y = x$, where $a > 0$ and $a \neq 1$, then y is called the *logarithm* of x to the base a, written as:

$$y = \log_a x \quad , \quad x > 0 \quad , \quad y \in \mathbb{R}$$

If $a = e$, where $e \approx 2.71828$, then the function

$$y = \log_e x = \ln x \text{ (whenever } e^y = x)$$

is called the *natural logarithm* of x.

Now, we will find the derivative of $f(x) = \log_a x$, using the definition.

Let $f(x) = \log_a x$ ... (1)

From equation (1) we have: $f(x+h) = \log_a (x+h)$ ... (2)

We know $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ (definition of derivative)

After substituting the expression for $f(x)$ and $f(x+h)$ from equations (1) and (2), we get:

$$= \lim_{h\to 0} \frac{\log_a (x+h) - \log_a x}{h} = \lim_{h\to 0} \frac{1}{h} \log_a \left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{x} . \frac{x}{h} \log_a \left(1 + \frac{h}{x}\right) = \lim_{h\to 0} \frac{1}{x} \log_a \left(1 + \frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \frac{1}{x} . \log_a e$$

$$= \frac{1}{x} . \frac{1}{\log_e a} \qquad \left(\because \log_a b = \frac{1}{\log_b a}\right)$$

$$= \frac{1}{x \ln a} \qquad (\because \log_e x = \ln x)$$

Hence, we have proved that $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$.

Recall!

$$\lim_{h\to 0} \left(1 + \frac{h}{a}\right)^{\frac{a}{h}} = e$$

Example 8 Find $\frac{dy}{dx}$ if $y = \frac{a^x}{\log_a x}$.

Solution: Differentiate $y = \frac{a^x}{\log_a x}$ with respect to 'x', and apply the Quotient Rule to obtain:

$$\frac{dy}{dx} = \frac{(a^x)' \log_a x - a^x (\log_a x)'}{(\log_a x)^2}$$

$$= \frac{a^x \ln a \log_a x - a^x \cdot \frac{1}{x \ln a}}{(\log_a x)^2}$$

$$\boxed{= \frac{a^x \left(x (\ln a)^2 \log_a x - 1\right)}{x \ln a (\log_a x)^2}}$$

Example 9

Find $\frac{dy}{dx}$, if $y = \ln(\cos(e^x))$.

Solution: Differentiate $y = \ln(\cos(e^x))$ with respect to 'x', and apply the Chain Rule to obtain:

$$\frac{dy}{dx} = \frac{1}{\cos(e^x)} \frac{d}{dx}(\cos e^x)$$

$$= \frac{1}{\cos(e^x)} (-\sin e^x) \frac{d}{dx}(e^x)$$

$$= -\frac{\sin(e^x)}{\cos(e^x)} e^x$$

$$\boxed{= -e^x \tan(e^x)}$$

Challenge!

Prove that $\frac{d}{dx}(\ln x) = \frac{1}{x}$

EXERCISE 2.2

1. Find $f'(x)$ if:

(i) $f(x)=(x^3+4x)e^{1/x}$ (ii) $f(x)=\sin(2^x)$

(iii) $f(x)=e^{x\cos x}$ (iv) $f(x)=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

2. Find $\dfrac{dy}{dx}$ if:

(i) $y=\ln(e^x-e^{-x})$ (ii) $y=\log_{10}\sqrt{\dfrac{1-x^2}{1+x^2}}$ (iii) $y=\ln\left(x-\sqrt{x^2-1}\right)$

(iv) $y=\ln(1-\cos^2 x)$ (v) $y=\dfrac{\ln x}{x^2}$ (vi) $y=\dfrac{\sqrt{x^2-1}(2x+1)}{(x^3-1)^{3/2}}$

2.7 Equations of Tangent and Normal Lines to a Curve

Tangent Line: Suppose that x_0 is a point in the domain of the function $f(x)$. The tangent line to the curve $y=f(x)$ at the point $(x_0, f(x_0))$ is the line with equation

$$y-y_0=m(x-x_0)$$

where $m=f'(x_0)=\left.\dfrac{dy}{dx}\right|_{x=x_0}$, provided $f'(x_0)$ exists.

Need to Know!

Vertical Tangent
If m is ∞ or $-\infty$ at $x=x_0$, then $x=x_0$ is tangent to the curve $y=f(x)$ is called vertical tangent

Normal Line: A line through point $(x_0, f(x_0))$ of the curve $y=f(x)$ and perpendicular to tangent line is called normal line to the curve $y=f(x)$.

Note

Slope of tangent line to a curve at a point is also known as slope of the curve at the point.

If $m=f'(x_0)$ is non zero, then equation of the normal line is $y-y_0=-\dfrac{1}{m}(x-x_0)$ and if $m=f'(x_0)=0$, then $x=x_0$ is the equation of the normal line.

Example 10 Find the equations of the tangent and normal to the curve $y=\sec x$ at the point where $x=\dfrac{\pi}{4}$.

Solution: Here, $y=\sec x$ ...(1)

Differentiating (1) with respect to x:

$$\frac{dy}{dx}=\sec x\tan x$$

$$m = \left.\frac{dy}{dx}\right|_{x=\pi/4} = \sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \sqrt{2} \quad \text{(Slope of tangent line)}$$

Put $x = \frac{\pi}{4}$ in (1), $y = \sec\left(\frac{\pi}{4}\right) = \sqrt{2}$

The equation of tangent at $\left(\frac{\pi}{4}, \sqrt{2}\right) = (x_0, y_0)$ with slope $m = \sqrt{2}$ is:

$$y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right) \qquad \because y - y_0 = m(x - x_0)$$

$$y - \sqrt{2} = \sqrt{2}x - \frac{\pi\sqrt{2}}{4}$$

$$4y - 4\sqrt{2} = 4\sqrt{2}x - \pi\sqrt{2}$$

$$4\sqrt{2}x - 4y + 4\sqrt{2} - \pi\sqrt{2} = 0$$

The equation of normal to given curve at $\left(\frac{\pi}{4}, \sqrt{2}\right) = (x_0, y_0)$ is:

$$y - \sqrt{2} = -\frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

$$x + 4\sqrt{2}y - 8 - \pi = 0$$

2.8 Higher-Order Derivatives

The derivative f' of a function f is itself a function and hence may have a derivative of its own. If f' is differentiable, then its derivative is denoted by f'' and is called the *second derivative* of f. As long as the new function remains differentiable, we can continue the process of differentiating to obtain the third, fourth, fifth, and even higher derivatives of f. These successive derivatives are called *higher-order* derivatives.
The following are the commonly used notations for higher-order derivatives:

1st derivative	2nd derivative	3rd derivative	...	n^{th} derivative
y'	y''	y'''	...	$y^{(n)}$
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	...	$\frac{d^ny}{dx^n}$
$f'(x)$	$f''(x)$	$f'''(x)$	...	$f^{(n)}(x)$
y_1	y_2	y_3	...	y_n
Dy	D^2y	D^3y	...	D^ny
$\frac{df}{dx}$	$\frac{d^2f}{dx^2}$	$\frac{d^3f}{dx^3}$	...	$\frac{d^nf}{dx^n}$

Example 11 Find the higher-order derivatives of the function

$$f(x)=\frac{1}{4}x^4-x^3+\frac{1}{2}x^2-3$$

Solution:

Differentiate $f(x)=\frac{1}{4}x^4-x^3+\frac{1}{2}x^2-3$ to find its higher-order derivatives.

$$f'(x)=\frac{1}{4}.4x^3-3x^2+\frac{1}{2}\cdot 2x-0$$

$$\begin{aligned} f''(x) &= 3x^2-6x+1\\ f'''(x) &= 6x-6\\ f^{(4)}(x) &= 6\\ f^{(5)}(x) &= 0\\ &\vdots\\ f^{(n)}(x) &= 0\ (n\geq 5) \end{aligned}$$

Example 13 Find all higher-order derivatives of the function $y=e^{ax}$.

Solution: We differentiate $y=e^{ax}$ to find its higher-order derivatives.

$$\frac{dy}{dx}=e^{ax}.\frac{d}{dx}(ax)$$

$$=e^{ax}\cdot a=ae^{ax}$$

Differentiate with respect to 'x', to obtain:

$$\frac{d^2y}{dx^2}=a(a^{ax}a)=a^2e^{ax}$$

$$\frac{d^3y}{dx^3}=a^2(e^{ax}a)=a^3e^{ax}$$

$$\frac{d^4y}{dx^4}=a^4e^{ax}$$

$$\vdots$$

$$\frac{d^ny}{dx^n}=a^ne^{ax}\qquad (n\geq 1)$$

$$=a^ny$$

Example 12 If $x=e^{at}$ and $y=e^{2at}$, show that $x^2\frac{d^2y}{dx^2}-2y=0$.

Solution: Given: $x=e^{at}$ ... (1)

and $y=e^{2at}$ ... (2)

Differentiate equations (1) and (2) with respect to 't', to get:

$$\frac{dx}{dt}=ae^{at} \text{ and } \frac{dy}{dt}=2ae^{2at}$$

Using equations (1) and (2), we have:

$$\frac{dx}{dt}=ax \text{ and } \frac{dy}{dt}=2ay$$

Apply the Chain Rule to obtain:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2ay}{ax}$$

$$\frac{dy}{dx}=\frac{2y}{x} \qquad \ldots(3)$$

Differentiate with respect to 'x', to obtain:

$$\frac{d^2y}{dx^2}=2\frac{\frac{dy}{dx}x-y.1}{x^2}$$

Using equation (3), we get:

$$=2\frac{\frac{2y}{x}x-y}{x^2}$$

$$=2\frac{y}{x^2}$$

$$x^2\frac{d^2y}{dx^2}=2y$$

$$\Rightarrow\quad x^2\frac{d^2y}{dx^2}-2y=0$$

Example 14: Find y_4 if $y = \ln\sqrt{\dfrac{2x-3}{x+1}}$.

Challenge!

Find y_{10} in example 16.

Solution: We are given: $y = \ln\sqrt{\dfrac{2x-3}{x+1}}$

Before differentiating we apply properties of logarithm:

$$y = \frac{1}{2}\ln\left(\frac{2x-3}{x+1}\right)$$

$$= \frac{1}{2}\left[\ln(2x-3) - \ln(x+1)\right]$$

$$y_1 = \frac{1}{2}\left(\frac{2}{2x-3} - \frac{1}{x+1}\right) \qquad \text{(By differentiating w.r.t. } x\text{)}$$

$$= \frac{1}{2}\left(2(2x-3)^{-1} - (x+1)^{-1}\right)$$

$$y_2 = \frac{1}{2}\left(-2(2x-3)^{-2} + (x+1)^{-2}\right) \qquad \text{(By differentiating w.r.t. } x\text{)}$$

$$y_3 = \frac{1}{2}\left(4(2x-3)^{-3} - 2(x+1)^{-3}\right) \qquad \text{(By differentiating w.r.t. } x\text{)}$$

$$y_4 = \frac{1}{2}\left(-12(2x-3)^{-4} + 6(x+1)^{-4}\right) \qquad \text{(By differentiating w.r.t. } x\text{)}$$

$$y_4 = \frac{1}{2}\left[\frac{-12}{(2x-3)^4} + \frac{6}{(x+1)^4}\right] = \frac{-6}{(2x-3)^4} + \frac{3}{(x+1)^4}$$

EXERCISE 2.3

1. Find the equation of the tangent line to the curve $y = \sin^{-1}\left(\dfrac{x}{2}\right) + \cos x$ at the point where $x = \dfrac{\pi}{2}$.
2. Find a point on the curve $y = \tan^{-1}\left(\dfrac{x}{3}\right) - \dfrac{x}{3}$, where tangent is horizontal. Also find the equations of the tangent and normal at that point.
3. Find a point on the curve $x + y - e^y = 0$ where the tangent line is vertical. Also, find the equations of the tangent and normal at that point.
4. Find all points on the ellipse $x^2 - xy + y^2 = 1$ at which the tangent line is horizontal.

5. Find y_2 if:

(i) $y = x^4 - 3x^2 + 4x - 5$ (ii) $y = \ln(2x+3)$ (iii) $y = x^3 e^{2x}$

(iv) $x^2 - y^2 = 4$ (v) $x^2 - xy + y^2 = 7$

(vi) $x = a\cos t,\quad y = b\sin t$ (vii) $x = t^2,\quad y = 2t^3$

6. Find y_4 and write expressions for y_{10} and y_{11} without evaluating them:

(i) $y = \sin\left(\frac{x}{2}\right)$ (ii) $y = \ln(x^2 - a^2)$

7. If $x^2 + y^2 = 25$, show that $y^3 \frac{d^2y}{dx^2} + 25 = 0$.

8. If $y = e^{ax}\cos bx$, show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$

9. If $x = \sin t$ and $y = \cos^2 t$, show that $\frac{d^2y}{dx^2} + 2 = 0$

10. If $x = \ln t$ and $y = \frac{1}{t}$, show that $\frac{d^2y}{dx^2} - y = 0$

11. If $y = a\cos(\ln x) + b\sin(\ln x)$, show that $x^2 y_2 + xy_1 + y = 0$

2.9 Increasing and Decreasing Functions

Increasing Function	Decreasing Function
Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f is increasing if for any $x_1, x_2 \in I$. $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$	Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f is decreasing if for any $x_1, x_2 \in I$. $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

Recall!

$(a,b), [a,b], (a,b], [a,b), (-\infty,a), (-\infty,\infty)(-\infty,a], (a,\infty), [a,\infty)$ all are intervals.

Let f be a differentiable function defined on an interval I. Then

(i) f is increasing on I, if $f'(x) \ge 0$ for each $x \in I$.

(ii) f is decreasing on I, if $f'(x) \le 0$ for each $x \in I$.

Example 15 Prove that $f(x) = e^x$ defined on $R = (-\infty, \infty)$ is increasing.

Solution: We differentiate $f(x) = e^x$ with respect to 'x' to obtain.

$$f'(x) = e^x > 0 \text{ for each } x \in (-\infty, \infty)$$

Hence, $f(x) = e^x$ is an increasing function on $R = (-\infty, \infty)$.

Example 16 Show that $f(x)=2^{-x}$ defined on $R=(-\infty,\infty)$ is decreasing.

Solution: We differentiate $f(x)=2^{-x}$ with respect to 'x' to obtain.

$$f'(x)=2^{-x}\frac{d}{dx}(-x)$$

$$f'(x)=-2^{-x}<0 \text{ for each } x\in(-\infty,\infty)$$

So, $f(x)=2^{-x}$ is decreasing function on $R=(-\infty,\infty)$.

2.9.1 Stationary Point

A point on the graph where its derivative is zero is called a stationary point.

Example 16 Find the intervals on which the function $f(x)=|x|$, defined on $\mathbb{R}=(-\infty,\infty)$, is increasing or decreasing.

Solution: We know that:

$$f(x)=|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}.$$

Critical Thinking

- Every stationary point is a critical point.
- Do you think every critical point is a stationary point?

Differentiating w. r. t. 'x':

$$f'(x)=\begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

We now evaluate $f'(0)$ by definition:

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \quad \text{(By definition of derivative)}$$

$$=\lim_{h\to 0}\frac{f(h)-0}{h}=\lim_{h\to 0}\frac{|h|}{h}$$

Since $f(x)=|x|$ is defined different on left and right of $x=0$, we compute one sided limits:

$$\lim_{h\to 0^-}\frac{|h|}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1 \quad \text{and} \quad \lim_{h\to 0^+}\frac{|h|}{h}=\lim_{h\to 0^+}\frac{h}{h}=1$$

Since left-hand and right-hand limits are different, the limit $\lim_{h\to 0}\frac{|h|}{h}$ does not exist.

Hence $f'(0)$ is not defined. ($\therefore$ x is a critical point but not stationary)

The function has no stationary point because there is no point where the derivative is equal to zero.

Since $f'(x)=-1<0$ for each $x\in(-\infty,0)$ $\Rightarrow$ the function is decreasing on $(-\infty,0)$ and $f'(x)=1>0$ for each $x\in(0,\infty)$ $\Rightarrow$ the function is increasing on $(0,\infty)$

Hence $f(x)=|x|$ is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$.

In the above example, the function has no stationary point, but it changes behavior around a point where its derivative is undefined.

Conclusion: From these examples, we conclude that a function may change its:

1. Increasing/decreasing behavior around:
   - Stationary points where $f'(x)=0$, or
   - Points where $f'(x)$ is undefined.

2.9.2 Critical Point

For a function $f(x)$ a point $x = c$, $c \in$ Dom f is called a critical point if:

$f'(c)=0$ or $f'(c)$ does not exist.

Examples:

1. For $f(x)=|x|$, the derivative does not exist at $x = 0$, so $x = 0$ is a critical point.
2. For $f(x)=x^2$, the derivative $f'(x)=2x$ is zero at $x=0$, so $x=0$ is a critical point.

Method to Find Intervals Where a Function is Increasing or Decreasing

To find the intervals where a function is increasing or decreasing, follow these steps:

Step 1: Differentiate the given function $f(x)$ to find $f'(x)$.

Step 2: Find critical points by solving $f'(x)=0$ and identifying points where $f'(x)$ does not exist.

Step 3: If critical points exist, divide the domain into sub-intervals using the critical points.

Step 4: Select a test point from each sub-interval and substitute it into $f'(x)$:

- If $f'(x)>0$ in a sub-interval, the function is increasing there.
- If $f'(x)<0$ in a sub-interval, the function is decreasing there.

Point to Note

In each sub-interval of the domain determined by the critical points, the sign of the derivative remains the same.

Example 12 Find the intervals in which the function $f(x)=2x^3-3x^2-12x$ is increasing or decreasing.

Solution: The function $f(x)=2x^3-3x^2-12x$ is defined for all $x \in R=(-\infty,\infty)$.

We differentiate $f(x)=2x^3-3x^2-12x$ with respect to x to obtain:

$$f'(x)=6x^2-6x-12 \quad \ldots (1)$$

It is clear from equation (1) that $f'(x)$ is defined for each $x \in (-\infty,\infty)$.

To find other critical points, we solve $f'(x)=0$:

$$6x^2-6x-12=0$$
$$6(x^2-x-2)=0$$
$$6(x+1)(x-2)=0$$
$$x=-1 \quad \text{and} \quad x=2$$

Hence $x=-1$ and $x=2$ are the critical points of the given function.
Mark these points on the real line. These critical points divide the domain into intervals $(-\infty,-1)$, $(-1,2)$ and $(2,\infty)$.



Figure 2.1

Now we check the sign of $f'(x)$ in each subinterval.

Interval	Test point	Sign of $f'(x)$	Conclusion
$(-\infty,-1)$	$x=-2$	$f'(-2)=6(-2)^2-6(-2)-12=24>0$	Increasing
$(-1,2)$	$x=0$	$f'(0)=6(0)^2-6(0)-12=12<0$	Decreasing
$(2,\infty)$	$x=3$	$f'(3)=6(3)^2-6(3)-12=24>0$	Increasing

Hence the function $f(x)=2x^3-3x^2-12x$:

- Increases on $(-\infty,-1)$ and $(2,\infty)$.
- Decreases on $(-1,2)$.

Example 19 Find the intervals where the function $f(x)=x^{\frac{2}{3}}$ is increasing or decreasing on the domain $x\in(-\infty,\infty)$.

Solution: We differentiate $f(x)=x^{\frac{2}{3}}$ with respect to 'x' to obtain:

$$f'(x)=\frac{2}{3}x^{-\frac{1}{3}}=\frac{2}{3x^{\frac{1}{3}}} \qquad \ldots (1)$$

From equation (1), we conclude that $f'(0)$ is not defined. Thus $x=0$ is a critical point of the function $f(x)=x^{\frac{2}{3}}$.

To find other critical points we solve $f'(x)=0$.

$$\frac{2}{3x^{\frac{1}{3}}}=0$$

The above equation has no solution. Thus, the only critical point is $x=0$. This critical point divides the domain in the following intervals.

$$(-\infty, 0) \text{ and } (0, \infty)$$

Choose $x = -1 \in (-\infty, 0)$ and substitute into equation (1).

$$f'(-1) = \frac{2}{3(-1)^{\frac{1}{3}}} = -\frac{2}{3} < 0 \Rightarrow f'(x) < 0 \text{ for all } x \in (-\infty, 0). \left(\because (-1)^{\frac{1}{3}} = -1\right)$$

Thus, the function is decreasing on $(-\infty, 0)$.

Choose $x = 1 \in (-\infty, 0)$, and substitute into equation (1).

$$f'(1) = \frac{2}{3(1)^{\frac{1}{3}}} = \frac{2}{3} > 0 \Rightarrow f'(x) > 0 \text{ for all } x \in (0, \infty).$$

Thus, the function is increasing on $(0, \infty)$.

Hence the function $f(x) = x^{\frac{2}{3}}$ defined on $(-\infty, \infty)$:

- Decreases on $(-\infty, 0)$.
- Increases on $(0, \infty)$.

2.10 Relative and Absolute Extrema

2.10.1 Global (Absolute) Maximum

Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f has a global (absolute) maximum at $x = c \in I$, if $f(c) \geq f(x)$ for all $x \in I$.

If $f(x)$ has a global (absolute) maximum at $x = c$, then $f(c)$ is called global (absolute) maximum value of the function.

2.10.2 Global (Absolute) Minimum

Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f has a global (absolute) minimum at $x = c \in I$, if $f(c) \leq f(x)$ for all $x \in I$.

If $f(x)$ has a global (absolute) minimum at $x = c$, then $f(c)$ is called global (absolute) minimum value of the function.

The global maximum and minimum of a function are collectively called its global extrema.

Example: (a) Let $f(x) = x^2$ be defined on the interval [0, 2].

The function f has a global maximum at $x = 2$ and a global minimum at $x = 0$. The corresponding maximum and minimum values of the function are:

$f(2) = 4$ (global maximum value)

and $f(0) = 0$ (global minimum value) (See Figure 2.2).

(b) The function $f(x) = x^2$ defined on $(-\infty, \infty)$ has an absolute minimum value at $x = 0$, where $f(0) = 0$. It has no absolute maximum value. (See Figure 2.3)

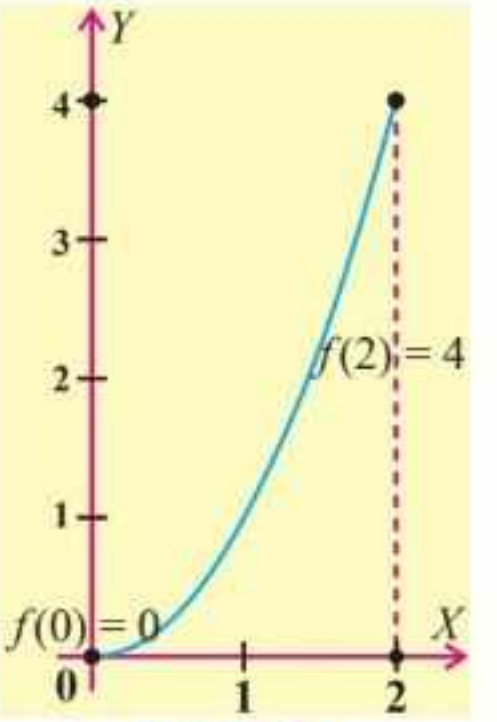


Figure 2.2

(c) The function $f(x) = -x^2 + 2$ defined on $(-\infty, \infty)$ has an absolute maximum value at $x = 0$, where $f(0) = 2$. It has no absolute minimum value. (See Figure 2.4)

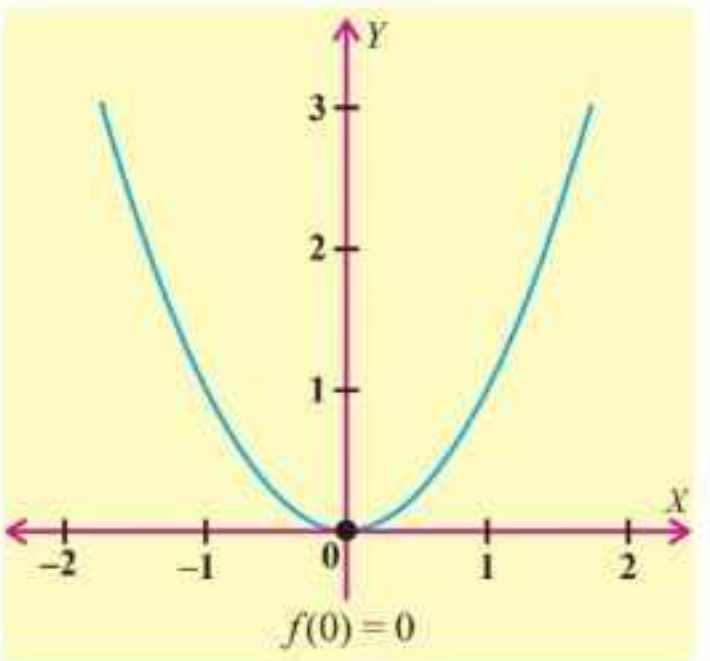


Figure 2.3

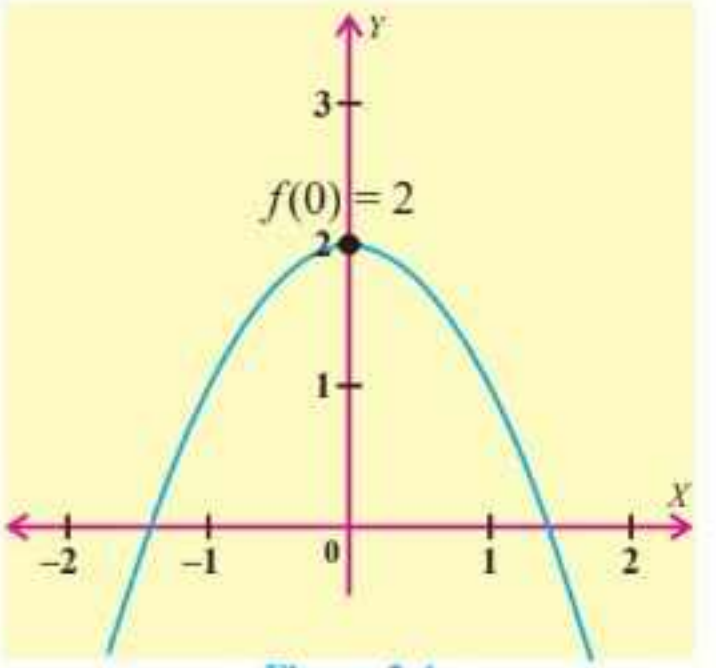


Figure 2.4

Local (Relative) Maxima	Local (Relative) Minima
Let I be an interval and let $f : I \to \mathbb{R}$ be a function. We say that f has a local (relative) maxima at $x = c \in I$, if there exists an open interval I_1 such that $c \in I_1$ and: $f(c) \geq f(x)$ for all $x \in I \cap I_1$ If $f(x)$ has a local (relative) maximum at $x = c$, then $f(c)$ is called local (relative) maximum value of the function. (See Figure 2.5)	Let I be an interval and let $f : I \to \mathbb{R}$ be a function. We say that f has a local (relative) minima at $x = e \in I$, if there exists an open interval I_1 such that $e \in I_1$ and: $f(e) \leq f(x)$ for all $x \in I \cap I_1$ If $f(x)$ has a local (relative) minimum at $x = e$, then $f(e)$ is called local (relative) minimum value of the function. (See Figure 2.5)

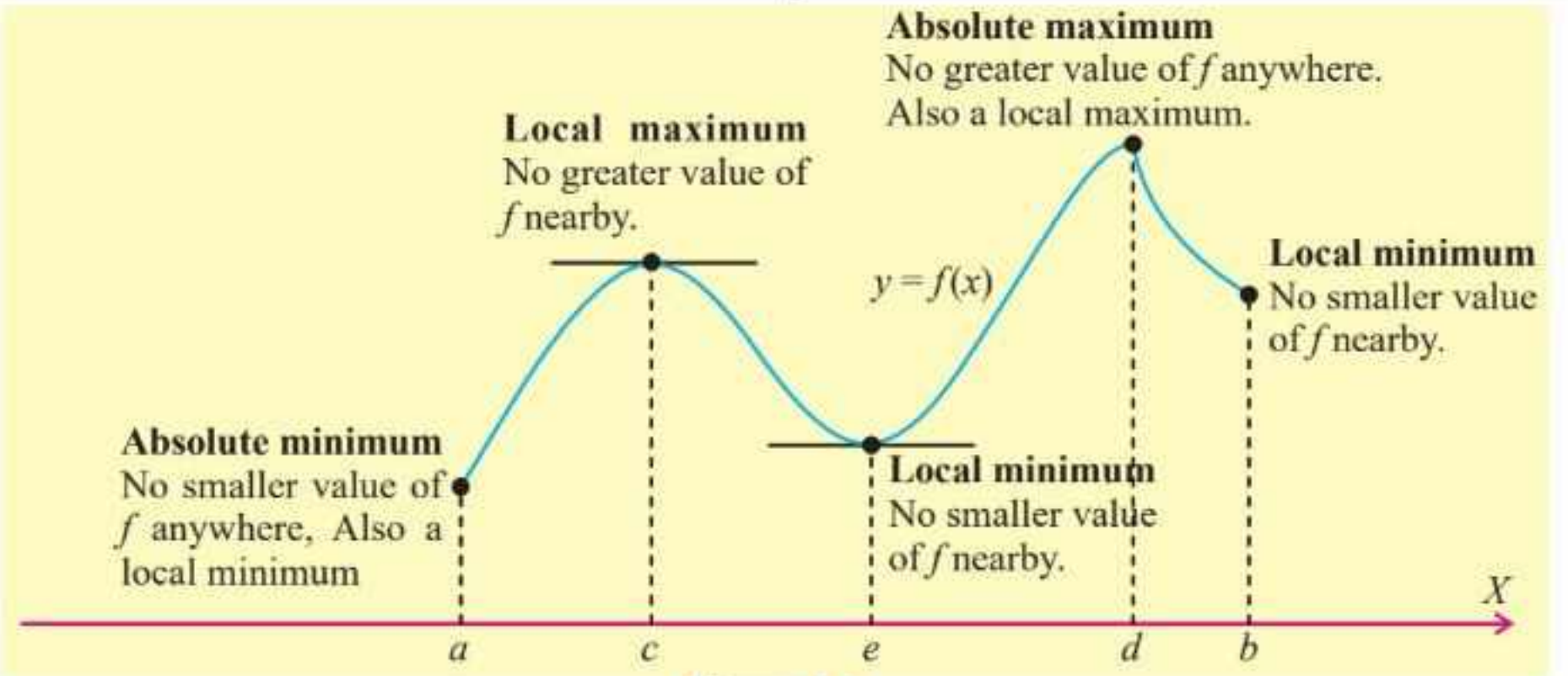


Figure 2.5

2.10.3 Method to Find Global Extrema of a Continuous Functions Defined on a Closed Interval $[a, b]$

1. Find all critical points of the function within the interval $[a, b]$.
2. Find values of the function at critical points and at end points of the interval.
3. The largest of these values is the global maximum value, and the smallest is the global minimum value.

Example 20 Find the global extreme values of $f(x) = x^4 - 4x$ on $[-2, 3]$.

Solution: We are given: $f(x) = x^4 - 4x$ ... (1)

We differentiate equation (1) with respect to 'x' to get:

$$f'(x) = x^4 - 4x \quad \text{... (2)}$$

From equation (2) it is clear that $f'(x)$ exists on $[-2,3]$.

For stationary points, we solve $f'(x) = 0$:

$$4x^3 - 4 = 0$$
$$4(x^3 - 1) = 0$$
$$(x-1)(x^2 + x + 1) = 0$$

Either: $x - 1 = 0$ or $x^2 + x + 1 = 0$

The equation $x^2 + x + 1 = 0$ has complex roots, so the only real stationary point is $x = 1$.

Critical Point Value: $f(1) = 1^4 - 4(1) = -3$

End Point Values: $f(-2) = (-2)^4 - 4(-2) = 16 + 8 = 24$

$f(3) = 3^4 - 4(3) = 81 - 12 = 69$

From the values –3, 24, 69:

- The global maximum is $f(3) = 69$
- The global minimum is $f(1) = -3$

2.10.4 Second Derivative Test

Let f be a twice differentiable function at $x = c$, where c lies in the domain of f. Suppose that $f'(c) = 0$. Then:

1. f has relative maximum at c if $f''(c) < 0$.
2. f has relative minimum at c, if $f''(c) > 0$.
3. If $f''(c) = 0$, then the test is inclusive, and the nature of the critical point cannot be determined using the second derivative test alone.

Point to Note

If first derivative is not defined at a critical point, can we apply second derivative test at that critical point.

Example 21 Examine the function $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ for relative extreme values.

Solution: We are given: $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ ...(1)

Differentiate w.r.t. 'x': $f'(x) = 2x^3 - 4x^2 - 2x + 4$ ... (2)

Set $f'(x) = 0$:

$$2x^3 - 4x^2 - 2x + 4 = 0$$
$$x^3 - 2x^2 - x + 2 = 0$$
$$x^3 - x - 2x^2 + 2 = 0$$
$$x(x^2 - 1) - 2(x^2 - 1) = 0$$
$$(x - 2)(x^2 - 1) = 0$$
$$(x - 2)(x - 1)(x + 1) = 0$$

Thus, the critical points are: $x = -1, x = 1, x = 2$

We differentiate the function $f'(x) = 2x^3 - 4x^2 - 2x + 4$ w.r.t. 'x':

$$f''(x) = 6x^2 - 8x - 2$$
$$= 2(3x^2 - 4x - 1) \quad \ldots(3)$$

Substitute $x = -1$ into equation (3):

$$f''(-1) = 2(3(-1)^2 - 4(-1) - 1) = 12 > 0$$

Thus, the function has a relative minimum at $x = -1$.

x = 1, substitute into equation (3): $f''(1) = 2(3(1)^2 - 4(1) - 1) = -4 < 0$

Thus, the function has a relative maximum at $x = 1$.

Substitute $x = 2$ into equation (3): $f''(2) = 2(3(2)^2 - 4(2) - 1) = 6 > 0$

Thus, the function has a relative minimum at $x = 2$.

Substitute $x = -1$, $x = 1$ and $x = 2$ into equation (1):

$$f(-1) = \frac{1}{2}(-1)^4 - \frac{4}{3}(-1)^3 - (-1)^2 + 4(-1) + 1 = \frac{1}{2} + \frac{4}{3} - 4 = -\frac{13}{6}$$

$$f(1) = \frac{1}{2}(1)^4 - \frac{4}{3}(1)^3 - (1)^2 + 4(1) + 1 = -\frac{5}{6} + 4 = \frac{19}{6}$$

$$f(2) = \frac{1}{2}(2)^4 - \frac{4}{3}(2)^3 - (2)^2 + 4(2) + 1 = \frac{7}{3}$$

Finally:

- The relative minimum value at $x = -1$ is $f(-1) = -\frac{13}{6}$.
- The relative maximum value at $x = 1$ is $f(1) = \frac{19}{6}$.
- The relative minimum value at $x = 2$ is $f(2) = \frac{7}{3}$.

EXERCISE 2.4

1. Determine the intervals on which f is increasing or decreasing for the domain mentioned in each case.

(i) $f(x) = \sin x \ ; \ x \in (0, 2\pi)$ (ii) $f(x) = 4 - x^2 \ ; \ x \in \mathbb{R} = (-\infty, \infty)$

(iii) $f(x) = x^2 + 3x + 2 \ ; \ x \in [-4, 1]$ (iv) $f(x) = \sin x + \frac{1}{4}\cos 2x \ ; x \in [0, 2\pi]$

2. Using the Second Derivative Test, find local extreme values of the following functions:

(i) $f(x)=8-2x^2$ (ii) $f(x)=x^2-x-2$ (iii) $f(x)=x^4-8x^2+2$

(iv) $f(x)=(x+2)^2(x-3)$ (v) $f(x)=(x^2-2x)e^x$

3. Discuss the global extrema of the following functions over specified domains.

(i) $f(x)=\sin x+\cos x$ on $[0,2\pi]$ (ii) $f(x)=x^{4/3}$ on $[-1,8]$

4. Find the global and local extrema of the following functions over the specified domains. (Hint: Use the increasing and decreasing behavior of the functions including the end points).

(i) $f(x)=4-x^2$ on $[-1,2]$ (ii) $f(x)=\cos x-\sin x$ on $[0,2\pi]$

5. Prove that $f(x)=\dfrac{\ln(2x)}{x^2}$ has local maximum at $x=\dfrac{\sqrt{e}}{2}$.

6. Show that $f(x)=x^{x^2}$ has a relative minimum at $x=\dfrac{1}{\sqrt{e}}$.

2.11 Local Linear Approximation (Linearization)

Sometimes, complicated functions are hard to work with, so we use simpler functions to approximate them. These approximations are often accurate enough for practical purposes.

In this section, we discuss linearization, which uses tangent lines to approximate a function near a specific point.

Linearization simplifies complex or nonlinear functions with a straight-line approximation, making calculations easier. It is widely used in machine learning, physics, engineering, and economics for quick, local estimates, especially for small changes around known values.

2.11.1 Linearization

If a function $f(x)$ is differentiable at $x=a$, then the approximating function

$$L(x)=f(a)+f'(a)(x-a)$$

is the linearization of $f(x)$ at $x=a$. The approximation

$$f(x)\approx L(x)$$

of $f(x)$ by $L(x)$ is called the local linear approximation of $f(x)$ at $x=a$. The point $x=a$ is referred to as the center of the approximation.

Recall!

A function of the form $y=mx+c$, where m is slope and c is the y- intercept, is called a linear function. The graph of a linear function is a straight line.

Let $(a,f(a))$ be a point on the graph of $y=f(x)$. If $f(x)$ is differentiable, then the equation of tangent line at $(a,f(a))$ is given by:

$$y=f(a)+f'(a()x-a).$$

We can see in Fig. 2.6, the tangent line $L(x) = 2x - 1$ to the graph of the curve $f(x) = x^2$ at $(1, f(1))$ lies close to the curve near the point of tangency. For a small interval on either side of $x = 1$, y-values of the tangent line provide good approximations to the y-values of the curve $f(x) = x^2$. We observe this phenomenon by observing the table of values for the difference between the values of $L(x)$ and $f(x)$. The phenomenon is true not just for this curve; every differentiable curve behaves locally like its tangent line.

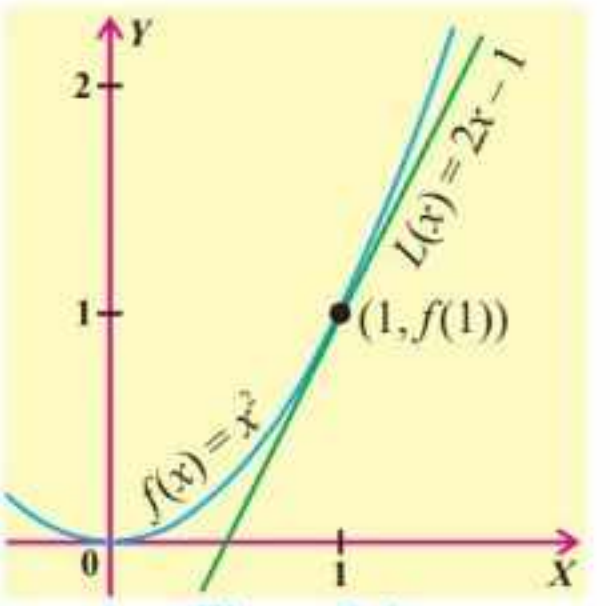


Figure 2.6

x	0.7	0.8	0.9	1	1.1	1.2	1.3
$f(x) = x^2$	0.49	0.64	0.81	1	1.21	1.44	1.69
$L(x) = 2x - 1$	0.4	0.6	0.8	1	1.2	1.4	1.6
$f(x) - L(x)$	0.09	0.04	0.01	0	0.01	0.04	0.09

Example 22 **(a)** Find the local linear approximation of $f(x) = \sqrt{2x-1}$ at $x = 5$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{9.2}$, and compare your approximation to the result produced directly by calculator.

Solution: **(a)** We are given: $f(x) = \sqrt{2x-1}$ ... (1)

Differentiate $f(x)$ w.r.t. x: $f'(x) = \frac{1}{2}(2x-1)^{-\frac{1}{2}}(2)$

$$= \frac{1}{\sqrt{2x-1}} \quad \text{... (2)}$$

Substitute $x = 5$ into equations (1) and (2):

$$f(5) = \sqrt{2(5)-1} = 3 \text{ and } f'(5) = \frac{1}{\sqrt{2(5)-1}} = \frac{1}{3}$$

Now linearization is: $L(x) = f(5) + f'(5)(x-5)$

$$= 3 + \frac{1}{3}(x-5) = \frac{1}{3}(9 + x - 5) = \frac{1}{3}(x+4)$$

Hence: $L(x) = \frac{1}{3}(x+4)$

Thus, the local linear approximation at $x = 5$ is:

$$\sqrt{2x-1} \approx \frac{1}{3}(x+4) \quad \text{... (3)}$$

(b) Substitute $x = 5.1$ into equation (3)

$$\sqrt{2(5.1)-1} \approx \frac{1}{3}(5.1+4)$$

$$\sqrt{9.2} \approx \frac{1}{3}(9.1)$$

$$\sqrt{9.2} \approx 3.0333 \qquad \ldots (4)$$

Calculator value $\sqrt{9.2} = 3.0331 \quad \ldots (5)$

From expressions (4) and (5), we see that the calculator value is very close to the value obtained by linear approximation.

Example 23: (a) Find the local linear approximation of $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

(b) Use the local linear approximation obtained in part (a) to approximate $\cos 91°$, and compare your approximation to the result produced directly by a calculator.

Solution: (a) We are given:

$$f(x) = \cos x \qquad \ldots (1)$$

Differentiate $f(x)$ with respect to x:

$$f'(x) = -\sin x \qquad \ldots (2)$$

Substitute $x = \frac{\pi}{2}$ into the equations (1) and (2)

$$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0 \quad \text{and} \quad f'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$$

The linearization $x = \frac{\pi}{2}$ is: $\quad L(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)$

$$= 0 + (-1)\left(x - \frac{\pi}{2}\right)$$

$$\Rightarrow \quad L(x) = -x + \frac{\pi}{2} \qquad \ldots (3)$$

Thus, the local linear approximation at $x = \frac{\pi}{2}$ is:

$$f(x) \approx L(x)$$

From equations (1) and (3): $\quad \cos x \approx -x + \frac{\pi}{2} \qquad \ldots (4)$

(b) To approximate $\cos 91°$, first convert degree into radians:

$$91° = 91 \times \frac{\pi}{180} = \frac{91\pi}{180}$$

Substitute $x = 91° = 1.588825$ into equation (4):

$$\cos 91° \approx -\frac{91\pi}{180} + \frac{\pi}{2}$$

$$\Rightarrow \cos 91° \approx -0.01745329 \quad \ldots(5)$$

Calculator value: $\cos 91° = -0.01745240 \quad \ldots(6)$

From expressions (4) and (5), we see that the calculator value is very close to the value obtained by linear approximation.

EXERCISE 2.5

1. Find the linear approximation of $f(x) = \sqrt{x+1}$ at $x = 3$, and use it to approximate $\sqrt{3.1}$ and $\sqrt{2.9}$.
2. (a) Find the local linear approximation of $f(x) = \sin x$ at $x = 0$.
   (b) Use the local linear approximation obtained in part (a) to approximate $\sin 2°$, and compare your approximation to the result produced directly by a calculator.
2. Find the linear approximation of the following functions at the given points:
   (i) $f(x) = \sqrt{x^2+16}$ at $x = 3$ (ii) $f(x) = \tan x$ at $x = \pi$
   (ii) $f(x) = \cos x$ at $x = 0$
3. Use a local linear approximation to estimate the following:
   (i) $(82)^{\frac{1}{3}}$ (ii) $\sin 29°$

2.12 Differentials

Assume that f is differentiable at a point x, if dx is an independent variable that can take values in the neighborhood of 0 and not 0, then we define dependent variable dy, by the formula:

$$dy = f'(x)dx \quad \ldots(1)$$

Since $dx \neq 0$, we can divide both sides of (1) by dx to obtain:

$$\frac{dy}{dx} = f'(x)$$

This achieves our goal of defining dy and dx so that their ratio is equal to the derivative $f'(x)$.

Example 24 **(a)** Find dy if $y = x^4 + 7x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.1$.

Solution: (a) Let $f(x) = x^4 + 7x$.

Differentiate $f(x)$ with respect to x:

$$f'(x) = 4x^3 + 7 \quad \ldots(1)$$

We know: $dy = f'(x)dx$

Substitute the value of $f'(x)$ from equation (1):

$$dy = (4x^3 + 7)dx \qquad \ldots (2)$$

(b) Substitute $x = 1$ and $dx = 0.1$ into equation (2):

$$dy = (4(1)^3 + 7)(0.1) = 11(0.1) = 1.1$$

Thus, the value of dy is 1.1.

2.12.1 Using dy to Approximate Δy

When a function $y = f(x)$ is differentiable at a point $P(x, y)$, we can use the differential dy to estimate the actual change Δy in y, for small change Δx in x.

The actual change Δy in y is given by:

$$\Delta y = f(x + \Delta x) - f(x) \qquad \ldots (1)$$

But calculating Δy exactly often requires complicated formulas. Instead, we use the differential dy, defined as:

$$dy = f'(x)dx$$

If we take $dx = \Delta x$, then: $dy = f'(x)\Delta x \qquad \ldots(2)$

Let $Q(x + \Delta x, f(x + \Delta x))$ be a neighboring point of $P(x, y)$ on $y = f(x)$.

- The slope of the tangent line at P is $f'(x)$
- The slope of the secant line connecting P and Q is $\dfrac{\Delta y}{\Delta x}$

As $\Delta x \to 0$, the point Q approaches P, and secant line PQ approaches the tangent line at P. (See Fig. 2.7)

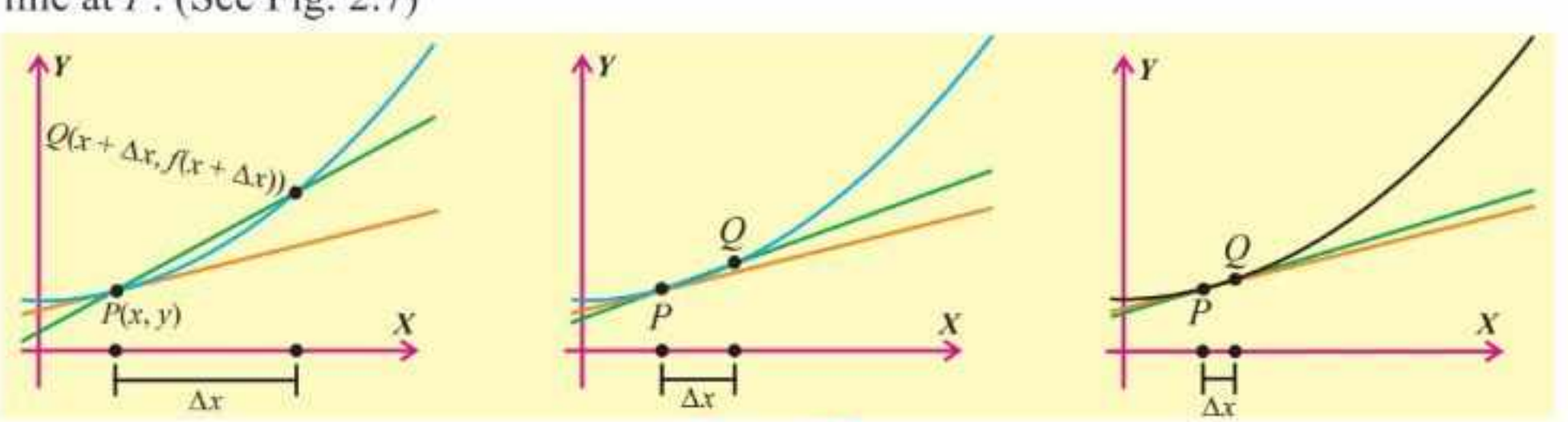


Figure 2.7

Therefore, for small Δx :

slope of tangent line at P $\approx$ Slope of secant line PQ

$$\Rightarrow \quad f'(x) \approx \frac{\Delta y}{\Delta x}$$

Substitute $f'(x) \approx \frac{\Delta y}{\Delta x}$ into equation (2):

$$dy \approx \frac{\Delta y}{\Delta x} \times \Delta x$$

$$dy \approx \Delta y \text{ (for } \Delta x \text{ is small)}$$

Thus, for small change Δx, the actual change Δy is approximately equal to the differential dy, that is:

$$\Delta y \approx dy = f'(x)dx$$

Using Differentials to Approximate:

Since $\Delta y = f(x+\Delta x) - f(x)$ and for small change Δx

$dx = \Delta x$ and $\Delta y \approx dy$.

$\Rightarrow \quad dy \approx f(x+\Delta x) - f(x)$

and hence: $f(x+\Delta x) \approx f(x) + dy$

2.13 Error in Differential Approximations

Suppose f is a differentiable function at a point x, and there is a small change Δx in x. If:

- Δy is actual corresponding change in y, given by

$$\Delta y = f(x+\Delta x) - f(x)$$

- dy is the approximation change in y obtained using differentials, given by

$$dy = f'(x)dx \quad \text{where } dx = \Delta x$$

Then:

Error in Δy is: Error $= \Delta y - dy$

Relative error in Δy is: Relative Error $= \frac{\Delta y - dy}{\Delta y} = \frac{\text{Error}}{\text{Original Value}}$, when $\Delta y \neq 0$

Percentage error Δy is:

$$\text{Percentage Error} = \left|\frac{\Delta y - dy}{\Delta y}\right| \times 100 = \left|\frac{\text{Error}}{\text{Original value}}\right| \times 100 \quad \text{when } \Delta y \neq 0$$

Example 25 Find dy and Δy for the function $f(x) = x^3$ when $x = 1$ and $dx = 0.1$.

Solution: We are given: $f(x) = x^3 \quad \ldots (1)$

Differentiate with respect to x: $f'(x) = 3x^2$

The formula for dy is: $dy = f'(x)dx$

Substituting expression for $f'(x)$: $dy = 3x^2 dx$

Substituting $x = 1$ and $dx = 0.1$: $dy = 3(1)^2(0.1) = 0.3$

We know: $\Delta y = f(x + \Delta x) - f(x)$

Substitute $x = 1$ and $\Delta x = dx = 0.1$: $\Delta y = f(1 + 0.1) - f(1) = f(1.1) - f(1)$

From equation (1), $f(x) = x^3$, so: $\Delta y = (1.1)^3 - (1)^3 = 1.331 - 1 = 0.331$

Example 26 Use differentials to approximate $\sqrt{24}$.

Solution: Let $f(x)=\sqrt{x}$ ... (1)

Differentiate with respect to x: $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$ ... (2)

Choose a real number that closest to 24 is a perfect square. Take $x=25$

We are approximating $\sqrt{24}$, so: $x+\Delta x=24$ and $x=25$

Thus: $24+\Delta x=25 \Rightarrow \Delta x=-1$

Since in differential approximation, $dx=\Delta x$, we take: $dx=-1$

We know: $dy=f'(x)dx$

From equation (2): $dy=\frac{1}{2\sqrt{x}}dx$

Substitute $x=25$ and $dx=-1$: $dy=\frac{1}{2\sqrt{25}}(-1)=-\frac{1}{10}=-0.1$

We know: $f(x+\Delta x)\approx f(x)+dy$

Substitute $dy=-0.1$, $x=25$ and $\Delta x=-1$:

$$f(25-1)\approx f(25)+(-0.1)$$

$$f(24)\approx f(25)-0.1$$

From equation (1): $\sqrt{24}=\sqrt{25}-0.1=5-0.1=4.9$

Example 27 Using differentials approximate the value of cos 61°3.

Solution: Let $f(x)=\cos x$ ... (1)

Differentiate with respect to x: $f'(x)=-\sin x$ ... (2)

Choose $x=60°$ because calculating cos 60° is easy.

We are approximating cos 61°, so: $x+\Delta x=61°$ and $x=60°$

Thus: $60°+\Delta x=61° \Rightarrow \Delta x=1°$

Since in differential approximation, $dx=\Delta x$, we take: $dx=1°=\frac{\pi}{180}$ radian

We know: $dy=f'(x)dx$

From equation (2): $dy=-\sin x\,dx$

Substitute $x=60°$ and $dx=\frac{\pi}{180}$: $dy=-\sin 60°\left(\frac{\pi}{180}\right)=-\frac{\sqrt{3}}{2}\times\frac{\pi}{180}=-0.0151$

We know: $f(x+\Delta x)\approx f(x)+dy$

Substitute $dy=-0.0151$, $x=60°$ and $x+\Delta x=61°$: $f(61°)\approx f(60°)+(-0.0151)$

From equation (1): $\cos 61°=\cos 60°-0.0151=0.5-0.0151=0.4849$

Example 28 The side of a cube is measured to be 25 cm, with a possible error of ±1 cm.

(a) Use differentials to estimate the error in the calculated volume.

(b) Estimate the percentage errors in the side and volume.

Solution: Let x be side of the cube: $x = 25$ cm, $dx = \Delta x = \pm 1$ cm

Let V be the volume of cube: $V = x^3$

(a) The differential of V is: $dV = \frac{d}{dx}\left(x^3\right)dx$

$$= 3x^2 dx \qquad \ldots (1)$$

Substitute $x = 25$ and $dx = \pm 1$ into equation (1):

$$dV = 3(25)^2(\pm 1) = \pm 1875\,\text{cm}^3$$

Thus, the estimated error in measuring volume is $\pm 1875\ cm^3$

(b) Percentage error in measuring the side is:

$$\left|\frac{dx}{x}\right| \times 100 = \frac{1}{25} \times 100 = 4\,\% \qquad \ldots (2)$$

Since: $V = x^3$ and $dV = 3x^2 dx$

The percentage error in measuring the volume is:

$$\left|\frac{dV}{V}\right| \times 100 = \left|\frac{3x^2 dx}{x^3}\right| \times 100 = 3 \times \left|\frac{dx}{x}\right| \times 100$$

Using equation (2): $\left|\frac{dV}{V}\right| \times 100 = 3 \times 4\% = 12\%$

Example 29 The radius r of a circle is measured with an error of at most 2%. What is the maximum corresponding percentage error in computing the circle's

(a) circumference? **(b)** area?

Solution: Since r is the radius. The percentage error in measuring the radius is given as:

$$\left|\frac{dr}{r}\right| \times 100 = 2\,\% \qquad \ldots (1)$$

(a) Let C be the circumference of the circle:

$$C = 2\,\pi r$$

The differential of C is: $dC = \frac{d}{dr}(2\pi r)\,dr = 2\pi dr \qquad \ldots (2)$

Percentage error in measuring the circumference:

$$\left|\frac{dC}{C}\right| \times 100 = \left|\frac{2\pi dr}{2\pi r}\right| \times 100 = \left|\frac{dr}{r}\right| \times 100$$

Using equation (1): $\left|\frac{dC}{C}\right| \times 100 = 2\,\%$

(b) Let A be the area of the circle: $A = \pi r^2$

The differential of A is: $dA = \frac{d}{dr}(\pi r^2)dr = 2\pi r dr$

Percentage error in measuring the area:

$$\left|\frac{dA}{A}\right| \times 100 = \left|\frac{2\pi r dr}{\pi r^2}\right| = 2 \times \left|\frac{dr}{r}\right| \times 100$$

Using equation (1): $\left|\frac{dA}{A}\right| \times 100 = 2 \times 2\,\% = 4\,\%$

EXERCISE 2.6

1. Find the differential dy in the following cases:
   (i) $y = x^3 - 3\sqrt{x}$ (ii) $y = \cos x^2$
2. Use dy to approximate Δy for the following functions:
   (i) $y = x^3 - x$ when x changes from 3 to 3.02.
   (ii) $y = \sin x + \cos x$ when x changes from $30°$ to $29°$.
   (iii) $y = \frac{1}{x}$ when $x = 0.5$ and $dx = -0.1$
3. Use differentials to approximate the value of:
   (i) $\sqrt{26}$ (ii) $(33)^{\frac{1}{5}}$ (iii) $\tan 46°$
4. The side of a cube is measured to be 30 cm with a maximum error of 0.14 cm in its measurement. Find the maximum error in the calculated volume of the cube.
5. The radius of a sphere is measured as 100 ± 0.5 cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

2.14 Applying Extrema to Real-World Problems

To solve extrema problems in real-life scenarios, follow these steps:

1. **Understand the Problem** – Identify what needs to be optimized.
2. **Define Variables** – Assign variables to unknown quantities.
3. **Formulate the Function** – Write an equation representing the quantity to optimize.
4. **Find Constraints** – If applicable, express limitations.
5. **Find Critical Points** – Take the derivative and find critical points.
6. **Verify Extrema** – Use the second derivative test to confirm maxima/minima.
7. **Interpret the Solution** – Relate the mathematical answer back to the real-world context.

Example 30 A rectangular garden is to be fenced on three sides with 100 meters of fencing. The fourth side is along a wall and requires no fence. Find the dimensions that maximize the area and the maximum area.

Solution: Let the side opposite to the wall be x meters.

Let each side perpendicular to the wall be y meters.

Total length of the garden to be fenced:

$$x + 2y = 100$$

$$y = \frac{100 - x}{2} \qquad \ldots (1)$$

Area of the garden $= A = xy$

Substituting the expression of y from (1):

$$A = x\left(\frac{100 - x}{2}\right)$$

$$A = \frac{1}{2}(100x - x^2) \qquad \ldots (2)$$

Differentiating w. r. t. 'x':

$$\frac{dA}{dx} = \frac{1}{2}(100 - 2x) = 50 - x \qquad \ldots (3)$$

Set $\frac{dA}{dx} = 0$: $\qquad 50 - x = 0$

$$x = 50$$

Thus, $x = 50$ is the critical point.

Differentiating equation (1) w. r. t. 'x':

Since $\frac{d^2A}{dx^2} = -1 < 0$, the area A is maximum at $x = 50$ meters

Substitute $x = 50$ into equations (1) and (2):

$$y = \frac{100 - 50}{2} = 25 \text{ meters and } A = \frac{1}{2}\left(100 \times 50 - 50^2\right) = 1250 \text{ m}^2$$

The maximum fenced area of the garden is 1250 m², which occurs when:

- The side opposite the wall is 50 meters
- Each of the sides perpendicular to the wall is 25 meters

Example 31 Find the dimensions of a rectangle of maximum area that can be inscribed under the curve $y = 4 - x^2$ and above the x-axis.

Solution: Let one vertex of the rectangle on the parabola in first quadrant be $A(x, y)$.

The other vertex in second quadrant is $B(-x,y)$.

Height of the rectangle $= y$

Width of the rectangle $= 2x$

Area of the rectangle: $A = 2xy$ ... (1)

The vertex $A(x, y)$ of the rectangle satisfies $y = 4 - x^2$.

Substitute this expression of y into equation (1):

$$A = 2x(4 - x^2) = 2(4x - x^3)$$

Differentiate with respect to x: $\frac{dA}{dx} = 2(4 - 3x^2)$ ... (2)

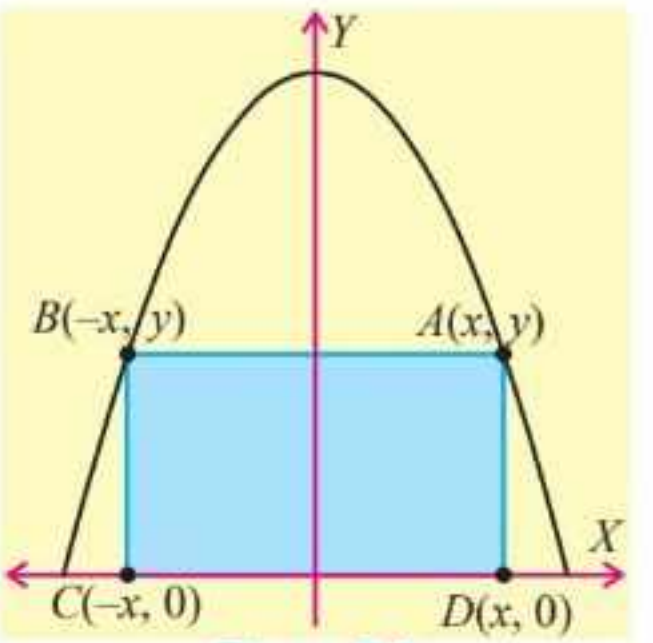


Figure 2.8

Set $\frac{dA}{dx} = 0$:

$$2(4 - 3x^2) = 0 \Rightarrow 3x^2 = 4 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \frac{2}{\sqrt{3}}$$

We consider only positive root since x in first quadrant is positive.

Differentiating $\frac{dA}{dx} = 2\left(4 - 3x^2\right)$ w.r.t. x:

$$\frac{d^2A}{dx^2} = 2(-6x) = -12x$$

At $x = \frac{2}{\sqrt{3}}$:

$$\frac{d^2A}{dx^2} = -12\left(\frac{2}{\sqrt{3}}\right) = -\frac{24}{\sqrt{3}} < 0$$

Negative second derivative implies that area is maximum at $x = \frac{2}{\sqrt{3}}$.

Substitute $x = \frac{2}{\sqrt{3}}$ into $y = 4 - x^2$:

$$y = 4 - \left(\frac{2}{\sqrt{3}}\right)^2 = 4 - \frac{4}{3} = \frac{8}{3}$$

The dimensions of the rectangle of maximum area are:

- Width $= 2x = \frac{4}{\sqrt{3}}$ units
- Height $= \frac{8}{3}$ units

Example 32 A cylindrical can must hold 500 cm³ of liquid. Find the radius and height that minimizes the material cost.

Solution: The cost of the material will be minimized if the surface area of the cylinder is minimized. Our objective is to minimize the surface area of the cylinder.

Let:
- Radius of the base = r cm
- Height of the cylinder = h cm

Volume of the cylinder: $V = \pi r^2 h = 500\,\text{cm}^3$

$$h = \frac{500}{\pi r^2} \qquad \ldots (1)$$

Surface area of cylinder:

- Surface area of two circular ends $= 2\pi r^2$
- Curved surface area $= 2\pi rh$

Total surface area of the cylinder:

$$S = 2\pi r^2 + 2\pi rh \qquad \ldots (2)$$

Substitute the expression of h from equation (1):

$$S = 2\pi r^2 + 2\pi r \times \frac{500}{\pi r^2} = 2\pi r^2 + \frac{1000}{r}$$

Differentiate 'S' w.r.t 'r':

$$\frac{dS}{dr} = 4\pi r - \frac{1000}{r^2} \qquad \ldots (3)$$

Set $\frac{dS}{dr} = 0$:

$$4\pi r - \frac{1000}{r^2} = 0 \;\Rightarrow\; 4\pi r = \frac{1000}{r^2} \;\Rightarrow\; r^3 = \frac{1000}{4\pi}$$

$$r = \left(\frac{250}{\pi}\right)^{1/3} \qquad \ldots (4)$$

Differentiate equation (3) w.r.t. r:

$$\frac{d^2S}{dr^2} = 4\pi + \frac{2000}{r^3}$$

Substitute $r = \left(\frac{250}{\pi}\right)^{1/3}$ into $\frac{d^2S}{dr^2}$:

$$\frac{d^2S}{dr^2} = 4\pi + \frac{2000}{\left(\left(\frac{250}{\pi}\right)^{1/3}\right)^3} > 0$$

Since $\frac{d^2S}{dr^2} > 0$, S is minimum at $r = \left(\frac{250}{\pi}\right)^{\frac{1}{3}}$.

Substitute $r=\left(\frac{250}{\pi}\right)^{\frac{1}{3}}$ into $h=\frac{500}{\pi r^2}$:

$$h=\frac{500}{\pi\left(\left(\frac{250}{\pi}\right)^{\frac{1}{3}}\right)^2}=\frac{500}{(\pi)^{\frac{1}{3}}(250)^{\frac{2}{3}}}$$

The material cost is minimized when:

- Radius $=r=\left(\frac{250}{\pi}\right)^{\frac{1}{3}}$ cm
- Height $=h=\frac{500}{(\pi)^{\frac{1}{3}}(250)^{\frac{2}{3}}}$ cm

Example 33 A car moves along the parabolic road $y=4-x^2$. A mobile tower is at the origin. Find the position of the car where the signal strength is strongest.

Solution: Signal strength is strongest when the distance between the car and the tower is minimum.

Let the position of the car be $P(x, y)$ on the parabola $y=4-x^2$

The distance D of the car from the tower at origin (0, 0) is:

$$D=\sqrt{x^2+y^2} \quad \ldots (1)$$

Since the car lies on the parabola $y=4-x^2$

$$\Rightarrow \quad x^2=4-y \qquad \ldots (2)$$

Substitute $x^2=4-y$ into equation (1):

$$D=\sqrt{4-y+y^2}$$

Since minimizing D is equivalent to minimizing D^2:

$$D^2=4-y+y^2$$

Let $D^2=f(y)$:

$$f(y)=y^2-y+4$$

Differentiate with respect to y:

$$\frac{df}{dy}=2y-1 \qquad \ldots (3)$$

Set $\frac{df}{dy}=0$:

$$2y-1=0$$

$$y=\frac{1}{2}$$

Differentiate equation (3) with respect to x:

$$\frac{d^2f}{dy^2} = 2 > 0$$

Since second derivative is positive, f is minimum at $y = \frac{1}{2}$.

Substitute $y = \frac{1}{2}$ into $x^2 = 4 - y$:

$$x^2 = 4 - \frac{1}{2} = \frac{7}{2}$$

$$x = \pm\sqrt{\frac{7}{2}}$$

The positions of the car where signal strength are $\left(\sqrt{\frac{7}{2}}, \frac{1}{2}\right)$ and $\left(-\sqrt{\frac{7}{2}}, \frac{1}{2}\right)$.

Example 34 A software company is designing a data processing algorithm for analyzing batches of customer transactions. The runtime $T(n)$ in milliseconds depends on the size of the batch n (the number of transactions processed together) and is modeled by the equation:

$$T(n) = n^3 - 12n^2 + 36n$$

- The term n^3 reflects the time complexity of a sorting step in the algorithm.
- The term $-12n^2$ accounts for optimizations that reduce processing time for moderate batch sizes.
- The term $36n$ represents overhead that scales linearly with batch size.

The company wants to find the optimal batch size n that minimizes the algorithm's total runtime.

Solution: The runtime of the algorithm is:

$$T(n) = n^3 - 12n^2 + 36n \quad \ldots (1)$$

We need to find the input size n (the number of transactions processed together) that minimizes the runtime $T(n)$.

Differentiate equation (1) with respect to n:

$$\frac{dT}{dn} = 3n^2 - 24n + 36 \quad \ldots (2)$$

To find critical points set $\frac{dT}{dn} = 0$:

$$3n^2 - 24n + 36 = 0$$

$$n^2 - 8n + 12 = 0$$

$$(n-2)(n-6) = 0$$

Thus, the critical points are:

$$n = 2 \text{ and } n = 6$$

Differentiate equation (2) with respect to n:

$$\frac{d^2T}{dn^2} = 6n - 24 \qquad \ldots (3)$$

Substitute $n = 2$ and $n = 6$ into equation (3):

$$\frac{d^2T}{dn^2} = 6\times 2 - 24 = -12 < 0 \quad \text{and} \quad \frac{d^2T}{dn^2} = 6\times 6 - 24 = 12 > 0$$

Since $\frac{d^2T}{dn^2} < 0$ at $n = 2$, the function $T(n)$ has a local maximum at $n = 2$.

Since $\frac{d^2T}{dn^2} > 0$ at $n = 6$, the function $T(n)$ has a local minimum at $n = 6$.

The runtime is minimized when the input size is $n = 6$.

EXERCISE 2.7

1. A window consists of a rectangle surmounted by a semicircular opening. If the total perimeter is 21 meter, find the dimensions that maximize the area.
2. A rectangular sheet of metal with dimensions 40 cm by 25 cm is to be made into an open box by cutting equal squares from the corners and folding up the sides. Find the size of the squares that maximize the volume.
3. A closed cylindrical tank is to be constructed to hold 54π liters of water. Find the dimensions that minimize the construction cost of the tank.
4. A wire of 20 meters is cut into two pieces. One piece is bent into a square and the other into a circle. How should the wire be cut to maximize the total area enclosed?
5. A team of environmental scientists is monitoring the oxygen level in a lake over time to ensure the water quality remains suitable for aquatic life. The oxygen level in the lake, measured in milligrams per liter (mg/L), is modeled by the equation:

   $$O(t) = t^3 - 18t^2 + 81t + 100$$

   Where $O(t)$ is the oxygen level at time t (in days after monitoring began).
   Scientists want to determine when during the monitoring period the oxygen level is at its lowest, as low oxygen levels can be harmful to fish and other aquatic organisms.
   Find the time t when the oxygen level is at its minimum.
6. The perimeter of a triangle is 16 cm. If one side is of length 6 cm, what are the lengths of the other sides for maximum area of the triangle.
7. A poster must have 200 cm^2 printed area with margins of 2 cm on top/bottom and 3 cm on sides. Find poster dimensions to minimize paper used.
8. Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.

Unit 03

Integration

INTRODUCTION

In previous units, we studied differentiation and learned how to determine rates of change, slopes of curves, maxima and minima, and solutions to various real-world problems. In this unit, we will study integration, which is the reverse process of differentiation and an important branch of calculus used to find accumulation, area, volume, and total change. We will learn the concepts of antiderivatives, indefinite and definite integrals, their notation, properties, and fundamental rules, including the power rule and the Fundamental Theorem of Calculus. We will also study the relationship between definite integrals and the net area under a curve, and use integration to find areas bounded by curves and lines, as well as volumes of solids of revolution. Furthermore, we will explore important integration techniques such as substitution, integration by parts, trigonometric substitution, partial fractions, and integration involving exponential and logarithmic functions. The concepts of integration have wide applications in mathematics, physics, engineering, economics, biology, and technology, including calculating the volume of containers, population growth, consumer and producer surplus, distance and velocity, drug dosage, moment of inertia, and sensor network measurements. Through this unit, we will develop both conceptual understanding and practical problem-solving skills related to integration and its real-life applications.

3.1 Antiderivatives and The Indefinite Integral

Definition: A function F is called an **antiderivative** of a given function f on an interval I if $F'(x) = f(x)$ for all x in I. In other words, the process of recovering a function F from its derivative f is called antidifferentiation.

To illustrate, let $f(x) = 2x$. If $F(x) = x^2$ then $F'(x) = 2x = f(x)$. Moreover, the function $G(x) = x^2 + 10$ also satisfies $G'(x) = 2x = f(x)$. Therefore, if $F(x)$ is any antiderivatives of $f(x)$ on a given interval, then for any value of c, the function $F(x)+c$ is also antiderivative of $f(x)$ on that interval.

Definition: Let F be an antiderivative of f on an interval I. The set (or family) of all antiderivatives of f is called the **indefinite integral** of $f(x)$ with respect to x, denoted by $\int f(x)\,dx$ and defined as $\int f(x)\,dx = F(x)+c$, where $F'(x) = f(x)$ and c is an arbitrary constant.

In the notation $\int f(x)\,dx = F(x)+c$.

(i) The symbol $\int$ is called the **integral sign.**

(ii) $\int f(x)\,dx$ is called the **indefinite integral** of $f(x)$.

(iii) The function $f(x)$ inside the integral sign is the **integrand**.

(iv) The symbol dx is called the **differential** of x. It indicates the **variable of integration**–the variable with respect to which the integration is performed.

(v) $F(x)$ is an **antiderivative** of $f(x)$.

(vi) c is the **constant of integration**.

3.1.1 Basic Rules of Integration

Rule-1: $\int k f(x)\,dx = k\int f(x)\,dx$ (Integration of function with constant (k)).

Rule-2: $\int [f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$ (Sum and Difference Rule).

Rule-3: $\int x^n dx = \dfrac{x^{n+1}}{n+1}+c$, where $n \neq -1$ (The Power Rule).

The **General Power Rule:** $\int [f(x)]^n f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1}+c$, where $n \neq -1$.

Rule-4: $\int \dfrac{f'(x)}{f(x)}dx = \ln|f(x)|+c$

Rule-5: If f' is the derivative of f, and $\int f'(x)\,dx = f(x)+c$, then for any constants $a \neq 0$ and b, $\int f'(ax+b)\,dx = \dfrac{f(ax+b)}{a}+c$. (Linear Substitution Rule).

3.1.2 Some Useful Formulae

From (the above Definitions), we obtain integration formulas directly from differentiation formulas, as shown in the following table.

Derivative $\left(\frac{d}{dx}(f(x))\right)$	Indefinite Integral $\left(\int \frac{d}{dx}(f(x))\,dx = f(x)+c\right)$
$\frac{d}{dx}(c)=0$	$\int 0\,dx = c$
$\frac{d}{dx}(x)=1$	$\int 1\,dx = \int dx = x+c$
$\frac{d}{dx}(x^n)=nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1}+c, n\neq -1$ (Power Rule)
$\frac{d}{dx}(\ln\lvert x\rvert)=\frac{1}{x}, x\neq 0$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + c, x\neq 0$
$\frac{d}{dx}(e^x)=e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}(a^x)=a^x \ln a$	$\int a^x dx = \frac{a^x}{\ln a}+c; a>0, a\neq 1$
$\frac{d}{dx}(\sin x)=\cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}(\cos x)=-\sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\tan x)=\sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}(\cot x)=-\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + c$
$\frac{d}{dx}(\sec x)=\sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + c$
$\frac{d}{dx}(\csc x)=-\csc x\cot x$	$\int \csc x\cot x\,dx = -\csc x + c$
$\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + c$
$\frac{d}{dx}(\sec^{-1} x)=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}\lvert x\rvert + c$
$\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = \tan^{-1} x + c$

Example 1 Integrate the following:

(i) $\int x^{99}\,dx$ (ii) $\int -98x^{-99}\,dx$

Solution: (i) $\int x^{99}\,dx = \frac{x^{99+1}}{99+1} + c = \frac{x^{100}}{100} + c$ (By using Rule 3: The Power Rule)

We can verify this result by differentiation: $\frac{d}{dx}\left(\frac{x^{100}}{100} + c\right) = \frac{100x^{99}}{100} + c = x^{99} + c$

(ii) $-98\int x^{-99}\,dx = -98\frac{x^{-98}}{-98} + c = x^{-98} + c = \frac{1}{x^{98}} + c$ (By using Rule 1 and Rule 3)

Example 2 Integrate: $\int\left(8x^7 - 6x^5 + 4x^3 - 2x - 3\right)dx$

Solution: By using Rule 2, we have:

$$\int\left(8x^7 - 6x^5 + 4x^3 - 2x - 3\right)dx = \int 8x^7\,dx - \int 6x^5\,dx + \int 4x^3\,dx - \int 2x\,dx - \int 3\,dx$$

$$= 8\int x^7\,dx - 6\int x^5\,dx + 4\int x^3\,dx - 2\int x\,dx - 3\int 1\,dx \qquad \text{(By using Rule 1)}$$

$$= 8\left(\frac{x^8}{8}\right) - 6\left(\frac{x^6}{6}\right) + 4\left(\frac{x^4}{4}\right) - 2\left(\frac{x^2}{2}\right) - 3x + c = x^8 - x^6 + x^4 - x^2 - 3x + c \quad \text{(By Rule 3)}$$

Example 3 Integrate: (i) $\int \frac{7}{\sqrt{x}}\,dx$ (ii) $\int \frac{(1+x^2)^2}{x^4}\,dx$

Solution: (i) $\int \frac{7}{\sqrt{x}}\,dx = 7\int x^{-1/2}\,dx = 7\frac{x^{1/2}}{1/2} + c = 14\sqrt{x} + c$ (By using Rule 1 & 3)

(ii) $\int \frac{\left(1+x^2\right)^2}{x^4}\,dx = \int \frac{1+2x^2+x^4}{x^4}\,dx = \int\left(\frac{1}{x^4} + \frac{2x^2}{x^4} + \frac{x^4}{x^4}\right)dx$ (By Rule 2)

$$= \int (x^{-4} + 2x^{-2} + 1)\,dx = \frac{x^{-3}}{-3} + 2\frac{x^{-1}}{-1} + x + c = -\frac{1}{3x^3} - \frac{2}{x} + x + c \text{ (By Rule 3)}$$

Example 4 (i) $\int (5x^2 + 7x - 3)^7 (10x + 7)\,dx = \frac{(5x^2 + 7x - 3)^8}{8} + c$

(ii) $\int\left(x + \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right)dx = \frac{1}{2}\left(x + \frac{1}{x}\right)^2 + c$

Example 5 Evaluate $\int \tan x\,dx$

Solution:

$$\int \tan x\,dx = \int \frac{\sec x \cdot \tan x}{\sec x}\,dx$$

$$= \ln|\sec x| + c \quad \text{(By Rule 4)}$$

or $$= -\ln|\cos x| + c$$

Example 6 Evaluate $\int \sec x\,dx$

Solution:

$$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\,dx$$

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x}\,dx$$

$$= -\ln|\cos x| + c \quad \text{(By Rule 4)}$$

Example 7 Evaluate $\int \tan^2 x\,dx$

Solution: $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \int \sec^2 x\,dx - \int 1\,dx = \tan x - x + c$

Example 8 Evaluate $\int \cos^2 x\,dx$

Solution: $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c$

Example 9 Evaluate $\int \frac{\tan x - \sec x}{1+\tan^2 x}\,dx$

Solution: $\int \frac{\tan x - \sec x}{1+\tan^2 x}\,dx = \int \frac{\tan x - \sec x}{\sec^2 x}\,dx = \int \left(\frac{\tan x}{\sec^2 x} - \frac{\sec x}{\sec^2 x}\right)dx$

$$= \int \left(\frac{\sin x}{\cos x}\cos^2 x - \frac{1}{\sec x}\right)dx = \int (\sin x \cos x - \cos x)\,dx = \frac{\sin^2 x}{2} - \sin x + c$$

Example 10 Evaluate $\int \sin 3x \cos 7x\,dx$

Solution: We know that, $\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$

Put $m = 3$ and $n = 7$, we get

$$\int \sin 3x \cos 7x\,dx = \frac{1}{2}\int [\sin(10x) + \sin(-4x)]\,dx$$

$$= \frac{1}{2}\int (\sin 10x - \sin 4x)\,dx = \frac{1}{2}\left[-\frac{\cos 10x}{10} - \left(-\frac{\cos 4x}{4}\right)\right] + c = -\frac{\cos 10x}{20} + \frac{\cos 4x}{8} + c$$

Example 11 Evaluate $\int \frac{\sin(x+a)}{\sin(x+b)}\,dx$

Solution: $\int \frac{\sin(x+a)}{\sin(x+b)}\,dx = \int \frac{\sin[(x+b)+(a-b)]}{\sin(x+b)}\,dx$

$$= \int \frac{\sin(x+b)\cos(a-b) + \cos(x+b)\sin(a-b)}{\sin(x+b)}\,dx$$

$$= \int \left[\cos(a-b) + \sin(a-b) \frac{\cos(x+b)}{\sin(x+b)} \right] dx = \cos(a-b) \int 1 \, dx + \sin(a-b) \int \frac{\cos(x+b)}{\sin(x+b)} dx$$

$$= \cos(a-b) \cdot x + \sin(a-b) \cdot \ln|\sin(x+b)| + c$$

Example 12 Evaluate $\int \frac{a^{2x} - b^{2x}}{(ab)^x} dx$

Solution: $\int \frac{a^{2x} - b^{2x}}{(ab)^x} dx = \int \frac{a^{2x} - b^{2x}}{a^x b^x} dx = \int \left[\frac{a^{2x}}{a^x b^x} - \frac{b^{2x}}{a^x b^x} \right] dx = \int \left[\frac{a^{2x-x}}{b^x} - \frac{b^{2x-x}}{a^x} \right] dx$

$$= \int \left[\frac{a^x}{b^x} - \frac{b^x}{a^x} \right] dx = \int \left(\frac{a}{b} \right)^x dx - \int \left(\frac{b}{a} \right)^x dx = \frac{\left(\frac{a}{b} \right)^x}{\ln\left(\frac{a}{b} \right)} - \frac{\left(\frac{b}{a} \right)^x}{\ln\left(\frac{b}{a} \right)} + c$$

EXERCISE 3.1

1. Find all the antiderivatives for each derivative.

   (i) $\frac{dy}{dx} = 100x^4$ (ii) $\frac{dy}{dt} = 2 + 24t^3 - 10t^4$

2. Find each indefinite integral. (Check answer by differentiating.)

   (i) $\int (1+2t) \, dt$ (ii) $\int (3u^2 + 4u) \, du$

   (iii) $\int 100x^8 \, dx$ (iv) $\int \left(2x^{-2} + 3x^{-3} \right) dx$

3. Evaluate the following:

   (i) $\int (5x^3 + 10x + 20) \, dx$ (ii) $\int x^2 \left(x + \frac{1}{x} \right) dx$

   (iii) $\int \left(\sqrt{y} + \frac{1}{y} \right)^2 dy, \ (y > 0)$ (iv) $\int \frac{\sqrt{x}(x+2)}{x} dx, \ (x > 0)$

   (v) $\int \frac{x+1}{\sqrt{x+2}} dx$ (vi) $\int \frac{ax+b}{\sqrt{ax^2 + 2bx + c}} dx$ (vii) $\int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx, \ (x > 0)$

   (viii) $\int x\sqrt{1+\sqrt{x}} \, dx$ (ix) $\int \frac{x}{(a+bx)^3} dx$ (x) $\int x^5 \sqrt{x^3 + 1} \, dx$

   (xi) $\int (3x + 3^x + x^3) \, dx$ (xii) $\int 7^{\ln x} dx$ (xiii) $\int \frac{1}{1 + \sqrt{2+x}} dx$

4. Evaluate the following:

   (i) $\int \csc x \, dx$ (ii) $\int \cot^2 x \, dx$ (iii) $\int \sec^4 x \, dx$ (iv) $\int \sin x \cdot \cos 7x \, dx$

(v) $\int \frac{\cos^3 x - \sin^3 x}{\cos x - \sin x} dx$ (vi) $\int \frac{\cos 2x}{\sin x + \cos x} dx$ (vii) $\int \sqrt{1-\cos 2x}\, dx$

(viii) $\int \frac{1}{1+\cos x} dx$ (ix) $\int \frac{\sin 2x}{\sqrt{1+\sin^2 x}} dx$ (x) $\int \frac{1-\cos 2x}{1+\cos 2x} dx$

3.2 Method of Substitution or Change of Variable

The method stems from reversing the Chain Rule for derivatives. Recall that for the differentiable functions $y = F(u)$ and $u = g(x)$ the Chain Rule states that

$$\frac{d}{dx}\left[F\left(g(x)\right)\right] = F'\left(g(x)\right)g'(x).$$

In terms of definite integrals $\int F'\left(g(x)\right)g'(x) = F\left(g(x)\right) + c.$

Theorem 3.1: Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f $\left(F' = f\right)$ on I, then $\int f(g(x)\cdot g'(x)dx = F(u) + c.$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and $\int f(u)\,du = F(u) + c.$

This equation $\int f(u)\,du = F(u) + c$ has the same form as the basic antiderivative formula, $\int f(x)\,dx = F(x) + c$, but the independent variable is now u.

The process of replacing x with a new variable $u = g(x)$ (and dx with the corresponding du) is called a **change of variable** or **the method of substitution**.

Recall! The Rule 4

$$\int \frac{f'(x)}{f(x)} dx = \ln\left|f(x)\right| + c$$

Let $u = f(x)$. Then $du = f'(x)\,dx$.

Substituting into the given integral, we have $\int \frac{f'(x)}{f(x)} dx = \int \frac{1}{u} du$

We know from basic integration rules that $\int \frac{1}{u} du = \ln|u| + c$

Finally, substituting $u = f(x)$, we obtain $\int \frac{f'(x)}{f(x)} dx = \ln\left|f(x)\right| + c$

Example 13 Evaluate $\int (5x^5 + 5)^3 x^4 dx$

Solution: When an integrand contains an expression raised to a power, such as $\left(5x^5 + 5\right)^3$, a useful strategy is to substitute u for that expression.

Let $u = 5x^5 + 5$. Then $du = 25x^4 dx$.

$$\int (5x^5+5)^3 x^4 dx = \frac{1}{25}\int (5x^5+5)^3 \cdot 25x^4 dx = \frac{1}{25}\int u^3 du = \frac{1}{25}\cdot\frac{u^4}{4} + c = \frac{1}{100}u^4 + c$$

Finally, substitute back $u = 5x^5 + 5$: $\int (5x^5+5)^3 x^4 dx = \frac{1}{100}\left(5x^5+5\right)^4 + c.$

Example 14 Integrate $\int \sqrt{x}\,\sqrt{1+\sqrt{x}}\,dx$

Solution: Let $1+\sqrt{x} = u$. Then, $\sqrt{x} = u-1$ and $x = (u-1)^2 \Rightarrow dx = 2(u-1)\,du$

Substitute $\sqrt{x} = u-1$ and $dx = 2(u-1)\,du$ into the integral:

$$\int \sqrt{x}\,\sqrt{1+\sqrt{x}}\,dx = \int (u-1)\sqrt{u}\;2(u-1)\;du = 2\int (u-1)^2 u^{\frac{1}{2}} du$$

$$= 2\int (u^2-2u+1)u^{\frac{1}{2}} du = 2\int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du = 2\left[\frac{2}{7}u^{\frac{7}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right] + c$$

$$= \frac{4}{7}u^{\frac{7}{2}} - \frac{8}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + c = \frac{4}{7}\left(1+\sqrt{x}\right)^{\frac{7}{2}} - \frac{8}{5}\left(1+\sqrt{x}\right)^{\frac{5}{2}} + \frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}} + c. \;\because u = 1+\sqrt{x}$$

Example 15 Evaluate $\int \frac{x}{\sqrt{1+x^2}+1+x^2}\,dx$

Solution: Let $u = \sqrt{1+x^2}$. Then $u^2 = 1+x^2 \Rightarrow 2u\,du = 2x\,dx \Rightarrow u\,du = x\,dx$

$$\int \frac{x}{\sqrt{1+x^2}+1+x^2}\,dx = \int \frac{u}{u+u^2}\,du = \int \frac{u}{u(1+u)}\,du = \int \frac{1}{1+u}\,du = \ln|1+u| + c$$

$$= \ln\left(1+\sqrt{1+x^2}\right) + c.$$

where we can remove the absolute value signs since $1+\sqrt{1+x^2} > 0$ for all x.

Example 16 Evaluate $\int \frac{\sin x}{e^{\cos x}}\,dx$

Solution: Let $u = \cos x$. Then $du = -\sin x\,dx$.

$$\int \frac{\sin x}{e^{\cos x}}\,dx = \int \frac{-du}{e^u} = -\int e^{-u}\;du = -\frac{e^{-u}}{-1} + c = e^{-u} + c = e^{-\cos x} + c$$

Example 17 Evaluate $\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

Solution: Let $u = \tan^{-1}x$, then $du = \frac{1}{1+x^2}\,dx$

$$\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx = \int e^u\;du = e^u + c = e^{\tan^{-1}x} + c$$

Example 18 Evaluate $\int \frac{e^{2x}+2e^{x}}{(e^{x}+1)^{2}}dx$

Solution: Let $e^{x}+1=u \Rightarrow e^{x}=u-1 \Rightarrow e^{x}dx=du$

$$\int \frac{e^{2x}+2e^{x}}{(e^{x}+1)^{2}}dx=\int \frac{e^{x}+2}{\left(e^{x}+1\right)^{2}}\cdot e^{x}dx=\int \frac{u-1+2}{u^{2}}du=\int \frac{u+1}{u^{2}}du=\int\left(\frac{u}{u^{2}}+\frac{1}{u^{2}}\right)du$$

$$=\int\left(\frac{1}{u}+u^{-2}\right)du=\ln|u|+\frac{u^{-1}}{-1}+c=\ln|u|-\frac{1}{u}+c=\ln\left|e^{x}+1\right|-\frac{1}{e^{x}+1}+c$$

Example 19 Evaluate $\int \frac{e^{x}\ln(1+e^{x})}{1+e^{x}}dx$

Solution: Let $u=\ln(1+e^{x})$, then $du=\frac{e^{x}}{1+e^{x}}dx$

$$\int \frac{e^{x}\ln(1+e^{x})}{1+e^{x}}dx=\int u\,du=\frac{u^{2}}{2}+c=\frac{1}{2}\left[\ln(1+e^{x})\right]^{2}+c$$

Example 20 Evaluate $\int \frac{\ln(1+x)-\ln x}{x(1+x)}dx$

Solution: Let $u=\ln(1+x)-\ln x$

Then, $\frac{du}{dx}=\frac{1}{1+x}-\frac{1}{x}=\frac{x-(1+x)}{x(1+x)}=\frac{x-1-x}{x(1+x)}=\frac{-1}{x(1+x)}$

$$\frac{du}{dx}=\frac{-1}{x(1+x)} \Rightarrow -du=\frac{1}{x(1+x)}dx$$

$$\int \frac{\ln(1+x)-\ln x}{x(1+x)}dx=\int u(-du)=-\int u\,du=-\frac{u^{2}}{2}+c=-\frac{1}{2}\left[\ln(1+x)-\ln x\right]^{2}+c$$

Example 21 Evaluate $\int \frac{x^{2}\tan^{-1}(x^{3})}{1+x^{6}}dx$

Solution: Let $u=\tan^{-1}(x^{3})$,

then $\frac{du}{dx}=\frac{1}{1+(x^{3})^{2}}\frac{d}{dx}(x^{3}) \Rightarrow du=\frac{3x^{2}}{1+x^{6}}dx \Rightarrow \frac{1}{3}du=\frac{x^{2}}{1+x^{6}}dx$

Thus, $\int \frac{x^{2}\tan^{-1}(x^{3})}{1+x^{6}}dx=\int u\frac{1}{3}du=\frac{1}{3}\int u\,du=\frac{u^{2}}{6}+c=\frac{1}{6}\left[\tan^{-1}(x^{3})\right]^{2}+c.$

3.3 Trigonometric Substitution

Integrals involving the radicals $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, and $\sqrt{x^2-a^2}$ (where $a>0$) can often be evaluated by making a substitution involving a trigonometric function. The objective of trigonometric substitution is to eliminate the radical in the integrand. We do this by using the Pythagorean identities:

$$\sin^2\theta+\cos^2\theta=1 \text{ and } \sec^2\theta-\tan^2\theta=1$$

Case 1: Integrals involving $\sqrt{a^2-x^2}$ (where, $a>0$) can frequently be reduced to a simpler form by using the substitution $x=a\sin\theta$.

Case 2: Integrals involving $\sqrt{a^2+x^2}$ (where, $a>0$) can frequently be reduced to a simpler form by using the substitution $x=a\tan\theta$.

Case 3: Finally, integrals involving $\sqrt{x^2-a^2}$ (where, $a>0$) can frequently be reduced to a simpler form by using the substitution $x=a\sec\theta$.

Some Special Integrals as Formulae

(i) $\displaystyle\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c$ **(ii)** $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+c$, where $a>0$

Proof: **(i)** We want to evaluate $\displaystyle\int\frac{1}{a^2-x^2}dx$

The denominator $a^2-x^2=(a-x)(a+x)$, so

$$\int\frac{1}{a^2-x^2}dx=\int\frac{1}{(a-x)(a+x)}dx$$

$$=\frac{1}{2a}\int\frac{(a-x)+(a+x)}{(a-x)(a+x)}\,dx$$

$$=\frac{1}{2a}\int\left[\frac{(a-x)}{(a-x)(a+x)}+\frac{(a+x)}{(a-x)(a+x)}\right]dx$$

$$=\frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx$$

$$=\frac{1}{2a}\int\left(\frac{1}{a+x}-\frac{-1}{a-x}\right)dx$$

$$=\frac{1}{2a}\left(\ln|a+x|-\ln|a-x|\right)+c$$

$$=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c$$

Challenge!

Prove the given formulae.

(i) $\displaystyle\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$

(ii) $\displaystyle\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

(iii) $\displaystyle\int\frac{dx}{\sqrt{a^2+x^2}}=\ln\left|x+\sqrt{a^2+x^2}\right|+c$, where $a>0$

(iv) $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+c$, where $a>0$

(v) $\displaystyle\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c$, where $a>0$

(ii) We want to evaluate $\int \frac{1}{\sqrt{a^2-x^2}}dx$, where a is a positive constant and the domain is $-a < x < a$ so that the square root is real.

Let $x = a\sin\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $dx = a\cos\theta\, d\theta$.

Also

$$\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin\theta)^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|$$

On the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\cos\theta > 0$, so $|\cos\theta| = \cos\theta$. Thus, $\sqrt{a^2-x^2} = a\cos\theta$.

Substitute into the integral: $\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{a\cos\theta}\cdot a\cos\theta\, d\theta = \int 1d = \theta + c$

We have $x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a}$, and since θ is in the principal range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

Thus, $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Example 22 Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}dx$

Solution: Let $x = 3\sin\theta$, where $\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, then $dx = 3\cos\theta\, d\theta$.

$$\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{3\cos\theta}{(3\sin\theta)^2}\cdot 3\cos\theta\, d\theta = \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta = -\cot\theta - \theta + c \qquad \ldots(1)$$

We have $x = 3\sin\theta \Rightarrow \sin\theta = \frac{x}{3}$. So, $\theta = \sin^{-1}\left(\frac{x}{3}\right)$.

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} = \frac{\sqrt{1-\left(\frac{x}{3}\right)^2}}{\frac{x}{3}} = \frac{\sqrt{9-x^2}}{x}$$

Thus, equation becomes $\int \frac{\sqrt{9-x^2}}{x^2}dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + c$

Example 23 Evaluate $\int \frac{1}{x\sqrt{x^4+1}}dx$

Solution: $\int \frac{1}{x\sqrt{x^4+1}}dx = \frac{1}{2}\int \frac{2x}{x^2\sqrt{(x^2)^2+1}}dx$

Let $x^2 = \tan\theta$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $2x\,dx = \sec^2\theta\,d\theta$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\tan\theta\sqrt{\tan^2\theta+1}}d\theta = \frac{1}{2}\int \frac{\sec^2\theta}{\tan\theta\sec\theta}d\theta$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}d\theta = \frac{1}{2}\int \csc\theta\,d\theta = \frac{1}{2}\ln|\csc\theta - \cot\theta| + c$$

$$= \frac{1}{2}\ln\left|\sqrt{1+\cot^2\theta} - \cot\theta\right| + c = \frac{1}{2}\ln\left|\sqrt{1+\left(\frac{1}{x^2}\right)^2} - \frac{1}{x^2}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{x^4+1}}{x^2} - \frac{1}{x^2}\right| + c = \frac{1}{2}\ln\left|\frac{\sqrt{x^4+1}-1}{x^2}\right| + c$$

Example 23 Evaluate $\int \frac{\sqrt{x^2-9}}{x}dx,\ (x>3)$

Solution: Let $x = 3\sec\theta$, then $dx = 3\sec\theta\tan\theta d\theta$

$$\int \frac{\sqrt{x^2-9}}{x}dx = \int \frac{\sqrt{(3\sec\theta)^2-9}}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \sqrt{9(\sec^2\theta-1)}\cdot\tan\theta\,d\theta = \int 3\tan\theta\tan\theta\,d\theta$$

$$= 3\int \tan^2\theta\,d\theta = 3\int(\sec^2\theta-1)\,d\theta = 3(\tan\theta-\theta)+c = 3\left[\sqrt{\sec^2\theta-1}-\theta\right]+c$$

$$= 3\left[\sqrt{\left(\frac{x}{3}\right)^2-1} - \sec^{-1}\left(\frac{x}{3}\right)\right]+c = \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right)+c$$

Example 24 Evaluate $\int \frac{(x+1)e^x}{1+x^2e^{2x}}dx$

Solution: Let $u = xe^x$, then $\frac{du}{dx} = xe^x + e^x(1) \Rightarrow du = (x+1)e^x\,dx$

$$\int \frac{(x+1)e^x}{1+x^2e^{2x}}dx = \int \frac{(x+1)e^x}{1+(xe^x)^2}dx = \int \frac{du}{1+u^2} = \tan^{-1}u + c = \tan^{-1}(xe^x) + c$$

Example 26 Evaluate $\int \frac{1}{x\sqrt{1-x^3}}dx,\ x \in (-\infty, 0)\cup(0, 1).$

Solution: $\int \frac{1}{x\sqrt{1-x^3}}dx$

$= -\frac{1}{3}\int \frac{-3x^2}{x^3\sqrt{1-x^3}}dx$ Put $u = \sqrt{1-x^3} \Rightarrow u^2 = 1-x^3 \Rightarrow 2u\,du = -3x^2\,dx$

$= -\frac{1}{3}\int \frac{2u}{(1-u^2)u}du = \frac{2}{3}\int \frac{1}{u^2-1^2}du = \frac{2}{3}\frac{1}{2(1)}\ln\left|\frac{u-1}{u+1}\right| + c = \frac{1}{3}\ln\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + c$

Example 27 Evaluate $\int \frac{2x}{\sqrt{x^4+9}}dx$

Solution: Let $u = x^2$, then $du = 2xdx$

The integral becomes:

$$\int \frac{1}{\sqrt{u^2+9}}du = \int \frac{1}{\sqrt{u^2+3^2}}du = \ln\left|u+\sqrt{u^2+3^2}\right| + c = \ln\left|u+\sqrt{u^2+9}\right| + c$$

Substitute u with x^2:

$$\int \frac{1}{\sqrt{u^2+9}}du = \ln\left|x^2+\sqrt{x^4+9}\right| + c$$

Example 28 $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Solution: We have x^2-4x+3

$$x^2-4x+3 = (x^2-4x+4)+3-4 = (x-2)^2-1$$

$$\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}$$

Let $u = x-2$, then $du = dx$, the integral becomes

$$\int \frac{du}{u\sqrt{u^2-1}} = \int \frac{du}{u\sqrt{u^2-1^2}} = \frac{1}{1}\sec^{-1}\left|\frac{u}{1}\right| + c = \sec^{-1}|u| + c$$

Substitute u with $x-2$

$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|x-2| + c$$

EXERCISE 3.2

1. Evaluate the following integrals: (By using suitable substitution)

(i) $\int x(7x^2+8)^9\,dx$ (ii) $\int \frac{x^2}{(x^3+2)^5}dx$ (iii) $\int\left(1+\frac{1}{x}\right)^{-2}\left(\frac{1}{x^2}\right)dx$

(iv) $\int \frac{(1+\sqrt{x})^5}{\sqrt{x}}dx$ (v) $\int (7-x^5)^4\,x^4dx$ (vi) $\int \frac{t^2-2t}{(t^3-3t^2+5)^4}dt$

(vii) $\int \frac{\sec^2 x}{\tan x\cdot \ln\tan x}dx$ (viii) $\int \frac{\sin x-\cos x}{\sqrt{1+\sin 2x}}dx$ (ix) $\int \frac{\tan^3 x}{\sec^4 x+\tan^4 x}dx$

(x) $\int \frac{x}{1+x\tan x}dx$ (xi) $\int \frac{\ln\sec x}{\cot x}dx$ (xii) $\int \frac{\ln(\sin x)}{\tan x}dx$

(xiii) $\int \frac{e^{2x}}{(2+e^{2x})^2}dx$

2. Evaluate the following integrals: (By change of variable)

(i) $\int \frac{e^x}{9-e^{2x}}dx$ (ii) $\int \frac{\cos x}{\sin^2 x-4}dx$ (iii) $\int \frac{e^x}{1+e^{2x}}dx$

(iv) $\int \frac{1}{x\sqrt{4-(\ln x)^2}}dx$ (v) $\int \frac{1}{9x^2-25}dx$ (vi) $\int \frac{dx}{x\sqrt{3x^2-12}}$

(vii) $\int \frac{x^2}{x^2+a^2}dx$ (viii) $\int \frac{1}{x\left[1+(\ln x)^2\right]}dx$ (ix) $\int \frac{1}{x^2\sqrt{x^2+4}}dx$

(x) $\int \frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}$ (xi) $\int \frac{dx}{x\sqrt{x^2-9}}$ (xii) $\int \sqrt{\frac{1+x}{1-x}}dx$

(xiii) $\int \frac{\sqrt{a^2-x^2}}{x}dx\ \ (a>0)$

3.4 Integration by Parts

The rule that corresponds to the Product Rule for differentiation is called **integration by parts**. Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx$$

Product Rule in Integral Form

Suppose that $f(x)$ and $h(x)$ are two differentiable functions. According to the product rule,

$$\frac{d}{dx}\left[f(x)h(x)\right]=f(x)\,h'(x)+h(x)\,f'(x)$$

Integrating both sides with respect to x gives:

$$f(x)\,h(x)=\int\left[f(x)h'(x)+h(x)f'(x)\right]dx$$

Splitting the integral, we have:

$$f(x)\,h(x)=\int f(x)h'(x)\,dx+\int h(x)f'(x)\,dx$$

Rearranging terms, we obtain:

$$\int f(x)\,h'(x)\,dx=f(x)\,h(x)-\int h(x)\,f'(x)\,dx \qquad \ldots(1)$$

Now, let $h'(x)=g(x)$, then $h(x)=\int g(x)\,dx$.

Substituting into (1), we get:

$$\int f(x)g(x)dx=f(x)\left(\int g(x)dx\right)-\int\left[\left(\int g(x)dx(f'(x)\right)\right]dx \qquad \ldots(2)$$

Example 29 Evaluate $\int x\sin nx\,dx,\ n\neq 0.$

Solution: $\int x\sin nx\,dx$; Integrating by parts taking x as first function, we have

$$=x\left(\frac{-\cos nx}{n}\right)-\int\left(\frac{-\cos nx}{n}\right)(1)dx=\frac{-x\cos nx}{n}+\frac{1}{n}\int\cos nx\,dx$$

$$=\frac{-x\cos nx}{n}+\frac{1}{n}\cdot\frac{\sin nx}{n}+c=\frac{-x\cos nx}{n}+\frac{\sin nx}{n^2}+c$$

Example 30 Evaluate $\int x\,e^{ax}\,dx$

Solution: Applying the formula for integration by parts, we have

$$\int x\,e^{ax}\,dx=x\frac{e^{ax}}{a}-\int\frac{e^{ax}}{a}\cdot 1\,dx=x\frac{e^{ax}}{a}-\frac{1}{a}\int e^{ax}\,dx=x\frac{e^{ax}}{a}-\frac{1}{a}\cdot\frac{e^{ax}}{a}+c$$

$$=e^{ax}\left(\frac{x}{a}-\frac{1}{a^2}\right)+c=e^{ax}\left(\frac{ax-1}{a^2}\right)+c$$

Example 30 Evaluate $\int\frac{x}{\cos^2 x}dx$

Solution: $\int\frac{x}{\cos^2 x}dx=\int x\sec^2 x\,dx$

Applying the formula for integration by parts, we have

$$\int x\sec^2 x\,dx=x\cdot\tan x-\int\tan x\cdot 1\,dx=x\cdot\tan x-\int\frac{\sin x}{\cos x}dx$$

$$=x\cdot\tan x+\int\frac{-\sin x}{\cos x}\,dx=x\cdot\tan x+\ln|\cos x|+c$$

Example 32 Evaluate $\int \ln(\sin x)\cdot \sin 2x\,dx$

Solution: $\int \ln(\sin x)\cdot \sin 2x\,dx = \int \ln(\sin x)\cdot 2\sin x\cdot\cos x\,dx$

$= 2\int \ln(\sin x)\cdot \sin x\cos x\,dx$ Put $u = \sin x$. $\frac{du}{dx} = \cos x \Rightarrow du = \cos x\,dx$

$= 2\int \ln u\cdot u\,du = 2\left[\ln u\cdot\frac{u^2}{2} - \int \frac{u^2}{2}\cdot\frac{1}{u}du\right] = \ln u\cdot u^2 - \int u\,du = u^2\ln u - \frac{u^2}{2} + c$

$= \sin^2 x\ln(\sin x) - \frac{\sin^2 x}{2} + c$

$= \sin^2 x\left[\ln(\sin x) - \frac{1}{2}\right] + c$

Note!

$\int e^{g(x)}[f(x)g'(x) + f'(x)]dx = e^{g(x)}f(x) + c$

Example 33 Evaluate $\int e^{x^2}(\cos x + 2x\sin x)\,dx$

Solution: $\int e^{x^2}(\cos x + 2x\sin x)\,dx = \int e^{x^2}\cos x\,dx + \int e^{x^2}\,2x\sin x\,dx$

$= e^{x^2}(\sin x) - \int (\sin x)e^{x^2}\,2x\,dx + \int e^{x^2}\,2x\sin x\,dx = e^{x^2}\sin x + c$

Example 34 Evaluate $\int \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)dx$

Solution: $\int \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)dx$; Putting $x = \tan\theta$; $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $dx = \sec^2\theta\,d\theta$

$= \int \sin^{-1}\left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\right)\cdot\sec^2\theta\,d\theta = \int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\cdot\sec^2\theta\,d\theta$

$= \int \sin^{-1}\left(\frac{\sin\theta}{\cos\theta}\ \cos\theta\right)\cdot\sec^2\theta\,d\theta = \int \sin^{-1}(\sin\theta)\cdot\sec^2\theta\,d\theta$

$= \int \theta\cdot\sec^2\theta\ d\theta = \theta\tan\theta - \int(\tan\theta)(1)\,d\theta = \theta\tan\theta - \ln|\sec\theta| + c$

$= \theta\tan\theta - \ln|\sqrt{\sec^2\theta}| + c = \theta\tan\theta - \frac{1}{2}\ln|1+\tan^2\theta| + c = \left(\tan^{-1}x\right)\cdot x - \frac{1}{2}\ln|1+x^2| + c$

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$

Example 35 Show that $\int e^{ax}\sin bx\,dx = \frac{1}{\sqrt{a^2+b^2}}\ e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

Solution: Let $I = \int e^{ax}\sin bx\,dx$ Integrating by parts, we have

$= e^{ax}\left(-\frac{\cos bx}{b}\right) - \int\left(-\frac{\cos bx}{b}\right)e^{ax}\cdot a\,dx$

$$= -\frac{e^{ax}\ \cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx\,dx \qquad \text{Again, integrating by parts, we have}$$

$$= -\frac{e^{ax}\ \cos\ bx}{b} + \frac{a}{b}\left[e^{ax}\left(\frac{\sin bx}{b}\right) - \int\left(\frac{\sin bx}{b}\right)e^{ax}\cdot a\ dx\right]$$

$$= -\frac{e^{ax}\cos\ bx}{b} + \frac{ae^{ax}\sin\ bx}{b^2} - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

$$I = -\frac{e^{ax}\cos bx}{b} + \frac{ae^{ax}\sin bx}{b^2} - \frac{a^2}{b^2}\ I$$

$$I + \frac{a^2}{b^2}I = -\frac{e^{ax}\cos bx}{b} + \frac{ae^{ax}\sin bx}{b^2}$$

$$\left(\frac{b^2+a^2}{b^2}\right)I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx) \ \Rightarrow\ I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Substituting $a = r\cos\theta$ and $b = r\sin\theta$, we have

$$I = \frac{e^{ax}}{a^2+b^2}(r\cos\theta\sin bx - r\sin\theta\cos bx)$$

$$= \frac{1}{a^2+b^2}e^{ax}\,r(\sin bx\cos\theta - \cos bx\sin\theta) = \frac{1}{a^2+b^2}e^{ax}\,r\sin(bx-\theta)$$

Since $a = r\cos\theta$ and $b = r\sin\theta$

Squaring and adding, we have $r^2 = a^2+b^2 \ \Rightarrow\ r = \sqrt{a^2+b^2}$ and $\frac{r\sin\theta}{r\cos\theta} = \frac{b}{a}$

$$\Rightarrow\ \tan\theta = \frac{b}{a}\ \Rightarrow\ \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore\quad I = \frac{1}{a^2+b^2}e^{ax}\sqrt{a^2+b^2}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) = \frac{1}{\sqrt{a^2+b^2}}e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Example 36 Evaluate $\int x\sin^2 x\,dx$

Solution: $\int x\sin^2 x\,dx = \int x.\frac{1-\cos 2x}{2}dx = \frac{1}{2}\int(x - x\cos\ 2x)\,dx$

$$= \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx \qquad \text{Integrating the second integral by parts}$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right) - \frac{1}{2}\left[x\left(\frac{\sin 2x}{2}\right) - \int\left(\frac{\sin 2x}{2}\right)\cdot 1\,dx\right] = \frac{x^2}{4} - \frac{x\sin 2x}{4} + \frac{1}{4}\int\sin 2x\,dx$$

$$= \frac{x^2}{4} - \frac{x\sin 2x}{4} + \frac{1}{4}\left(-\frac{\cos 2x}{2}\right) + c = \frac{x^2}{4} - \frac{x\sin 2x}{4} - \frac{1}{8}\cos 2x + c$$

Special Integral

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$$

Proof: Let $I = \int \sqrt{x^2+a^2}\,dx$

$= \int \sqrt{x^2+a^2}\,(1)\,dx$ Integrating by parts, we have

$$= \sqrt{x^2+a^2}\,(x) - \int x\frac{d}{dx}\left[\left(x^2+a^2\right)^{\frac{1}{2}}\right]dx = x\sqrt{x^2+a^2} - \int x\cdot\frac{1}{2}\left(x^2+a^2\right)^{-\frac{1}{2}}\cdot(2x)\,dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}dx = x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2}{\sqrt{x^2+a^2}} + \int \frac{a^2}{\sqrt{x^2+a^2}}dx$$

$$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,dx + a^2\int \frac{1}{\sqrt{x^2+a^2}}\,dx$$

$$I = x\sqrt{x^2+a^2} - I + a^2\int \frac{1}{\sqrt{x^2+a^2}}dx$$

$$2I = x\sqrt{x^2+a^2} + a^2\ln\left|x+\sqrt{x^2+a^2}\right| + c_1$$

$$I = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$$

Thus, $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$

Challenge!

Prove that: (i) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

(ii) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$

Example 37 Evaluate $\int \sqrt{x^2-4x+13}\,dx$

Solution: $\int \sqrt{x^2-4x+13}\,dx = \int \sqrt{(x^2-4x+4)+13-4}\,dx = \int \sqrt{(x-2)^2+9}\,dx$

$$= \int \sqrt{(x-2)^2+(3)^2}\,dx$$

Let $x-2=u$, so $du=dx$

$$\int \sqrt{(x-2)^2+(3)^2}\,dx = \int \sqrt{u^2+3^2}\,du = \frac{u}{2}\sqrt{u^2+3^2} + \frac{3^2}{2}\ln\left|u+\sqrt{u^2+3^2}\right| + c$$

Replace u with $x-2$

$$=\frac{x-2}{2}\sqrt{(x-2)^2+9}+\frac{9}{2}\ln\left|(x-2)+\sqrt{(x-2)^2+9}\right|+c$$

$$=\frac{x-2}{2}\sqrt{x^2-4x+13}+\frac{9}{2}\ln\left|(x-2)+\sqrt{x^2-4x+13}\right|+c$$

Example 38 Evaluate $\int\sqrt{8x-x^2-7}\,dx$

Solution: $\int\sqrt{8x-x^2-7}\,dx=\int\sqrt{-(x^2-8x)-7}\,dx$

$$=\int\sqrt{-(x^2-8x+16-16)-7}\,dx=\int\sqrt{-[(x-4)^2-16]-7}\,dx$$

$$=\int\sqrt{3^2-(x-4)^2}\,dx \qquad \text{Let } x-4=u, \text{ so } dx=du$$

$$=\int\sqrt{3^2-u^2}\,du=\frac{u}{2}\sqrt{3^2-u^2}+\frac{3^2}{2}\sin^{-1}\left(\frac{u}{3}\right)+c$$

Substitute back $u=x-4$

$$=\frac{x-4}{2}\sqrt{9-(x-4)^2}+\frac{9}{2}\sin^{-1}\left(\frac{x-4}{3}\right)+c=\frac{x-4}{2}\sqrt{8x-x^2-7}+\frac{9}{2}\sin^{-1}\left(\frac{x-4}{3}\right)+c$$

Example 39 Evaluate $\int\sqrt{x^2-8x+7}\,dx$

Solution: $\int\sqrt{x^2-8x+7}\,dx=\int\sqrt{(x^2-8x+16)+7-16}\,dx$

$$=\int\sqrt{(x-4)^2-9}\,dx=\int\sqrt{(x-4)^2-3^2}\,dx \quad \text{Let } u=x-4, \text{ so } du=dx$$

$$=\int\sqrt{u^2-3^2}\,du=\frac{u}{2}\sqrt{u^2-3^2}-\frac{3^2}{2}\ln\left|u+\sqrt{u^2-3^2}\right|+c$$

Substitute back $u=x-4$

$$=\frac{x-4}{2}\sqrt{(x-4)^2-9}-\frac{9}{2}\ln\left|x-4+\sqrt{(x-4)^2-9}\right|+c$$

$$=\frac{x-4}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\ln\left|x-4+\sqrt{x^2-8x+7}\right|+c$$

EXERCISE 3.3

Evaluate the following integrals:

1. $\int\ln x\,dx$
2. $\int x^7\cdot\ln x\,dx$
3. $\int x\cos 2x\,dx$
4. $\int\frac{x}{\sin^2 x}dx$
5. $\int xe^{2x}\,dx$
6. $\int x^2e^{ax}\,dx$

7. $\int x\tan^2 x\,dx$ 8. $\int \ln(\sin x)\cdot\cos x\,dx$ 9. $\int e^x \ln(1+e^x)\,dx$

10. $\int (2x^3+2x)e^{x^2}\,dx$ 11. $\int e^x \tan^{-1}(e^x)\,dx$ 12. $\int \frac{x}{1+\cos x}\,dx$

13. $\int \frac{x}{1-\sin x}\,dx$ 14. $\int \frac{x\cos x-\sin x}{x^2}\,dx$ 15. $\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\,dx$

16. $\int \sqrt{x}\sin\sqrt{x}\,dx$ 17. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ 18. $\int \frac{e^{\tan^{-1}x}}{(1+x^2)^2}\,dx$

19. $\int \sin^{-1} x\,dx$ 20. $\int \frac{\ln x\tan^{-1}(\ln x)}{x}\,dx$ 21. $\int \csc^3 x\,dx$

22. $\int e^{5x}(x^5+x^4+1)\,dx$ 23. $\int \frac{e^{\sqrt{x}}(1+\sqrt{x})}{\sqrt{x}}\,dx$ 24. $\int e^{3x}(\sin x-3\cos 3x)\,dx$

3.5 Integration by Using Partial Fractions

If $P(x), Q(x)$ are polynomial functions and the denominator $Q(x)$ $(x\neq 0)$, in the rational function $\frac{P(x)}{Q(x)}$, can be factorized into linear and quadratic (irreducible) factors, then the rational function $\frac{P(x)}{Q(x)}$ is written as a sum of simpler rational functions, each of which can be integrated by methods already known to us.

Here, we will give examples of the following four cases when the denominator $Q(x)$ contains

Case-I: Non-repeated linear factors.
Case-II: Repeated and non-repeated linear factors.
Case-III: Linear and non repeated irreducible quadratic factors.
Case-IV: Linear and repeated irreducible quadratic factors.

Case-I: Integration of $\int \frac{P(x)}{Q(x)}dx$ where $Q(x)$ has non-repeated linear factors

Example 40 Integrate $\int \frac{3x-2}{x(x-1)(x-2)}dx$

Solution: $\int \frac{3x-2}{x(x-1)(x-2)}dx$

Let $\frac{3x-2}{x(x-1)(x-2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-2}$

Multiplying both sides, by $x(x-1)(x-2)$ we have

$3x-2=A(x-1)(x-2)+B(x)(x-2)+C(x)(x-1)$ ...(1)

Putting $x=0$ in (1): $3(0)-2=A(0-1)(0-2)+B(0)(0-2)+C(0)(0-1)$

$-2=A(-1)(-2)+B(0)+C(0) \Rightarrow -2=2A \Rightarrow \boxed{A=-1}$

Putting $x=1$ in (1): $3(1)-2=A(1-1)(1-2)+B(1)(1-2)+C(1)(1-1)$

$3-2=A(0)+B(-1)+C(0) \Rightarrow 1=-B \Rightarrow \boxed{B=-1}$

Putting $x=2$ in (1): $3(2)-2=A(2-1)(2-2)+B(2)(2-2)+C(2)(2-1)$

$6-2=A(0)+B(0)+C(2)(1) \Rightarrow 4=2C \Rightarrow \boxed{C=2}$

$\therefore \quad \dfrac{3x-2}{x(x-1)(x-2)}=\dfrac{-1}{x}+\dfrac{-1}{x-1}+\dfrac{2}{x-2}$

Thus, $\int \dfrac{3x-2}{x(x-1)(x-2)}dx=\int\left[\dfrac{-1}{x}+\dfrac{-1}{x-1}+\dfrac{2}{x-2}\right]dx$

$=-\int\dfrac{1}{x}dx-\int\dfrac{1}{x-1}dx+2\int\dfrac{1}{x-2}dx=-\ln|x|-\ln|x-1|+2\ln|x-2|+C$

Case-II Integration of $\int\dfrac{P(x)}{Q(x)}dx$ where $Q(x)$ has repeated linear factors:

Example 41 Evaluate $\int\dfrac{6x-10}{(x+1)(x-3)^2}dx$

Solution: Let $\dfrac{6x-10}{(x+1)(x-3)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-3}+\dfrac{C}{(x-3)^2}$

Multiplying both sides by $(x+1)(x-3)^2$:

$6x-10=A(x-3)^2+B(x+1)(x-3)+C(x+1)$ ...(1)

From equation (1),

$0.x^2+6x-10=A(x^2-6x+9)+B(x^2-2x-3)+C(x+1)$ ...(2)

Let $x+1=0 \Rightarrow x=-1$ Putting $x=-1$ in (1)

$6(-1)-10=A(-1-3)^2+B(0)+C(0)$

$-6-10=A(-4)^2 \Rightarrow -16=16A \Rightarrow \boxed{A=-1}$

Let $x-3=0 \Rightarrow x=3$ Putting $x=3$ in (1)

$6(3)-10=A(0)^2+B(0)+C(3+1)$

$18-10=4C \Rightarrow 8=4C \Rightarrow \boxed{C=2}$

Comparing coefficient of x^2 on both sides of equation (2)

$A+B=0 \Rightarrow B=-A \Rightarrow B=-(-1) \Rightarrow \boxed{B=1}$

So (1) becomes

$$\frac{6x-10}{(x+1)(x-3)^2}=\frac{-1}{x+1}+\frac{1}{x-3}+\frac{2}{(x-3)^2}$$

Thus, $\int\frac{6x-10}{(x+1)(x-3)^2}dx=\int\left[\frac{-1}{x+1}+\frac{1}{x-3}+\frac{2}{(x-3)^2}\right]dx$

$$=-\int\frac{1}{x+1}dx+\int\frac{1}{x-3}dx+2\int(x-3)^{-2}dx=-\ln|x+1|+\ln|x-3|+2\frac{(x-3)^{-2+1}}{-2+1}+c$$

$$=\ln|x-3|-\ln|x+1|+2\frac{(x-3)^{-1}}{-1}+c=\ln\left|\frac{x-3}{x+1}\right|-\frac{2}{x-3}+c$$

Case-III Integration of $\int\frac{P(x)}{Q(x)}dx$ where $Q(x)$ has at least one non-repeated irreducible quadratic factor.

Example 42 Find $\int\frac{5x^2+3x+2}{(x+1)(x^2+1)}dx$

Solution: $\frac{5x+3x+2}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$

Multiplying both sides by $(x+1)(x^2+1)$ to clear fractions:

$5x^2+3x+2=A(x^2+1)+(Bx+C)(x+1)$ ...(1)

Put $x+1=0 \Rightarrow x=-1$ in equation (1)

$5(-1)^2+3(-1)+2=A[(-1)^2+1]+[B(-1)+C](-1+1)$

$5-3+2=A(2)+0 \Rightarrow 4=2A \Rightarrow \boxed{A=2}$

From (1)

$5x^2+3x+2=Ax^2+A+Bx^2+Bx+Cx+C$

$5x^2+3x+2=(A+B)x^2+(B+C)x+C+A$

Equating coefficients of x^2, we get

$5=A+B \Rightarrow 5=2+B \Rightarrow \boxed{B=3}$

Equating constants, we get

$2=A+C \Rightarrow 2=C+2 \Rightarrow \boxed{C=0}$

Now, the integrand becomes,

$$\frac{5x+3x+2}{(x+1)(x^2+1)}=\frac{2}{x+1}+\frac{3x+0}{x^2+1}$$

$$\int\frac{5x+3x+2}{(x+1)(x^2+1)}=2\int\frac{1}{x+1}dx+\int\frac{3x}{x^2+1}dx=2\int\frac{1}{x+1}dx+\frac{3}{2}\int\frac{2x}{x^2+1}dx$$

$$=2\ln|x+1|+\frac{3}{2}\ln|x^2+1|+c=2\ln|x+1|+\frac{3}{2}\ln(x^2+1)+c$$

Case-IV Integration of $\int \frac{P(x)}{Q(x)}dx$ where $Q(x)$ has repeated irreducible quadratic factors:

Example 43 Evaluate $\int \frac{x^3+2x+1}{(x^2+1)^2}\,dx$

Solution: $\frac{x^3+2x+1}{(x^2+1)^2} = \frac{x^3+x+x+1}{(x^2+1)^2} = \frac{x(x^2+1)+x+1}{(x^2+1)^2} = \frac{x(x^2+1)}{(x^2+1)^2} + \frac{x+1}{(x^2+1)^2}$

$$= \frac{x}{x^2+1} + \frac{x}{(x^2+1)^2} + \frac{1}{(x^2+1)^2}$$

Thus, $\int \frac{x^3+2x+1}{(x^2+1)^2}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx + \int \frac{1}{(x^2+1)^2}\,dx$

$$= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int (x^2+1)^{-2}(2x)\,dx + \int \frac{1}{(x^2+1)^2}\,dx$$

$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\frac{(x^2+1)^{-1}}{-1} + \int \frac{1}{(x^2+1)^2}\,dx \quad \ldots(1)$$

To evaluate $\int \frac{1}{(x^2+1)^2}\,dx$, put $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{1}{(\tan^2\theta+1)^2}\cdot\sec^2\theta\,d\theta = \int \frac{1}{(\sec^2\theta)^2}\cdot\sec^2\theta\,d\theta = \int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \int \cos^2\theta\,d\theta = \int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] = \frac{1}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right]$$

$$= \frac{1}{2}[\theta + \sin\theta\cos\theta] = \frac{1}{2}\left[\theta + \frac{\sin\theta\cos^2\theta}{\cos}\right] = \frac{1}{2}\left[\theta + \frac{\tan\theta}{\sec^2\theta}\right] = \frac{1}{2}\left[\theta + \frac{\tan\theta}{1+\tan^2\theta}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}x + \frac{x}{1+x^2}\right]$$ (1) becomes

Finally: $\int \frac{x^3+2x+1}{(x^2+1)^2}\,dx = \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\frac{1}{x^2+1} + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{1+x^2} + c$

$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right) + c$$

$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{x-1}{2(x^2+1)} + c$$

EXERCISE 3.4

Evaluate the following integrals:

1. $\int \frac{5x+7}{(x+1)(x+2)}dx$
2. $\int \frac{3x+5}{x^2+4x+3}dx$
3. $\int \frac{x^3+2x^2-x-1}{x^2-1}dx$
4. $\int \frac{x^2-12x-21}{x^2-6x-7}dx$
5. $\int \frac{3-16x}{4+5x-6x^2}dx$
6. $\int \frac{6x^2-22x+18}{x(x-1)(x-2)(x-3)}dx$
7. $\int \frac{x^2-3}{(x+2)^3}dx$
8. $\int \frac{e^x-1}{e^x\left(e^x+1\right)}dx$
9. $\int \frac{25(x+2)}{(x-2)(x+3)^2}dx$
10. $\int \frac{x^3+4x^2+5x+2}{(x+1)^3(x+2)}dx$
11. $\int \frac{x^2+2x-1}{(x-1)(x^2+1)}dx$
12. $\int \frac{x^3+2x+1}{(x^2+1)(x^2+4)}dx$
13. $\int \frac{x^4+3x^2+x+1}{(x^2+1)^2(x-1)}dx$
14. $\int \frac{\cos x}{\sin^3 x-1}dx$
15. $\int \frac{x^2+1}{(x^2-2x+2)^2}dx$

3.6 The Definite Integral

Definition: Let f be a continuous function on the closed interval $[a, b]$. The **definite integral** of f from a to b is defined as $\int_a^b f(x)\,dx = F(b)-F(a)$, where $F'(x) = f(x)$.

Here, a is the **lower limit** and b the **upper limit** of the integration.

Remark: The value of a definite integral is a real number, not a family of functions, as is the case for indefinite integrals. The numbers a and b are the **limits of integration**.

Theorem 3.2: First Fundamental Theorem of Calculus (FTC)

If f is continuous on $[a, b]$ and $F(x)=\int_a^x f(t)\,dt$ for every x in $[a, b]$,

then $F'(x) = f(x)$, on $[a, b]$.

The derivative with respect to upper limit of the integral of f is just $f(x)$.

Theorem 3.3: Second Fundamental Theorem of Calculus (FTC)

If f is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

Example 44 Evaluate: $\int_{-1}^{1}(x^{\frac{1}{5}}+1)\,dx$

Solution: $$\int_{-1}^{1}(x^{\frac{1}{5}}+1)\,dx=\left(\frac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1}+x\right)_{-1}^{1}=\left(\frac{5}{6}x^{\frac{6}{5}}+x\right)_{-1}^{1}$$

$$=\left(\frac{5}{6}(1)^{\frac{6}{5}}+1\right)-\left(\frac{5}{6}(-1)^{\frac{6}{5}}+(-1)\right)=\frac{5}{6}+1-\frac{5}{6}+1=2$$

Example 45 Evaluate $\int_{0}^{\pi/4}\frac{\sec\theta}{\sin\theta+\cos\theta}d\theta$

Solution: $$\int_{0}^{\pi/4}\frac{\sec\theta}{\sin\theta+\cos\theta}d\theta=\int_{0}^{\pi/4}\frac{\sec\theta}{\cos\theta(\tan\theta+1)}d\theta=\int_{0}^{\pi/4}\frac{\sec^2\theta}{1+\tan\theta}d\theta$$

$$=\left[\ln\left|1+\tan\theta\right|\right]_{0}^{\pi/4}=\ln\left|1+\tan\frac{\pi}{4}\right|-\ln|1+0|=\ln|1+1|-0=\ln 2$$

Example 46 Evaluate $\int_{0}^{\sqrt{7}}\frac{x}{\sqrt{x^2+9}}dx$

Solution: Let $f(x)=x^2+9$, then $f'(x)=2x$, so

$$\int_{0}^{\sqrt{7}}\frac{x}{\sqrt{x^2+9}}dx=\frac{1}{2}\int_{0}^{\sqrt{7}}(x^2+9)^{-\frac{1}{2}}(2x)\,dx=\frac{1}{2}\left|\frac{(x^2+9)^{\frac{1}{2}}}{\frac{1}{2}}\right|_{0}^{\sqrt{7}}$$

$$=\left[\sqrt{x^2+9}\right]_{0}^{\sqrt{7}}=\left[\sqrt{7+9}-\sqrt{0+9}\right]=\sqrt{16}-\sqrt{9}=4-3=1$$

Substitution in a Definite Integral (Change of variables)

Suppose that g is differentiable function on $[a, b]$. Also suppose that f is continuous on the range of g. Then $\int_{a}^{b}f(g(x))g'(x)\,dx=\int_{g(a)}^{g(b)}f(u)\,du.$

Example 47 Evaluate $\int_{0}^{2}2xe^{x^2}dx$

Solution: Let $g(x)=x^2$. Then $g'(x)=2x$

The integrand is $e^{x^2}\cdot 2x=f(g(x))\cdot g'(x)$ with $f(u)=e^u$

When $x=0$, $u=g(0)=0^2=0$

When $x=2$, $u=g(2)=2^2=4$

Therefore, $\int_0^2 2xe^{x^2}\,dx=\int_{u=0}^{u=4} e^u\,du=\int_0^4 e^u\,du=\left[e^u\right]_0^4=e^4-e^0=e^4-1$

Integration by Parts Formula for Definite Integrals

Let $f(x)$ and $g(x)$ be functions with continuous derivatives on the interval $[a, b]$. Then the integration by parts formula for definite integrals is:

$$\int_a^b f(x)g(x)\,dx=\left[f(x)\left(\int g(x)dx\right)\right]_a^b-\int_a^b\left[\left(\int g(x)dx\right)\left(f'(x)\right)\right]dx$$

Example 48 If $f(0)=f(\pi)=0$, show that $\int_0^\pi f(x)\sin x\,dx=-\int_0^\pi f''(x)\sin x\,dx$

Solution: $\int_0^\pi f(x)\sin x\,dx$ Applying integration by parts:

$$=\left|f(x)(-\cos x)\right|_0^\pi-\int_0^\pi f'(x)(-\cos x)\,dx\ =-f(\pi)\cos\pi+f(0)\cos 0+\int_0^\pi f'(x)\cos x\,dx$$

$$=-(0)(-1)+(0)(1)+\left|f'(x)\sin x\right|_0^\pi-\int_0^\pi \sin x\cdot f''(x)dx$$

$$=0+0+f'(\pi)\sin(\pi)-f'(0)\sin(0)-\int_0^\pi f''(x)\sin x dx$$

$$=f'(\pi)(0)-f'(0)\ (0)-\int_0^\pi f''(x)\sin x\,dx=-\int_0^\pi f''(x)\sin x\,dx$$

3.6.1 Properties of the Definite Integrals

Throughout this section, assume f is a continuous function on the given interval.

Property-I: $\int_a^b f(x)\,dx=\int_a^b f(t)\,dt.$

Property-II: $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx.$

Property-III: $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx,$ where $a<c<b.$

This result can be generalized as follows:

$$\int_a^b f(x)\,dx=\int_a^{c_1} f(x)\,dx+\int_{c_1}^{c_2} f(x)\,dx+\int_{c_2}^{c_3} f(x)\,dx+\cdots+\int_{c_{n-1}}^{c_n} f(x)\,dx+\int_{c_n}^b f(x)\,dx$$

where $a<c_1<c_2<c_3<\cdots<c_{n-1}<c_n<b$

Remember!

$\int_a^a f(x)dx=0$

Property-IV: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$

(i) As a special case, if we take $a = 0$, we obtain: $\int_0^b f(x)\,dx = \int_0^b f(b-x)\,dx$

(ii) As another special case, if we take $a = -b$, we obtain: $\int_{-b}^b f(x)\,dx = \int_{-b}^b f(-x)\,dx.$

(iii) For any integrand of the form $\frac{f(x)}{f(x)+g(x)}$, if there exists a substitution $x \to a+b-x$ such that $f(a+b-x) = g(x)$ and $g(a+b-x) = f(x)$, then

$$\int_a^b \frac{f(x)}{f(x)+g(x)}\,dx = \frac{b-a}{2}.$$

Property-V: $\int_{-a}^a f(x)\,dx = \int_0^a \big(f(x)+f(-x)\big)\,dx$

In particular, if $f(-x) = f(x)$ (that is, f is an even function), then

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

and if $f(-x) = -f(x)$ (that is, f is an odd function), then $\int_{-a}^a f(x)\,dx = 0$

Property-VI: $\int_0^a f(x)\,dx = \int_0^{a/2} \big(f(x)+f(a-x)\big)\,dx$

In particular, $\int_0^a f(x)\,dx = \begin{cases} 0 & \text{if } f(a-x) = -f(x) \\ 2\int_0^{a/2} f(x)\,dx & \text{if } f(a-x) = f(x) \end{cases}$

Property-VII: If $f(x) \geq 0$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$ and if $f(x) \leq 0$ on the interval $[c, d]$, then $\int_c^d f(x)\,dx \leq 0.$

Property-VIII: If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \leq \int_c^b g(x)\,dx.$

Property-IX: $\int_a^b \big(\alpha f_1(x) + \beta f_2(x)\big)\,dx = \alpha\int_a^b f_1(x)\,dx + \beta\int_a^b f_2(x)\,dx.$

Example 49 Let $f(x)=\begin{cases} x^2 & \text{if} \quad 0\le x<2 \\ 4-x & \text{if} \quad 2\le x\le 4 \end{cases}$ Compute: $\int_0^4 f(x)dx.$

Solution: $\int_0^4 f(x)\,dx=\int_0^2 f(x)\,dx+\int_2^4 f(x)\,dx=\int_0^2 x^2\,dx+\int_2^4 (4-x)\,dx$

$$=\left[\frac{x^3}{3}\right]_0^2+\left[4x-\frac{x^2}{2}\right]_2^4=\left[\frac{(2)^3}{3}-\frac{(0)^3}{3}\right]+\left[\left(4(4)-\frac{(4)^2}{2}\right)-\left(4(2)-\frac{(2)^2}{2}\right)\right]$$

$$=\left[\frac{8}{3}-0\right]+\ (16-8)-(8-2)=\frac{8}{3}+8-6=\frac{8}{3}+2=\frac{8+6}{3}=\frac{14}{3}.$$

Example 50 Evaluate: $\int_{-6} f(x)\,dx$ where $f(x)=|x-1|+|x+2|$.

Solution: To find where the expressions inside the absolute values changes:

$$x-1=0 \ \Rightarrow \ x=1 \quad \text{and} \quad x+2=0 \ \Rightarrow \ x=-2$$

So, break the interval $[-6, 4]$ at $x=-6, x=-2, x=1, x=4$.

These points divide the interval $[-6, 4]$ into three intervals $[-6, 2]$, $[-2, 1]$, $[1, 4]$.

Now, express $f(x)$ without absolute values and compute the integral over each subinterval:

(a) For $x \in [-6, -2]$: $|x-1|=-(x-1)=-x+1$ and $|x+2|=-(x+2)=-x-2$

$$f(x)=|x-1|+|x+2|=-x+1-x-2=-2x-1$$

$$\int_{-6}^{-2} f(x)\,dx=\int_{-6}^{-2}(-2x-1)\,dx=\left[-\frac{2x^2}{2}-x\right]_{-6}^{-2}=\left[-x^2-x\right]_{-6}^{-2}$$

$$=(-4+2)-(-36+6)=-2-(-30)=-2+30=28$$

(b) For $x \in [-2, 1]$: $|x-1|=-(x-1)=-x+1$ and $|x+2|=x+2$

$$f(x)=|x-1|+|x+2|=-x+1+x+2$$

$$\int_{-2}^{1} f(x)\,dx=\int_{-2}^{1} 3\,dx=\left|3x\right|_{-2}^{1}=3(1)-3(-2)=3+6=9$$

(c) For $x \in [1, 4]$: $|x-1|$ and $|x+2|=x+2$

$$f(x)=|x-1|+|x+2|=x-1+x+2=2x+1$$

$$\int_1^4 f(x)\,dx=\int_1^4 (2x+1)\,dx=[x^2+x]_1^4=[(4)^2+4]-[(1)^2+1]=16+4-2=18$$

From (a), (b) and (c), we have:

$$\int_{-6}^{4} f(x)\,dx=\int_{-6}^{-2} f(x)\,dx+\int_{-2}^{1} f(x)\,dx+\int_{1}^{4} f(x)\,dx=28+9+18=57.$$

Example 51 Evaluate $\int_0^1 x^2(1-x)^{\frac{3}{2}}\,dx$.

Solution: Let $I=\int_0^1 x^2(1-x)^{\frac{3}{2}}\,dx$

$$I=\int_0^1 (1-x)^2\left(1-(1-x)\right)^{\frac{3}{2}}\,dx \qquad \left(\text{By using } \int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right)$$

$$=\int_0^1 (1-2x+x^2)(1-1+x)^{\frac{3}{2}}\,dx=\int_0^1 (1-2x+x^2)\,x^{\frac{3}{2}}dx=\int_0^1\left(x^{\frac{3}{2}}-2x\,x^{\frac{3}{2}}+x^2x^{\frac{3}{2}}\right)dx$$

$$=\int_0^1\left(x^{\frac{3}{2}}-2x^{\frac{5}{2}}+x^{\frac{7}{2}}\right)dx=\left[\frac{2}{5}x^{\frac{5}{2}}-\frac{4}{7}x^{\frac{7}{2}}+\frac{2}{9}x^{\frac{9}{2}}\right]_0^1=\left(\frac{2}{5}(1)^{\frac{5}{2}}-\frac{4}{7}(1)^{\frac{7}{2}}+\frac{2}{9}(1)^{\frac{9}{2}}\right)-(0)$$

$$=\frac{2}{5}-\frac{4}{7}+\frac{2}{9}=\frac{126-180+70}{315}=\frac{16}{315}$$

Example 52 $\int_0^{2\pi}\cos^5 x\,dx$

Solution: Let $I=\int_0^{2\pi}\cos^5 x\,dx$

Here $f(x)=\cos^5 x \Rightarrow f(2\pi-x)=\left[\cos(2\pi-x)\right]^5=\left(\cos x\right)^5=\cos^5 x=f(x)$

$\therefore\ I=2\int_0^{\pi}\cos^5 x\,dx$

Also $f(\pi-x)=\left(\cos(\pi-x)\right)^5=(-\cos x)^5=-\cos^5 x=-f(x)$

Thus, $I=\int_0^{2\pi}\cos^5 x\,dx=0.$

Example 53 Show that $\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2}+\sin^2 x\ \cos^2 x}\,dx=0.$

Solution: Let $I=\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2}+\sin^2 x\ \cos^2 x}\,dx$

Using the property: $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$ of definite integrals, we substitute x

with $\frac{\pi}{5}+\frac{3\pi}{10}-x=\frac{\pi}{2}-x$:

$$I=\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}-\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{2}+\sin^2\left(\frac{\pi}{2}-x\right)\cos^2\left(\frac{\pi}{2}-x\right)}dx=\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\cos x}-\sqrt{\sin x}}{\sqrt{2}+\sin^2 x\ \cos^2 x}dx$$

$$=-\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2}+\sin^2 x\ \cos^2 x}dx.\ \text{Clearly } I=-I\Rightarrow 2I=0\Rightarrow I=0.$$

That is, $\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2}+\sin^2 x\ \cos^2 x}dx=0.$

Example 54 Given $\int_{-3}^{3}\frac{f(x)}{1+e^x}dx=4$ and $\int_{-3}^{3}f(x)\,dx=10$. Evaluate $\int_{-3}^{3}\frac{f(x)}{1+e^{-x}}dx.$

Solution: Key identity: $\frac{1}{1+e^x}+\frac{1}{1+e^{-x}}=1.$

Multiplying the identity by $f(x)$, we get: $\frac{f(x)}{1+e^x}+\frac{f(x)}{1+e^{-x}}=f(x)$

Integrate over [–3, 3], we get: $\int_{-3}^{3}\frac{f(x)}{1+e^x}dx+\int_{-3}^{3}\frac{f(x)}{1+e^{-x}}dx=\int_{-3}^{3}f(x)\,dx$

Substitute the given values: $4+\int_{-3}^{3}\frac{f(x)}{1+e^{-x}}dx=10\Rightarrow\int_{-3}^{3}\frac{f(x)}{1+e^{-x}}dx=10-4=6$

Thus, $\int_{-3}^{3}\frac{f(x)}{1+e^{-x}}dx=6.$

Example 55 Evaluate $\int_{-a}^{a}\frac{x^3\sin^2 x}{x^4+2x^2+1}dx.$

Solution: Let $f(x)=\frac{x^3\sin^2 x}{x^4+2x^2+1}$

$$f(-x)=\frac{(-x)^3\sin^2(-x)}{(-x)^4+2(-x)^2+1}=\frac{-x^3\sin^2 x}{x^4+2x^2+1}=-f(x)$$

This shows that $f(x)$ is odd over the symmetric interval $[-a, a]$.

Hence, $\int_{-a}^{a}\frac{x^3\sin^2 x}{x^4+2x^2+1}dx=0$

Example 56 Evaluate $\int_1^3 f(x)\,dx$.

Solution: Since $f(2-x)=f(x) \Rightarrow \int_0^2 f(x)dx = 4 \Rightarrow 2\int_0^1 f(x)\,dx = 4 \Rightarrow \int_0^1 f(x)\,dx = 2$

Given $\int_0^2 f(x)dx = 4$ and $f(2-x)=f(x)=f(-x)$.

To evaluate $\int_1^3 f(x)\,dx$. Put $x=2-t$, $dx=-dt$. When $x=1$, $t=1$; When $x=3$, $t=-1$

Therefore, $\int_1^3 f(x)\,dx = \int_1^{-1} f(2-t)(-dt) = -\int_{-1}^{1} f(2-t)(-dt) = \int_{-1}^{1} f(2-t)\,dt$

$= \int_{-1}^{1} f(2-x)\,dx = \int_{-1}^{1} f(x)\,dx = 2\int_0^1 f(x)\,dx = 2(2) = 4$. Since $f(2-x)=f(x)=f(-x)$

EXERCISE 3.5

1. Evaluate the following definite integrals:

(i) $\int_1^2 (x^2+x)\,dx$ (ii) $\int_{-2}^3 (4x^3+3x^2+2x)\,dx$. (iii) $\int_0^{\pi/2} \sin^2 x\,dx$

(iv) $\int_0^{\pi/4} \frac{1}{1+\sin x}\,dx$ (v) $\int_0^{\pi/2} \cos^3\theta\,d\theta$ (vi) $\int_0^1 \frac{\sqrt{x}}{1+x}\,dx$

(vii) $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$ (viii) $\int_0^1 \tan^{-1} x\,dx$ (ix) $\int_0^{2\pi} \frac{\sin^2 x}{1+e^{\cos x}}\,dx$

2. Evaluate the following definite integrals:

(i) $\int_{-1}^{1} f(x)dx$, where $f(x)=\begin{cases} 5-3x & \text{if} \quad x\le 0 \\ 5+3x & \text{if} \quad x>0 \end{cases}$

(ii) $\int_{-3}^{5} f(x)dx$, where $f(x)=|x-4|+|x|$ (iii) $\int_{-3}^{4} |4-x^2|\,dx$.

3. Let $f(x)=\begin{cases} x^2 & \text{if} \quad 0\le x\le 2 \\ 2x+1 & \text{if} \quad 2<x\le 4 \end{cases}$, Here $g(x)$ is defined as: $g(x)=f(x)+f(4-x)$.

Compute: $\int_0^4 g(x)dx$.

4. Simplify: (i) $\int_0^1 f(x)\,dx - \int_0^2 f(x)\,dx - \int_3^1 f(x)\,dx + \int_3^2 f(x)\,dx$.

(ii) $\int_0^3 5f(x)\,dx - \int_1^3 5f(4-x)\,dx - \int_1^5 5f(x)\,dx - \int_0^5 4f(5-x)\,dx$.

5. Prove that $\int_0^1 x^m(1-x)^n\,dx = \int_0^1 x^n(1-x)^m\,dx$ and hence evaluate $\int_0^1 x(1-x)^{10}\,dx$.

6. Evaluate the following definite integrals:

   (i) $\int_0^4 x\sqrt{4-x}\,dx$ (ii) $\int_0^1 \frac{x^2}{\sqrt{1-x}}\,dx$ (iii) $\int_{-1}^1 \frac{e^x-1}{e^x+1}\,dx$

7. Let $f(x)$ be a function satisfying $f(x)+f(-x)=x^2+x^3$ for all x. Evaluate the definite integral $\int_{-2}^2 f(x)dx$.

8. Let $\int_0^3 f(x)\,dx = 20$, $\int_0^{10} f(x)\,dx = 45$. Evaluate $\int_3^{10} f(x)\,dx$.

9. Given that: $\int_0^9 f(x)\,dx = 16$ and $\int_0^3 f(x)\,dx = 12$. Evaluate the integral $\int_3^9 (2f(x)+5)\,dx$.

10. If $f(4-x)=f(x)$, show that $\int_0^2 f(2x)\,dx = \int_0^2 f(x)\,dx$.

11. Show that $\int_{-1}^1 (1-x)f\left(1-x^2\right)dx = \int_{-1}^1 f\left(1-x^2\right)dx$.

12. If $f(a+x)+f(a-x)=0$, show that $\int_0^{2a} f(x)\,dx = 0$.

13. Show that $\int_{-a}^a \frac{f(x)}{1+e^x}\,dx = \int_0^a f(x)dx$, where $f(x)$ is an even function. Hence or otherwise evaluate $\int_{-a}^a \frac{x^2}{1+e^x}\,dx$.

14. Given the function $f(x)$ satisfies the following conditions:

   (i) $\int_0^4 f(x)dx = 12$ (ii) $f(4-x)=f(x)$ (iii) $f(x+2)=f(x)$

   Evaluate $\int_{-1}^5 f(x)\,dx$.

15. Show that:

   (i) $\int_{\pi/6}^{\pi/3} \ln(\tan x)\,dx = 0$ (ii) $\int_0^{\pi} \frac{\sin x\cos x}{1+e^{\sin x}}\,dx = 0$

(iii) $\int_0^{\pi} x(\pi - x)\cos(\pi - x)\,dx = 0.$ (iv) $\int_1^3 \frac{(x-2)^3 + 4(x-2)}{1+e^{(x-2)^2}}\,dx = 0.$

16. Evaluate the following definite integrals:

(i) $\int_1^3 \frac{x^2}{x^2 + (4-x)^2}\,dx$ (ii) $\int_0^{\ln 2} \frac{e^x}{e^x + 2e^{-x}}\,dx.$

17. Find the indicated derivatives:

(i) $\frac{d}{dt}\int_{\pi}^{t} \frac{\sin y}{1+\sqrt{y}}\,dy$ (ii) $\frac{d}{dx}\int_{ax}^{bx} \frac{1}{1+t^2}\,dt$

3.7 Application of Definite Integrals

3.7.1 Area Under the Curve

A remarkable result in calculus is that the area under the graph of $y = f(x)$, where $f(x) \geq 0$, can be obtained simply by evaluating the definite integral of $f(x)$ over the interval $[a, b]$.

If f is continuous and $f(x) \geq 0$ over the interval $[a, b]$, then the area A of the region under the curve of f from a to b is $A = \int_a^b f(x)\,dx$

Notes:

1. If $f(x) \leq 0$ on the interval $[a, b]$, then the area bounded by the curve $y = f(x)$ and x-axis is given by $\left|\int_a^b f(x)\,dx\right|$ or equivalently $\int_a^b |f(x)|\,dx$ or equivalently $-\int_a^b f(x)\,dx.$

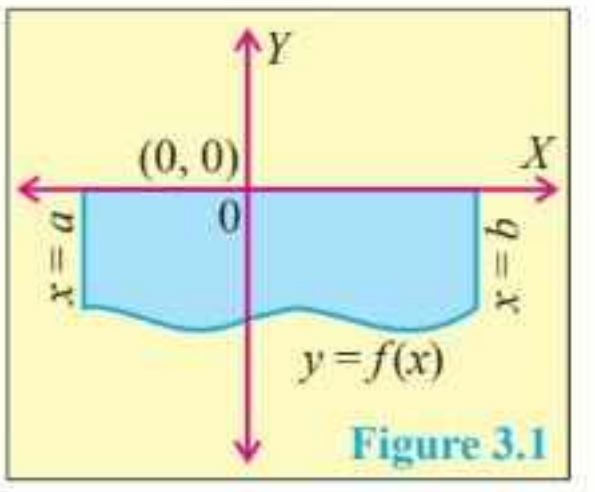


Figure 3.1

2. If $f(x) \geq 0$ on the interval $[a, c]$ and $f(x) \leq 0$ on the interval $[c, b]$, then the total area bounded by the curve $y = f(x)$ and x-axis on the interval $[a, b]$ is

$$\int_a^c f(x)dx + \left|\int_c^b f(x)dx\right|$$

or equivalently $\int_a^c f(x)\,dx + \int_c^b |f(x)|\,dx$

or equivalently $\int_a^c f(x)dx - \int_c^b f(x)dx$

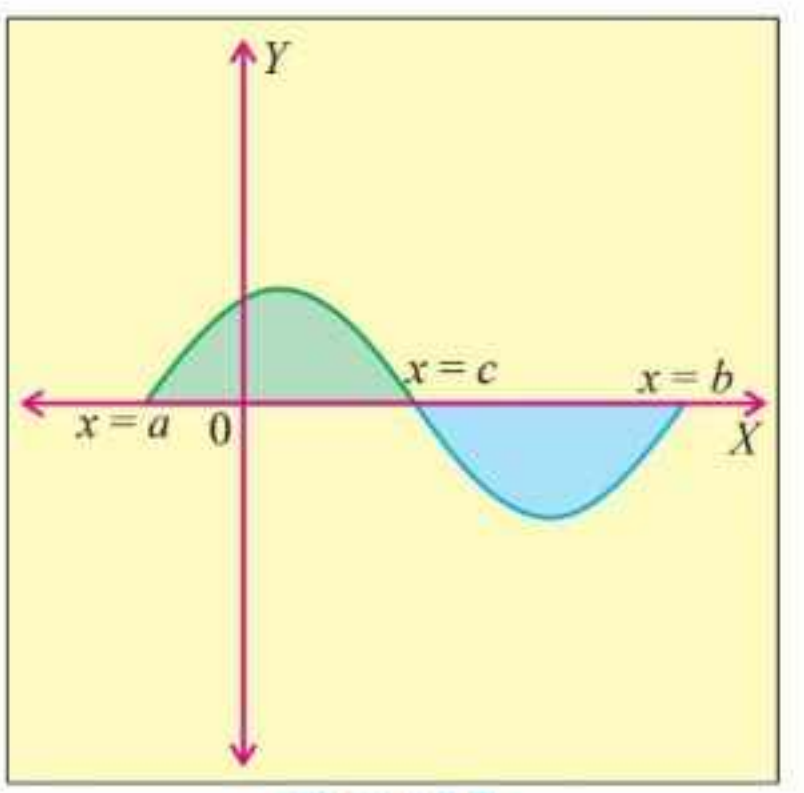


Figure 3.2

3. If $f(x)$ changes sign at $c_1, c_2, c_3, \ldots, c_n$ in (a, b) where $c_1 < c_2 < \cdots < c_n$ are the roots of $f(x) = 0$.

$$\text{Total Area} = \int_a^b |f(x)|\, dx = \left|\int_a^{c_1} f(x)\, dx\right| + \left|\int_{c_1}^{c_2} f(x)\, dx\right| + \cdots + \left|\int_{c_n}^{b} f(x)\, dx\right|$$

4. The area bounded by the curve $x = f(y)$, the y-axis, the horizontal lines $y = c$ and $y = d$ $(d > c)$ is given by

$$\int_c^d f(y)\, dy.$$

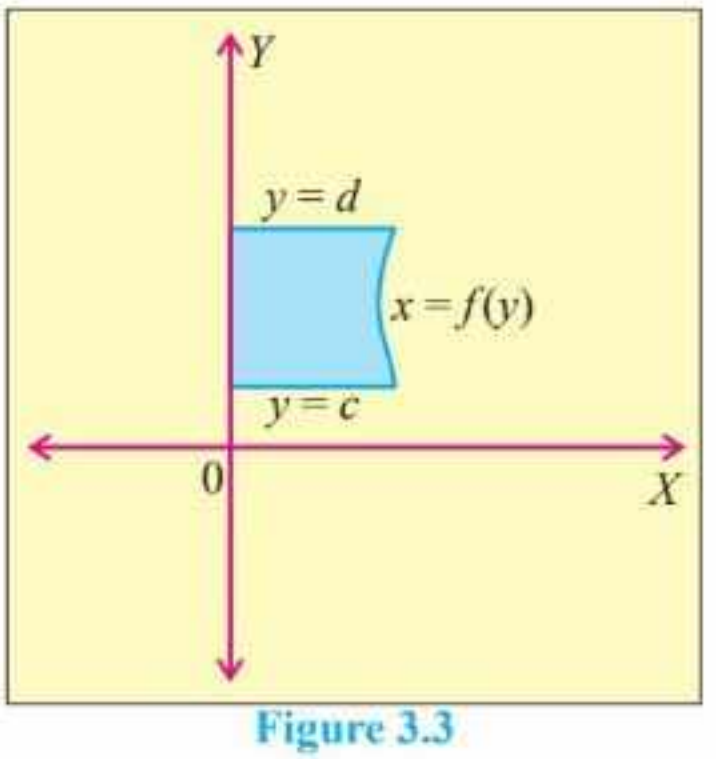


Figure 3.3

5. **Area between two curves**

If f and g are continuous and $f(x) \geq g(x)$ over the interval $[a, b]$, then the area A of the region bounded by the graphs of f, g, $x = a$, and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

The above result does not require $f(x)$ or $g(x)$ to remain positive over the interval $[a, b]$.

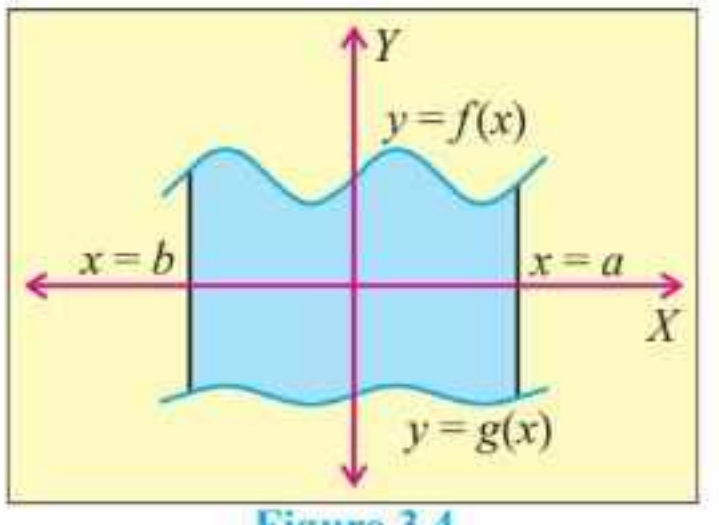


Figure 3.4

Signed Area and Net Signed Area

The definite integral $\int_a^b f(x)\, dx$ gives the area between the curve and the x-axis, but the interpretation depends on whether the curve crosses the axis.

- If the **entire curve** lies **above** the x-axis, then the definite integral directly gives the **signed area**—and it is **positive**.
- If the **entire curve** lies **below** the x-axis, then the definite integral directly gives the **signed area**—and it is **negative**.
- If **some part** of the curve is above the axis and some part is below the axis, and we add these areas **with their signs** (positive for above, negative for below), then we get the **net signed area**.

Thus:

- **Signed area** refers to the case when the curve is entirely above or entirely below the axis.
- **Net signed area** refers to the result after canceling positive and negative contributions when the curve crosses the axis.
  In both cases, the formula is the same:

$$\text{Signed Area (or Net Signed Area)} = \int_a^b f(x)\,dx$$

The result may be:

- **Positive** (more area above or entirely above),
- **Negative** (more area below or entirely below), or **Zero** (equal area above and below).

Example 57 Find the area bounded by the parabola $y = x^2 - 5x + 4$ and the x-axis.

Solution: Given the parabola: $y = x^2 - 5x + 4$

Set $y = 0$ to find x-intercepts:

$$x^2 - 5x + 4 = 0 \Rightarrow (x-1)(x-4) = 0$$

So, the x-intercepts are at $x = 1$ and $x = 4$.

The parabola opens upwards (positive leading coefficient).

Between the roots $x = 1$ and $x = 4$, y(function) is negative. The vertical lines are $x = 1$ and $x = 4$, which are exactly the intercepts.

Therefore, the region bounded by the curve, the x-axis, and these lines is from $x = 1$ to $x = 4$, where the curve is below the x-axis.

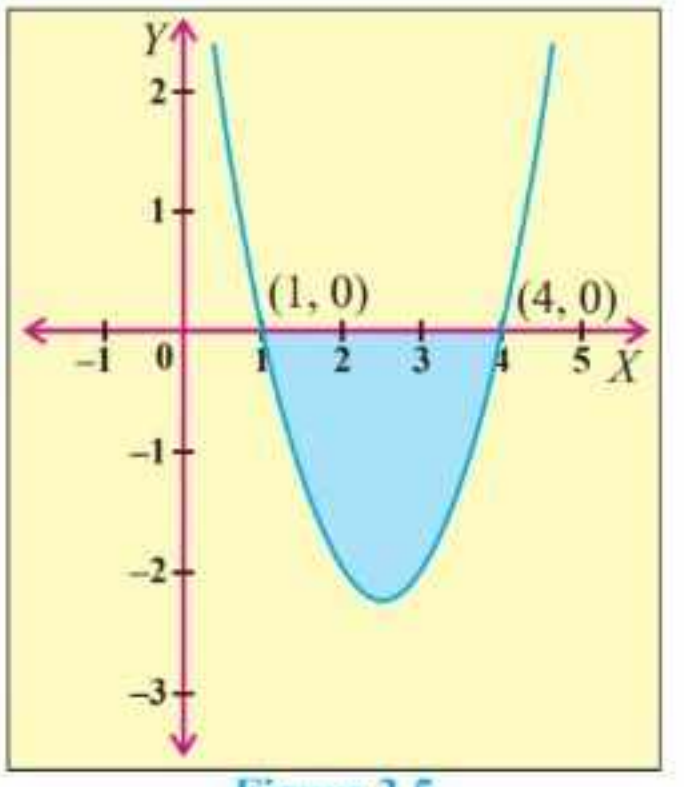


Figure 3.5

Since the curve is below the x-axis from $x = 1$ to $x = 4$, the area is:

$$\text{Area} = \int_1^4 |y|\,dx = \int_1^4 -(x^2 - 5x + 4)\,dx = \int_1^4 (-x^2 + 5x - 4)\,dx$$

$$= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x\right]_1^4 = \left(-\frac{(4)^3}{3} + \frac{5(4)^2}{2} - 4(4)\right) - \left(-\frac{(1)^3}{3} + \frac{5(1)^2}{2} - 4(1)\right)$$

$$= \frac{8}{3} - \left(-\frac{11}{6}\right) = \frac{16+11}{6} = \frac{27}{6} = \frac{9}{2}$$

The area is $\frac{9}{2}$ square units.

Example 58 Find the area bounded by the curves $y = 13 - x^2$ and $y = 2x + 5$.

Solution: First, we find points of intersection of the curves $y = 13 - x^2$ and $y = 2x + 5$ as:

$$13 - x^2 = 2x + 5 \Rightarrow 13 - x^2 - 2x - 5 = 0 \Rightarrow 8 - x^2 - 2x = 0$$

$$\Rightarrow x^2 + 2x - 8 = 0 \Rightarrow x^2 + 4x - 2x - 8 = 0$$

$$\Rightarrow x(x+4) - 2(x+4) = 0 \Rightarrow (x-2)(x+4) = 0$$

So, $x = -4$ or $x = 2$

Determine which curve is on the top:

Test $x = 0$:

$$y(0) = 13 - 0^2 = 13$$

$$y(0) = 2(0) + 5 = 5$$

So, $y = 13 - x^2$ is above $y = 2x + 5$ in the interval $[-4, 2]$.

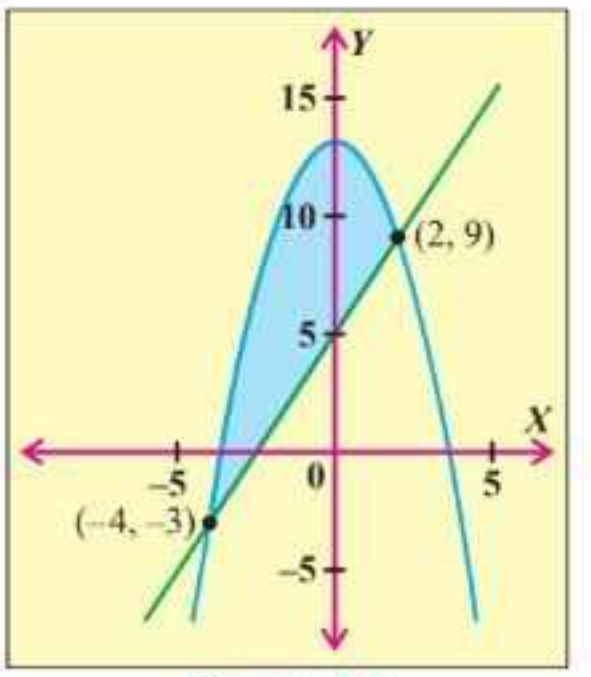


Figure 3.6

$$\text{Area} = \int_{-4}^{2} [\text{upper} - \text{lower}]\,dx = \int_{-4}^{2} [(13 - x^2) - (2x + 5)]\,dx$$

$$= \int_{-4}^{2} (8 - x^2 - 2x)\,dx = \left[8x - \frac{x^3}{3} - x^2 \right]_{-4}^{2}$$

$$= \left(8(2) - \frac{2^3}{3} - 2^2 \right) - \left(8(-4) - \frac{(-4)^3}{3} - (-4)^2 \right)$$

$$= 60 - \left(\frac{8}{3} + \frac{64}{3} \right) = 60 - \frac{8+64}{3} = 60 - \frac{72}{3} = 60 - 24 = 36 \text{ square units}$$

The area between the two curves is 36 square units.

Example 59 Find the area of the region bounded by the parabolas $y = 4 - x^2$ and $y = x^2 - 2x$.

Solution: To find the point of intersection equating both equations.

$$4 - x^2 = x^2 - 2x$$

$$4 - x^2 - x^2 + 2x = 0$$

$$4 - 2x^2 + 2x = 0$$

$$-2x^2 + 2x + 4 = 0$$

Divide by -2:

$$x^2 - x - 2 = 0$$

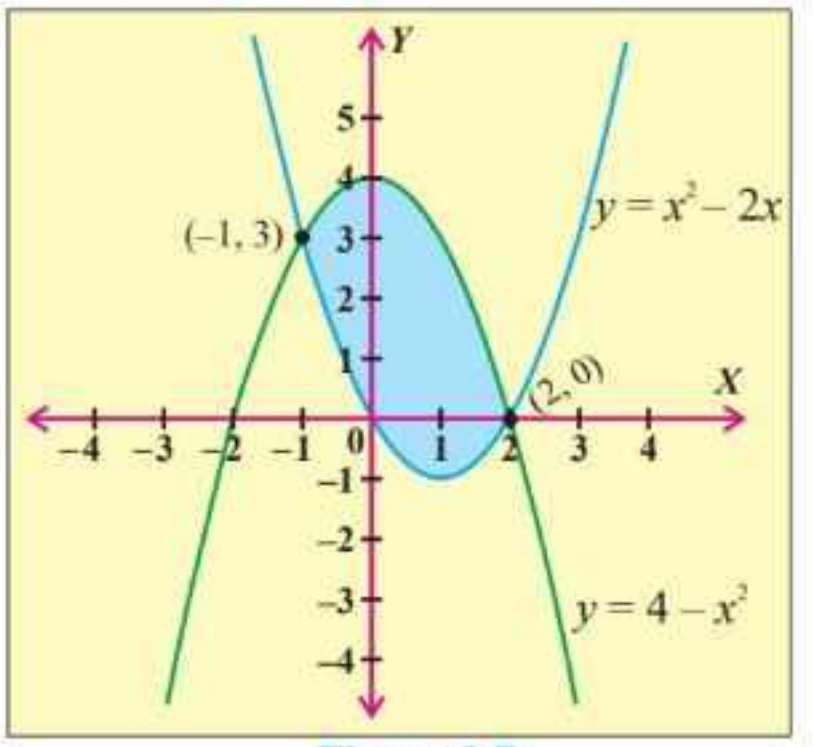


Figure 3.7

$$(x+1)(x-2)=0$$

So, $x=-1$ and $x=2$

Intersection points: $(-1,\ 3)$ and $(2,\ 0)$

Determine which curve is on top at $[-1, 2]$

Take $x=0 \in [-1, 2]$. Put it on both curves, we have

Test $x=0$: $y(0)=4-0^2=4$ and $y(0)=0^2-2(0)=0$

So, $4-x^2>x^2-2x$ for $x=0$

Thus, $f(x)=4-x^2$ is the upper curve, $g(x)=x^2-2x$ is the lower curve.

$$\text{Area}=\int_{-1}^{2}[f(x)-g(x)]dx=\int_{-1}^{2}\left[(4-x^2)-(x^2-2x)\right]dx$$

$$=\int_{-1}^{2}(4-2x^2+2x)\,dx=\left[4x-\frac{2}{3}x^3+x^2\right]_{-1}^{2}$$

$$=\left(4(2)-\frac{2(8)}{3}+2^2\right)-\left(4(-1)-\frac{2(-1)}{3}+(-1)^2\right)=\frac{20}{3}-\left(-\frac{7}{3}\right)=9 \text{ square units}$$

3.7.2 Consumer and Producer Surplus

One place in economics where area between two curves is important is in the calculation of a quantity called **consumer surplus** and a quantity called **producer surplus**.

Demand Function

The demand curve for a commodity shows the quantity consumers are willing to buy at any given price. Let p be the price that a consumer is willing to pay for a quantity x of a certain commodity. Then p and x are related to each other through the demand function $p = D(x)$. The graph of the demand function generally slopes downwards, as the demand for a commodity decreases when its price increases; that is, with an increase in price, the consumer is likely to buy less. [Figure 3.8]

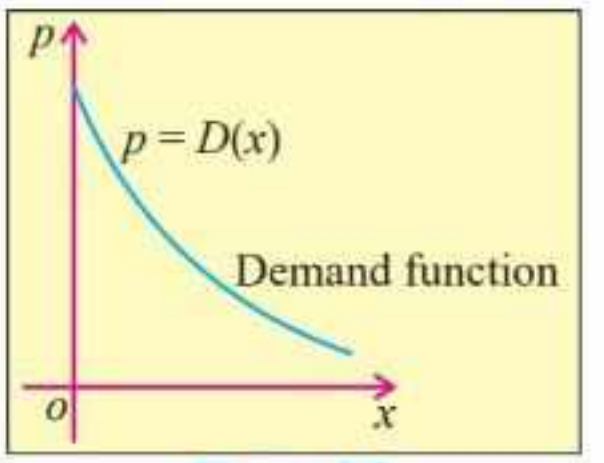


Figure 3.8

Supply Function

Let p be the price at which a producer is willing to sell a quantity x of a certain commodity. Then p and x are related to each other through the supply function $p = S(x)$. The graph of the supply function generally slopes upwards, as the supply of a commodity increases when its price increases; that is, with an increase in price, the producer is willing to supply more. [Figure 3.9]

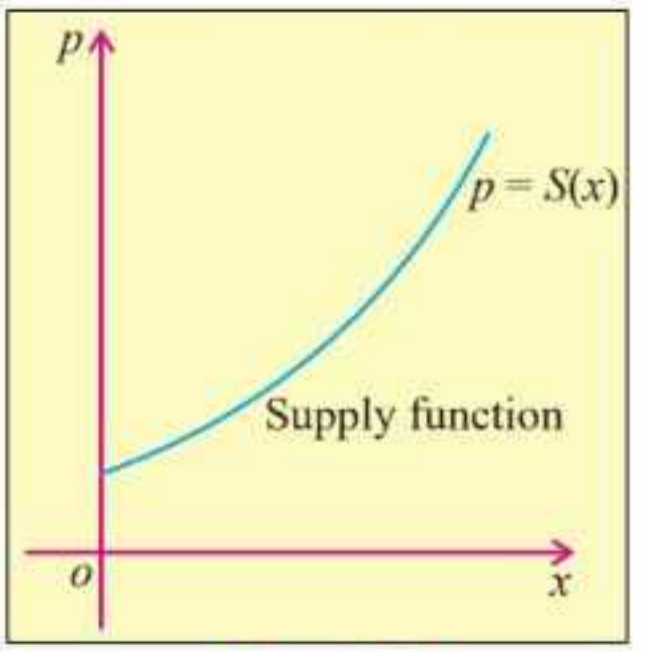


Figure 3.9

Market Equilibrium

The point where the demand and supply curves intersect is called the market equilibrium. At this point the quantity that consumers want to buy equals the quantity that producers want to sell.

If (x_e, p_e) is the equilibrium point, the quantity x_e is called the equilibrium quantity and the price p_e is called the equilibrium price.

Algebraically to find the equilibrium point, we solve the system of equations formed by the demand equation and supply equation.

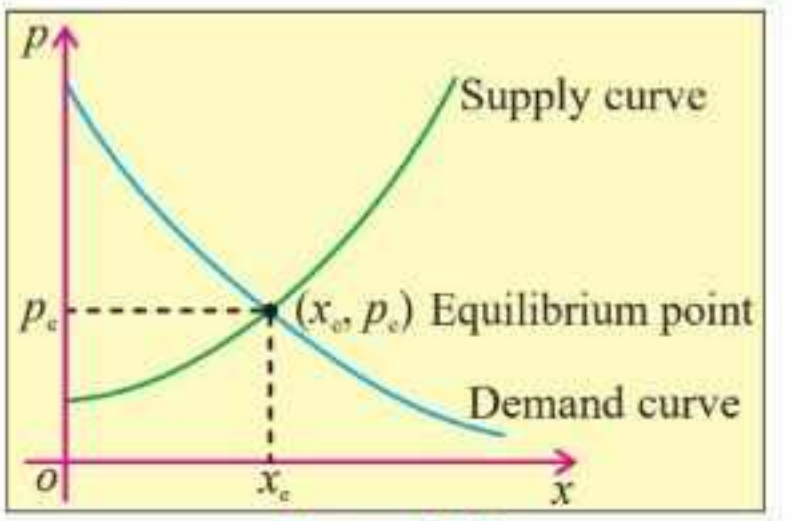


Figure 3.10

Consumer Surplus (CS)

Let the demand function be $p = D(x)$, and let x_e be the quantity demanded when the market price is p_e. However, some consumers who are unaware of equilibrium price are willing to pay more than p_e for the quantity x_e of the commodity. The difference between what a consumer is willing to pay and what they actually pay is a benefit to the consumer.

Thus, consumer surplus can be defined as:

3.4 Consumer Surplus (CS)

Consumer surplus is the total benefit gained by consumers when they purchase an item at the equilibrium price, rather than at the higher price they would have been willing to pay.

Geometrically, it is the area under the demand curve and above the horizontal line drawn through the equilibrium point.

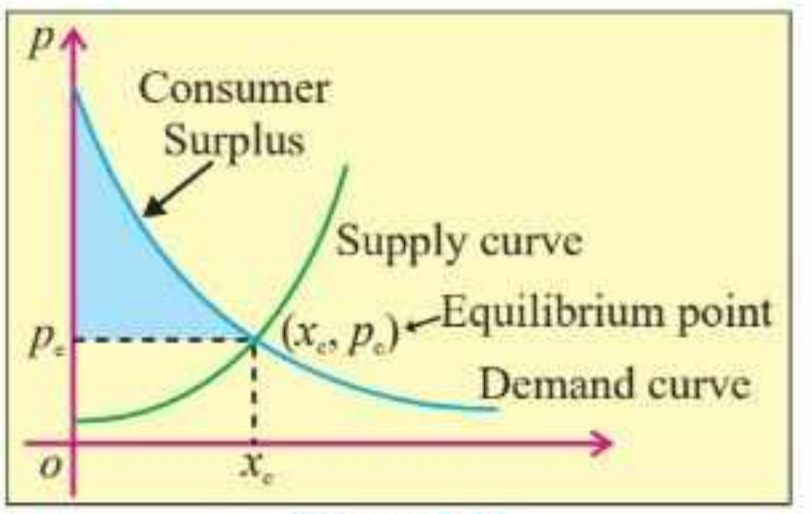


Figure 3.11

$$\text{Thus, CS} = \begin{pmatrix}\text{Area under demand function}\\ \text{from } x=0 \text{ to } x=x_e\end{pmatrix} - \begin{pmatrix}\text{Area under equilibrium price}\\ \text{from } x=0 \text{ to } x=x_e\end{pmatrix}$$

$$= \int_0^{x_e} (\text{Demand function})\,dx - \int_0^{x_e} (\text{Equilibrium price})\,dx$$

$$= \int_0^{x_e} D(x)\,dx - \int_0^{x_e} p_e\,dx = \int_0^{x_e} (D(x) - p_e)\,dx$$

Producer Surplus (PS)

Let the supply function be $p = S(x)$, and let x_e be the quantity supplied when the market price is p_e. However, some producers are willing to accept less than p_e for the quantity x_e of the commodity. The difference between what producers actually receive and what they would have been willing to accept is a benefit to the producer.
The great economist Alfred Marshall called this benefit producer surplus.

3.5 Producer Surplus (PS)

Producer surplus is the total benefit gained by producers when they sell an item at the equilibrium price, rather than at the lower price they would have been willing to accept.
Geometrically, it is the area above the supply curve and below the horizontal line drawn through the equilibrium point.

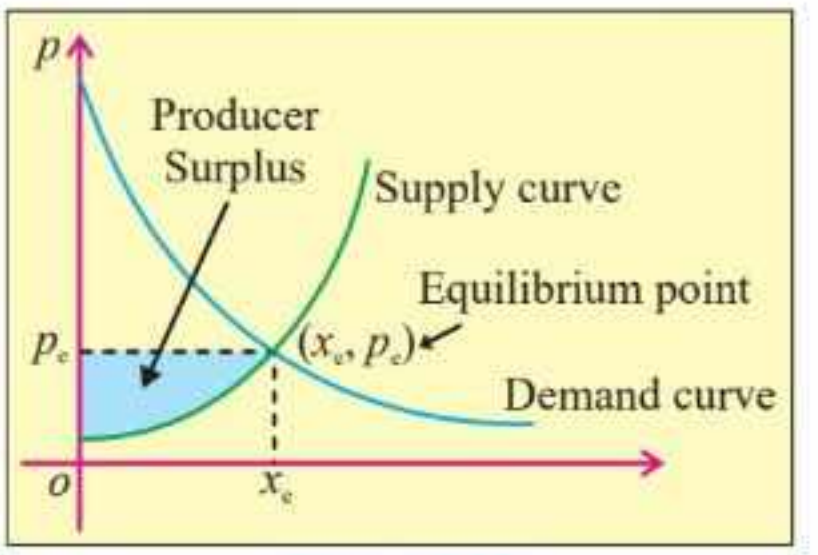


Figure 3.12

$$\text{Thus, PS} = \begin{pmatrix}\text{Area under equilibrium price}\\ \text{from } x=0 \text{ to } x=x_e\end{pmatrix} - \begin{pmatrix}\text{Area above supply function}\\ \text{from } x=0 \text{ to } x=x_e\end{pmatrix}$$

$$= \int_0^{x_e} (\text{Equilibrium price})\,dx - \int_0^{x_e} (\text{Supply function})\,dx$$

$$= \int_0^{x_e} p_e\,dx - \int_0^{x_e} S(x)\,dx = \int_0^{x_e} (p_e - S(x))\,dx$$

Consumer Surplus (CS) and Producer Surplus (PS) Formulae

1. With Respect to Quantity (x)

Let:
- $p = D(x)$ = demand price function
- $p = S(x)$ = supply price function
- x_e = equilibrium quantity
- x_e = equilibrium price $= D(x_e) = S(x_e)$

Consumer Surplus (CS) = area between demand curve and equilibrium price line, from $x = 0$ to $x = x_e$: $\text{CS} = \int_0^{x_e} (D(x) - p_e)\,dx$

Producer Surplus (PS) = area between demand curve and equilibrium price line, from $x = 0$ to $x = x_e$: $\text{PS} = \int_0^{x_e} (p_e - S(x))\,dx$

Total surplus (TS): $\text{TS} = \text{CS} + \text{PS} = \int_0^{x_e} (D(x) - S(x))\,dx$

2. With Respect to Price (p)

Let:
- $x_d(p) = D^{-1}(p)$ = quantity demanded at price p
- $X_s(p) = S^{-1}(p)$ = quantity supplied at price p
- $P_{\max}$ = choke price (demand = 0 for higher prices)
- $P_{\min}$ = minimum price (supply = 0 for lower prices)
- P_e = equilibrium price

Consumer Surplus (CS) = area between p_e and $p_{\max}$, under demand curve $x_d(p)$:

$$CS = \int_{p=p_e}^{p_{max}} x_d(p)\,dp$$

Producer Surplus (PS) = area between $p_{\min}$ and p_e, under supply curve $x_s(p)$:

$$PS = \int_{p=p_{min}}^{p_e} x_s(p)\,dp$$

3. Summary Table

Surplus	Integration w.r.t. Quantity	Integration w.r.t. price
CS	$\int_0^{x_e} (D(x) - p_e)\,dx$	$\int_{p=p_e}^{p_{max}} x_d(p)\,dp$
PS	$\int_0^{x_e} (p_e - S(x))\,dx$	$\int_{p=p_{min}}^{p_e} x_s(p)\,dp$

4. Key Notes

- Choke price $p_{\max} = D(0)$ (price when $x = 0$ in demand).
- Minimum supply price $p_{\min} = S(0)$ (price when $x = 0$ in supply)
- When integrating w.r.t. x, limits are quantities, integrand is price difference.

Example 60 The supply curve for a good is $p_s = \sqrt{16+x}$. The demand curve is linear with vertical intercept 73 and passes through the equilibrium point. If the quantity sold is $x_e = 33$, find the equation of the demand curve (p_d).

Solution: Supply curve: $p_s = \sqrt{16+x}$

At $x_e = 33$, $p_e = \sqrt{16+33} = \sqrt{49} = 7$. So intersecting point is $(x_e, p_e) = (33, 7)$.

Demand is linear $p_d = mx + c$.

Vertical intercept $c = 73 \Rightarrow p_d = mx + 73$

It passes through (33, 7): $7 = m \cdot 33 + 73 \Rightarrow 33m = -66 \Rightarrow m = -2$

So, $p_d = -2x + 73$.

Example 61 If the supply function for a particular commodity is $25p=(x+10)^2$ and the market price is *Rs.*64, find the producer's surplus by two methods:

(i) Integrate with respect to x (ii) Integrate with respect to p

Solution: Given: $25p=(x+10)^2 \Rightarrow p=\dfrac{(x+10)^2}{25}$

At $p_e=64: 25\times 64=(x_e+10)^2 \Rightarrow (x_e+10)^2=5^2\times 8^2$

Take positive root: $x_e+10=5\times 8 \Rightarrow x_e+10=40 \Rightarrow x_e=30$

Method (i) Integrate w.r.t. x

$$\text{PS}=\int_0^{x_e}(p-p(x))\,dx=\int_0^{30}\left[64-\frac{(x+10)^2}{25}\right]dx=\left[64x-\frac{(x+10)^3}{25\times 3}\right]_0^{30}$$

$$=\left(64\times 30-\frac{(30+10)^3}{75}\right)-\left(0-\frac{10^3}{75}\right)=1920-\frac{(40)^3}{75}+\frac{(10)^3}{75}=1920-840=\text{Rs. }1080$$

Method (ii) Integrate w.r.t. p

From: $25p=(x+10)^2 \Rightarrow x+10=5\sqrt{p} \Rightarrow x=5\sqrt{p}-10$

Find $p_{\min}$ when $x=0$:

$$0=5\sqrt{p}-10 \Rightarrow 5\sqrt{p}=10 \Rightarrow \sqrt{p}=\frac{10}{5} \Rightarrow \sqrt{p}=2 \Rightarrow p_{\min}=4$$

$$\text{PS}=\int_{p_{\min}}^{p_e} x_s(p)\,dp=\int_4^{64}(5\sqrt{p}-10)\,dp=\left[5\frac{p^{3/2}}{3/2}-10p\right]=\left[\frac{10}{3}p^{3/2}-10p\right]$$

$$=\left[\frac{10}{3}(64)^{3/2}-10(64)\right]-\left[\frac{10}{3}(4)^{3/2}-10(4)\right]=\left[\frac{10}{3}(8^2)^{3/2}-640\right]-\left[\frac{10}{3}(2^2)^{3/2}-10(4)\right]$$

$$=\frac{5120}{3}-\frac{80}{3}-640+40=\frac{5120-80}{3}-600=\frac{5040}{3}-600=1680-600=1080$$

Both methods give PS = Rs.1080.

3.7.3 Solids of Revolution

The volume of an object plays an important role in many problems in the physical sciences. In this section, we consider methods for computing volumes of solid of revolution.

If a region in a plane is revolved about a line in the plane, the resulting solid is a **solid of revolution**, and we say that the solid is **generated** by the region. The line about which we revolve is an axis of **revolution**. In particular, if the region shown in Figure 3.13(a) is revolved about the x-axis, we obtain the solid illustrated in

Figure 3.13(b). If a plane perpendicular to the x-axis intersects the solid shown in Figure 3.13(b), a circular section is obtained and volume in this case can be found by disk method which is explained below in theorem 3.10 and 3.11.

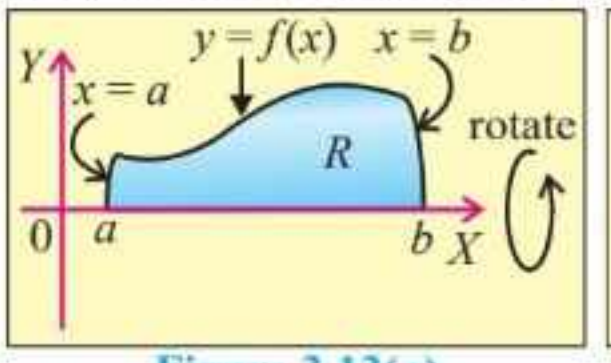


Figure 3.13(a)

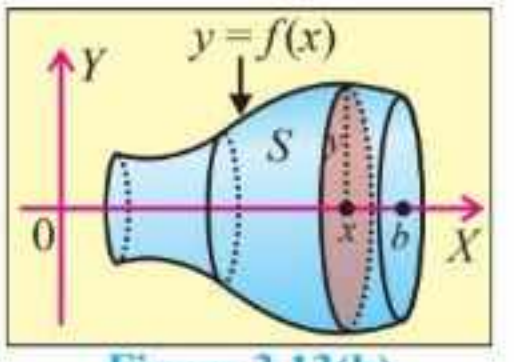


Figure 3.13(b)

Let f be continuous and non-negative on $[a, b]$, and let R be the region bounded by the graph of $y = f(x)$, the x-axis and the vertical lines $x = a$, and $x = b$. The **volume** V of the solid of revolution generated by revolving R about the x-axis is $V = \int_a^b \pi [f(x)]^2 \, dx$.

Similarly, let g be continuous and non-negative on $[c, d]$, and let R be the region bounded by the graph of $x = g(x)$, the y-axis and the horizontal lines $y = c$, and $y = d$. The **volume** V of the solid of revolution generated by revolving R about the y-axis is

$$V = \int_c^d \pi [g(y)]^2 \, dy$$

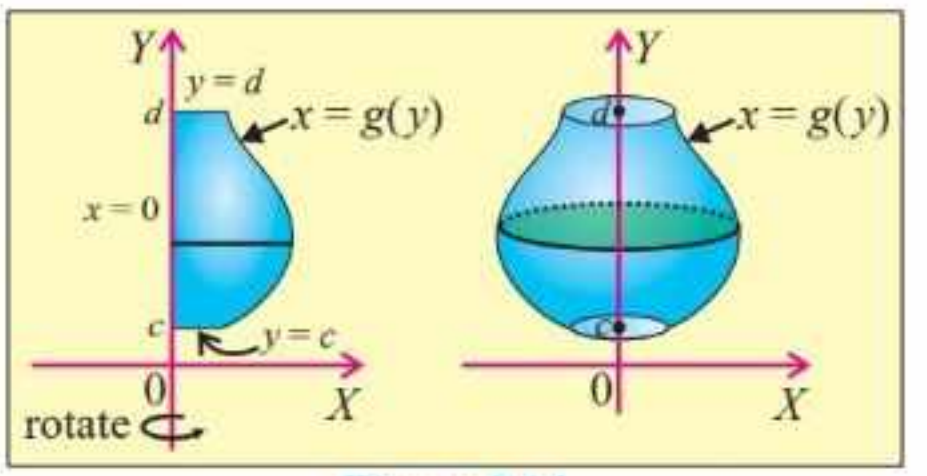


Figure 3.14

Volume of Solid of Revolution by Cross-sectional Method

Let S be a solid that lies between the planes P_a and P_b. If the cross-sectional area of S in the plane P_x is $A(x)$, where A is a continuous function, then **volume** of S is

$$V = \int_a^b A(x) \, dx$$

Example 62 A solid has a circular base of radius 2. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

Solution: Let us take the circle $x^2 + y^2 = 4$. The solid, its base, and a typical cross-section at a distance x from the origin are shown in Figure 3.15 (b).

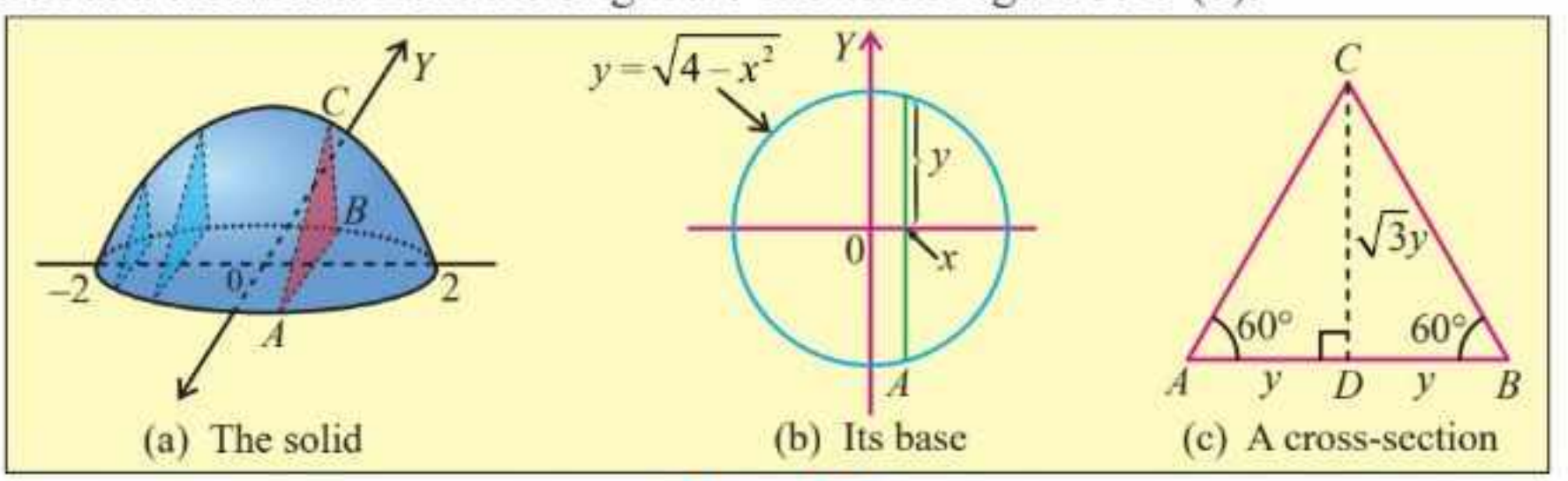


Figure 3.15

Since B lies on the circle, we have $y=\sqrt{4-x^2}$ and so the base of the triangle ABC is $|AB|=2y$. Since the triangle is equilateral, we see from the Figure 3.15 that its height is $\sqrt{3}y$. Therefore, the cross-sectional area is

$$A(x)=\frac{1}{2}(\text{Base})(\text{Height})=\frac{1}{2}\cdot 2y\cdot\sqrt{3}y$$
$$=\sqrt{3}y^2=\sqrt{3}\left(4-x^2\right)$$

and the volume of the solid is

$$V=\int_{-2}^{2}A(x)\,dx=\int_{-2}^{2}\sqrt{3}\left(4-x^2\right)dx=\sqrt{3}\left[4x-\frac{x^3}{3}\right]_{-2}^{2}$$
$$=\sqrt{3}\left[\left(4(2)-\frac{2^3}{3}\right)-\left(4(-2)-\frac{(-2)^3}{3}\right)\right]=\sqrt{3}\left[8-\frac{8}{3}+8-\frac{8}{3}\right]=\frac{32\sqrt{3}}{3}\text{ cubic units}$$

Example 63 Find the volume of the solid obtained by rotating the region bounded by $y=x^3$, $y=64$, and $x=0$ around the y-axis.

Solution: Since the axis of rotation is the y-axis, we write the curve $y=x^3$ in the form of $x=y^{1/3}$. The region and the solid with a typical disk are sketched in the Figure 16. Using the Formula

$V=\int_c^d \pi\left[g(y)\right]^2 dy$ with $c=0$, $d=64$ and

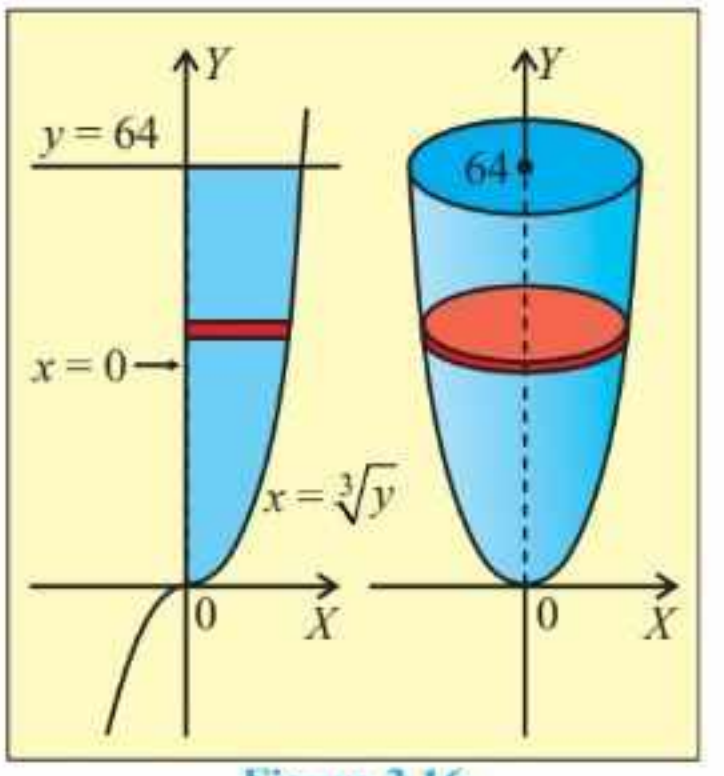


Figure 3.16

$g(y)=y^{1/3}$, we have

$$V=\int_0^{64}\pi\left(y^{1/3}\right)^2 dy=\pi\int_0^{64}y^{2/3}dy=\pi\left[\frac{3}{5}\,y^{5/3}\right]_0^{64}$$
$$=\frac{3\pi}{5}\left[(64)^{5/3}-0\right]=\frac{3\pi}{5}\left[(4^3)^{5/3}\right]=\frac{3\pi}{5}(4^5)=\frac{3\pi}{5}(1024)=\frac{3072\pi}{5}\text{ cubic units}$$

Example 64 A cone is formed by rotating the straight line $y=2x$ around the x-axis from $x=0$ to $x=5$ cm. Find the volume of the cone.

Solution: The volume V of a solid formed by rotating $y=f(x)$ around the x-axis from $x=a$ to $x=b$ is: $V=\int_a^b \pi\left[f(x)\right]^2 dx$

Here $f(x)=2x, a=0$ and $b=5$.

So, $V = \pi\int_0^5 (2x)^2\,dx = \pi\int_0^5 4x^2\,dx = 4\pi\int_0^5 x^2\,dx = 4\pi\left[\frac{x^3}{3}\right]_0^5$

$= 4\pi\left(\frac{5^3}{3}-0\right) = 4\pi\left(\frac{125}{3}\right) = \frac{500\pi}{3} \approx 523.6 \text{ cm}^3$

Example 65: A mug is designed by rotating the line $y = 3x+1$ around the x-axis from $x = 0$ to $x = 4$ cm. Find the volume of the mug.

Solution: The Volume V is given by the formula: $V = \int_a^b \pi[f(x)]^2\,dx$

Here $f(x) = 3x+1$, $a = 0$, and $b = 4$

$V = \pi\int_0^4 (3x+1)^2 dx = \pi\int_0^4 (9x^2+6x+1)\,dx = \pi\left[\frac{9x^3}{3}+\frac{6x^2}{2}+x\right]_0^4$

$= \pi\left[3x^3+3x^2+x\right]_0^4 = \pi\left[\left(3(4)^3+3(4)^2+4\right)-(0)\right] = \pi[192+48+4] = 244\pi \text{ cm}^3$

EXERCISE 3.6

1. Find the area of the region that is above the x-axis, but below the curve $y = (x-2)(3-x)$.
1. Find the total area and the signed area between the graph of $y = x^2-4$ and the x-axis over the interval (i) $[-3,-1]$ (ii) $[-3,3]$.
2. Sketch the region bounded by the given curves and find the area of the region.
   (i) $y = 2x,\ y = x^2$ (ii) $y = \sqrt{x},\ y = x^2$
   (iii) $y = x^4-x^2,\ y = 1-x^2$ (iv) $y = x^2$ and $y = |x|$.
3. Find the area of the region bounded by $y = |x-3|, x = 2, x = 4$ and the x-axis.
4. Find the area bounded by the parabola $y = x^2-6x+8, x = 2, x = 4$ and the x-axis.
5. Find the area bounded by the parabola $x = 9-y^2$, and the y-axis.
6. Find the ratio of the areas of the regions bounded by the parabola $y = x^2-4x+3$ and the coordinate axes.
7. For the function $f(x) = 2x$ over the interval $[0,3]$:
   (a) Find the signed area (b) Find the total (physical) area
   (c) Explain why they are equal in this case.
8. Find the area of the region bounded by the curve $y = (x-1)(x+1)(x-2)$ and the x-axis.

9. Find the area of the region bounded by the line $y = x+1$ and the parabola $y^2 = 2x+10$.
10. The demand function is $p = 30 - 2x - x^2$ and supply function is $p = -6 + 4x + x^2$. Find consumer's surplus and producer's surplus.
11. At equilibrium, the quantity sold is $x_e = 4$ and the price is $p_e = 16$. Consumer's surplus is 16 and producer's surplus is 8. Both the demand and supply curves are linear. Find the demand function $D(x)$ and the supply function $S(x)$.
12. The supply function is $p = \ln(x + 2)$ and the quantity sold is $x_e = 2$. Consumer's surplus is $6 - 2 \ln 2$. Find:
    (a) Equilibrium price (b) Linear demand function $p = a - bx$.
13. If the demand function for a particular commodity is $p = 100 - 2x$ and the market price is p_e = Rs.64, find the consumer's surplus using two methods:
    (a) Integrate with respect to x. (b) Integrate with respect to p.
15. Find the volume of the solid obtained by rotating the region bounded by the given curves about the indicated axis.
    (i) $y = x^2, x = 2, y = 0$; x-axis (ii) $x + y = 2, x = 0, y = 0$; x-axis
    (iii) $y = \sqrt{x-5}, x = 6, x = 8, y = 0$; x-axis (iv) $y = x^2 - 2x, y = 0$; x-axis

3.8 Applying Concepts of Integration to Real-Life World Problems

3.8.1 Moment of Inertia

Definition: The moment of inertia of a single particle of mass m about an axis AB is defined to be mr^2, where r is its distance from the particle to the axis AB. For a system of particles $m_1, m_2, \ldots, m_n$ placed at distances $r_1, r_2, \ldots, r_n$ respectively from a given axis, the moment of inertia of the system is defined as

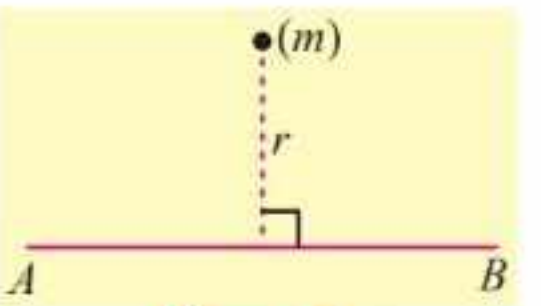


Figure 17

$$m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2$$

The moment of inertia of a rigid body having continuous matter is:

$$\lim_{\delta m \to 0} \sum r^2 \cdot \delta m = \int r^2 \, dm$$

the integral being taken all over the body.

1. If the moment of inertia of a body of mass M about any axis be MR^2, then R is called the radius of gyration of the body about that axis.
2. The above definitions and the following two theorems enable us to determine the moment of inertia of many other bodies.

Example 66 **A uniform Rod of Length $2a$:**

Find the moment of inertia of a uniform rod of length $2a$ and mass M about a line through one end and perpendicular to the rod.

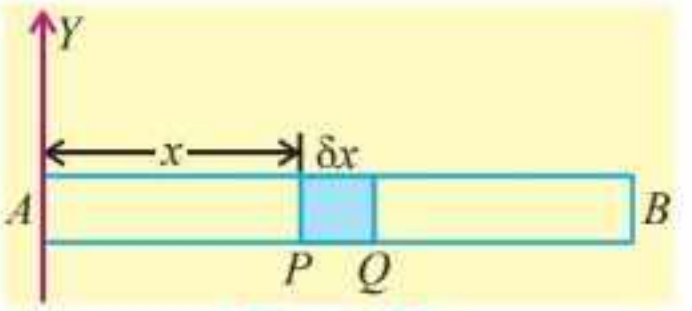


Figure 20

Solution: Let M be the mass of a uniform rod AB of length $2a$, then the mass per unit length of the rod is $\frac{M}{2a}$.

Let us consider an element PQ of length δx at a distance x from the line AY. Let AY be a line through A and perpendicular to AB.

$$\therefore\ \delta m = \text{Mass of an element } PQ = (\text{Mass per unit length}) \times (\delta x) = \frac{M}{2a} \times \delta x$$

The moment of inertia of this element PQ about the line AY is

$$x^2 \delta m = x^2 \cdot \frac{M}{2a} \cdot \delta x$$

Thus, the moment of inertia of the whole rod about AY is

$$\int_0^{2a} x^2 \frac{M}{2a} dx = \frac{M}{2a} \int_0^{2a} x^2 dx = \frac{M}{2a}\left[\frac{x^3}{3}\right]_0^{2a} = \frac{M}{2a}\left(\frac{(2a)^3}{3} - \frac{(0)^3}{3}\right) = \frac{M}{2a}\left(\frac{8a^3}{3}\right) = \frac{4}{3}Ma^2.$$

Example 67 **A Uniform Rectangular Lamina:**

Find the moment of inertia of a uniform rectangular lamina about a line through the centre and parallel to a side.

Solution: Let O be the centre of the rectangular lamina $ABCD$ with $AB = 2a, BC = 2b$ and M be the mass of this lamina.

Then the mass per unit area (density) is $\frac{\text{Mass}}{\text{Area}} = \frac{M}{(2a)(2b)} = \frac{M}{4ab}$.

Let OX be a line parallel to AB about which the moment of inertia is to be required. Take an elementary strip PQ of breadth δx perpendicular to OX at a distance x from O.

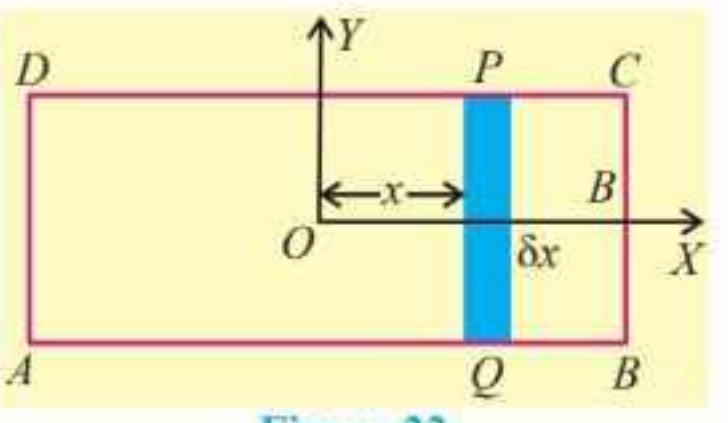


Figure 22

$\therefore\ \delta m =$ The mass of this elementary strip

$$= \text{density} \times \text{area} = \frac{M}{4ab}(2b \cdot \delta x) = \frac{M}{2a}\delta x.$$

The moment of inertia of this strip about OX is

$$\delta m \cdot \frac{b^2}{3} = \frac{M}{2a}\delta x \cdot \frac{b^2}{3} \text{ by previous example.}$$

Thus, the moment of inertia of the rectangular lamina about OX is

$$\int_{-a}^{a} \frac{M}{2a}\cdot\frac{b^2}{3}\,dx = \frac{Mb^2}{6a}\int_{-a}^{a} 1\,dx = \frac{Mb^2}{6a}\,[x]_{-a}^{a} = \frac{Mb^2}{6a}\big(a-(-a)\big) = \frac{Mb^2}{6a}(2a) = \frac{1}{3}Mb^2$$

Hence the moment of inertia of the rectangular lamina about a line through the centre and parallel to the side with length $2a$ is $\frac{1}{3}Mb^2$.

3.8.2 Distance, Velocity, Acceleration

Suppose a particle is moving along a straight line with position function $s(t)$ velocity function $v(t)$, and acceleration function $a(t)$, Since $s'(t) = v(t)$, the Fundamental theorem of Calculus gives:

$$\int_{t_1}^{t_2} v(t)\,dt = \int_{t_1}^{t_2} s'(t)\,dt = [s(t)]_{t_1}^{t_2} = s(t_2)-s(t_1) \qquad \ldots(1)$$

Similarly, since $v'(t) = a(t)$, the Fundamental Theorem of Calculus gives:

$$\int_{t1}^{t2} a(t)\,dt = v(t_2)-v(t_1) \qquad \ldots(2)$$

If we want to calculate the distance traveled during the time interval $[t_1, t_2]$, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore, total

distance traveled $= \int_{t_1}^{t_2} v(t)\,dt.$

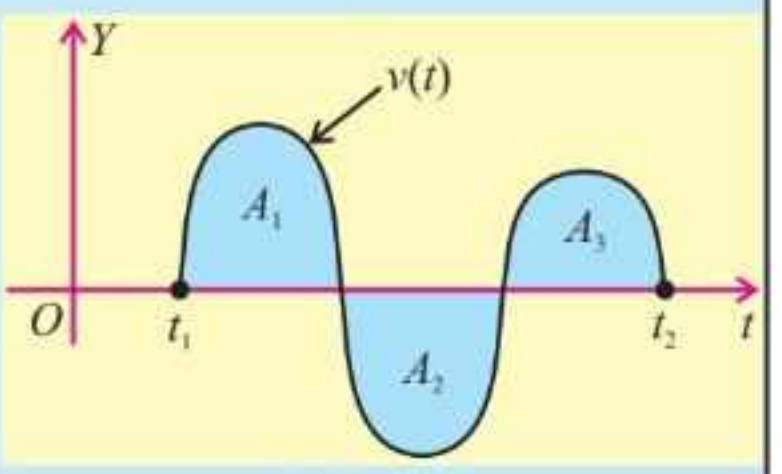


Figure 23

Given figure shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.

$$\text{Displacement} = \int_{t_1}^{t_2} v(t)dt = A_1 - A_2 + A_3$$

$$\text{Distance} = \int_{t_1}^{t_2} v(t)\,dt = A_1 + A_2 + A_3$$

Example 68 A particle moves along a straight line with acceleration $a(t) = 6t - 18(\text{m/s}^2)$ and initial velocity $v(0) = 15$ m/s.

(a) Find the velocity function $v(t)$.

(b) Find the displacement of the particle during $0 \leq t \leq 3$.

(c) Find the total distance traveled during $0 \leq t \leq 3$.

Solution: (a) Find $v(t)$

We have $a(t) = 6t - 18$

Velocity is the integral of acceleration:

$$v(t) = \int_0^t a(t)\,dt = \int_0^t (6t-18)\,dt = 6\frac{t^2}{2} - 18t + c = 3t^2 - 18t + c$$

where c is constant of integration.

As $v(0) = 15$ m/s

So, $v(0) = 3(0)^2 - 18(0) + c$

$\boxed{15 = c}$

Hence $v(t) = 3t^2 - 18t + 15$

(b) Displacement on $0 \le t \le 3$

$$S = \int_0^3 (3t^2 - 18t + 15)\,dt = [t^3 - 9t^2 + 15t]_0^3$$

$= 27 - 81 + 45 = -9$ This means that the particle moved 9 m to the left.

(c) Total distance traveled

Note that $v(t) = 3(t^2 - 6t + 5) = 3(t-1)(t-5)$

and so $v(t) \ge 0$ on the interval [0, 1] and $v(t) \le 0$ on [1, 3]. Thus the total distance traveled is

$$\int_0^3 |v(t)|\,dt = \int_0^1 |v(t)|\,dt + \int_1^3 |v(t)|\,dt = \int_0^1 v(t)\,dt + \int_1^3 (-v(t))\,dt$$

$$= \int_0^1 (3t^2 - 18t + 15)\,dt - \int_1^3 (3t^2 - 18t + 15)\,dt$$

$$= [t^3 - 9t^2 + 15t]_0^1 - [t^3 - 9t^2 + 15t]_1^3$$

$$= [(1 - 9 + 15) - (0)] - [(27 - 81 + 45) - (1 - 9 + 15)]$$

$$= 7 - (-16) = 7 + 16 = 23 \text{ m}$$

Example 69 **Growth rate of population:** A town's population growth rate (in people per year) is given by

$$P'(t) = 5000 + 200t$$

where t is the number of years after 2020. If the population in 2020 ($t = 0$) was 100,000:

(a) Find the population function $P(t)$ (b) What will the population be in 2030?

(c) Find the total population increase from 2022 to 2028.

Solution Given

A town's population growth rate is

$P'(t) = 5000 + 200t$ people/year, $t =$ years after 2020,

$P(0) = 100,000$

(a) Population function $P(t)$

$$P(t) = \int P'(t)\,dt = \int (5000 + 200t)\,dt$$

$P(t) = 5000t + 100t^2 + C$

Given $P(0) = 100000$

$100000 = 0 + 0 + C \Rightarrow C = 100000$

$P(t) = 100000 + 5000t + 100t^2$

(b) Population in 2030

$2030 \rightarrow t = 10$ (i.e. 10 years)

$P(10) = 100000 + 5000(10) + 100(100)$

$= 100000 + 50000 + 10000$

$= 160000$ people

(c) Population increase from 2022 to 2028

$2022 \rightarrow t_1 = 2$ and $2028 \rightarrow t_2 = 8$

Increase $= P(8) - P(2)$

$P(8) = 100000 + 5000(8) + 100(64) = 146400$

$P(2) = 100000 + 5000(2) + 100(4) = 110400$

Increase $= 146400 - 110400 = 36000$ people

3.8.3 Average Value of a Function

finition: If f is continuous on $[a, b]$, then its average value on $[a, b]$ (also called its mean value) is $f_{av} = \dfrac{1}{b-a}\int_a^b f(x)\,dx$.

Example 70 Find the average value of the function $f(x) = x^2 + 2$ over the interval $[-1, 2]$.

Solution: With $a = -1$ and $b = 2$, we have

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx = f_{av} = \frac{1}{2-(-1)}\int_{-1}^{2}\left(x^2+2\right)dx = \frac{1}{3}\left[\frac{x^3}{3} + 2x\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[\left(\frac{2^3}{3} + 2(2)\right) - \left(\frac{(-1)^3}{3} + 2(-1)\right)\right] = \frac{1}{3}\left(\frac{8}{3} + \frac{1}{3} + 4 + 2\right) = \frac{1}{3}\left(\frac{9}{3} + 6\right) = 3$$

Thus, the average value of $f(x) = x^2 + 2$ over $[-1, 2]$ is 3.

Example 71 A temperature sensor moves slowly along a 10-meter track, continuously measuring temperature. The temperature at position x is given by

$$T(x) = 20 + x, \text{ where } T(x) \text{ is in } °C$$

Find the average temperature measured by the sensor as it moves from $x = 0$ to $x = 10$ meters.

Solution: Function: $T(x)=20+x$, Limits are from $x=0$ to $x=10$.

$$\text{Average Temperature} = \frac{1}{10-0}\int_0^{10}(20+x)\,dx$$

$$= \frac{1}{10}\left[20x+\frac{x^2}{2}\right]_0^{10} = \frac{1}{10}\left[\left(20(10)+\frac{(10)^2}{2}\right)-\left(20(0)+\frac{(0)^2}{2}\right)\right] = 25\ ^\circ\text{C}$$

3.8.4 Drug Dosage Required by Integrating Concentration

Core Concept: Drug Concentration and Integration

Think of fighting an infection as a **battle** between the drug and the bacteria.

Drug Concentration $C(t)$: This is like the "number of soldiers" in your bloodstream at time t.
Right after a dose, you have many soldiers. Over time, the body removes them (metabolism/excretion), so the number decreases.
Minimum Inhibitory Concentration (MIC): This is the **minimum number of soldiers** needed to hold the battlefield and prevent the bacteria from growing. If the number of soldiers drops below this, the infection can start winning again.
Area Under the Curve (AUC): If you imagine a graph of "number of soldiers" (concentration) vs. time, the area under that curve represents the total fighting effort over time.
A short time with many soldiers might give the same total "fighting effort" as a long time with fewer soldiers.
This "total fighting effort" (AUC) is what matters for killing the bacteria effectively.
Why Integrate?
Integration is just a way to **add up** the total "soldier-hours" (concentration × time).
If $C(t)$ is concentration at time t, then:

$$\text{AUC} = \int_{t_1}^{t_2} C(t)\,dt$$

This AUC (Area Under the Curve) measures **total drug exposure**.
Doctors know from experiments:

- If AUC is too small $\rightarrow$ drug won't work well.
- If AUC is too large $\rightarrow$ risk of side effects.

So they calculate the AUC to decide the right dose and frequency.

Example 72 **Basic AUC Calculation**

A patient is given an intravenous dose of an antibiotic. The concentration in the bloodstream over time is modeled by:

$C(t)=8e^{-0.15t}\,(\text{mg/L})$ where t is in hours.

(a) Find the total drug exposure (AUC) from $t=0$ to $t=6$ hours.
(b) What is the AUC from $t=0$ to infinity?
(c) If the minimum required total exposure over the first 6 hours is 40 mg hr/L, is the drug exposure from this dose sufficient for a 6-hour treatment?

Solution: (a) AUC from $t=0$ to $t=6$:

$$\text{AUC}_{0\to6}=\int_0^6 8e^{-0.15t}\,dt=8\left[\frac{e^{-0.15t}}{-0.15}\right]_0^6=\frac{8}{-0.15}(e^{-0.15\times6}-e^0)$$

$$=\frac{8}{0.15}(1-e^{-0.9})=53.333\times0.5934\approx31.65\ \text{mg}\cdot\text{hr/L}$$

(b) AUC from $t=0$ to infinity:

$$AUC_{0\to\infty}=\int_0^\infty 8e^{-0.15t}\,dt8\left[\frac{e^{-0.15t}}{-0.15}\right]_0^\infty=\frac{8}{0.15}(1-0)=53.33\ \text{mg hr / L}$$

(c) Required total exposure over first 6 hours $=40\,\text{mg}\cdot\text{hr/L}$

Actual AUC from $t=0$ to $t=6=31.65\,\text{mg}\cdot\text{hr/L}$

$$31.65<40$$

Conclusion: No

The drug exposure from this dose over 6 hours is insufficient to meet the required minimum of $40\,\text{mg}\cdot\text{hr/L}$

EXERCISE 3.7

1. Find the moment of inertia I of a uniform rectangular plate $ABCD$ about the edge AD when $AD=3$, $AB=8$, $BC=3$, $CD=8$ and the mass per unit area is 4.
2. Find the moment of inertia of a uniform triangular lamina about its base.
3. A particle moves along a straight line with acceleration $a(t)=12t-36$ (m/s^2) and initial velocity $v(0)=30$ m/s.
   (a) Find the velocity function $v(t)$.
   (b) Find the displacement of the particle during the time period $1\le t\le6$.
   (c) Find the distance traveled during this time period.
4. A particle moving along a coordinate line at time $t=0$ is at a position $2cm$ from the origin and traveling at a velocity of $5cm/\text{sec}$. If the acceleration of the particle is given by: $a(t)=3-4(t+2)^{-3}$. Find the velocity and the position of the particle as functions of t.
5. A projectile is launched vertically upward from an initial height of s_o feet with an initial velocity of $v_o ft/s$. Neglecting air resistance, show that its height $s(t)$ above the ground at time t is given by $s(t)=-\frac{1}{2}gt^2+v_ot+s_o$ where, $g>0$ is the magnitude of the acceleration due to gravity.

6. In a bacterial culture, the growth rate is proportional to the number of bacteria present. Initially there are 200 bacteria. After 2 hours, there are 1800 bacteria.
   (a) Find the number of bacteria at time t hours.
   (b) How many bacteria will there be after 4 hours?
7. The rate of population growth of a small city (in people per year) is modeled by $P'(t) = 800 + 60t$, where t is the number of years after 2025. If the population in 2025 ($t = 0$) was 25,000 people:
   (a) Find the population function $P(t)$.
   (b) Estimate the population in 2035.
   (c) Calculate the total population growth from 2028 to 2033.
8. Find the integer b such that the average value of $f(x) = 2 + 4x - 3x^2$ on the interval [2, b] equals –7.
9. Let f be continuous, $\int_0^2 f(x)\,dx = 5$. If the average value of f on $[0,\ 4]$ is twice the average value on $[0,\ 2]$, find $\int_2^4 f(x)\,dx$.
10. Let f be continuous, and suppose: $\int_1^3 f(x)\,dx = 8$. If the average value of f on $[1,\ 4]$ is three times the average value on $[3,\ 4]$, find $\int_3^4 f(x)\,dx$.
11. Let f be continuous and suppose: $\int_0^2 f(x)\,dx = 10$ and $\int_2^4 f(x)\,dx = 6$. If the average value of f on $[0,\ 3]$ equals the average value on $[3,\ 4]$, find $\int_3^4 f(x)\,dx$.
12. A patient receives an IV bolus dose of a drug. The concentration in the blood is modeled by:
    $C(t) = 12e^{-0.3t}$ (mg/L) where t is the time in hours since that dose was taken.
    The minimum effective concentration (MEC) is 3 mg/L. The drug is considered effective if total AUC in a 24-hour period is at least 150 mg·h/L.
    (a) Find the time T (in hours) when the concentration reaches the MEC.
    (b) Calculate the AUC from $t = 0$ to $t = T$ for a single dose.
    (c) Determine if one dose provides enough total AUC over 24 hours. If not, how many doses (given at intervals of T hours) are needed in 24 hours to meet the requirement?

13. A patient takes a drug every 8 hours. The single-dose concentration (in mg/L) is $C(t)=12e^{-0.2t}$ where t is time in hours since that dose was taken.

(a) Understanding Contributions

If the patient takes doses at $t=0$, $t=8$ and $t=16$. Write the contribution from each dose separately at time $t=10$ hours. Then write the total concentration at $t=10$.

(b) Common Mistake

A student calculates concentration at $t=10$ as $12e^{-0.2\times10}\approx1.624$ mg/L.

What is wrong with this calculation?

(c) Just Before V/S Just After a Dose

Calculate

(i) Concentration just before the third dose at $t=16$ hours.

(ii) Concentration just after the third dose at $t=16$ hours.

(d) General Formula

Write the general formula for $C_{\text{total}}(t)$ for $16\le t\le24$, i.e., after 3 doses have been given.

Unit 04 Differential Equations

INTRODUCTION

The idea of differential equations was first proposed by Isaac Newton and Gottfried Leibniz during the development of calculus. The term "differential equation" was introduced by Leibniz in 1676. In the 18th century, mathematicians like Euler and Lagrange made significant contributions to the theory of these equations. Over the centuries, research conducted by hundreds of mathematicians, physicists, economists, chemists and other scientists around the world has broadened the range of applications and methods to solve differential equations.

For example, if a particle covers a distance s in time t, then its velocity v and acceleration a can be represented by the differential equations as follows:

$$v = \frac{ds}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt}$$

To begin our study with this important topic, we will first define differential equation and classify their various types.

4.1 Differential equations: Definition, Order and Degree

An equation that involves at least one derivative of a dependent variable with respect to an independent variable is called a differential equation and this particularly called ordinary differential equation. For example:

(i) $\frac{dy}{dx} + y\cos y = \sin x$ (ii) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$ (iii) $\sqrt{\frac{dy}{dx}} = \frac{d^2y}{dx^2}$

(iv) $\frac{d^2y}{dx^2} + 1 = \cos\left(\frac{d^3y}{dx^3}\right)$ (v) $\sqrt{xy}\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = \sin y$ (iv) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

The **order** of a differential equation is the order of the highest derivative that appears in the differential equation. For example, the order of the equations (i) and (v) is 1, the order of equations (ii) and (iii) is 2, whereas equation (iv) has order 3. A differential equation of order 1 is called **first-order differential equation,** such as (i) and (v).

The **degree** of a differential equation is the greatest exponent of the highest-order derivative appearing in the differential equation. For example, differential equation (v) is a polynomial in $\frac{dy}{dx}$, and its degree is 2.

4.1.1 Construction of First-Order and Differential Equations from Practical Situations

First-order differential equations model problems where the rate of change of a quantity depends on the quantity itself or on time. For example, motion of a particle, population growth or decay, Newton's law of cooling or heating, electric circuits, economic models and radioactive decay etc.

Example 1 Population Growth/Decay

Solution: The rate of change of population is proportional to the population itself.

Mathematical Model:

Let P represent the population at time t . Since the rate of change of population is proportional to the population at that time, we can write:

$$\frac{dP}{dt} \propto P$$

If k is the constant of proportionality, this becomes:

$$\boxed{\frac{dP}{dt} = kP}$$

This is a first order differential equation.

Observation: From this equation, we observe:

(i) If $k > 0$, the population increases (growth).

(ii) If $k < 0$, the population decreases (decay).

(iii) If $k = 0$, population remains constant.

By solving this differential equation, we can find the population at any specific time t.

Example 2 Temperature Problem (Cooling/Heating Law)

Solution: Let T represent the temperature of a body at time t, and let T_S be the constant temperature of the surrounding medium. According to Newton's law of cooling (or heating), the rate of change of body's temperature with respect to time is proportional to the difference between body's temperature and the surrounding temperature.

Mathematical Model:

$$\frac{dT}{dt} \propto (T_S - T)$$

Introducing a constant of proportionality $k > 0$, the relation becomes:

$$\boxed{\frac{dT}{dt} = k\ (T_S - T)}$$

This is the required first ordered differential.

Observations: (i) If $T < T_S$, the body is warming up and $\frac{dT}{dt}$ is positive.

(ii) If $T > T_S$, the body is cooling down and $\frac{dT}{dt}$ is negative.

By solving this differential equation, we can find the temperature T in the body at any time t.

Example 3 Inflation/Deflation Problem in Economics

Inflation means the price of goods is increasing over time, while **deflation** means the price of goods is decreasing over time. In this example, we construct a first-order differential equation to describe how the price of a good changes over time due to inflation or deflation.

Solution: Let P represent the price of a good at time t. We assume that the rate of change of price is proportional to the current price.

Mathematical Model:

$$\frac{dP}{dt} \propto P$$

Let k be the constant of proportionality representing the rate of inflation or deflation, then the desired differential equation can be written as:

$$\boxed{\frac{dP}{dt} = kP}$$

Example 4 Flow of Electricity in Series Circuits

In daily life, many problems involve the current flowing in series circuits. Here, we discuss two simple cases connected to first-order differential equations.

(i) Charging a Capacitor Through a Resistor (RC Circuit Model)

Solution: An RC circuit consists of a resistor of resistance R ohms and a capacitor of capacitance C farad connected in a series with a DC voltage source of constant emf V_S volts. The initial current is assumed to be zero (see Figure 4.1).

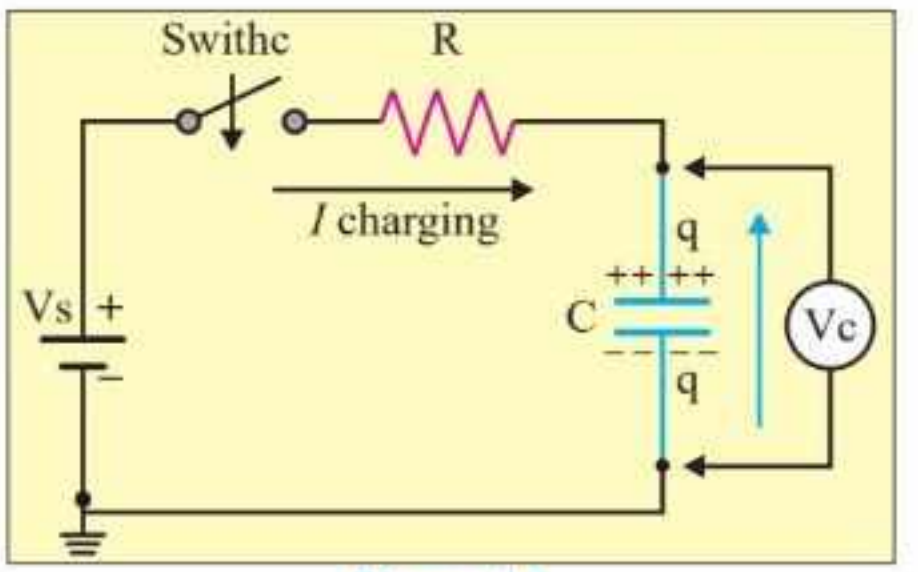


Figure 4.1

At time $t = 0$, the switch is closed, and the capacitor begins to charge. Let q be the charge on the capacitor at time t, and I the current flowing through the circuit.

Mathematical Model:

The voltage across the capacitor is:

$$V_C = \frac{q}{C}$$

According to Ohm's law, the voltage across the resistor is:

$$V_R = IR$$

By Kirchhoff's voltage law, the sum of voltages across the resistor and capacitor equals to the total voltage from the battery. The source voltage denoted by V_S that is given as:

$$V_S = V_r + V_C$$

$$V_S = \frac{q}{C} + V_R = \frac{q}{C} + IR$$

Substituting $I = \frac{dq}{dt}$, the equation becomes:

$$R\frac{dq}{dt} + \frac{q}{C} = V_S$$

or simply,

$$\boxed{RC\frac{dq}{dt} = CV_S - q}$$

(ii) Charging an Inductor Through a Resistor (RL Circuit Model)

Situation: An RL circuit consists of a resistor of resistance R ohms and an inductor of induction L henries connected in a series with a DC voltage source of constant emf V_S volts. The initial current is assumed to be zero (see Figure 4.2).

At time $t = 0$, the switch is closed and the inductor begins to charge. Let I be the current flowing through the inductor at time t.

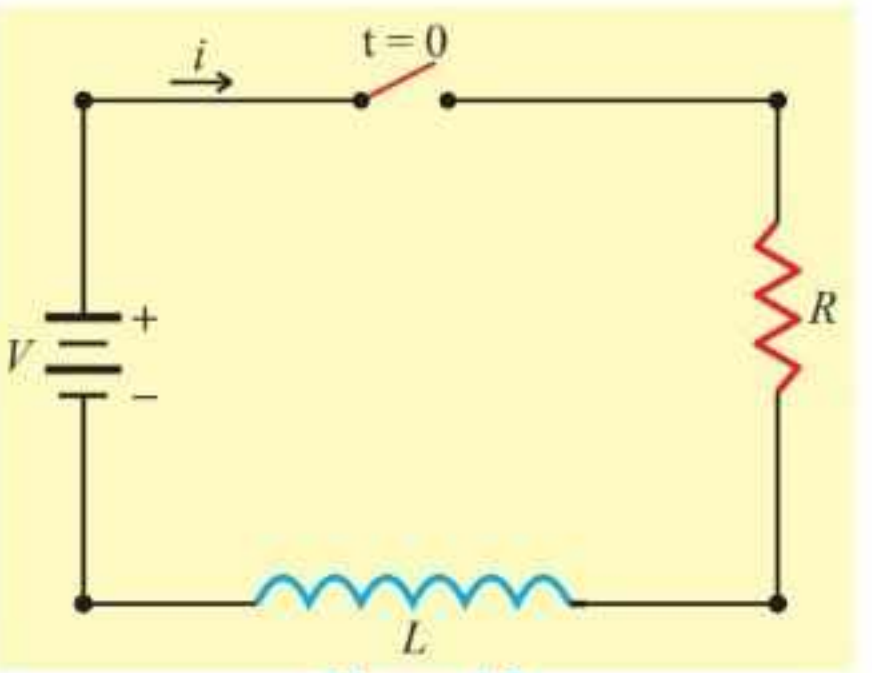


Figure 4.2

Mathematical Model

The voltage across the inductor is:

$$V_L = L\frac{dI}{dt}$$

According to Ohm's law, the voltage across the resistor is:

$$V_R = RI$$

By Kirchhoff's voltage law, the sum of voltages across the resistor and inductor equals to the applied voltage:

$$V_S = V_L + V_R$$

$$L\frac{dI}{dt} + RI = V_S$$

In this way, differential equation is:

$$\boxed{L\frac{dI}{dt} = V_S - RI}$$

EXERCISE 4.1

1. Write the order and degree (if defined) of the following differential equations:

(i) $\frac{d^3y}{dx^2}+7\frac{d^2y}{dx^2}-9xy\frac{dy}{dx}+x^2y=0$ (ii) $\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}=1$

(iii) $\frac{d^2y}{dx^2}+\sin\left(\frac{dy}{dx}\right)=0$ (iv) $\frac{dy}{dx}+e^y\sin(x)=0$

(v) $\cos x\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{3}}+\frac{dy}{dx}+y=0$

2. Let A denote the amount of money deposited in a bank at time t, which grows over time with constant compound interest rate $r>0$. Construct a differential equation which represents the growth of amount over time.
(Hint: The rate of change of amount with respect to time is proportional to amount itself).
3. Let T denote the temperature of a body at time t. Suppose that the effect of the surrounding medium is negligible. Construct a differential equation that represents the temperature of the body at any time t, assuming the body is losing heat due to some internal processes.
4. An object of mass m is thrown vertically upward. Assuming that the air resistance is proportional to its velocity at any time t, construct a differential equation representing the velocity v. Also, deduce the equation in the case when air resistance is negligible.
5. Let N denote the amount of radioactive substance present at time t. Construct a differential equation that models the radioactive decay of the substance.
Hint: (**Radioactive decay** is a natural process by which an unstable atomic nucleus loses energy by emitting radiation. The rate at which a radioactive substance decays is proportional to the amount of the substance present at any given time).
6. For an RC circuit, recall the differential equation representing the charging of a capacitor. Deduce the differential equation that represents the discharging of the capacitor.
7. For an RL circuit, recall the differential equation representing the charging of an inductor. Deduce the differential equation that represents the discharging of the inductor.

4.2 Solution of First-Order and First-Degree Differential Equations

General form of the first order and first degree differential equations.

$$\frac{dy}{dx} = f(x, y)$$

where f is a function of two variables x and y, defined on some region of the xy-plane. It is always possible to write the above equation in differential form as follows:

$$M(x, y)dx + N(x, y)dy = 0$$

where $M(x, y)$ and $N(x, y)$ are the functions of x and y.

Examples: (i) $\frac{dy}{dx} = \frac{x+y}{y-x}$ or $(x+y)dx + (x-y)dy = 0$

(ii) $\frac{dy}{dx} = -\frac{x}{y}$ or $x\,dx + y\,dy = 0$

A solution of a differential equation is a function defined on some interval I that reduces the equation to an identity. A condition of the form

$$y = y_0 \ \ at \ \ x = x_0 \text{ written as } y(x_0) = y_0$$

is called an **initial value condition** or **one point boundary condition**. A differential equation together with an initial value condition is called an **Initial Value Problem (IVP)**.

4.2.1 Separable Differential Equation

A first-order, first-degree differential equation is called **Separable** if it can be written as:

$$\frac{dy}{dx} = g(x)\,h(y) \quad \text{or} \quad g(x)dx = \frac{1}{h(y)}dy$$

For example, the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ is separable can be written as $y\,dy = -x\,dx$. However, the differential equation $\frac{dy}{dx} = \frac{x+y}{y-x}$ is not separable.

Method to Solve a Separable Differential Equation

Note that the separable differential equation:

$$\frac{dy}{dx} = g(x)\,h(y)$$

can be written as: $\frac{1}{h(y)}dy = g(x)\,dx$

Now, integrate both sides:

$$\int \frac{1}{h(y)}dy = \int g(x)\,dx$$

By solving the integrals on both sides, we obtain the general solution of separable differential equation.

Example 2 Find the general solution of the differential equation $x\,dx + y\,dy = 0$.

Solution: The given differential equation is separable. It can be written as:

$$y\,dy = -x\,dx$$

Integrating both sides:

$$\int y\,dy = -\int x\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\Rightarrow \quad y^2 = -x^2 + 2c$$

Since the constant c is arbitrary, it can be replaced by $\frac{c}{2}$. Hence, the general solution of the given differential equation can be written as:

$$x^2 + y^2 = c$$

Example 3 Find the general solution for the differential equation $x^2(2y+1)\frac{dy}{dx} - 1 = 0$

Solution: The given differential equation can be written as:

$$\frac{dy}{dx} = \frac{1}{x^2(2y+1)}$$

By separating variables, we have

$$(2y+1)\,dy = \frac{1}{x^2}dx$$

Integrating both sides we get:

$$\int (2y+1)dy = \int \frac{1}{x^2}dx$$

$$\Rightarrow \quad 2\int y\,dy + \int dy = \int x^{-2}dx \quad \Rightarrow \quad y^2 + y = -\frac{1}{x} + C$$

Hence, $y^2 + y + \frac{1}{x} = C$ represents the general solution for the given differential equation where C is an arbitrary constant.

4.2.2 Homogeneous Differential Equation

A first-order, first degree differential equation is called **Homogeneous** if it can be written in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

where $F\left(\frac{y}{x}\right)$ is a function of the ratio $\frac{y}{x}$ only. For example, $\frac{dy}{dx} = \frac{x^3 + xy^2}{y^3 - x^2 y}$ is a

homogeneous differential equation because, it can be written as:

$$\frac{dy}{dx}=\frac{\frac{x^3+xy^2}{x^3}}{\frac{y^3-x^2y}{x^3}}=\frac{1+\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3-\left(\frac{y}{x}\right)}=F\left(\frac{y}{x}\right)$$

On the other hand, the differential equation $\frac{dy}{dx}=x+y$ is not homogeneous because $x+y$ cannot be expressed as a function of $\frac{y}{x}$.

Method to Solve a Homogeneous Differential Equation

The homogeneous differential equation:

$$\frac{dy}{dx}=F\left(\frac{y}{x}\right) \qquad \ldots(1)$$

can be converted into a separable differential equation in variables x and v by making the substitution:

$$y=vx \qquad \ldots(2)$$

Differentiating both sides of equation (2) w.r.t x:

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \qquad \ldots(3)$$

Substituting the values of $\frac{y}{x}$ and $\frac{dy}{dx}$ from equations (2) and (3), the equation (1) can be written as:

$$v+x\frac{dv}{dx}=F(v)$$

Rewriting, we obtain:

$$x\frac{dv}{dx}=F(v)-v$$

It becomes separable differential equation.

$$\frac{1}{F(v)-v}dv=\frac{1}{x}dx$$

On integrating both sides:

$$\int\frac{1}{F(v)-v}dv=\int\frac{1}{x}dx$$

By solving integrals of the above equation and then substituting back $v=\frac{y}{x}$, we obtain the general solution for the homogeneous differential equation (1).

Note

If the equation $\frac{dy}{dx} = f(x, y)$ is homogeneous, then on substituting $y = vx$, $\frac{dy}{dx}$ is replaced by $v + x\frac{dv}{dx}$ and $f(x, y)$ can easily be simplified by replacing x with 1 and y with v. For example, in $\frac{dy}{dx} = \frac{x+y}{y-x}$, replacing x with 1 and y with v, in the right-hand side, we get $v + x\frac{dv}{dx} = \frac{1+v}{v-1}$.

Example 4 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{y-x}$.

Solution: The given differential equation is:

$$\frac{dy}{dx} = \frac{x+y}{y-x} \qquad \ldots(1)$$

Note that every term of the numerator and denominator in $f(x, y) = \frac{x+y}{y-x}$ is of degree 1. Thus, the differential equation (1) is homogeneous. On substituting $y = vx$, we have:

$$v + x\frac{dv}{dx} = \frac{x+vx}{vx-x} = \frac{1+v}{v-1} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{1+v}{v-1} - v$$

$$x\frac{dv}{dx} = \frac{-v^2+2v+1}{v-1} \quad \Rightarrow \quad \frac{v-1}{-v^2+2v+1}dv = \frac{1}{x}dx$$

Integrating both sides we get:

$$\int \frac{v-1}{-v^2+2v+1}dv = \int \frac{1}{x}dx$$

$$\Rightarrow \quad -\frac{1}{2}\int \frac{-2v+2}{-v^2+2v+1}dv = \ln|x| + \ln|c| \quad \Rightarrow \quad -\frac{1}{2}\ln\left|-v^2+2v+1\right| = \ln|cx|$$

$$\Rightarrow \quad \ln\left|-v^2+2v+1\right| = -2\ln|cx| \quad \Rightarrow \quad \ln\left|-v^2+2v+1\right| = \ln|cx|^{-2}$$

$$\Rightarrow \quad \left|-v^2+2v+1\right| = \frac{1}{c^2x^2}$$

Substituting $v = \frac{y}{x}$ back in the above equation:

$$\left|-\frac{y^2}{x^2}+2\frac{y}{x}+1\right| = \frac{1}{c^2x^2} \quad \Rightarrow \quad \frac{\left|-y^2+2xy+x^2\right|}{x^2} = \frac{1}{c^2x^2}$$

$$\Rightarrow \quad \left|-y^2+2xy+x^2\right| = \frac{1}{c^2} \quad \Rightarrow \quad -y^2+2xy+x^2 = \pm\frac{1}{c^2}$$

Renaming the arbitrary constant as $c \neq 0$, we get the required general solution of the given differential equation as follows:

$$-y^2 + 2xy + x^2 = c \text{ or } x^2 + 2xy - y^2 = c$$

Example 5 Solve the differential equation $x\sin\left(\frac{y}{x}\right)dy - \left[y\sin\left(\frac{y}{x}\right) - x\right]dx = 0$.

Solution: Rewrite the given differential equation as follows:

$$\frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - x}{x\sin\left(\frac{y}{x}\right)} = \frac{\frac{y}{x}\sin\left(\frac{y}{x}\right) - 1}{\sin\left(\frac{y}{x}\right)}$$

Note that the right-hand side of the above equation is a function of $\frac{y}{x}$ form only. Thus, it is a homogeneous differential equation. On substituting $y = vx$, we have:

$$v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v = \frac{v\sin v - 1 - v\sin v}{\sin v} = -\frac{1}{\sin v}$$

$$\Rightarrow \quad -\sin v\, dv = \frac{1}{x}dx$$

Integrating both sides:

$$\int(-\sin v)\,dv = \int\frac{1}{x}dx \quad \Rightarrow \quad \cos v = \ln|x| + \ln|c|$$

Replacing v by $\frac{y}{x}$ in the above equation, we get the general solution for the given differential equation as follows:

$$\cos\left(\frac{y}{x}\right) = \ln|cx|$$

EXERCISE 4.2

1. Which of the following differential equations are separable and which of them are homogeneous?

(i) $x\frac{dy}{dx} = 1 + y$ (ii) $x^2(2y+1)dy - dx = 0$

(iii) $(2xy + x^2)dy - y^2dx = 0$ (iv) $\frac{dy}{dx} = e^{x-y}$

(v) $\frac{dT}{dt} + k\,(T - T_m) = 0$ (vi) $\frac{dy}{dx} = \ln(x) - \ln(y)\,;\ x>0,\ y>0$

Find the general solutions of the differential equations given in problems (2 to 21):

2. $\frac{dy}{dx}=\left(\frac{1-y^2}{1-x^2}\right)^{\frac{1}{2}}$

3. $\frac{dy}{dx}=e^{x-y}$

4. $\frac{dy}{dx}=\sqrt{y}\cos^2\sqrt{y}$

5. $\sqrt{x}\frac{dy}{dx}=e^{y+\sqrt{x}}$

6. $\frac{dy}{dx}=2x\sqrt{1-y^2}$

7. $\frac{dy}{dx}=\frac{1+x}{(1-x)(1+x^2)}$

8. $x^2\,dx+(1-x^2)dy=0$

9. $y-x\frac{dy}{dx}=2\left(y^2+\frac{dy}{dx}\right)$

10. $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$

11. $(e^{2x}+e^{-2x})dy+(e^{-2x}-e^{2x})dx=0$

12. $x\sin x\,dx+\ln y\,dy=0$

13. $\frac{dy}{dx}=\cos^2\left(\frac{y}{x}\right)+\frac{y}{x}$

14. $(y^2+2xy)dx+x^2dy=0$

15. $(x^2+3y^2)dx-2xydy=0$

16. $(x^2+xy+y^2)dx-x^2dy=0$

17. $x\frac{dy}{dx}=xe^{-\frac{y}{x}}+y$

18. $x\frac{dy}{dx}-y=\frac{x^4+x^3y}{y^3-x^3}$

19. $y-x\frac{dy}{dx}=3\left(1+x\frac{dy}{dx}\right)$

20. $y\,dx+x\,dy=0$

21. $\frac{dy}{dx}+\frac{2xy}{2y+1}=x$

Solve the initial value problems (22 to 25):

22. $\frac{dy}{dx}-x=xy^2;\ y(0)=1$

23. $\frac{ds}{dt}+2st=0;\ s=4e$ when $t=0$

24. $\left(y+\sqrt{x^2+y^2}\right)dy-x\,dy=0;\ y=0$ when $x=0$

25. $\frac{dy}{dx}=\frac{x+y}{x};\ y(1)=1$

4.3 Applications of First Order Differential Equations in Real-Life Problems

First-order differential equations play a vital role in modeling various real-life phenomena in fields such as physics, biology, chemistry, economics, and engineering. These equations involve the rate of change of a quantity and help describe how that quantity evolves over time.

Example 6 (Population Growth)

In a culture, there are 100 bacteria initially and are doubled in 2 hours. Find the number of bacteria 7 hours later.

Solution: Let N represent the number of bacteria at time t, then we have to find $N(7)$ using the information:

$$N(0) = 100$$
$$N(2) = 200$$

We have to solve the differential equation:

$$\frac{dN}{dt} = kN$$

Where $k > 0$, because the population is increasing. Note that the above differential equation is separable. It can be written as follows:

$$\frac{1}{N}dN = k\,dt$$

On integrating both sides:

$$\int \frac{1}{N}dN = \int k\,dt$$

$$\Rightarrow \qquad \ln|N| = kt + C \qquad \ldots(1)$$

Using the condition $N(0) = 100$ in equation (1), we get $C = \ln(100)$.
Now, equation (1) becomes:

$$\ln|N| = kt + \ln(100) \quad \text{or} \quad \ln\left|\frac{N}{100}\right| = kt$$

$$\Rightarrow \qquad \frac{N}{100} = e^{kt}$$

$$\Rightarrow \qquad N = 100e^{kt} \qquad \ldots(2)$$

Applying the condition $N(2) = 200$, we get:

$200 = 100e^{2k}$ or $e^{2k} = 2$. which gives $2k = \ln 2$ or $k = \frac{1}{2}\ln 2 = \ln\sqrt{2}$

Thus, equation (2) is rewritten as:

$$N = 100e^{(\ln\sqrt{2})t}$$

Substituting $t = 7$ in the above equation, we get:

$$N(7) = 100e^{7\ln\sqrt{2}} = 100e^{\ln(2)^{\frac{7}{2}}} = 100(2)^{\frac{7}{2}} \approx 1131.371$$

Hence, the number of bacteria 7 hours later is 1131 (approximately).

Example 7 (Radioactive Decay)

A certain radioactive substance has a mass of 80 grams initially. After 10 hours, only 50 grams remain.

(a) Find decay constant.

(b) How much of the substance will remain after 24 hours?

(c) After how many hours will only 20 grams remain?

Solution: Let M represents the mass of radioactive decay at time t. Then we have, $M(0) = 80$ and $M(10) = 50$. The differential equation for the radioactive decay is given by:

$$\frac{dM}{dt} = -kM$$

Which is clearly a separable differential equation. On separating the variables and then integrating both sides, we get:

$$M(t) = ce^{-kt} \qquad \ldots(1)$$

Where c is the constant of integration and $k > 0$ is the constant of proportionality (decay constant).

(a) Applying initial condition $M(0) = 80$ on equation (1), we get $c = 80$.

Now, imposing the condition $M(10) = 50$ on equation (1), we get $50 = 80e^{-10k}$ which gives:

$$k = -\frac{1}{10}\ln\left(\frac{5}{8}\right) \approx 0.047 .$$

Hence the decay constant is 0.047 approximately.

(b) Substituting the values of c and k in equation (1), we get:

$$M(t) = 80e^{-0.047t} \qquad \ldots(2)$$

$$\Rightarrow \quad M(24) = 80e^{-0.047(24)} \approx 25.8944 \approx 26 \text{ grams}$$

Hence, there will be approximately 26 grams substance after 24 hours.

(c) Substituting $M(t) = 20$ in equation (2), we get $20 = 80e^{-0.047t}$. Solving this equation for t, we have $t \approx 29.496$. Hence, 20 grams substance will be left after approximately 29.5 hours.

Example 8 (Deflation Problem in Economics)

Suppose an economy is experiencing deflation at 4% per year. If the current price of an item is Rs. 500, find its expected price after 3 years.

Solution: Let P denote the price of item at time t. Since there is a deflation, the differential equation concerning the problem is given by:

$$\frac{dp}{dt} = -(4\%)P = -0.04P$$

On solving the above differential equation, we get:

$$P(t) = Ce^{-0.04t} \qquad \ldots(1)$$

Since the current price of item is Rs. 500, we have initial condition $P(0) = 500$. Using this condition in equation (1), we get:

$$P(t) = 500e^{-0.04t}$$

Substituting $t = 3$ in the above equation:

$$P(3) = 500e^{-0.04(3)} \approx 443.46$$

Hence the expected price of the item after three years will be Rs. 443.5.

Example 9 (Cooling/Heating Problem)

A cup of coffee is served at 90° C in a room where the temperature is 25° C. After 10 minutes, the coffee cools to 60° C. Find the temperature of the coffee after 20 minutes.

Solution: Let T denote the temperature of the coffee at time t. Using the conditions $T(0) = 90$ and $T(10) = 60$, we have to calculate $T(20)$ by solving the differential equation:

$$\frac{dT}{dt} = -k(T - 25)$$

Note that the above differential equation is separable and can be rewritten as:

$$\frac{1}{(T-25)}dT = -k\,dt$$

On integrating, we have:

$$\int \frac{1}{(T-25)}dT = -k\int dt \quad \Rightarrow \quad \ln|T-25| = -kt + \ln|c|$$

$$\Rightarrow \quad \ln\left|\frac{T-25}{c}\right| = -kt \quad \Rightarrow \quad \frac{T-25}{c} = e^{-kt}$$

$$\Rightarrow \quad T(t) = 25 + ce^{-kt} \qquad \ldots(1)$$

Applying initial condition $T(0) = 90$ on above equation, we get $90 = 25 + c$ or $c = 65$. For the second condition $T(10) = 60$, we get $60 = 25 + 65e^{-10k}$ from equation (1). On solving this equation for k, we have: $e^{10k} = \frac{13}{7}$ or $k \approx 0.0619$. Now, the equation (1) can be written as:

$$T(t) = 25 + 65e^{-0.0619t}$$

Substituting, $t = 20$ in the above equation:

$$T(20) = 25 + 65e^{-0.0619(20)} \quad \Rightarrow \quad T(20) \approx 43.85$$

Hence the approximate temperature of the coffee after 20 minutes will be 43.85°C.

Example 10 (RL Circuit Problem)

An RL circuit has an emf of 5 volts, a resistance 50 ohms, an inductor of 1 henry and no initial current. Find the current in the circuit at any time t.

Solution: We have to find the current I flowing in the circuit at time t using the information:

$$V_S = 5 \text{ volts}, \; R = 50\ \Omega, \text{ and } I(0) = 0.$$

The differential equation governing the RL circuit at time t is given by: (see Example 4, Section 4.1.2)

$$L\frac{dI}{dt} = V_S - RI \qquad \because \quad L = 1 \text{ henry}$$

Substituting the given values:

$$\frac{dI}{dt} = 5 - 50I$$

The above differential equation is separable. On separating the variables and integrating, we have:

$$\frac{1}{5}\int \frac{1}{1-10I}\,dI = \int dt \quad \Rightarrow \quad -\frac{1}{50}\int \frac{-10}{1-10I}\,dI = \int dt$$

$$\Rightarrow \quad -\frac{1}{50}\ln|1-10I| = t + C \qquad \ldots(1)$$

Applying the initial condition $I(0) = 0$:

$$-\frac{1}{50}\ln|1-10(0)| = 0 + C$$

$$\Rightarrow \quad -\frac{1}{50}\ln|1| = C \quad \Rightarrow \quad C = 0$$

For $C = 0$, the equation (1) becomes:

$$-\frac{1}{50}\ln|1-10I| = t \quad \Rightarrow \quad \ln|1-10I| = -50t \quad \Rightarrow \quad |1-10I| = e^{-50t}$$

$$\Rightarrow \quad \pm(1-10I) = e^{-50t}$$

On checking the initial condition $I(0) = 0$ for the above equation we have, $\pm(1-10(0)) = e^{-50(0)}$ or $\pm 1 = 1$. This means that the negative sign is to be ignored.

Thus, we have:

$$1-10I = e^{-50t}$$

$$\Rightarrow \quad I(t) = \frac{1}{10}\left(1-e^{-50t}\right)$$

EXERCISE 4.3

1. A particle is moving in a straight line and its acceleration is given by $a = 2t - 7$. Find
   (i) velocity v in terms of t if $v = 10$ m/sec, when $t = 0$.
   (ii) distance s in terms of t if $s = 0$ m/sec, when $t = 0$.
2. In a culture, there were 1000 bacteria initially and were found to be 900 after five hours. Use differential equation to find the number of bacteria 7 hours later.
3. The number of bacteria in a culture currently is 1000. Three hours ago, it was 750. What was the number of bacteria five hours ago?
4. A radioactive isotope has a half-life of 12 hours. The initial mass is 160 grams.
   (a) What is the decay constant?
   (b) How much mass will remain after 36 hours?
   (c) After how many hours will only 10 grams remain?

Half-Life

Half-life is the time required for a quantity of a radioactive substance to reduce to half of its initial amount due to radioactive decay.

5. Suppose the inflation rate for the cost of a basket of goods is 8% per year. If the current price level is Rs. 1000, use differential equation to find the price level after 5 years.
6. Use a differential equation to find the current price of a good if the deflation rate is 11% per year and its price after ten years is expected to be Rs. 5000.
7. Aslam deposited Rs. 5000 on opening an account in a bank 15 years ago. Using differential equation, find the constant annual rate of growth if he closes his account now by receiving Rs. 15,000.
8. A can of soda at 4°C is placed in a room at 24 °C. After 30 minutes, its temperature is 12 °C. Find the temperature after 1 hour.
9. A fully insulated battery overheats during operation and reaches 80 °C. After shutdown, it begins to cool without any external influence. If it cools to 50 °C in 2 hours, how long will it take to cool to 15 °C?
10. An RL circuit has a battery of 12 volts, a resistor of 60 Ohms, and an inductor of 2 Henries. If the circuit is closed at time $t = 0$ with initial current zero, find the current $I(t)$ flowing in the circuit at any time t.
11. An RC circuit has an emf of 12 volts, resistance $R = 6\Omega$ and capacitance $C = 3$ farad. Initially, the capacitor is uncharged. Find the current $I(t)$ at any time t.

Unit 05 Analytical Geometry

INTRODUCTION



Figure 5.1

Analytical geometry, also called coordinate geometry, is the branch of mathematics that combines algebra and geometry to study shapes, lines and figures using a coordinate system. Instead of just drawing or imagining figures, we describe them with equations and points on the *xy*-plane, which makes it easier to calculate distances, midpoints, slopes and intersections. This approach allows us to solve practical problems, like finding the shortest path, checking if two lines are parallel or locating the exact position of an object. Analytical geometry was first developed in the 17^{th} century by the French mathematician René Descartes, who introduced the idea of using coordinates to represent geometric figures. His work laid the foundation for modern mathematics, as analytical geometry became the link between algebra and geometry, making it possible to study space and shapes with greater precision.

In this unit, we will study about straight lines, angles between coplanar intersecting lines, concurrency of medians, right bisectors and altitudes, area of triangular region and applications of analytical geometry to real life problems.

5.1 Equation of Straight Lines

We have learnt that in coordinate geometry, a straight line can be represented in different forms depending on the information given.

General Form

The general form of a straight line in two variables is written as $ax + by + c = 0$, where a, b and c are real numbers and a and b are not simultaneously zero.

Slope-Intercept Form

If m is the slope and c is the y-intercept of a line, then equation of this straight line in slope-intercept form is $y = mx + c$.

Point-Slope Form

If a line passes through a point (x_1, y_1) and has slope m, then equation of straight line in point slope form is $y - y_1 = m(x - x_1)$.

Two-Point Form

If a line passes through two points (x_1, y_1) and (x_2, y_2), then equation of straight line in two-point form is $\frac{y-y_2}{y_2-y_1}=\frac{x-x_2}{x_2-x_1}$ or $\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}$ or $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

Intercept Form

If a and b are the x-intercept and y-intercept of the line, then equation of straight line in intercept form is $\frac{x}{a}+\frac{y}{b}=1$.

Normal Form

$x\cos\theta + y\sin\theta = p$ is an equation of the straight line, where p is the perpendicular distance of the line from the origin and θ is the angle made by perpendicular with the positive x-axis.

Example 1 Find an equation of the straight line if its slope is 3 and y-intercept is 7.

Solution: The slope and y-intercept of the line are respectively:

$$m = 3,\ c = 7$$

Thus, slope-intercept form of a line is:

$$y = mx + c$$
$$y = 3x + 7$$

Example 2 Write down an equation of the straight line passing through (–4, 3) and having slope –5.

Solution: As the point (–4, 3) lies on the required line having slope –5, so by point-slope form of equation of the straight line, we have

$y-3=-5(x-(-4))$ or $y-3=-5(x+4)$ or $y-3=-5x-20$ or $5x+y+17=0$ which is an equation of the required line.

Example 3 Find an equation of the straight line passing through the points (–3, –2) and (5, 4).

Solution: Using two-point form of an equation of the straight line, the required equation is

$$\frac{y-(-2)}{4-(-2)}=\frac{x-(-3)}{5-(-3)}$$
$$\frac{y+2}{6}=\frac{x+3}{8}$$

or $y+2=\frac{6}{8}(x+3)$ or $8(y+2)=6(x+3)$

or $8y+16=6x+18$ or $6x-8y+2=0$ or $3x-4y+1=0$

Example 4 Write down an equation of the straight line which cuts the x-axis at $(3, 0)$ and y-axis at $(0, -6)$.

Solution: As 3 and -6 are x and y-intercepts respectively of the required line, so by two-intercepts form of an equation of the straight line, we have

$$\frac{x}{3}+\frac{y}{-6}=1$$

$$2x-y=6$$

$$2x-y-6=0$$

which is the required equation.

Example 5 The length of perpendicular from the origin to a line is 3 units and the inclination of this perpendicular is 120°. Find the equation of the straight line.

Solution: Here, $p = 3$, $\theta = 120^\circ$.

Equation of the straight line in normal form is:

$$x\cos\theta + y\sin\theta = p$$

$$x\cos 120^\circ + y\sin 120^\circ = 3$$

$$-\frac{1}{2}x+\frac{\sqrt{3}}{2}y=3$$

$$x-\sqrt{3}y=-6$$

$$x-\sqrt{3}y+6=0$$

5.2 A Linear Equation in Two Variables Represents a Straight Line

Theorem 1: The linear equation $ax + by + c = 0$ in two variables x and y represents a straight line, where a, b and c are real numbers and a and b are not simultaneously zero.

Proof: Here, a and b cannot be both zero. So, the following cases arise:

Case-I: When $a = 0$

Then $ax + by + c = 0$ becomes

$$by+c=0 \quad \text{or} \quad y=-\frac{c}{b}$$

which is an equation of the straight line parallel to x-axis.

Case-II: When $b = 0$

Then $ax + by + c = 0$ becomes

$$ax+c=0 \quad \text{or} \quad x=-\frac{c}{a}$$

which is an equation of straight line parallel to y-axis.

Case-III: When $a \neq 0$, $b \neq 0$

Then $ax + by + c = 0$ takes the form

$$by=-ax-c \quad \text{or} \quad y=-\frac{a}{b}x-\frac{c}{b}=mx+k$$

which is slope-intercept form of the straight line with slope $-\frac{a}{b}$ and y-intercept $-\frac{c}{b}$.

Thus, equation $ax + by + c = 0$ always represents a straight line.

5.2.1 Transform the General Linear Equation to Standard Forms

Theorem 2: To transform the equation $ax + by + c = 0$ in the standard forms.

(i) Slope-Intercept Form

We have $a\,x+b\,y+c=0$

or $by=-ax-c$

or $y=-\frac{a}{b}x-\frac{c}{b}=mx+k$ (where, $b\neq 0$)

where $m=-\frac{a}{b}, k=-\frac{c}{b}$

(ii) Point-Slope Form

We note from above that slope of the line $ax+by+c=0$ is $-\frac{a}{b}$ and y–intercept is $-\frac{c}{b}$.

So, the point $\left(-\frac{c}{a},0\right)$ lies on the line.

Equation of the straight line in point-slope form is $y-0=-\frac{a}{b}\left(x+\frac{c}{a}\right)$

(iii) Symmetric Form

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r\text{ (say)}$$

$$m=\tan\theta=-\frac{a}{b},\ \sin\theta=\frac{a}{\pm\sqrt{a^2+b^2}},\ \cos\theta=\frac{b}{\pm\sqrt{a^2+b^2}}$$

A point on $ax + by + c = 0$ is $\left(-\frac{c}{a}, 0\right)$.

So, equation of the straight line in symmetric form is:

$$\frac{x-\left(-\frac{c}{a}\right)}{\frac{b}{\pm\sqrt{a^2+b^2}}}=\frac{y-0}{\frac{a}{\pm\sqrt{a^2+b^2}}}=r$$

Sign of the radical to be properly chosen.

(iv) Two-Point Form

We choose two arbitrary points on $ax + by + c = 0$. Two such points are $\left(-\frac{c}{a}, 0\right)$ and $\left(0,-\frac{c}{b}\right)$. Equation of the straight line through these points is

$$\frac{y-0}{0+\frac{c}{b}}=\frac{x+\frac{c}{a}}{-\frac{c}{a}-0} \quad \text{or} \quad y-0=\frac{\frac{c}{b}}{-\frac{c}{a}}\left(x+\frac{c}{a}\right) \quad \text{or} \quad y=-\frac{a}{b}\left(x+\frac{c}{a}\right)$$

(v) Two-Intercept Form

$$ax+by=-c$$

or $\frac{ax}{-c}+\frac{by}{-c}=1$ or $\frac{x}{\left(-\frac{c}{a}\right)}+\frac{y}{\left(-\frac{c}{b}\right)}=1$

where x-intercept $=-\frac{c}{a}$, y-intercept $=-\frac{c}{b}$

which is an equation of straight line in two-intercept form.

(vi) Normal Form

The equation $ax+by+c=0$ ...(i)

can be written in the normal form as:

$$\frac{ax+by}{\pm\sqrt{a^2+b^2}}=\frac{-c}{\pm\sqrt{a^2+b^2}} \quad \text{...(ii)}$$

The sign of the radical to be such that the right hand side of (ii) is positive.

Proof: We know that an equation of a straight line in normal form is:

$$x\cos\theta+y\sin\theta=p \quad \text{...(iii)}$$

If (i) and (iii) are identical, we must have

$$\frac{a}{\cos\theta}=\frac{b}{\sin\theta}=-\frac{c}{p}$$

i.e. $\frac{p}{-c}=\frac{\cos\theta}{a}=\frac{\sin\theta}{b}=\frac{\sqrt{\cos^2\theta+\sin^2\theta}}{\pm\sqrt{a^2+b^2}}=\frac{1}{\pm\sqrt{a^2+b^2}}$

Hence, $\cos\theta=\frac{a}{\pm\sqrt{a^2+b^2}}$ and $\sin\theta=\frac{b}{\pm\sqrt{a^2+b^2}}$ and $p=\frac{-c}{\pm\sqrt{a^2+b^2}}$

Substituting values of $\cos\theta$, $\sin\theta$ and p in (iii), we have:

$$\frac{ax+by}{\pm\sqrt{a^2+b^2}}=\frac{-c}{\pm\sqrt{a^2+b^2}}$$

Thus (i) can be reduced to the form (ii) by dividing it by $\pm\sqrt{a^2+b^2}$. The sign of the radical to be chosen so that the right hand side of (ii) is positive.

Example 6 Transform the equation $3x - 10y + 13 = 0$ into

(i) Slope-intercept form (ii) Two-intercept form (iii) Normal form

(iv) Point-slope form (v) Two-point form (vi) Symmetric form

Solution: **(i)** $3x - 10y + 13 = 0$

$$10y = 3x + 13$$

$$y = \frac{3}{10}x + \frac{13}{10} \quad \left(\text{where, } m = \frac{3}{10}, c = \frac{13}{10}\right)$$

(ii) $3x - 10y + 13 = 0$

or $3x - 10y = -13$

or $\frac{3x}{-13} + \frac{10}{13}y = 1$

or $\frac{x}{-\frac{13}{3}} + \frac{y}{\frac{13}{10}} = 1$

(iii) $3x - 10y = -13$

Dividing both sides by $\pm\sqrt{(3)^2 + (-10)^2} = \pm\sqrt{109}$.

Since R.H.S is to be positive, we have to take negative sign.

Hence, $\frac{-3x}{\sqrt{109}} + \frac{10y}{\sqrt{109}} = \frac{13}{\sqrt{109}}$ is the normal form of the given equation.

(iv) A point on the line is $\left(-\frac{13}{3}, 0\right)$ and its slope is $\frac{3}{10}$.

Equation of line in point slope-form is:

$$y - 0 = \frac{3}{10}\left(x + \frac{13}{3}\right)$$

(v) We take any two points $\left(-\frac{13}{3}, 0\right)$ and $\left(0, \frac{13}{10}\right)$ on the line.

So, equation of the line in two-point form is:

$$\frac{y-0}{\frac{13}{10}-0} = \frac{x+\frac{13}{3}}{0+\frac{13}{3}}$$

(vi) We have $\tan\theta = \frac{3}{10} = m$, $\sin\theta = \frac{3}{\sqrt{109}}$, $\cos\theta = \frac{10}{\sqrt{109}}$ and a point on the line is $\left(-\frac{13}{3}, 0\right)$.

Equation of the line in symmetric form is:

$$\frac{x+\frac{13}{3}}{\frac{10}{\sqrt{109}}} = \frac{y-0}{\frac{3}{\sqrt{109}}} = r \text{ (say)}$$

Challenge!

Transform the equation $2x + 5y - 10 = 0$ into slope-intercept form, two-intercept form, normal form, point-slope form, two-point form and symmetric form.

5.3 Equations of Medians, Altitudes and Right Bisectors

Example 7 Triangle ABC has vertices $A(2, 3)$, $B(4, 5)$ and $C(-2, 7)$. Find the equations of medians of the triangle ABC.

Solution: Given that $A(2, 3)$, $B(4, 5)$ and $C(-2, 7)$

Let D, E and F be the mid-points of the sides BC, AC and AB respectively.

$$D\left(\frac{4+(-2)}{2}, \frac{5+7}{2}\right) = D(1,6)$$

$$E\left(\frac{2+(-2)}{2}, \frac{3+7}{2}\right) = E(0,5)$$

$$F\left(\frac{2+4}{2}, \frac{3+5}{2}\right) = F(3,4)$$

Remember!

A median joins the vertex of a triangle to the mid-point of the opposite side.

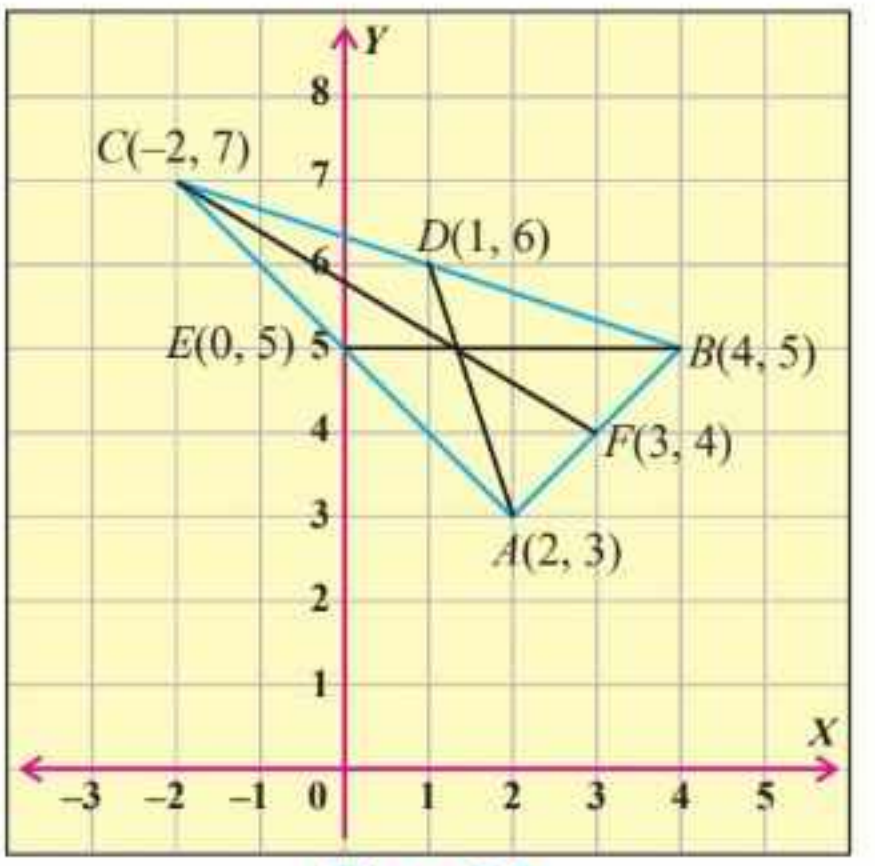


Figure 5.2

Equation of the median AD is

$$\frac{y-3}{3-6} = \frac{x-2}{2-1}$$

or $$y-3 = -3(x-2)$$

or $$3x + y - 9 = 0$$

Equation of the median BE is

$$\frac{y-5}{5-5} = \frac{x-4}{4-0}$$

or $$y-5 = 0$$

Equation of the median CF is

$$\frac{y-7}{7-4} = \frac{x+2}{-2-3}$$

or $$-5(y-7) = 3(x+2)$$

or $$3x + 5y - 29 = 0$$

Example 8 $A(1, 2)$, $B(5, -1)$ and $C(-2, -4)$ are the vertices of ΔABC. Find the equations of altitudes of triangle ABC.

Solution: Given that: $A(1, 2)$, $B(5, -1)$, $C(-2, -4)$.

Let $\overline{AE}, \overline{BF}$ and $\overline{CD}$ be the altitudes of the sides BC, CA and AB of the triangle respectively.

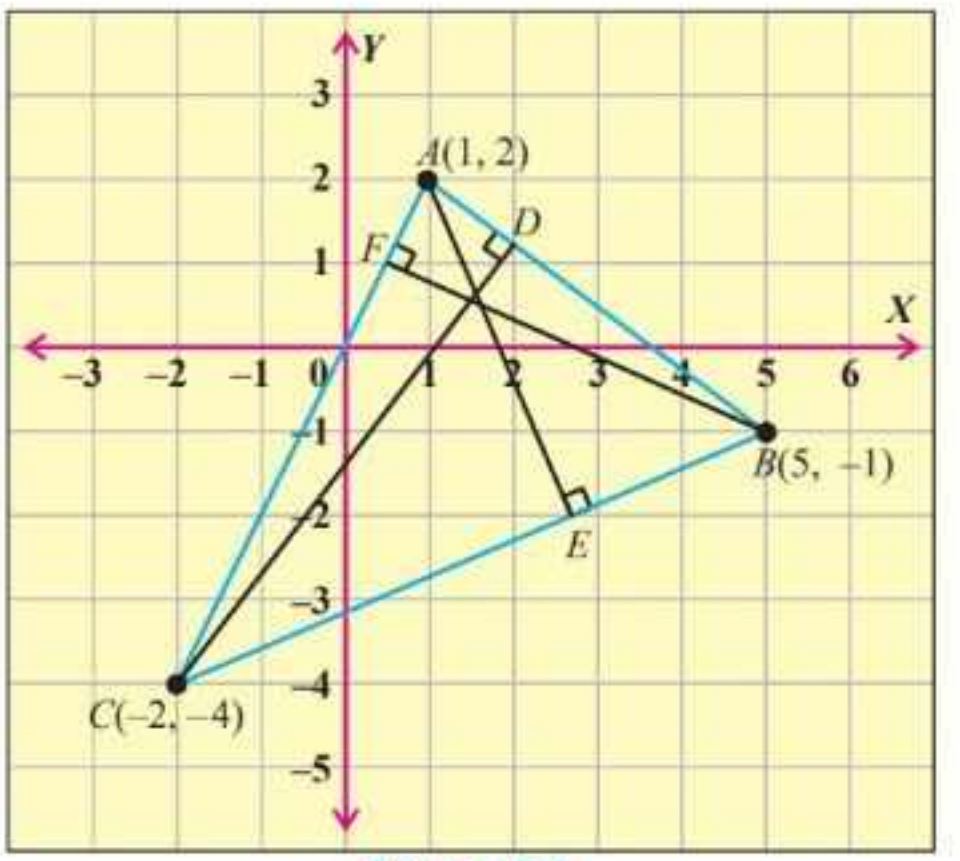


Figure 5.3

$$\text{Slope of } \overline{BC} = \frac{-4-(-1)}{-2-5} = \frac{3}{7}$$

$$\text{Slope of } \overline{AE} = \frac{-1}{\text{Slope of } \overline{BC}} = \frac{-7}{3}$$

Equation of the altitude AE is

$$y-2 = \frac{-7}{3}(x-1)$$

or $3y-6 = -7x+7$

$7x+3y-13 = 0$

$$\text{Slope of } \overline{AB} = \frac{-1-2}{5-1} = -\frac{3}{4}$$

$$\text{Slope of } \overline{CD} = \frac{-1}{\text{Slope of } \overline{AB}} = \frac{4}{3}$$

Equation of the altitude CD is

$$y+4 = \frac{4}{3}(x+2)$$

or $4x-3y-4 = 0$

Example 9: Triangle ABC has vertices $A(-3, 2)$, $B(5, 4)$ and $C(3, -8)$. Find the equations of right bisectors of triangle ABC.

Solution: Given that

$A(-3, 2)$, $B(5, 4)$, $C(3, -8)$

Let D, E and F be the mid-points of the sides BC, CA and AB respectively.

$$D\left(\frac{5+3}{2}, \frac{4+(-8)}{2}\right) = D(4,-2)$$

$$E\left(\frac{-3+3}{2}, \frac{2+(-8)}{2}\right) = E(0,-3)$$

$$F\left(\frac{-3+5}{2}, \frac{2+4}{2}\right) = F(1,3)$$

$$\text{Slope of } \overline{AB} = \frac{4-2}{5-(-3)} = \frac{2}{8} = \frac{1}{4}$$

$$\text{Slope of } \overline{CA} = \frac{2-(-4)}{1-(-2)} = \frac{6}{3} = 2$$

$$\text{Slope of } \overline{BF} = \frac{-1}{\text{Slope of } \overline{CA}} = \frac{-1}{2}$$

Equation of the altitude BF is

$$y+1 = \frac{-1}{2}(x-5)$$

or $x+2y-3 = 0$

Remember!

An altitude is the line drawn from a vertex perpendicular to the opposite side.

Remember!

If m_1 and m_2 be the slopes of perpendicular lines then $m_1 m_2 = -1$.

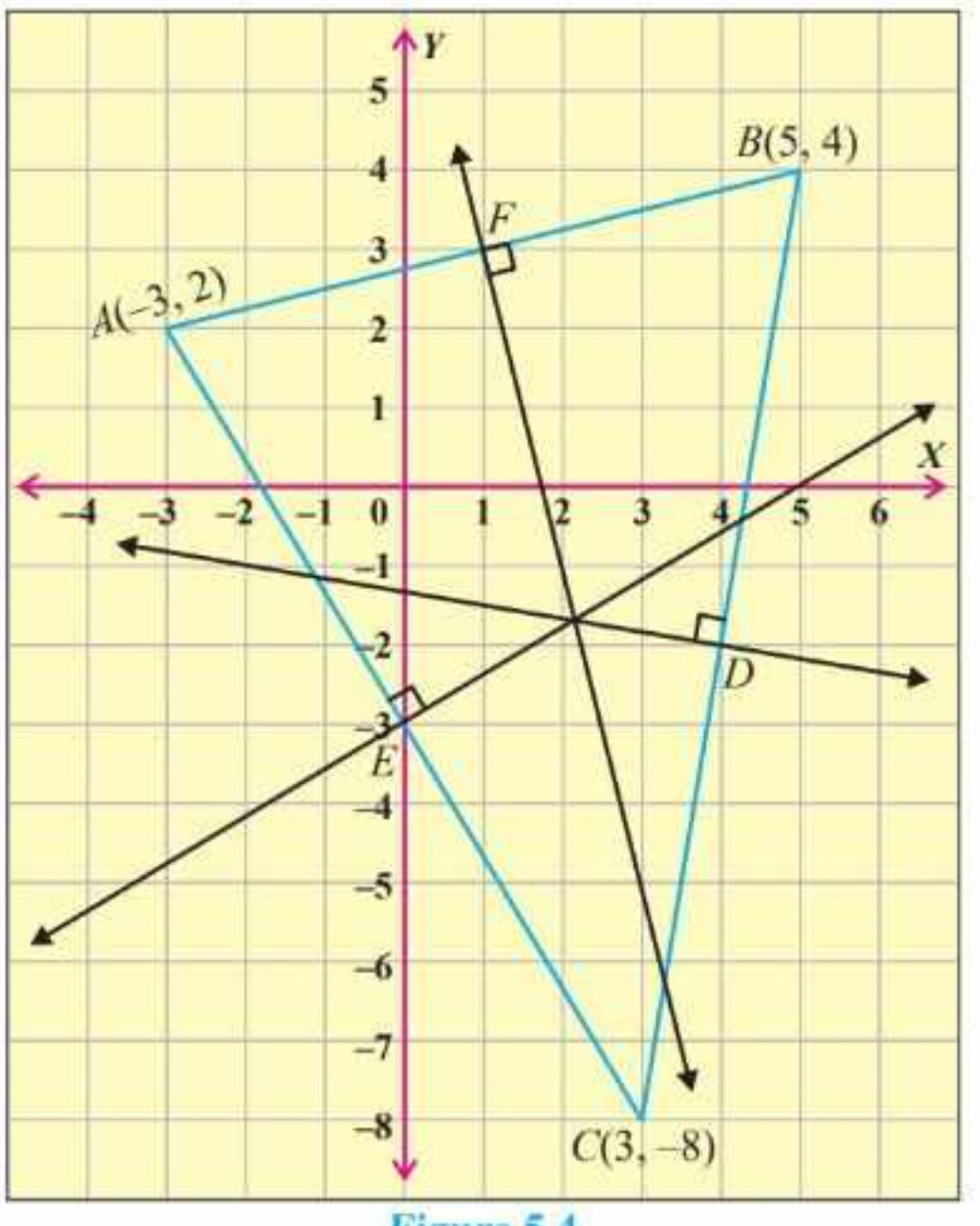


Figure 5.4

Slope of right bisector to $\overline{AB} = \dfrac{-1}{\text{Slope of } \overline{AB}} = -4$

Equation of the right bisector to $\overline{AB}$ is:

$$y - 3 = -4(x - 1) \quad \text{or} \quad 4x + y - 7 = 0$$

$$\text{Slope of } \overline{BC} = \frac{-8-4}{3-5} = \frac{-12}{-2} = 6$$

Slope of right bisector to $\overline{BC} = \dfrac{-1}{\text{Slope of } \overline{BC}} = -\dfrac{1}{6}$

Equation of the right bisector to BC is:

$$y + 2 = -\frac{1}{6}(x - 4)$$

or $\quad x + 6y + 8 = 0$

$$\text{Slope of } \overline{CA} = \frac{2-(-8)}{-3-3} = \frac{10}{-6} = -\frac{5}{3}$$

Slope of right bisector to $\overline{CA} = \dfrac{-1}{\text{Slope of } \overline{CA}} = \dfrac{3}{5}$

Equation of the right bisector to $\overline{CA}$ is

$$y + 3 = \frac{3}{5}(x - 0) \quad \text{or} \quad 3x - 5y - 15 = 0$$

Remember!

A perpendicular or right bisector of a triangle is a line that is perpendicular to one side of the triangle and passes through the mid point of that side.

5.4 Condition of Concurrency of Three Straight Lines

Theorem 3: Three non-parallel lines

$$\ell_1 : a_1x + b_1y + c_1 = 0 \qquad \ldots(i)$$

$$\ell_2 : a_2x + b_2y + c_2 = 0 \qquad \ldots(ii)$$

$$\ell_3 : a_3x + b_3y + c_3 = 0 \qquad \ldots(iii)$$

are concurrent iff $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Proof: If the lines are concurrent, then they have a common point of intersection $P(x_1, y_1)$ say. As $\ell_1 \nparallel \ell_2$, so their point of intersection (x, y) is:

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},\ y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} \qquad (\because a_1b_2 - a_2b_1 \neq 0)$$

This point also lies on (iii), so

$$a_3\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}\right)+c_3=0$$

or $\quad a_3(b_1c_2-b_2c_1)+b_3(a_2c_1-a_1c_2)+c_3(a_1b_2-a_2b_1)=0$

An easier way to write the above equation is in the following determinant form:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

This is necessary and sufficient condition iff of concurrency of given three lines.

Example 10: Check whether the following lines are concurrent or not. If concurrent, then find the point of concurrency.

$$3x-2y-2=0$$
$$x+y-4=0$$
$$2x-y-2=0$$

Solution: The determinant of the coefficients of the given equations is:

$$\begin{vmatrix} 3 & -2 & -2 \\ 1 & 1 & -4 \\ 2 & -1 & -2 \end{vmatrix}=3\begin{vmatrix} 1 & -4 \\ -1 & -2 \end{vmatrix}-(-2)\begin{vmatrix} 1 & -4 \\ 2 & -2 \end{vmatrix}-2\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$$

$$=3(-2-4)+2(-2+8)-2(-1-2)$$
$$=3(-6)+2(6)-2(-3)=-18+12+6=0$$

Thus, the given lines are concurrent.

As we know that the point of intersection of any two lines is the point of concurrency.

From $x+y-4=0$ and $2x-y-2=0$, we have

$3x-6=0 \Rightarrow \quad x=2$

Put $x=2$ in $x+y-4=0$, we have

$2+y-4=0 \quad \Rightarrow \quad y=2$

So, (2, 2) is the point of concurrency.

Challenge!

If the lines $2x \quad 3y+5=0$, $3x+4y-1=0$ and $kx-7y+6=0$ are concurrent, then find the value of k.

5.5 Equation of Lines Through the Point of Intersection of Two Lines

We can find a family of lines through the point of intersection of two non-parallel lines ℓ_1 and ℓ_2.

Let $\quad \ell_1: a_1x+b_1y+c_1=0 \quad$ ...(i)

and $\quad \ell_2: a_2x+b_2y+c_2=0 \quad$ ...(ii)

For a non-zero real k, consider the equation

$$a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0 \qquad \text{...(iii)}$$

This, being a linear equation, represents a straight line. For different values of k, (iii) represents different lines. Thus (iii) is a family of lines.

If (x_1, y_1) is any point lying on both (i) and (ii), then it is their point of intersection.

Since, (x_1, y_1) lies on both (i) and (ii), we have

$$a_1x_1+b_1y_1+c_1=0 \quad \text{and} \quad a_2x_1+b_2y_1+c_2=0$$

From the above two equations, we note that (x_1, y_1) also lies on (iii).

Thus, (iii) is the required family of lines through the point of intersection of (i) and (ii).

Since, k can assume an infinite number of values, (iii) represents an infinite number of lines.

A particular line of the family (iii) can be determined if one more condition is given.

Example 11 Find the family of lines through the point of intersection of the lines

$$2x-3y-14=0 \quad \text{and} \quad 2x+y-10=0$$

Find the member of the family which is:

(a) parallel to the line with slope $-\frac{1}{2}$

(b) perpendicular to the line $2x-3y+2=0$

Solution: (a) Given that $2x-3y-14=0$ ...(i)

$$2x+y-10=0 \qquad \text{...(ii)}$$

A family of lines through the point of intersection of equations (i) and (ii) is:

$$2x-3y-14+k(2x+y-10)=0$$

or $$(2+2k)x+(k-3)y+(-14-10k)=0 \qquad \text{...(iii)}$$

Slope of (iii) is given by: $m=-\frac{2+2k}{k-3}$

This is the slope of any member of the family (iii). If (iii) is parallel to the line with slope $-\frac{1}{2}$, then

$$-\frac{2+2k}{k-3}=-\frac{1}{2} \quad \text{or} \quad 4+4k=k-3 \quad \text{or} \quad 3k=-7 \quad \text{or} \quad k=-\frac{7}{3}$$

Put $k=-\frac{7}{3}$ in (iii), we have

$$\left(2+2\left(-\frac{7}{3}\right)\right)x+\left(-\frac{7}{3}-3\right)y+\left(-14-10\left(-\frac{7}{3}\right)\right)=0$$

$$\left(\frac{6-14}{3}\right)x+\left(\frac{-7-9}{3}\right)y+\left(\frac{-42+70}{3}\right)=0$$

$$\left(-\frac{8}{3}\right)x+\left(-\frac{16}{3}\right)y+\left(\frac{28}{3}\right)=0$$

$$-8x-16y+28=0$$

$$2x+4y-7=0$$

which is the required equation of the member of the family.

(b) Slope of $2x-3y+2=0$ ...(iv)

is $\frac{2}{3}$. Since (iii) is perpendicular to (iv) given,

we have $-\frac{2+2k}{k-3}\times\frac{2}{3}=-1$ or $4+4k=3k-9$

$$k=-13$$

Substituting $k=-13$ in (iii), we get

$-24x-16y+116=0$ or $6x+4y-29=0$

which is the required equation of line.

Theorem 4: Altitudes of a triangle are concurrent.

Proof: Let the coordinates of the vertices of ΔABC be as shown in figure.

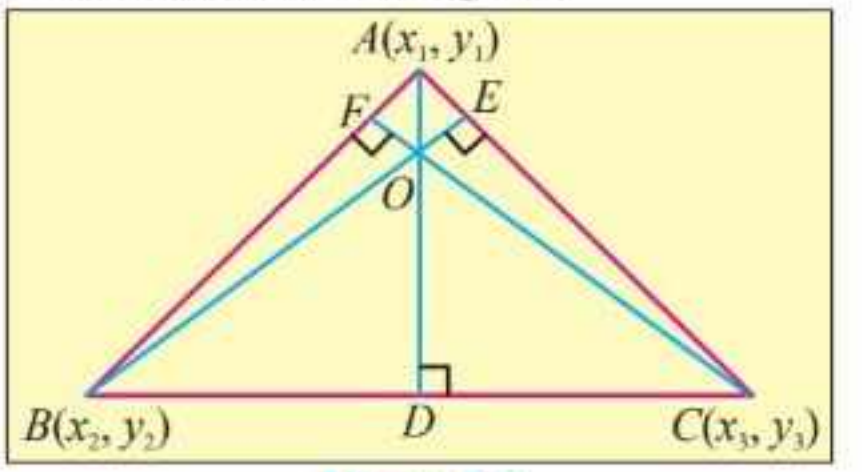


Figure 5.5

Then, slope of $\overline{BC}=\frac{y_2-y_3}{x_2-x_3}$

Therefore, slope of the altitude $AD=-\frac{x_2-x_3}{y_2-y_3}$

Equation of the altitude AD is

$y-y_1=-\frac{x_2-x_3}{y_2-y_3}(x-x_1)$ (point-slope form)

or $x(x_2-x_3)+y(y_2-y_3)-x_1(x_2-x_3)-y_1(y_2-y_3)=0$ ...(i)

Equations of the altitudes BE and CF are respectively (by symmetry)

$x(x_3-x_1)+y(y_3-y_1)-x_2(x_3-x_1)-y_2(y_3-y_1)=0$ ...(ii)

and $x(x_1-x_2)+y(y_1-y_2)-x_3(x_1-x_2)-y_3(y_1-y_2)=0$ ...(iii)

The three lines (i), (ii) and (iii) are concurrent if and only if

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -x_1(x_2-x_3)-y_1(y_2-y_3) \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-x_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix}$$

Adding 2nd and 3rd rows to the 1st row of the determinant, we have

$$\begin{vmatrix} 0 & 0 & 0 \\ x_3 - x_1 & y_3 - y_1 & -x_2(x_3 - x_1) - y_2(y_3 - y_1) \\ x_1 - x_2 & y_1 - y_2 & -x_3(x_1 - x_2) - y_3(y_1 - y_2) \end{vmatrix} = 0$$

Thus, the altitudes of the triangle are concurrent.

Theorem 5: Right bisectors of a triangle are concurrent.

Proof: Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC.

The mid-point D of BC has coordinates $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$.

Since the slope of $\overline{BC}$ is $\frac{y_2 - y_3}{x_2 - x_3}$, the slope of the right bisector DO of BC is $-\frac{x_2 - x_3}{y_2 - y_3}$.

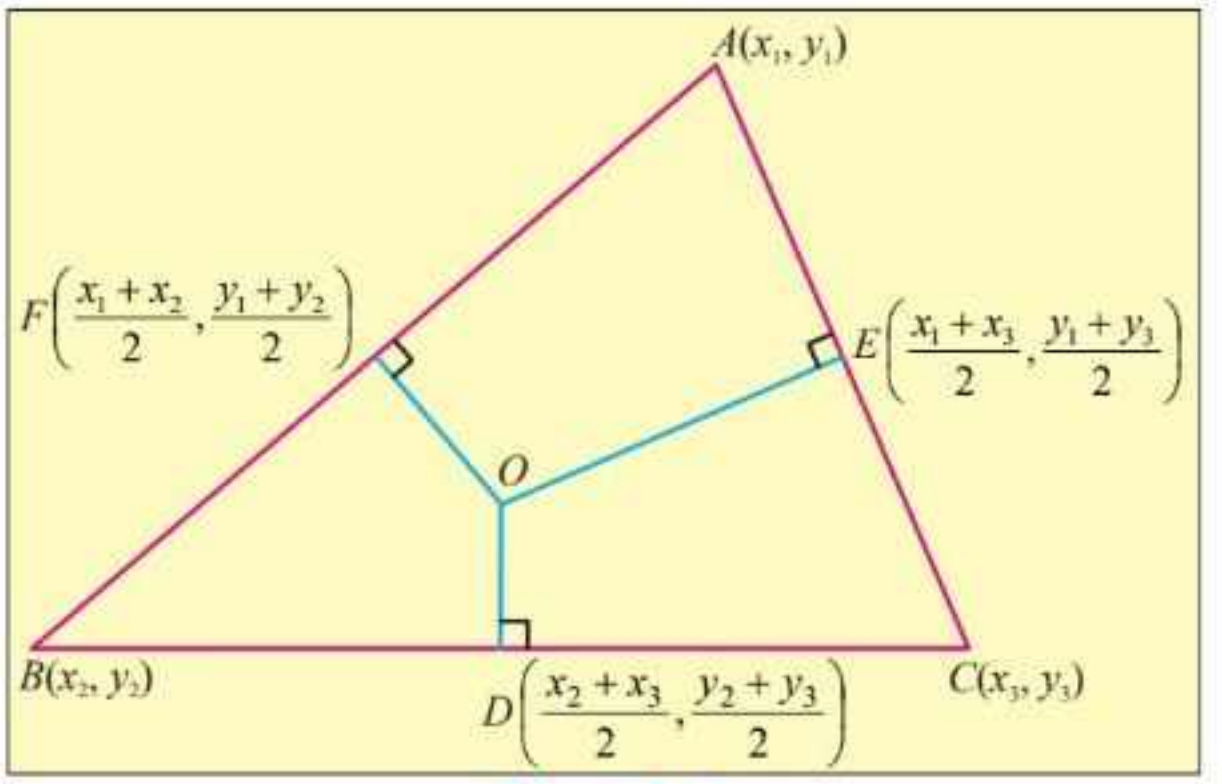


Figure 5.6

Equation of the right bisector DO of BC is:

$$y - \frac{y_2 + y_3}{2} = -\frac{x_2 - x_3}{y_2 - y_3}\left(x - \frac{x_2 + x_3}{2}\right), \text{ (By using, point-slope form)}$$

or $$x(x_2 - x_3) + y(y_2 - y_3) - \frac{1}{2}\left(y_2^2 - y_3^2\right) - \frac{1}{2}\left(x_2^2 - x_3^2\right) = 0$$

By symmetry, equations of the other two right bisectors EO and FO are respectively:

$$x(x_3 - x_1) + y(y_3 - y_1) - \frac{1}{2}\left(y_3^2 - y_1^2\right) - \frac{1}{2}\left(x_3^2 - x_1^2\right) = 0 \quad \text{...(ii)}$$

and $$x(x_1 - x_2) + y(y_1 - y_2) - \frac{1}{2}\left(y_1^2 - y_2^2\right) - \frac{1}{2}\left(x_1^2 - x_2^2\right) = 0 \quad \text{...(iii)}$$

The lines (i), (ii) and (iii) will be concurrent if and only if

$$\begin{vmatrix} x_2 - x_3 & y_2 - y_3 & -\frac{1}{2}\left(y_2^2 - y_3^2\right) - \frac{1}{2}\left(x_2^2 - x_3^2\right) \\ x_3 - x_1 & y_3 - y_1 & -\frac{1}{2}\left(y_3^2 - y_1^2\right) - \frac{1}{2}\left(x_3^2 - x_1^2\right) \\ x_1 - x_2 & y_1 - y_2 & -\frac{1}{2}\left(y_1^2 - y_2^2\right) - \frac{1}{2}\left(x_1^2 - x_2^2\right) \end{vmatrix} = 0$$

Adding 2nd and 3rd row to 1st row of the determinant, we have

$$\begin{vmatrix} 0 & 0 & 0 \\ x_3 - x_1 & y_3 - y_1 & -\frac{1}{2}(y_3^2 - y_1^2) - \frac{1}{2}(x_3^2 - x_1^2) \\ x_1 - x_2 & y_1 - y_2 & -\frac{1}{2}(y_1^2 - y_2^2) - \frac{1}{2}(x_1^2 - x_2^2) \end{vmatrix} = 0$$

Thus, the right bisectors of a triangle are concurrent.

Theorem 6: Medians of the triangle are concurrent.

Proof: Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC.

The mid-point D of BC has coordinates $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$.

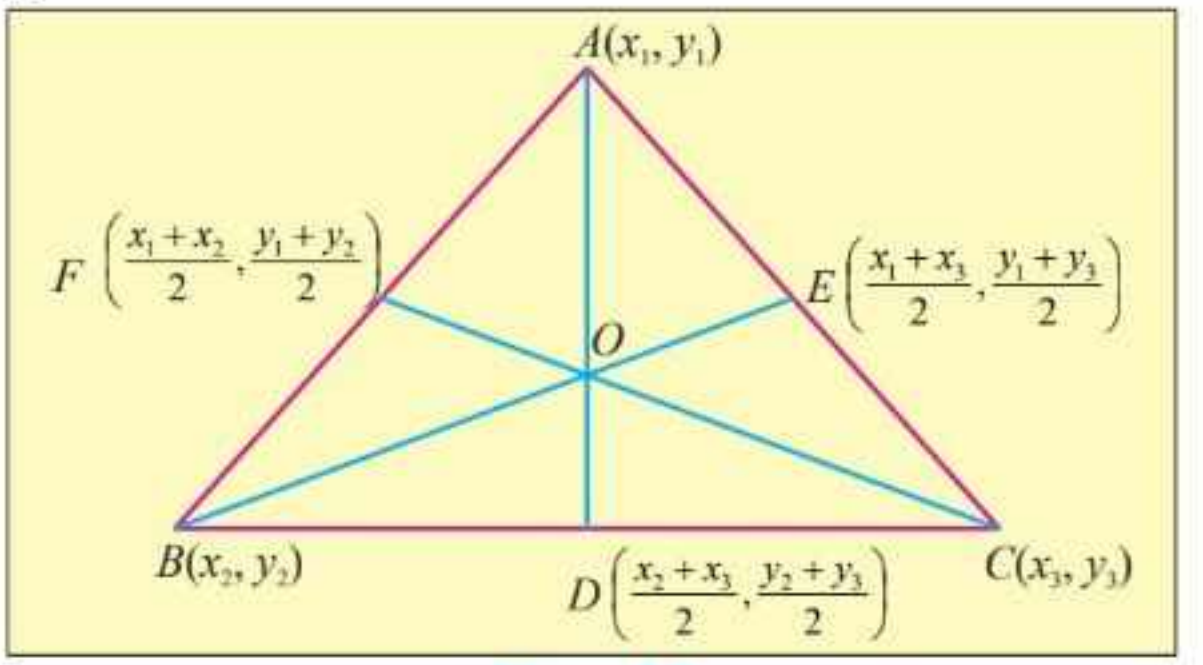


Figure 5.7

Equation of the median AD is

$$\frac{y - y_1}{\frac{y_2 + y_3}{2} - y_1} = \frac{x - x_1}{\frac{x_2 + x_3}{2} - x_1} \text{ (Two-point form)}$$

or $x(y_2 + y_3 - 2y_1) + y(2x_1 - x_2 - x_3) - x_1(y_2 + y_3 - 2y_1) + y_1(x_2 + x_3 - 2x_1) = 0$ ...(i)

By symmetry, equations of the other two medians BE and CF are respectively:

$$x(y_3 + y_1 - 2y_2) + y(2x_2 - x_3 - x_1) - x_2(y_3 + y_1 - 2y_2) + y_2(x_3 + x_1 - 2x_2) = 0 \text{ ...(ii)}$$

$$x(y_1 + y_2 - 2y_3) + y(2x_3 - x_1 - x_2) - x_3(y_1 + y_2 - 2y_3) + y_3(x_1 + x_2 - 2x_3) = 0 \text{ ...(iii)}$$

The lines (i), (ii) and (iii) will be concurrent if and only if

$$\begin{vmatrix} y_2 + y_3 - 2y_1 & 2x_1 - x_2 - x_3 & -x_1(y_2 + y_3 - 2y_1) + y_1(x_2 + x_3 - 2x_1) \\ y_3 + y_1 - 2y_2 & 2x_2 - x_3 - x_1 & -x_2(y_3 + y_1 - 2y_2) + y_2(x_3 + x_1 - 2x_2) \\ y_1 + y_2 - 2y_3 & 2x_3 - x_1 - x_2 & -x_3(y_1 + y_2 - 2y_3) + y_3(x_1 + x_2 - 2x_3) \end{vmatrix} = 0$$

Adding 2nd and 3rd row to 1st row, we have

$$\begin{vmatrix} 0 & 0 & 0 \\ y_3 + y_1 - 2y_2 & 2x_2 - x_3 - x_1 & -x_2(y_3 + y_1 - 2y_2) + y_2(x_3 + x_1 - 2x_2) \\ y_1 + y_2 - 2y_3 & 2x_3 - x_1 - x_2 & -x_3(y_1 + y_2 - 2y_3) + y_3(x_1 + x_2 - 2x_3) \end{vmatrix} = 0$$

Thus, the medians of a triangle are concurrent.

EXERCISE 5.1

1. Find equation of the altitude of triangle XYZ from vertex X (1, 3), Y (5, 7) and $Z(-3, 9)$.
2. $A(-3, -7)$, $B(4, 5)$ and $C(-1, 2)$ are the vertices of ΔABC. Find the equation of right bisector of side BC.
3. The vertices of ΔPQR are $P(1, 2)$, $Q(5, -6)$ and $R(7, -2)$. Find the equations of all three medians.
4. Show that the medians of the triangle with vertices $A(3, 2)$, $B(-1, -5)$ and $C(7, 2)$ are concurrent.
5. The vertices of a triangle are $A(2, -1)$, $B(4, 3)$ and $C(6, -1)$. Find the equations of altitudes and show that they are concurrent.
6. Find the equations of medians, altitudes and right bisectors of the triangle whose vertices are $A(2, 5)$, $B(-3, 4)$ and $C(6, -8)$.
7. Convert each of the following equation into
   (a) $2x + 3y - 6 = 0$ (b) $x - 2y + 4 = 0$
   (c) $4x + 5y - 20 = 0$ (d) $3x + 4y - 12 = 0$
   (i) Slope-intercept form (ii) Two-point form
   (iii) Intercept form (iv) Point-slope form
   (v) Normal form (vi) Symmetric form
8. Find the equation of family of lines through the point of intersection of the lines $x + 2y - 3 = 0$, $3x - y + 4 = 0$. Find the member of the family which is:
   (i) parallel to a line with slope $\frac{3}{2}$ (ii) perpendicular to the line $3x + 4y = 0$
9. Find the family of lines through the point of intersection of the lines $2x - 3y + 5 = 0$, $x + 4y - 7 = 0$. Find the member of the family which is
   (i) parallel to $y = 2x + 1$ (ii) perpendicular to $y = x$
10. Check whether the following lines are concurrent or not. If concurrent, then find the point of concurrency $x + 2y - 7 = 0$; $3x - y - 4 = 0$; $2x + 35y - 145 = 0$
11. Find the condition that the lines $y = m_1 x + c_1$; $y = m_2 x + c_2$ and $y = m_3 x + c_3$ are concurrent.
12. Find the value of k if the lines $x + y - 4 = 0$, $2x - y + 1 = 0$ and $kx + 2y - 10 = 0$ are concurrent.
13. If the lines $2x + 3y - 7 = 0$, $4x - y - 5 = 0$ and $6x + ay - 9 = 0$ meet at a point, then find the value of a.

5.6 Angle Between Two Lines

Let ℓ_1 and ℓ_2 be two intersecting lines $(\ell_1 \not\parallel \ell_2)$, which meet at a point P. At the point P two supplementary angles are formed by the lines ℓ_1 and ℓ_2.

Unless $\ell_1 \perp \ell_2$, one of the two angles is acute. The angle from ℓ_1 to ℓ_2 is the angle θ through which ℓ_1 is rotated counterclockwise about the point P so that it coincides with ℓ_2.

In the figure below, θ is the angle of intersection. It is clear that the inclination of a line is the angle measured in the counterclockwise direction from the positive x-axis to the line.

Theorem 7: Let ℓ_1 and ℓ_2 be two non-vertical lines such that they are not perpendicular to each other. If m_1 and m_2 are the slopes of ℓ_1 and ℓ_2 respectively, then the angle θ from ℓ_1 to ℓ_2 is given by $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

Proof: From the figure, we have

$$\alpha_2 = \alpha_1 + \theta$$

or $$\theta = \alpha_2 - \alpha_1$$

$$\tan\theta = \tan(\alpha_2 - \alpha_1)$$

$$\tan\theta = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{m_2 - m_1}{1 + m_1 m_2}$$

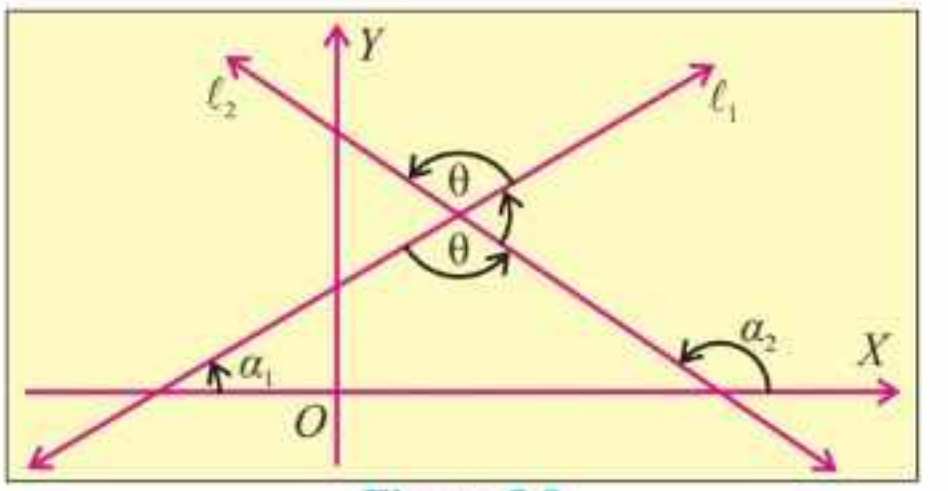


Figure 5.8

Corollary 1: $\ell_1 \parallel \ell_2$ if and only if $m_1 = m_2$. Snice, $\ell_1 \parallel \ell_2$ so, $\theta = 0$

$$\Leftrightarrow \quad \tan\theta = 0 = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\Leftrightarrow \quad m_2 = m_1$$

Corollary 2: $\ell_1 \perp \ell_2$ iff $1 + m_1 m_2 = 0$. Since, $\ell_1 \perp \ell_2$ so, $\theta = \dfrac{\pi}{2}$

$$\Leftrightarrow \quad \tan\frac{\pi}{2} = \frac{m_2 - m_1}{1 + m_1 m_2} = \infty$$

$$\Leftrightarrow \quad \infty = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad \boxed{\because \tan\frac{\pi}{2} = \infty}$$

$$\Leftrightarrow \quad 1 + m_1 m_2 = 0$$

Example 12 Find the angle from the line with slope $-\frac{7}{3}$ to the line with slope $\frac{5}{2}$.

Solution: Here, $m_1 = -\frac{7}{3}$ and $m_2 = \frac{5}{2}$

Let θ be the required angle, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{5}{2} - \left(-\frac{7}{3}\right)}{1 + \left(\frac{5}{2}\right)\left(-\frac{7}{3}\right)} = \frac{29}{-29} = -1 \qquad \Rightarrow \tan\theta = -1$$

Thus, $\theta = 135^{o}$

Example 13 : Find the angles of the triangle whose vertices are $A(1, 2)$, $B(7, 3)$ and $C(4, 9)$.

Solution: Let the slopes of the sides AB, BC and CA be denoted by m_c, m_a and m_b respectively. Then

$$m_a = \frac{9-3}{4-7} = -2,\ m_b = \frac{2-9}{1-4} = \frac{7}{3},\ m_c = \frac{3-2}{7-1} = \frac{1}{6}$$

The angle A is measured from $\overline{AB}$ to $\overline{AC}$.

$$\tan A = \frac{m_c - m_b}{1 + m_c m_b} = \frac{\frac{1}{6} - \frac{7}{3}}{1 + \left(\frac{1}{6}\right)\left(\frac{7}{3}\right)} = -\frac{39}{25}$$

$$\Rightarrow \quad A = 57.35^{o}$$

The angle B is measured from $\overline{BC}$ to $\overline{AB}$.

$$\tan B = \frac{m_c - m_a}{1 + m_c m_a} = \frac{\frac{1}{6} - (-2)}{1 + \left(\frac{1}{6}\right)(-2)} = \frac{13}{4}$$

$$\Rightarrow \quad B = 72.87^{o}$$

The angle C is measured from $\overline{AC}$ to $\overline{BC}$.

$$\tan C = \frac{m_a - m_b}{1 + m_a m_b} = \frac{-2 - \frac{7}{3}}{1 + (-2)\left(\frac{7}{3}\right)} = \frac{13}{11}$$

$$\Rightarrow \quad C = 49.78^{o}$$

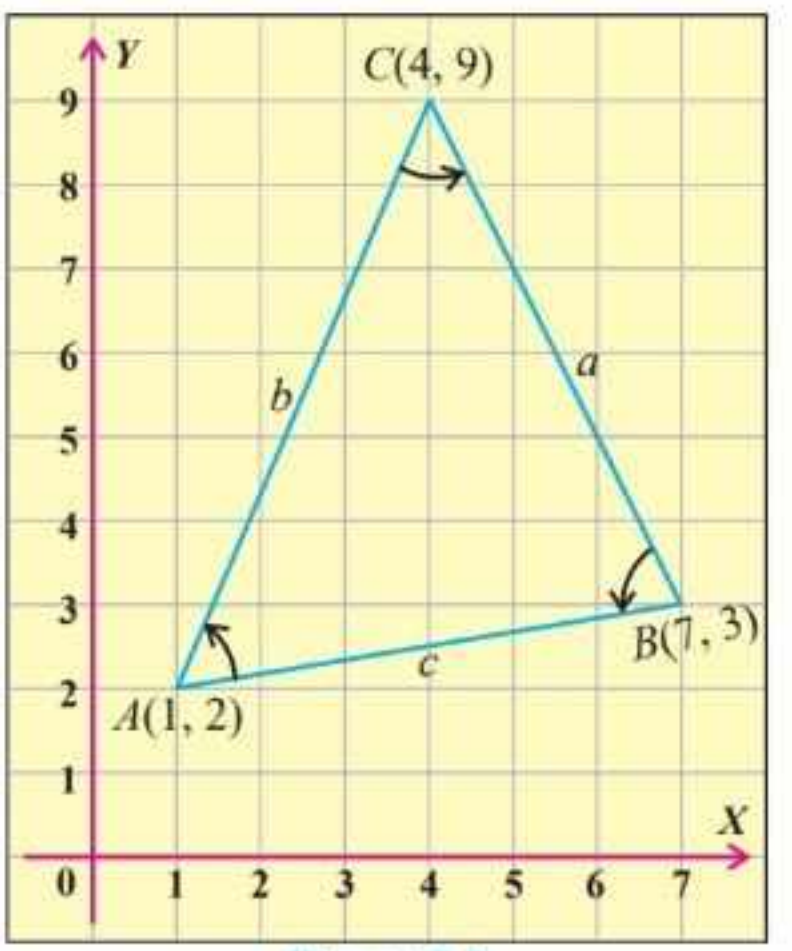


Figure 5.9

5.7 Area of a Triangular Region Whose Vertices are Given

To find the area of triangular region whose vertices are: $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$.

Draw perpendicular $\overline{PL}, \overline{QN}$ and $\overline{RM}$ on x-axis.

Area of triangular region PQR = Area of trapezoidal region $PLMR$ + Area of trapezoidal region $RMNQ$ – Area of trapezoidal region $PLNQ$

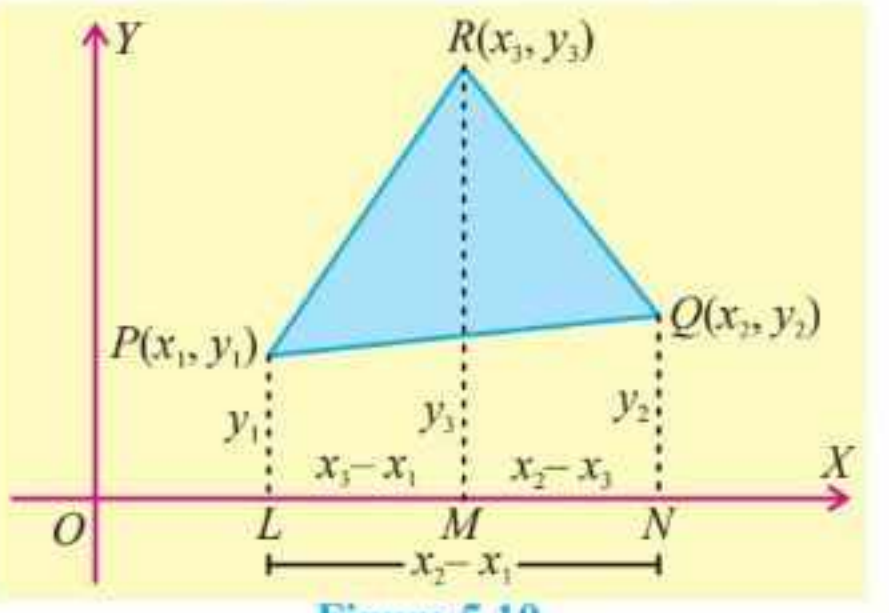


Figure 5.10

Note

A trapezium is a quadrilateral having two parallel and two non-parallel sides.

Area of trapezoidal region $= \frac{1}{2}$ (sum of $\parallel$ sides) (distance between $\parallel$ sides).

$$= \frac{1}{2}\left(|\overline{PL}|+|\overline{RM}|\right)\left(|\overline{LM}|\right)+\frac{1}{2}\left(|\overline{RM}|+|\overline{QN}|\right)\left(|\overline{MN}|\right)-\frac{1}{2}\left(|\overline{PL}|+|\overline{QN}|\right)\left(|\overline{LN}|\right)$$

$$= \frac{1}{2}\left[(y_1+y_3)(x_3-x_1)+(y_3+y_2)(x_2-x_3)-(y_1+y_2)(x_2-x_1)\right]$$

$$= \frac{1}{2}\left(x_3y_1+x_3y_3-x_1y_1-x_1y_3+x_2y_3+x_2y_2-x_3y_3-x_3y_2-x_2y_1-x_2y_2+x_1y_1+x_1y_2\right)$$

$$= \frac{1}{2}\left(x_3y_1-x_1y_3+x_2y_3-x_3y_2-x_2y_1+x_1y_2\right)$$

Thus, the required area is given by: $\Delta = \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Corollary: If the points P, Q and R are collinear, then $\Delta = 0$

Example 14: Find the area of the region bounded by the triangle with vertices $A(a, b)$, $B(-a, b)$ and $C(0, -b)$.

Solution: $\Delta = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ -a & b & 1 \\ 0 & -b & 1 \end{vmatrix}$

$$= \frac{1}{2}\left[a(b+b)-b(-a-0)+1(ab-0)\right] = \frac{1}{2}\left[2ab+ab+ab\right] = 2ab$$

Thus, the area of ΔABC is $2ab$

Example 15: Find the area of the region bounded by the triangle with vertices $A(1, 2)$, $B(3, 6)$ and $C(5, 10)$.

Solution: Area (Δ) of the region bounded by the triangle ABC is

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 5 & 10 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 4 & 8 & 0 \end{vmatrix} \quad \text{by} \begin{pmatrix} R_2 - R_1 \\ R_3 - R_1 \end{pmatrix}$$

$$= \frac{1}{2}[16-16] \text{ (Expanding by } 3^{rd} \text{ column)} = 0$$

Thus, the given points are collinear.

EXERCISE 5.2

1. Find the angle measured from the line ℓ_1 to the line ℓ_2, where

   (i) ℓ_1: Joining (2, 7) and (−4, 5)
   ℓ_2: Joining (−7, 2) and (1, 3)

   (ii) ℓ_1: Joining (1, 2) and (3, 5)
   ℓ_2: Joining (2, −1) and (4, 3)

   Also find the acute angle in each case.
2. Find the angle from the line with slope m_1 to the line with slope m_2, where

   (i) $m_1 = 2, m_2 = -\frac{1}{2}$ (ii) $m_1 = 3, m_2 = -2$
3. Find the angle from line ℓ_1 to the line ℓ_2, where

   (i) $\ell_1: 2x + y - 3 = 0$; $\ell_2: x + y - 5 = 0$ (ii) $\ell_1: 2x - 3y + 2 = 0$; $\ell_2: x + y - 4 = 0$
4. The angle from the line $ax - y + 2 = 0$ to the line $x + y - 4 = 0$ is 30°. Find the value of a.
5. Find the interior angles of the triangle whose vertices are:

   (i) $A(2, 3), B(6, 3), C(4, 7)$ (ii) $L(1, 2), M(5, 5), N(7, 2)$
6. Find the interior angles of the quadrilateral whose vertices are $A(2, 1)$, $B(3, 1)$, $C(4, 5)$ and $D(-1, 5)$.
7. Find the interior angles of the kite whose vertices are $A(0, 0)$, $B(2, 3)$, $C(4, 0)$ and $D(2, -3)$.
8. Find the area of the triangle formed by the points $A(-3, 2)$, $B(4, -1)$, $C(1, 5)$.
9. By computing the area of triangle ABC check whether the points are collinear where $A(0, 4)$, $B(3, 2)$ and $C(6, 0)$.
10. The coordinates of the points are $A(0, \lambda)$, $B(-2, 1)$ and $C(-3, -2)$. By computing the area bounded by ABC, find the value of λ if points are collinear.

5.8 Homogeneous Equation of the Second Degree in Two Variables

We know that if a graph is a straight line in a plane, then its equation is a linear equation in two variables x and y. Conversely, the graph of any linear equation in x and y is a straight line.

Suppose we have two straight lines represented by

$$a_1x + b_1y + c_1 = 0 \qquad \ldots(i)$$

and $$a_2x + b_2y + c_2 = 0 \qquad \ldots(ii)$$

Multiplying equations (i) and (ii), we have

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0 \qquad \ldots(iii)$$

It is a second degree equation in x and y.

Equation (iii) is called **joint equation of the pair of lines** (i) and (ii). On the other hand, given an equation of second degree in x and y, say

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \ldots(iv)$$

where $a \neq 0$ represents equations of a pair of lines if (iv) can be resolved into two linear factors. In this section, we shall study special joint equations of pairs of lines which pass through the origin.

Let $y = m_1x$ and $y = m_2x$ be two lines passing through the origin. Their joint equation is:

$$(y - m_1x)(y - m_2x) = 0$$

or $$y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 = 0 \qquad \ldots(v)$$

Equation (v) is a special type of a second degree homogeneous equation.

5.8.1 Homogeneous Equation

Let $f(x, y) = 0$ be any equation in two variables x and y is called a homogeneous equation of degree n (a positive integer) if the function $f(x, y)$ is homogeneous of degree n, then, $f(kx, ky) = k^n f(x, y)$ for some real number k.

For example, in equation (v) above if we replace x and y by kx and ky respectively, we have:

$$f(kx, ky) = k^2 y^2 - k^2 (m_1 + m_2) xy + k^2 m_1 m_2 x^2$$

$$= k^2[y^2 - (m_1 + m_2)xy + m_1m_2x^3] = 0 \Rightarrow f(kx, ky) = k^2 f(x, y)$$

Thus (v) is a homogeneous equation of degree 2. A general second degree homogeneous equation can be written as: $ax^2 + 2hxy + by^2 = 0$

provided a, h and b are not simultaneously zero.

Theorem 8: Every homogeneous second degree equation

$$ax^2 + 2hxy + by^2 = 0 \qquad \ldots(1)$$

represents a pair of lines through the origin. The lines are

(i) real and distinct, if $h^2 > ab$ (ii) real and coincident, if $h^2 = ab$

(iii) imaginary, if $h^2 < ab$

Proof: Multiplying (1) by b and re-arranging the terms, we have:

$$b^2y^2 + 2bhxy + abx^2 = 0$$

$$b^2y^2 + 2bhxy + h^2x^2 - h^2x^2 + abx^2 = 0$$

$$(by + hx)^2 - x^2(h^2 - ab) = 0$$

$$\left(by + hx + x\sqrt{h^2 - ab}\right)\left(by + hx - x\sqrt{h^2 - ab}\right) = 0$$

Thus equation (1) represents a pair of lines whose equations are:

$$by + x\left(h + \sqrt{h^2 - ab}\right) = 0 \qquad \ldots(2)$$

and $$by + x\left(h - \sqrt{h^2 - ab}\right) = 0 \qquad \ldots(3)$$

Clearly the lines (2) and (3) are (i) real and distinct if $h^2 > ab$ (ii) real and coincident if $h^2 = ab$ (iii) imaginary, if $h^2 < ab$. It is interesting to note that even in case the lines are imaginary, they intersect in a real point viz (0,0) since this point lies on their joint equation (1).

Example 16 Find an equation of each of the lines represented by $3x^2 - 10xy + 3y^2 = 0$

Solution:

$$3x^2 - 10xy + 3y^2 = 0$$

$$3x^2 - 9xy - xy + 3y^2 = 0$$

$$3x(x - 3y) - y(x - 3y) = 0$$

$$(x - 3y)(3x - y) = 0$$

$$x - 3y = 0 \text{ and } 3x - y = 0$$

5.9 Angle Between the Lines

$$ax^2 + 2hxy + by^2 = 0 \qquad \ldots(i)$$

We have already seen that the lines represented by (i) are:

$$by + x\left(h + \sqrt{h^2 - ab}\right) = 0 \qquad \ldots(ii)$$

and $$by + x\left(h - \sqrt{h^2 - ab}\right) = 0 \qquad \ldots(iii)$$

Now, slopes of (ii) and (iii) are respectively given by:

$$m_1 = \frac{-\left(h + \sqrt{h^2 - ab}\right)}{b} \text{ and } m_2 = \frac{-\left(h - \sqrt{h^2 - ab}\right)}{b}$$

Therefore, $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$

If θ is the measure of the angle between the lines (ii) and (iii), then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}}$$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b} \qquad \ldots(iv)$$

Thus, equation (iv) gives angle between the lines represented by equation (i).

Special Cases:

- If two lines are parallel, then $\theta = 0$, so that $\tan\theta = 0$ this implies $h^2 - ab = 0$, which is the condition for the lines to be coincident.
- If the lines are perpendicular, then $\theta = 90^\circ$, so that $\tan\theta$ is not defined. This implies $a + b = 0$.

Hence, the condition for (i) to represent a pair of perpendicular lines is that sum of the coefficients of x^2 and y^2 is 0.

Example 17 Find measure of the angle between the lines represented by:

$$5x^2 + 6xy + y^2 = 0.$$

Solution: Here $a = 5, h = 3, b = 1$

If θ is the measure of the angle between the given lines, then it is given by:

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b} = \frac{2\sqrt{9 - 5}}{5 + 1} = \frac{2}{3}$$

$$\theta = 33.69^\circ$$

5.10 Application of Analytical Geometry to Real Life

Analytical geometry connects mathematics to the real life. It allows us to analyze positions, distances, slopes and angles making it essential in aviation, astronomy, engineering and many fields of science and technology.

Example 17 Planet A is at (1, 0) AU and planet B is at (–0.5, 0.866) AU. Find the distance between them. Write answer in km.

Solution: Distance $= \sqrt{[1 - (-0.5)]^2 + (0 - 0.866)^2}$

$$= \sqrt{2.25 + 0.75} = \sqrt{3} = 1.732\ AU$$

We know that $|AU| = 149.6$ million km, then

Distance $= 1.732 \times 149.6 = 259.1$ million km

Remember!

An astronomical unit (AU) is a unit of length and exactly equal to 149597870700 m.

Example 19 Aircraft flies from $A(100, 200)$ km to $B(500, 800)$ km. Find slope of the track and straight line distance.

Solution: Slope $= \dfrac{800-200}{500-100} = \dfrac{600}{400} = 1.5$

$$\text{Distance} = \sqrt{(500-100)^2 + (800-200)^2}$$

$$= \sqrt{(400)^2 + (600)^2} = \sqrt{520000} = 721.11 \text{ km}$$

EXERCISE 5.3

Find the equation of lines represented by each of the following and also find measure of the angle between them (**Problem 1–5**):

1. $6x^2 - 5xy - 4y^2 = 0$
2. $2x^2 - 7xy + 3y^2 = 0$
3. $5x^2 + 6xy + y^2 = 0$
4. $x^2 + 2xy \sec\alpha + y^2 = 0$
5. $7x^2 - 8xy + y^2 = 0$
6. Find the value of k such that the lines represented by the equation $(k-8)x^2 - 9xy + 3ky^2 = 0$ are perpendicular to each other.
7. Prove that the lines represented by the equation $9y^2 + 6xy + x^2 = 0$ are coincident.
8. Find a joint equation of the lines through the origin and perpendicular to the lines $3x^2 + 5xy - 2y^2 = 0$.
9. $A(-2,-1)$ and $B(3, 4)$ are two points on the solar panel roof. Find slope and tilt angle.
10. The paths of two rays from a star are represented by the equation

    $$2x^2 + 5xy - 3y^2 = 0$$

    Find the equations of the two lines and determine the angle between the rays.
11. The trajectories of two projectiles in a physics experiment are modeled by the homogeneous equation $9x^2 - 12xy + 4y^2 = 0$. Prove that this represents two coincident lines and find angle between them.
12. The stresses in a metal plate are represented by the homogeneous equation $3x^2 - 8xy - 3y^2 = 0$. Show that this equation represents a pair of straight lines through the origin. Find:
    i. the equations of lines represented by this equation.
    ii. angle between them.

Unit 06

Conics

INTRODUCTION

The Greek mathematicians **Apollonius** (260-200 B.C.) and **Pappus** (early fourth century A.D.) made significant contributions to the study of conic sections using Euclidean geometry. Here, we will employ the methods of **analytic geometry** to investigate their properties.

The theory of conics has important applications in various fields, including **space mechanics**, **oceanography**, and many other branches of science and technology.

Consider a circle, with its centre C, draw a line RS passes through c and perpendicular to the circle's plane. Choose a fixed point A on this line. Now, from A, draw every possible straight line that touches some point on the circle. These lines together form a surface known as a **right circular cone**. Each of these straight lines is called a **generator** (or a **ruling**) of the cone. The complete surface has two separate parts, called **nappes**, which meet only at point A. This meeting point is the **vertex** of the cone and the original line RS is the **axis** of come.

When you slice this cone with a plane, the intersection curve is called a **conic section** (or simply **conics**).

There are also special cases called **degenerate conics**. These happen when the slicing plane passes through the vertex A. Instead of getting a nice curve, the intersection might be just a single point, a single straight line, or a pair of intersecting lines.

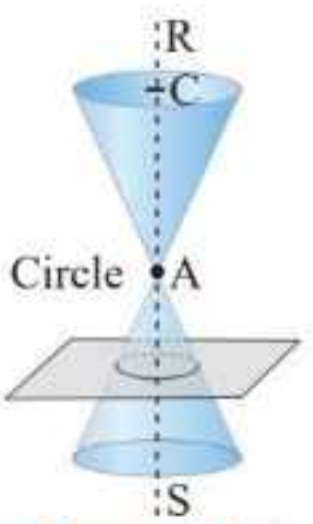


Figure 6.1(a)

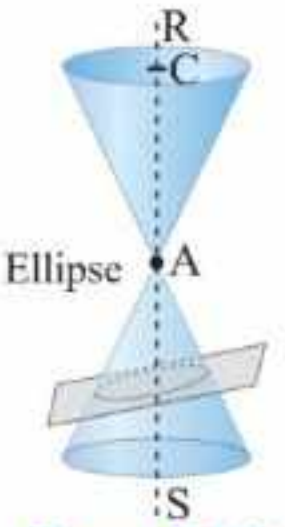


Figure 6.1(b)

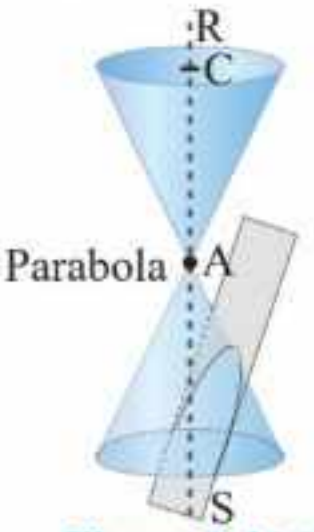


Figure 6.1(c)

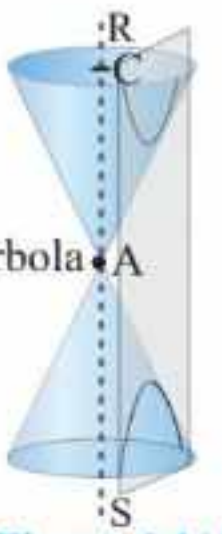


Figure 6.1(d)

Depending on the angle and position of the plane, you can get a circle, an ellipse, a parabola, or a hyperbola.

When this cone is intersected by a plane, different curves are obtained depending on the angle and position of the cutting plane:

- If the cutting plane is perpendicular to the axis, the section is a circle, see Figure 6.1(a).
- If the plane is slightly inclined and cuts only one nappe, the resulting curve is an ellipse, see Figure 6.1(b).

- If the plane is parallel to a generator and cuts only one nappe, the curve is a parabola, see Figure 6.1(c).
- If the plane is parallel to the axis and cuts both nappes, the curve is a hyperbola, see Figure 6.1(d).

We will being our study with the circle. Other conic sections will be discussed in subsequent section.

6.1 Circle

A circle is the set of all points in a plane that are at a fixed distance from a given fixed point. The fixed point is called the centre, and the fixed distance is called the radius of the circle.

Theorem 1: Equation of a circle with centre at the point (h, k) and radius r is $(x-h)^2+(y-k)^2=r^2$.

Proof: Let $C(h, k)$ be the centre of the circle and r be its radius. Let $P(x, y)$ be any point on the circle. Then by definition, the distance $|\overline{CP}|$ is equal to r. Using the distance formula between the points $P(x, y)$ and $C(h, k)$, we obtain

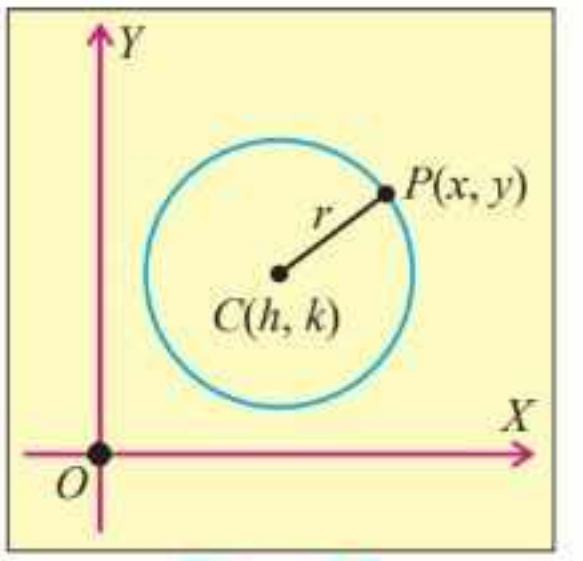


Figure 6.2

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

Squaring both sides gives

$$(x-h)^2+(y-k)^2=r^2 \qquad \ldots(1)$$

which is the required equation of the circle.

Note:

(i) Equation (1) is known as the **standard form** of the equation of a circle.

(ii) If $r = 0$, equation (1) reduces to $(x-h)^2+(y-k)^2=0$. This represents a **point circle** centered at (h, k).

(iii) If $(h, k) = (0, 0)$, then equation (1) reduces to $x^2+y^2=r^2$. This equation represents a circle whose centre is at origin.

6.1.1 Parametric Equations of a Circle

Let $P(x, y)$ be any point on the circle $x^2+y^2=r^2$ and let the inclination of $\overline{OP}$ be θ, as shown in the Figure 6.3. Draw $\overline{PM}$ perpendicular to the x-axis. From right triangle OMP, we have

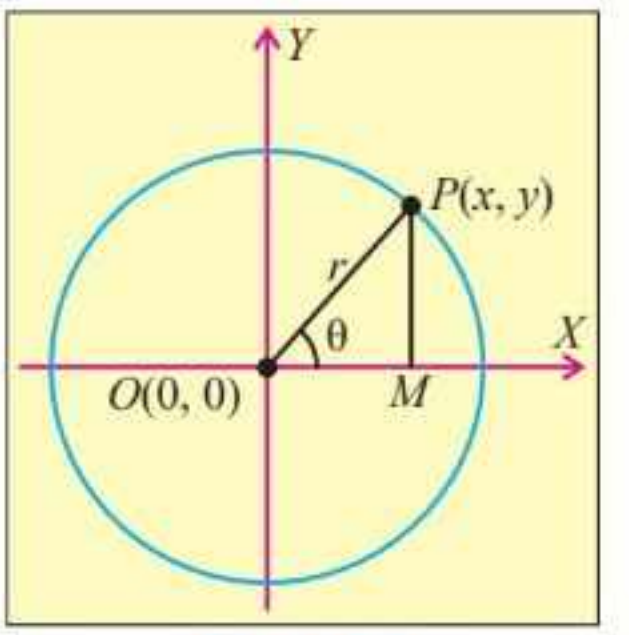


Figure 6.3

$$\cos\theta=\frac{|\overline{OM}|}{|\overline{OP}|}, \quad \sin\theta=\frac{|\overline{PM}|}{|\overline{OP}|}$$

Since $|\overline{OM}|=x$ and $|\overline{PM}|=y$, and $|\overline{OP}|=r$,

it follows that

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}$$

Thus, the parametric equations of the circle are:

$$x = r\cos\theta, y = r\sin\theta$$

The point $P(r\cos\theta, r\sin\theta)$ lies on the circle $x^2 + y^2 = r^2$ for all values of θ.

Example 1 Write an equation of the circle with centre (–2, 1) and radius 5.

Solution: Using the standard form $(x-h)^2 + (y-k)^2 = r^2$ with $h = -2, k = 1$, and $r = 5$, we have

$$(x-(-2))^2 + (y-1)^2 = 5^2$$
$$(x+2)^2 + (y-1)^2 = 25$$

Hence, the required equation is $x^2 + y^2 + 4x - 2y - 20 = 0$.

Example 2 Prove that the locus of the point of intersection of the lines

$$x\cos\theta + y\sin\theta = a, \quad x\sin\theta - y\cos\theta = b$$

is a circle with centre at (0, 0) and radius $\sqrt{a^2+b^2}$, Where θ is a parameter.

Solution: The locus of the point of intersection of the given lines is obtained by eliminating θ between the two equations. To eliminate θ, we square both equations and add them:

$$(x\cos\theta + y\sin\theta)^2 + (x\sin\theta - y\cos\theta)^2 = a^2 + b^2$$

Expanding,

$$(x^2\cos^2\theta + 2xy\cos\theta\sin\theta + y^2\sin^2\theta) + (x^2\sin^2\theta - 2xy\sin\theta\cos\theta + y^2\cos^2\theta) = a^2 + b^2$$

Simplifying, $x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2$

Since $\cos^2\theta + \sin^2\theta = 1$, we obtain

$$x^2(1) + y^2(1) = a^2 + b^2 \quad \text{or} \quad x^2 + y^2 = \left(\sqrt{a^2+b^2}\right)^2$$

This is the equation of a circle with centre (0, 0) and radius $\sqrt{a^2+b^2}$.

6.1.2 General Form of an Equation of a Circle

Theorem 2: The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ …(1) where $g, f, c \in \mathbb{R}$ represents a circle.

Proof: Equation (1) can be written as: $x^2 + 2gx + y^2 + 2fy = -c$

Completing the square for x and y, we add $g^2 + f^2$ to both sides:

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$$

This gives $(x+g)^2 + (y+f)^2 = \left(\sqrt{g^2+f^2-c}\right)^2$.

Alternatively, this can be written as $[x-(-g)]^2+[y-(-f)]^2=\left(\sqrt{g^2+f^2-c}\right)^2$, which is the general form of the equation of a circle with centre $(-g,-f)$ and radius $r=\sqrt{g^2+f^2-c}$, provided $g^2+f^2-c>0$.

Hence equation (1) represents a circle under the condition $g^2+f^2-c>0$.

Note:

(i) If $g^2+f^2-c=0$, equation (1) represents a point circle with centre $(-g,-f)$.

(ii) If $g^2+f^2-c<0$, equation (1) represents no real locus (an imaginary circle).

6.1.3 Working Rule to Find the Centre and Radius of a Circle in General Form

To find the centre and radius of a circle from its general equation, follow these steps:

Step 1: If necessary, make the coefficients of x^2 and y^2 equal to 1 by dividing the entire equation by the common coefficient of x^2 and y^2.

Step 2: Centre of the circle $=\left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$

Step 3: Radius of the circle

$$=\sqrt{\left(\frac{\text{coefficient of } x}{2}\right)^2+\left(\frac{\text{coefficient of } y}{2}\right)^2-\text{constant term}}$$

Example 3 Equation $5x^2+5y^2+20x+25y+10=0$ represents a circle. Find its centre and radius.

Solution: The given equation can be written as:

$x^2+y^2+4x+5y+2=0$ ...(i) (by dividing both sides by 5).

(i) is the general form of the equation of a circle. Comparing with the standard general form *i.e.*

$x^2+y^2+2gx+2fy+c=0$ ...(ii)

we obtain $2g=4\Rightarrow g=2, \quad 2f=5\Rightarrow f=\frac{5}{2}, \quad c=2$

Therefore, centre is $(-g,-f)=\left(-2,-\frac{5}{2}\right)$.

The radius is given by: Radius $=\sqrt{g^2+f^2-c}=\sqrt{(2)^2+\left(\frac{5}{2}\right)^2-2}=\sqrt{4+\frac{25}{4}-2}$

$$=\sqrt{2+\frac{25}{4}}=\sqrt{\frac{8+25}{4}}=\frac{\sqrt{33}}{2}$$

Thus, the given equation represents a circle with centre $\left(-2,-\frac{5}{2}\right)$ and radius $\frac{\sqrt{33}}{2}$.

Example 4 A point moves so that the sum of the squares of its distances from two points $A(1, 1)$ and $B(2, 2)$ is equal to 4. Prove that its locus is a circle.

Solution: Let the moving point be $P(x, y)$. Then the geometric condition is:

$$|\overline{PA}|^2 + |\overline{PB}|^2 = 4$$

Using the distance formula:

$$\left(\sqrt{(x-1)^2+(y-1)^2}\right)^2 + \left(\sqrt{(x-2)^2+(y-2)^2}\right)^2 = 4$$

$(x-1)^2+(y-1)^2+(x-2)^2+(y-2)^2=4$ (Simplifying)

$(x^2-2x+1)+(y^2-2y+1)+(x^2-4x+4)+(y^2-4y+4)=4$ (Expanding)

$2x^2+2y^2-6x-6y+10=4$ (Collecting like terms)

$x^2+y^2-3x-3y+5=2$ (Dividing both sides by 2)

$x^2+y^2-3x-3y+3=0$ (Subtracting 2 from both sides)

It represents a circle with centre $\left(\frac{3}{2},\frac{3}{2}\right)$ and $r=\frac{\sqrt{6}}{2}$

6.1.4 Equation of a Circle with Given End points of a Diameter

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of the diameter. Let $P(x, y)$ be any point on the circle.

From plane geometry, the angle subtended by a diameter at any point on the circle is a right angle. Therefore,

$$APB = 90°$$

Thus, the lines $\overline{AP}$ and $\overline{BP}$ are perpendicular to each other.

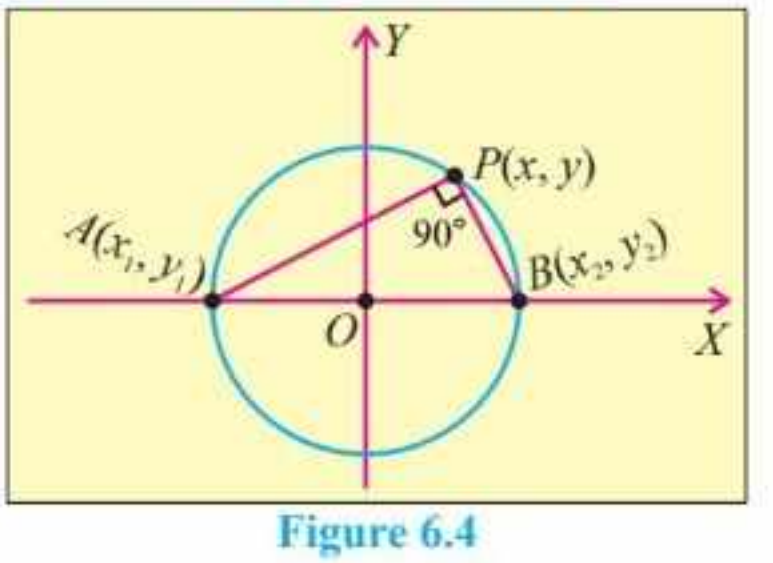


Figure 6.4

The slope of $\overline{AP}$ is $m_1=\frac{y-y_1}{x-x_1}$ and the slope of $\overline{BP}$ is $m_2=\frac{y-y_2}{x-x_2}$.

Using the condition for perpendicularity of two lines, $m_1 \cdot m_2 = -1$, we obtain:

$$\frac{y-y_1}{x-x_1}\cdot\frac{y-y_2}{x-x_2}=-1$$

Do You Know?

A circle has the maximum area for a given perimeter among all closed shapes.

Multiplying both sides by $(x-x_1)(x-x_2)$ gives:

$$(y-y_1)(y-y_2)=-(x-x_1)(x-x_2)$$

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

This is the required equation of the circle.

6.1.5 Equations of Circles Determined by Given Conditions

The general equation of a circle $x^2+y^2+2gx+2fy+c=0$ contains three independent constants g, f and c. These can be determined if the equation satisfies three given conditions. We will discuss four different cases.

Case-1: A Circle Passing through Three Non-collinear Points

If three non-collinear points through which a circle passes are given, we can find the three independent constants g, f and c in the general equation.

Example 5 Find the equation of the circle passing through the points $(-3, 3)$, $(-1, 5)$ and $(5, 1)$.

Solution: Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(A)$$

Since the circle passes through $(-3, 3)$:

$$(A) \Rightarrow (-3)^2+(3)^2+2g(-3)+2f(3)+c=0$$
$$9+9-6g+6f+c=0$$
$$-6g+6f+c+18=0 \quad \ldots(1)$$

Since the circle passes through $(-1, 5)$:

$$(A) \Rightarrow (-1)^2+(5)^2+2g(-1)+2f(5)+c=0$$
$$1+25-2g+10f+c=0$$
$$-2g+10f+c+26=0 \quad \ldots(2)$$

Since the circle passes through $(5, 1)$:

$$(5)^2+(1)^2+2g(5)+2f(1)+c=0$$
$$25+1+10g+2f+c=0$$
$$10g+2f+c+26=0 \quad \ldots(3)$$

Subtracting equation (2) from equation (1):

$$(-6g+6f+c+18)-(-2g+10f+c+26)=0$$
$$-6g+6f+c+18+2g-10f-c-26=0$$
$$-4g-4f-8=0$$
$$g+f+2=0 \quad \ldots(4)$$

Subtracting equation (3) from equation (2):

$$(-2g+10f+c+26)-(10g+2f+c+26)=0$$
$$-2g+10f+c+26-10g-2f-c-26=0$$
$$-12g+8f=0$$
$$3g-2f=0 \quad \ldots(5)$$

From equation (4), $g=-f-2$; Substituting into equation (5):

$$3(-f-2)-2f=0$$

$$-3f-6-2f=0 \quad \Rightarrow \quad -5f=6 \quad \Rightarrow \quad f=-\frac{6}{5}$$

Do You Know?

Infinitely many circles can pass through two points, but exactly one circle passes through three non-collinear points.

Equation of the circle in case of 3 non collinear points (x, y) (x_2, y_2) and (x_3, y_3) is given by

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Substituting $f=-\frac{6}{5}$ into equation (5):

$$3g-2\left(-\frac{6}{5}\right)=0 \quad \Rightarrow \quad 3g=-\frac{12}{5} \quad \Rightarrow \quad g=-\frac{4}{5}$$

Substituting $g=-\frac{4}{5}$ and $f=-\frac{6}{5}$ into equation (1):

$$-6\left(-\frac{4}{5}\right)+6\left(-\frac{6}{5}\right)+c+18=0 \quad \Rightarrow \quad \frac{24}{5}-\frac{36}{5}+c+18=0$$

$$\Rightarrow \quad -\frac{12}{5}+c+18=0 \quad \Rightarrow \quad c=\frac{12}{5}-18 \quad \Rightarrow \quad c=-\frac{78}{5}$$

Thus, the required equation of the circle is

$$x^2+y^2+2\left(-\frac{4}{5}\right)x+2\left(-\frac{6}{5}\right)y-\frac{78}{5}=0$$

Multiplying both sides by 5:

$$5x^2+5y^2-8x-12y-78=0$$

Case-2: A Circle Passing through Two Points and Having its Centre on a Given Line

Example 6 A circle with its centre on y-axis passes through the origin and the point (a, b). Find its equation.

Solution: Since the circle passes through the origin and has its centre on the y-axis, the centre is of the form $(0, r)$, where r is the radius (distance from centre to origin).
The circle also passes through (a, b). Therefore, the distance from the centre $(0, r)$ to (a, b) equals the radius r:

$$r=\sqrt{(a-0)^2+(b-r)^2}$$

Squaring both sides: $r^2=a^2+(b-r)^2$

$$r^2=a^2+b^2-2br+r^2 \quad \Rightarrow \quad r=\frac{a^2+b^2}{2b},\ b\neq 0$$

The equation of the circle with centre $(0, r)$ and radius r is:

$$(x-0)^2+(y-r)^2=r^2$$

$$x^2+y^2-2yr=0$$

$x^2+y^2-2y\left(\frac{a^2+b^2}{2b}\right)=0$ Substituting $r=\frac{a^2+b^2}{2b}$:

Multiplying through by b:

$$b(x^2+y^2)-(a^2+b^2)y=0$$

Case 3: A Circle Passing through Two Points and having Tangent at One of These Points

Example 7 Find the equation of the circle passing through (1, 2) and touching the line $x-2y+4=0$ at (2, 3).

Solution: Let the equation of circle be:

$$x^2+y^2+2gx+2fy+c=0 \qquad \ldots(A)$$

Through (1, 2):

$$1^2+2^2+2g(1)+2f(2)+c=0$$

$$2g+4f+c+5=0 \qquad \ldots(1)$$

Through (2, 3):

$$2^2+3^2+2g(2)+2f(3)+c=0$$

$$4g+6f+c+13=0 \qquad \ldots(2)$$

Given line $x-2y+4=0$ has slope $\frac{1}{2}\cdot$ The normal through (2, 3) has slope –2.

$\Rightarrow$ equation of normal line is $y-3=-2(x-2) \Rightarrow y-3=-2x+4$

$\Rightarrow$ $2x+y-7=0$ is the equation of the normal line.

This line being a normal through (2, 3) passes through the centre $(-g, -f)$ of the circle.
Therefore:

$$2(-g)+(-f)-7=0 \Rightarrow 2g+f=-7 \qquad \ldots(3)$$

Subtracting (2) from (1):

$$[2g+4f+c+5]-[4g+6f+c+13]=0$$

$$-2g-2f-8=0 \Rightarrow g+f=-4$$

$$\Rightarrow f=-4-g \qquad \ldots(4)$$

Subtracting value of f in (3)

$$2g+(-4-g)=-7 \Rightarrow 2g-4-g=-7 \Rightarrow \boxed{g=-3}$$

Then, from (4) $f=-4-(-3)=-4+3 \Rightarrow \boxed{f=-1}$

Put values of f and g in (1)

$$2(-3)+4(-1)+c+5=0 \Rightarrow c=5$$

Thus, the required substiting the values of f, gernd c is equation (A)

$x^2+y^2+2(-3)x+2+(-1)+\Leftarrow 0$ equation of circle is: $x^2+y^2-6x-2y+5=0$.

Case 4: A Circle Passing through Two Points and Touching a Given Line

Example 8 Find the equations of the circles passing through the points $A(3, 4)$ and $B(3, -4)$ and touching the line $x+2y+9=0$.

Solution: Let $O(h, k)$ be the centre of the required circle.

Since $\overline{OA}=\overline{OB} \Rightarrow \left|\overline{OA}\right|^2=\left|\overline{OB}\right|^2$, we have

$$(h-3)^2+(k-4)^2=(h-3)^2+(k+4)^2$$

$$(k-4)^2=(k+4)^2$$

$$k^2-8k+16=k^2+8k+16 \Rightarrow -8k=8k \Rightarrow 16k=0 \Rightarrow k=0$$

Thus, the centre is $O(h, 0)$. The radius is

$$r=\left|\overline{OA}\right|=\sqrt{(h-3)^2+(0-4)^2}=\sqrt{(h-3)^2+16}$$

The distance from the centre $(h, 0)$ to the line $x+2y+9=0$ is

$$\frac{|h+2(0)+9|}{\sqrt{1^2+2^2}}=\frac{|h+9|}{\sqrt{5}}$$

Since the circle touches the line. This distance equals to the radius:

$$\frac{|h+9|}{\sqrt{5}}=\sqrt{(h-3)^2+16}$$

Squaring both sides:

$$\frac{(h+9)^2}{5}=(h-3)^2+16$$

$$(h+9)^2=5[(h-3)^2+16]$$

$$h^2+18h+81=5(h^2-6h+9+16)$$

$$h^2+18h+81=5(h^2-6h+25)$$

$$h^2+18h+81=5h^2-30h+125$$

$$0=5h^2-30h+125-h^2-18h-81$$

$$0=4h^2-48h+44$$

$$h^2-12h+11=0$$

$$h^2-11h-h+11=0$$

$$h(h-11)-1(h-11)=0$$

$\Rightarrow (h-11)(h-1)=0$

$\Rightarrow h=1, h=11$

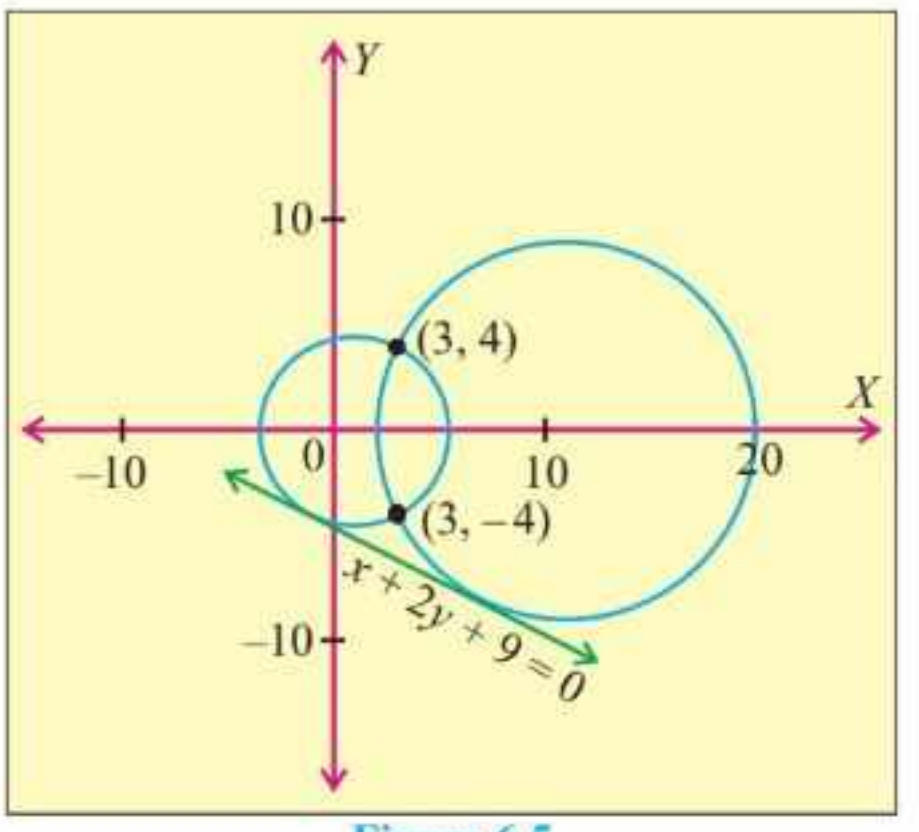


Figure 6.5

Thus, the centres of two circles are at (1, 0) and (11, 0)

Radius of circle with centre (1, 0):

$$r=\sqrt{(1-3)^2+(0-4)^2}=\sqrt{(-2)^2+16}=\sqrt{4+16}=\sqrt{20}$$

Radius of circle with centre (11, 0):

$$r=\sqrt{(11-3)^2+(0-4)^2}=\sqrt{(8)^2+16}=\sqrt{64+16}=\sqrt{80}$$

Equation of circle with centre at (1, 0) and radius $=\sqrt{20}$

$$(x-1)^2+(y-0)^2=(\sqrt{20})^2$$
$$(x-1)^2+y^2=20$$

Equation of circle with centre at (11, 0) and radius $=\sqrt{80}$

$$(x-11)^2+(y-0)^2=(\sqrt{80})^2$$

or $$(x-11)^2+y^2=80$$

Example 9 Find the equation of the circle which touches the y-axis at a distance of 3 units from the origin and cuts off an intercept of 8 units on the x-axis.

Solution: Let the point of tangency be (0, 3). Since the circle touches the y-axis at this point, the centre lies on the horizontal line through (0, 3), that is $y=3$, and its x-coordinate is $\pm r$ where r is radius.

Take the centre to the right of the y-axis:

Centre $C=(r, 3)$, radius r

Equation of circle:

$$(x-r)^2+(y-3)^2=r^2$$
$$(x^2-2rx+r^2)+(y^2-6y+9)=r^2$$
$$x^2+y^2-2rx-6y+9=0$$

For intercept on the x-axis, set $y=0$:

$$x^2-2rx+9=0$$

Let the roots be x_1, x_2. Then:

$$x_1+x_2=2r,\ x_1\cdot x_2=9$$

Intercept length $=|x_1-x_2|(\ )$

$$=\sqrt{(x_1+x_2)^2-4x_1x_2}=\sqrt{4r^2-36}$$

Given intercept = 8

$$\sqrt{4r^2-36}=8\ \Rightarrow\ 4r^2-36=64$$

$$\Rightarrow 4r^2=100\Rightarrow r^2=25\Rightarrow r=5$$

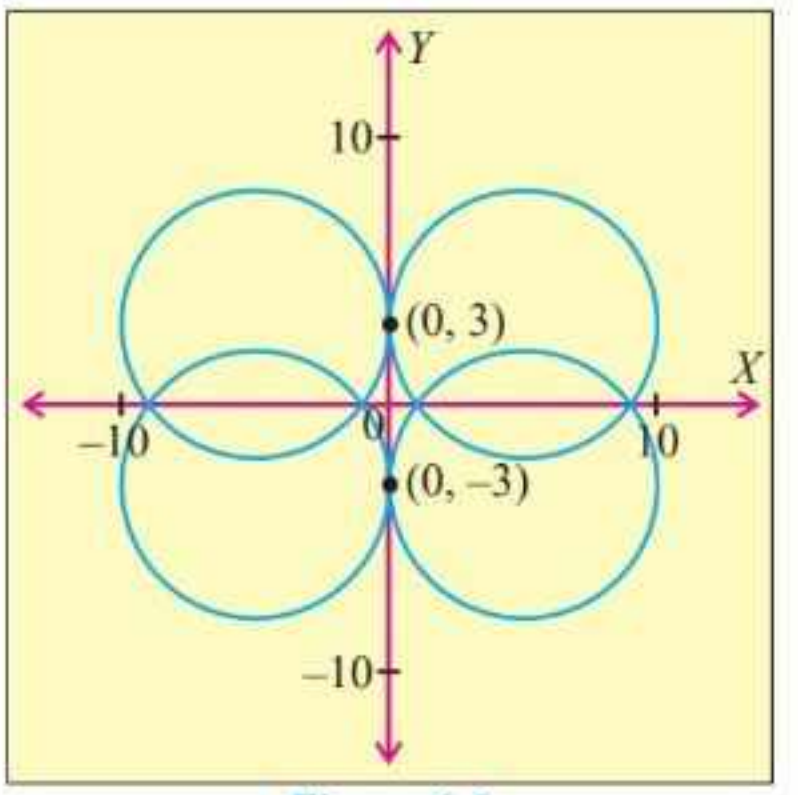


Figure 6.6

Thus, one circle with centre (5, 3) and radius 5 is:

$$(x-5)^2+(y-3)^2=25$$

By symmetry (tangency at (0, –3) and centre on left side of y-axis), we obtain four circles:

$$(x\pm 5)^2+(y\pm 3)^2=25 \text{ where } \pm \text{ signs are independent.}$$

EXERCISE 6.1

1. In each of the following, find an equation of the circle with
   (a) centre at (5, 7) and radius 2
   (b) centre at $(a\cos\theta, a\sin\theta)$ and radius a
   (c) ends of diameter (3, 4) and (2, –7).
2. Express each of the following, in the standard form and hence find its centre and radius:
   (a) $2x^2 + 2y^2 - 4x - 12y + 11 = 0$ (b) $x^2 + y^2 - 2ax + 2by - 2ab = 0$
3. Find an equation of circle passing through the points (2, 8), (5, –1) and (–3, 3), also find the centre and radius.
4. Find the equation of a circle which passes through the origin and cuts off intercepts 2 and 3 on the x-axis and y-axis respectively.
5. Find the equation of a circle passing through the points (1, 3), (2, –1) and having its centre on the line $x + y = 3$. Does the point (4, 1) lie on the circle?
6. Find the equation of the circle passing through the point (–3, – 4) and touching the line $3x - y - 1 = 0$ at the point (1, 2).
7. Find an equation of a circle of radius a lying in the second quadrant and tangent to both axes.
8. Find the equation of the circle which touches the x-axis at the point (5, 0) and passes through another point (3, 2).
9. Find the equation of the circle passing through the point of intersection of the lines $x - y - 5 = 0$, $2x + y - 4 = 0$ and it has its centre at (5, 0).
10. Find the equation of the circle touching the y-axis and also the straight line $3x - 4y = 0$ at the point (8, 6).
11. Find the equations of the circles passing through the points $A(3, 2)$ and $B(3, -2)$ and touching the line $x + 2y + 3 = 0$.
12. Find the equations of the circles of radius 5 that are tangent to the line $4x + 3y - 10 = 0$ at the point (1, 2).

6.2 Tangent and Normal to a Curve

If a line touches a curve at exactly one point, it is called a **tangent** to the curve, and the point is called the point of contact. In Figure 6.7, $\overline{LM}$ is a tangent to the curve at the point P.

The line perpendicular to the tangent at its point of contact is called the normal to the curve at that point. For example, in the Figure 6.7, $\overrightarrow{PQ} \perp \overrightarrow{LM}$ at the point P; thus, $\overrightarrow{PQ}$ is the normal to the curve at P.

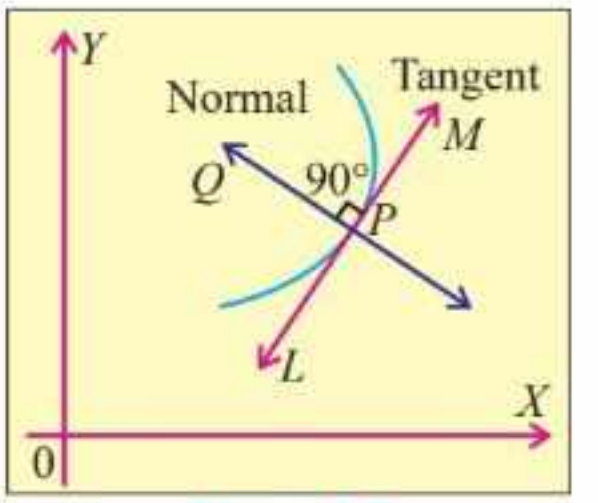


Figure 6.7

6.2.1 Equation of Tangent to a Circle

(a) Circle centered at origin: $x^2 + y^2 = r^2$

Let $P(x_1, y_1)$ be a point on the circle $x^2 + y^2 = r^2$ ...(i)

Differentiating both sides of the equation with respect of x, we have:

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

The slope of the tangent at $P(x_1, y_1)$ is the value of $\frac{dy}{dx}$ at this point:

$$m_T = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{x_1}{y_1}$$

Therefore, the equation of the tangent at $P(x_1, y_1)$ is: $y - y_1 = -\frac{x_1}{y_1}(x - x_1),\ y_1 \neq 0$

$$xx_1 + yy_1 = x_1^2 + y_1^2 \qquad \ldots(ii)$$

Since $P(x_1, y_1)$ lies on the circle $x^2 + y^2 = r^2$, we have

$$x_1^2 + y_1^2 = r^2 \qquad \ldots(iii)$$

From equations (ii) and (iii), we obtain the required equati on of the tangent:

$$xx_1 + yy_1 = r^2$$

(b) General circle: $x^2 + y^2 + 2gx + 2fy + c = 0$

Let $P(x_1, y_1)$ be a point on the circle

Differentiating both sides with respect to x:

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} + 2f\frac{dy}{dx} = -2x - 2g$$

$$2(y + f)\frac{dy}{dx} = -2(x + g) \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x + g}{y + f}$$

The slope of the tangent at $P(x_1, y_1)$ is:

$$m_T = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{x_1 + g}{y_1 + f}$$

Hence, the equation of the tangent is:

$$y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$

Multiplying both sides by $(y_1 + f)$:

$$(y - y_1)(y_1 + f) = -(x_1 + g)(x - x_1)$$

$$yy_1 + fy - y_1^2 - fy_1 = -xx_1 + x_1^2 - gx + gx_1$$

$$xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1 \text{ (Rearranging)}$$

Adding both sides $gx_1 + fy_1$:

$$xx_1 + yy_1 + gx + gx_1 + fy + fy_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 \quad \ldots(3)$$

Since $P(x_1, y_1)$ lies on the circle, we have:

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

$$\Rightarrow \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 = -c \quad \ldots(4)$$

Substituting equation (4) into equation (3) and rearranging gives the required equation:

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

6.2.2 Rule for Writing the Equation of a Tangent to a Circle

We have proved above that the equation of the tangent to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) is:

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

This can be obtained from the equation of the circle by making the following substitutions:

(i) $x^2 \to xx_1$ (ii) $y^2 \to yy_1$

(iii) $x \to \frac{x + x_1}{2}$ (iv) $y \to \frac{y + y_1}{2}$

(v) The constant term c remains unchanged.

Note

The rule holds for all second-degree curves (conics) as well.

6.2.3 Equation of Normal to a Circle at P(x_1, y_1)

The tangent to the circle $x^2 + y^2 = r^2$ at $P(x_1, y_1)$ is $xx_1 + yy_1 = r^2$. The normal is

$xy_1 - yx_1 = 0$ or $\frac{x}{x_1} = \frac{y}{y_1}$.

Example 10 Find the equation of the tangent and normal to the circle $x^2 + y^2 - 4x + 6y + 12 = 0$ at the point on the circle whose ordinate is -2.

Solution: The equation of the circle is:

$$x^2 + y^2 - 4x + 6y + 12 = 0$$

Here, $y = -2$, Substituting into the given equation of circle.

$$x^2 + (-2)^2 - 4x + 6(-2) + 12 = 0 \Rightarrow x^2 + 4 - 4x - 12 + 12 = 0$$

$$\Rightarrow \quad x^2 - 4x + 4 = 0 \Rightarrow (x - 2)^2 = 0 \Rightarrow x = \pm 2$$

Note

The centre of the circle, (0, 0), satisfies the equation of the normal, confirming that the normal to a circle always passes through its centre.

Thus, point of contact is $P(2, -2)$. The equation of the tangent at (x_1, y_1) is:

$$xx_1 + yy_1 - 2(x + x_1) + 3(y + y_1) + 12 = 0$$

Substituting $x_1 = 2$ and $y_1 = -2 \Rightarrow 2x + (-2)y - 2(x + 2) + 3(y - 2) + 12 = 0$

$$2x - 2y - 2x - 4 + 3y - 6 + 12 = 0 \Rightarrow y + 2 = 0$$

So, the equation of the tangent is $y = -2$ and the equation of the normal is $x = 2$.

6.2.4 Position of a Point with Respect to a Circle

Theorem 1: The point $P(x_1, y_1)$ lies outside, on, or inside the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \gtreqless 0$

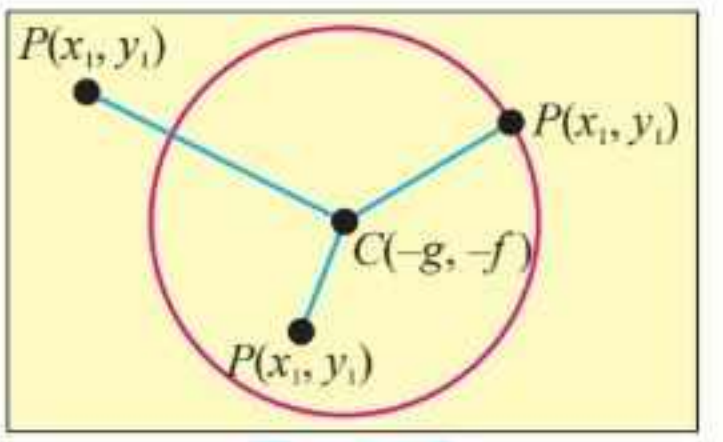


Figure 6.8

Proof: The centre of the circle is $C(-g, -f)$ and radius is $r = \sqrt{g^2 + f^2 - c}$. The point $P(x_1, y_1)$ lies outside, on, or inside the circle according as

$|CP| > r$, $|CP| = r$, or

$CP < r$, that is $|CP| \gtreqless r$

that is, according as:

$$\sqrt{(x_1 + g)^2 + (y_1 + f)^2} \gtreqless \sqrt{g^2 + f^2 - c}$$

Squaring both sides (since all quantities are non-negative):

$$(x_1 + g)^2 + (y_1 + f)^2 \gtreqless g^2 + f^2 - c$$

$$x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 \gtreqless g^2 + f^2 - c$$

Cancelling g^2 and f^2:

$$x_1^2 + 2gx_1 + y_1^2 + 2fy_1 \gtreqless -c$$

Thus: $$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \gtreqless 0$$

Example 11 Determine whether the point $P(2, 3)$ lies outside, on, or inside the circle:

$$x^2 + y^2 + 2x - 3y - 10 = 0$$

Solution: Substituting $x = 2$ and $y = 3$ into the left-hand side of the circle's equation:

$$(2)^2 + (3)^2 + 2(2) - 3(3) - 10 = 4 + 9 + 4 - 9 - 10 = -2 < 0$$

Thus, the point $P(2, 3)$ lies inside the circle.

6.2.5 Intersection of a Line and a Circle

Let the equation of the line be

$$y = mx + c \qquad \ldots(1)$$

and the equation of the circle be

$$x^2 + y^2 = a^2 \qquad \ldots(2)$$

The coordinates of the points of intersection are the simultaneous solutions of (1) and (2). Substituting the value of y from (1) into (2):

$$x^2 + (mx + c)^2 = a^2$$
$$x^2 + m^2x^2 + 2mxc + c^2 - a^2 = 0$$
$$x^2(1 + m^2) + 2mcx + (c^2 - a^2) = 0 \qquad \ldots(3)$$

Equation (3) is a quadratic in x and gives two values of x say x_1 and x_2. Corresponding to these values of x_1 and x_2, we get values of y as y_1 and y_2. Thus, the line $y = mx + c$ intersects the circle $x^2 + y^2 = a^2$ in at the most two points (x_1, y_1) and (x_2, y_2).

For the nature of the points of the intersection, we examine the discriminant, D, of (3).

$$D = (2mc)^2 - 4(1 + m^2)(c^2 - a^2) = 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2)$$
$$= 4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4m^2a^2 = -4c^2 + 4a^2 + 4m^2a^2$$
$$= 4a^2(1 + m^2) - 4c^2 = = 4[a^2(1 + m^2) - c^2]$$

Case-I: Two distinct real points

The two points of intersection of the line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ are real and distinct, if the discriminant > 0

that is, if $4[a^2(1 + m^2) - c^2] > 0 \quad \Rightarrow \quad a^2(1 + m^2) - c^2 > 0 \quad \Rightarrow \quad a^2(1 + m^2) > c^2$

Case-II: One real point (tangent)

The line $y = mx + c$ touch the circle $x^2 + y^2 = a^2$ at only one real point, that is, case of coincident points if the discriminant $= 0$

that is, if $4[a^2(1 + m^2) - c^2] = 0 \quad \Rightarrow \quad a^2(1 + m^2) - c^2 = 0 \quad \Rightarrow \quad a^2(1 + m^2) = c^2$

Case-III: No real intersection

The line $y = mx + c$ cuts the circle $x^2 + y^2 = a^2$ at two imaginary points (that is, does not cut the circle) if the discriminant < 0

that is, if $4\left[a^2(1+m^2) - c^2\right] < 0 \quad \Rightarrow \quad a^2(1+m^2) - c^2 < 0 \quad \Rightarrow \quad a^2(1+m^2) < c^2$

Corollary-1: Condition of Tangency of a Line to a Circle

From Case II, the line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ if:

$$|c| = a\sqrt{1+m^2} \quad \text{or} \quad c = \pm a\sqrt{1+m^2}$$

Therefore, the family of tangents to the circle is: $\boxed{y = mx \pm a\sqrt{1+m^2}}$

Note that $\dfrac{|c|}{\sqrt{1+m^2}}$ is the perpendicular distance from the centre (0, 0) to the line.

Hence, the condition for tangency is that the distance $\dfrac{|c|}{\sqrt{1+m^2}}$ equals the radius a.

Corollary 2: Coordinates of Point of Contact

For the tangent $y = mx + a\sqrt{1+m^2}$, the quadratic equation (3) has equal roots given by:

$$x = \frac{-2mc}{2(1+m^2)} = \frac{-mc}{1+m^2}$$

$$= \frac{-m}{1+m^2}\left(a\sqrt{1+m^2}\right) = \frac{-am}{\sqrt{1+m^2}}$$

Substituting into the equation $y = mx + a\sqrt{1+m^2}$

$$y = m\left(\frac{-am}{\sqrt{1+m^2}}\right) + a\sqrt{1+m^2} = \frac{-am^2}{\sqrt{1+m^2}} + a\sqrt{1+m^2} = \frac{-am^2 + a(1+m^2)}{\sqrt{1+m^2}} = \frac{a}{\sqrt{1+m^2}}$$

Thus, the point of contact is $\left(\frac{-am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$

Similarly, for the tangent $y = mx - a\sqrt{1+m^2}$, the point of contact $\left(\frac{am}{\sqrt{1+m^2}}, \frac{-a}{\sqrt{1+m^2}}\right)$.

6.2.6 Tangents from a Point to a Circle

Theorem 2: Two tangents can be drawn to a circle from any point $P(x_1, y_1)$. The tangents are **real and distinct**, **coincident**, or **imaginary** according as the point lies **outside**, **on**, or **inside** the circle.

Proof: Consider the circle $x^2 + y^2 = a^2$. The line $y = mx + a\sqrt{1+m^2}$ is a tangent for all m. If it passes through $P(x_1, y_1)$, then:

$$y_1 = mx_1 + a\sqrt{1+m^2} \quad \text{or} \quad y_1 - mx_1 = a\sqrt{1+m^2}$$

Squaring both sides:

$$(y_1 - mx_1)^2 = a^2(1+m^2) \Rightarrow y_1^2 - 2mx_1y_1 + m^2x_1^2 = a^2 + a^2m^2$$

or $(x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 - a^2) = 0$

This is a quadratic in m. Its discriminant, D, is:

$$D = (-2x_1y_1)^2 - 4(x_1^2 - a^2)(y_1^2 - a^2)$$

$$= 4x_1^2y_1^2 - 4(x_1^2y_1^2 - a^2x_1^2 - a^2y_1^2 + a^4) = 4a^2(x_1^2 + y_1^2 - a^2)$$

The two values of m are:

- **Real and distinct** if $D > 0 \Rightarrow x_1^2 + y_1^2 > a^2$ (point lies outside the circle)
- **Real and coincident** if $D = 0 \Rightarrow x_1^2 + y_1^2 = a^2$ (point lies on the circle)
- **Imaginary** if $D < 0 \Rightarrow x_1^2 + y_1^2 < a^2$ (point lies inside the circle).

Example 12 Tangents are drawn from (4, 5) to the circle $x^2+y^2=29$. Find an equation of the line joining the points of contact (The line is called the chord of contact).

Solution: Let the points of contact of the two tangents be $P(x_1, y_1)$ and $Q(x_2, y_2)$.

The equation of the tangent at P is $xx_1+yy_1=29$ and at Q is $xx_2+yy_2=29$. Since both tangents pass through (4, 5): $4x_1+5y_1=29$ and $4x_2+5y_2=29$.

Thus, both $P(x_1, y_1)$ and $Q(x_2, y_2)$ satisfy the equation $4x+5y=29$. Hence, the equation of the chord of contact is: $4x+5y=29$.

Example 13 Find a combined equation of the pair of tangents drawn from (5, 0) to the circle $x^2+y^2=9$.

Solution: Let $P(h,k)$ be any point on either of the two tangents drawn from $A(5,\ 0)$ to the given circle. The equation of line AP is:

$$y-0=\frac{k-0}{h-5}(x-5) \Rightarrow y=\frac{k}{h-5}(x-5) \Rightarrow y(h-5)=k(x-5)$$
$$\Rightarrow kx-(h-5)y-5k=0$$

This line is tangent to the circle $x^2+y^2=9$. The perpendicular distance from the centre (0, 0) to the line must equal the radius 3: $\dfrac{|5k|}{\sqrt{k^2+(h-5)^2}}=3$

Squaring both sides: $25k^2=9[k^2+(h-5)^2] \Rightarrow 25k^2=9k^2+9(h-5)^2$

$$9(h-5)^2-16k^2=0$$

Since (h, k) is any point on the two tangents, the combined equation of the pair of tangents is:

$$9(x-5)^2-16y^2=0$$

Example 14 Find the equations of the pair of tangents drawn from $(4, -1)$ to the circle $x^2+y^2-2x+4y=0$. Also find the points of contact.

Solution: Given: $x^2+y^2-2x+4y=0 \quad \ldots(1)$

The circle has centre $(1, -2)$ and radius $\sqrt{(-1)^2+(2)^2}=\sqrt{5}$. The equation of a line through $(4, -1)$ with slope m is: $y+1=m(x-4) \Rightarrow mx-y-4m-1=0$

The perpendicular distance from the centre $(1, -2)$ to this line must equal the radius:

$$\frac{|m(1)-(-2)-4m-1|}{\sqrt{m^2+(-1)^2}}=\sqrt{5} \Rightarrow \frac{|m+2-4m-1|}{\sqrt{m^2+1}}=\sqrt{5} \Rightarrow \frac{|-3m+1|}{\sqrt{m^2+1}}=\sqrt{5}$$

Squaring both sides:

$$\Rightarrow (3m-1)^2=5(m^2+1) \Rightarrow 9m^2-6m+1=5m^2+5 \Rightarrow 4m^2-6m-4=0$$
$$\Rightarrow 2m^2-3m-2=0 \Rightarrow 2m^2-4m+m-2=0 \Rightarrow 2m(m-2)+1(m-2)=0$$

$\Rightarrow (m-2)(2m+1)=0 \Rightarrow m=2,\ m=-\frac{1}{2}$

Substituting these slopes into the line equation gives the tangents:

For $m=2$: $2x-y-8-1=0 \Rightarrow 2x-y-9=0$ ...(2)

For $m=-\frac{1}{2}$: $-\frac{1}{2}x-y-4\left(-\frac{1}{2}\right)-1=0$

$\Rightarrow -\frac{1}{2}x-y+1=0 \Rightarrow x+2y-2=0$ ...(3)

Now we proceed to find the points of contact.

From (2) $y=2x-9$ putting in (1), we have

$$x^2+(2x-9)^2-2x+4(2x-9)=0$$
$$x^2+4x^2-36x+81-2x+8x-36=0$$
$$5x^2-30x+45=0$$
$$x^2-6x+9=0 \Rightarrow (x-3)^2=0 \Rightarrow x-3=0 \Rightarrow x=3$$

Putting $x=3$ in $y=2x-9$, we have $y=2(3)-9=-3$

Therefore, one point of contact is $(3,-3)$

From (3) $y=\frac{2-x}{2}$ putting in (1), we have $x^2+\left(\frac{2-x}{2}\right)^2-2x+4\left(\frac{2-x}{2}\right)=0$

$$x^2+\frac{4-4x+x^2}{4}-2x+4-2x=0$$
$$x^2+\frac{x^2-4x+4}{4}-4x+4=0$$
$$4x^2+x^2-4x+4-16x+16=0$$
$$4x^2+x^2-4x+4-16x+16=0$$
$$5x^2-20x+20=0$$
$$x^2-4x+4=0$$
$$(x-2)^2=0 \Rightarrow x-2=0 \Rightarrow x=2$$

Putting $x=2$ in $y=\frac{2-x}{2}$, we have $y=\frac{2-2}{2}=0$

Therefore, other point of contact is $(2,0)$.

6.2.7 Length of the Tangent from a Point to a Circle

Theorem 3: The length of the tangent drawn from a point $P(x_1, y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$.

Proof: Let the centre of the circle be $C(-g, -f)$ and its radius be $r = \sqrt{g^2 + f^2 - c}$. Let PT be a tangent from P to the circle, touching it at T. Then, $CT \perp PT$. In right triangle PTC:

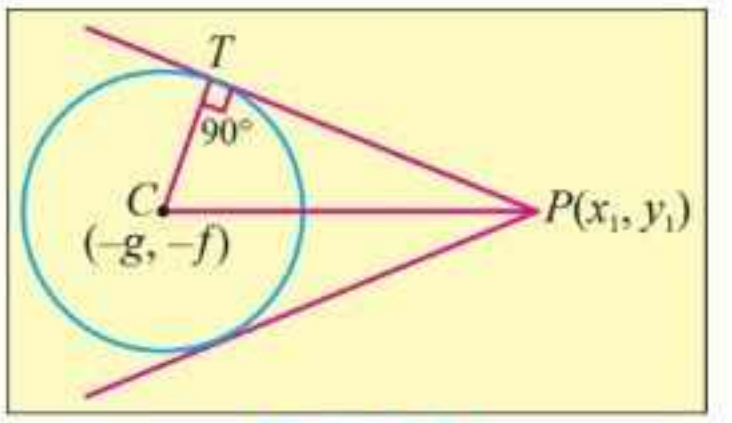


Figure 6.9

$$|PT|^2 = |PC|^2 - |CT|^2$$

Where, $|PC|^2 = (x_1 + g)^2 + (y_1 + f)^2$

$$|CT|^2 = r^2 = g^2 + f^2 - c$$

Therefore, $|PT|^2 = (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$

$$= x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - g^2 - f^2 + c$$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

Hence, $|PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

6.2.8 Rule to Find the Length of the Tangent

(i) Write the equation of the circle with coefficients of x^2 and y^2 as 1.
(ii) Make the right-hand side of the equation zero.
(iii) Substitute the coordinates of the point into the left-hand side expression.
(iv) Take the square root of the result.

Example 15 Find the length of the tangent from point $P(2, -5)$ to the circle $5x^2 + 5y^2 + 10x + 14y - 15 = 0$.

Solution: Divide the equation by 5 to get the standard form:

$$x^2 + y^2 + 2x + \frac{14}{5}y - 3 = 0$$

The length of the tangent is:

$$\sqrt{(2)^2 + (-5)^2 + 2(2) + \frac{14}{5}(-5) - 3} = \sqrt{4 + 25 + 4 - 14 - 3} = \sqrt{16} = 4$$

Example 16 Length of a Chord of a Circle

Find the length of intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$.

Solution: The circle is $x^2 + y^2 = a^2$ and the line is $y = mx + c$.

Let the intersection points be $P(x_1, y_1)$ and $Q(x_2, y_2)$. Substituting the line into the circle:

$$x^2 + (mx + c)^2 = a^2$$

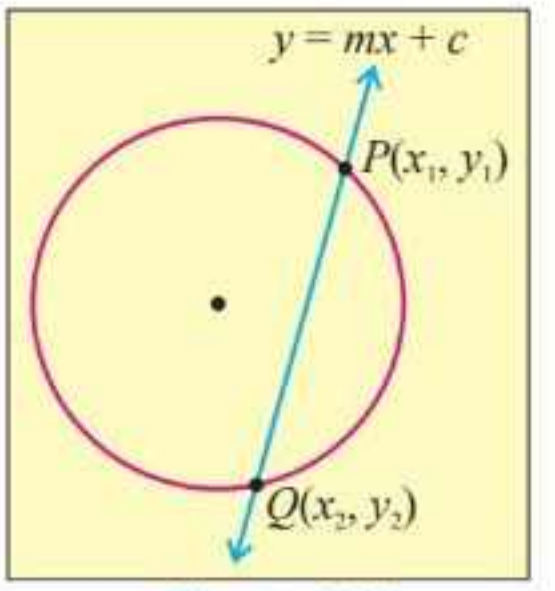


Figure 6.10

$$x^2 + m^2x^2 + 2mcx + c^2 = a^2$$

$$(1 + m^2)x^2 + 2mcx + (c^2 - a^2) = 0$$

The roots x_1 and x_2 satisfy:

$$x_1 + x_2 = -\frac{2mc}{1+m^2},\ x_1 x_2 = \frac{c^2 - a^2}{1+m^2}$$

Since the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the line,

$$y_1 = mx_1 + c \text{ and } y_2 = mx_2 + c, \text{ so } y_1 - y_2 = m(x_1 - x_2)$$

The length of the chord PQ is:

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2 + m^2(x_1 - x_2)^2}$$

$$= \sqrt{(1+m^2)(x_1 - x_2)^2} = \sqrt{1+m^2}\ \sqrt{(x_1 + x_2)^2 - 4x_1x_2}$$

$\because$ Substituting $x_1 - x_2$ and $x_1 x_2$

$$= \sqrt{1+m^2}\ \sqrt{\frac{4m^2c^2}{(1+m^2)^2} - \frac{4(c^2 - a^2)}{1+m^2}}$$

$$= \sqrt{1+m^2}\ \sqrt{\frac{4m^2c^2 - 4(1+m^2)(c^2 - a^2)}{(1+m^2)^2}}$$

$$= 2\sqrt{1+m^2}\ \sqrt{\frac{m^2c^2 - (c^2 - a^2 + m^2c^2 - m^2a^2)}{(1+m^2)^2}}$$

$$= 2\sqrt{1+m^2}\ \frac{\sqrt{m2c2 - c2 + a2 - m2c2 + m2a2}}{1 + \text{m2}}$$

Hence, the length of the chord is: $PQ = \frac{2}{\sqrt{1+m2}}\sqrt{a2(1+m2)c2}$

Note

In numerical problems, it is often more convenient to find the points of intersection directly and then compute the distance between them.

EXERCISE 6.2

1. Find the equations of the tangent and normal to the circles
   (a) $x^2 + y^2 = 8$ at $(2, 2)$ and at $(2\sqrt{2}\cos\theta, 2\sqrt{2}\sin\theta)$.
   (b) $x^2 + y^2 - 6x - 2y - 7 = 0$ at $(4, 5)$
2. Show that the line $4x + 3y - 25 = 0$ is tangent to the circle $x^2 + y^2 = 25$ and find the point of contact.
3. Find the equations of tangents to the circle $x^2 + y^2 = 4$,
   (a) which are inclined at an angle of $45°$ to the axis of x.
   (b) which are parallel to the line $3x + 4y + 1 = 0$.
   (c) which are perpendicular to the line $6x - 8y + 3 = 0$.

4. Find the equations of the tangents to the circle $x^2+y^2-4x-2y-4=0$, which are parallel to the line $3x-4y+2=0$.
5. Find the equations of the tangents drawn from:
   (a) the point $(-3, 7)$ to the circle $x^2+y^2=8$.
   (b) the point $(12, -4)$ to the circle $x^2+y^2-4x-2y-20=0$.
   Also find the points of contact.
6. A circle has the parametric equations: $x=3+5\cos\theta, y=-1+5\sin\theta$
   (a) Find the Cartesian equation of the circle.
   (b) Find the centre and radius of the circle.
   (c) Find the parametric coordinates of the point P on the circle corresponding to $\theta=\frac{\pi}{2}$.
   (d) Find the equation of the tangent to the circle at point P.
   (e) Find the equation of the normal to the circle at point P.
   (f) Find the equation of the tangent to the circle at the point where $\theta=\pi$.

6.3 Parabola

A **Parabola** is the locus of a point which moves in a plane such that its distance from a fixed point, called the **focus**, is equal to its distance from a given fixed straight line, called the **directrix**.

6.3.1 Standard Equation of the Parabola

Let F be the focus and CD, the directrix of the parabola. Draw FZ perpendicular from the focus F to the directrix CD. Let V be the mid-point of FZ and let $FZ=2a$. Take V as the origin, the x-axis along the line ZF and y-axis along the perpendicular to VF at V, as shown in the figure.

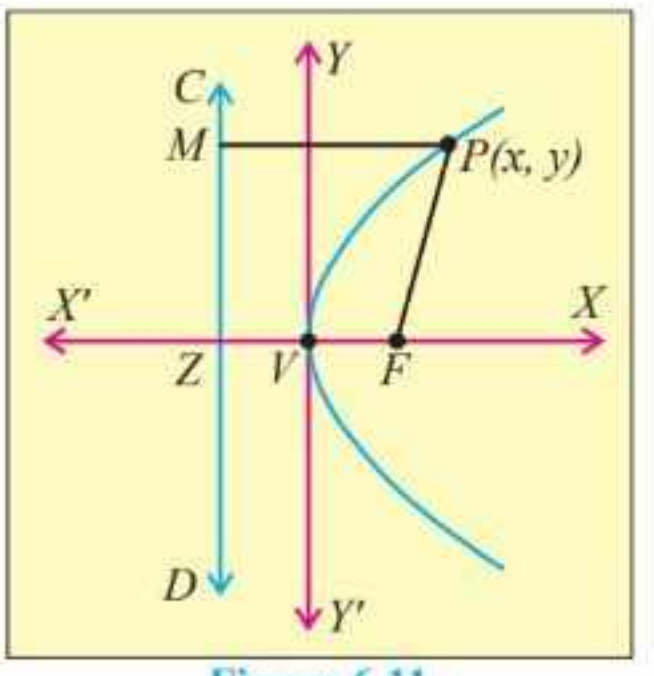


Figure 6.11

Since V is equidistant from the focus and the directrix, by definition, the point V lies on the parabola. The point V is called the vertex of the parabola.

Thus, the focus of the parabola is $F(a, 0)$ and the directrix is $x=-a$.

Let $P(x, y)$ be any point on the parabola and draw PM perpendicular to CD.

Therefore, $|PF|=\sqrt{(x-a)^2+(y-0)^2}=\sqrt{x^2-2ax+a^2+y^2}$

Distance of P from the directrix $x=-a$ is $PM=\frac{|x+a|}{\sqrt{1^2+0^2}}=|x+a|$

By the definition of a parabola, $|PF|=|PM|$

Therefore, $\sqrt{x^2-2ax+a^2+y^2}=|x+a|$

Squaring both sides, we get

$$x^2-2ax+a^2+y^2=(x+a)^2$$

$x^2-2ax+a^2+y^2=x^2+2ax+a^2$ which simplifies to $y^2=4ax$

This is the standard equation of the parabola.

6.3.2 Elements of the Parabola

1. **Axis:** The straight line through the focus and perpendicular to the directrix is called axis of the parabola. In the figure, the axis is x-axis.
2. **Vertex:** The point where the axis meets the parabola is called the vertex. In the figure, V is the vertex with coordinates $(0, 0)$.
3. **Chord and Focal Chord:** A line segment joining two distinct points on a parabola is called a chord. A chord passing through the focus is called a focal chord.
4. **Focal Distance:** The distance of a point on the parabola from the focus is called its focal distance. In the figure, $|FP|$ is the focal distance.
5. **Latus Rectum:** The focal chord perpendicular to the axis of the parabola is called the latus rectum. Its equation is $x = a$. It intersects the parabola $y^2=4ax$ at points where $y^2=4a^2$ that is, $y=\pm 2a$. Thus, the end points L and L' of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$.

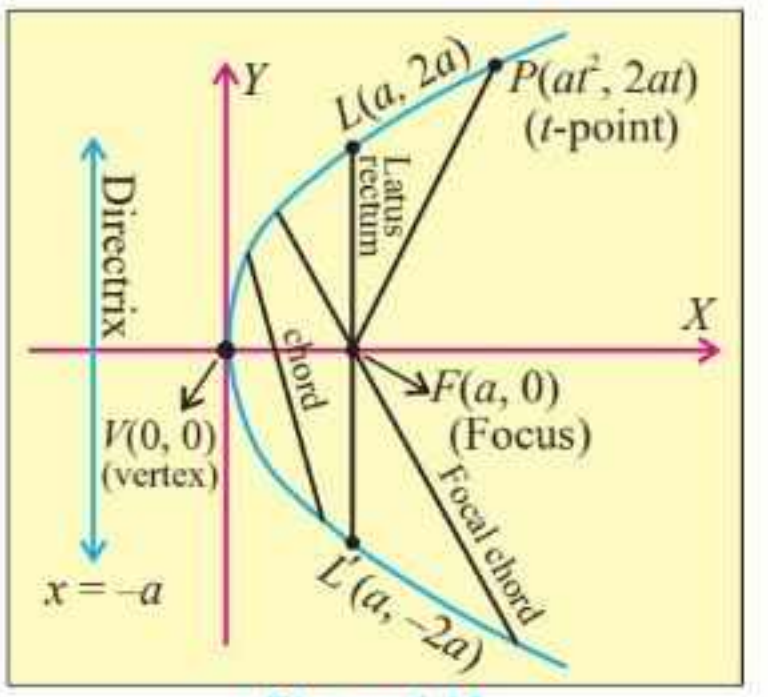


Figure 6.12

6. **Parametric Equations:** The equation $y^2=4ax$ can be expressed parametrically. Let

$$\frac{y}{2a}=t \text{ and } \frac{2x}{y}=t.$$

Then $y=2at$ and $2x=yt=(2at)t=2at^2$,

So, $x=at^2$

Hence, the parametric equations are

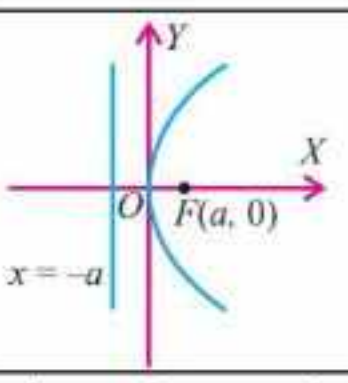
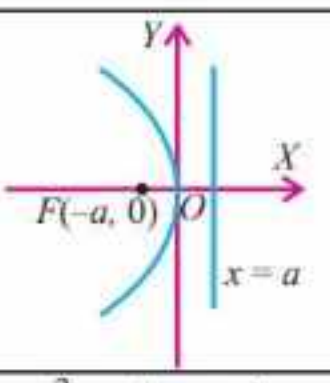
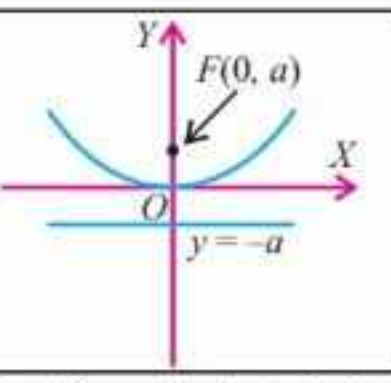
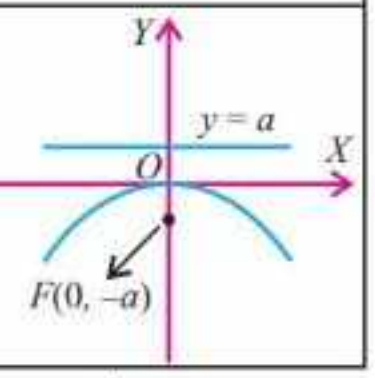
$$x=at^2,\ y=2at$$

The point $(at^2, 2at)$ lies on the parabola for all real t and is often referred to as the t-point.

6.3.3 Other Standard Forms of the Parabola

By taking the directrix on the right side of the focus, or below or above the focus, we obtain three more standard forms.

The table below summarizes the key features of all four forms.

Parabola in standard form	$y^2=4ax$	$y^2=-4ax$	$x^2=4ay$	$x^2=-4ay$
Focus	$(a,0)$	$(-a,0)$	$(0,a)$	$(0,-a)$
Vertex	$(0,\ 0)$	$(0,\ 0)$	$(0,\ 0)$	$(0,\ 0)$
Equation of directrix	$x=-a$	$x=a$	$y=-a$	$y=a$
Equation of axis	$y=0$	$y=0$	$x=0$	$x=0$
Equation of latus rectum	$x=a$	$x=-a$	$y=a$	$y=-a$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Sketch of parabola in standard form				

Example 17 Analyze the parabola $x^2-4x-4y-2=0$ and sketch its graph.

Solution: $x^2-4x-4y-2=0$ …(1)

Rewrite the equation as: $x^2-4x=4y+2$

Adding 4 on both sides, we have

$$x^2-4x+4=4y+6$$

$$(x-2)^2=4\left(y+\frac{3}{2}\right) \quad \ldots(2)$$

Let $x-2=X$ and $y+\frac{3}{2}=Y$. Then the equation (2) becomes $X^2=4Y$ …(3)

which is a parabola whose focus lies on $X=0$ and whose length of latus rectum is equal to $4a=4$, thus coordinates of the focus of (3) are:

$$X=0, Y=1, \text{ that is, } x-2=0 \text{ and } y+\frac{3}{2}=1 \quad \text{or} \quad x=2,\ y=-\frac{1}{2}$$

Thus, coordinates of the focus of the parabola (1) are $\left(2, -\frac{1}{2}\right)$

Axis of (3) is $X=0 \Rightarrow x-2=0$ $x=2$ is the axis of (1)

Vertex of (3) has coordinates

$$X=0, Y=0$$

that is, $x-2=0$, $y+\frac{3}{2}=0$

or $x=2$, $y=-\frac{3}{2}$ are coordinates of the vertex of (1).

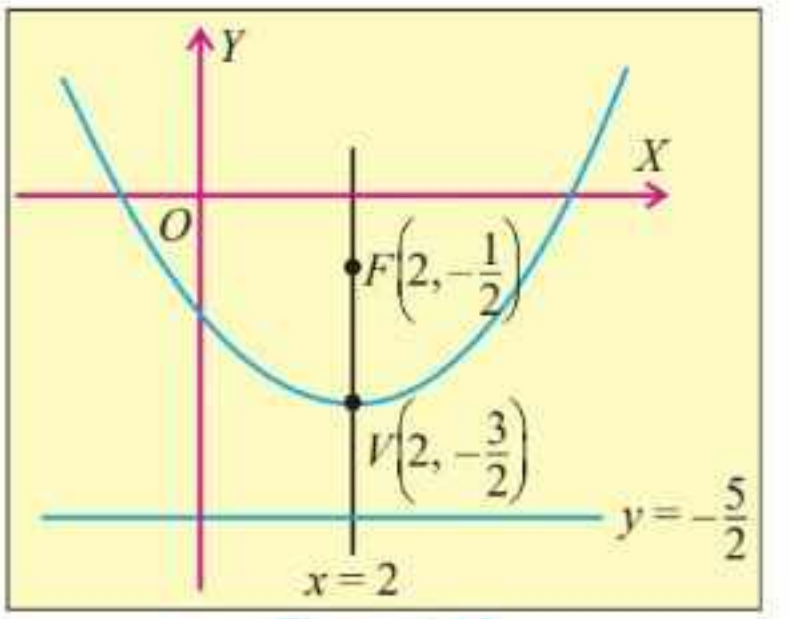


Figure 6.13

Equation of the directrix of (3) is:

$Y=-1$ that is, $y+\frac{3}{2}=-1$ or $y=-\frac{5}{2}$ is equation of the directrix of (1).

Length of the latus rectum of the parabola is 4.

The graph of (1) can easily be sketched and is as shown in the figure.

6.3.4 Tangents and Normals to a Parabola (Condition of the tangency to Parabola)

To find an equation of the tangent to a parabola, we first determine the slope of the tangent at the given point by calculating $\frac{dy}{dx}$ from the equation of the parabola. Using the point-slope form of a line, the equation of the tangent can then be written.

Since the normal to a curve at a point is perpendicular to the tangent at that point, its equation can be obtained readily from the equation of the tangent.

Let us consider the equation of a line

$$y = mx + c \qquad \ldots(1)$$

and the parabola

$$y^2 = 4ax \qquad \ldots(2)$$

To find their points of intersection, we solve equations (1) and (2) simultaneously. Substituting the value of y from (1) into (2), we obtain:

$$(mx + c)^2 = 4ax$$
$$m^2x^2 + 2mcx + c^2 = 4ax$$
$$m^2x^2 + 2mcx - 4ax + c^2 = 4ax$$
$$m^2x^2 + 2(mc - 2a)x + c^2 = 0 \qquad \ldots(3)$$

Line (1) is tangent to (3) if roofs of (3) are equal. This implies discriminant D, of (3) is zero.

$$[2(mc-2a)]^2-4(m^2)(c^2)=0$$
$$4(mc-2a)^2-4m^2c^2=0$$

Dividing both sides by 4, we get:

$$(mc-2a)^2-m^2c^2=0$$
$$m^2c^2-4amc+4a^2-m^2c^2=0$$

$$-4amc+4a^2=0 \Rightarrow 4amc=4a^2 \Rightarrow mc=a \Rightarrow c=\frac{a}{m}, m\neq 0$$

Thus, the required condition for the line $y = mx + c$ to be a tangent to the parabola $y^2=4ax$ is $c=\frac{a}{m}$.

Consequently, for any non-zero real number m, the line $y=mx+\frac{a}{m}$ is a tangent to the parabola $y^2=4ax$. Similarly, we can find condition of tangency for other parabolas

6.3.5 Similarly, we find condition of Tangency for other parabolas

Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point (x_1, y_1).

Solution: Differentiating $y^2 = 4ax$ with respect to x, we obtain

$$2y\frac{dy}{dx}=4a \Rightarrow \frac{dy}{dx}=\frac{2a}{y}$$

Thus, the slope of the tangent at (x_1, y_1) is $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{2a}{y_1}$

Using the point-slope form, the equation of the tangent at (x_1, y_1) is

$$y-y_1=\frac{2a}{y_1}(x-x_1).$$

Multiplying both sides by y_1, we get

$$yy_1-y_1^2=2ax-2ax_1.$$

Rearranging terms,

$$yy_1-2ax=y_1^2-2ax_1$$

Since the point (x_1, y_1) lies on the parabola, it satisfies $y_1^2=4ax_1$. Substituting this value, we have

$$yy_1-2ax=4ax_1-2ax_1=2ax_1.$$

Bringing all terms to one side,

$$yy_1-2ax-2ax_1=0.$$

Which simplifies to $yy_1 = 2a(x + x_1)$.

This is the required equation of the tangent.

The slope of the normal is the negative reciprocal of the slope of the tangent. Hence,

$$\text{Slope of the normal} = -\frac{y_1}{2a}.$$

Using the point-slope form, the equation of the normal is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

EXERCISE 6.3

1. Find the focus, vertex and directrix of the following parabolas also sketch the graph:
   (i) $y^2 = 12x$ (ii) $x^2 = 8(y - 1)$
   (iii) $y^2 = -8(x - 3)$ (v) $x^2 - 4x - 8y + 4 = 0$
2. Write the equation of the parabola with given elements.
   (i) Focus $(-3, 2)$; directrix $x = 5$ (ii) Focus $(3, 5)$, vertex $(3, 2)$
   (iii) Directrix $x = 4$ vertex $(6, 3)$ (iv) Axis $y = 0$ through $(2, 2)$ and $(6, -3)$
3. Find the equation of the parabola having its focus at the origin and directrix is parallel to: (i) x-axis (ii) y-axis.
4. Show that the an equation of parabola with focus at $(a\cos\theta, a\sin\theta)$ and directrix $x\cos\theta + y\sin\theta + a = 0$ is $(x\sin\theta - y\cos\theta)^2 = 4a(x\cos\theta + y\sin\theta)^2$.
5. Prove that in parabola a circle described on latus rectum as diameter touches the directrix.
6. A parabola is defined by the equation $y = x^2 - 4x + 3$. Find the coordinates of its vertex, focus, and the equation of its directrix.
7. The parabola $y = ax^2 + bx + c$ has a vertex at $(3, -2)$ and passes through the point $(1, 6)$. Find the values of a, b, and c
8. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.
9. Find the equations of the tangent and normal to the parabola $x^2 - 6x - 2y + 11 = 0$ at the point $(5, 3)$.
10. Find the equation of the tangent to the parabola $y^2 = 12x$ which is parallel to the line $3x - y + 4 = 0$. Also find the point of tangency.

6.4 Ellipse

An **ellipse** is the set of all points in a plane for which the sum of distances from two fixed points, called the **foci** (plural of focus), is constant (see Figure 14).

Kepler's first law states that the orbits of the planets in the solar system are elliptical.

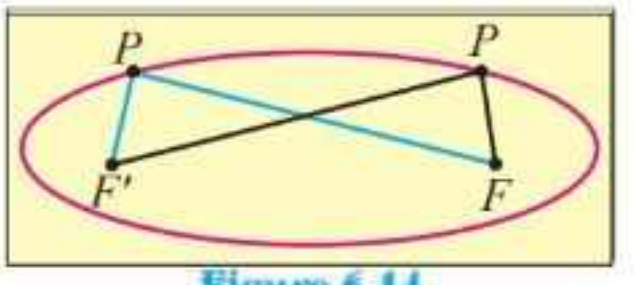


Figure 6.14

Derivation of the Standard Equation

To obtain the simplest equation for an ellipse, we place the foci on the x-axis at the points $F'(-c, 0)$ and $F(c, 0)$, so that the origin is the midpoint of the segment joining them (see Figure 17). Let the constant sum of distances from any point $P(x, y)$ on the ellipse to the foci be $2a$, where $a > 0$.

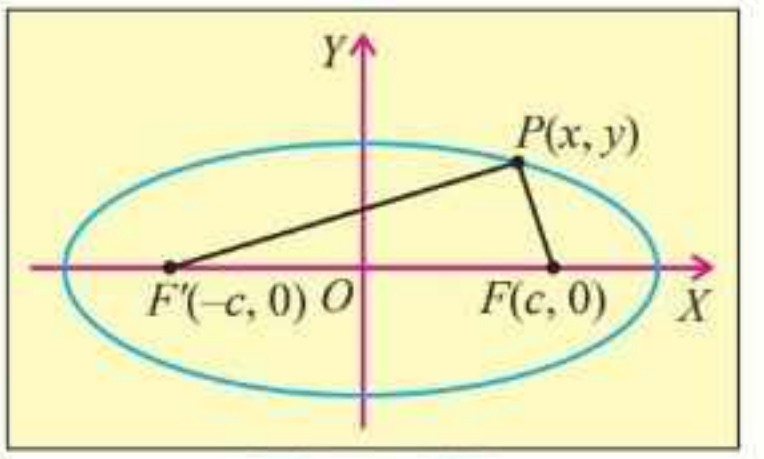


Figure 6.15

Thus, for any point $P(x, y)$ on the ellipse,

$$|PF'|+|PF|=2a$$

Using the distance formula,

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

To simplify, isolate one radical and square both sides:

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}\ \sqrt{}$$

Squaring, $$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

Expanding the squares,

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$$

Cancelling the common terms x^2, c^2, and y^2 from both sides,

$$2cx=4a^2-4a\sqrt{(x-c)^2+y^2}-2cx$$

Bring the terms involving cx together:

$$4cx=4a^2-4a\sqrt{(x-c)^2+y^2}$$

Divide by 4 and isolate the remaining radical:

$$a\sqrt{(x-c)^2+y^2}=a^2-cx$$

Now we square both sides again:

$$a^2\left[(x-c)^2+y^2\right]=a^4-2a^2cx+c^2x^2$$

Expanding the left side:

$$a^2(x^2-2cx+c^2+y^2)=a^4-2a^2cx+c^2x^2$$

$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$$

$$a^2x^2+a^2c^2+a^2y^2=a^4+c^2x^2$$

Rearranging, $a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

From triangle $PF'F$ in Figure 17, we note that $|PF'|+|PF|>|F'F|$, that is, $2a>2c$, so $a > c$ and thus $a^2 - c^2 > 0$. For convenience, we define a new constant b such that $b^2=a^2-c^2$

Do You Know?

The word "focus" means "fireplace" in Latin.

Substituting, the equation of the ellipse becomes

$$b^2x^2 + a^2y^2 = a^2b^2$$

Finally, dividing both sides by a^2b^2 gives the standard form of the equation of an ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Since $b^2 = a^2 - c^2$ and $a > c$, it follows that $a > b$.

6.4.1 Key Features of the Ellipse

- **Foci**: The foci are at $(\pm c,\ 0)$, where $c^2 = a^2 - b^2$.
- **Centre**: The midpoint C of the segment joining the foci is called the **centre**. For the equation above, the centre is at the origin (0, 0).
- **Vertices**: Setting $y = 0$ gives $x = \pm a$. The points $A'(-a,\ 0)$ and $A(a,\ 0)$ are the **vertices**. The line segment $A'A$ of length $2a$ is the **major axis**.
- **Co-vertices**: Setting $x = 0$ gives $y = \pm b$. The points $B'(0,\ -b)$ and $B(0,\ b)$ are called the **co-vertices**. The line segment $B'B$ of length $2b$ is the **minor axis**. Since $a > b$, the major axis is longer than the minor axis.
- **Symmetry**: The equation is unchanged if x is replaced by $-x$ and y by $-y$; hence the ellipse is symmetric about the x-axis, y-axis, and the origin.
- **Eccentricity:** The **eccentricity** e of an ellipse is defined as $e = \frac{c}{a}$.

  Since $a > c > 0$, we have $0 < e < 1$.
- **Directrices:** For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a > b$, there are two directrices given by the vertical lines $x = \pm\frac{a}{e}$.
- **Latus Rectum:** Each of the focal chords perpendicular to the major axis is called a **latus rectum** (plural: latera recta). The length of each latus rectum is $\frac{2b^2}{a}$.
- **Parametric Equations:** For all real θ, the point $(a\cos\theta,\ b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Thus, the **parametric equations** of the ellipse are: $x = a\cos\theta, y = b\sin\theta$.
- **Foci on the y-axis:** If the foci are on the y-axis at $(0, \pm c)$, the roles of a and b are interchanged, and the standard equation becomes $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, a > b$: with vertices $(0,\ \pm a)$, and covertices $(\pm b,\ 0)$.

- **Relation between *a*, *b* and *e*:**
  Since $b^2 = a^2 - c^2$, setting $c = ae \Rightarrow b^2 = a^2 - a^2e^2 = a^2(1-e^2)$.
- **Circle as a Special Case:** If the foci coincide, then $c = 0$ and $a = b$, so the ellipse becomes a circle with radius $r = a = b$, and has equation: $x^2 + y^2 = a^2$. Clearly for circle $e = 0$.
- **Second definition of ellipse:**
  The ellipse can also be defined as, if a point P moves such that the ratio of its distance from a fixed point F (the focus) to its distance from a fixed straight line (the directrix) is equal to a constant, e (less than 1), the locus of P is an ellipse of eccentricity e.

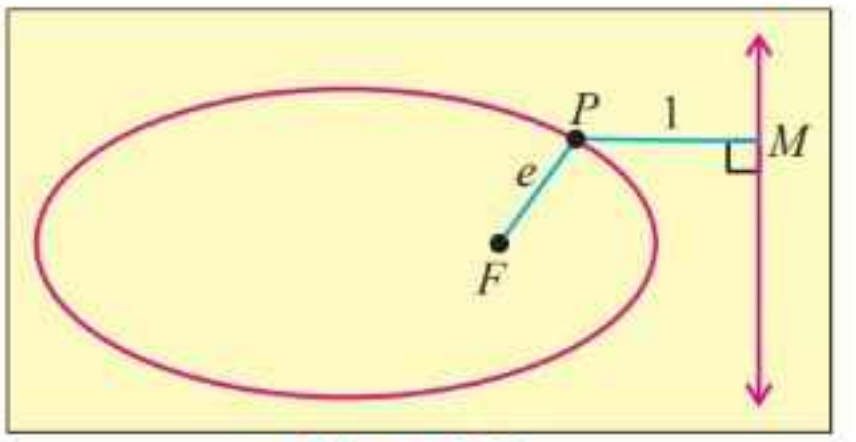


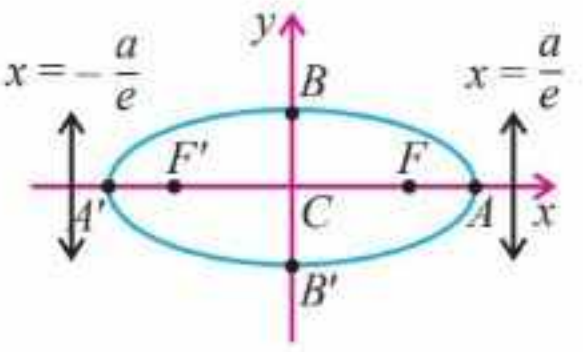
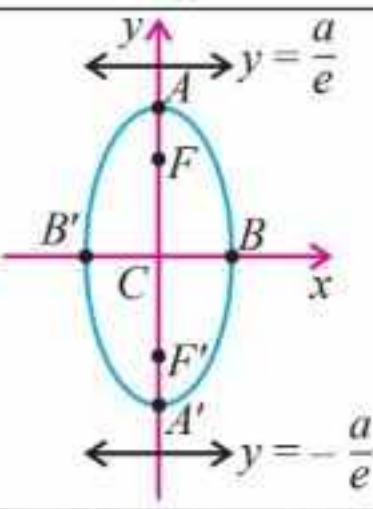
Figure 6.16

That is, if $\frac{|PF|}{|PM|} = e$, where $0 < e < 1$, the locus of P is an ellipse.

Note The equation of ellipse can also be derived using the above definition.

6.4.2 Summary of Important Results

Ellipse in standard form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,\ a>b$
Centre	$(0, 0)$	$(0, 0)$
Equation of major axis	$y=0, -a \le x \le a$	$x=0, -a \le y \le a$
Length of major axis	$2a$	$2a$
Equation of minor axis	$x=0, -b \le y \le b$	$y=0, -b \le x \le b$
Length of minor axis	$2b$	$2b$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Covertices	$(0, \pm b)$	$(\pm b, 0)$
Foci	$(\pm c, 0), c = ae$	$(0, \pm c), c = ae$
Eccentricity	$e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}<1$	$e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}<1$
Equations of directrices	$x=\pm\frac{a}{e}=\pm\frac{c}{e^2}$	$y=\pm\frac{a}{e}=\pm\frac{c}{e^2}$
Equations of latera recta	$x=\pm ae$	$y=\pm ae$

Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Sketch of ellipse in standard form		
Note : In both cases of ellipses, the major axis is the longer line segment and foci lie on the major axis.		

Theorem 1: (Translated forms of Ellipses)

The equation $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, with $a>b$ describes a horizontal ellipse with foci at $F'(h-c, k)$ and $F(h+c, k)$, where $c=\sqrt{a^2-b^2}$.

The centre of the ellipse is at the point $C(h,k)$ and the vertices are located at $(h\pm a,k)$ on the major axis.

The co-vertices are located at $(h,k\pm b)$.

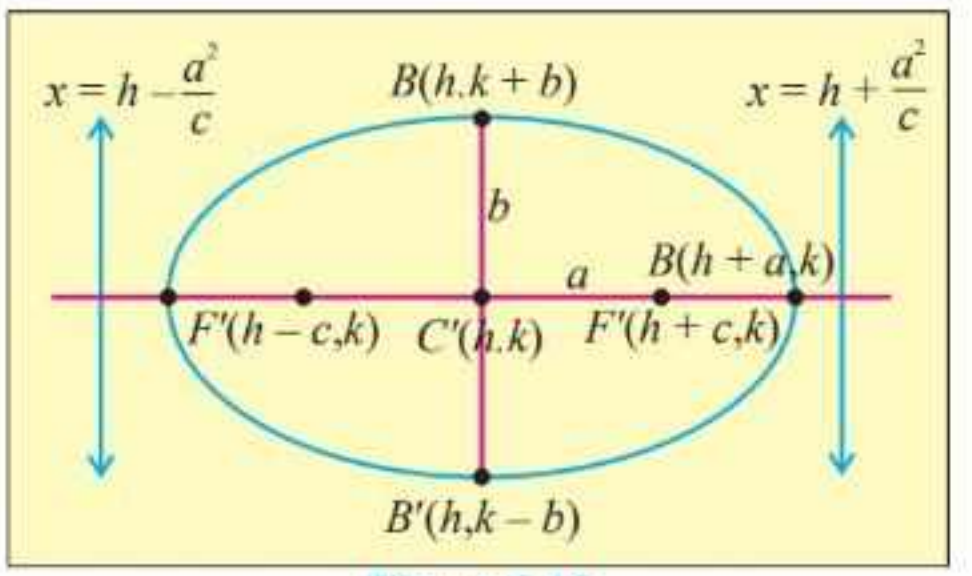


Figure 6.16

Theorem 2:

The equation $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, with $a>b$ describes a vertical ellipse with foci at $F'(h, k-c)$ and $F(h, k+c)$, where $c=\sqrt{a^2-b^2}$. The centre of the ellipse is at the point $C(h,k)$ and the vertices are located at $(h,k\pm a)$ on the major axis. The co-vertices are located at $(h\pm b,k)$.

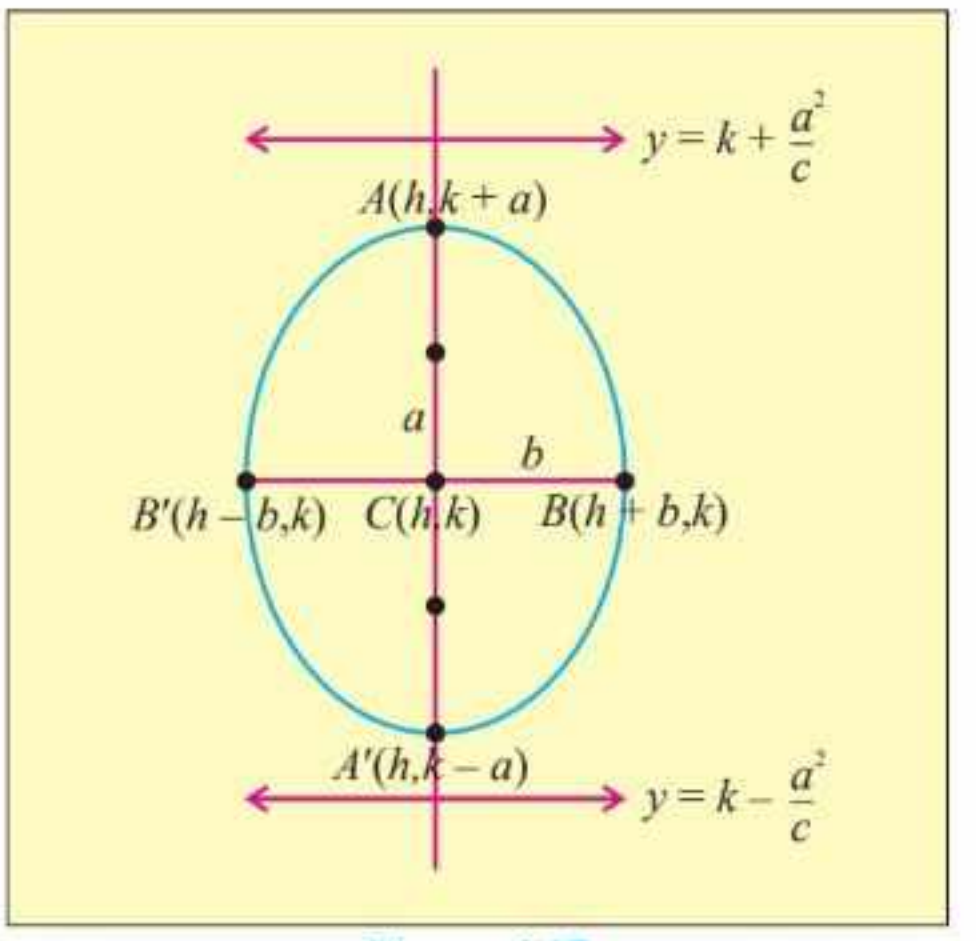


Figure 6.17

Example 18 Find an equation of the ellipse with foci (2, 1), (2, 7), major axis of length 10 and sketch its graph.

Solution: Given the foci: (2, 1) and (2, 7).

Both foci have the same x-coordinate

$\Rightarrow$ Major axis is vertical.

Major axis length $= 10 \Rightarrow 2a = 10 \Rightarrow a = 5$.

Centre is the midpoint of the foci:

$$\text{Centre} = \left(\frac{2+2}{2}, \frac{1+7}{2}\right) = (2,4).$$

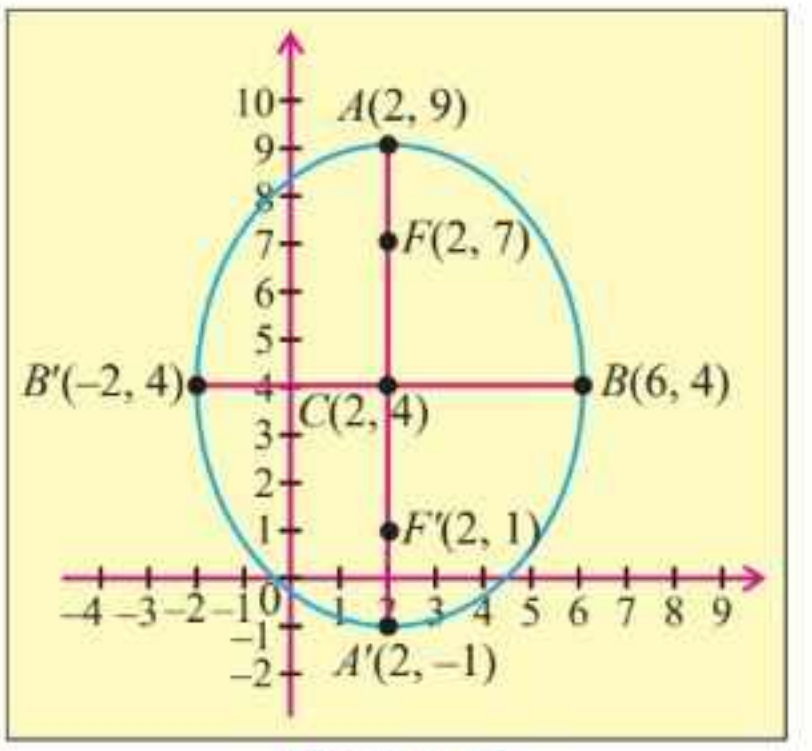


Figure 6.18

Distance from centre to either focus:

$$c = |4 - 1| = 3$$

For an ellipse: $c^2 = a^2 - b^2 \Rightarrow 3^2 = 5^2 - b^2 \Rightarrow 9 = 25 - b^2 \Rightarrow b^2 = 16 \Rightarrow b = 4$.

Thus, required equation of the ellipse with centre $(h, k) = (2, 4)$, major axis vertical, $a = 5$, and $b = 4$ is: $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, that is, $\frac{(x-2)^2}{16} + \frac{(y-4)^2}{25} = 1$.

Example 19 Find an equation of the ellipse with centre (0, 0), focus (5, 0) and vertex (6, 0). Sketch its graph.

Solution: Given centre is at (0, 0).

Focus: (5, 0) $\Rightarrow$ one focus is to the right; the other focus is (–5, 0) by symmetry.

Vertex: (6, 0) $\Rightarrow$ one vertex is to the right; the other vertex is (–6, 0).

All points have same y-coordinate 0, so major axis horizontal.

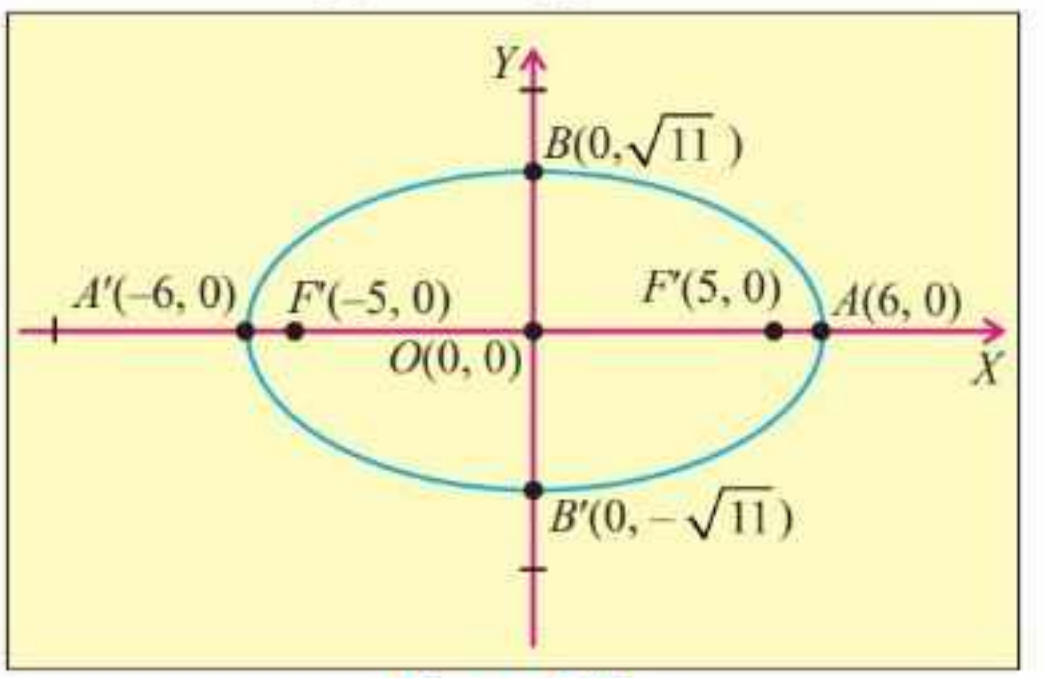


Figure 6.19

From vertices: $(\pm 6, 0) \Rightarrow a = 6$, so $a^2 = 36$.

From foci: $(\pm 5, 0) \Rightarrow c = 5, c^2 = 25$.

To find b, using the ellipse relation:

$c^2 = a^2 - b^2 \Rightarrow 25 = 36 - b^2 \Rightarrow b^2 = 11 \Rightarrow b = \sqrt{11}$

Required equation with horizontal major axis, centre at origin, $a^2 = 36$ and $b^2 = 11$ is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, that is, $\frac{x^2}{36} + \frac{y^2}{11} = 1$.

Example 20 Find the centre, foci, eccentricity, vertices and directrices of the ellipse whose equation is $16x^2 + 9y^2 - 64x + 54y + 1 = 0$. Also sketch its graph.

Solution:

$$16x^2 + 9y^2 - 64x + 54y + 1 = 0$$

$$(16x^2 - 64x) + (9y^2 + 54y) + 1 = 0$$

$$16(x^2 - 4x) + 9(y^2 + 6y) + 1 = 0$$

$$16[(x^2 - 4x + 4) - 4] + 9[(y^2 + 6y + 9) - 9] + 1 = 0$$

$$16[(x-2)^2 - 4] + 9[(y+3)^2 - 9] + 1 = 0$$

$$16(x-2)^2 - 64 + 9(y+3)^2 - 81 + 1 = 0$$

$$16(x-2)^2 + 9(y+3)^2 - 144 = 0$$

$$16(x-2)^2 + 9(y+3)^2 = 144$$

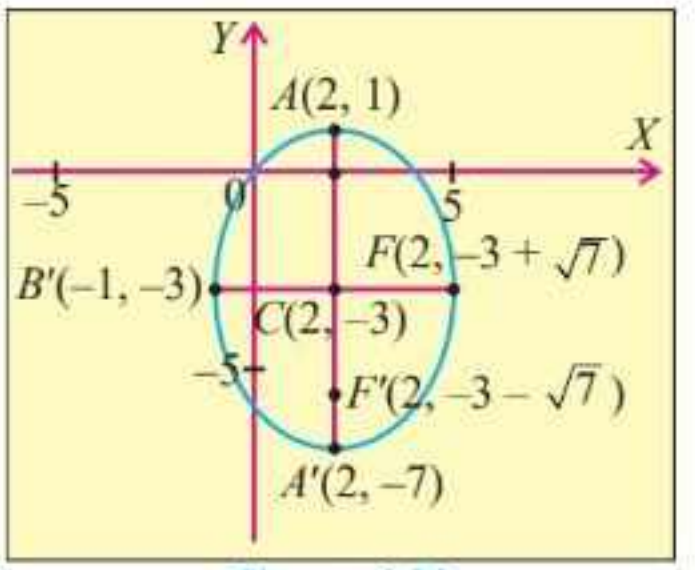


Figure 6.20

Divide both sides by 144:

$$\frac{16(x-2)^2}{144} + \frac{9(y+3)^2}{144} = 1$$

$$\frac{(x-2)^2}{9} + \frac{(y+3)^2}{16} = 1$$

Comparing with $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, we obtain

$h = 2, k = -3, b^2 = 9 \Rightarrow b = 3, a^2 = 16 \Rightarrow a = 4.$

Since $a > b$ and a^2 is under y-term, major axis is vertical.

Centre: $C(2, -3)$.

Using the ellipse relation: $c^2 = a^2 - b^2$; $c^2 = 16 - 9 = 7 \Rightarrow c = \sqrt{7}$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{7}}{4}$

Vertices: Vertices are $a = 4$ units above and below centre along vertical axis:

First vertex: $A(2, -3+4) = A(2,1)$.

Second vertex: $A'(2, -3-4) = A'(2, -7)$.

Foci: Foci are $c = \sqrt{7}$ units above and below centre:

First focus: $F(2, -3+\sqrt{7})$, Second focus: $F'(2, -3-\sqrt{7})$.

Directrices: For vertical major axis, directrices are horizontal lines: $y = k \pm \frac{a}{e}$

$$y = k \pm \frac{a}{e} = -3 \pm \frac{4}{\sqrt{7}/4} = -3 \pm \frac{16}{\sqrt{7}}$$

Example 21 An ellipse has the following properties:

(i) Its major axis is vertical and has a length of 10 units.

(ii) The distance between its foci is 6 units.

(iii) The centre of the ellipse is at the point (3, –2).

Find the standard equation of this ellipse.

Solution:

(i) The major axis is vertical and has a length 10, so: $2a=10\Rightarrow a=5\Rightarrow a^2=25$

(ii) The distance between the foci is 6, so: $2c=6\Rightarrow c=3\Rightarrow c^2=9$

(iii) The centre is at $(h,k)=(3,-2)$

For an ellipse, the relationship between a, b and c is: $c^2=a^2-b^2$

Substitute $a^2=25$ and $c^2=9$

$$9=25-b^2 \Rightarrow b^2=25-9 \Rightarrow b^2=16$$

Since the major axis is vertical, the standard form of the ellipse equation is:

$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ Substitute $h=3,\ k=-2,\ a^2=25,$ and $b^2=16$:

$$\frac{(x-3)^2}{16}+\frac{(y+2)^2}{25}=1.$$

6.4.3 Tangents and Normals to Ellipse

We are given the ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1) and the line: $y=mx+c$ ...(2)

We want to find their **points of contact** and the **condition for tangency**.

Substitute $y=mx+c$ into the ellipse equation:

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

Multiplying through by a^2b^2 to clear denominators:

$$b^2x^2+a^2(mx+c)^2=a^2b^2$$

Expand: $b^2x^2+a^2(m^2x^2+2mcx+c^2)=a^2b^2$

$$b^2x^2+a^2m^2x^2+2a^2mcx+a^2c^2-a^2b^2=0$$

$$(b^2+a^2m^2)x^2+2a^2mcx+a^2(c^2-b^2)=0 \qquad ...(3)$$

The equation is quadratic in x and it gives the x-coordinates of the two points where line and ellipse intersect. The corresponding values of y are obtained by setting the values of x obtained from last equation into $y=mx+c$. Thus line and ellipse intersect in two points.

Tangency Condition

Tangency occurs when the quadratic in x has equal roots.
This means the discriminant $D = 0$:

$$(2a^2mc)^2 - 4(b^2 + a^2m^2)\cdot a^2(c^2 - b^2) = 0$$

$$4a^4m^2c^2 - 4a^2(b^2 + a^2m^2)(c^2 - b^2) = 0$$

Factor out $4a^2$:

$$4a^2\left[a^2m^2c^2 - (b^2 + a^2m^2)(c^2 - b^2)\right] = 0$$

Since $a \neq 0$:

$$a^2m^2c^2 - (b^2 + a^2m^2)(c^2 - b^2) = 0$$

Expand the second term:

$$a^2m^2c^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2m^2b^2 = 0$$

$$a^2m^2b^2 - b^2c^2 + b^4 = 0$$

Divide through b^2 $(b \neq 0)$:

$$a^2m^2 - c^2 + b^2 = 0$$

So, the tangency condition is:

$$\boxed{c^2 = a^2m^2 + b^2} \Rightarrow c = \pm\sqrt{a^2m^2 + b^2} \quad \ldots(4)$$

Putting value of c in $y = mx + c$, we obtain $y = mx \pm \sqrt{a^2m^2 + b^2}$ two tangents to ellipse for all values of m.

Point of Tangency

For the tangent $y = mx + \sqrt{a^2m^2 - 5}$ the Q.E (3) has equal roots given by

$$x = \frac{-2a^2mc}{2(b^2 + a^2 + m^2)} = \frac{-a^2mc}{c^2}$$

We have equation of $= \dfrac{-a^2m}{c}$

$$= \frac{-a^2m}{\sqrt{a^2m^2 + b^2}}$$

Substituting $x = \dfrac{-a^2m}{\sqrt{a^2m^2 + b^2}}$

Into 0 (5) we have

So, point of contact of (5) and (1) $y = \dfrac{b^2}{\sqrt{a^2m^2 + b^2}}$ is $\left(\dfrac{-a^2m}{\sqrt{a^2m^2 + b^2}}, \dfrac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$.

Similarly, point of contact of $y = mx - \sqrt{a^2m^2 + b^2}$ and (1) is

$$\left(\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$$

Do You Know?

If a glass of water is tilted, The surface of the water forms an ellipse.

Example 22 Find equation of the tangent and normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1).

Solution: Equation of the ellipse is: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Differentiating implicitly with respect to x: $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$

Thus, the slope at (x_1, y_1) is: $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=-\frac{b^2x_1}{a^2y_1}$

Equation of the tangent:

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

Multiplying through $\frac{y_1}{b^2}$: $\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=-\frac{xx_1}{a^2}+\frac{x_1^2}{a^2}$

Rearranging: $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$

Since (x_1, y_1) lie on ellipse, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

Hence, the equation of the tangent is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Equation of the normal:

The slope of the normal is $\frac{a^2y_1}{b^2x_1}$

Thus, $y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$

Multiplying $b^2x_1y-b^2x_1y_1=a^2y_1x-a^2x_1y_1$

Rearranging: $a^2y_1x-b^2x_1y=x_1y_1(a^2-b^2)$

Dividing both sides by by x_1y_1 (assuming $x_1\neq 0$, $y_1\neq 0$): $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.

EXERCISE 6.4

1. Find an equation of the ellipse with given data.
   (i) Foci $(0, \pm 5)$ and minor axis of length 8.
   (ii) Vertices $(-2, 5)$, $(-2, -3)$; foci $(-2, 4)$ and $(-2, -2)$.

(iii) Vertices $(\pm 6, 0)$, eccentricity $= \frac{2}{3}$.

(iv) Centre $(-3, 1)$, major axis parallel to the x-axis and of length 10 units, minor axis parallel to the y-axis and of length 6 units.

2. Find the centre, foci, eccentricity, vertices and directrices of the ellipse whose equation is given and sketch its graph.
(i) $4x^2 + 9y^2 = 36$ (ii) $4x^2 + 24x + 9y^2 - 36y + 36 = 0$

3. Find the equation of the ellipse as the locus of points $P(x, y)$ such that the sum of the distances from P to the points $(-3, 0)$ and $(3, 0)$ is 10.

4. Find the equation of an ellipse centered at $(5, 3)$ that is tangent to both the x-axis and y-axis.

5. What are the dimensions of the smallest rectangle aligned to the axes that completely contains the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$?

6. Find the dimensions of the rectangle with greatest perimeter that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

7. Find the minimum distance of the point $P(3\cos\theta, 2\sin\theta)$ on the ellipse from the centre of the ellipse.

8. Find the minimum distance from the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ to the line $x + y = 6$.

9. Find the value of k for which the line $y = 2x + k$ is tangent to the ellipse $\frac{x^2}{9} + y^2 = 1$.

10. Determine whether the line $y = 2x - 3$ is a tangent, secant, or external to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

11. Find the equation of an ellipse with centre at $(0, 0)$, a horizontal major axis of length 8, and that touches the line $y = x + 5$.

12. Show that the family of lines given by: $x\cos\theta + y\sin\theta = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$ where θ is a parameter and $a > b$, are all tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

13. If a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre C cuts the major and minor axes at P and Q respectively. Prove that $\frac{a^2}{|CP|^2} + \frac{b^2}{|CQ|^2} = 1$.

6.5 Hyperbola

The definition of a hyperbola is similar to that of an ellipse. For an ellipse, the sum of the distances between the foci and a point on the ellipse is fixed, whereas for a hyperbola, the absolute value of the difference between these distances is fixed. One distinguishing feature of a hyperbola is that its graph has two separate branches (see Figure 21).

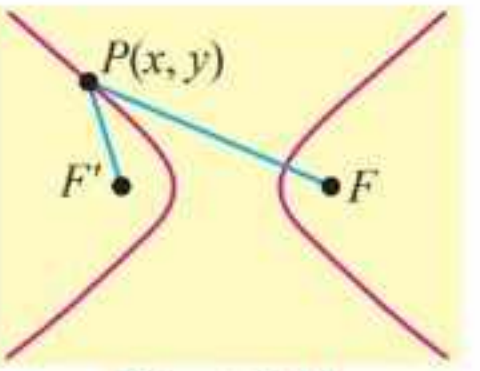


Figure 6.21

Derivation of the Standard Equation of Hyperbola

Thus, we can define hyper bola as a **hyperbola** is the set of all points (x, y) for which the absolute value of the difference between the distances from two distinct fixed points called **foci** is constant. The hyperbolas with simplest equations are those with foci on one of the coordinate axes. To obtain the simplest equation for a hyperbola, we place the foci on the x-axis at the points $F'(-c, 0)$ and $F(c, 0)$, so that the origin is the midpoint of the segment joining them (see Figure 22).

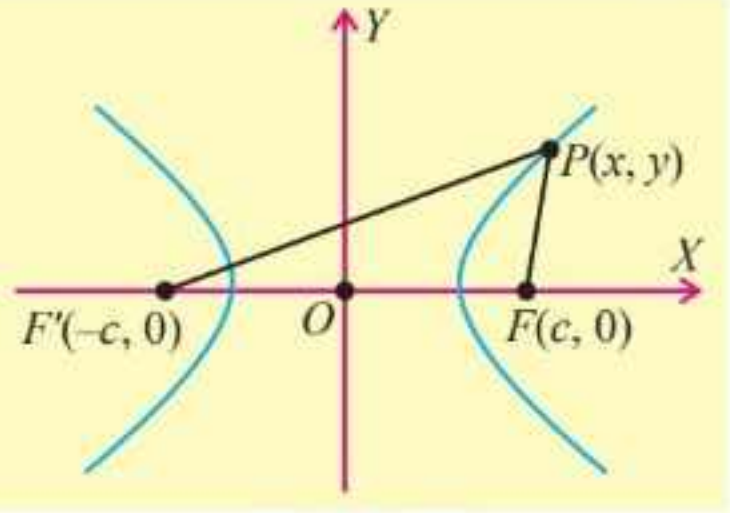


Figure 6.22

Let the absolute value of constant difference of distances from any point $P(x, y)$ on the hyperbola to the foci be $2a$, where $a > 0$. Thus,

$$\big||PF'|-|PF|\big| = 2a \quad \text{or} \quad |PF'|-|PF| = \pm 2a$$

Using the distance formula,

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a$$

To simplify, isolate one radical and square both sides:

$$\sqrt{(x+c)^2+y^2} = \pm 2a + \sqrt{(x-c)^2+y^2}$$

Squaring, $(x+c)^2+y^2 = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$

Expanding the squares,

$$x^2+2cx+c^2+y^2 = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2} + x^2-2cx+c^2+y^2$$

Cancelling the common terms x^2, c^2, and y^2 from both sides,

$$2cx = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2} - 2cx$$

Bring the terms involving cx together: $4cx = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2}$

Divide by 4 and isolate the remaining radical:

$$\mp a\sqrt{(x-c)^2+y^2} = a^2 - cx$$

Now we square both sides again:

$$a^2[(x-c)^2+y^2]=a^4-2a^2cx+c^2x^2$$

Expanding the left side: $a^2(x^2-2cx+c^2+y^2)=a^4-2a^2cx+c^2x^2$

$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$$

Adding $2a^2cx$ to both sides:

$$a^2x^2+a^2c^2+a^2y^2=a^4+c^2x^2$$

Rearranging: $a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

From triangle $PF'F$ in Figure 2, we note that $|PF'|-|PF|<|F'F|$, that is, $2a<2c$, so $a<c$ and thus $a^2-c^2<0$. For convenience, we define a new constant b such that $-b^2=a^2-c^2$.

Substituting, the equation of the hyperbola becomes $(-b^2)x^2+a^2y^2=a^2(-b^2)$

Finally, dividing both sides by $-a^2b^2$ gives the standard form of the equation of a hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $c^2=a^2+b^2$.

6.5.1 Key Features of the Hyperbola

- **Vertices**: Setting $y=0$ in the equation of hyperbola, gives $x=\pm a$. The points $A'(-a, 0)$ and $A(a, 0)$ are the **vertices** of the hyperbola.
- **Transverse Axis**: The line segment $A'A$ connecting the vertices is the **transverse axis** (or **focal axis**) of the hyperbola. Length of the transverse axis is $2a$.
- **Conjugate Axis:** A line passes through the centre and perpendicular to the transverse axis is called conjugate axis. Setting $x=0$ in the equation of hyperbola, gives $y^2=-b^2$, which is impossible, so hyperbola does not meet y-axis in real points. If we take two points $B'(0, -b)$ and $B(0, b)$ on the y-axis, then the line segment $B'B$ is called the **conjugate axis** of the hyperbola. Length of the **conjugate axis** is $2b$.
- **Foci**: The foci are at $(\pm c, 0)$, they always lie on the transverse axis. In case of hyperbola $c^2=a^2+b^2$, so that, unlike the ellipse, we may have $a>b$ or $a<b$ or $a=b$.
- **Centre**: The midpoint $C(0, 0)$ of the segment joining the foci is called the **centre** of the hyperbola. Note that it is same as the midpoint of the transverse axis.
- **Symmetry**: The equation is unchanged if x is replaced by $-x$ and y by $-y$; hence the hyperbola is symmetric with respect to both the axis, and the origin.
- **Eccentricity:** The **eccentricity** e of a hyperbola is defined as $e=\frac{c}{a}$.

  Since $c>a$, we have $e>1$.

- **Directrices:** For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, there are two directrices given by the vertical lines $x=\pm\frac{a}{e}$.
- **Latus Rectum:** Each of the focal chords perpendicular to the transverse axis is called a **latus rectum** (plural: latera recta). The length of each latus rectum is $\frac{2b^2}{a}$.
- **Parametric Equations:** For all real θ such that $\theta\neq(2n+1)\frac{\pi}{2}$ (where n is an integer), the point $(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Thus, the **parametric equations** of the hyperbola are: $x=a\sec\theta, y=b\tan\theta$.
- **Foci on the *y*-axis:** If the foci are on the y-axis at $(0,\pm c)$, the roles of x and y are interchanged, and the standard equation becomes $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, vertices $(0,\pm a)$.
- **Relation between *a*, *b* and *e*:**

  Since $b^2=c^2-a^2$, setting $c=ae \Rightarrow b^2=a^2e^2-a^2 \Rightarrow b^2=a^2(e^2-1)$.
- **Branches:** To analyze the hyperbola further, we write $\frac{x^2}{a^2}=1+\frac{y^2}{b^2}\geq 1$. This shows that $x^2\geq a^2$ so $|x|\geq a$. Therefore, we have $x\geq a$ or $x\leq -a$. This means that the hyperbola consists of two parts, called its branches. Note that no portion of the curve lies between $-a<x<a$, because in this interval y is imaginary.
- **Asymptotes:** An important feature of hyperbola that is not shared by ellipses is the presence of asymptotes. An asymptote of a curve $y=f(x)$ is a line $y=g(x)$ such that

$$\lim_{x\to\pm\infty}\left[f(x)-g(x)\right]=0$$

For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we have

$\frac{y^2}{b^2}=\frac{x^2}{a^2}-1$ or $y^2=\frac{b^2}{a^2}x^2-b^2$. Notice that

$$\lim_{x\to\pm\infty}\frac{y^2}{x^2}=\lim_{x\to\pm\infty}\left(\frac{b^2}{a^2}-\frac{b^2}{x^2}\right)=\frac{b^2}{a^2}.$$

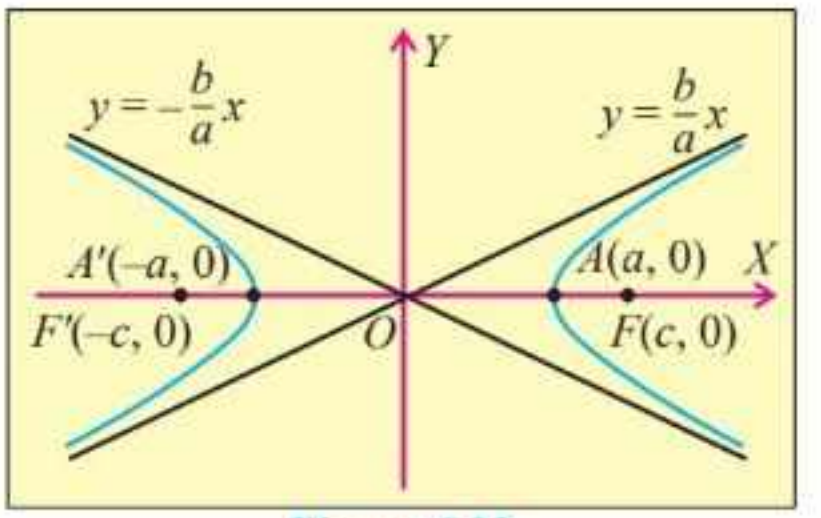


Figure 6.23

That is, as $x \to \pm\infty, \frac{y^2}{x^2} \to \frac{b^2}{a^2}$, so that $\frac{y}{x} \to \pm\frac{b}{a}$ and so, $y = \pm\frac{b}{a}x$ are the (slant) asymptotes as shown in Figure 6.23. The asymptotes of a curve do not meet the curve but distance of any point on the curve from any of the two lines approaches zero.

Note. The joint equation of the two asymptotes is

$$\left(y + \frac{b}{a}x\right)\left(y - \frac{b}{a}x\right) = 0, \text{ that is, } y^2 - \frac{b^2}{a^2}x^2 = 0 \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

Rectangular Hyperbola as a special case: If in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the lengths a and b become equal then the resulting hyperbola $x^2 - y^2 = a^2$ is called rectangular hyperbola. In this case asymptotes are perpendicular to each other.

This is obtained by replacing the constant term 1 with 0 in the standard equation.

- **Second definition of Hyperbola**

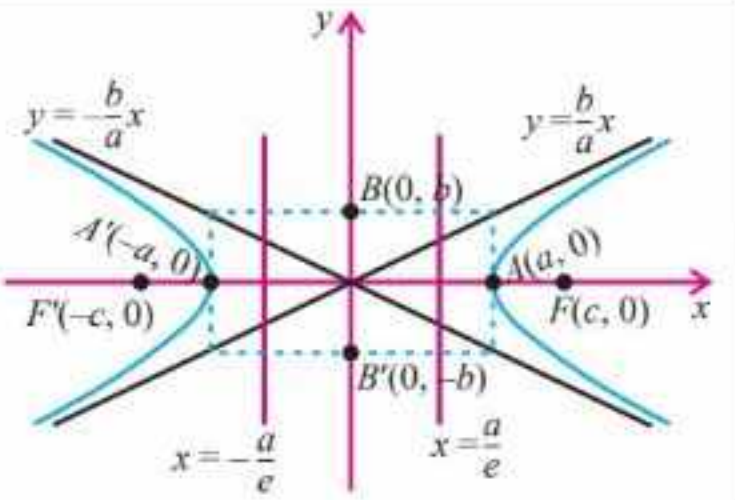
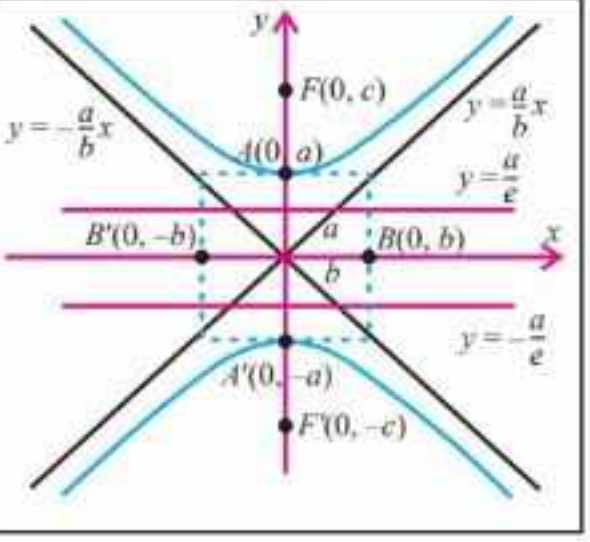
The hyperbola can also be defined as, if a point P moves so that the ratio of its distance from a fixed point F (the focus) to its distance from a fixed straight line (the directrix) is equal to a constant, e (greater than 1), the locus of P is a hyperbola of eccentricity e.

That is, if $\frac{|PF|}{|PM|} = e$, where $e > 1$, the locus of P is a hyperbola.

Note. The equation of hyperbola can also be derived using the above definition.

6.5.2 Summary of Important Results

Hyperbola in standard form	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Centre	$(0, 0)$	$(0, 0)$
Equation of transverse axis	$y = 0, -a \le x \le a$	$x = 0, -a \le y \le a$
Length of transverse axis	$2a$	$2a$
Equation of conjugate axis	$x = 0, -b \le y \le b$	$y = 0, -b \le x \le b$
Length of conjugate axis	$2b$	$2b$
Vertices	$(\pm a, 0)$	$(0, \pm a)$

Conjugate points	$(0, \pm b)$	$(\pm b, 0)$
Foci	$(\pm c, 0), c = ae$	$(0, \pm c), c = ae$
Eccentricity	$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} > 1$	$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} > 1$
Equations of directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{a}{e}$
Equations of latera recta	$x = \pm ae$	$y = \pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Equations of asymptotes	$y = \pm \frac{b}{a} x$	$y = \pm \frac{a}{b} x$
Sketch of hyperbola in standard form		

Theorem 1: Translated Forms of Hyperbola's

The equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

describes a hyperbola with foci at the points $F'(h-c, k)$ and $F(h+c, k)$, where $c = \sqrt{a^2 + b^2}$.

The centre of the hyperbola is at the point $C(h, k)$ and the vertices are located at $(h \pm a, k)$.

The asymptotes are $y = \pm \frac{b}{a}(x-h) + k$.

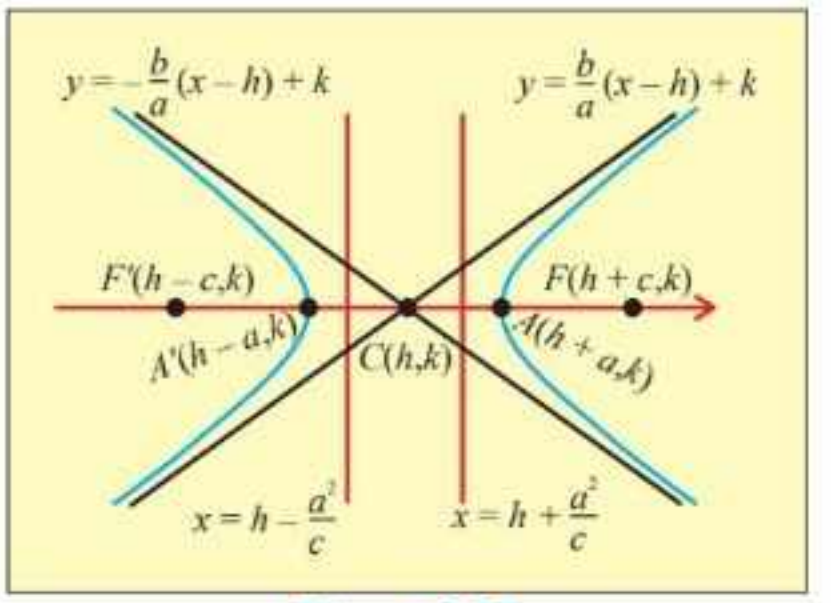


Figure 6.25

Theorem 2: The equation

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

describes a hyperbola with foci at the points $F'(h, k-c)$ and $F(h, k+c)$, where $c=\sqrt{a^2+b^2}$.

The centre of the hyperbola is at the point (h, k) and the vertices are located at $(h, k\pm a)$.

The asymptotes are $x=\pm\frac{b}{a}(y-h)+k$.

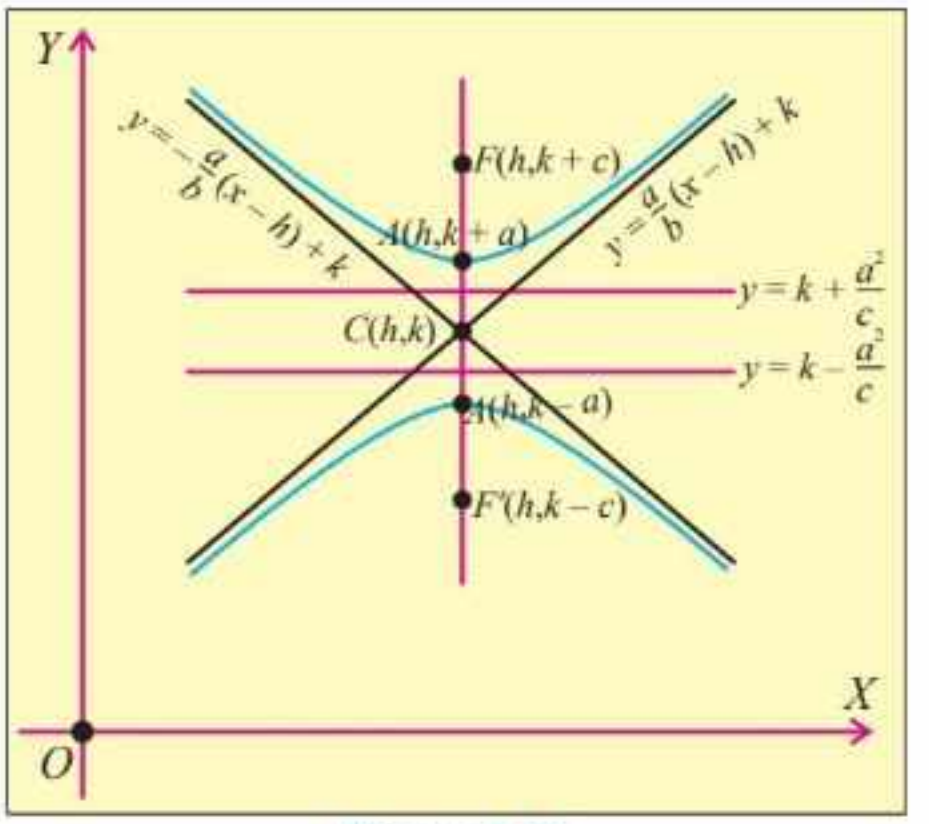


Figure 6.26

Example 23 A hyperbola has a horizontal transverse axis of length 12, centre at (–2, 5), and eccentricity $\frac{5}{3}$. Find the equation of the hyperbola.

Solution: Since the transverse axis is horizontal, the hyperbola opens left and right.

The standard form is: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, where (h, k) is the centre.

Given centre: $(h, k) = (-2, 5)$.

Length of transverse axis $= 2a = 12 \Rightarrow a = 6$

Eccentricity: $e=\frac{c}{a}=\frac{5}{3} \Rightarrow \frac{c}{6}=\frac{5}{3} \Rightarrow c=10$

Using the relationship $c^2=a^2+b^2 : 100=36+b^2 \Rightarrow b^2=64$.

Substitute $h=-2$, $k=5$, $a^2=36$, $b^2=64$ into the standard form:

$$\frac{(x+2)^2}{36}-\frac{(y-5)^2}{64}=1$$

Example 24 Find an equation of the hyperbola with given data and sketch the graph.

(i) Foci: $(-3,\ 5\pm5\sqrt{2})$ (ii) Length of transverse axis $= 6$.

Solution: Midpoint of foci $(-3,5+5\sqrt{2})$, $(-3,5-5\sqrt{2})$ is the centre: $C(-3,\ 5)$.

Transverse axis is vertical, so hyperbola has the form: $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

$2a=6 \Rightarrow a=3$

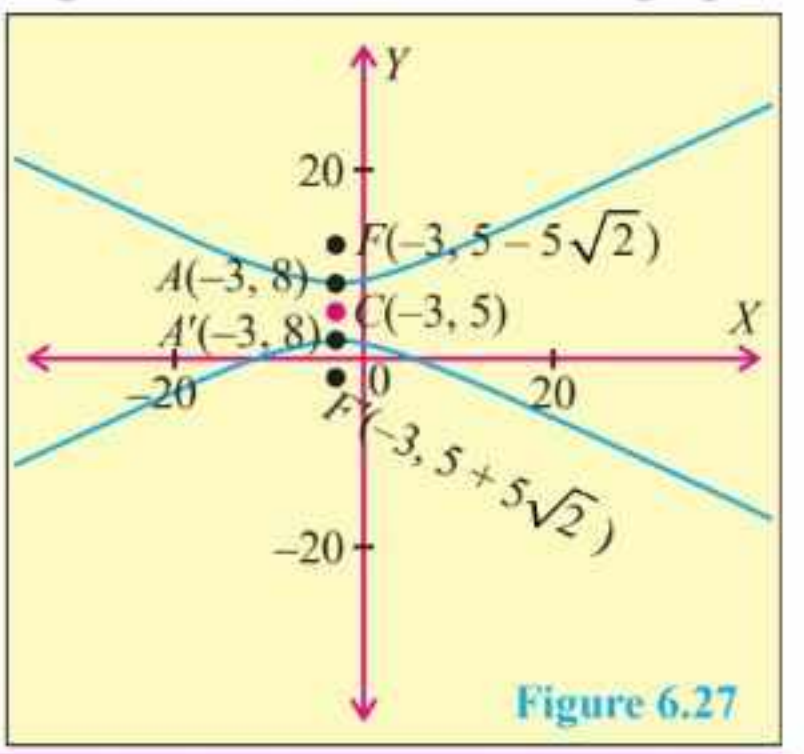


Figure 6.27

$2c$ = distance between foci $= 10\sqrt{2}$

$c = 5\sqrt{2} \Rightarrow c^2 = 50$

$c^2 = a^2 + b^2 \Rightarrow 50 = 9 + b^2 \Rightarrow b^2 = 41$

Required equation: $\frac{(y-5)^2}{9} - \frac{(x+3)^2}{41} = 1.$

Example 25 Find an equation of the hyperbola with given data and sketch the graph. (i) Foci $(4,-3)$, $(4, 7)$ and (ii) one vertex $(4, 5)$.

Solution:

Centre (h,k) is the midpoint of foci (4, –3) and (4, 7)

$$h = \frac{4+4}{2} = 4,\ k = \frac{-3+7}{2} = \frac{4}{2} = 2$$

So centre: $(4,\ 2)$.

Both foci have same x-coordinate, this implies transverse axis is vertical.

c = Distance from centre (4, 2) to a focus (4, 7)

$= 7 - 2 = 5.$

Given vertex: (4, 5).

a = Distance from centre (4, 2) to vertex (4, 5)

$= 5 - 2 = 3.$

$c^2 = a^2 + b^2$

$5^2 = 3^2 + b^2$

$b^2 = 25 - 9$

$= 16$

$$\frac{(y-2)^2}{9} - \frac{(x-4)^2}{16} = 1$$

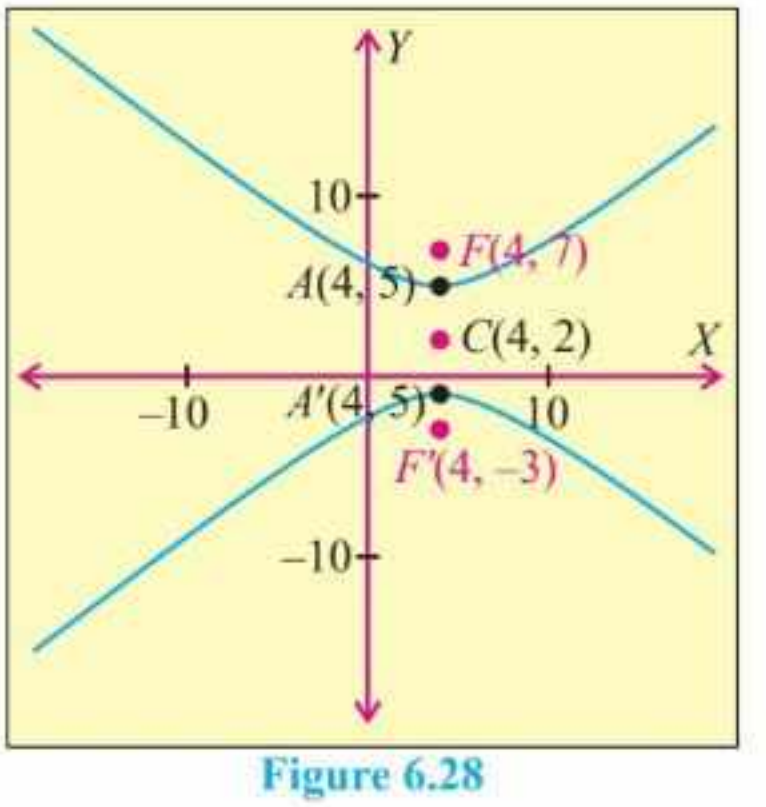


Figure 6.28

Example 29 Find the centre, foci, eccentricity, vertices and equations of the directrices of the hyperbola:

$9x^2 - 36x - 4y^2 + 8y - 4 = 0.$ Also sketch the graph.

Solution:

$9x^2 - 36x - 4y^2 + 8y - 4 = 0$

$9x^2 - 36x - 4y^2 + 8y = 4$

$9(x^2 - 4x) - 4(y^2 - 2y) = 4$

$$9\left(x^2-4x+4-4\right)-4\left(y^2-2y+1-1\right)=4$$

$$9\left[(x-2)^2-4\right]-4\left[(y-1)^2-1\right]=4$$

$$9(x-2)^2-36-4(y-1)^2+4=4$$

$$9(x-2)^2-4(y-1)^2=36$$

Divide by 36

$$\frac{9(x-2)^2}{36}-\frac{4(y-1)^2}{36}=1$$

$$\frac{(x-2)^2}{4}-\frac{(y-1)^2}{9}=1$$

This is a horizontal hyperbola:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

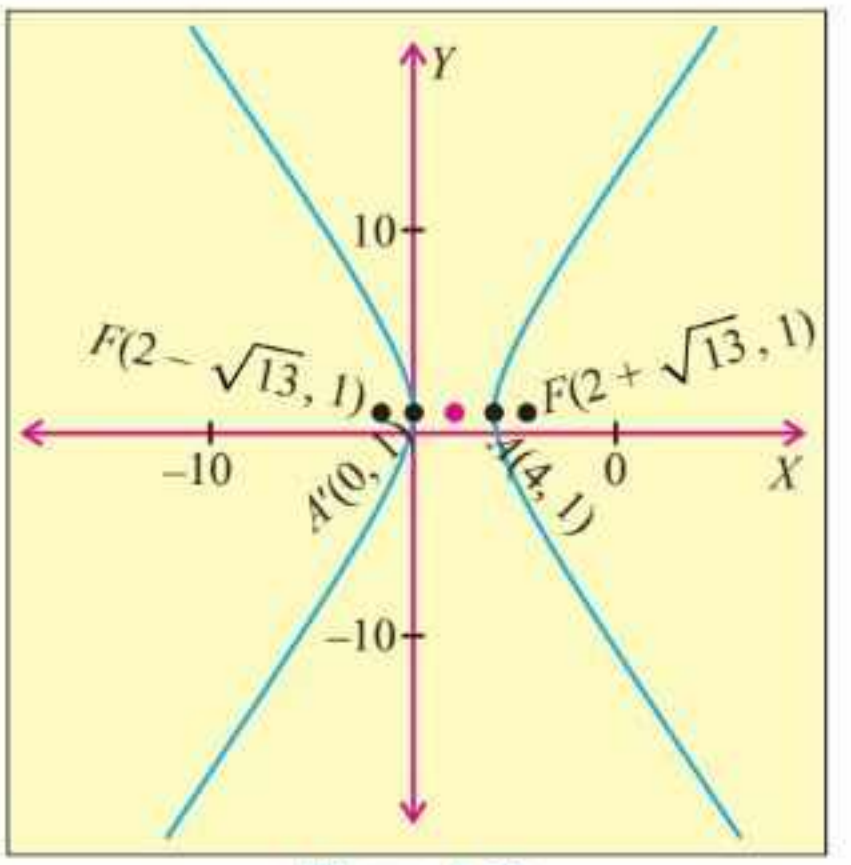


Figure 6.29

Here: $h=2,\ k=1,\ a^2=4\Rightarrow a=2,\ b^2=9\Rightarrow b=3.$

Using the relation: $c^2=a^2+b^2:\ c^2=4+9=13\Rightarrow c=\sqrt{13}$

Centre: $(h,\ k)=(2,\ 1).$

For horizontal hyperbola vertices are: $(h\pm a,k)$

That is vertices are $A(2+2,1)=A(4,1),\ A'(2-2,1)=A'(0,\ 1)$

Foci: $(h\pm c,k)$

That is foci are $F(2+\sqrt{13},1),\ F'(2-\sqrt{13},1)$

Eccentricity $e=\frac{c}{a}=\frac{\sqrt{13}}{2}$

Directrices: For a horizontal hyperbola: $x=h\pm\frac{a}{e}$

$\frac{a}{e}=\frac{2}{\frac{\sqrt{13}}{2}}=\frac{4}{\sqrt{13}}=\frac{4\sqrt{13}}{13}$. So directrices: $x=2\pm\frac{4\sqrt{13}}{13}$

6.5.3 Tangents and Normals to Hyperbola

We are given the hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ …(1)

and the line: $y=mx+c$ …(2)

We want to find their **points of intersection** and the **condition for tangency**.

Substitute $y=mx+c$ into the hyperbola equation:

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

Multiplying through by a^2b^2 to clear denominators: $b^2x^2 - a^2(mx+c)^2 = a^2b^2$

Expand:

$$b^2x^2 - a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$$

$$b^2x^2 - a^2m^2x^2 - 2a^2mcx - a^2c^2 - a^2b^2 = 0$$

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0 \qquad \ldots(3)$$

The equation is quadratic in x and it gives the x-coordinates of the two points where line and hyperbola intersect. The corresponding values of y are obtained by setting the values of x obtained from last equation into $y = mx + c$. Thus line and hyperbola intersect in two points.

Tangency Condition

Tangency occurs when the quadratic in x has equal roots.

This means the discriminant $D = 0$:

$$(-2a^2mc)^2 - 4(b^2 - a^2m^2)\left[-a^2(c^2 + b^2)\right] = 0$$

$$4a^4m^2c^2 + 4a^2(b^2 - a^2m^2)(c^2 + b^2) = 0$$

Factor out $4a^2$: $\quad 4a^2\left[a^2m^2c^2 + (b^2 - a^2m^2)(c^2 + b^2)\right] = 0$

Since $a \neq 0$: $\quad a^2m^2c^2 + (b^2 - a^2m^2)(c^2 + b^2) = 0$

Expand the second term:

$$a^2m^2c^2 + b^2c^2 + b^4 - a^2m^2c^2 - a^2m^2b^2 = 0$$

$$-a^2m^2b^2 + b^2c^2 + b^4 = 0$$

Divide through b^2 $(b \neq 0)$:

$$-a^2m^2 + c^2 + b^2 = 0$$

So the tangency condition is:

$$\boxed{c^2 = a^2m^2 - b^2} \Rightarrow \; c = \pm\sqrt{a^2m^2 - b^2} \quad \text{provided } a^2m^2 > b^2$$

Putting value of c in $y = mx + c$, we obtain $y = mx \pm \sqrt{a^2m^2 - b^2}$ two tangents to hyperbola for all values of m.

Points of Contact

The line $y = mx + c$ intersects the hyperbola in two distinct points given by solving:

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0$$

$$x = \frac{-(-2a^2mc) \pm \sqrt{(2a^2mc)^2 - 4(b^2 - a^2m^2)\cdot(-a^2(c^2 + b^2))}}{2(b^2 - a^2m^2)}$$

$$= \frac{2a^2mc \pm \sqrt{4a^4m^2c^2 + 4a^2(b^2 - a^2m^2)(c^2 + b^2)}}{2(b^2 - a^2m^2)}$$

$$= \frac{a^2mc \pm \sqrt{a^4m^2c^2 + a^2(b^2 - a^2m^2)(c^2 + b^2)}}{b^2 - a^2m^2}$$

The y-coordinates follow from $y = mx + c$.

Example 27 Find equation of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1).

Solution: Equation of the hyperbola is: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Differentiating implicitly with respect to x:

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}$$

Thus the slope at (x_1, y_1) is: $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{b^2x_1}{a^2y_1}$

Equation of the tangent: $y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$

Multiplying through $\frac{y_1}{b^2}$: $\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$

Rearranging: $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$

Since (x_1, y_1) lie on hyperbola, so $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

Hence, the equation of the tangent is: $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Hence, equation of normal at $(x - y_1)$ is:

The slope of the normal is $-\frac{a^2y_1}{b^2x_1}$

Thus, $y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1)$

Multiplying: $b^2x_1y - b^2x_1y_1 = -a^2y_1x + a^2x_1y_1$

Rearranging: $a^2y_1x + b^2x_1y = x_1y_1(a^2 + b^2)$

Dividing both sides by $x_1\, y_1$ (assuming $x_1 \neq 0,\ y_1 \neq 0$): $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

Example 28 Find the equations of the tangents to the hyperbola $2x^2 - 3y^2 = 24$ that pass through the point $(2, -4)$.

Solution: $2x^2 - 3y^2 = 24$

Divide through by 24: $\frac{x^2}{12} - \frac{y^2}{8} = 1$. So $a^2 = 12,\ b^2 = 8.$

For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, tangent with slope m is: $y = mx \pm \sqrt{a^2m^2 - b^2}$

Substitute $a^2 = 12,\ b^2 = 8$: $y = mx \pm \sqrt{12m^2 - 8}$

Substitute $x = 2, y = -4$: $-4 = 2m \pm \sqrt{12m^2 - 8} \Rightarrow -4 - 2m = \pm\sqrt{12m^2 - 8}$

Square: $16 + 16\,m + 4m^2 = 12m^2 - 8 \Rightarrow 8\,m^2 - 16\,m - 24 = 0 \Rightarrow m^2 - 2m - 3 = 0$

$\Rightarrow (m - 3)(m + 1) = 0 \Rightarrow m = 3$ or $m = -1$.

For $m = 3$:

$$y = 3x \pm \sqrt{12(9) - 8} \Rightarrow y = 3x \pm \sqrt{108 - 8} \Rightarrow y = 3x \pm \sqrt{100} \Rightarrow y = 3x \pm 10.$$

Check which sign satisfies the point $(2, -4)$:

Check: $y = 3x + 10$:

At $x = 2, y = 3(2) + 10 = 16$. So $(2, -4)$ do not lie on $y = 3x + 10$.

Check: $y = 3x - 10$:

At $x = 2, y = 3(2) - 10 = -4$. So $(2, -4)$ lie on $y = 3x - 10$.

So tangent is $3x - y - 10 = 0$.

For $m = -1$:

$$y = -x \pm \sqrt{12(1) - 8} \Rightarrow y = -x \pm \sqrt{12 - 8} \Rightarrow y = -x \pm \sqrt{4} \Rightarrow y = -x \pm 2.$$

Check which sign satisfies the point $(2, -4)$:

Check: $y = -x + 2$:

At $x = 2, y = -2 + 2 = 0$. So $(2, -4)$ do not lie on $y = -x + 2$

Check: $y = -x - 2$::

At $x = 2, y = -2 - 2 = -4$. So $(2, -4)$ lie on $y = -x - 2$

So tangent is $x + y + 2 = 0$.

EXERCISE 6.5

1. Find an equation of the hyperbola with the given data. Also sketch the graph.
   (i) Centre $(0, 0)$, Focus $(0, 10)$, Vertex $(0, 8)$
   (ii) Foci $(0, \pm 8)$, Eccentricity $= 4$ (iii) Foci $(0, \pm 10)$, Directrices $y = \pm 5$
2. Find the centre, foci, eccentricity, vertices and equations of directrices of each of the following. Also sketch the graph.

   (i) $\frac{x^2}{36} - \frac{y^2}{64} = 1$ (ii) $\frac{(x+3)^2}{5} - \frac{(y-2)^2}{4} = 1$ (iii) $\frac{y^2}{25} - \frac{x^2}{16} = 1$

3. A hyperbola has its centre at the origin, transverse axis of length 14 along the y-axis, and conjugate axis of length 10 along the x-axis. Find the equation of the hyperbola.
4. A hyperbola has a horizontal transverse axis of length 8. One of its foci is at the point $(1+2\sqrt{5}, 3)$ and its eccentricity is $\frac{\sqrt{5}}{2}$. Find the equation of the hyperbola.
5. A hyperbola has a vertical transverse axis of length 8, centre at $(2, -1)$, and eccentricity $\frac{\sqrt{13}}{2}$. Find the equation of the hyperbola.
6. A hyperbola has its centre at $(-1, 4)$, a focus at $(5, 4)$, and the corresponding directrix given by the line $x = -3$. Find the equation of the hyperbola.
7. A hyperbola has a vertical transverse axis, eccentricity $\frac{\sqrt{19}}{4}$, and latus rectum of length $\frac{3}{2}$. The hyperbola passes through the points $(2, 8)$ and $(2, 0)$. Find the equation of the hyperbola.
8. Find the equation of the hyperbola in standard form (centred at the origin) that passes through the points $(\sqrt{80}, 4)$ and $(\sqrt{32}, 2)$.
9. Find equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$.
10. Find the length of the latus rectum of the hyperbola $9x^2-16y^2=144$.
11. Find the eccentricity of the hyperbola whose asymptotes are perpendicular.
12. If the transverse axis of a hyperbola is 8 and the latus rectum is 9, find its eccentricity.
13. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. If the eccentricity of the hyperbola is 2, find its equation.
14. Find the eccentricity of the hyperbola whose latus rectum is half its transverse axis.
15. Find the equations of the normals to the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ which are parallel to the line $2x - y = 5$.
16. If the distance between the foci of a hyperbola is 16 and the distance between its vertices is 12, find its eccentricity.
17. Find the length of the transverse axis of the hyperbola $4x^2 - 9y^2 = 36$.

18. A hyperbola for which $a = b$ is called rectangular. Show that a hyperbola is rectangular if and only if its asymptotes are perpendicular to one another.
19. Find the equations of the tangents to the hyperbola $7x^2 - 3y^2 = 105$ that pass through the point (3, 1).
20. Find the condition that the line $lx + my = 1$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

6.6 Applying Concepts of Conics To Real Life World Problems

Example 29 A comet has a parabolic orbit with the earth at the focus. When the comet is h km from the earth, the line joining the comet and the earth makes an angle of 30° with the axis of the parabola. How close will the comet come to the earth?

Solution: Let equation of the parabolic orbit be $y^2 = 4ax$. Let the centre of Earth $E(a, 0)$ be the focus. As point a parabola that is closest to the focus is it vertex. So, we need to find $|OE| = a$

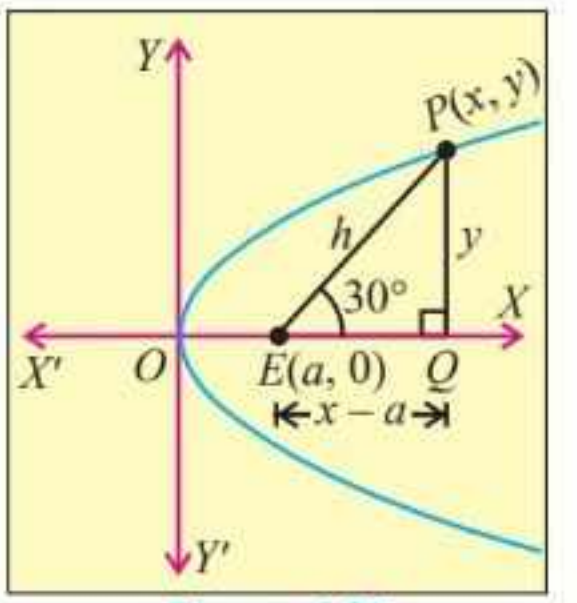


Figure 6.30

From right ΔEQP, $\cos 30° = \frac{x-a}{h} \Rightarrow \frac{\sqrt{3}}{2} = \frac{x-a}{h}$

$$\Rightarrow x - a = \frac{\sqrt{3}}{2}h \Rightarrow x = a + \frac{\sqrt{3}}{2}h$$

also $\sin 30° = \frac{y}{h} \Rightarrow \frac{1}{2} = \frac{y}{h} \Rightarrow y = \frac{h}{2}$

Since $\left(a + \frac{\sqrt{3}}{2}h, \frac{h}{2}\right)$ lies on the parabola $y^2 = 4ax$, it must satisfy it.

$$\left(\frac{h}{2}\right)^2 = 4a\left(a + \frac{\sqrt{3}}{2}h\right) \Rightarrow \frac{h^2}{4} = 4a^2 + 2\sqrt{3}ah$$

$$4a^2 + 2\sqrt{3}ah - \frac{h^2}{4} = 0$$

$$\Rightarrow \quad a = \frac{-2\sqrt{3}h \pm \sqrt{\left(2\sqrt{3}h\right)^2 - 4(4)\left(-\frac{h^2}{4}\right)}}{2(4)} = \frac{-2\sqrt{3}h \pm \sqrt{12h^2 + 4h^2}}{8}$$

$$= \frac{-2\sqrt{3}h \pm \sqrt{16h^2}}{8} = \frac{-2\sqrt{3}h \pm 4h}{8}$$

$$= \frac{2\left(-\sqrt{3} \pm 2\right)h}{8} = \frac{\left(-\sqrt{3} \pm 2\right)h}{4} \text{ Since, } \frac{-\sqrt{3}+2}{4} < 0 \text{ so, neglect it.}$$

Since minimum value of a is required, we have $a = \frac{\left(-\sqrt{3}+2\right)h}{4}$ km

Example 30 Elliptical Electron Orbit

In the Bohr–Sommerfeld model of the atom, electrons move in elliptical orbits around the nucleus, which is located at one focus. Consider an electron in an elliptical orbit with a semi-major axis $a = 5.29 \times 10^{-11}$ m (the Bohr radius) and an eccentricity $e = 0.20$.

(a) Find the distance from the nucleus to the centre of the ellipse.

(b) Determine the minimum distance (perihelion) and maximum distance (apoapsis) of the electron from the nucleus.

Solution: (a) For an ellipse, the distance from the centre to each focus is $c = ae$.

$$c = (5.29 \times 10^{-11}) \times 0.20 = 1.058 \times 10^{-11} \text{m}$$

Thus, the nucleus is 1.058×10^{-11} m from the centre of the ellipse.

(b) The minimum distance (periapsis) occurs when the electron is closest to the nucleus at the focus:

$$r_{min} = a(1-e) = (5.29 \times 10^{-11}) \times (1 - 0.20) = (5.29 \times 10^{-11}) \times 0.80 = 4.232 \times 10^{-11} \text{ m}$$

The maximum distance (apoapsis) occurs when the electron is farthest from the nucleus:

$$r_{max} = a(1+e) = (5.29 \times 10^{-11}) \times (1 + 0.20) = (5.29 \times 10^{-11}) \times 1.20 = 6.348 \times 10^{-11} \text{ m}$$

Example 30 The cable of a suspension bridge is parabolic. The roadway is 10 m, below the lowest point of the cable. The span of the bridge is 200 m, and the tops of the piers are 50 m above the roadway. Find the equation of the parabola. If the load is supported by vertical chains at intervals of 10 m, find the length of the chain that is 20 m away from either pier.

Solution: Let BAC be the cable, $A(0, 0)$ the origin at the lowest point of the cable (vertex of the parabola), and DE be the road way.

Taking Ax as the x-axis the parabola opens upwards with vertex at (0, 0) and its equation is of the form:

$$x^2 = 4ay \qquad \ldots(1)$$

Since $HB = 100$ m and $AH = 50 - 10 = 40$ m, the coordinates of B are $(100, 40)$ which lie on (1).

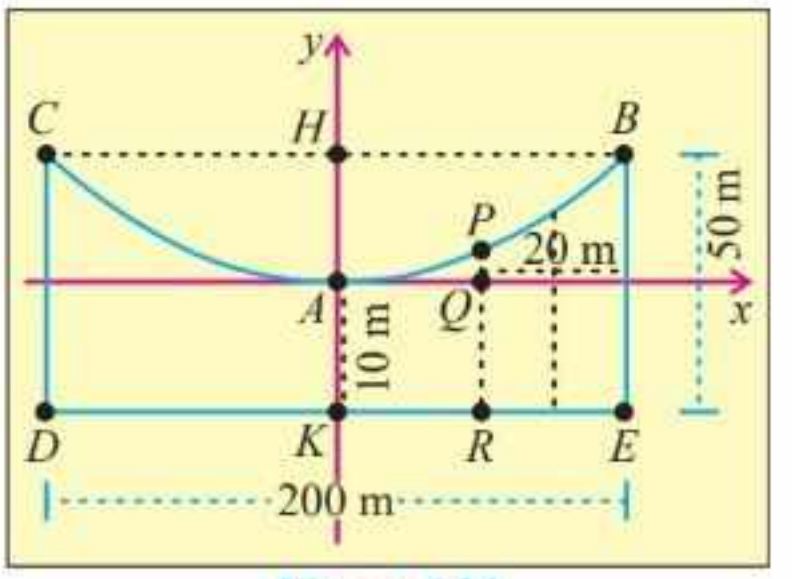


Figure 6.31

Therefore, $(100)^2 = 4a(40) \Rightarrow 10000 = 4a(40)$

$$4a = \frac{10000}{40} = 250.$$

Hence equation of the parabola is $x^2 = 250y$ ...(2)

If PR be the chain at a distance 20 m from the pier BE, then

$$AQ = 100 - 20 = 80\text{m}$$

Since $P(80, PQ)$ lies on (2), $(80)^2 = 250 \cdot PQ \Rightarrow 6400 = 250 \cdot PQ$

$$\Rightarrow PQ = \frac{6400}{250} = 25.6\,\text{m}$$

Thus, the required length of the chain ($= PR$) is the vertical distance from the cable to the road way:

Length $= PR = PQ + QR = 25.6 + 10 = 35.6\,\text{m}.$

Example 32 The moon revolves around Earth in an elliptical orbit, with the centre of Earth located at one focus. The major and minor axis of the orbit have lengths of 768,800 km and 767,600 km, respectively. Find the greatest and least distances (the apogee and perigee) from Earth's centre to the moon's centre.

Solution: Begin by solving for a and b:

$2a = 768,800 \Rightarrow a = 384,400$ and $2b = 767,600$

$$\Rightarrow \quad b = 383,800$$

$$c = \sqrt{a^2 - b^2} = \sqrt{(384400)^2 - (383800)^2}$$

$$= \sqrt{147763360000 - 147302440000}$$

$$= \sqrt{460920000}$$

$$= 21469.047 \approx 21469 \text{ km}$$

The greatest distance between the centre of Earth and the centre of the moon is

$$a + c \approx 384,400 + 21,469 = 405,869 \text{ km}$$

and the least distance is

$$a - c \approx 384,400 - 21,469 = 362,931\text{km}.$$



Figure 6.32

Example 33 Satellite Orbit

The **apogee** (the point in orbit farthest from Earth) and the **perigee** (the point in orbit closest to Earth) of an elliptical orbit of an Earth satellite are given by A and P, respectively. Show that the eccentricity of the orbit is $e = \frac{A-P}{A+P}$.

Solution: Consider an elliptical orbit with Earth at one focus. Let:

a = semi-major axis of the ellipse, c = distance from the centre to each focus,

$$e = \frac{c}{a} = \text{eccentricity}$$

The **perigee** P is the closest distance from the satellite to Earth. Since Earth is at one focus, this distance occurs when the satellite is at the vertex nearest that focus: $P = a - c$

The **apogee** A is the farthest distance from the satellite to Earth. This occurs when the satellite is at the opposite vertex: $A = a + c$

We have the system: $\begin{cases} A = a + c \\ P = a - c \end{cases}$

Add the two equations: $A + P = (a + c) + (a - c) = 2a \Rightarrow a = \dfrac{A + P}{2}$

Subtract the second equation from the first:

$$A - P = (a + c) - (a - c) = 2c \Rightarrow c = \frac{A - P}{2}$$

By definition, eccentricity is: $e = \dfrac{c}{a}$

Substitute the expressions for c and a: $e = \dfrac{\frac{A - P}{2}}{\frac{A + P}{2}} = \dfrac{A - P}{A + P}$

Thus, the eccentricity e of the elliptical orbit is given by: $e = \dfrac{A - P}{A + P}$.

Remark

- This formula holds for any elliptical orbit where the central body is at one focus

Example 34 A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 20 cm.

(a) Find an equation of the parabola.

(b) Find the diameter of the opening $|CD|$, 20 cm from the vertex.

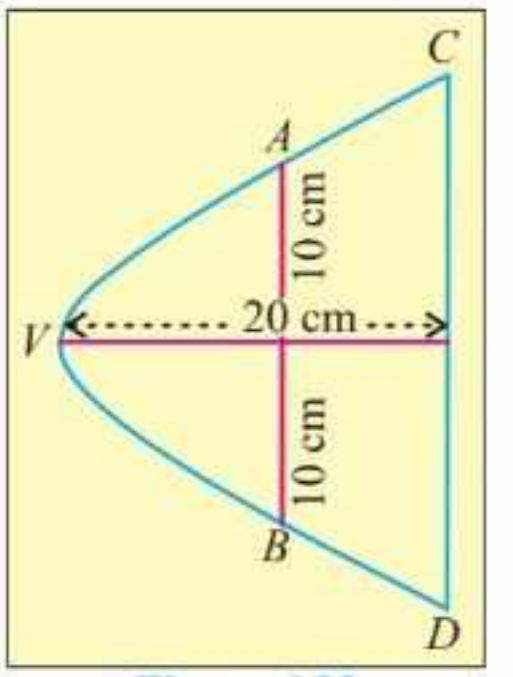


Figure 6.33

Solution: From Figure description:

- Cross-section of a parabolic reflector is a parabola
- The bulb is at the focus F
- The opening at the focus is 20 cm
  This means the width perpendicular to the axis at the focus is 20 cm
- 10 cm appears on each side of the axis in the diagram at the focus level
- Take vertex at the origin, axis horizontal (opening to the right)

Thus, the standard form is: $y^2 = 4ax$

where $4a = |AB| =$ length of latus rectum

Equation of parabola is $y^2 = 20x$

20 cm from vertex means $x = 20$ (since vertex at origin, axis along x-axis)

From $y^2 = 20x$: At $x = 20$: $y^2 = 20 \times 20 = 400 \Rightarrow y = \pm 20$

Half-width $= 20$ cm

Diameter $|CD| = 2(20) = 40$ cm.

Example 35 A parabolic flashlight reflector has the shape given by the equation:

$$x = \frac{1}{20} y^2$$

where should the lightbulb be placed so that the reflected rays emerge parallel to the axis?

Solution: The equation is given as: $x = \frac{1}{20} y^2 \Rightarrow y^2 = 20x$

This is a parabola that opens to the right with its vertex at the origin (0, 0).

The standard form of a parabola opening to the right is: $y^2 = 4ax$

Where:

- a is the distance from the vertex to the focus
- The focus is at $(a, 0)$
- The directrix is the vertical line $x = -a$

Compare $y^2 = 20x$ with $y^2 = 4ax$: $4a = 20 \Rightarrow a = 5$

For a parabola $y^2 = 4ax$, the focus is at $(a, 0)$

Substituting $a = 5$: Focus = (5, 0)

The lightbulb should be placed at the focus of the parabola. By the reflective property of parabolas, any light ray emitted from the focus will reflect off the parabolic surface and travel outward parallel to the axis (the x-axis).

Thus, the bulb should be located at: $(5,\ 0)$.

Remark:

The reflective property ensures that light from the focus reflects parallel to the axis, a principle used in flashlights, headlights, and satellite dishes.

EXERCISE 6.6

1. A comet has a parabolic orbit with the sun at the focus. When the comet is 100 million km from the sun, the line joining the sun and the comet makes an angle of 60° with the axis of the parabola. How close will the comet get to the sun?
2. Find an equation of the parabola formed by the cables of a suspension bridge whose span is ℓ m and the vertical height of the supporting towers is h m.
3. A parabolic arch has 120 m base and height 25 m. Find the height of the arch at a point 42 m from the centre of the base.

4. An athletic field features an elliptical running track. The track is 120 meters long along its major axis and 80 meters wide along its minor axis.
   (a) Find the distance from the centre to each focus of the ellipse.
   (b) If two water fountains are placed at the foci, how far apart are they?
5. A landscaper wants to create an elliptical flower bed with a major axis of 10 meters and a minor axis of 6 meters. Using the gardener's method, two stakes are placed at the foci, and a loop of string is stretched around them to trace the ellipse.
   (a) How far apart should the two stakes be placed?
   (b) What should be the total length of the string loop?
6. An architect designs a bridge with an elliptical arch. The arch has a span (width) of 24 meters and a height of 8 meters. The ellipse is centred at the origin with the major axis horizontal.
   (a) Write the equation of the ellipse representing the arch.
   (b) A boat needs to pass under the bridge. The boat is 6 meters wide. What is the maximum height of the boat that can pass through the arch if it must stay centred?
7. In the Bohr–Sompmerfeld model, an electron in an elliptical orbit around a nucleus obeys a modified form of Kepler's third law. For a hydrogen-like atom, the square of the orbital period T is proportional to the cube of the semi-major axis a.

   Consider an electron in an elliptical orbit with a semi-major axis $a = 2.12 \times 10^{-10}$ m (four times the Bohr radius) and an eccentricity $e = 0.30$.

   The constant of proportionality for hydrogen is $k = 2.96 \times 10^{-16} \text{s}^2 / \text{m}^3$.
   (a) Determine the distance between the nucleus (at one focus) and the centre of the ellipse. (b) Calculate the orbital period T of the electron.
   (c) Find the maximum and minimum distances of the electron from the nucleus during its orbit.
8. LORAN (long distance radio navigation) for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci.

Assume that two stations, 500 miles apart, are positioned on a rectangular coordinate system at $(-250, 0)$ and $(250, 0)$ and that a ship is traveling on a path with coordinates $(x,120)$ (a horizontal line 120 miles above the x-axis). Find the x-coordinate of the position of the ship when the time difference between the pulses from the transmitting stations is 1200 microseconds $(0.0008$ second$)$.

9. A satellite orbits Earth in an elliptical path. The apogee (farthest distance from Earth) is $A = 42000$ km, and the perigee (closest distance to Earth) is $P = 30000$ km.

   (a) Find the semi-major axis a of the orbit.

   (b) Find the distance from the centre to each focus c.

   (c) Calculate the eccentricity e of the orbit.

10. The end towers of a suspension bridge are 100 meters high and 540 meters apart. The suspension cable hangs from the top of the towers in the form of a parabola and the lowest point of the cable is 25 meters above the horizontal road. Find the length of the supporting vertical cable at a horizontal distance of 81 meters from the centre of the bridge.

11. An architectural arch is built in the shape of the upper half of an ellipse with semi-major axis 8 meters and semi-minor axis 5 meters. A horizontal beam is to be placed across the arch at a height where the width of the arch is exactly 10 meters. Calculate the exact height above ground level where this beam should be installed.

Unit 07 Kinematics of Motion in a Straight Line

INTRODUCTION

Kinematics is the branch of mechanics that describes the motion of objects without considering the causes of motion. In this unit, we will explore how objects move along a straight line by studying key physical quantities such as displacement, velocity, and acceleration. Through sketching and interpreting displacement–time and velocity–time graphs, we will develop a visual understanding of motion. The unit also introduces the use of rates of change to analyze how displacement, velocity, and acceleration vary over time. Using differentiation and integration with respect to time, we will solve simple problems related to motion. For situations involving constant acceleration, we will apply appropriate kinematic formulae to determine unknown quantities. Finally, the concepts learned will be applied to real-life situations, such as the motion of vehicles, free fall, vertical projectile motion, relative motion, and the creation of realistic movement in animation.

7.1 Rectilinear Motion

The motion of a particle along a straight line is called rectilinear motion.

7.1.1 Scalars

Scalar quantities are those that are completely described by magnitude only (size or numerical value). Scalars do not have direction.

Examples:

- **Distance:** Total length of the path travelled, regardless of direction.
- **Speed:** Rate at which distance is covered per unit time, irrespective of direction.

7.1.2 Vectors

Vector quantities are described by both magnitude and direction.

Examples:

- **Displacement:** Displacement is change in position of a particle represented by a vector from initial to final position.
- **Velocity:** Rate of change of displacement with respect to time in a specified direction (i.e., speed in a given direction). It is vector quantity and its magnitude gives speed.
- **Acceleration:** Rate of change of velocity with respect to time, including direction.

7.1.3 Displacement, Average Velocity and Acceleration

When a particle undergoes rectilinear motion, its displacement, velocity, and acceleration can have only one of two possible directions, along the line of motion.

In such cases, we do not use arrows or other vector symbols to represent direction; instead, we use positive and negative signs to indicate it:

- One direction along the line is taken as positive.
- The opposite direction is taken as negative.

This simplifies the representation of displacement, velocity, and acceleration in rectilinear motion.

Average Velocity: The average velocity (v) of an object is defined as the total displacement divided by the total time taken:

$$v_{avg} = \text{Average velocity} = \frac{\text{Total displacement}}{\text{Total Time}} = \frac{\Delta x}{\Delta t}$$

Average Speed: The average speed of an object is defined as the total distance travelled divided by the total time taken:

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total Time}}$$

Recall!

a body goes and come back starting position

- Displacement = 0
- Distance 0

Example 2 A man walks 100 meters east in 2 minutes, then turns around and walks 60 meters west in 1 minute. Find his speed and velocity.

Solution: Total distance = 100 + 60 = 160 meters

Total time = 2 + 1 = 3 minutes

$$\text{Average Speed} = \frac{\text{Total distance travelled}}{\text{Total Time}} = \frac{160}{3} = 53.3 \text{ meters/min}$$

Taking east as positive and west as negative:

Displacement eastward x_1 = 100 meters

Displacement westward x_2 = – 60 meters

Total displacement = $x_1 + x_2$ = 100 – 60 = 40 meters

$$v_{avg} = \text{Average velocity} = \frac{\text{Total displacement}}{\text{Total Time}} = \frac{40}{3} = 13.33 \text{ meters/min (eastward)}$$

Instantaneous Velocity: The instantaneous velocity is the rate of change of displacement with respect to time at a given instant. If $x(t)$ is the displacement of a particle at any time t, then instantaneous velocity $v(t)$ at any time t is given by:

$$v(t) = \frac{dx}{dt}$$

Instantaneous Speed: Instantaneous speed is the magnitude of instantaneous velocity:

$$\text{Instantaneous speed} = |v(t)| = \left|\frac{dx}{dt}\right|$$

Example 2 Let $x(t) = t^3 - 6t^2$ be the position of a particle moving along a straight line at any time t, where x is in meters and t is in seconds. Find the velocity and the speed of the particle at any time t and at $t = 1$ second.

Solution: Given: $x(t) = t^3 - 6t^2$

Differentiate with respect to t:

$$\frac{dx}{dt} = 3t^2 - 12t$$

Therefore, the velocity at any time t is:

$$v(t) = 3t^2 - 12t \quad \ldots (1)$$

Speed at any time t is:

$$\text{Speed} = |v(t)| = |3t^2 - 12t| \quad \ldots (2)$$

Substitute $t = 1$ into equations (1) and (2):

$$v(1) = 3 - 12 = -9 \text{ meter/second and Speed} = |v(1)| = |-9| = 9 \text{ m/s}$$

Acceleration: Rate of change of velocity with respect to time is called acceleration. If $\boldsymbol{v(t)}$ is velocity of a particle at any time $\boldsymbol{t}$, then its acceleration $\boldsymbol{a(t)}$ is given by:

$$a(t) = \frac{dv}{dt}$$

Since $v = \frac{dx}{dt}$, the above expression can also be written as:

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

If velocity $\boldsymbol{v}$ of the particle is given as a function of displacement $\boldsymbol{x}$:

Using the Chain Rule:

$$a = \frac{dv(x)}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$$

But since $v = \frac{dx}{dt}$, we get:

$$a = v\frac{dv}{dx}$$

Units of Acceleration: If displacement is measured in meters and time is in seconds, then unit of acceleration is meters per second square *i.e* m/s^2.

7.1.4 Displacement-Time Graph

Displacement-time graphs are important tools in kinematics, because they provide a visual representation of an object's motion. Here's why we discuss them:

- They give a clear picture of how an object's position changes over time.
- They help us easily determine whether an object is moving, stationary or accelerating.

In a displacement-time graph, the displacement (x) is plotted on the y-axis, and time (t) is plotted on the x-axis.

Key Features of a Displacement-Time Graph

1. Slope of the graph $= \left(\frac{dx}{dt}\right)$, represents velocity of the particle.

- If velocity is uniform i.e.,

  $v = \frac{dx}{dt}$ = Slope of the graph = constant,

  the displacement-time graph is a straight line.

  - If $v = \frac{dx}{dt} = 0$, the particle is at rest and the displacement-time graph is a horizontal line, which reflects that displacement remains constant with time. In Figure 7.1, the particle is at rest from point B to point C.

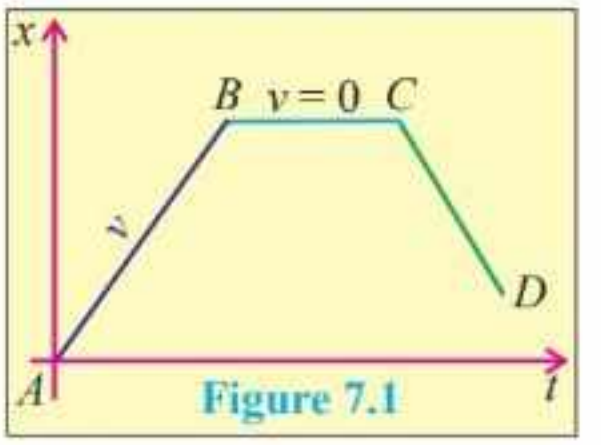


Figure 7.1

  - If $v = \frac{dx}{dt}$ = constant > 0, the particle moves forward, and the displacement-time graph is a rising straight line; that is, displacement increases as time increases. In Figure 7.1, the particle is moving with positive uniform velocity from point A to point B.

Recall!
- Positive slope forward motion,
- Negative slope backward motion.

  - If $v = \frac{dx}{dt}$ = constant < 0, the particle moves in the opposite direction, and the displacement time graph is a falling straight line; that is, displacement decreases as time increases. In Figure 7.1, the particle is moving with uniform negative velocity from point C to point D.

Example 3 A cyclist rides in a straight line for 20 minutes. He then stops and waits for 30 minutes, after which he returns in a straight line to the starting point in 10 minutes. Figure 7.2 shows the displacement-time graph for his journey.

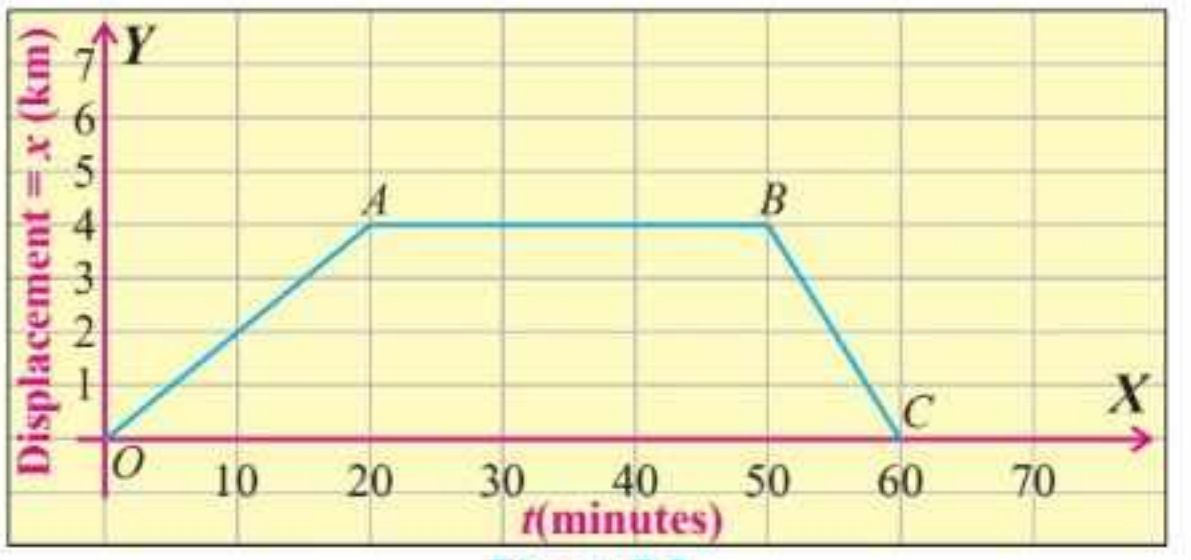


Figure 7.2

(a) Calculate the average velocity for each stage of the journey in km/h.
(b) Calculate the average velocity for the whole journey.
(c) Find the average speed for the whole journey.

Solution: (a) Journey from O to A:

Time = 20 mins, displacement = 4 km

$$\text{Average velocity} = \frac{\text{Displacment}}{\text{Time}} = \frac{4}{20} = \frac{1}{5} \text{ km/min} = 12 \text{ km/h}$$

Journey from A to B:

Time = 30 mins, displacement = 0 km

$$\text{Average velocity} = \frac{\text{Displacment}}{\text{Time}} = \frac{0}{30} = 0 \text{ km/min}$$

$\Rightarrow 0 \times 60 = 0$ km/h $\Rightarrow$ object is stationary.

Journey from B to C:

Time = 10 mins, displacement = -4 km

$$\text{Average velocity} = \frac{\text{Displacment}}{\text{Time}} = \frac{-4}{10} = -\frac{2}{5} \text{ km/min} = -24 \text{ km/h}$$

(b) The displacement for the whole journey is **0**, so average velocity is also **0**.

(c) Total time = 60 mins = 1 h ; Total distance = 4 + 4 = 8 km

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{8}{1} \text{ km/h} = 8 \text{ km/h}$$

Example 4 Javed drives from his home to Murree. He drives for 2 hours at an average velocity of 50 km/h. He then stops for lunch before continuing to Murree. Figure 7.3 shows a displacement-time graph for his journey.

(a) Work out the displacement of Murree from Javed's home.

(b) Work out his average velocity for his whole journey.

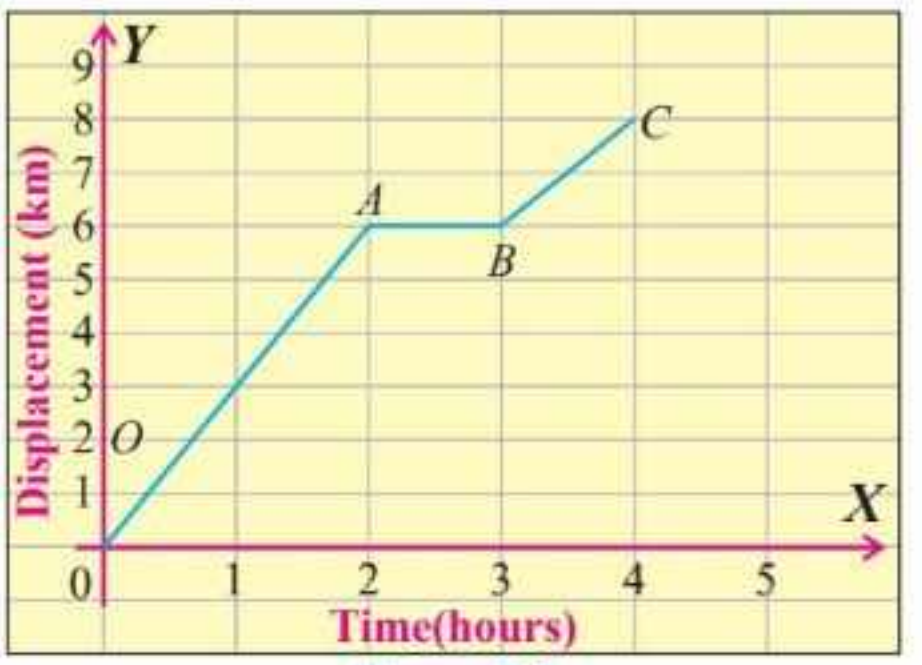


Figure 7.3

Solution: (a) Journey from O to A:

Time = 2 h ; average velocity = 50 km/h

Since, $\text{average velocity} = \frac{\text{Displacement}}{\text{Time}}$

$$50 = \frac{\text{Displacement}}{2}$$

Displacement $= 50 \times 2 = 100$ km

Scale on vertical axis from graph:

$$6 \text{ units} = 100 \text{ km}$$

$$\text{So,} \quad 1 \text{ unit} = \frac{100}{6} = \frac{50}{3} \text{ km}$$

Journey from B to C:

$$\text{vertical rise} = 2 \text{ units}$$

$$\text{Displacement} = 2 \times \frac{50}{3} = \frac{100}{3} \text{ km}$$

$$\text{Total displacement of Murree from Javed's home} = 100 + \frac{100}{3}$$

$$= \frac{400}{3} \simeq 133.3 \text{ km}$$

(b) Average velocity:

Time = 2 h (first drive) + 1 h(lunch stop) + 1 h(second drive) = 4 h

Total displacement = 133.3 km

$$\text{Average velocity} = \frac{133.3}{4}$$

$$\approx 33.33 \text{ km/h}$$

7.1.5 Velocity-Time Graph

Velocity-time-graph helps us visually understand how the velocity of an object changes with time. It is used to:

- Analyze uniform or non-uniform motion
- Determine acceleration
- Calculate displacement
- Study motion in different time intervals clearly and effectively

In a velocity-time graph, the velocity (v) is plotted on the y-axis, and time (t) is plotted on the x-axis.

Key Features of Velocity-Time Graph

1. Slope of the graph $\left(\frac{dv}{dt}\right)$, represents acceleration of the particle.

- If a particle is moving with constant acceleration, its velocity-time graph is a straight line.
  - If $a = \frac{dv}{dt} = 0$, the velocity-time graph is a horizontal straight line. In Figure 7.4, the particle is moving with uniform velocity from point B to point C.

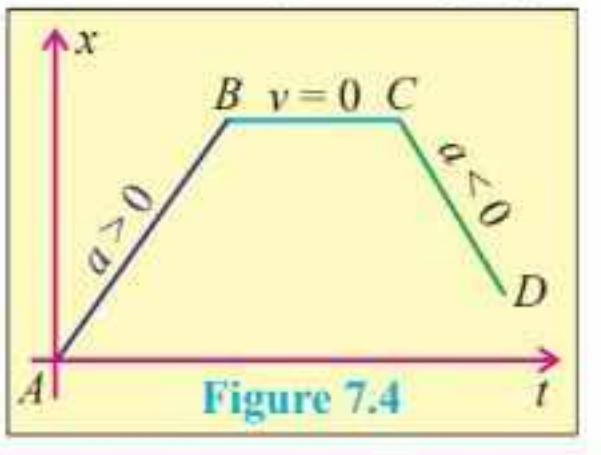


Figure 7.4

- If $a = \frac{dv}{dt} =$ constant > 0, the velocity of the particle is increasing with increase in time, and the velocity-time graph is a rising straight line. In Figure 7.4, the particle is moving with positive uniform acceleration from point A to point B.

Recall!

- Displacement = Area under velocity- time graph
- Acceleration = Slope of velocity time graph

- If $a = \frac{dv}{dt} =$ constant < 0, the velocity of the particle decreases with increase in time, and the velocity-time graph is a falling straight line. In Figure 7.4, the particle is moving with uniform negative acceleration (retardation) from point C to point D.

- If a particle is moving with variable acceleration, then its velocity-time graph is not a straight line. (See Figure 7.5)

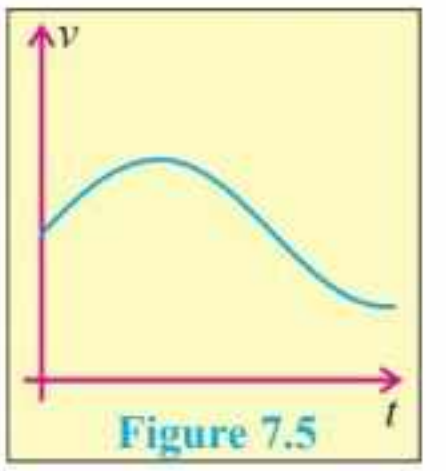


Figure 7.5

2. If A is the area under the velocity-time graph and above the time-axis from $t = t_1$ to $t = t_2$, then from integral calculus:

$$A = \int_{t_1}^{t_2} v(t)\,dt$$

Recall!

If $y = f(x) \geq 0$ for $a \leq x \leq b$, then the area under the curve $y = f(x)$, above the x-axis, and between the vertical lines $x = a$, $x = b$ is equal to $\int_a^b y\,dx$.

Since $v = \frac{dx}{dt}$,

we have:

$$A = \int_{t_1}^{t_2} \frac{dx}{dt}\,dt = \int_{t_1}^{t_2} dx = x(t_2) - x(t_1)$$

Thus, A = displacement of the particle during the time interval from t_1 to t_2.

Note: If velocity-time graph lies both above and below the time axis, and A_1 is area above time axis and graph, while A_2 is the area below time axis and graph, then:

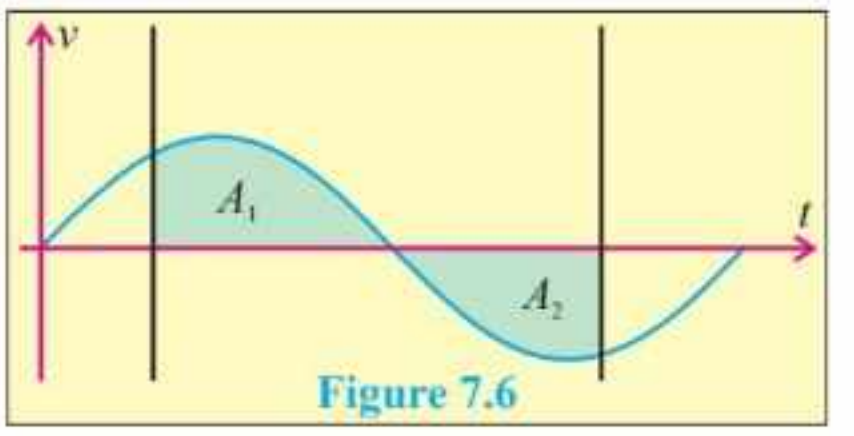


Figure 7.6

Displacement $= A_1 - A_2$

Distance $= A_1 + A_2$

Example 5 Figure 7.7 is the velocity-time graph for a cyclist traveling in a straight line. (The time is in minutes and velocity is in meters per minute).

(a) Calculate the total displacement of the cyclist from his starting position at the end of the 12 minutes.

(b) Determine the uniform deceleration during the final 4 minutes of motion.

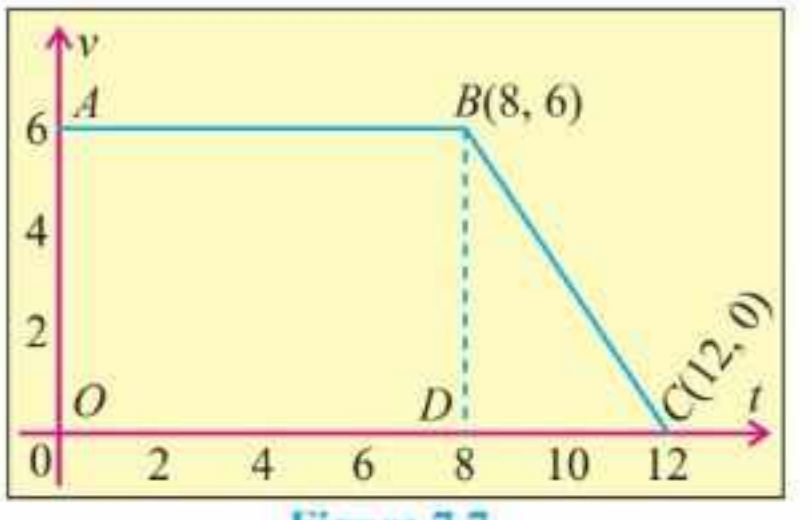


Figure 7.7

Solution: Let x be the total displacement.

The displacement is equal to the area under the velocity–time graph, which is the area of trapezium $OABC$:

x = Area of trapezium $OABC = \frac{1}{2}(a+b)h$ …(1)

Area of Trapezium

Area of Trapezium

A trapezium (trapezoid) is a four-sided plane figure that has exactly one pair of parallel sides. If a and b are the lengths of parallel sides and h is the perpendicular distance between them, then the area is given by:

Area $= \frac{1}{2}(a+b)\times h$

Where:

a = length of lower side $\overline{OC} = 12$,

b = length of upper side $\overline{AB} = 8$, and

h = vertical distance between $\overline{OC}$ and $\overline{AB} = 6$

Substitute the values of a, b and h into equation (1):

$$x = \frac{1}{2}(12+8)\times 6 = 60 \text{ meters}$$

(b) Let a be the acceleration during the final 4 minutes of motion, then:

$$a = \text{slope of } \overline{BC} = \frac{0-6}{12-8}$$

$$= -\frac{6}{4} = -\frac{3}{2} \text{ m/min}^2$$

Example 6 Ali drives a car 400 km in a straight line. Figure 7.8 is a sketch of a velocity–time graph representing his motion. Ali starts with a velocity of u km/h and accelerates to a velocity of 70 km/h in 3 hours. He then maintains his velocity for 3 hours, after which he slows down uniformly to rest over the next $\frac{1}{2}$ hour. Find:

(a) The value of u

(b) The acceleration of Ali during the first 3 hours of his drive.

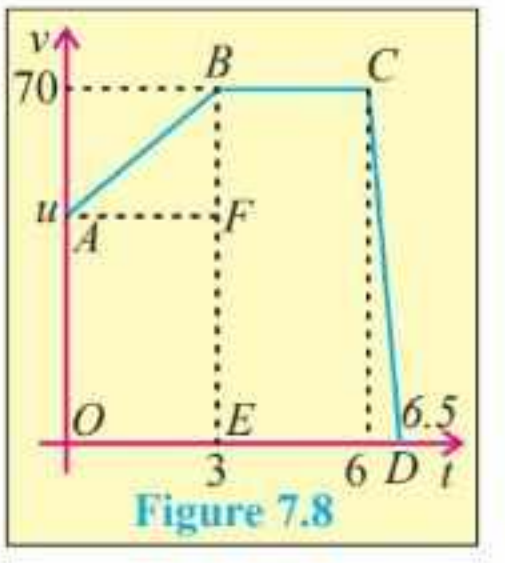


Figure 7.8

Solution: (a) Given that the total displacement is 400 km.

Since the displacement is the area under the velocity-time graph:

x = area of trapezium OABE + area of trapezium BCDE

$$x = \frac{1}{2}(a+b)\times h_1 + \frac{1}{2}(c+d)\times h_2 \qquad \ldots (1)$$

Where: $a = |OA| = u$ $\quad b = |BE| = 70$ $\quad h_1 = |OE| = 3$

$c = |BC| = 6 - 3 = 3$ $\quad d = |ED| = 6.5 - 3 = 3.5$ $\quad h_2 = b = 70$

Substitute these values into equation (1):

$$x = \frac{1}{2}(u+70)\times 3 + \frac{1}{2}(3+3.5)\times 70$$

$$= \frac{1}{2}(3u+210+455) \quad = \frac{1}{2}(3u+665)$$

From equation (1) substitute $x = 400$:

$$400 = \frac{1}{2}(3u+665)$$

$$3u = 135 \qquad u = \frac{135}{3} \qquad = 45\,\text{km/h}$$

(b) Let a be the acceleration during the first 3 hours of his drive:

$$a = \text{slope of } \overline{AB} = \frac{|BF|}{|AF|} \qquad \ldots (1)$$

Where $|BF| = |BE| - |EF| = 70 - u$, $|BF| = 70 - 45 = 25$, $|AF| = |OE| = 3$

Substitute $|BF| = 25$ and $|AF| = 3$ into equation (1):

$$a = \frac{25}{3} \approx 8.33 \text{ km/h}^2$$

7.1.6 Motion with Constant Acceleration

To study the motion of a particle moving with constant acceleration, we consider four quantities: displacement, time, velocity, and acceleration. Usually, some of these quantities are known and the remaining are to be determined. To relate these quantities, we use the equations of motion. Suppose a particle moves along a straight line with constant acceleration a. At time $t = 0$, the particle is at point O (i.e, $x = 0$) and has initial velocity u. Since acceleration is the instantaneous rate of change in velocity with respect to time, we have:

$$\frac{dv}{dt} = a \;\Rightarrow\; dv = adt$$

Integrating both sides:

$$\int dv = \int a dt$$

$$v = at + c \text{ (where } c \text{ is the constant of integration)} \qquad \ldots (1)$$

Using the initial condition $v = u$ at $t = 0$, substitute into equation (1):

$$u = 0 + c \quad \Rightarrow \quad c = u$$

Substitute back into equation (1):

$$v = u + at \qquad \ldots(2)$$

Since velocity is the instantaneous rate of change of displacement with respect to time:

$$v = \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = u + at \Rightarrow dx = (u + at)dt$$

Integrating both sides:

$$\int dx = \int (u + at)dt$$

$$x = ut + a\frac{t^2}{2} + c_1 \qquad \ldots(3)$$

Using the initial condition $x = 0$ at $t = 0$, substitute into equation (3):

$$0 = 0 + 0 + c_1 \quad \Rightarrow \quad c_1 = 0$$

Substitute back into equation (3):

$$x = ut + \frac{1}{2}at^2 \qquad \ldots(4)$$

Now from equation (2):

$$t = \frac{v - u}{a}$$

Substitute this into equation (4) to eliminate t:

$$x = u\left(\frac{v-u}{a}\right) + \frac{1}{2}a\left(\frac{v-u}{a}\right)^2$$

By simplification, we get:

$$2ax = v^2 - u^2 \qquad \ldots(5)$$

The three equations of motion for a particle moving with constant acceleration are:

- $v = u + at$
- $x = ut + \frac{1}{2}at^2$
- $2ax = v^2 - u^2$

These equations are frequently used to describe and analyze the motion of a particle under constant acceleration.

Example 7 A car starts from rest and accelerates uniformly at 3 m/s^2. Find the velocity of the car after 5 seconds.

Solution: Given that $u = 0$ (car stats from vert) $a = 3m/s^2$ $t = 5$, $v = ?$

To find v, we use $v = u + at$

By substituting the values of u, a and t, we get:

$$v = 0 + 3 \times 5 = 15 \text{ m/s.}$$

Example 8 A car accelerates from 10 m/s to 30 m/s in 4 seconds. Find the distance covered during this time.

Solution: Given that $v = 30$ m/s, $u = 10$ m/s, $t = 4$ s.

First, we find its acceleration, using:

$$v = u + at$$

Substitute the known values: $30 = 10 + a \times 4$

To find distance x, we use

$$2ax = v^2 - u^2 \qquad \ldots (1)$$

$$\Rightarrow 4a = 20 \quad \Rightarrow \quad a = \frac{20}{4} = 5 \text{ m/s}^2$$

By substituting the values of u, a and v in (1), we get:

$$2 \times 5x = 30^2 - 10^2 = 800 \Rightarrow x = \frac{800}{10} = 80 \text{ meter}$$

Example 9 Find the position and the velocity of a particle moving along a straight line at any time t seconds, given that starts from rest at $t = 0$ and is subject to an acceleration $a(t) = t^2 + \cos t$.

Solution: Given: $a(t) = t^2 + \cos t$

$$\frac{dv}{dt} = t^2 + \cos t \Rightarrow dv = (t^2 + \cos t)dt \Rightarrow \int dv = \int (t^2 + \cos t)dt \quad \boxed{\because a(t) = \frac{dv}{dt}}$$

$$v(t) = \frac{1}{3}t^3 + \sin t + c$$

Using the initial condition $v = 0$ at $t = 0$:

$$0 = 0 + 0 + c \Rightarrow c = 0$$

So, $$v(t) = \frac{1}{3}t^3 + \sin t$$

$$\frac{dx}{dt} = \frac{1}{3}t^3 + \sin t \Rightarrow \int dx = \int \left(\frac{1}{3}t^3 + \sin t\right)dt \quad \boxed{\because a(t) = \frac{dv}{dt}}$$

$$x(t) = \frac{1}{12}t^4 - \cos t + c_1$$

Example 9 $= 0$ at $t = 0$:

$0 = 0 - 1 + c_1 \Rightarrow c_1 = 1$ So, $x(t) = \frac{1}{12}t^4 - \cos t + 1$

Example 10 The acceleration (in m/s^2) of a particle is given by $a(t) = \cos t + \sin t$.

Given that the velocity is 2 m/s at $t = 0$, find the expression for velocity $v(t)$.

Solution: Given: $a(t) = \cos t + \sin t$

$$\frac{dv}{dt} = \cos t + \sin t \Rightarrow \int dv = \int (\cos t + \sin t)\, dt \qquad \boxed{\because a(t) = \frac{dv}{dt}}$$

$$v(t) = \sin t - \cos t + c$$

Using initial condition $v = 2$ at $t = 0$: $2 = 0 - 1 + c \Rightarrow c = 3$

So, $v(t) = \sin t - \cos t + 3$

7.1.7 Motion of a Free Particle Moving Along a Vertical Line

When a particle moves under the influence of gravity alone and air resistance is neglected, the motion is called free fall or free vertical motion. This motion takes place along a vertical straight line, either upward or downward. The acceleration due to gravity is denoted by ***g*** and is approximately:

$$g = 9.8 \text{ m/s}^2$$

If a particle is projected upward, its velocity decreases uniformly due to gravity and becomes zero at the highest point. If the particle moves downward, it accelerates toward the ground at the rate g. The equations of motion for constant acceleration can also be used in vertical motion by replacing ***a*** with ***g*** for downward motion and with $-$ ***g*** for upward motion.

Example 11 (Free Fall from Rest)

A stone is dropped from the top of a tower 45 m high. Find the time taken to reach the ground and the final velocity on impact.

Solution: Given: $u = 0, x = 45$ m, $a = g = 9.8 \text{ m/s}^2$

Use the equation of motion:

$$x = ut + \frac{1}{2} gt^2$$

Substitute values: $45 = 0 + \frac{1}{2}(9.8)t^2$

Solve for t:

$$t = \sqrt{\frac{45 \times 2}{9.8}} \approx 3.03 \text{ s}$$

To find the final velocity on impact, use the equation:

$$v = u + gt$$

Substitute the known values:

$$v = 0 + 9.8 \times 3.03 \approx 29.7 \text{ m/s}$$

Example 12 (Upward Throw)

A ball is thrown vertically upward with a speed of 20 m/s. Find, the maximum height attained and the time to reach the maximum height.

Solution: Given: $u = 20$ m/s, $v = 0$ m/s (at top), $a = g = -9.8 \text{ m/s}^2$

Use the equation of motion:

$$v = u + gt$$

By substituting the values of u, v and g, we get:

$$0 = 20 + (-9.8)t$$

$$t = \frac{20}{9.8} \approx 2.04 \text{ s}$$

To find maximum height, use the equation of motion:

$$x = ut + \frac{1}{2} gt^2$$

By substituting the values of u, t and g, we get:

$$x = 20 \times 2.04 + \frac{1}{2}(-9.8)(2.04)^2$$

$$\approx 20.4 \text{ m}$$

EXERCISE 7.1

1. Junaid covers 100 km in the first 2 hours, stops for 1 hour, and then drives 60 km in the next 2 hours. The displacement-time graph is shown in Figure 7.9.
   (a) What is Junaid's average velocity?
   (b) What is his velocity during the last phase of the journey?

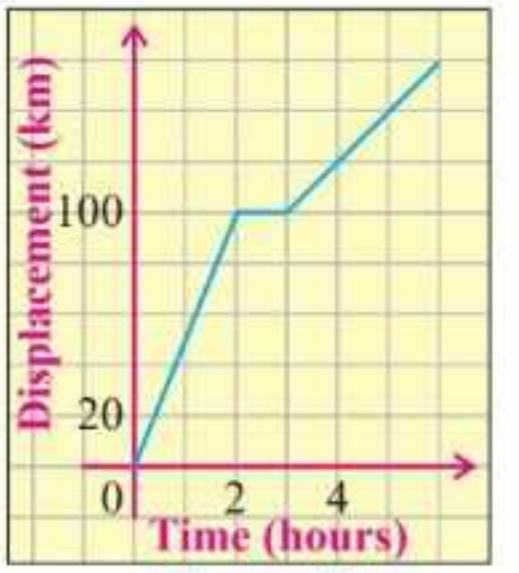


Figure 7.9

2. The displacement-time graph of a moving car is shown in Fig. 7.10. The car changes its direction during the journey, and its motion occurs in three distinct phases.
   (a) What is the car's velocity in each phase?
   (b) When did the car pass its starting point again?
   (c) What is the total distance travelled?
   (d) What is the average velocity?

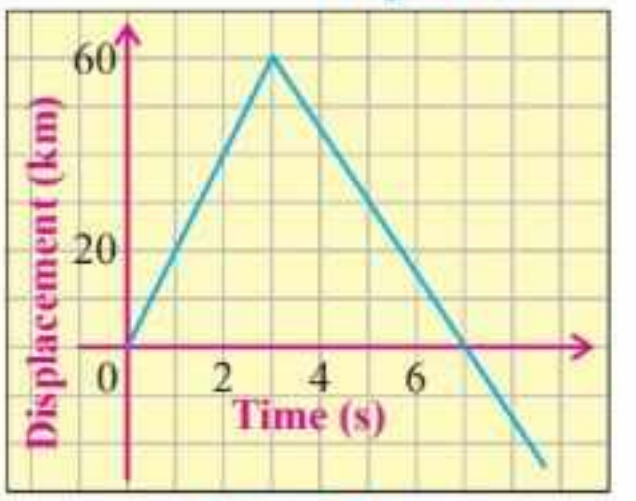


Figure 7.10

3. Figure 7.11 is the velocity-time graph of Aslam's motion as he runs along a straight track.
   For the first 5 seconds, he accelerates uniformly from rest to a velocity of 8 m/s. He then maintains this velocity for a further 8 seconds. Find:
   (a) The rate at which Aslam accelerates.
   (b) The displacement from the starting point of Aslam after 13 seconds.

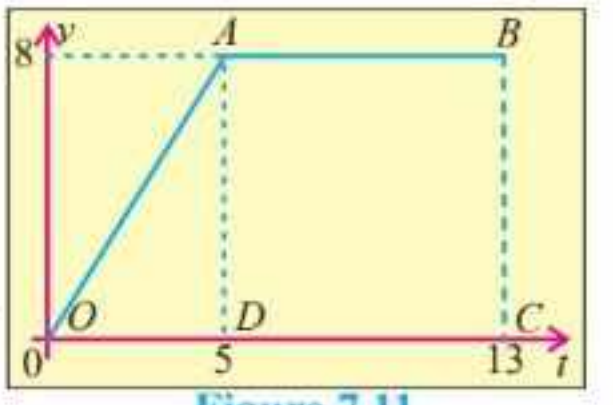


Figure 7.11

4. A car is travelling along a straight road. At $t = 0$ seconds, the car passes a point A with a velocity of 12 m/s, and this velocity is maintained up to $t = 35$ seconds. The driver then applies the brakes, and the car decelerates uniformly, coming to rest at the point B at $t = 40$ seconds.
   (a) Sketch a velocity–time graph to illustrate the motion of the car.
   (b) Find the displacement from A to B.

5. Figure 7.12 shows the velocity–time graph of an object moving in a straight line over a 12-second time interval.
   (a) At what time did the object change its direction of motion?
   (b) Calculate the total distance travelled and the displacement of the object during the 12 seconds.

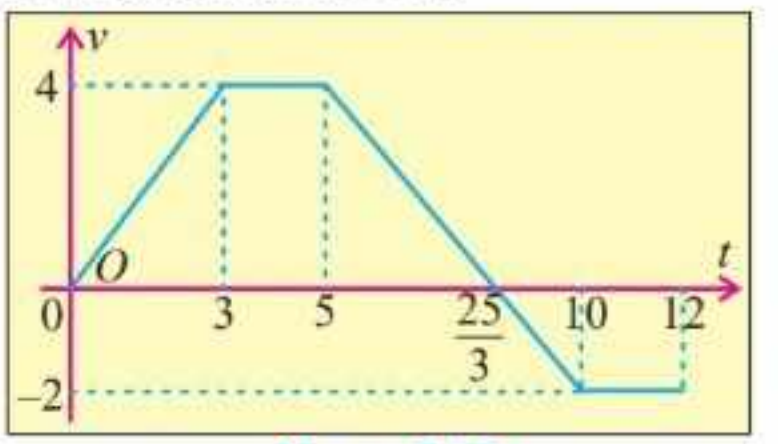


Figure 7.12

6. A vehicle moving with a speed of 36 km/h is stopped by applying brakes and comes to rest in 10 seconds. Find the retardation and distance covered during this time.
7. A train moving at 54 km/h is accelerated uniformly to 90 km/h in 20 s. Find the acceleration and the distance covered in this time.
8. A particle moving in a straight line starts from rest and is accelerated uniformly to attain a velocity of 30 km/h in 5 seconds. Find the distance travelled by the particle in the last three seconds.
9. A particle is projected vertically upwards with a velocity 100 m/s, and another is let fall from a height of 200 m at the same time. Find the height of the point where they meet each other.
10. A ball is thrown vertically upward with a velocity of 20 m/s. Find, the maximum height reached, and the time taken to return to the thrower's hand.
11. A rock is dropped from a cliff. One second later, another rock is dropped from the same height.
    (a) What is the distance between the two rocks 2 seconds after the first rock is dropped?
    (b) At what time will the distance between them be 14.7 m?
12. The acceleration of a particle is $a(t) = \frac{1}{1+t}$ for $t \geq 0$. If the velocity is 0 m/s when $t = 0$, find the velocity at $t = 5$ seconds.
13. The motion of a robotic arm moving along a straight track is described by the velocity function
$$v(t) = 6 + t - t^2$$
where $v(t)$ is in meters per second and t is the time in seconds, valid for $0 \leq t \leq 5$. Find the distance travelled by the arm.

Unit 08 Numerical Solutions of Nonlinear Equations

INTRODUCTION

Nonlinear equations, where variables appear with powers exceeding one or within trigonometric, exponential, or logarithmic functions, are ubiquitous in mathematics, science, and engineering. Since most such equations resist exact algebraic solutions, numerical techniques are essential for approximating their real roots. The Graphical Method offers a visual foundation by plotting functions and identifying x-axis intersections. More sophisticated numerical methods—the Bisection Method, Regula-Falsi Method, and Newton-Raphson Method—systematically refine approximations to achieve desired accuracy through iterative procedures grounded in function continuity and the intermediate value theorem. This chapter explores solution techniques for nonlinear equations and introduces the Trapezium Rule, a numerical integration method for estimating definite integrals when analytical integration is impractical. Beyond theoretical importance, these methods have substantial practical applications in determining chemical equilibrium concentrations, regulating cardiac rhythms, analyzing nonlinear electronic circuits with diodes and transistors, and solving cryptographic problems.

8.1 Basic Principles of Approximating Real Roots of Nonlinear Equations

Before learning methods to find roots of nonlinear equations, it is important to understand some basic definitions. A **root** (or **zero**) of a function $y = f(x)$ is a value of x for which $y = 0$. In other words, a root is the solution of the equation $f(x) = 0$. Since the equation $y = 0$ represents the $.x.$-axis and a root is a common solution of the equations $y = f(x)$ and $y = 0$, the **geometric representation of a root** of the function $y = f(x)$ is the point on the Cartesian plane where the graph of the function intersects the x-axis. For example, if $f(x) = x^2 - 4$ then $f(x) = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = 2$ or $x = -2$. So, 2 and -2 are roots of the function (see Figure 8.1). A **nonlinear equation** is an equation of the form, $y = f(x) =$

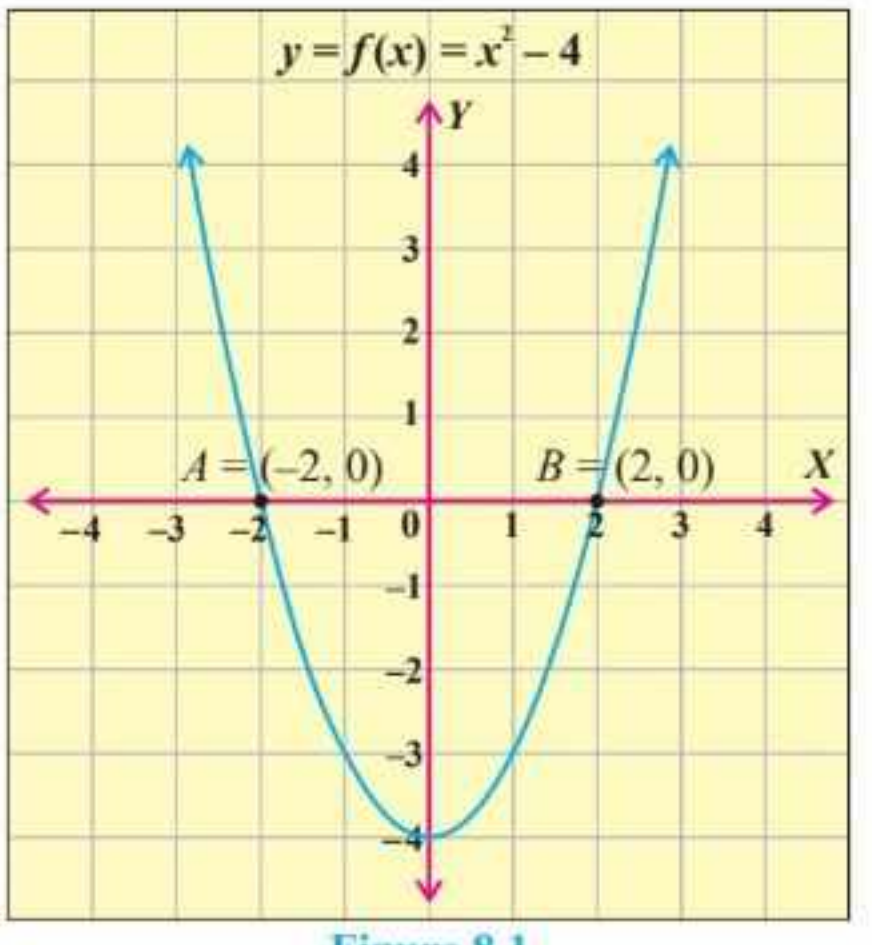


Figure 8.1

0, where f is not a linear function in the variable x, that is, $f(x)$ cannot be expressed in the form $ax+b$ for some real numbers a and b with $a \neq 0$. In such equations, the variable may appear with powers greater than one or inside trigonometric, exponential, logarithmic or other nonlinear functions. For example:

(i) $x^3-1=0$ (ii) $x^3+2x+5=0$ (iii) $x-\cos x=0$

Note that $x=1$ is a root of the cubic function $y=x^3-1$ which can easily be found analytically. Consider another cubic function $y=x^3+2x+5$. This equation has an exact root given by

$$x=\sqrt[3]{-\frac{5}{2}+\sqrt{\frac{707}{108}}}+\sqrt[3]{-\frac{5}{2}-\sqrt{\frac{707}{108}}}$$

which can be obtained analytically using algebraic methods but the expression is complicated and difficult to handle in practice. On the other hand, its approximate root (correct to two decimal places) $x \approx -1.33$ can easily be obtained using graphical or numerical methods. In fact, formulas for solving quadratic, cubic, and quartic equations do exist but no general algebraic formula is available for quintic or higher-degree polynomials. Therefore, numerical methods are essential for finding approximate roots in such cases. Before exploring these methods, we need to introduce a basic result known as the **Location of Root Theorem (LRT)**.

8.1.1 Location of Root Theorem (LRT)

Theorem 1: Let f be a real-valued function which is continuous on a closed interval $[a,b]$ If $f(a)f(b)<0$ then f has at least one real root c in the open interval $[a,b]$

The geometric idea of the Location of Root Theorem (LRT) comes from the behavior of a continuous graph. If a function is continuous on an interval $[a,b]$, its graph can be drawn without lifting the pencil from the paper. Since $f(a)f(b)<0$, the values of $f(a)$ and $f(b)$ have opposite signs. Without loss of generality, assume that $f(a)<0$ and $f(b)>0$. In this case the point $(a, f(a))$ lies below the x-axis and the point $(b, f(b))$ lies above the x-axis as shown in Figure 8.2. Since f is continuous on $[a,b]$, the graph can be drawn continuously

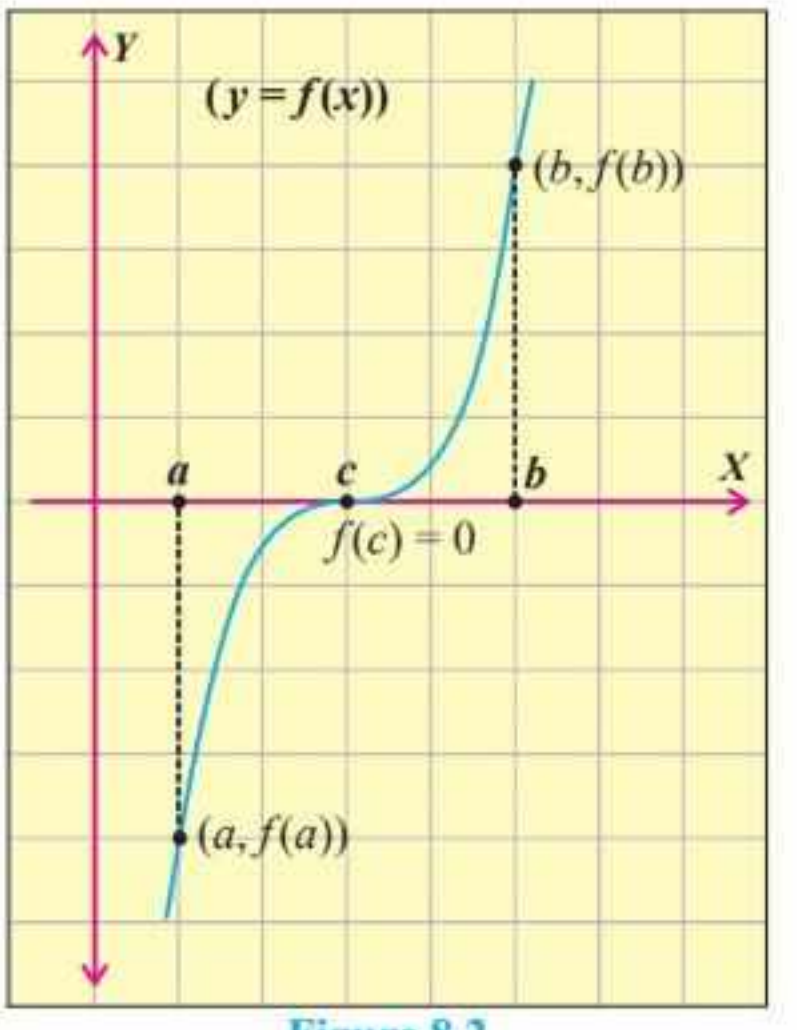


Figure 8.2

from the point $(a, f(a))$ to the point $(b, f(b))$ without any jump or break, in particular, at x-axis.

As the graph moves from below the x- axis to above the x- axis, it must cross the x-axis at some point c in the interval $[a, b]$. At this point, $f(c)=0$. Which shows that c is a real root in the interval $[a, b]$, Which shows that at least one root must exist. We call the interval $[a,b]$ an **interval of sign change for the function f**, because the function changes sign from positive to negative or vice versa.

Example 1 Show that the function $f(x)=x^4-200x-50$ has at least two real roots by finding its intervals of sign change.

Solution: The given function is $f(x)=x^4-200x-50$

Continuity:

Note that, $\lim_{x\to c} f(x)=\lim_{x\to c}(x^4-200x-50)=f(c) \ \forall c\in\mathbb{R}$

$\Rightarrow f$ is continuous at every real number.

Interval of sign change:

At $x=-3$, $f(-3)=81+600-50=631>0$

At $x=-2$, $f(-2)=16+400-50=366>0$

Similar calculations are given in the table below.

x	–3	–2	–1	0	1	2	3	4	5	6
$f(x)$	631	366	151	–50	–249	–434	–569	–594	–425	46
Sign of $f(x)$	+	+	+	–	–	–	–	–	–	+

Note that, $f(-1)f(0)<0$. It means that, $[-1,0]$ is an interval of sign change for f.

Similarly, $f(5)f(6)<0$. Thus, [5, 6] is another interval of sign change. We have shown two intervals of sign change for f. Since f is continuous on these intervals, it has at least two real roots.

8.2 Graphical Method for Approximating Real Roots

After learning the Location of Root Theorem (LRT), we can use a graphical method to approximate roots of nonlinear equations. This method provides a visual way to locate where a function crosses the x-axis.

The graphical method is mainly suitable for obtaining the root correct up to one or two decimal places and is used to provide an initial estimate for numerical methods such as the Bisection Method, Regula-Falsi Method, and Newton–Raphson Method.

Steps of Graphical Method:

1. Identify the interval of sign change where $f(a)\cdot f(b)<0$, if not provided.
2. Draw the graph of $y=f(x)$ on xy-plane within the interval of sign change only.
3. Observe where the graph crosses the x-axis.

Example 2 Find a root of the equation $x-\cos x=0$ graphically, approximate to two decimal places.

Solution:

Fixing Calculator: Fix the calculator to two decimal places.

Let $y=f(x)=x-\cos x$

Continuity:

Note that, $f(c)=\lim_{x\to c}(x-\cos x)\ \ \forall\, c\in\mathbb{R}$

$\Rightarrow$ f is continuous at every real number.

Identify Interval of Sign Change

Note that $f(0)=0-\cos 0=-1<0$ and $f(1)=1-\cos(1)=0.46>0$. Thus, [0,1] is the interval of sign change and hence by LRT, a root lies within this interval.

Draw the graph

Calculate some functional values on some points inside the interval [0, 1] as given in the following table.

x	f
0.1	–0.89
0.2	–0.78
0.3	–0.66
0.4	–0.52
0.5	–0.38
0.6	–0.23
0.7	–0.06
0.8	0.10
0.9	0.28

Draw the graph of $y=x-\cos x$ on the interval [0, 1] as shown in Figure 8.3.

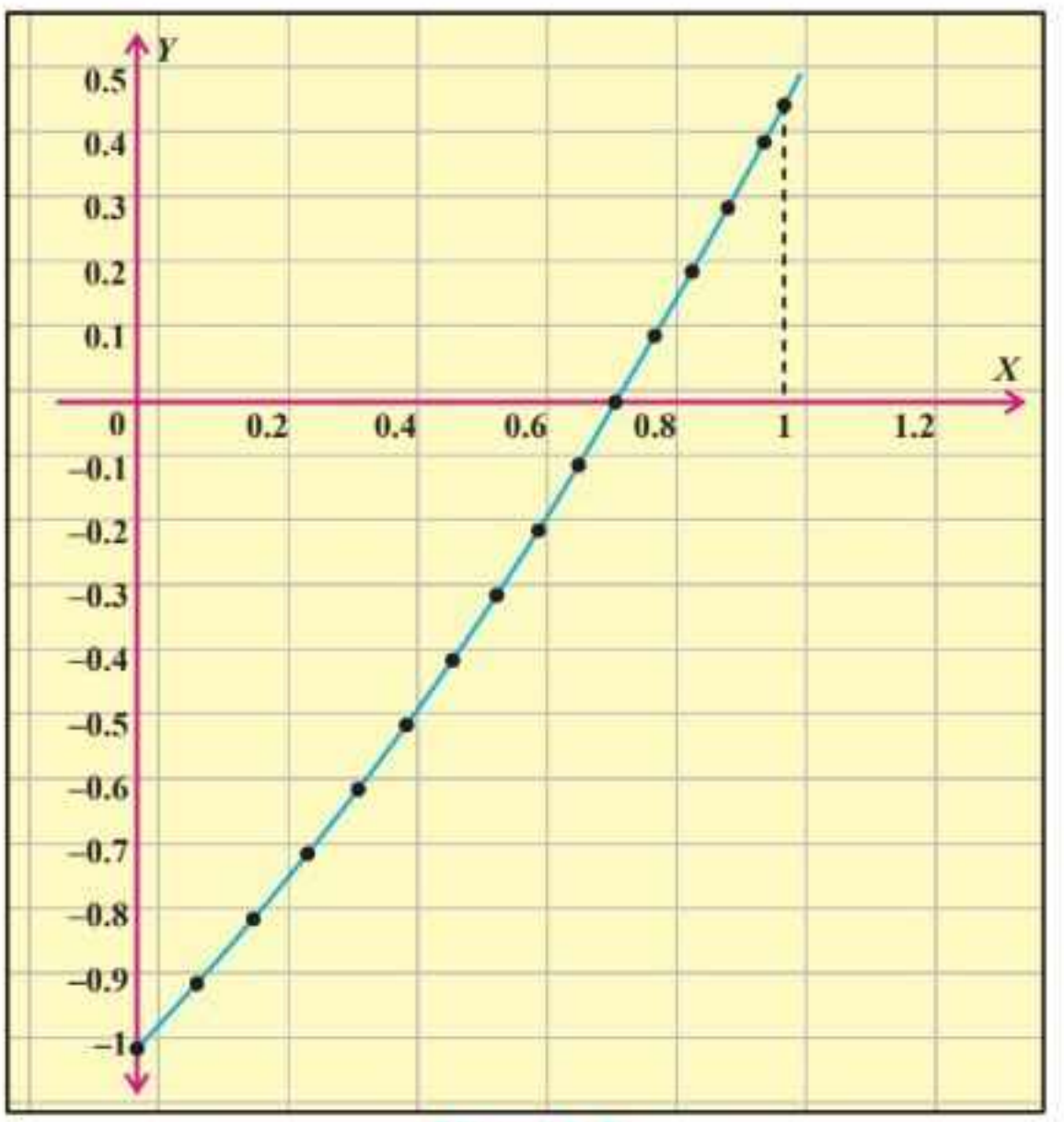


Figure 8.3

Read the root

It is clear from the figure, that the curve is crossing the x-axis at $x \approx 0.74$. Hence, the required root approximate to two decimal places is

$$\boxed{x \approx 0.74}$$

Note

While performing calculations with trigonometric equations, always keep your calculator in radian mode.

EXERCISE 8.1

1. Find the real roots of the following functions correct to two decimal places using the graphical method, within the specified interval of sign change.
   (i) $f(x) = 3x + \sin(x) - e^x$ [0, 1] (ii) $f(x) = x^4 - x - 10$ [1, 2]
2. Find the real roots of the following functions correct to two decimal places using the graphical method, by finding some interval of sign change.
   (i) $f(x) = x^3 - x - 1$ (ii) $f(x) = xe^x - 2$ (iii) $f(x) = \ln(x) - 2$

8.3 Numerical Methods for Solving Nonlinear Equations

To find roots accurate to two or more decimal places, we need systematic numerical methods. These methods use the concept of intervals where the function changes sign to progressively narrow down the root. In this section, we will study three important numerical techniques—Bisection Method, Regula-Falsi Method, and Newton–Raphson Method.

8.3.1 General Rule of Accuracy, Tolerance and Calculator Setup

Before applying any numerical method for solving nonlinear equations, it is important to decide the desired accuracy of the root. This is done using a tolerance value, which determines how close the approximate root should be to the actual root to achieve the required number of decimal places.

Decide tolerance based on desired decimal places:

There are several methods to check the accuracy of a root in numerical methods, such as reducing the interval length and checking the function value. In this chapter, we will follow the function value criterion and consider the root accurate when $|f(x)|$ is sufficiently close to zero.

If n is the number of decimal places required for the root c of the function $f(x)$, then

$$|f(c)| < \frac{10^{-n}}{2}.$$

Examples:

- For 2 decimal places ($n = 2$): $|f(c)| < 0.005 \approx 0.01$
- For 3 decimal places ($n = 3$): $|f(c)| < 0.0005 \approx 0.001$

Calculator Setup

- To obtain a root correct to n decimal places, fix the calculator to at least $n + 1$ decimal places through all the intermediate calculations and round the final answer to the required accuracy.
- Use radian mode if trigonometric functions are involved.

8.3.2 Bisection Method

The Bisection Method is a simple and reliable numerical technique to find roots of nonlinear equations in one variable. It is based on the Location of Root Theorem (LRT): if a continuous function $f(x)$ changes sign over an interval $[a,b]$, that is, $f(a)f(b)<0$, then at least one root exists within that interval. The method systematically halves the interval to narrow down the location of the root until the desired accuracy is reached.

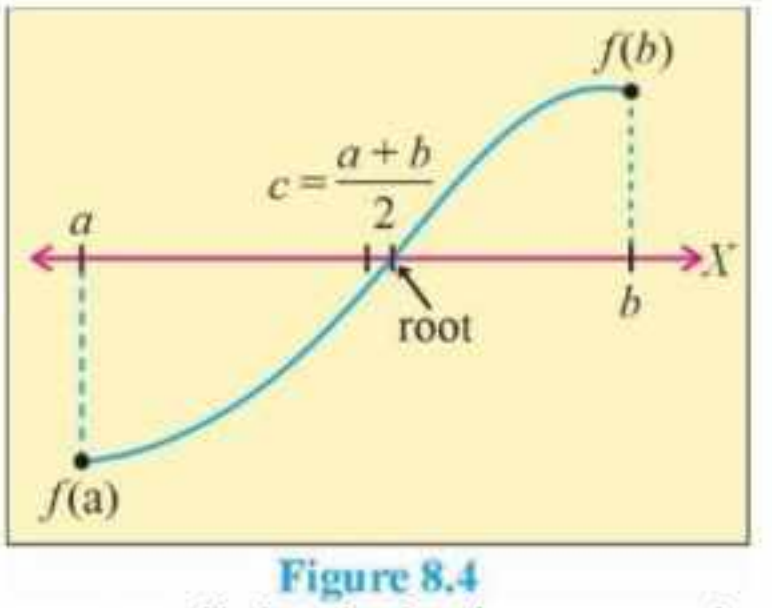


Figure 8.4

Steps of Bisection Method:

1. Check that continuity of f on $[a, b]$ and the condition $f(a)\ f(b)<0$.
2. Compute the midpoint: $c=\dfrac{a+b}{2}$.
3. Check the function value $f(c)$:
   - If $|f(c)|\approx 0$, then c is the desired root.
   - Otherwise, determine the subinterval containing the root:
     - If $f(a)\cdot f(c)<0$, the root lies in $[a, c]$. Set $b=c$.
     - If $f(c)\cdot f(b)<0$, the root lies in $[c, b]$. Set $a=c$.
4. Repeat Steps 2–3 until $|f(c)|$ is near zero.

Note: the by section method guarantees that, after a finite number of steps, there must exists $c\in\ ]a, b[$ such that $|f(c)|\approx 0$ up to the desired accuracy.

Example 3 Use bisection method to find a root of $f(x)=x^3-3x-10$ within the interval $[2, 3]$ accurate up to two decimal places.

Solution: Check Initial Interval and Continuity:

Note that, $f(c)=\lim\limits_{x\to c}(x^3-3x-10)\ \ \forall\ c\in[2,3]$.

$\Rightarrow$ f is continuous on $[2, 3]$.

Clearly, f being a polynomial in x is continuous on $[2, 3]$.

$f(2)=2^3-3(2)-10=-8<0, f(3)=3^3-3(3)-10=8>0$

Sign change occurs: $f(2)f(3)<0\ \Rightarrow$ root exists in $[2, 3]$.

Apply Bisection Method: (Intermediate values rounded to 4 decimal places)

Iteration 1: $a=2,\ b=3$

$$c=\frac{2+3}{2}=2.5 \qquad f(2.5)=15.625-7.5-10=-1.875<0$$

Since $f(2.5)<0$ and $f(3)=8>0$, the root lies in $]2.5,3[$. thus, consider $a=c$.

Iteration 2: $a=2.5,\ b=3$

$$f(2.75)=20.7969-8.25-10=2.5469>0$$

Root lies in $]2.5,2.75[$ and so, consider $b=c$.

Iteration 3:

$$a=2.5,\ b=2.75 \qquad c=\frac{2.5+2.75}{2}=2.625$$

$$f(2.625)=18.087-7.875-10=0.212>0$$

Root lies in$[2.5,\ 2.625]$ and so, set $b=c$.

Note

In the iteration table, the signs written inside the brackets along with the values of a, b and c means their functional values are negative or positive accordingly.

Continuing the same process, similar calculations are performed for the next iterations. The results are summarized in the following table.

Iteration	a	b	$c=\frac{a+b}{2}$	$f(c)$	New Interval
1	2 (–)	3 (+)	2.50 (–)	–1.875	[2.5,3]
2	2.5 (–)	3 (+)	2.75 (+)	2.547	[2.5,2.75]
3	2.5 (–)	2.75 (+)	2.625 (+)	0.212	[2.5,2.625]
4	2.5 (–)	2.625 (+)	2.5625 (–)	–0.8611	[2.5625,2.625]
5	2.5625 (–)	2.625 (+)	2.5938 (–)	–0.3308	[2.5938,2.625]
6	2.5938 (–)	2.625 (+)	2.6094 (–)	–0.0609	[2.6094,2.625]
7	2.6094 (–)	2.625 (+)	2.6172 (+)	0.0755	[2.6094,2.6172]
8	2.6094 (–)	2.6172 (+)	2.6133 (+)	0.0072	Stop

Approximate Root: $|f(2.6133)|=0.0072\approx 0.01$

Final Root: $x\approx 2.61$ (correct to 2 decimal places)

8.3.3 Regula-Falsi Method (False Position Method)

The Regula-Falsi Method (False Position) is a numerical method used to find a real root of a nonlinear equation $f(x) = 0$. Like bisection method, this method also based directly on the Location of Root Theorem (LRT). Suppose the function $y = f(x)$ is continuous on an interval $[a, b]$ and $f(a)f(b) < 0$, then there exists at least one root in $[a, b]$. Without loss of generality, we assume that $f(a) < 0$ and $f(b) > 0$. The idea of Regula-Falsi is to approximate the root by drawing a straight line through the points $(a, f(a))$ and $(b, f(b))$ (called **secant line**) and finding where this line crosses the x-axis. This intersection is used often as a better approximation of the root than the simple midpoint $c = \frac{a+b}{2}$ used in the Bisection Method (see Figure 8.5).

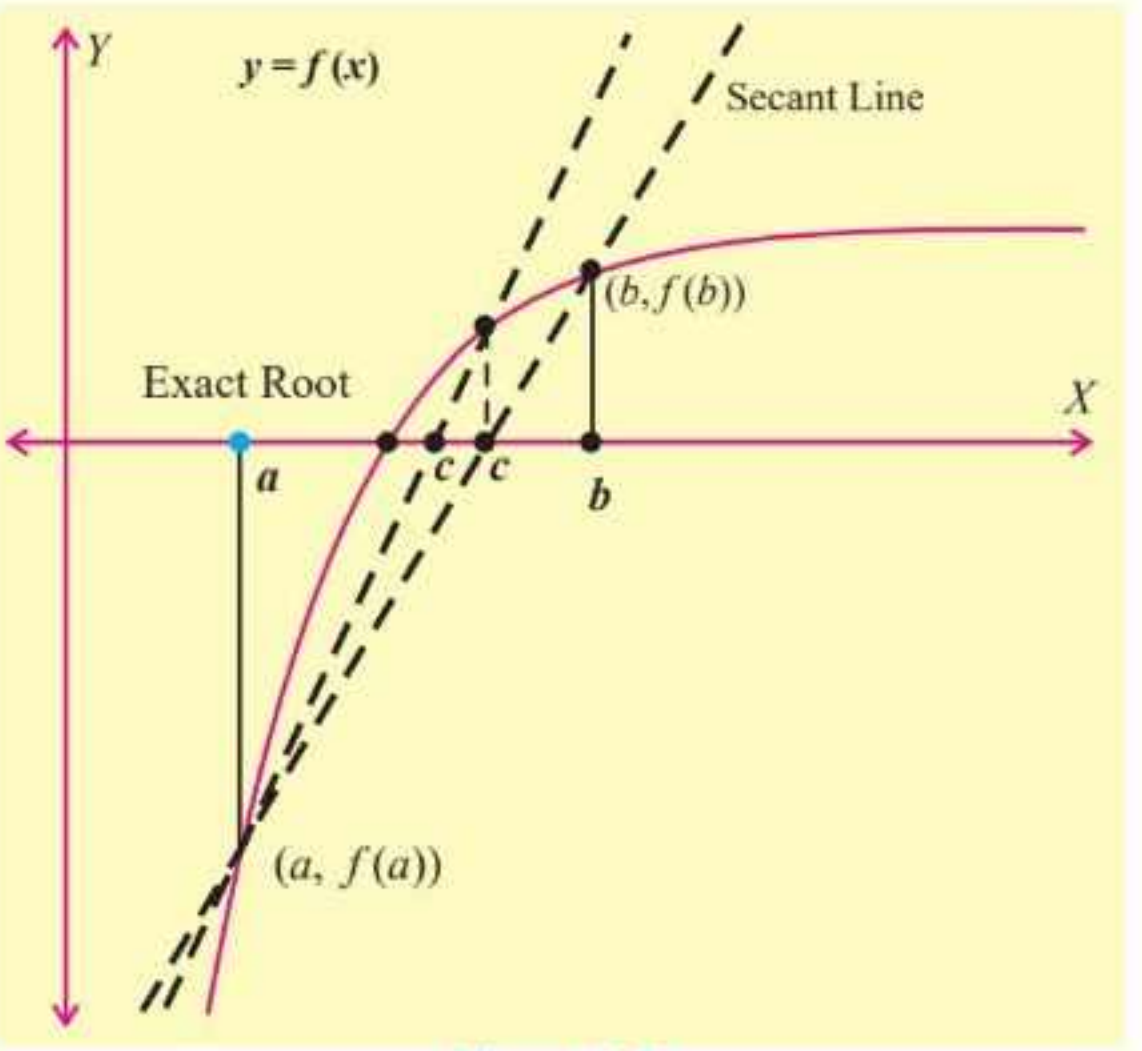


Figure 8.5

Slope of the secant line passing through the points $(a, f(a))$ and $(b, f(b))$ is $\frac{f(b)-f(a)}{b-a}$. Thus, the equation of secant line is given by:

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

For the x-intercepts of the secant line, we substitute $y = 0$ and $x = c$ in the above equation to get:

$$c = a - \frac{f(a)(b-a)}{f(b)-f(a)}$$

On simplifying:

$$c = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

Steps of the Regula-Falsi Method:

1. Check that continuity of f on $[a, b]$ and the condition $f(a)f(b) < 0$.

2. Compute the value of c using the Regula-Falsi formula:

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

3. Check the function value $f(c)$:
   - If $|f(c)| \approx 0$, then c is the desired root.
   - Otherwise, determine the subinterval containing the root:
     - If $f(a) \cdot f(c) < 0$, the root lies in $[a, c]$. Set $b = c$.
     - If $f(c) \cdot f(b) < 0$, the root lies in $[c, b]$. Set $a = c$.
4. Repeat Steps 2–3 until $|f(c)|$ is less than the desired tolerance.

Note

The Regula-Falsi method guarantees that, after a finite number of steps, there exists some $c \in]a, b[$ such that $|f(c)| \approx 0$ upto the desired accuracy.

Example 4 Use Regula-Falsi method to find a root of $f(x) = x^3 - 3x - 10$ within the interval [2, 3] accurate up to two decimal places.

Solution: We set $a = 2$, $b = 3$ and apply the Regula-Falsi Formula:

$f(2) = -8 < 0$ and $f(3) = 8 > 0 \Rightarrow f$ has a root in [2, 3]

Continuity: Note that, $f(c) = \lim_{x \to c}(x^3 - 3x - 10) \quad \forall\ c \in [2,3]$.

$\Rightarrow f$ is continuous on [2, 3].

Regula-Falsi Formula: $c = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

Iterations (Intermediate 4 decimals)

Iteration 1: $a = 2$, $b = 3$

$$f(a) = f(2) = 2^3 - 3(2) - 10 = -8 < 0$$
$$f(b) = f(3) = 3^3 - 3(3) - 10 = 8 > 0$$
$$c = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{(2)(8) - (3)(-8)}{(8) - (-8)} = 2.5$$
$$f(c) = f(2.5) = (2.5)^3 - 3(2.5) - 10 = -1.8750 < 0$$

Since $f(2.5)\, f(3) < 0$, root lies in the interval [2.5, 3].

Iteration 2: Now, $a = 2.5$, $b = 3$

$$f(a) = f(2.5) = -1.8750 < 0, \quad f(b) = f(3) = 8 > 0$$
$$c = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{(2.5)(8) - (3)(-1.875)}{(8) - (-1.875)} = 2.5949$$
$$f(c) = f(2.5949) = -0.3119 < 0$$

Since $f(2.5949)\, f(3) < 0$, root lies in the interval [2.5949, 3].

Iteration 3: Now we take, $a = 2.5949,\ b = 3$

$$f(a) = f(2.5949) = -0.3119 < 0,\quad f(b) = f(3) = 8 > 0$$

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{(2.5949)(8) - (3)(-0.3119)}{(8) - (-0.3119)} = 2.6101$$

$$f(c) = f(2.6101) = -0.0487 < 0$$

Since $f(2.6101)f(3) < 0$, root lies in the interval [2.6101, 3].

Iteration 4: Take, $a = 2.6101,\ b = 3$

$$f(a) = f(2.6101) = -0.0487 < 0,\quad f(b) = f(3) = 8 > 0$$

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{(2.6101)(8) - (3)(-0.0487)}{(8) - (-0.0487)} = 2.6125$$

$$f(c) = f(2.6125) = -0.0068 < 0$$

Since $|f(2.6125)| = |-0.0068| = 0.0068 \approx 0.01 \Rightarrow 2.6125$ is a root.

Final Root: $\boxed{x \approx 2.61 \text{ (correct to 2 decimal places)}}$

8.3.4 Newton–Raphson Method

The Newton–Raphson Method is a numerical method used to find an approximate real root of a nonlinear equation $f(x) = 0$. This method works by drawing a tangent to the curve $y = f(x)$ at a point near the root and using the point where this tangent crosses the x-axis as a better approximation of the root. The initial guess chosen is based on a sign-change interval.

Since this method requires the function to be differentiable (having a slope of tangent), it usually reaches a good approximation faster than the Bisection or Regula-Falsi methods. The derivative of f should not be zero near the root. This method does not rely directly on the Location of Root Theorem (LRT).

Suppose x_0 is an initial guess for the root of $f(x) = 0$, where:

1. f is differentiable near x_0, and
2. $f'(x) \neq 0$ near x_0.

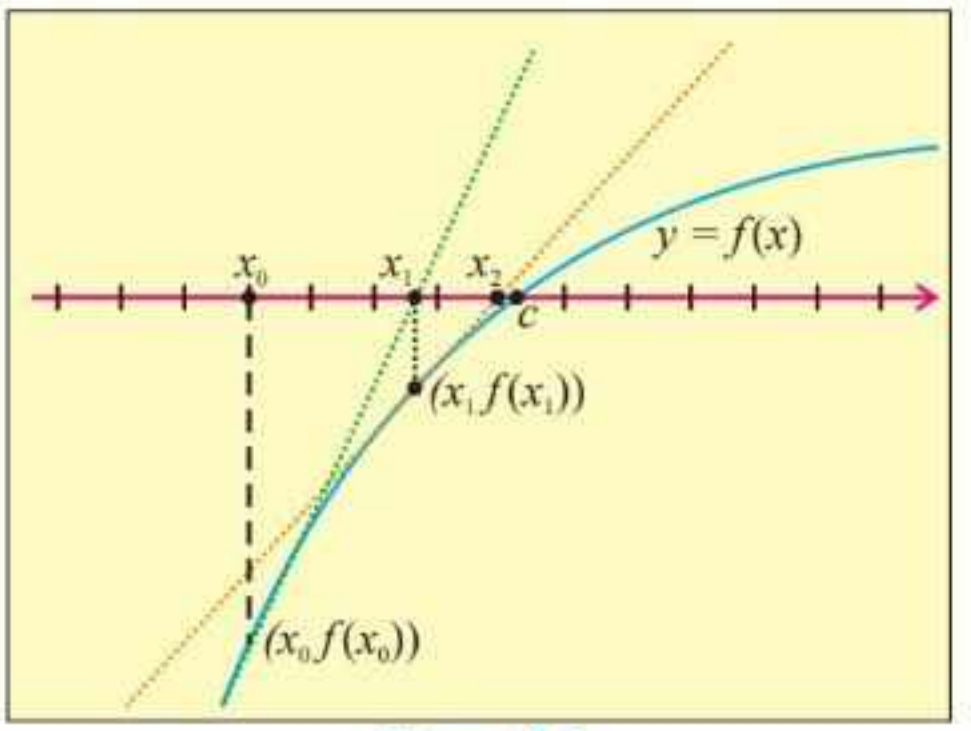


Figure 8.6

The slope of the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is given by $f'(x_0)$. Draw a tangent to the curve at this point (see Figure 8.6).

The equation of the tangent line is:

$$y-f(x_0)=f'(x_0)(x-x_0)$$

This tangent intersects the x-axis at $x=x_1$. Substituting $y=0$ and $x=x_1$ gives:

$$0-f(x_0)=f'(x_0)(x_1-x_0)$$

Solving for x_1:

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$

Similarly, if we draw a tangent at $x=x_1$, its x-intercept x_2 is:

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

Repeating this process produces a sequence of x-intercepts:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, n=0,1,2,\ldots, \quad f'(x_n)\neq 0$$

This is the Newton–Raphson formula, used to approximate the root of $f(x)=0$ iteratively.

From Figure 8.5, it is clear that each new approximation x_{n+1} is closer to the root, $x = c$, than the previous one x_n. If there exists some non-negative integer m such that $x_{m+1} - x_m$ is closed to zero, then x_m will be the desired root, i.e. $|f(x_m)| \approx 0$

Steps of Newton-Raphson Method:

1. **Prepare the equation:** Write the given equation in the form $f(x)=0$
   Choose an initial approximation x_0 from the interval where the function changes sign and ensure that:
   (i) $f(x)$ is differentiable within this interval. (ii) $f'(a) \neq 0$ near x_0.
2. **Set up the formula:** $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, n=0,1,2,\ldots$
   Use this to compute successive approximations.
3. **Check for convergence:** If $|f(x_{n+1})|\approx 0$, then stop.
4. **Conclude the root:** $x=x_{n+1}$ is the root correct to the desired accuracy.

Example 5 Use Newton-Raphson method to find a root of $f(x)=x^3+2x^2+10x-20$ correct to two decimal places.

Solution: Prepare the function, Given:

$$f(x)=x^3+2x^2+10x-20$$

Choose initial approximation:

Note that: $f(1) = -7 < 0$ and $f(2) = 16 > 0$.

- Note that, $f(c) = \lim_{x \to c}(x^3 + 2x^2 + 10x - 20) \quad \forall c \in [1, 2]$.

  $\Rightarrow$ is continuous on the interval of sign change [1, 2].

  $\Rightarrow$ Root lies in the interval [1, 2].
- Take initial guess: $x_0 = 1.500$.
- Calculator fixed to 3 decimal places (for accuracy).

Derivative: $f' = 3x^2 + 4x + 10$

- Note that, f' exists and $f'(x) \neq 0$ around $x_0 = 1.5$.
- $\Rightarrow$ Newton–Raphson method can be applied.

Newton–Raphson formula: $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$, $n = 0, 1, 2, \ldots$

Perform iterations: We have, $x_0 = 1.5$.

$$f(x_0) = f(1.5) = (1.500)^3 + 2(1.500)^2 + 10(1.500) - 20 = 3.375 + 4.500 + 15 - 20 = 2.875 \neq 0$$

Iteration 1: $f'(1.5) = 3(1.500)^2 + 4(1.500) + 10 = 6.750 + 6 + 10 = 22.750$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.500 - \frac{2.875}{22.750} = 1.374$$

$$f(x_1) = f(1.374) \approx (1.374)^3 + 2(1.374)^2 + 10(1.374) - 20 \approx 0.110 \neq 0$$

Iteration 2: $f'(1.374) \approx 3(1.374)^2 + 4(1.374) + 10 \approx 21.160$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.374 - \frac{0.110}{21.160} = 1.369$$

$$f(x_2) = f(1.369) \approx 1.369^3 + 2(1.369)^2 + 10(1.369) - 20 \approx 0.002$$

Stop iterations: $|f(1.369)| \approx 0.002 < 0.01$

Conclude the root: $x \approx 1.37$ (accurate to two decimal places)

EXERCISE 8.2

1. Find the real roots of the following functions correct to 3 decimal places using the bisection method, within the specified interval of sign change.

   (i) $f(x) = x^2 \sin(x) - 1 : [1, 2]$ (ii) $f(x) = 3x - \cos x - 1 : [0, 1]$

   (iii) $f(x) = x^4 - 2x - 10 : [1, 2]$ (iv) $f(x) = x^3 - 3x - 10 : [2, 3]$

2. Find the real roots of nonlinear equations correct to 2 decimal places using the Regula -Falsi method, by finding some interval of sign change if not specified.

(i) $3x^3 - 9x^2 + 8 = 0$; $[2, 3]$ (ii) $x^2 + 4\sin x = 0$; $[-2.2, -1]$

(iii) $xe_x = 2$ (iv) $3x + \sin x = e^x$

3. Find the real roots of nonlinear equations using the Newton-Raphson method.

(i) $x^3 + x^2 = 3\ (x + 1)$ near $x = 1$ accurate to 2 decimal places.

(ii) $e^x = 4x$ near $x = 2$ accurate to 3 decimal places.

(iii) $x^3 + 2x - 5 = 0$ accurate to 4 decimal places.

(iv) $3x + \sin x = e^x$ accurate to 3 decimal places.

(v) $\tan(x) = x$ around $x_0 = 4.3$ accurate to 4 decimal places.

8.4 Estimating the Value of a Definite Integral

In many practical problems, however, the objective is to evaluate a definite integral of a function over a given interval. Similar to nonlinear equations, definite integrals are not always easy to evaluate analytically, and in some cases exact solutions are impossible to obtain using standard integration techniques.

For example, the integrals

(i) $\int_0^1 e^{-x^2} dx$ and (ii) $\int_0^{\pi} \sin(x^2) dx$

cannot be evaluated easily by elementary methods. In such situations, numerical methods are used to estimate the value of definite integrals with reasonable accuracy. One of the important numerical methods used for this purpose is the Trapezium Rule, which will be developed and applied in this section.

8.4.1 Trapezium (Trapezoidal) Rule

Consider a continuous function $f(x) \geq 0$ on $[a,b]$.

Let $h = \dfrac{b-a}{n}$ for some positive integer n. Divide the interval $[a,b]$ into n subintervals of equal length h as follows:

$$[a, a+h], [a+h, a+2h], [a+2h, a+3h], \ldots, [a+(n-1)h, a+nh = b]$$

Let $x_i = a + ih$; $i = 0, 1, 2, \ldots, n$. In this way the above subintervals can be written as follows:

$$[a = x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n = b].$$

The function values $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ at the end points of each subinterval represent the lengths of parallel sides of n trapeziums as shown in the

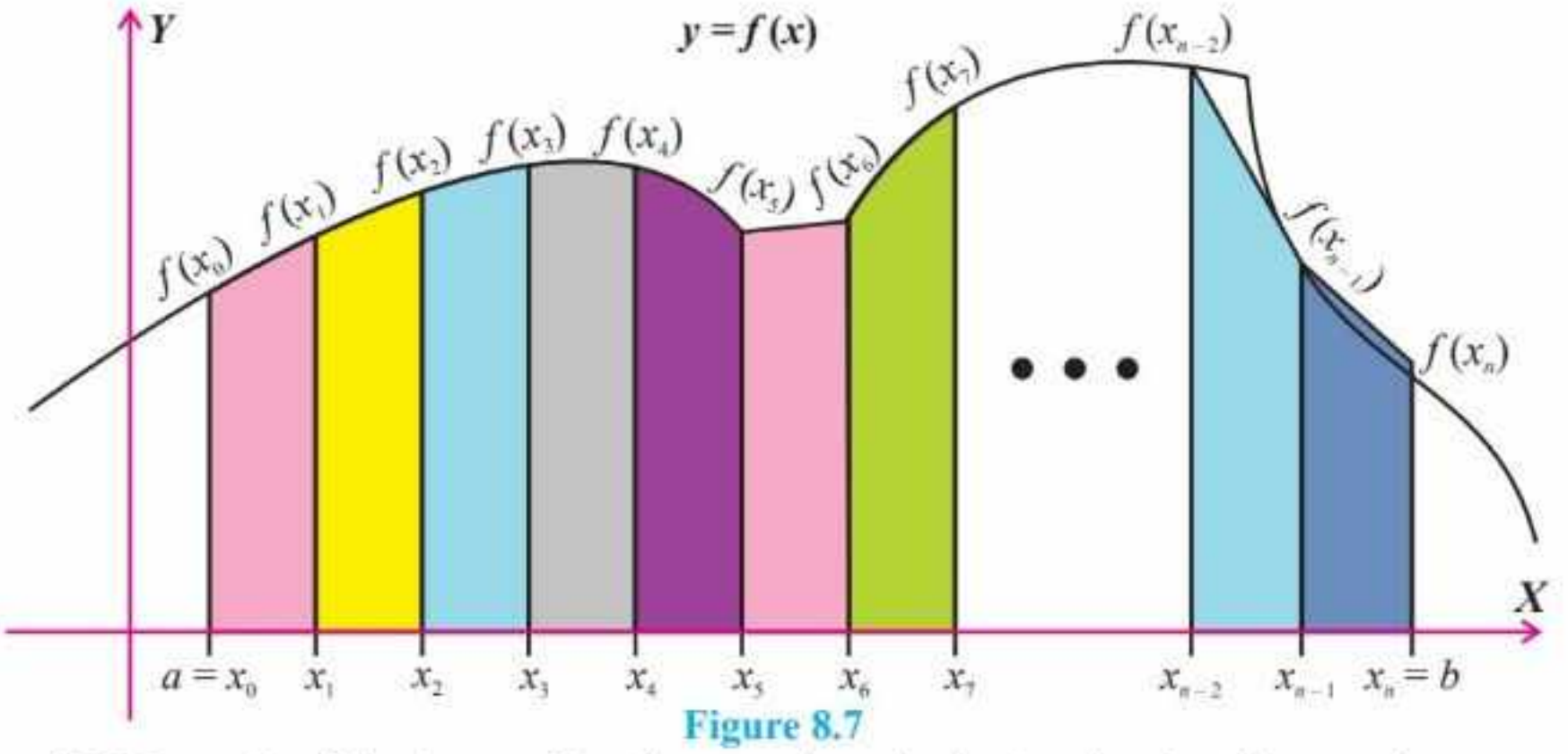


Figure 8.7

Figure 8.7. Length of the base of each trapezium is h, that is, the distance between the parallel sides of these trapeziums. Thus, the areas of these trapeziums are given by:

$$A_i = \frac{h}{2}[f(x_{i-1}) + f(x_i)]\ ; i = 1,2,\ldots,n$$

Since the value of the definite integral $\int_a^b f(x)dx$ is approximately equals to sum of areas of the trapeziums, we have:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} A_i = A_1 + A_2 + A_3 + \cdots + A_n$$

$$= \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] + \frac{h}{2}[f(x_2) + f(x_3)] + \ldots + \frac{h}{2}[f(x_{n-1}) + f(x_n)]$$

$$= \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Hence,

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Note:

1. From Figure 8.6, it is clear that the accuracy of the approximation improves as n is taken sufficiently large, because increasing the number of trapeziums reduces the gap between the exact area and the approximated area.
2. While applying the Trapezium Rule, the values obtained from the calculator should be rounded to at least four decimal places to maintain accuracy in the final result.
3. Although the rule is motivated using a positive function only to represent area, it is generally true for any continuous function on $[a,b]$ including those that

take negative values. In such cases, each trapezium approximates the signed area, and the sum provides a reliable estimate of the integral. The proof of this generalization is beyond the scope of the book.

Example 1 Use trapezium rule to approximate the definite integral $\int_0^1 e^{-x^2}dx$ by dividing the closed interval $[0,1]$ into 7 subintervals of equal length.

Solution: Here, $f(x)= e^{-x^2}$

Given: $a=0,\ b=1,\ n=7$

Continuity:

Note that, $f(c)=\lim_{x\to c}(e^{-x^2})\ \ \forall\, c\in[0,1]$.

$\Rightarrow\ f(x)=e^{-x^2}$ is continuous on $[0,1]$.

Step size: $h=\dfrac{b-a}{n}=\dfrac{1-0}{7}=\dfrac{1}{7}\cong 0.142$

Trapezium Rule: $\int_a^b f(x)dx\approx\dfrac{h}{2}\left[f(x_0)+2\left(f(x_1)+f(x_2)+\cdots+f(x_6)\right)+f(x_7)\right]$.

We have, $f(x)=e^{-x^2}$.

Compute function values (round to 4 decimal places):

$x_0=0,\qquad f(0)=1$

$x_1=0+\dfrac{1}{7}=\dfrac{1}{7},\ f\left(\dfrac{1}{7}\right)=e^{-1/49}=0.9798\ x_2=\dfrac{1}{7}+\dfrac{1}{7}=\dfrac{2}{7},\qquad f\left(\dfrac{2}{7}\right)=e^{-4/49}=0.9216$

$x_3=\dfrac{2}{7}+\dfrac{1}{7}=\dfrac{3}{7},\qquad f\left(\dfrac{3}{7}\right)=e^{-9/49}=0.8322\ x_4=\dfrac{3}{7}+\dfrac{1}{7}=\dfrac{4}{7},\ f\left(\dfrac{4}{7}\right)=e^{-16/49}=0.7214$

$x_5=\dfrac{4}{7}+\dfrac{1}{7}=\dfrac{5}{7},\qquad f\left(\dfrac{5}{7}\right)=e^{-25/49}=0.6004\ x_6=\dfrac{5}{7}+\dfrac{1}{7}=\dfrac{6}{7},\ f\left(\dfrac{6}{7}\right)=e^{-36/49}=0.4797$

$x_7=\dfrac{6}{7}+\dfrac{1}{7}=1,\qquad f(1)=0.3679$

Apply Trapezium Rule:

$$\int_0^1 e^{-x^2}dx\approx\frac{1}{14}\left[1+2(0.9798+0.9216+0.8322+0.7214+0.6004+0.4797)+0.3679\right]$$

$$=\frac{1}{14}\left[1+2(4.5351)+0.3679\right]=0.7456$$

Hence, $\boxed{\int_0^1 e^{-x^2}dx\approx 0.7456}$

Example 2 Use Trapezium Rule to approximate the definite integral

$$\int_0^6 \frac{x}{1+x^2}\,dx$$

by subdividing the closed interval $[0,6]$ each of length 1.

Also compare the answer by solving the integral using the Fundamental Theorem of Calculus.

Solution: Given: $a=0,\ b=6,\ h=1$

Number of subintervals:

$$n=\frac{b-a}{h}=\frac{6-0}{1}=6$$

Points: $x_0=0, x_1=1, x_2=2, x_3=3, x_4=4, x_5=5, x_6=6$

Function: $f(x)=\dfrac{x}{1+x^2}$

Continuity: Note that, $f(c)=\lim\limits_{x\to c}\left(\dfrac{x}{1+x^2}\right)\ \ \forall\, c\in[0,6]$.

$\Rightarrow f(x)=\dfrac{x}{1+x^2}$ is continuous on $[0,6]$.

Trapezium Rule Formula:

$$\int_a^6 f(x)dx\approx\frac{h}{2}\left[f(x_0)+2\left(f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)\right)+f(x_6)\right]$$

Compute function values (4 decimal places):

$f(0)=0/(1+0)=0.0000$

$f(1)=1/(1+1)=0.5000$

$f(2)=2/(1+4)=0.4000$

$f(3)=3/(1+9)=3/10=0.3000$

$f(4)=4/(1+16)=4/17\approx 0.2353$

$f(5)=5/(1+25)=5/26\approx 0.1923$

$f(6)=6/(1+36)=6/37\approx 0.1622$

Apply Trapezium Rule:

$$=\int_0^6\frac{x}{1+x^2}dx\approx\frac{1}{2}\left[0+2(0.5+0.4+0.3+0.2353+0.1923)+0.1622\right]$$

$$=0.5\left[2(1.6276)+0.1622\right]$$

$$=0.5(3.4174)=1.708\,7$$

Trapezium Rule Answer: $\boxed{\int_0^6 \frac{x}{1+x^2}dx \approx 1.7087}$

Exact Value using Fundamental Theorem of Calculus:

We have:

$$\int_0^6 \frac{x}{1+x^2}dx = \frac{1}{2}\int_0^6 \frac{2x}{1+x^2}dx = \frac{1}{2}ln\left|1+x^2\right|_0^6 \qquad \left(\because \int \frac{\frac{du}{dx}}{u}dx = ln|u| + c\right)$$

$$= \frac{1}{2}\ln(1+36) - \frac{1}{2}\ln(1+0)$$

(By Fundamental Theorem of Calculus)

$$= \frac{1}{2}\ln(37) = 0.5 \times 3.6109 \approx 1.8055$$

Comparison:

$\boxed{\int_0^6 \frac{x}{1+x^2}dx \approx 1.7087}$ (By Trapezium Rule) $\boxed{\int_0^6 \frac{x}{1+x^2}dx \approx 1.8055}$ (Exact Value)

Note: The difference between the approximate value and the exact value of the definite integral is $|1.708 - 1.806| = 0.098$ which can be decreased by increasing the number of subintervals.

EXERCISE 8.3

Use Trapezoidal Rule to approximate the value of the following definite integrals.

1. $\int_0^2 \cos(x)dx$ taking $n=4$
2. $\int_0^1 e^x dx$ taking $n=7$
3. $\int_1^7 \ln(x)dx$ taking $n=6$
4. $\int_0^2 x^3 dx$ taking $n=8$
5. $\int_0^4 \sqrt{x}dx$ taking $n=4$
6. $\int_{-1}^1 e^{-x^2}dx$ taking $h=0.4$
7. $\int_{-1}^1 \left(x^4 - x + 1\right)dx$ taking $h=0.2$
8. $\int_0^\pi \sin(x^2)dx$ taking $h=\frac{\pi}{6}$
9. Show by the trapezium rule that the approximate area bounded by the curve $y=\sqrt{8-2x-x^2}$ and the $x-$axis using $n=8$ is 13.5 square units. Calculate the exact area by using the formula for finding the area of semi-circle.
10. Setting $h=0.5$, use the trapezium rule to find the approximate area bounded by the curve $y=e^{x^2}$ from the line $x=0$ to the line $x=4$.

8.5 Real-Life Applications of Nonlinear Equations

In real life, many physical, biological, chemical, and technological problems lead to approximate real solutions to nonlinear equations. In such situations, solving equations by algebraic methods becomes complicated, and sometimes no exact formula exists. Therefore, numerical methods provide a practical and systematic way to obtain approximate solutions with desired accuracy.

For example, in chemical reactions, scientists need to determine equilibrium concentrations of substances, which often leads to nonlinear equations. In biology and medicine, the regulation of heart beats and nerve signals involves nonlinear mathematical models. Similarly, in electronic circuits, components such as diodes and transistors follow exponential laws, producing nonlinear equations that must be solved numerically. In cryptography, nonlinear equations play an important role in designing secure communication systems and encryption algorithms.

8.5.1 Chemical Reactions

In many chemical reactions, equilibrium concentration or reaction rate leads to nonlinear equations. These equations are often difficult to solve analytically, so numerical methods are used to find approximate values.

Example 1 In a chemical reaction, the concentration x (in moles per liter) of a product at equilibrium satisfies the relation $xe^x = 1$.

Find the equilibrium concentration correct to two decimal places using Newton-Raphson Method.

Solution: Define the function: $f(x) = xe^x - 1$

Derivative: $f'(x) = e^x + xe^x$ exists at every real number x.

Newton-Raphson formula: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{x_n e^{x_n} - 1}{e^{x_n} + x_n e^{x_n}}$$

$$f(0) = -1$$
$$f(1) = i.e^1 - 1 = 1.718$$

So, root lies between 0 and 1

Initial guess:

Note that $f(0.5) = 0.5e^{0.5} - 1 = -0.1756$, which is closed to zero.

Also f is differentiable near $x_0 = 0.5$ and hence continuous as well. Thus, we take $x_0 = 0.5$ as the initial guess and Newton-Raphson Formula is applicable.

First iteration:

$$f(0.5)=0.5\cdot e^{0.5}-1=-0.1757, \qquad f'(0.5)=e^{0.5}+0.5.e^{0.5}=2.4730$$

$$x_1=0.5-\frac{-0.1757}{2.4730}=0.5711, \quad f(x_1)=f(0.5711)=0.5711\cdot e^{0.5711}-1\approx 0.0107$$

Second iteration:

$$f'(x_1)=f'(0.5711)=e^{0.5711}+0.5711\cdot e^{0.5711}\approx 2.7813$$

$$x_2=0.5711-\frac{0.0107}{2.7813}\approx 0.5673$$

$$f(x_2)=f(0.5673)=0.5673\cdot e^{0.5673}-1\approx 1.0004-1\approx 0.0004\approx 0$$

That is, $|f(0.5673)|\approx 0$

$$\boxed{x\approx 0.57 \text{ moles per liter}}$$

8.5.2 Regulation of Heartbeats

The human heartbeat is regulated by electrical signals that control the contraction and relaxation of the heart muscles. The variation of these signals over time can often be modeled by nonlinear equations.

Example 2 The electrical activity of the heart is represented by a variable x, which measures the signal strength of the heartbeat. At steady-state, the signal satisfies the nonlinear equation:

$$x^3-x-1=0$$

Find the real steady-state value of the heartbeat signal near $x=1$ using the Bisection Method, correct to two decimal places.

Solution: Identify the function: $f(x)=x^3-x-1$

Continuity: Note that, $f(c)=\lim_{x\to c}(x^3-x-1), \quad \forall c\in\mathbb{R}$

$\Rightarrow f$ is continuous at every real number.

Interval of sign change:

$$f(1)=1^3-1-1=-1<0 \qquad f(2)=8-2-1=5>0$$

$\Rightarrow [1,2]$ is an interval of sign change

$\Rightarrow$ f has a root in [1, 2].

Iterations Table: (Intermediate 4 decimal places)

Iteration	a	b	$c=\frac{a+b}{2}$	$f(c)$	New Interval
1	1 (–)	2 (+)	1.5 (+)	0.875	[1,1.5]
2	1 (–)	1.5 (+)	1.25 (–)	–0.2969	[1.25,1.5]
3	1.25 (–)	1.5 (+)	1.375 (+)	0.2246	[1.25,1.375]
4	1.25 (–)	1.375 (+)	1.3125 (–)	–0.0515	[1.3125,1.375]
5	1.3125 (–)	1.375 (+)	1.3438 (+)	0.0828	[1.3125, 1.3438]
6	1.3125 (–)	1.3438 (+)	1.3282 (+)	0.0149	[1.3125, 1.3282]
7	1.3125 (–)	1.3282 (+)	1.3204 (–)	–0.0183	[1.3204, 1.3282]
8	1.3204 (–)	1.3282 (+)	1.3243 (–)	–0.0018	Stop

Note that, $|f(1.3243)|=|-0.0018|<0.01$

Final Answer: $\boxed{x\approx 1.32}$ (correct to two decimal places)

Thus, the steady-state value of the heartbeat signal is approximately 1.32 units.

8.5.3 Electronic Circuits

Electronic components such as diodes and transistors often produce nonlinear voltage-current relationships. In circuits, the voltage or current at certain points may satisfy nonlinear equations.

Example 3 Diode/AC Circuit

In a circuit, the voltage V across a component satisfies the equation:

$$\sin V-0.5V=0$$

Find a non-zero real value of V correct to two decimal places using Regula-Falsi method.

Note: V represents the voltage in volts.

Solution: Define Function: $f(V)=\sin V-0.5V$

Select Interval: $f(1)=\sin 1-0.5\approx 0.3415>0$

$f(2)=\sin 2-1\approx -0.0907<0$

$\Rightarrow$[1, 2] is an interval of sign change.

Continuity: Note that, $f(c)=\lim_{V\to c}(\sin V-0.5V)\ \ \forall c\in[1,2]$.

$\Rightarrow$ f is continuous on the interval of sign change [1, 2].

$\Rightarrow$ Regula-Falsi Method is applicable with the interval [1, 2].

Regula-Falsi Iterations: $V_c=\frac{af(b)-bf(a)}{f(b)-f(a)}$

Iterations Table: (4 decimal places)

Iteration	a	b	$V_c = \dfrac{af(b)-bf(a)}{f(b)-f(a)}$	$f(V_c)$	New Interval
1	1 (+)	2 (–)	1.7901 (+)	0.0810	[1.7901, 2]
2	1.7901 (+)	2 (–)	1.8891 (+)	0.0052	[1.8891, 2]
3	1.8891 (+)	2 (–)	1.8951 (+)	0.0003	Stop

Checking: $|f(1.8951)| = 0.0003 < 0.01$

Final Answer: $\boxed{V \approx 1.90 \text{ volts}}$ (two decimal places)

8.5.4 Cryptography

Cryptographic algorithms sometimes involve high-degree nonlinear equations to generate secure keys or pseudo-random numbers.

Example 4 Key Parameter

In a cryptosystem, a key parameter x satisfies the equation:

$$x^5 - 10x + 5 = 0$$

Find a real value of x correct to two decimal places using the Regula-Falsi method.

Note: x represents a component of the encryption key; its approximate value is crucial for secure key generation.

Solution: Define Function: $f(x) = x^5 - 10x + 5$

Select Interval: $f(1) = 1 - 10 + 5 = -4 < 0,\ f(2) = 32 - 20 + 5 = 17 > 0$

Continuity: Note that, $f(c) = \lim\limits_{x \to c}\left(x^5 - 10x + 5\right)\ \forall c \in [1, 2]$.

f is continuous on the interval of sign change [1, 2].

Regula-Falsi Iteration Table: (4 decimal places)

Iteration	a	b	$x_c = \dfrac{af(b)-bf(a)}{f(b)-f(a)}$	$f(x_c)$	New Interval
1	1.0 (–)	2.0 (+)	1.1905 (–)	–4.5136	[1.1905, 2]
2	1.1905 (–)	2.0 (+)	1.3603 (–)	–3.9453	[1.3603, 2]
3	1.3603 (–)	2.0 (+)	1.4808 (–)	–2.6880	[1.4808, 2]
4	1.4808 (–)	2.0 (+)	1.5523 (–)	–1.5098	[1.5523, 2]
5	1.5523 (–)	2.0 (+)	1.5888 (–)	–0.7641	[1.5888, 2]
6	1.5888 (–)	2.0 (+)	1.6065 (–)	–0.3645	[1.6065, 2]
7	1.6065 (–)	2.0 (+)	1.6148 (–)	–0.1682	[1.6148, 2]
8	1.6148 (–)	2.0 (+)	1.6186 (–)	–0.0764	[1.6186, 2]
9	1.6186 (–)	2.0 (+)	1.6203 (–)	–0.0350	[1.6203, 2]
10	1.6203 (–)	2.0 (+)	1.6581 (+)	+0.9520	[1.6203, 1.6581]
11	1.6203 (–)	1.6581 (+)	1.6216 (–)	–0.0031	Stop

Checking: $|f(1.6216)| = 0.0031 < 0.01$

Final Answer: $\boxed{x \approx 1.62}$

EXERCISE 8.4

1. In a certain chemical reaction, the concentration of a substance at equilibrium is governed by the relation $x^2+0.2x-0.04=0$, where x is the equilibrium concentration (in moles per liter). Use bisection method to find the positive equilibrium concentration of the substance accurate to two decimal places.
2. The flow rate of blood in an artery is modeled by the nonlinear equation $Q=k\left(P-\frac{Q^2}{R}\right)$, where Q is the flow rate (mL/s), P = 100 mmHg is the blood pressure, k = 0.05, and R = 10. Determine the possible real values of the flow rate Q accurate to two decimal places.
3. In a nonlinear LC circuit, the equilibrium charge q on the capacitor satisfies $q^3-4q+2=0$. Find the equilibrium values of charge stored on the capacitor lying in the interval [0, 1] using Regula-Falsi method accurate to three decimal places.
4. In a certain cryptographic system, the decoding speed v (in units per second) is related to the security delay function $d(v)=\frac{v^2}{20+v}$, where $d(v)$ represents the decoding delay in milliseconds. For secure transmission, the system requires a decoding delay of 50 milliseconds. Use Newton–Raphson Method with initial guess $v_0=35$ to determine the decoding speed v that satisfies $d(v)=50$, correct to three decimal places.
5. In a secure communication system, the phase shift θ (in radians) of an encrypted signal is modeled by the nonlinear function
$$f(\theta)=\sin\theta-0.5.$$
For proper synchronization of the encrypted signal, the phase shift must satisfy $f(\theta)=0$. Use the Newton–Raphson Method with initial guess $\theta_0=0.5$ to determine the required phase shift θ, correct to three decimal places.
6. In a nonlinear electric circuit, the voltage x (in volts) across a component is modeled by
$$f(x)=-0.1x^2+2x+1.$$
Use the Newton–Raphson Method with initial guess $x_0=15$ to estimate the operating voltage at which the circuit reaches equilibrium. The equilibrium voltage is the voltage at which the net change in the circuit becomes zero, i.e., the circuit reaches a stable operating point.

Unit 09 Inverse Trigonometric

INTRODUCTION

In earlier classes, we studied trigonometric functions and their real-life applications. Now, we extend that knowledge by exploring inverse trigonometric functions which help us to find angles when function values are known. This is essential in solving equations and modeling periodic behavior in calculus of inverse trigonometric functions. In this unit, we will define inverse trigonometric functions, their domains and ranges, examine their graphs within the given interval and understand their key properties and uses.

9.1 Inverse Trigonometric Functions and their Graph

We have studied about inverse of a function. We learned that only a one-to-one and onto function will have an inverse. If a function is not one-to-one, it may be possible to restrict its domain to make it one-to-one so that its inverse can be found.

Recall!

Let f be a bijective function from a set X to a set Y. The **inverse function** of f, written as f^{-1}, is a bijective function from a set Y back to set X such that

$$x = f^{-1}(y),\ \forall\, y \in Y \quad \Leftrightarrow \quad y = f(x),\ \forall\, x \in X$$

Here domain of f^{-1} = range of f and range of f^{-1} = domain of f. This mean that f^{-1} reverses the correspondence of f i.e.,

$f^{-1}(y) = x$, when $f(x) = y$, also their composition gives:

$$\left(f^{-1} \circ f\right)(x) = f^{-1}\left(f(x)\right) = f^{-1}(y) = x,\ \ \forall\, x \in X$$

and $$\left(f \circ f^{-1}\right)(y) = f\left(f^{-1}(y)\right) = f(x) = y,\ \ \forall\, y \in Y$$

The equation $\sin 90° = 1$ clearly shows that $90°$ is the angle whose sine is 1. Sometimes, we know the value of sine, cosine or tangent, but not the angle itself. In such cases, we use the inverse trigonometric functions to find the angle. For example, the inverse of $\sin 90° = 1$ can be written as $90° = \sin^{-1}(1)$ or $90° = \arcsin 1$, and read it as $90°$ is equal to sine inverse of 1 or arcsin of 1. Generally, the equation $x = \sin y$ gives value of x for a given angle of y. To find the angle y when x is known, we use the inverse function, that is,

$$y = \sin^{-1} x$$

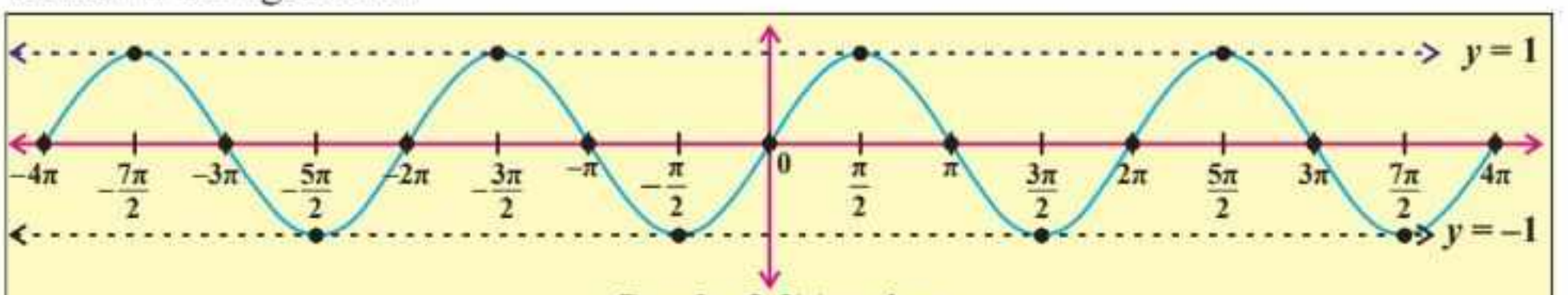
This means y is an angle whose sine is x and the symbol $\sin^{-1} x$ is read as sine inverse of x or arcsine of x. In other words, $y = \sin^{-1} x$ iff $x = \sin y$. Similarly, we can write the other equations as $y = \cos^{-1} x$ iff $x = \cos y$ and $y = \tan^{-1} x$ iff $x = \tan y$, etc.

We know that trigonometric functions sine, cosine and tangent are periodic, meaning they repeat their values at regular intervals. Due to this periodicity, trigonometric

functions are generally not bijective (i.e., they are not both one-to-one and onto over their entire domain). As a result, their inverses do not naturally exist without restrictions. However, by carefully restricting their domains and codomains, we can make these functions bijective that allows us to define their inverses, known as inverse trigonometric functions. Next, we discuss these functions one by one in details.

9.1.1 Inverse of Sine Function

Consider the function $f : \mathbb{R} \to [-1, 1]$ define by $f(x) = \sin x$. The graph of the function is shown in Figure 9.1.



Graph of $f(x) = \sin x$

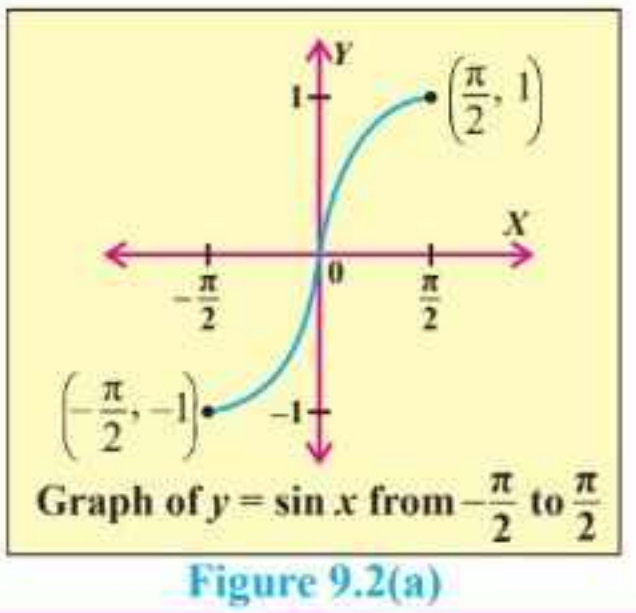
Figure 9.1

Clearly the graph of sine function is not bijective because it touches the line $y = 1$ and $y = -1$ at many points due to the domain of sine function which is the set of all real numbers. However, if we restrict the domain of $f(x) = \sin x$ to the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then the function with restricted domain i.e., $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ becomes one-one and onto (bijective) that is invertible and is called **principal sine function**. Then the inverse sine function is defined by

$$y = \sin^{-1} x \text{ iff } x = \sin y$$

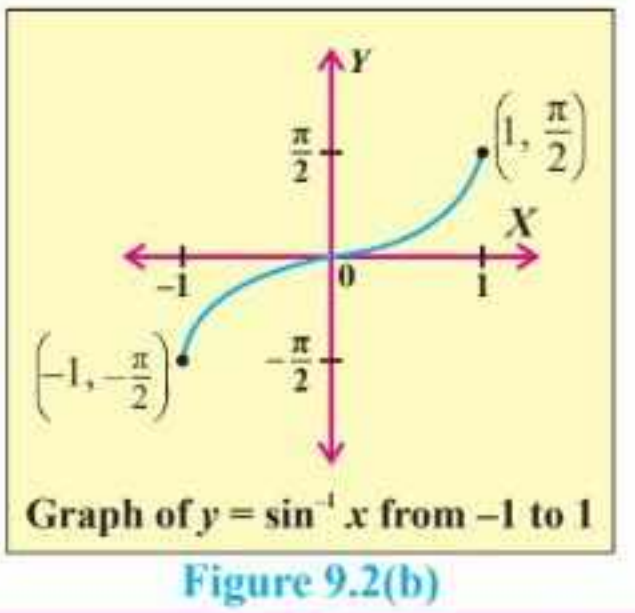
where the domain of $\sin^{-1} x$ is $[-1, 1]$ and range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The graph of the function $\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ is shown in Figure 9.2(a). and graph of $\sin^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is shown in Figure 9.2(b).



Graph of $y = \sin x$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

Figure 9.2(a)



Graph of $y = \sin^{-1} x$ from −1 to 1

Figure 9.2(b)

The graph of $\sin^{-1}x$ is also obtain by reflecting the graph of principal sine function $y = \sin x$ along the line $y = x$ as a mirror image shown in Figure 9.3. Observe that the graph of $y = \sin x$ is symmetric about the line $y = x$.

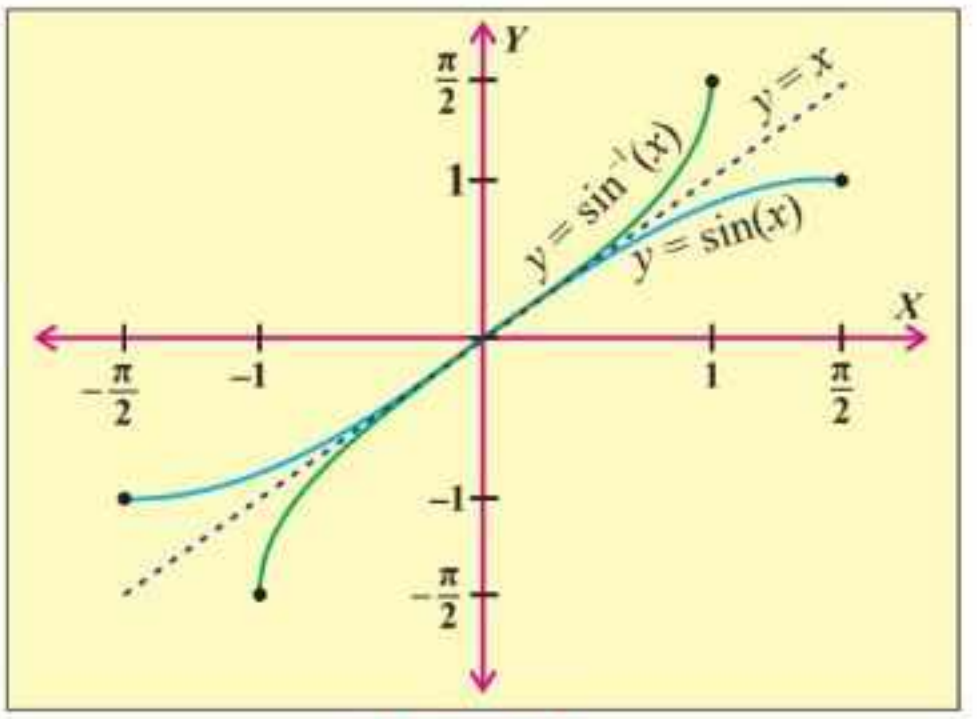


Figure 9.3

The sine function can be restricted to any one of the infinitely many intervals $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ or generally $\left[n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right], n \in Z$ with range $[-1, 1]$ and length of each interval π where it is strictly increasing or decreasing, to make the sine function one-to-one. If interval is not specified, then $\sin^{-1} x$ is a function with domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Note

$\sin^{-1} x \neq (\sin x)^{-1}$ or $\frac{1}{\sin x}$

9.1.2 Inverse of Cosine Function

Consider the function $f: \mathbb{R} \to [-1, 1]$ define by $f(x) = \cos x$. The graph of the function is shown in Figure 9.4.

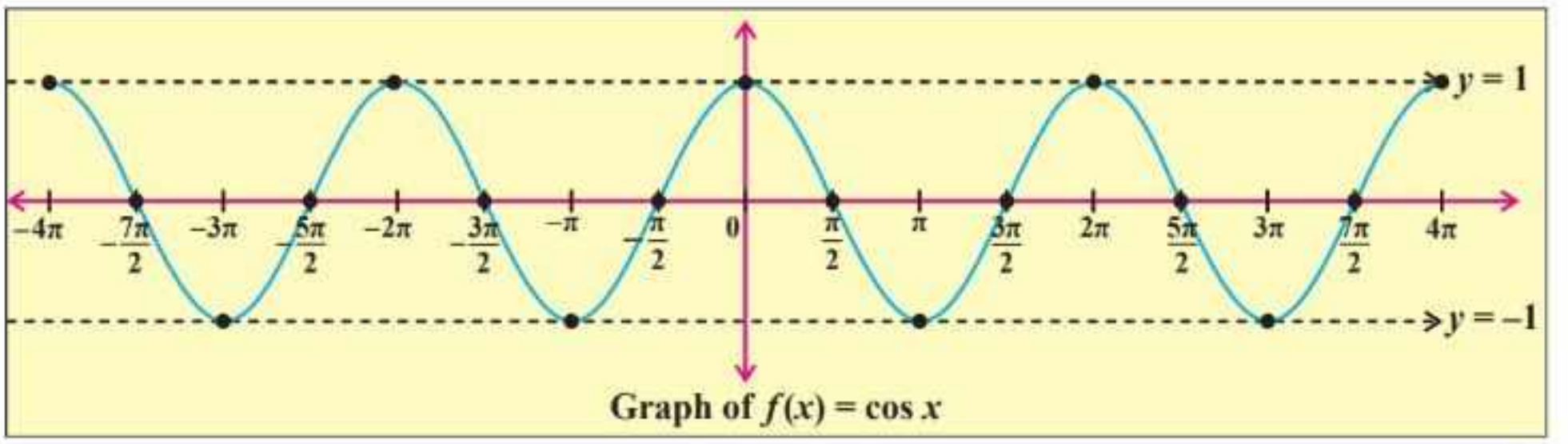


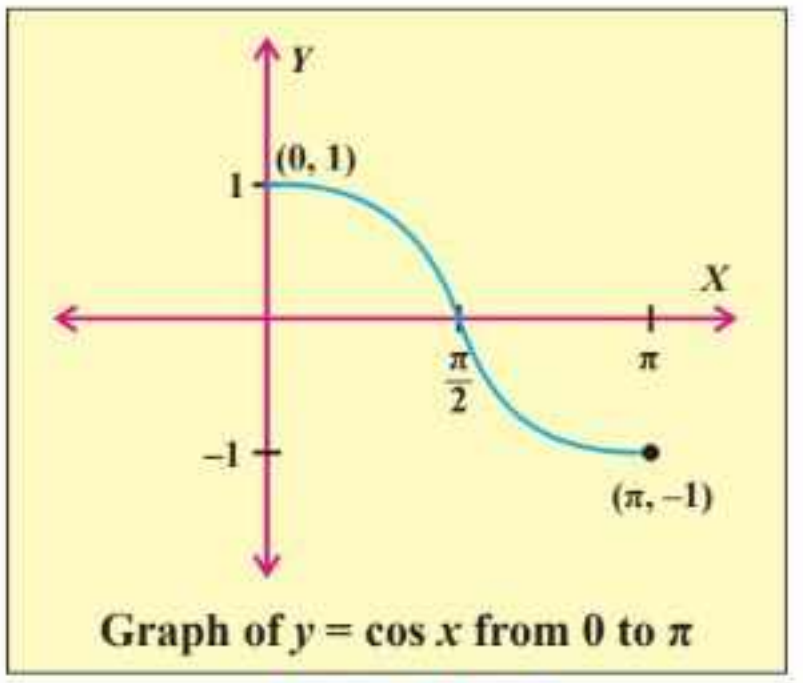
Figure 9.4

Like the graph of sine function, the graph of cosine function is not bijective because again it fails the horizontal line test. It touches the line $y = 1$ and $y = -1$ at multiple points. However, if we carefully restrict the domain of $f(x) = \cos x$ to the closed interval $[0, \pi]$, then the function with restricted domain i.e., $f : [0, \pi] \to [-1, 1]$ becomes one-one and onto that is invertible and is called **principal cosine function**. The inverse cosine function is defined by

$$y = \cos^{-1} x \text{ iff } x = \cos y$$

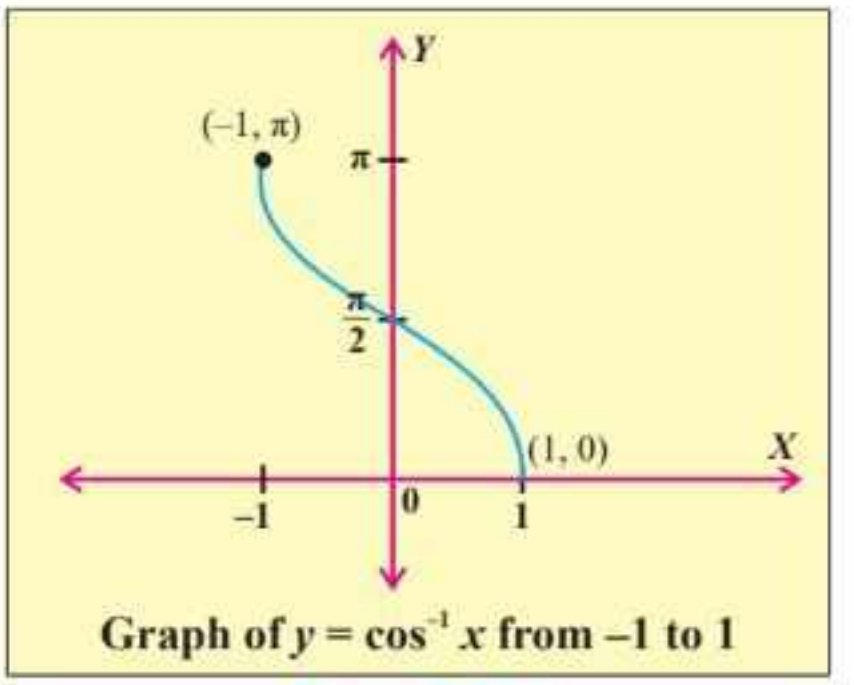
The domain of $\cos^{-1} x$ is $[-1, 1]$ and range of $\cos^{-1} x$ is $[0, \pi]$.

The graph of the function $\cos : [0, \pi] \to [-1, 1]$ is shown in Figure 9.5(a) and graph of $\cos^{-1} : [-1, 1] \to [0, \pi]$ is shown in Figure 9.5(b).



Graph of $y = \cos x$ from 0 to π

Figure 9.5(a)



Graph of $y = \cos^{-1} x$ from –1 to 1

Figure 9.5(b)

The graph of $\cos^{-1} x$ is also obtain by reflecting the graph of principal cosine function $y = \cos x$ along the line $y = x$ as a mirror image shown in Figure 9.6. Observe that the graph of $y = \cos x$ is symmetric about the line $y = x$.

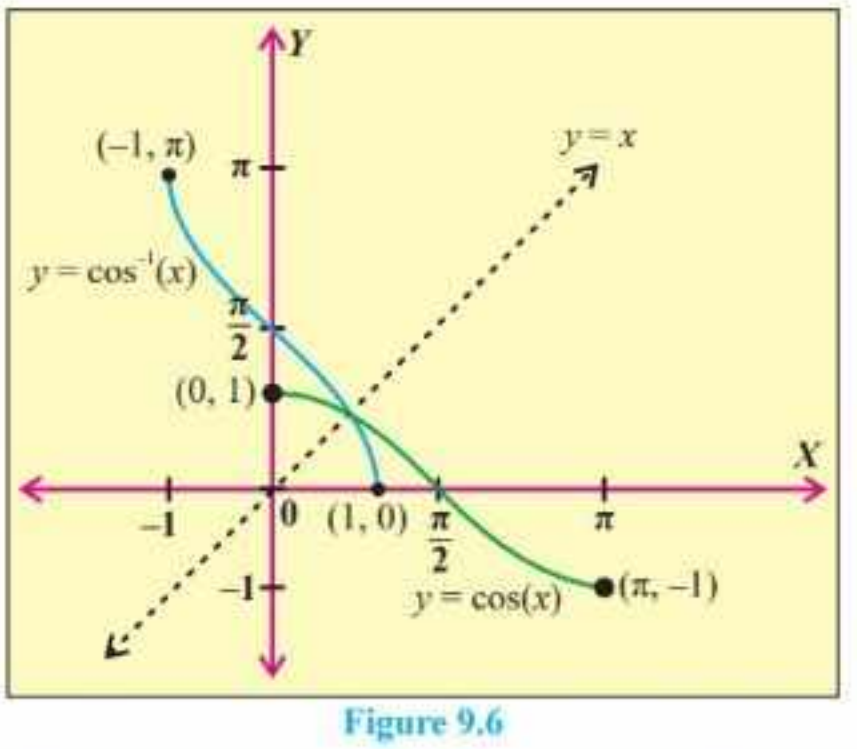


Figure 9.6

Like, sine function, the cosine function with range $[-1, 1]$ can be restricted to any one of the infinitely many intervals $[-2\pi, -\pi], [-\pi, 0], [0, \pi], [\pi, 2\pi]$ or generally $[n\pi, n\pi+\pi], n \in Z$ to make the cosine function one-to-one. If no interval is specified, then $\cos^{-1} x$ is a function with domain $[-1, 1]$ and range $[0, \pi]$.

Note

$\cos^{-1} x \neq (\cos x)^{-1}$ or $\dfrac{1}{\cos x}$

9.1.3 Inverse of Tangent Function

Consider the function define by $f(x) = \tan x$ with domain is the set $\left\{x \mid x \in \mathbb{R} \wedge x \neq (2n+1)\dfrac{\pi}{2}, n \in \mathbb{Z}\right\}$ and range is the set of all real numbers. The graph of the tangent function is shown in Figure 9.7.

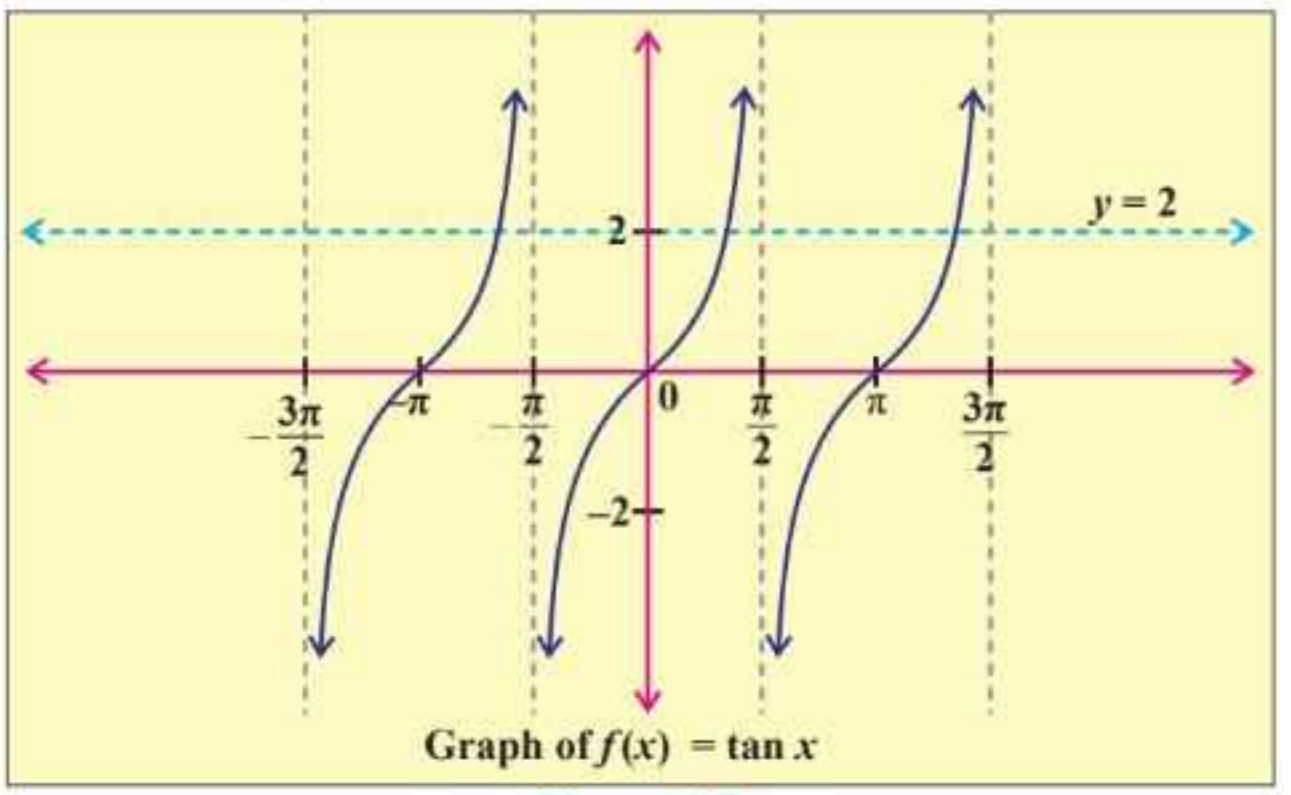


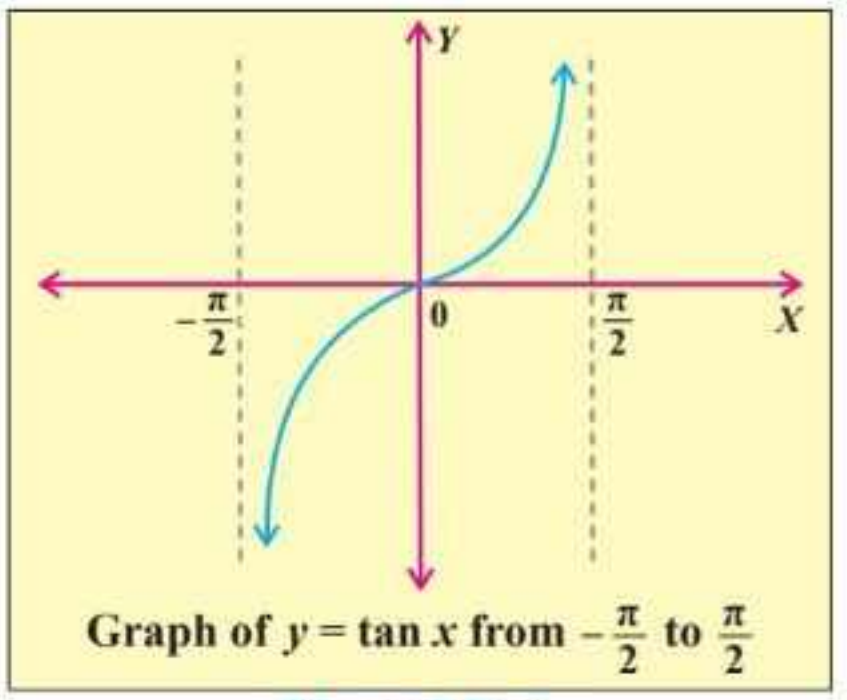
Figure 9.7

Clearly the graph of tangent function is not bijective because the line $y = 2$ intersect the graph at multiple points. However, if we restrict the domain of $f(x) = \tan x$ to the open interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, then the function with restricted domain i.e. $f: \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R}$ becomes one-one and onto that is invertible. This function with domain $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and range $(-\infty, \infty)$ is called **principal tangent function**. The inverse tangent function is defined by $y = \tan^{-1} x$ iff $x = \tan y$.

The domain of $\tan^{-1} x$ is $(-\infty, \infty)$ and range of $\tan^{-1} x$ is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

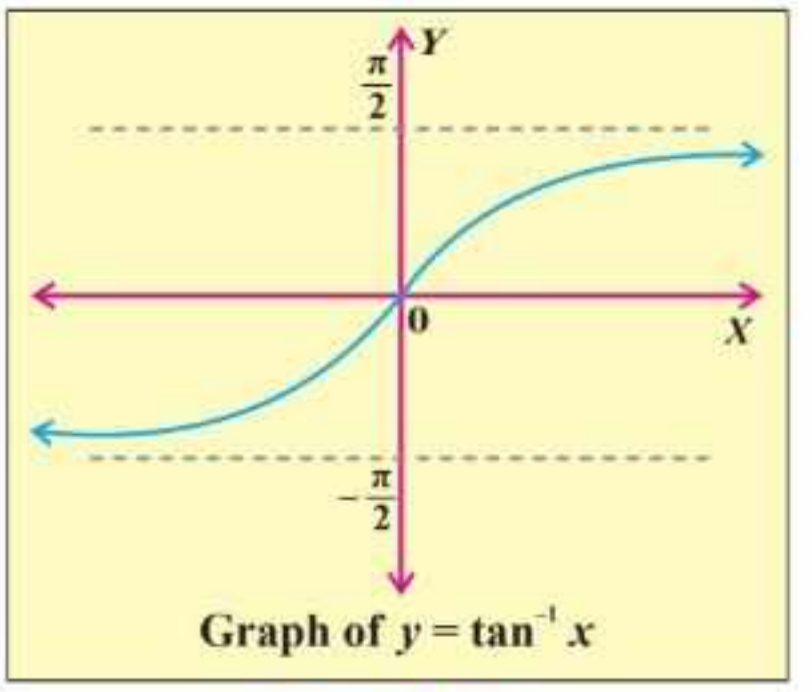
The graph of the function $\tan: \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R}$ is shown in Figure 9.8(a) and graph of

$\tan^{-1}:(-\infty, \infty)\to\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is shown in Figure 9.8(b).



Graph of $y = \tan x$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

Figure 9.8(a)



Graph of $y = \tan^{-1} x$

Figure 9.8(b)

The graph of $\tan^{-1}x$ is also obtain by reflecting the graph of principal tangent function $y = \tan x$ along the line $y = x$ as a mirror image shown in Figure 9.9. Observe that the graph of $y = \tan x$ is symmetric about the line $y = x$.

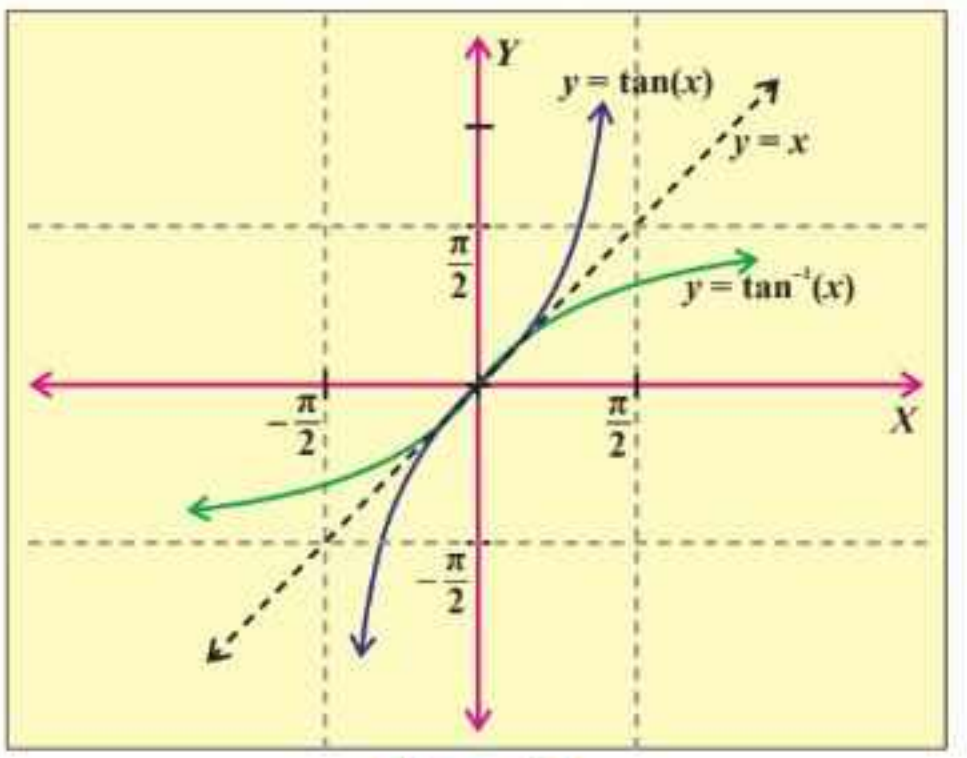


Figure 9.9

Like sine and cosine functions, the domain of tangent function can be restricted to any one of the infinitely many intervals $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ or generally $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$, $n \in \mathbb{Z}$ with range $\mathbb{R}$ to make the tangent function one-to-one. If no interval is specified then $\tan^{-1} x$ is a function with domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Remember!

$\tan^{-1} x \neq (\tan x)^{-1}$, or $\frac{1}{\tan x}$

Example 1 Find the principal value of

(i) $\sin^{-1}\left(\frac{1}{2}\right)$ (ii) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution:

(i) Let $\sin^{-1}\left(\frac{1}{2}\right)=y$

We want to find the angle y, whose sine is $\frac{1}{2}$

$\Rightarrow \quad \sin y=\frac{1}{2}$, where the principal value of $\sin^{-1}x$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

As $\sin y=\frac{1}{2}=\sin\frac{\pi}{6}$

$\Rightarrow \quad y=\frac{\pi}{6}$

Thus, $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

(ii) Let $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=y$

We want to find the angle y, whose cosine is $-\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \cos y=-\frac{\sqrt{3}}{2}$, where the principal value of $\cos^{-1}x$ lies in $[0, \pi]$

As $\cos y=-\frac{\sqrt{3}}{2}=\cos\frac{5\pi}{6}$

$\Rightarrow \quad y=\frac{5\pi}{6}$

Thus, $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$

(iii) Let $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=y$

Then, $\tan y=-\frac{1}{\sqrt{3}}$, where the principal value of $\tan^{-1}x$ lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

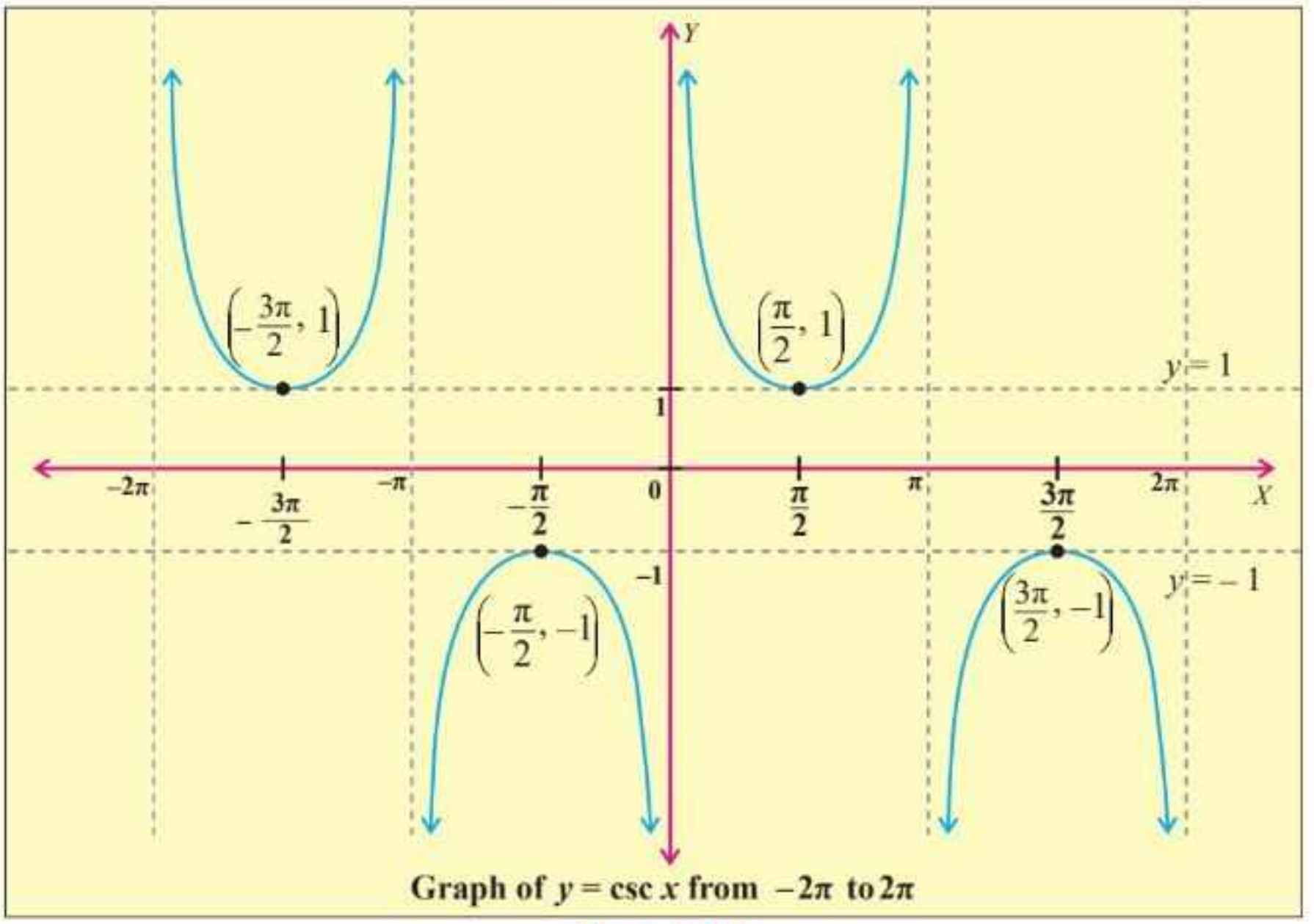
As $\tan y=-\frac{1}{\sqrt{3}}=\tan\left(-\frac{\pi}{6}\right)$

$\Rightarrow \quad y=-\frac{\pi}{6}$

Hence, $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$

9.1.4 Inverse of Cosecant Function

The graph of the function $f(x) = \csc x$ with domain $\{x \mid x \in \mathbb{R} \wedge x \neq n\pi, n \in \mathbb{Z}\}$ and range is the set $\mathbb{R} - (-1, 1)$ is shown in Figure 9.10.



Graph of $y = \csc x$ from -2π to 2π

Figure 9.10

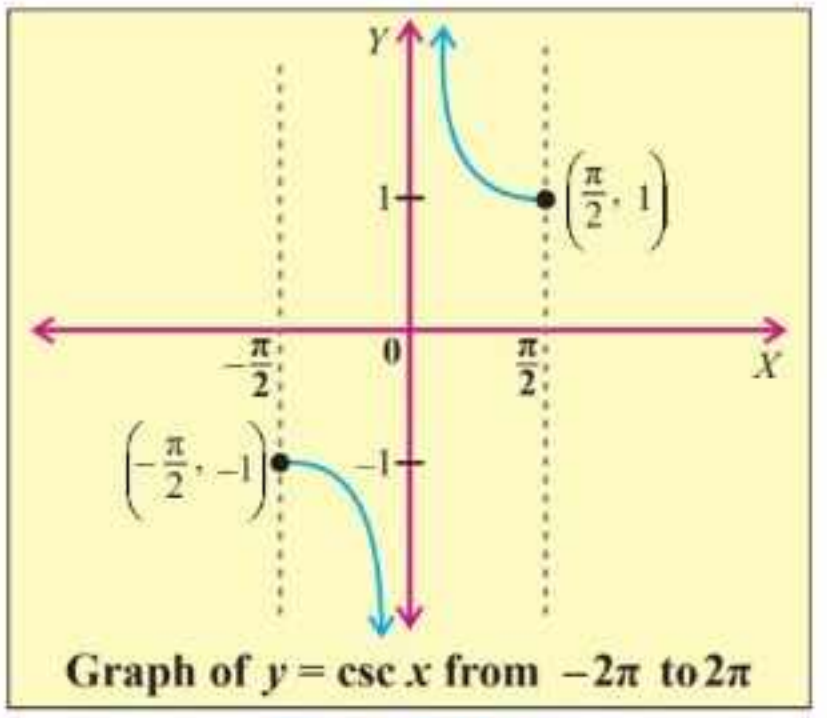
Clearly from the graph it follows that cosecant function is not bijective because the line $y = 1$ and $y = -1$ touches the graph at many points. However, if we restrict the domain of $f(x) = \csc x$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, then the function with restricted domain

i.e. $f : \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \to (-\infty, 1) \cup (1, \infty)$ becomes one-one and onto that is invertible and is called **principal cosecant function**. The inverse cosecant function is defined by

$$y = \csc^{-1} x \text{ iff } x = \csc y$$

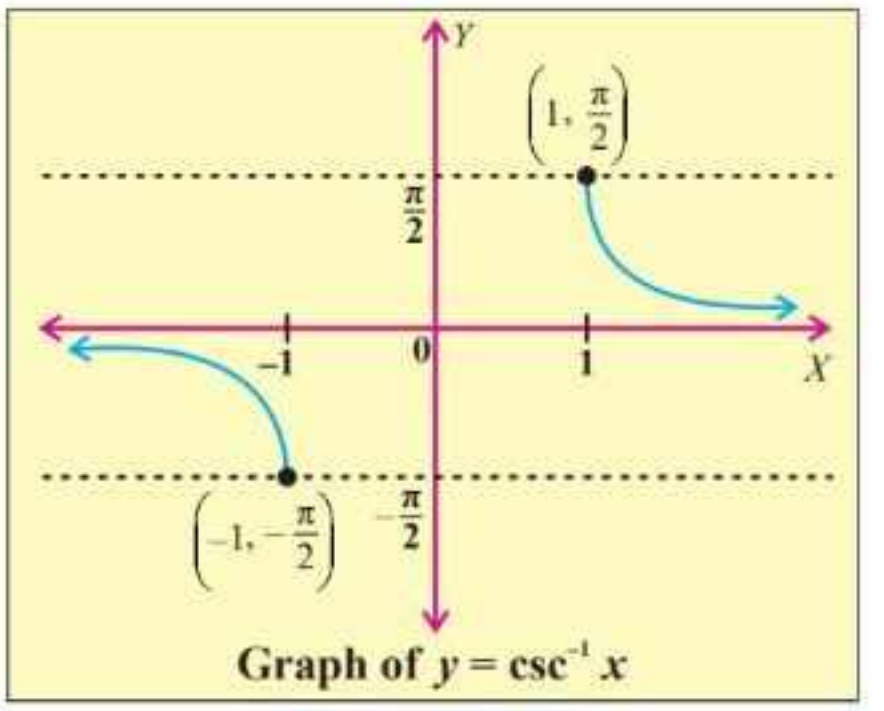
The domain of $\csc^{-1} x = \mathbb{R} - (-1, 1)$ and range of $\csc^{-1} x = \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

The graph of the function $\csc : \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \to (-\infty, -1) \cup (1, \infty)$ is shown in Figure 9.11(a) and graph of $\csc^{-1} : (-\infty, -1) \cup (1, \infty) \to : \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ is shown in Figure 9.11(b).



Graph of $y = \csc x$ from -2π to 2π

Figure 9.11(a)

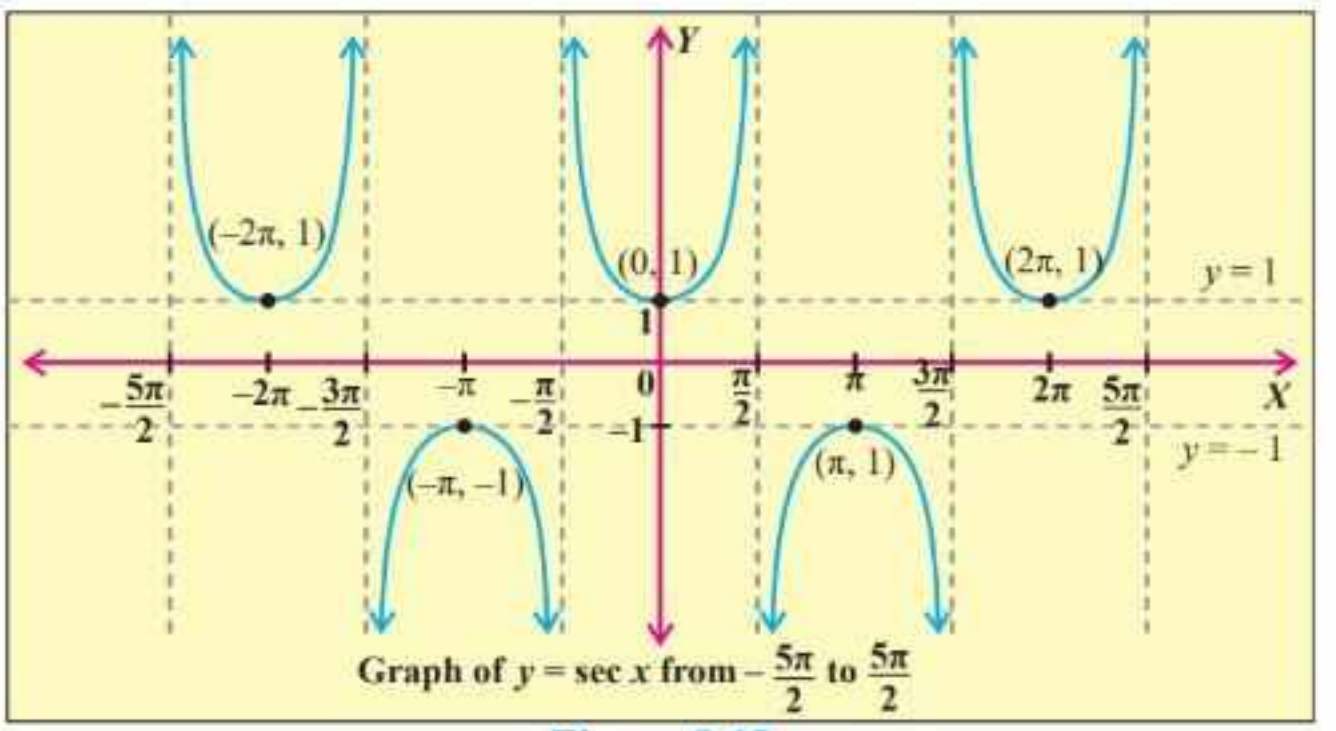


Graph of $y = \csc^{-1} x$

Figure 9.11(b)

9.1.5 Inverse of Secant Function

The graph of the function $f(x) = \sec x$ with domain $\left\{x \mid x \in \mathbb{R} \wedge x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$ and range $\mathbb{R} - (-1, 1)$ is shown in Figure 9.12.



Graph of $y = \sec x$ from $-\frac{5\pi}{2}$ to $\frac{5\pi}{2}$

Figure 9.12

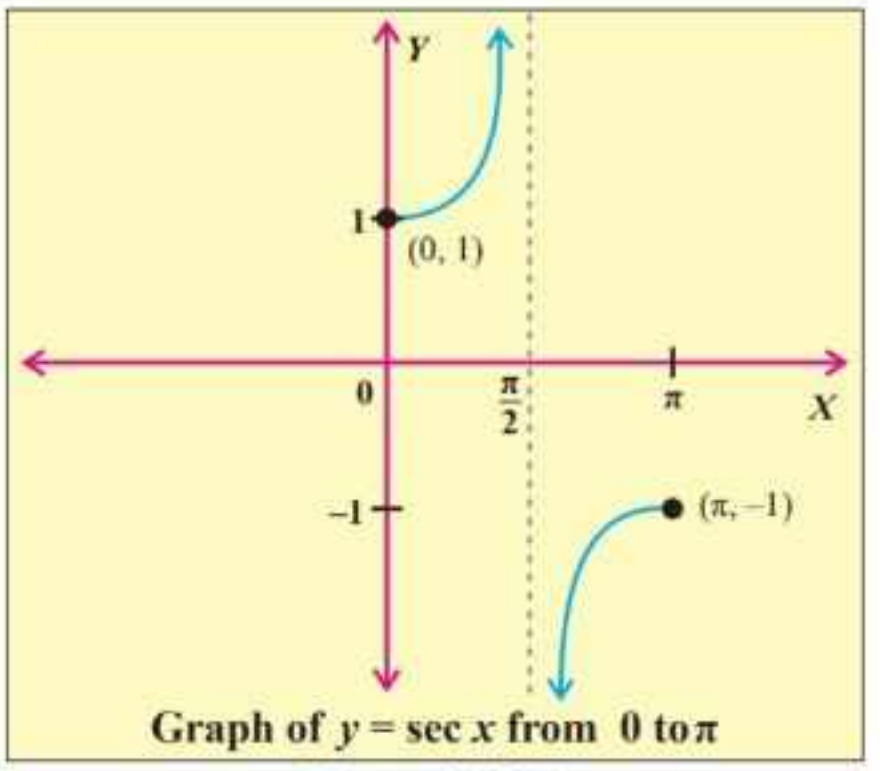
Clearly from the graph it follows that secant function is not bijective because the line $y = 1$ and $y = -1$ touches the graph at many points. However, if we restrict the domain of $f(x) = \sec x$ to the interval $[0, \pi] - \left\{\frac{\pi}{2}\right\}$, then the function with restricted domain

i.e. $f : \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \to (-\infty, -1] \cup [1, \infty)$ becomes one-one and onto (bijective) that is invertible and is called **principal secant function**. The inverse secant function is defined by

$$y = \sec^{-1} x \text{ if } x = \sec y$$

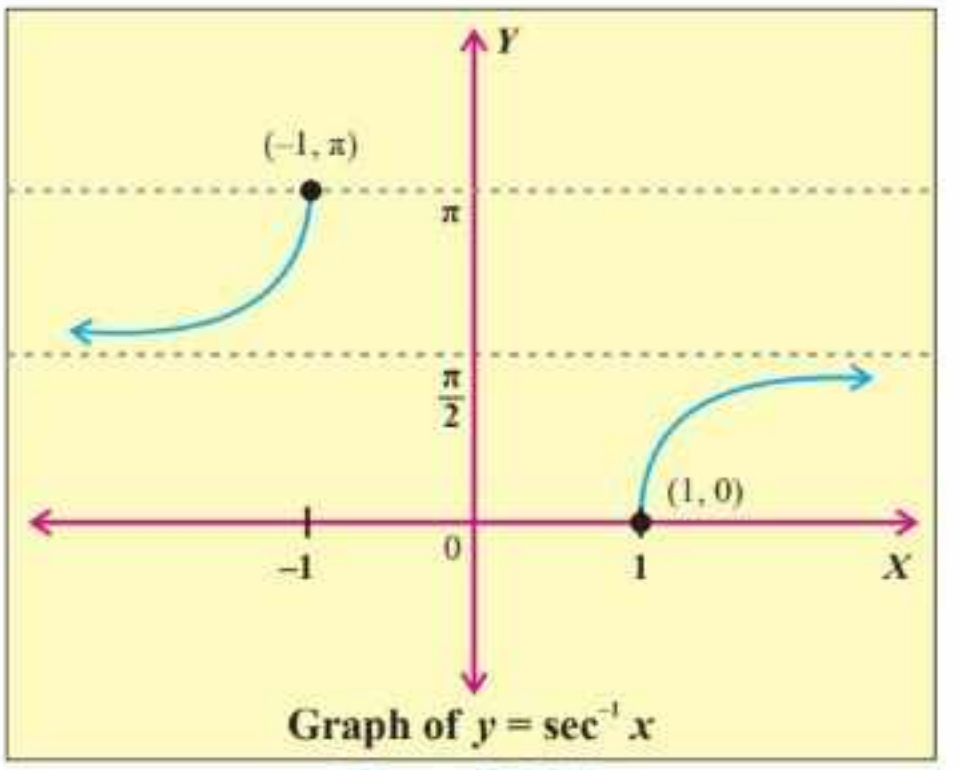
The domain of $\sec^{-1}x$ is $\mathbb{R}-(-1, 1)$ and range of $\sec^{-1}x$ is $\left[0, \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2}, \pi\right]$.

The graph of the function $\sec:\left[0, \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2}, \pi\right]\to(-\infty, -1]\cup[1, \infty)$ is shown in Figure 9.13(a) and graph of $\sec^{-1}:(-\infty, -1]\cup[1, \infty)\to\left[0, \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2}, \pi\right]$ is shown in Figure 9.13(b).



Graph of $y = \sec x$ from 0 to π

Figure 9.13(a)

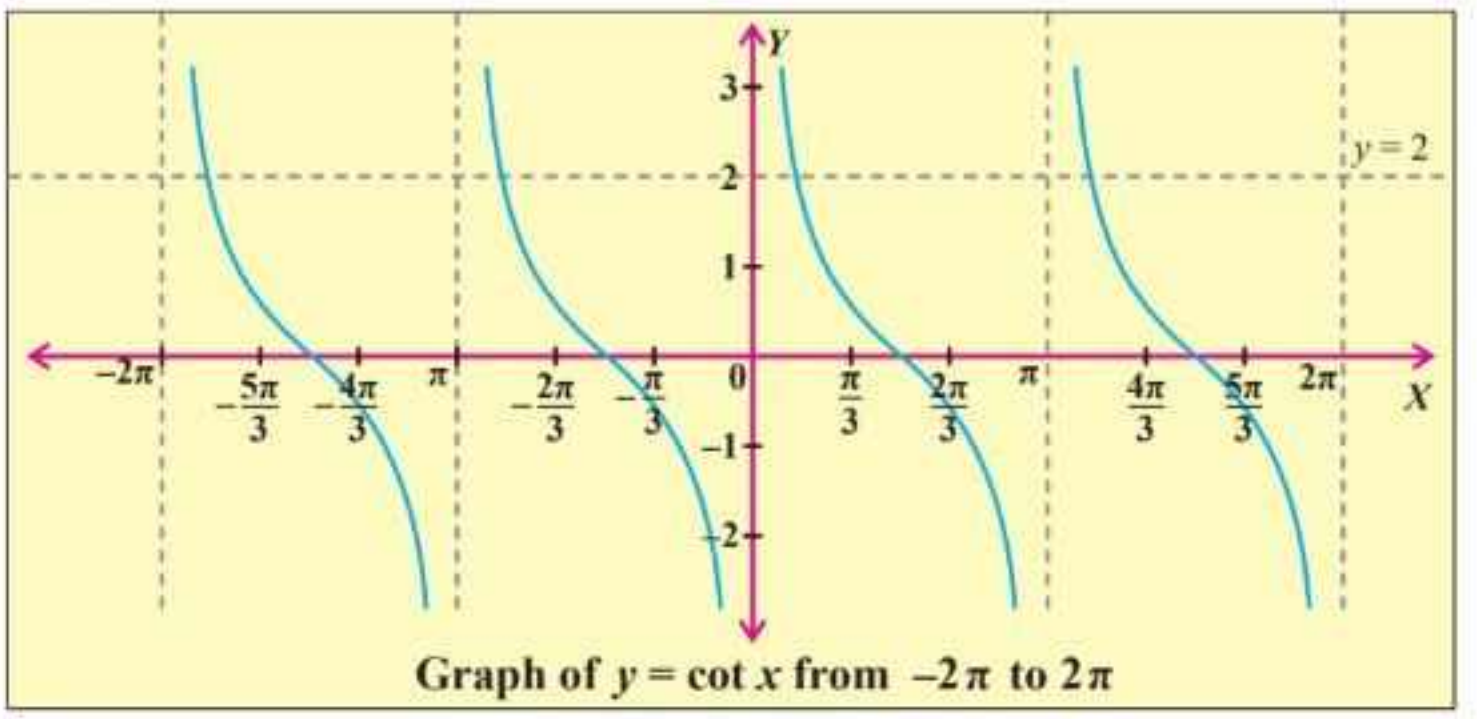


Graph of $y = \sec^{-1} x$

Figure 9.13(b)

9.1.6 Inverse of Cotangent Function

The graph of the function $f(x) = \cot x$ with domain $\{x \mid x \in \mathbb{R} \wedge x \neq n\pi, n \in \mathbb{Z}\}$ and range $(-\infty, \infty)$ is shown in Figure 9.14.



Graph of $y = \cot x$ from -2π to 2π

Figure 9.14

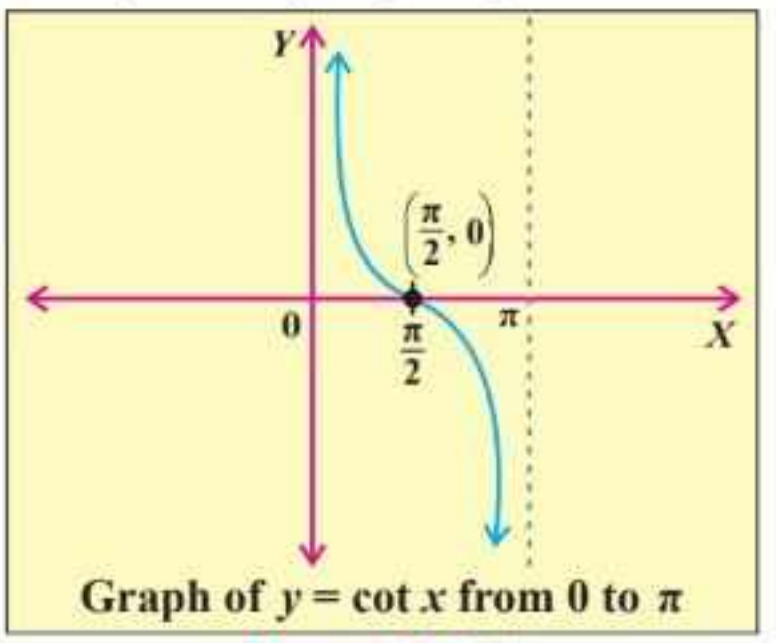
Clearly from the graph it follows that cotangent function is not bijective because the line $y = 2$ intersects the graph at many points. So, we restrict the domain of $f(x) = \cot x$ to the

interval $(0, \pi)$, then the function with restricted domain i.e., $f:(0, \pi) \to (-\infty, \infty)$ becomes one-one and onto that is invertible and is called **principal cotangent function**. The inverse cot x function is defined by

$$y = \cot^{-1} x \text{ iff } x = \cot y$$

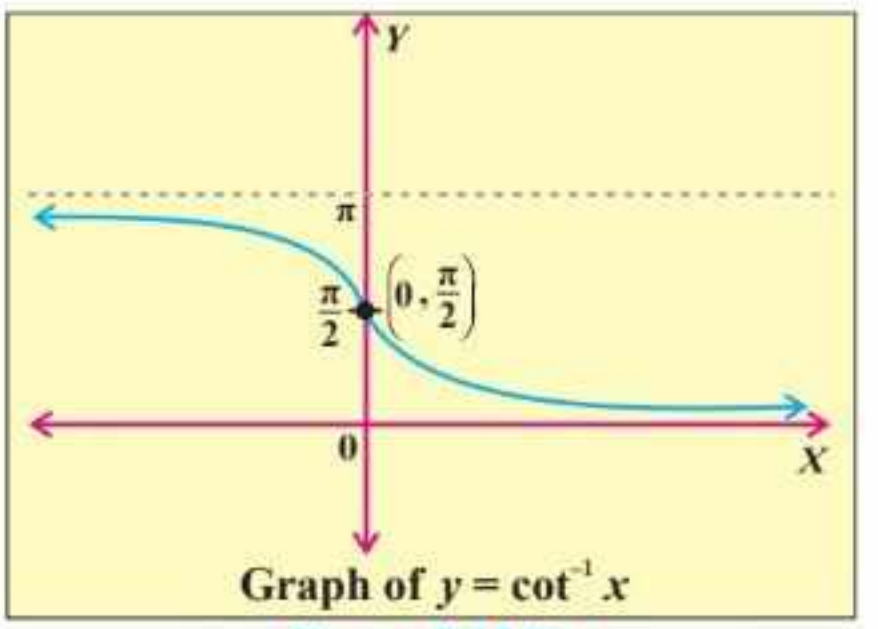
The domain of $\cot^{-1} x$ is $(-\infty, \infty)$ and range of $\cot^{-1} x$ is $(0, \pi)$.

The graph of the function $\cot:(0, \pi) \to (-\infty, \infty)$ is shown in Figure 15(a) and graph of $\cot^{-1}:(-\infty, \infty) \to (0, \pi)$ is shown in Figure 9.15(b).



Graph of $y = \cot x$ from 0 to π

Figure 9.15(a)



Graph of $y = \cot^{-1} x$

Figure 9.15(b)

Example 2 Find value of:

(i) $\sec\left[\tan^{-1}\left(-\sqrt{3}\right)\right]$ (ii) $\cot\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$ (iii) $\csc\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$

Solution: (i) $\sec\left[\tan^{-1}\left(-\sqrt{3}\right)\right]$

Let $\tan^{-1}\left(-\sqrt{3}\right) = y$

First, we find the angle y, whose tangent is $-\sqrt{3}$

$\tan y = -\sqrt{3}$, where $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Now $\tan y = -\sqrt{3} = \tan\left(-\frac{\pi}{3}\right) \quad \Rightarrow \quad y = -\frac{\pi}{3}$

So, $\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$

Now, $\sec\left[\tan^{-1}\left(-\sqrt{3}\right)\right] = \sec\left(-\frac{\pi}{3}\right) = 2$

Therefore $\sec\left[\tan^{-1}\left(-\sqrt{3}\right)\right] = 2$

(ii) $\cot\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$

Let $\cos^{-1}\left(-\frac{1}{2}\right)=y$

First, we find the angle *1*, whose cosine is $-\sqrt{3}$.

$\cos y=-\frac{1}{2}$, where $y\in[0,\pi]$

Now, $\cos y=-\frac{1}{2}=\cos\left(\frac{2\pi}{3}\right)\Rightarrow y=\frac{2\pi}{3}$

So, $\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$

Now, $\cot\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]=\cot\left(\frac{2\pi}{3}\right)=-\frac{1}{\sqrt{3}}$

Therefore $\cot\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]=-\frac{1}{\sqrt{3}}$

(iii) $\csc\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$

Let $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=y$

$\Rightarrow \sin y=\frac{\sqrt{3}}{2}$, where $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Now, $\sin y=\frac{\sqrt{3}}{2}=\sin\left(\frac{\pi}{3}\right)\Rightarrow y=\frac{\pi}{3}$

So, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$

Now, $\csc\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]=\csc\left(\frac{\pi}{3}\right)=\frac{2}{\sqrt{3}}$

Therefore, $\csc\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]=\frac{2}{\sqrt{3}}$

The table below summarizes the domains and ranges of the principal trigonometric and inverse trigonometric functions.

Function	Domain	Range	Function	Domain	Range
$\sin x$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[-1,1]$	$\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos x$	$[0,\pi]$	$[-1,1]$	$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$\tan x$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$(-\infty,\infty)$	$\tan^{-1}x$	$(-\infty,\infty)$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$\csc x$	$\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$	$(-\infty,-1]\cup[1,\infty)$	$\csc^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
$\sec x$	$\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$	$(-\infty,-1]\cup[1,\infty)$	$\sec^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$
$\cot x$	$(0,\pi)$	$(-\infty,\infty)$	$\cot^{-1}x$	$(-\infty,\infty)$	$(0,\pi)$

Example 3 Find the domain of the following functions

(i) $f(x)=\cos^{-1}(3x-1)$ (ii) $g(x)=\sin^{-1}\left(\frac{x^2-5}{4}\right)$ (iii) $h(x)=\sec^{-1}(\ln x)$

Solution: (i) The inverse cosine function, $f(x)=\cos^{-1}x$ is defined for all x such that $-1\le x\le 1$.

Thus, the domain of $f(x)=\cos^{-1}(3x-1)$ is defined for all x that satisfy

$$-1\le 3x-1\le 1 \quad\Rightarrow\quad 0\le 3x\le 2 \quad\Rightarrow\quad 0\le x\le\frac{2}{3} \quad\Rightarrow\quad x\in\left[0,\frac{2}{3}\right]$$

Hence, the domain of $f(x)=\cos^{-1}(3x-1)$ is $\left[0,\frac{2}{3}\right]$ such that $-1\le x\le 1$.

(ii) The inverse sine function, $f(x)=\sin^{-1}x$ is defined for all x in the interval $[-1,\ 1]$.

Thus, the domain of $g(x)=\sin^{-1}\left(\frac{x^2-5}{4}\right)$ is defined for all x that satisfy

$$-1\le\frac{x^2-5}{4}\le 1 \quad\Rightarrow\quad -4\le x^2-5\le 4 \quad\Rightarrow\quad 1\le x^2\le 9$$

$\Rightarrow$ $x^2\ge 1$ and $x^2\le 9$

$\Rightarrow$ $x\ge 1$ or $x\le -1$ and $(x-3)(x+3)\le 0$

$\Rightarrow$ $x\in(-\infty,-1]\cup[1,\infty)$ and $x\in[-3,3]$

The intersection of two result gives $[-3,-1]\cup[1,3]$ $\Rightarrow$ $x\in[-3,-1]\cup[1,3]$

Thus, the domain of $f(x)=\sin^{-1}\left(\frac{x^2-5}{4}\right)$ is $[-3,-1]\cup[1,3]$

(iii) The inverse secant function $f(x)=\sec^{-1}x$ is defined for all x in the interval $(-\infty,-1]\cup[1,\infty)$.

Thus, the domain of $h(x)=\sec^{-1}(\ln x)$ is defined for all x that satisfy

$\Rightarrow$ $\ln x\ge 1$ or $\ln x\le -1$ $\Rightarrow$ $x\ge e$ or $x\le e^{-1}$ $\Rightarrow$ $x\ge e$ or $x\le\frac{1}{e}$

$\Rightarrow$ $x\in\left(-\infty,\frac{1}{e}\right]\cup[e,\infty)$

Hence, the domain of $h(x)=\sec^{-1}(\ln x)$ is $\left(-\infty,\frac{1}{e}\right]\cup[e,\infty)$

Example 4 Prove that:

(i) $\csc^{-1}x=\sin^{-1}\frac{1}{x}$, $x\in(-\infty,-1]\cup[1,\infty)$

(ii) $\sec^{-1}x=\cos^{-1}\frac{1}{x}$, $x\in(-\infty,-1]\cup[1,\infty)$ (iii) $\cot^{-1}x=\tan^{-1}\frac{1}{x}$, $x\in(0,\infty)$

Solution:

Proof: **(i)** Let $\csc^{-1}x = y$ then $x = \csc y$

$$\Rightarrow \quad \sin y = \frac{1}{x}$$

$$\Rightarrow \quad y = \sin^{-1}\left(\frac{1}{x}\right)$$

Hence, $\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

Proof: (ii) Let $\sec^{-1}x = y$ then $x = \sec y$

$$\Rightarrow \quad \cos y = \frac{1}{x}$$

$$\Rightarrow \quad y = \cos^{-1}\left(\frac{1}{x}\right)$$

Hence, $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

Proof: (iii) Let $\cot^{-1}x = y$ then $x = \cot y$

$$\Rightarrow \quad \tan y = \frac{1}{x} \quad \Rightarrow \quad y = \tan^{-1}\left(\frac{1}{x}\right)$$

Hence, $\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$

Example 5 Prove that

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $x \in [-1, 1]$ (ii) $\csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$, $x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, $x \in (-\infty, \infty)$

Solution: **(i)** Let $\sin^{-1}x = y$

$$\Rightarrow \quad x = \sin y = \cos\left(\frac{\pi}{2} - y\right) \qquad \left(\because \quad \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha\right)$$

$$\Rightarrow \quad x = \cos\left(\frac{\pi}{2} - y\right) \quad \Rightarrow \quad \cos^{-1}x = \left(\frac{\pi}{2} - y\right)$$

$$\Rightarrow \quad \cos^{-1}x + y = \frac{\pi}{2} \quad \Rightarrow \quad \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$$

Hence, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Similarly, we can prove (ii) and (iii)

Example 6 Show that $\cos^{-1}\frac{7}{25} = \csc^{-1}\frac{25}{24}$

Solution: Let $\cos^{-1}\frac{7}{25} = \alpha \quad \Rightarrow \quad \cos\alpha = \frac{7}{25}, \quad \alpha \in \left[0, \frac{\pi}{2}\right]$ and

$$\sin\alpha = \pm\sqrt{1-\left(\frac{7}{25}\right)^2} = \pm\sqrt{\frac{625-49}{625}} \qquad \left(\because \quad \sin\alpha = \pm\sqrt{1-\cos^2\alpha}\right)$$

$$= \pm\sqrt{\frac{576}{625}} = \pm\frac{24}{25}$$

$\Rightarrow \quad \sin\alpha = \frac{24}{25}$ $\quad \left(\because \ \alpha \in \left[0, \frac{\pi}{2}\right]\right)$

$\Rightarrow \quad \csc\ \alpha = \frac{25}{24} \ \Rightarrow \ \alpha = \csc^{-1}\frac{25}{24}$

Thus, $\cos^{-1}\frac{7}{25} = \csc^{-1}\frac{25}{24}$

Example 7: Without using calculator find the value of $\sin\left[\frac{1}{2}\tan^{-1}\frac{4}{3}\right]$.

Solution: Let $\tan^{-1}\frac{4}{3} = \alpha \quad \Rightarrow \quad \tan\alpha = \frac{4}{3}$

Here, $x = 3, y = 4$

$\Rightarrow \quad r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

So, $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$

Using half angle identity

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}$$

Thus, $\sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right) = \frac{1}{\sqrt{5}}$

EXERCISE 9.1

1. Find the principal value of the following:

(i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (ii) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ (iii) $\tan^{-1}\left(-\sqrt{3}\right)$

(iv) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (v) $\csc^{-1}\left(-\sqrt{2}\right)$ (vi) $\sin^{-1}\left[\cos\left(\frac{3\pi}{4}\right)\right]$

(vii) $\tan^{-1}\left[\cot\left(\frac{\pi}{6}\right)\right]$ (viii) $\sec^{-1}\left[2\cos\left(\frac{3\pi}{4}\right)\right]$ (ix) $\csc^{-1}\left(\sqrt{2}\cos\frac{\pi}{4}\right)$

(x) $\tan^{-1}\left[3\cos\left\{3\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}\right]$ (xi) $\csc^{-1}\left[\frac{4}{3}\cos\left\{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right\}\right]$

2. Find the domain of the following:

(i) $f(x)=\sin^{-1}(5x-3)$ (ii) $g(x)=\cos^{-1}\sqrt{x-3}$

(iii) $h(x)=\csc^{-1}(x^2-3)$ (iv) $k(x)=\sin^{-1}(\ln x)$

(v) $p(x)=\tan^{-1}\sqrt{4-x^2}$ (vi) $q(x)=\sec^{-1}(x+2)$

(vii) $u(x)=\sec^{-1}\left(\frac{3x-1}{2}\right)$ (viii) $v(x)=\sin^{-1}(\ln x)+\csc^{-1}(e^x)$

3. Show that:

(i) $\cos^{-1}\frac{8}{17}=\sin^{-1}\frac{15}{17}$ (ii) $\csc^{-1}\frac{5}{4}=\cos^{-1}\frac{3}{5}$

(iii) $2\tan^{-1}\frac{3}{4}=\tan^{-1}\frac{24}{7}$ (iv) $2\sin^{-1}\frac{7}{25}=\sin^{-1}\frac{336}{625}$

4. Without using calculator find the value of the following:

(i) $\cos\left(\frac{1}{2}\cot^{-1}\frac{5}{12}\right)$ (ii) $\sec\left(\frac{1}{2}\sin^{-1}\frac{7}{25}\right)$ (iii) $\tan\left(\frac{1}{2}\cot^{-1}\frac{24}{7}\right)$

(iv) $\tan\left(\frac{1}{2}\sin^{-1}\frac{\sqrt{3}}{2}\right)$ (v) $\cot\left(\frac{1}{2}\sec^{-1}\frac{2\sqrt{6}}{3}\right)$ (v) $\csc\left(\frac{1}{2}\cot^{-1}\frac{4}{3}\right)$

9.2 Sum and Difference Formulas for Inverse Trigonometric Functions

In this section, we will derive the addition and subtraction formulas for inverse trigonometric functions. These identities can be derived by applying the fundamental trigonometric identities along with algebraic manipulation. These identities are valid only within the principal value of the respective inverse trigonometric functions. We will now establish the sum and difference formulas for the inverse sine, cosine and tangent functions that are given below.

(i) $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

(ii) $\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$

(iii) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left(xy-\sqrt{(1-x^2)(1-y^2)}\right)$

(iv) $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left(xy+\sqrt{(1-x^2)(1-y^2)}\right)$

Remember!

$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta)\pm\cos\alpha\sin\beta$

$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\pm\sin\alpha\sin\beta$

$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\pm\tan\alpha\tan\beta}$

(v) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ (vi) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$

To prove **(i)**, we proceed as follow:

Let $\sin^{-1}x=\alpha \Rightarrow \sin\alpha=x$ and $\sin^{-1}y=\beta \Rightarrow \sin\beta=y$

Now $\cos\alpha=\pm\sqrt{1-\sin^2\alpha}=\pm\sqrt{1-x^2}$

Since $\alpha,\beta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so $\cos\alpha,\cos\beta\geq 0$

$\therefore \quad \cos\alpha=\sqrt{1-x^2}$

Similarly, $\cos\beta=\sqrt{1-y^2}$

Now $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$\sin(\alpha+\beta)=x\sqrt{1-y^2}+y\sqrt{1-x^2} \quad \Rightarrow \quad \alpha+\beta=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$$

Thus, $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

Similarly, we can prove (ii), (iii) and (iv).

Next, we proceed to prove formula (v) as follows:

Let $\tan^{-1}x=\alpha \Rightarrow \tan\alpha=x$ and $\tan^{-1}y=\beta \Rightarrow \tan\beta=y$

Now $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$\tan(\alpha+\beta)=\frac{x+y}{1-xy} \quad \Rightarrow \quad \alpha+\beta=\tan^{-1}\frac{x+y}{1-xy}$$

Thus, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\dfrac{x+y}{1-xy}$

Further by putting $y=x$ in (v), we get

$$\tan^{-1}x+\tan^{-1}x=\tan^{-1}\frac{x+x}{1-xx} \quad \Rightarrow \quad 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$$

Similarly, we can prove (vi).

In a similar way, the following results can be derived from (i) and (ii)

$$2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right) \quad \text{and} \quad 2\cos^{-1}x=\cos^{-1}\left(2x^2-1\right)$$

Example 8 Prove that $\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}$.

Solution: Let $\alpha=\cos^{-1}x$, then $\cos\alpha=x$

Now $\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-x^2}$

$$\Rightarrow \alpha=\sin^{-1}\left(\sqrt{1-x^2}\right) \qquad \left(\because \quad \cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\right)\right)$$

Example 9 Prove that $\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{12}{13}\right)=\frac{\pi}{2}$.

Solution: Let $x=\frac{5}{13}$ and $y=\frac{12}{13}$

Using the identity $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$, we have

$$\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{12}{13}=\sin^{-1}\left(\frac{5}{13}\sqrt{1-\left(\frac{12}{13}\right)^2}+\frac{12}{13}\sqrt{1-\left(\frac{5}{13}\right)^2}\right)$$

$$=\sin^{-1}\left(\frac{5}{13}\sqrt{1-\frac{144}{169}}+\frac{12}{13}\sqrt{1-\frac{25}{169}}\right)$$

$$=\sin^{-1}\left(\frac{5}{13}\sqrt{\frac{25}{169}}+\frac{12}{13}\sqrt{\frac{144}{169}}\right)=\sin^{-1}\left(\frac{5}{13}\cdot\frac{5}{13}+\frac{12}{13}\cdot\frac{12}{13}\right)$$

$$=\sin^{-1}\left(\frac{25}{169}+\frac{144}{169}\right)=\sin^{-1}\left(\frac{169}{169}\right)=\sin^{-1}(1)=\frac{\pi}{2}$$

Therefore, $\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{12}{13}\right)=\frac{\pi}{2}$

Example 10 Prove that $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)=\frac{\pi}{4}$.

Solution: L.H.S $=\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$

$$=\tan^{-1}\left(\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\frac{x}{y}\cdot\frac{x-y}{x+y}}\right)=\tan^{-1}\left(\frac{\frac{x^2+xy-yx+y^2}{y(x+y)}}{\frac{yx+y^2+x^2+xy}{y(x+y)}}\right)$$

$$=\tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2+xy}\right)=\tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)$$

$$=\tan^{-1}(1)=\frac{\pi}{4}=\text{R.H.S}$$

Thus, $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)=\frac{\pi}{4}$

Example 11 Solve the equation $\tan^{-1}\left(\frac{x-1}{x+1}\right)+\tan^{-1}\left(\frac{2x}{x+1}\right)=\frac{\pi}{4}$.

Solution: Given that $\tan^{-1}\left(\frac{x-1}{x+1}\right)+\tan^{-1}\left(\frac{2x}{x+1}\right)=\frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{x-1}{x+1}+\frac{2x}{x+1}}{1-\frac{x-1}{x+1}\cdot\frac{2x}{x+1}}\right)=\frac{\pi}{4} \quad \Rightarrow \quad \tan^{-1}\left(\frac{\frac{3x-1}{x+1}}{1-\frac{2x(x-1)}{(x+1)^2}}\right)=\frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{\frac{3x-1}{x+1}}{\frac{-x^2+4x+1}{(x+1)^2}}\right)=\frac{\pi}{4} \quad \Rightarrow \quad \left(\frac{(3x-1)}{(x+1)}\cdot\frac{(x+1)^2}{(-x^2+4x+1)}\right)=\tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{(3x-1)}{(x+1)}\cdot\frac{(x+1)^2}{(-x^2+4x+1)}=1$$

$$\Rightarrow \quad 3x^2+2x-1=-x^2+4x+1 \quad \Rightarrow \quad 2x^2-x-1=0$$

$$\Rightarrow \quad (x-1)(2x+1)=0 \quad \Rightarrow \quad x-1=0 \text{ or } 2x+1=0$$

$$\Rightarrow \quad x=1 \text{ or } x=-\frac{1}{2}$$

Here, $x=-\frac{1}{2}$ does not satisfy the equation.

Thus, $x = 1$ is the only solution.

EXERCISE 9.2

Prove the following:

1. $\sin^{-1}\frac{7}{25}+\sin^{-1}\frac{15}{17}=\cos^{-1}\frac{87}{425}$
2. $\cos^{-1}\frac{7}{25}-\cos^{-1}\frac{5}{13}=\cos^{-1}\frac{323}{325}$
3. $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$
4. $\sin^{-1}\frac{12}{13}-\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{33}{65}$
5. $\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{63}{65}$
6. $\sin^{-1}\frac{3}{5}+2\tan^{-1}\frac{1}{4}=\cos^{-1}\frac{36}{85}$
7. $\sin^{-1}\frac{1}{\sqrt{2}}+\sin^{-1}\frac{1}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{10}}=\frac{\pi}{2}$
8. $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$

9. $\tan^{-1}\frac{10}{3}-\tan^{-1}\frac{7}{13}+\tan^{-1}\frac{9}{5}-\tan^{-1}\frac{2}{7}=\frac{\pi}{2}$

10. $\sec^2\left(\tan^{-1}\frac{1}{4}\right)+\csc^2\left(\cot^{-1}\frac{2}{5}\right)=\frac{889}{400}$

11. $\tan^2\left(\sec^{-1}3\right)+\cot^2\left(\csc^{-1}4\right)=23$

12. $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right),\ -1\le x\le 1$

13. $\tan(\sin^{-1}x)=\frac{x}{\sqrt{1-x^2}},\ -1<x<1$

14. $2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

14. $\cos(\sin^{-1}x)=\sqrt{1-x^2}$

15 $\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1)$

16. $2\tan^{-1}\left(\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)=\cos^{-1}\left(\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}\right)$

17. $\cot^{-1}\left(\frac{\cos x}{1-\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}$

18. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then show that $x+y+z=xyz$

19. Find the value of x if, $\tan^{-1}\left(\frac{2x-1}{3x-1}\right)+\tan^{-1}\left(\frac{2x+1}{3x+1}\right)=\frac{\pi}{4}$

20. If $\sin^{-1}\frac{2p}{1+p^2}+\cos^{-1}\frac{1-q^2}{1+q^2}=\tan^{-1}\frac{2y}{1-y^2}$, show that $y=\frac{p+q}{1-pq}$

Unit 10 Solution of Trigonometric Equations

INTRODUCTION

Trigonometric equations involve functions like $\sin\theta$, $\cos\theta$ and $\tan\theta$, and are solved to find the angle(s) that satisfy a given condition. In this unit, we solve such equations using the periodic, even/odd, and translation properties of trigonometric functions. For instance, equations such as $\sin\theta = k$, $\cos\theta = k$ or $\tan\theta = k$ are solved by considering the periodic nature of these functions. Next, we also explore graphical methods to solve equations like $\sin\theta = \frac{\theta}{2}$, $\cos\theta = \theta$ or $\tan\theta = 2\theta$ within specific intervals. The general solution of these equations incorporates the periodic behavior of trigonometric functions to identify all possible values of θ. Finally, we apply the concepts of inverse trigonometric functions to real-life problems in engineering and architecture, such as determining heights, calculating angles of elevation and depression, and designing accurate scale models.

10.1 Trigonometric Equation

A trigonometric equation is an equation that involves one or more trigonometric functions such as sine, cosine, tangent etc. and their inverses. For example,

$$\sin x = \frac{1}{\sqrt{2}},\ 2\cos^2\theta - 5\cos\theta + 1 = 0,\ 2\tan^{-1}x = \frac{\pi}{2} \text{ and } \sqrt{3}\cot x - \sec x - 1 = 0$$

are trigonometric equations. An unknown value of the angle that satisfy the given trigonometric equation is called a **solution** of the equation. The solutions that lie within the interval $[0, 2\pi]$ are called the **principal solutions**. Due to the periodic nature of trigonometric functions, these equations have infinitely many solutions. Once a solution is found within one period, adding or subtracting multiples of the period generates all possible solutions called the **general solution**. For example, Consider the equation:

$$\cos x = \frac{1}{2}$$

Since cosine is positive, so x lies in the I and IV quadrant.

$$x = \frac{\pi}{3} \text{ and } x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \quad \text{where } x \in [0, 2\pi]$$

So, $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ are the solution of the equation $\cos x = \frac{1}{2}$ and are called the **principal**

Solutions. Since 2π is the period of $\cos x$.

So, all possible solutions of $\cos x = \frac{1}{2}$ are $\frac{\pi}{3} + 2n\pi$ and $\frac{5\pi}{3} + 2n\pi$, $n \in \mathbb{Z}$. Hence, **general Solution** is given by $x = \left\{\frac{\pi}{3} + 2n\pi\right\} \cup \left\{\frac{5\pi}{3} + 2n\pi\right\}$.

Example 1 Solve $\sin x = -\frac{\sqrt{3}}{2}$

Solution: Since sine is negative. So x lies in III and IV quadrants with reference angle $\frac{\pi}{3}$

Hence, $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ and $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ where $x \in [0, 2\pi]$

as 2π is the period of $\sin x$.

So, all possible solutions of $\sin x = -\frac{\sqrt{3}}{2}$ are $\frac{2\pi}{3} + 2n\pi$ and $\frac{5\pi}{3} + 2n\pi$, $n \in \mathbb{Z}$

General solution $= \left\{\frac{4\pi}{3} + 2n\pi\right\} \cup \left\{\frac{5\pi}{3} + 2n\pi\right\}$, $n \in \mathbb{Z}$

Example 2 Solve $2\sec^2 x - \frac{8}{3} = 0$.

Solution: $2\sec^2 x - \frac{8}{3} = 0$

$$\Rightarrow \quad 2\sec^2 x = \frac{8}{3} \quad \Rightarrow \quad \sec x = \pm\frac{2}{\sqrt{3}}$$

$$\Rightarrow \quad \sec x = \frac{2}{\sqrt{3}} \quad \text{or} \quad \sec x = -\frac{2}{\sqrt{3}}$$

$$\Rightarrow \quad \cos x = \frac{\sqrt{3}}{2} \quad \text{or} \quad \cos x = -\frac{\sqrt{3}}{2}$$

When $\cos x = \frac{\sqrt{3}}{2}$

Since cosine is positive, so x lies in I and IV quadrants with reference angle $\frac{\pi}{6}$

Hence, $x = \frac{\pi}{6}$ and $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ where $x \in [0, 2\pi]$. Period of $\cos x$ is 2π.

So, all possible solutions of $\cos x = \frac{\sqrt{3}}{2}$ are $\frac{\pi}{6} + 2n\pi$ and $\frac{11\pi}{6} + 2n\pi$, $n \in \mathbb{Z}$.

When $\cos x = -\frac{\sqrt{3}}{2}$

Since cosine is negative, so x lies in II and III quadrants with reference angle $\frac{\pi}{6}$.

Hence, $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ and $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$. Period of $\cos x$ is 2π.

So, all possible solutions of $\cos x = -\frac{\sqrt{3}}{2}$ are $\frac{5\pi}{6} + 2n\pi$ and $\frac{7\pi}{6} + 2n\pi$, $n \in \mathbb{Z}$

General solution $= \left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi\right\} \cup \left\{\frac{7\pi}{6} + 2n\pi\right\} \cup \left\{\frac{11\pi}{6} + 2n\pi\right\}$, $n \in \mathbb{Z}$

Example 3 Solve $\sqrt{3}\cot x - \csc x - 1 = 0$.

Solution: $\sqrt{3}\cot x - \csc x - 1 = 0$ ...(1)

$$\Rightarrow \quad \sqrt{3}\frac{\cos x}{\sin x} - \frac{1}{\sin x} - 1 = 0$$

$$\Rightarrow \quad \sqrt{3}\cos x - 1 - \sin x = 0$$

$$\Rightarrow \quad \sqrt{3}\cos x - 1 = \sin x$$

Taking square on both sides

$$\left(\sqrt{3}\cos x - 1\right)^2 = \sin^2 x$$

$$\Rightarrow \quad 3\cos^2 x + 1 - 2\sqrt{3}\cos x = \sin^2 x$$

$$\Rightarrow \quad 3\cos^2 x + 1 - 2\sqrt{3}\cos x = 1 - \cos^2 x$$

$$\Rightarrow \quad 4\cos^2 x - 2\sqrt{3}\cos x = 0$$

$$\Rightarrow \quad 2\cos x\left(2\cos x - \sqrt{3}\right) = 0$$

$$\Rightarrow \quad 2\cos x = 0 \quad \text{or} \quad 2\cos x - \sqrt{3} = 0$$

$$\Rightarrow \quad \cos x = 0 \quad \text{or} \quad \cos x = \frac{\sqrt{3}}{2}$$

When $\cos x = 0$

$$\Rightarrow \quad x = \frac{\pi}{2} \text{ and } x = \frac{3\pi}{2} \quad \text{where } x \in [0, 2\pi]$$

To check extraneous roots, put $x = \frac{\pi}{2}$ in equation (1)

$$\sqrt{3}\cot x - \csc x - 1 = 0$$

Remember!

Sometimes it is necessary to square both sides of a trigonometric equation to solve it. However, this operation can introduce extraneous roots i.e., solutions that appear valid but don't actually satisfy the original equation. Therefore, it is essential to verify every potential solution by substituting it back into the original equation before accepting it as valid.

L.H.S$=\sqrt{3}\cot\frac{\pi}{2}-\csc\frac{\pi}{2}-1=\sqrt{3}(0)-1-1=-2\neq 0=$ R.H.S

$\Rightarrow \quad x=\frac{\pi}{2}$ does not satisfy the equation, so $\frac{\pi}{2}$ is extraneous root.

Now put $x=\frac{3\pi}{2}$ in equation (1)

L.H.S$=\sqrt{3}\cot\frac{3\pi}{2}-\csc\frac{3\pi}{2}-1=\sqrt{3}(0)-(-1)-1=-1+1=0=$ R.H.S

$\Rightarrow \quad x=\frac{3\pi}{2}$ is the solution of the equation. Period of $\cos x$ is 2π.

So, general values of x are $\frac{3\pi}{2}+2n\pi,\ n\in\mathbb{Z}$.

When $\cos x=\frac{\sqrt{3}}{2}$

$\Rightarrow$ Since cosine is positive, so x lies in I and IV quadrants with reference angle $\frac{\pi}{6}$.

So, $x=\frac{\pi}{6}$ and $x=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$ where $x\in[0,\ 2\pi]$

again, to check extraneous roots, put $x=\frac{\pi}{6}$ in equation ...(1)

$$\sqrt{3}\cot x-\csc x-1=0$$

L.H.S$=\sqrt{3}\cot\frac{\pi}{6}-\csc\frac{\pi}{6}-1=\sqrt{3}\cdot\sqrt{3}-2-1=3-3=0=$ R.H.S

$\Rightarrow \quad x=\frac{\pi}{6}$ is solution of equation (1).

Now put $x=\frac{11\pi}{6}$ in equation (1)

L.H.S$=\sqrt{3}\cot\frac{11\pi}{6}-\csc\frac{11\pi}{6}-1=\sqrt{3}(-\sqrt{3})-(-2)-1=-3+2-1=-2\neq 0=$ R.H.S

$\Rightarrow \quad x=\frac{11\pi}{6}$ does not satisfy the equation (1), so $\frac{11\pi}{6}$ is an extraneous root. Period of $\cos x$ is 2π.

So, general values of x are $\frac{\pi}{6}+2n\pi,\ n\in\mathbb{Z}$.

Hence, general solution $=\left\{\frac{3\pi}{2}+2n\pi\right\}\cup\left\{\frac{\pi}{6}+2n\pi\right\},\ n\in\mathbb{Z}$

EXERCISE 10.1

Solve the following trigonometric equation:

1. $\sin x = \frac{1}{\sqrt{2}}$
2. $2\cos x = -1$
3. $3\tan^2 x = 1$
4. $9 - 3\csc^2 x = 5$
5. $2\cos^2 x - 3\cos x + 1 = 0$
6. $\sin^2 x - 3\cos^2 x = 0$
7. $\sqrt{2}\cos x \sin x - \sin x = 0$
8. $3\cot^2 x + 3\csc^2 x = 5$
9. $\sin 2x - \sqrt{3}\cos x = 0$
10. $\cos 2x = \cos x$
11. $\tan^2 x - \left(\sqrt{3}+1\right)\tan x = -\sqrt{3}$
12. $\tan x = \sin 2x$
13. $\sin 3x = \cos 2x$
14. $\sqrt{2\sin^2 x + 1} = \sin x + \cos x$
15. $\sin^3 x \cos x + \cos^3 x \sin x = \frac{1}{2}$
16. $\sin^3 2x \cos 2x - \cos^3 2x \sin 2x = \frac{\sqrt{2}}{8}$
17. $\sin 3x \cos x + \cos 3x \sin x = \frac{1}{\sqrt{2}}$
18. $\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 4$
19. $\frac{\tan\left(7x+\frac{\pi}{6}\right)+\tan 3x}{1-\tan\left(7x+\frac{\pi}{6}\right)\tan 3x} = 1$
20. $\frac{\cot\left(2x+\frac{\pi}{12}\right)-\cot\left(5x+\frac{\pi}{12}\right)}{1+\cot\left(2x+\frac{\pi}{12}\right)\cot\left(5x+\frac{\pi}{12}\right)} = \frac{1}{\sqrt{3}}$
21. $\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} - \frac{\sin x}{\sqrt{1+\tan^2 x}} = \frac{1}{2}$
22. $\sin 3x - \sin 5x + \sin 7x = 0$
23. $\cos 2x - \cos 6x = \sin 4x$
24. $\sin x + \sin 3x + \sin 5x + \sin 7x = 0$

10.2 Graphical Solution of Trigonometric Equation

The graphical solution of trigonometric equations involves plotting the graphs of both sides of the equation and identifying their points of intersection. Each point where the curves intersect represents a solution to the equation.

When dealing with trigonometric equations of the form $f(x) = g(x)$, it can be challenging to find exact solutions. While some equations can be solved algebraically using trigonometric identities and properties, many are too complex for such methods. In these cases, graphical methods serve as a valuable alternative. By plotting both functions $y = f(x)$ and $y = g(x)$ on the coordinate plane, we can visually observe their points of intersection. The x-coordinates of these intersection points are the solutions that satisfy $f(x) = g(x)$. This graphical approach not only helps in determining the number of solutions but also provides deeper insight into the behavior of the functions and the nature of their interactions.

Example 4 Solve the equation $\sin x = \frac{x}{2}$ graphically over the interval $[-\pi, \pi]$

Solution: To find the values of x that satisfy $\sin x = \frac{x}{2}$, we identify the points where the graphs of $y = \sin x$ and $y = \frac{x}{2}$ intersect.

Now, graph of both the functions in the interval $[-\pi, \pi]$ represented as follows:

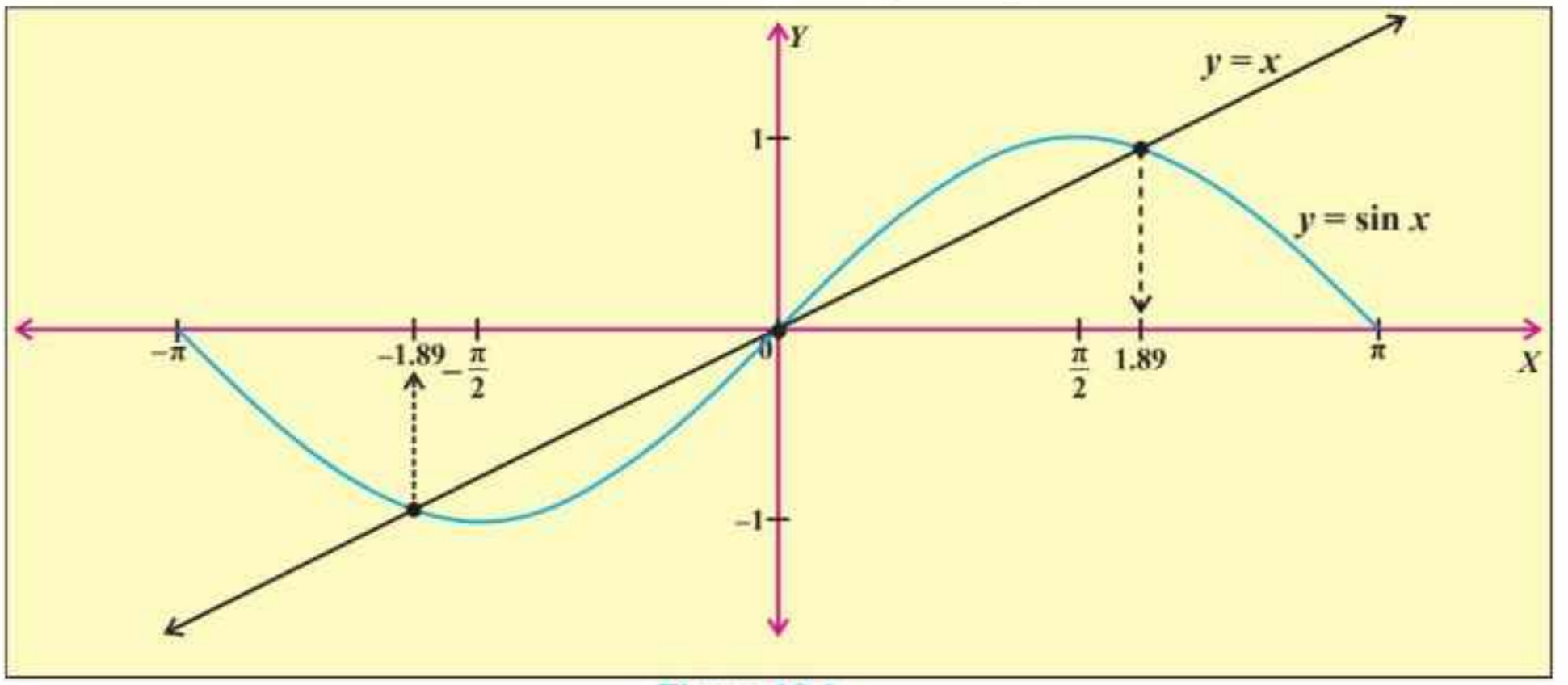


Figure 10.1

From the graph (see Figure 10.1), we observe that the curves intersect at three points within the interval $[-\pi, \pi]$. Thus, the solutions to $\sin x = \frac{x}{2}$ in $[-\pi, \pi]$ are:

$x = -1.89, 0$ or 1.89 radians

Note

The exact values can be refined using numerical methods, but the graphical approach provides a clear visual approximation

Example 5 Solve the equation $\cos x = \frac{x}{8}$ graphically.

Solution: To find the values of x that satisfy $\cos x = \frac{x}{8}$, we identify the points where the graphs of $y = \cos x$ and $y = \frac{x}{8}$ intersect.

First, note that the range of $\cos x$:

$$-1 \le \cos x \le 1 \Rightarrow -1 \le \frac{x}{8} \le 1 \Rightarrow -8 \le x \le 8$$

Next, we plot both functions over the interval $[-8, 8]$

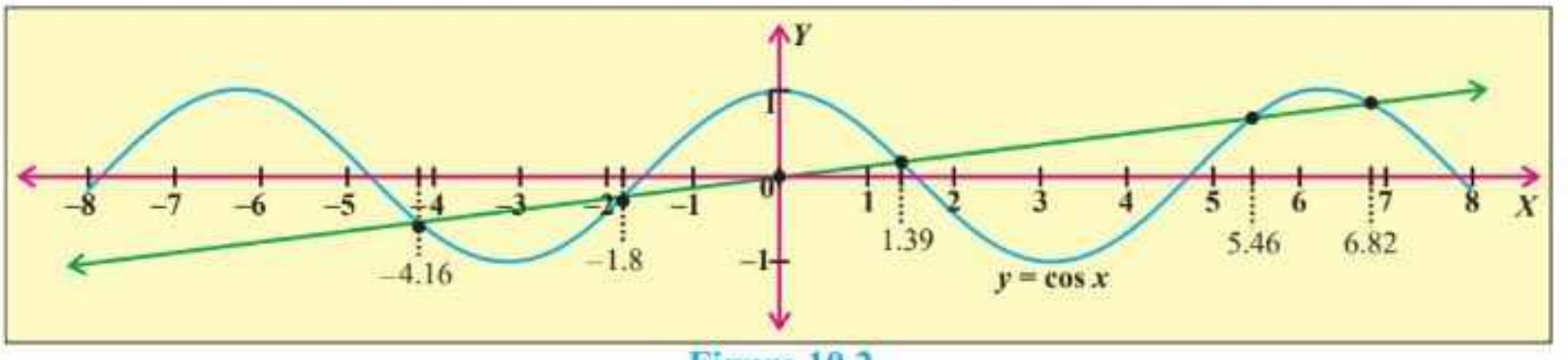


Figure 10.2

From the graph (see Figure 10.2), we observe that the curves intersect at six points within the interval $[-8, 8]$.

Thus, the solutions to $\cos x = \frac{x}{8}$ in $[-\pi, \pi]$ are approximately:

$x = -4.16,\ -1.8,\ 0,\ 1.39,\ 5.46$ and 6.82 radians

Example 6: Solve the equation $\tan x = \frac{x}{2}$ graphically in $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$.

Solution: To find the values of x that satisfy $\tan x = \frac{x}{2}$, we identify the points where the graphs of $y = \tan x$ and $y = \frac{x}{2}$ intersect.

Now, we plot both functions over the interval $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

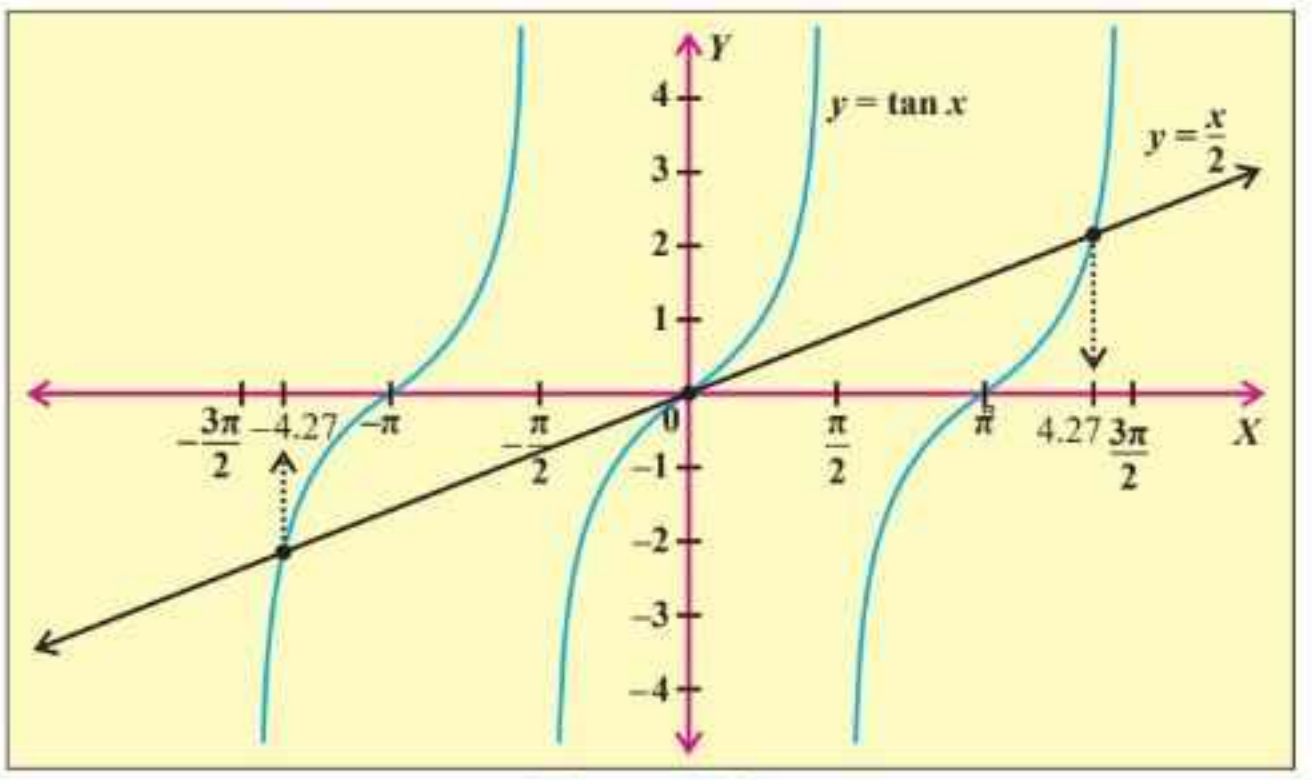


Figure 10.3

From the graph (see figure 3), we observe that the curves intersect at three points within the interval $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$.

Thus, the solutions to $\tan x = \frac{x}{2}$ in $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$ are approximately:

$x = -4.27,\ 0$ and 4.27 radians

Example 7 Solve the equation $3\cos x = 2\sin x - 1$ graphically in $[-2\pi, 2\pi]$.

Solution: To find the values of x that satisfy $3\cos x = 2\sin x - 1$, we identify the points where the graphs of $y = 3\cos x$ and $y = 2\sin x - 1$ intersect.

Now, we plot both functions over the interval $[-2\pi, 2\pi]$.

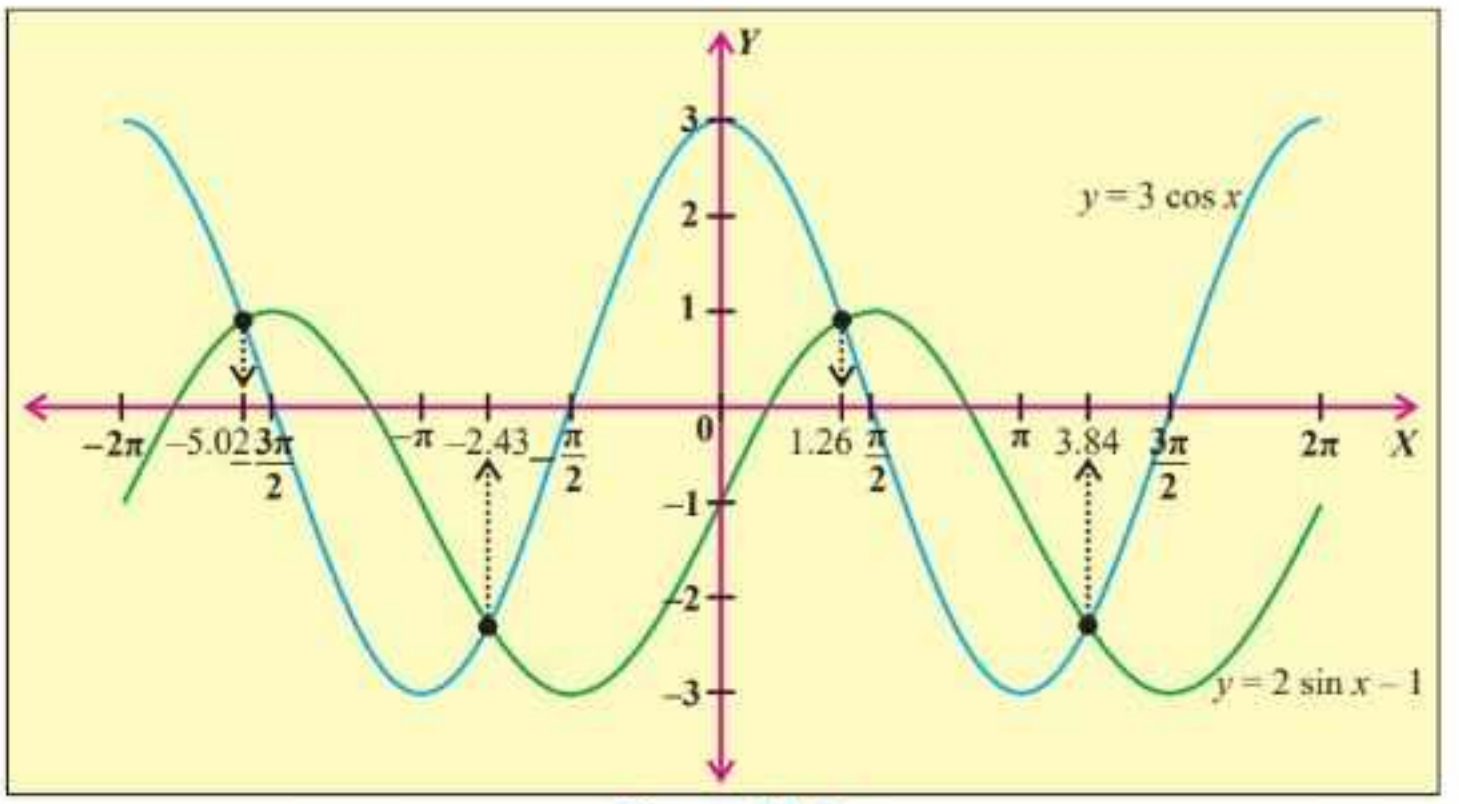


Figure 10.4

From the graph (see Figure 10.4), we observe that the curves intersect at four points within the interval $[-2\pi, 2\pi]$.

Thus, the solutions to $3\cos x = 2\sin x - 1$ in $[-2\pi, 2\pi]$ are approximately:

$x = -5.02,\ -2.43,\ 1.26$ and 3.84 radians.

Example 8 Solve the equation $2\sin^2 x = 1 - 3\cos x$ graphically in $[0, 2\pi]$.

Solution: To find the values of x that satisfy $2\sin^2 x = 1 - 3\cos x$, we identify the points where the graphs of $y = 2\sin^2 x$ and $y = 1 - 3\cos x$ intersect.

Now, we plot both functions over the interval $[0, 2\pi]$.

From the graph (see figure 5), we observe that the curves intersect at four points within the interval $[0, 2\pi]$.

Thus, the solutions to $2\sin^2 x = 1 - 3\cos x$ in $[0, 2\pi]$ are approximately:

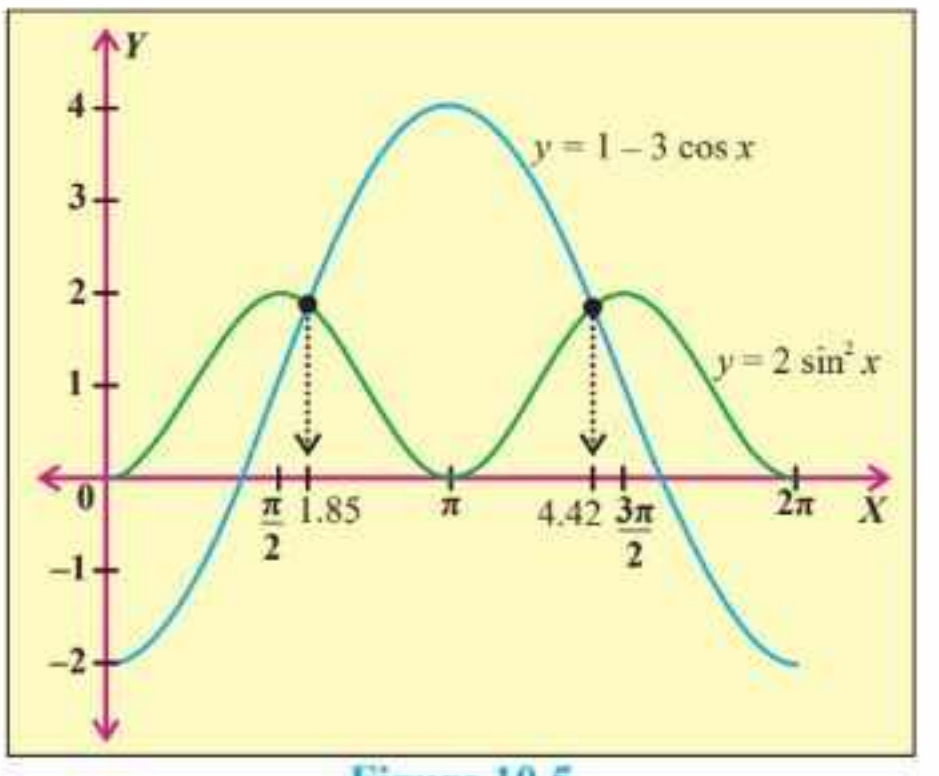


Figure 10.5

$$x = 1.855 \text{ and } 4.427 \text{ radians}$$

10.3 Real Life Problems

Example 9 A lighthouse with a 15 meter tall lantern stands atop a cliff. From a ship 120 meters offshore, the lantern subtends the same angle as a 1.5 meter buoy placed at the base of the cliff.

(a) Find the exact height of the cliff.

(b) Calculate the subtended angle (θ).

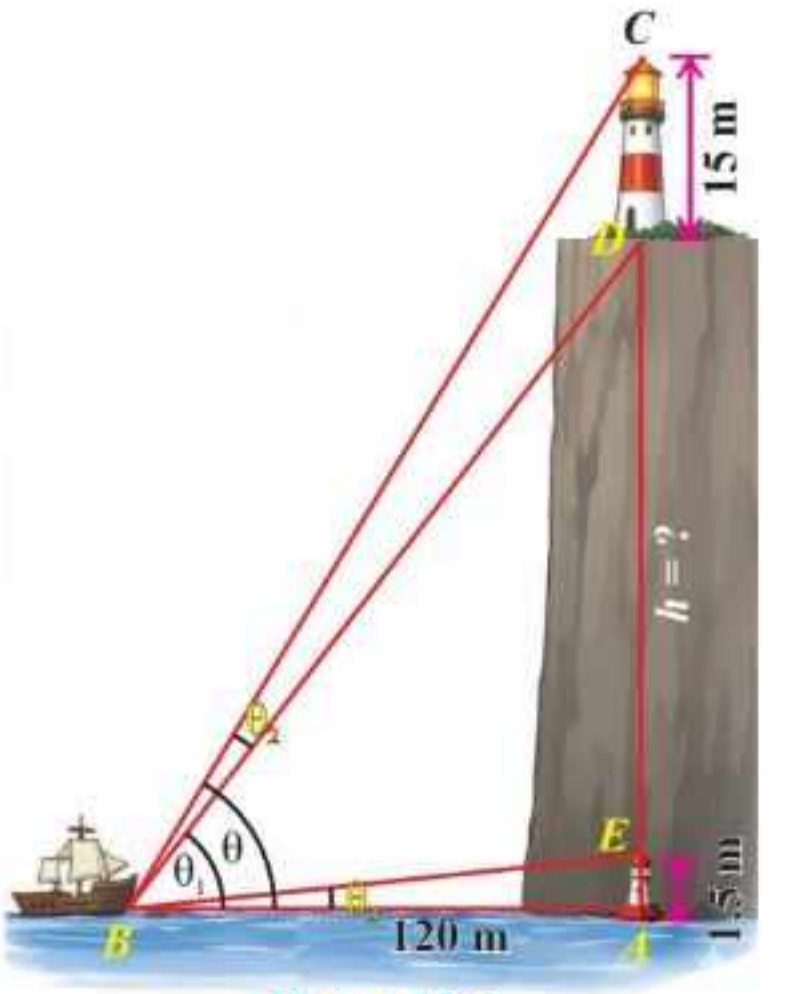


Figure 10.6

Solution: Let h be the height of the cliff then from triangle ABD

$$\tan\theta_1 = \frac{h}{120} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{h}{120}\right)$$

Also, from triangle ABC

$$\tan\theta = \frac{h+15}{120} \Rightarrow \theta = \tan^{-1}\left(\frac{h+15}{120}\right)$$

The subtended angle for the lantern

$$\theta_2 = \theta - \theta_1 = \tan^{-1}\left(\frac{h+15}{120}\right) - \tan^{-1}\left(\frac{h}{120}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{h+15}{120}\right) - \tan^{-1}\left(\frac{h}{120}\right)$$

height of the buoy = 1.5 m, then from triangle ABE

$$\tan\theta_3 = \frac{1.5}{120} \Rightarrow \theta_3 = \tan^{-1}\left(\frac{1.5}{120}\right)$$

given that the lantern subtends the same angle as a 1.5-meter buoy placed at the base of the cliff, that is, $\theta_2 = \theta_3$

$$\tan^{-1}\left(\frac{h+15}{120}\right) - \tan^{-1}\left(\frac{h}{120}\right) = \tan^{-1}\left(\frac{1.5}{120}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{h+15}{120} - \frac{h}{120}}{1+\frac{h+15}{120}\cdot\frac{h}{120}}\right) = \tan^{-1}\left(\frac{1.5}{120}\right)$$

$$\left(\because \quad \tan^{-1}\alpha - \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha-\beta}{1+\alpha\cdot\beta}\right)\right)$$

$$\Rightarrow \quad \frac{\frac{15}{120}}{\frac{14400+h^2+15h}{14400}}=0.0125$$

$$\Rightarrow \quad \frac{15}{120}\cdot\frac{14400}{14400+h^2+15h}=0.0125$$

$$\Rightarrow \quad h^2+15h-129600=0$$

Using quadratic formula

$$\Rightarrow \quad h=\frac{-15\pm\sqrt{15^2-4(1)(-129600)}}{2(1)}=\frac{-15\pm\sqrt{225+518400}}{2}$$

$$\Rightarrow \quad h=\frac{-15\pm\sqrt{518625}}{2}=\frac{-15\pm720.2}{2} \quad \Rightarrow \quad h=\frac{705.2}{2}=352.6$$

Thus, height of the cliff is 352.6 m.

(b) The subtended angle for the lantern

$$\theta_2=\tan^{-1}\left(\frac{367.6}{120}\right)-\tan^{-1}\left(\frac{352.6}{120}\right)=71.93^\circ-71.21^\circ=0.72^\circ$$

$$\Rightarrow \quad \theta_2=0.72^\circ$$

Thus, the subtended angle for the lantern is 0.72°.

Example 10 A projectile is launched from the ground with an initial velocity $v_\circ=30\,m/s$. The range R of the projectile is given by the formula $R=\frac{v_\circ^2}{g}\sin 2\theta$, where θ is the launch angle to the horizontal and $g=9.8$ m/s^2 is the acceleration due to gravity.

(a) Find the angle θ (in degrees) required for the projectile to hit a target exactly 80 meters away.

(b) If the maximum range achievable is $R_{max}=\frac{v_\circ^2}{g}$, determine the angle θ that gives the maximum range. Justify your answer mathematically.

Solution: (a) Given that: $v_\circ=30\,m/s$, $R=80\,m$, $g=9.8\,m/s^2$.

Substituting these values in $R=\frac{v_\circ^2}{g}\sin 2\theta$, we have

$$80=\frac{(30)^2\sin 2\theta}{9.8} \qquad \Rightarrow \qquad 80\times 9.8=900\sin 2\theta$$

$$\Rightarrow \quad 784 = 900\sin 2\theta \quad \Rightarrow \quad \frac{784}{900} = \sin 2\theta$$

$$\Rightarrow \quad \sin 2\theta = 0.8711 \quad \Rightarrow \quad 2\theta = \sin^{-1}(0.8711)$$

$$\Rightarrow \quad 2\theta = 60.5^\circ \quad \Rightarrow \quad \theta = \frac{60.5^\circ}{2} = 30.25^\circ$$

Since $\sin\theta = \sin(180^\circ - \theta,$, second possible solution in $[0,\ 2\pi]$ is:

$$\Rightarrow \quad 2\theta = 180^\circ - 60.5^\circ = 119.5^\circ$$

$$\text{or} \quad \theta = \frac{119.5^\circ}{2} = 59.75^\circ$$

Thus, the projectile will hit the ground 80 meters away at angles 30.25° or 59.75°.

(b) The range is maximized when $\sin 2\theta$ is maximized. Since the maximum value of $\sin 2\theta$ is 1, so we have:

$$\sin 2\theta = 1 \quad \Rightarrow \quad 2\theta = \sin^{-1}(1)$$

$$\Rightarrow \quad 2\theta = 90^\circ \quad \Rightarrow \quad \theta = \frac{90^\circ}{2}$$

$$\Rightarrow \quad \theta = 45^\circ$$

Now, the maximum range can be calculated as:

$$R_{\max} = \frac{v_\circ^2}{g} = \frac{900}{9.8} = 91.84\,m$$

Thus, the maximum range is achieved at 45°, as sin 2θ reaches its maximum value of 1 at this angle, and no other angle yields a greater range.

EXERCISE 10.2

Solve graphically:

1. $3\sin x = \frac{x}{2} + 1,\ x \in [-\pi,\ \pi]$
2. $\cos(2x-1) = \frac{x}{5},\ x \in [0,\ \pi]$
3. $3\tan x = 2x + 1,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
4. $\sec(x+3) = -2x + \frac{1}{3},\ x \in [-\pi,\ \pi]$
5. $2\cos x = x^2 + 2x - 1,\ x \in [-\pi,\ \pi]$
6. $\sin 3x = \frac{x}{2}$
7. $\sin 2x = 2\cos 3x,\ x \in [0,\ \pi]$
8. $\sin x + \cos x = -2\cos x,\ x \in [0,\ \pi]$
9. A hiker pulls a sled using a rope that exerts a force of 80 newtons over a distance of 25 meters. The total work done is 1200 joules. At what angle to the direction of motion is the force being applied?

10. A radio tower stands on flat ground with a small antenna mounted at its top. From a point 50 meters away from the base of the tower. The antenna subtends the same angle as a 1 meter vertical rod placed at the base of the tower. The total height of the tower and antenna is 10 meters.
    (a) Find the height of the tower (without the antenna).
    (b) Calculate the subtended angle.
11. A person standing 20 meters away from the base of a tree measures the angle of elevation to the top of the tree as 30°.
    (a) Find the height of the tree.
    (b) If the person walks 10 meters closer to the tree, what will be the new angle of elevation to the top of the tree?

Unit 11 Vector Valued Functions and Their Derivatives

INTRODUCTION

Vector-valued functions return vectors with both magnitude and direction to model multidimensional quantities such as position, velocity, or force in 2D or 3D, unlike scalar functions which give a single value. Their derivative provides the instantaneous rate of change, yielding the velocity vector essential for analyzing motion, with applications in engineering, computer graphics, and physics, from robotic arms to planetary orbits. This chapter covers definitions, component forms, operations, and differentiation techniques for analyzing curves and solving real-world problems involving velocity and acceleration.

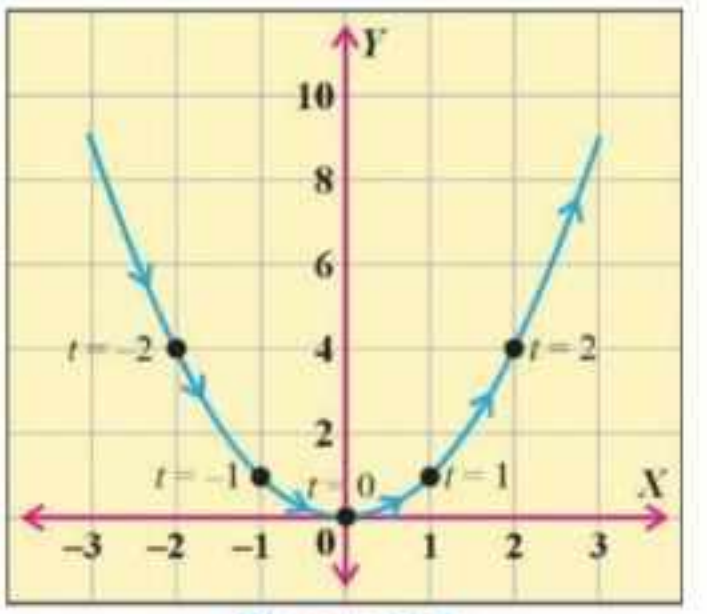


Figure 11.1

For example, a particle moving along a path in 2D might have its position at time t given by:

$\mathbf{r}(t) = t\hat{i} + t^2\hat{j}$. This means at time t, the particle is at the point (t, t^2) (Figure 11.1).

A real-world example of a vector-valued function in three-dimensional space is the position of a satellite orbiting Earth. Suppose the vector-valued function describes a satellite's position at time t (in hours): $\mathbf{r}(t) = (4000\cos t)\hat{i} + (4000\sin t)\hat{j} + 3000t\hat{k}$

Here, the components represent the satellite's coordinates in space: $x = 4000\cos t$, $y = 4000\sin t$, and $z = 3000t$ (in kilometers). The x- and y-components model the satellite's circular orbit in the xy-plane with a radius of 4000 km, while the z-component indicates a linear ascent along the z-axis at a rate of 3000 km per hour. This function maps time to a position vector, describing the satellite's trajectory through space, which is essential for tracking and communication in aerospace engineering (Figure 11.2).

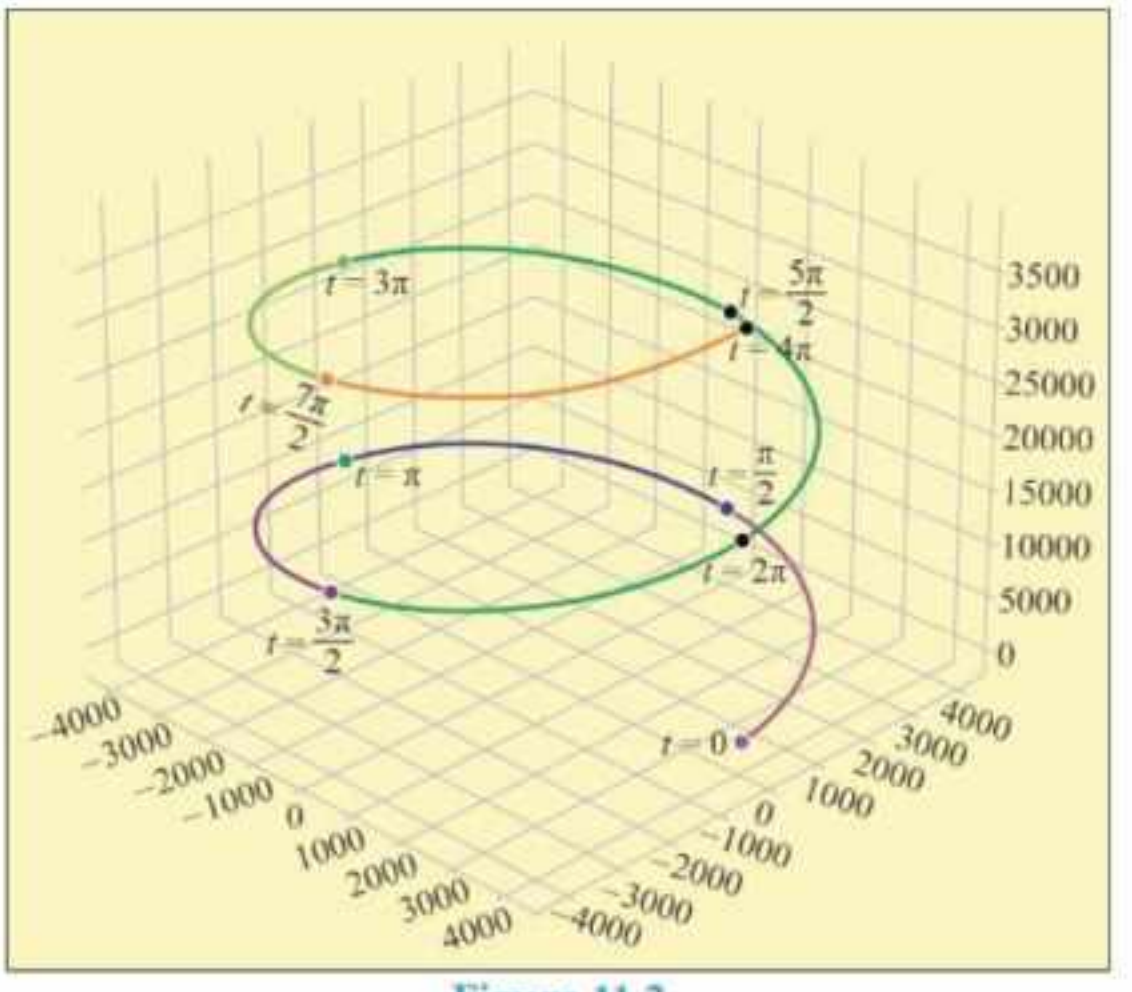


Figure 11.2

11.1 Scalar And Vector-Valued Functions Of A Single Variable

11.1.1 Scalar Function

A scalar function is a function that assigns a single real number (a scalar) to each point in its domain. Think of it as a rule that assigns one value to a position in space, like measuring temperature at a specific point.

Mathematically, if to each value of a scalar variable t in the domain of a function $f(t)$, there corresponds a unique scalar f, then f is called a scalar function of t. A real-valued function is commonly called a scalar function. For example, $f(t)=3t+2\cos t$ is a scalar function.

11.1.2 Vector-Valued Function (Vector Function)

A vector-valued function in a single variable is a function that takes one variable as input and returns a vector as output. Each component of the output vector is itself a scalar function of the input. Typically, a vector-valued function is written as:

$$\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle=f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$$

Where $\hat{i},\hat{j},\hat{k}$ are the unit vectors associated with the rectangular coordinates $f(t),g(t)$, and $h(t)$ are scalar functions of t, are the components of the vector-valued function $\mathbf{r}(t)$.

11.1.3 Constructing A Vector-Valued Function

Step-1: Determine the Components: Identify how each coordinate changes with respect to t.

Step-2: Combine into Vector Form:

Express the components as a single vector using either component notation $\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle$ or unit vectors $\mathbf{r}(t)=f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$.

Step-3: Specify the Domain: Define the range of t where the function is valid (e.g., $t\geq 0$) for time-dependent motion.

Example 1: Projectile Motion

When a ball is thrown at an angle, its position over time follows a parabolic path under the influence of gravity. If the initial velocity is $v_\circ$ at an angle θ, the position vector is:

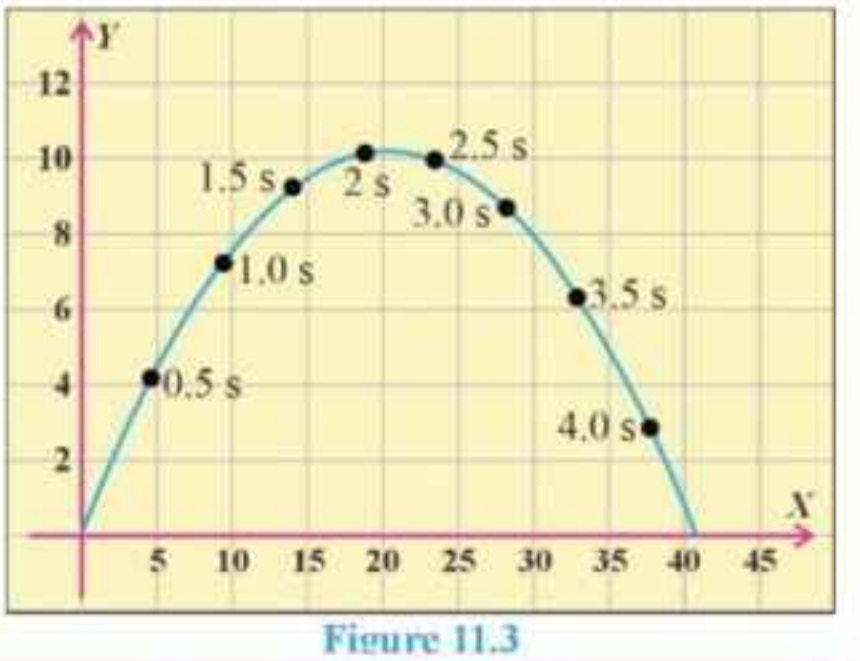


Figure 11.3

$$\mathbf{r}(t)=(v_\circ\cos\theta)t\hat{i}+\left((v_\circ\sin\theta)t-\frac{1}{2}gt^2\right)\hat{j}$$

Where, $(v_\circ\cos\theta)t$ =x-component (horizontal motion at constant speed) and

$(v_\circ\sin\theta)t-\frac{1}{2}gt^2=y$-component (vertical motion with gravity deceleration).

This model simulates the trajectories of objects like soccer balls, arrows, or rockets in 2D.

Example 2: Flight Path of a Drone

A drone taking off and moving in a helical search pattern can be modeled as:

$$\mathbf{r}(t)=5\cos t\hat{i}+5\sin t\hat{j}+0.5t\hat{k}$$

Where x- and y-components represent the circular motion with a 5-meter radius, and the z–component represents the steady ascent at 0.5 meters per second. This describes real-world applications like surveillance, search missions, or aerial photography.

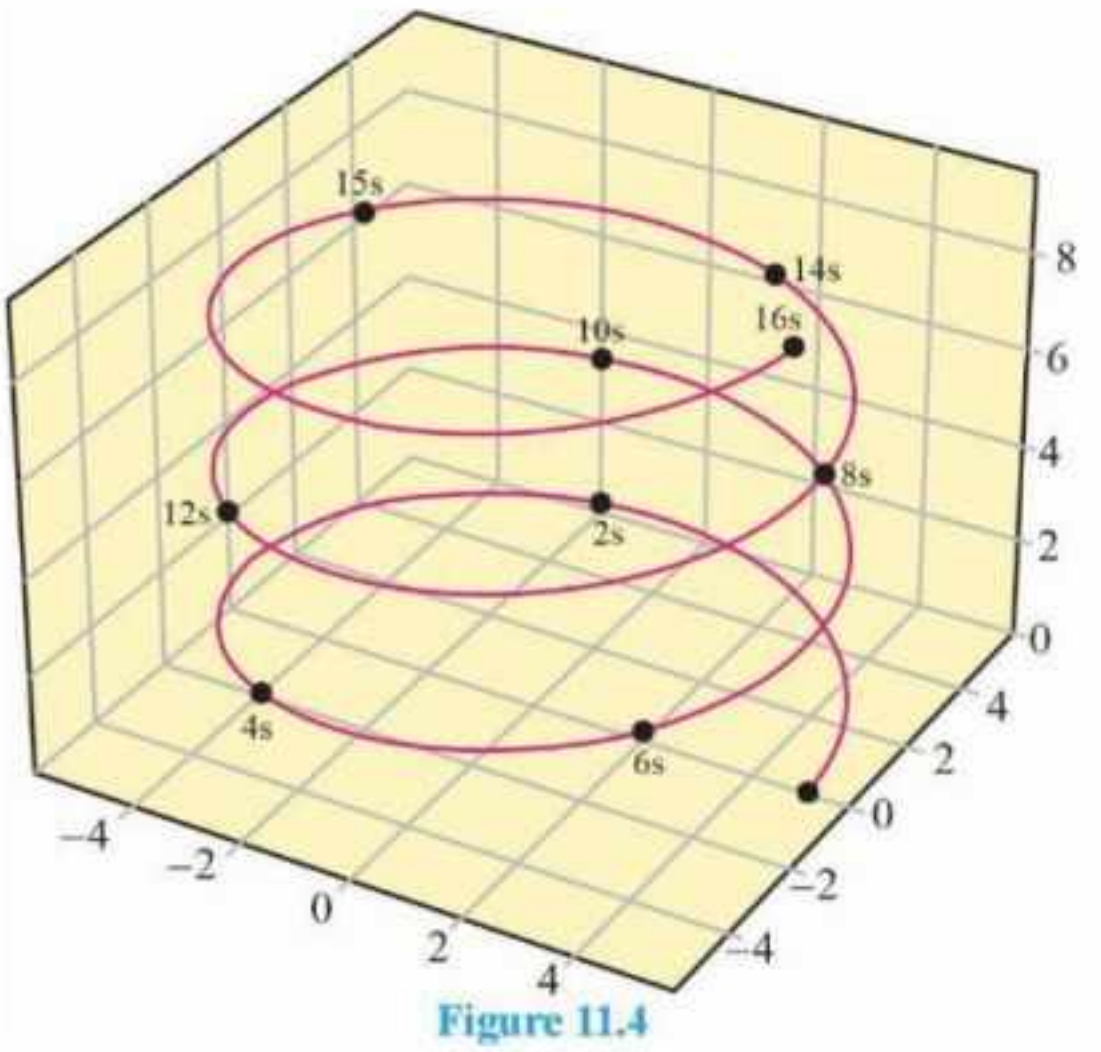


Figure 11.4

11.2 Domain and Range of a Vector-Valued Function

The domain of a vector–valued function $\mathbf{r}(t)=f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$ is the set of all real numbers t for which every component function $f(t),g(t)$, and $h(t)$ is defined.

11.2.1 Steps to Find the Domain of a Vector-Valued Function

Step-1: Determine the domain of each component separately.

Step-2: Find the intersection of the domains of each component $f(t),g(t)$, and $h(t)$. The domain of $\mathbf{r}(t)$ is the intersection of the domains of $f(t),g(t)$, and $h(t)$

The range of a vector-valued function is the set of all possible output vectors $\mathbf{r}(t)$ as t varies over the domain.

Example 1 Find the domain and range of the vector-valued function

$\mathbf{r}(t)=\sqrt{4-t^2}\hat{i}+\frac{1}{t-1}\hat{j}$.

Solution: Step 1: The domain of $\mathbf{r}(t)=\sqrt{4-t^2}\,\hat{i}+\dfrac{1}{t-1}\,\hat{j}$ is the set of all t for which both components are defined.

For the first component $\sqrt{4-t^2}$, the square root requires $4-t^2\geq 0$.

$\Rightarrow t^2\leq 4\Rightarrow -2\leq t\leq 2 \Rightarrow t\in[-2,2]$.

For the second component $\dfrac{1}{t-1}$, the denominator cannot be zero.

i.e. $t-1\neq 0\Rightarrow t\neq 1$

Hence, the final domain: $t\in[-2,1)\cup(1,2]$.

The range is the set of all possible output vectors $\mathbf{r}(t)=\langle x(t),y(t)\rangle$.

For $4-t^2$, range is $[0,4]$. Thus, the range of $x(t)$ is $[0,2]$, and $y(t)$ can be any real number except 0 (since $\dfrac{1}{t-1}$ never equals zero).

Finally, range: $\{\langle x,y\rangle \mid 0\leq x\leq 2, y\in\mathbb{R}, y\neq 0, x\neq\sqrt{3}\}$

11.3 Identifying Scalar and Vector Valued Functions

Understanding the distinction between scalar and vector-valued functions is crucial for advanced studies in calculus, physics, and engineering. These functions are fundamental in describing quantities that vary with respect to one or more variables, such as time or position.

11.3.1 Key Identification Criteria

Scalar Function

(i) Scalar-valued functions produce a single numerical value (e.g., a real number).

(ii) Scalar-valued functions are typically denoted by lowercase letters (e.g., $f(x)$).

(iii) Scalar-valued functions describe quantities like temperature or potential energy.

Example 2 In a heat transfer problem, the temperature $T(x,y,z)=100-x^2-y^2-z^2$ describes the temperature at a point (x,y,z) in a three-dimensional space.

Identification: The function takes a position (x,y,z) and outputs a single number T (temperature in degrees), making it scalar-valued.

Engineers use such functions to model heat distribution in materials, such as in designing cooling systems for electronic devices.

Vector-valued function:

(i) Vector-valued functions produce a vector, often written as an ordered tuple or in terms of component functions (e.g., $\mathbf{f}(t)=\langle f_1(t), f_2(t)\rangle$ (2D) or $\mathbf{g}(t)=\langle g_1(t), g_2(t), g_3(t)\rangle$ (3D)).

(ii) Vector-valued functions are often denoted by boldface letters or with an arrow (e.g., $\mathbf{f}(t)$ or $\vec{f}(t)$).

(iii) Vector-valued functions describe quantities such as position, velocity, and force.

Example 3 Consider the function $\mathbf{r}(t)=\langle \cos(t), \sin(t), t\rangle$. Here, the input is a single scalar variable t (often representing time), and the output is a 3-dimensional vector. For instance, if $t=\frac{\pi}{2}$ then $\mathbf{r}\left(\frac{\pi}{2}\right)=\left\langle \cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right), \frac{\pi}{2}\right\rangle=\left\langle 0,1,\frac{\pi}{2}\right\rangle$. The output is a vector, indicating a point in 3D space. This function traces out a helix.

EXERCISE 11.1

1. Give two real-life examples where vector-valued functions are necessary.
2. Construct a vector function for a particle moving in a circle.
3. Write a vector function for a car moving along a parabolic path.
4. Create a 3D vector function for a drone flying upward in a spiral.
5. Given $x(t)=t^2+1, y(t)=\sqrt{t}$, construct $\mathbf{r}(t)$.
6. Given $x(t)=\frac{1}{t+4}, y(t)=\ln\sqrt{t}, z(t)=\sec^{-1}(t)$, construct $\mathbf{r}(t)$.
7. Find the domain and range of the following vector-valued functions.
   (i) $\mathbf{r}(t)=\sqrt{t}\hat{i}+\ln t\hat{j}$ (ii) $\mathbf{r}(t)=\cos(t)\hat{i}+\sin(t)\hat{j}$
   (iii) $\mathbf{r}(t)=t\hat{i}+t^2\hat{j}+e^t\hat{k}$ (iv) $\mathbf{r}(t)=\sin(t)\hat{i}+\cos(t)\hat{j}+t\hat{k}$
8. Determine which of the following functions are scalar functions or vector-valued functions. Give appropriate reasoning to support the answer.
   (i) $f(t)=\left(3t^2-5\right)^2$ (ii) $\mathbf{h}(t)=\left\langle t^2, e^t, \ln\sqrt{t}\right\rangle$
   (iii) $g(t)=\tan(t)+\csc(t)$ (iv) $\mathbf{j}(t)=\left\langle \frac{1}{t^2+1}, \frac{1}{t^2+1}\right\rangle$

11.4 Derivative Of A Vector-Valued Function

A vector-valued function of a single variable is a function that maps a scalar (usually time t) to a vector in $\mathbb{R}^n$. For a function in three-dimensional space, it is typically written as:

$$\mathbf{r}(t)=\langle f_1(t), f_2(t), f_3(t)\rangle = f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}$$

Where $\hat{i},\hat{j},\hat{k}$ are the unit vectors associated with the rectangular coordinates and $f_1(t), f_2(t)$, and $f_3(t)$ are scalar functions of t, are the components of the vector-valued function $\mathbf{r}(t)$.

The derivative of a vector-valued function $\mathbf{r}(t)$ with respect to t is defined as:

$$\frac{d\mathbf{r}}{dt}=\mathbf{r}'(t)=\lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}$$

If the limit exists, then $\mathbf{r}'(t)$ is computed by differentiating each component:

$$\mathbf{r}'(t)=\langle f_1'(t), f_2'(t), f_3'(t)\rangle$$
$$=f_1'(t)\hat{i}+f_2'(t)\hat{j}+f_3'(t)\hat{k}$$

Example 4 Find the first derivative of $\mathbf{r}(t)=t^2\hat{i}+\sin t\hat{j}+e^t\hat{k}$ with respect to t.

Solution: Differentiating each component separately, we have,

$$\frac{d}{dt}(t^2)=2t, \frac{d}{dt}(\sin t)=\cos t, \frac{d}{dt}(e^t)=e^t$$

$$\Rightarrow \mathbf{r}'(t)=2t\hat{i}+\cos t\hat{j}+e^t\hat{k}$$

Example 5 Find the first derivative of $\mathbf{r}(t)=\sec t\hat{i}+\tan t\hat{j}-\cos t\hat{k}$ at $t=\frac{\pi}{4}$.

Solution: Differentiating each component separately and finding the value at $t=\frac{\pi}{4}$, we have,

$$\frac{d}{dt}(\sec t)=\sec t.\tan t \Rightarrow \left.\frac{d(\sec t)}{dt}\right|_{t=\frac{\pi}{4}}=\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)=\sqrt{2}.1=\sqrt{2}$$

$$\frac{d}{dt}(\tan t)=\sec^2 t \Rightarrow \left.\frac{d(\tan t)}{dt}\right|_{t=\frac{\pi}{4}}=\sec^2\left(\frac{\pi}{4}\right)=\left(\sqrt{2}\right)^2=2$$

$$\frac{d}{dt}(-\cos t)=\sin t \Rightarrow \left.\frac{d(-\cos t)}{dt}\right|_{t=\frac{\pi}{4}}=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \mathbf{r}'\left(\frac{\pi}{4}\right)=\sqrt{2}\hat{i}+2\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$$

EXERCISE 11.2

1. Find the first-order derivatives of the following vector-valued functions with respect to t.

   (i) $\mathbf{r}(t)=\left(t^2+1\right)\hat{i}+\left(t^3+t^2+3t\right)\hat{j}$

   (ii) $\mathbf{r}(t)=(t\sin t-\cos t)\hat{i}+\left(\cot(t)+\ln t\right)\hat{j}+\left(\ln\left(1+t^2\right)\right)\hat{k}$

   (iii) $\mathbf{r}(t)=\frac{2}{\sqrt{1-t^2}}\hat{i}+\frac{\sqrt{3}}{1+t^2}\hat{k}$

2. Find the first-order derivatives of the following vector-valued functions with respect to t at the specified values of t.

   (i) $\mathbf{r}(t)=\left(\frac{t^2+1}{\sin t}\right)\hat{i}+\left(\sqrt{t}e^t\right)\hat{j}$ at $t=\frac{\pi}{2}$

   (ii) $\mathbf{r}(t)=\left(t\sqrt{t+2}\right)\hat{i}+\ln\left(t+\sqrt{t^2+1}\right)\hat{j}$ at $t=0$

   (iii) $\mathbf{r}(t)=\arctan(t)\hat{i}+\frac{\sin\left(t^2\right)}{t}\hat{j}+\sqrt[3]{1+t}\hat{k}$ at $t=0$

   (iv) $\mathbf{r}(t)=te^{-t}\hat{i}+\cos(3t)\hat{j}+\ln\left(1+t^2\right)\hat{k}$ at $t=0$

11.5 Velocity and Acceleration

If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then $\frac{d}{dt}\mathbf{r}(t)$ represent the velocity vector of the particle, which is tangent to the curve, i.e. $\frac{d}{dt}\mathbf{r}(t)=\mathbf{v}(t)$, and the derivative of the velocity vector with respect to time t is the acceleration of the particle, denoted by $\mathbf{a}$ i.e. $\frac{d}{dt}\mathbf{v}(t)=\mathbf{a}(t)=\frac{d^2}{dt^2}\mathbf{r}(t)$.

Example 6 Vibrating conveyor belt in a factory sorting system

A small metal part moves on a conveyor belt in a factory. The belt transports the part eastward (x-direction) while simultaneously vibrating sideways (north-south, y-

direction) to help sort items into bins. The position of the part at time t seconds is given by $\mathbf{r}(t)=3t^2\hat{i}+2\sin t\hat{j}$ in meters. At a specific moment, $t=\frac{\pi}{2}$ an engineer needs to analyze the part's motion to ensure it enters the correct bin. Hence, find the following quantities at $t=\frac{\pi}{2}$.

(a) velocity (b) acceleration (c) speed (d) direction of motion.

Solution:

(a) Velocity: The velocity vector is the first derivative of the position function.

$\mathbf{v}(t)=\mathbf{r}'(t)=6t\hat{i}+2\cos t\hat{j}$, at $t=\frac{\pi}{2}$ is, $\mathbf{v}\left(\frac{\pi}{2}\right)=6\left(\frac{\pi}{2}\right)\hat{i}+2\cos\left(\frac{\pi}{2}\right)\hat{j}=3\pi\hat{i}+0\hat{j}$

At this instant, the part is moving purely eastward with no north–south velocity.

(b) Acceleration: The acceleration vector is the derivative of the velocity vector.

$$\mathbf{a}(t)=\mathbf{v}'(t)=6\hat{i}-2\sin t\hat{j}\text{, at } t=\frac{\pi}{2} \text{ is, } \mathbf{a}\left(\frac{\pi}{2}\right)=6\hat{i}-2\sin\left(\frac{\pi}{2}\right)\hat{j}=6\hat{i}-2\hat{j}$$

The part is accelerating eastward at 6 m/sec^2 and southward at 2 m/sec^2. Even though its north-south velocity is momentarily zero, it is starting to move southward.

(c) Speed: The speed is the magnitude of the velocity vector.

$|\mathbf{v}(t)|=\sqrt{36t^2+4\cos^2 t}$, at $t=\frac{\pi}{2}$,

$$\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right|=\sqrt{\left(6.\frac{\pi}{2}\right)^2+\left(2\cos\left(\frac{\pi}{2}\right)\right)^2}=\sqrt{(3\pi)^2+0}=3\pi$$

The part is moving along its path at about 9.42 meters per second.

(d) Direction of Motion: The direction is given by the unit tangent vector,

$$\hat{\mathbf{v}}(t)=\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}\text{, at } t=\frac{\pi}{2}\text{ is, } \hat{\mathbf{v}}\left(\frac{\pi}{2}\right)=\frac{3\pi\hat{i}+0\hat{j}}{3\pi}=\hat{i}+0\hat{j}$$

The direction of motion is due east (positive x-direction). The part has momentarily stopped its north-south oscillation and is moving purely horizontally along the belt.

Example 7 Drone inspecting a cylindrical water tower

A drone is programmed to fly in a helical (spiral) path around a large cylindrical water tower to inspect its outer surface for cracks or rust. The drone maintains a constant horizontal distance of 2 meters from the tower's center while climbing upward at a steady rate. The drone's position in 3D space at time t seconds is given by: $\mathbf{r}(t)=2\cos t\hat{i}+2\sin t\hat{j}+3t\hat{k}$

Here:

- $x(t)=2\cos t$: east-west position (amplitude 2 m)
- $y(t)=2\cos t$: north-south position (amplitude 2 m)
- $z(t)=3t$: Vertical height (climbs 3 m/s)

At a specific moment $t=\pi$ seconds, the drone operator needs to analyze its motion to ensure smooth inspection. Find the following quantities at $t=\pi$ (a) velocity, (b) acceleration, (c) speed, and (d) direction of motion at $t=\pi$.

Solution:

(a) Velocity: Compute the first derivatives of each component.

$$\mathbf{v}(t)=\mathbf{r}'(t)=-2\sin t\hat{i}+2\cos t\hat{j}+3\hat{k}$$

at $t=\pi$, $\mathbf{v}(\pi)=-2\sin(\pi)\hat{i}+2\cos(\pi)\hat{j}+3\hat{k}=0\hat{i}-2\hat{j}+3\hat{k}$

At this instant, the drone has no east–west motion, is moving southward at 2 m/s, and is climbing at 3 m/s.

(b) Acceleration: Compute the second derivative.

$$\mathbf{a}(t)=\mathbf{v}'(t)=-2\cos t\hat{i}-2\sin t\hat{j}+0\hat{k}$$

at $t=\pi$ $\mathbf{a}(\pi)=-2\cos(\pi)\hat{i}-2\sin(\pi)\hat{j}+0\hat{k}=2\hat{i}+0\hat{j}+0\hat{k}$

The drone is accelerating eastward at 2 m/s². This centripetal acceleration points toward the center of the circular path (the tower's axis), keeping the drone in its circular horizontal trajectory.

(c) Speed: Calculate the magnitude of the velocity.

$$|\mathbf{v}(t)|=\sqrt{4\cos^2 t+4\sin^2 t+9}=\sqrt{13}$$

at $t=\pi$, $|\mathbf{v}(\pi)|=\sqrt{13}\approx 3.6056$ m/sec

The drone's speed is constant at about 3.61 m/s throughout its helical climb, ideal for capturing stable inspection footage.

(d) Direction of Motion: The unit tangent vector is,

$$\hat{\mathbf{v}}(t)=\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}=\frac{-2\sin t\hat{i}+2\cos t\hat{j}+3\hat{k}}{\sqrt{13}}$$

at $t=\pi$ $\hat{\mathbf{v}}(\pi)=\frac{0\hat{i}-2\hat{j}+3\hat{k}}{\sqrt{13}}=0\hat{i}-\frac{2}{\sqrt{13}}\hat{j}+\frac{3}{\sqrt{13}}\hat{k}$

The drone is moving in the south-up vertical plane. The direction has no east-west component.

EXERCISE 11.3

In the following exercises (Questions 1-5), each problem involves a vector-valued function representing the particle's position in the *xy*-plane or in 3D space. Find the velocity, acceleration, speed, and direction of motion at a specified value of ***t***. Interpret the results by describing the particle's motion at that instant, including how fast and in what direction it is moving, whether it is speeding up or slowing down, and how its path is curving.

1. $\mathbf{r}(t)=2\sin t\hat{i}+3t\hat{j}$ at $t=\dfrac{\pi}{2}$

2. $\mathbf{r}(t)=e^{t}\hat{i}+e^{-t}\hat{j}$ at $t=0$

3. $\mathbf{r}(t)=\cos t\hat{i}+\sin t\hat{j}+2t\hat{k}$ at $t=\pi$

4. $\mathbf{r}(t)=t^{2}\hat{i}+t\hat{j}+t^{3}\hat{k}$ at $t=2$

5. $\mathbf{r}(t)=e^{t}\sin t\hat{i}+\left(t^{3}-2t\right)\hat{j}+\arctan(2t)\hat{k}$ at $t=0$

6. A cannonball is fired from the ground at an angle 30° with an initial velocity of 100 m/s. Assuming gravity acts downward ($g=9.8$ m/s²) and neglecting air resistance.
   (a) Formulate the position vector $\mathbf{r}(t)$ of the cannonball as a function of time.
   (b) Find the velocity vector at $t = 5$ sec.
   (c) Determine the time of the flight (when the cannonball hits the ground).

7. A comet orbits the Sun in an elliptical path such that its position (in AU) at time **t** (in years) is described by, $x=2\cos t, y=\sin t$.
   (a) Write the position vector $\mathbf{r}(t)$
   (b) Find the velocity and speed at $t=\dfrac{\pi}{2}$
   (c) Show that the acceleration vector always points toward the origin.

8. A robotic arm moves such that: Its *x*-position increases linearly at 2 m/s, Its *y*-position follows $y(t)=t^{2}$, Its *z*-position is given by $z(t)=1-e^{-t}$.
   (a) Construct the position vector $\mathbf{r}(t)$
   (b) Find the velocity vector at t=1 sec
   (c) Compute the acceleration at t=*1* sec

9. A particle moves along a circular path, described by the position vector $\mathbf{r}(t)=R\cos\omega t\hat{i}+R\sin\omega t\hat{j}$. Prove that its velocity is perpendicular to its position vector.

10. A Ferris wheel of radius 20 m rotates with an angular speed of 0.2 rad/sec.
    (a) Construct its position vector $\mathbf{r}(t)$
    (b) Find the passenger's velocity at t=10sec
    (c) Show that the acceleration is centripetal (toward the center)
11. A quarterback throws a football with a position vector

$$\mathbf{r}(t) = 20t\hat{i} + \left(20t - 4.9t^2\right)\hat{j}$$

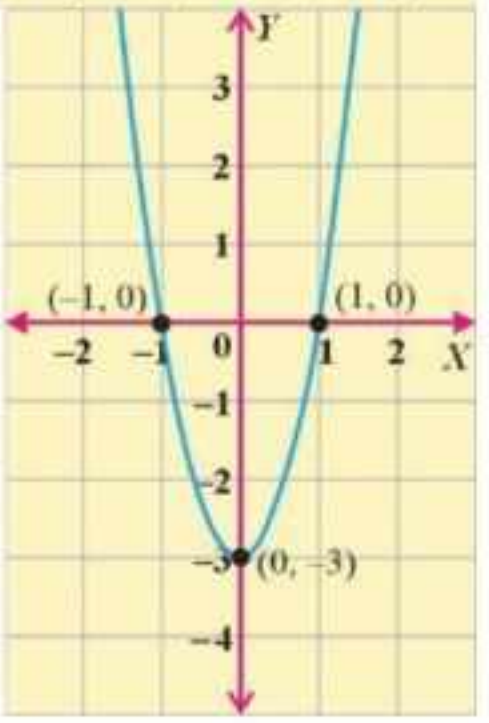
    (a) Find the time of flight before it hits the ground.
    (b) What is the maximum height reached?
    (c) At which time did football reach its maximum height?

Answers

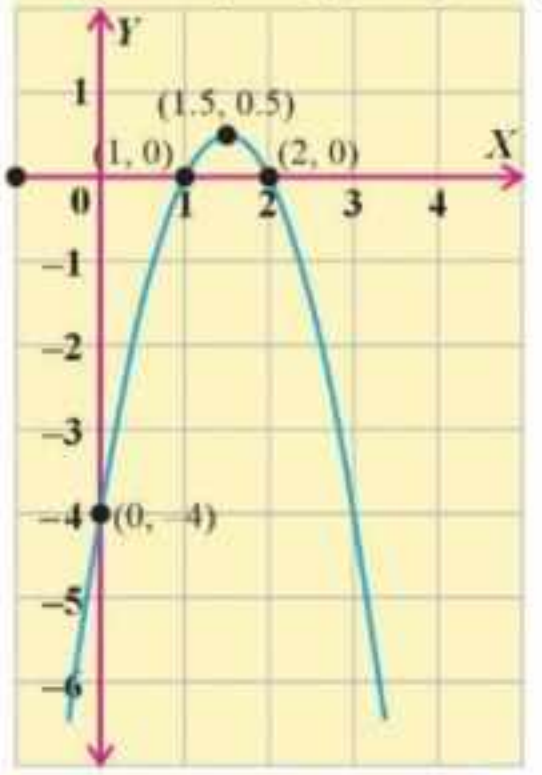
Answers to Exercise 1.1

1.

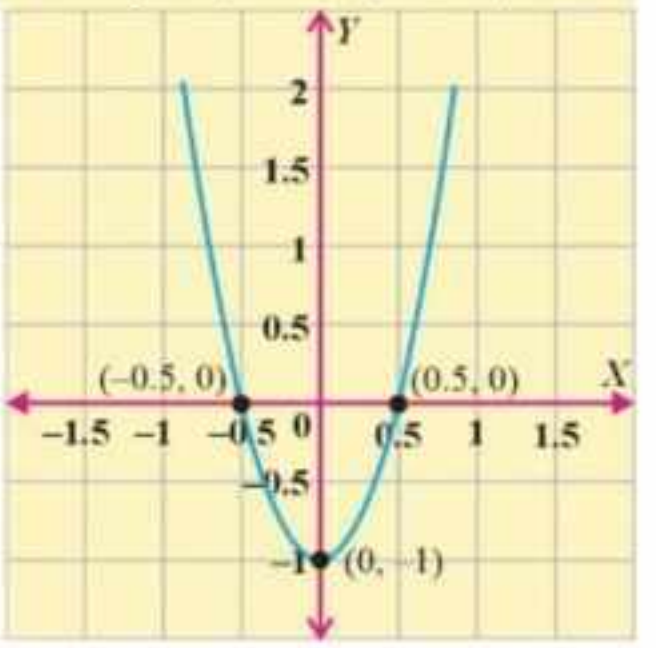
(i) $y = 3(x + 1)(x - 1)$



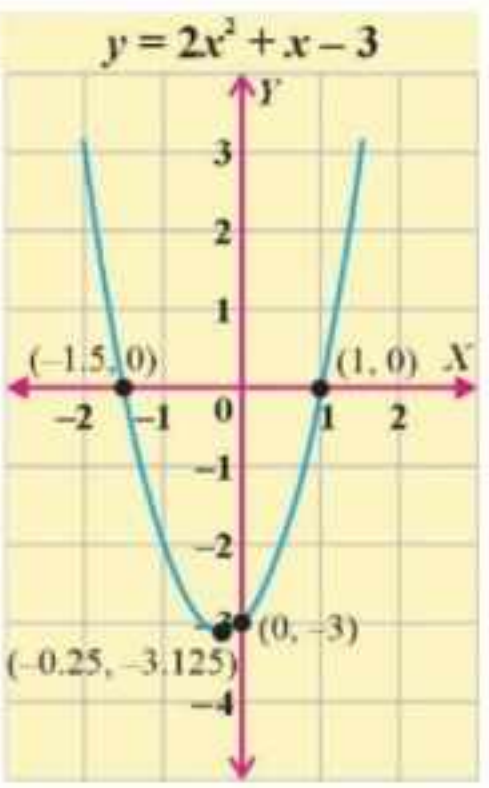
(ii) $y = -2(x - 1)(x - 2)$



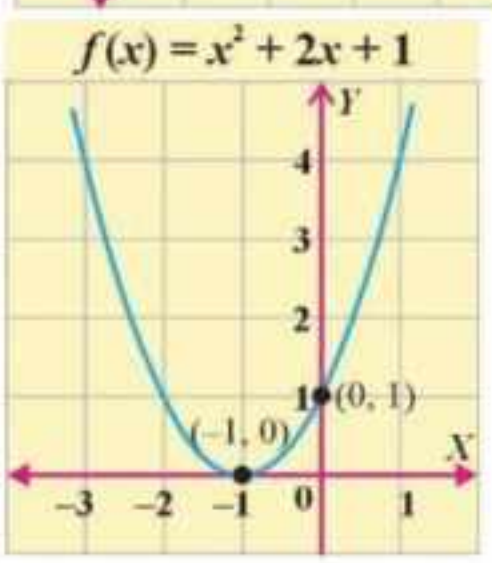
(iii) $y = (2x - 1)(2x + 1)$



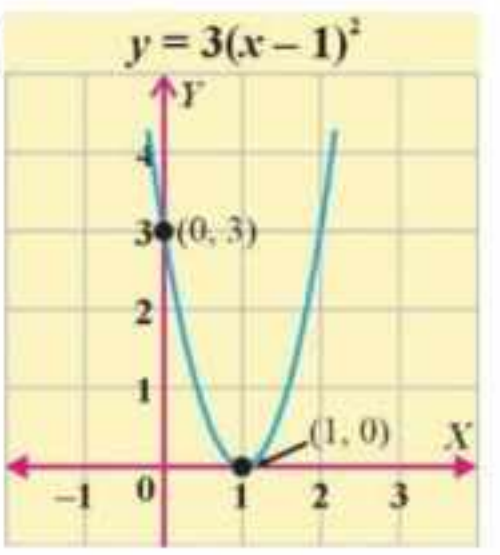
(iv) $y = 2x^2 + x - 3$



(v) $f(x) = x^2 + 2x + 1$



(vi) $y = 3(x - 1)^2$



2. **(i)** $y = -(x - 1)(x + 2)$ **(ii)** $y = (x - 2)^2$

3. $y = (x + 1)(x + 2)$ **4.** $y = -2(x + 1)^2$

5. $y = 2x^2 - 2$

ANSWERS TO EXERCISE 1.2

1. Algebraic Functions: $g(x) = \dfrac{2x-3}{x+1}$, $h(x) = \sqrt{x^2 + 2x}$, $r(x) = (x - 1)^3$

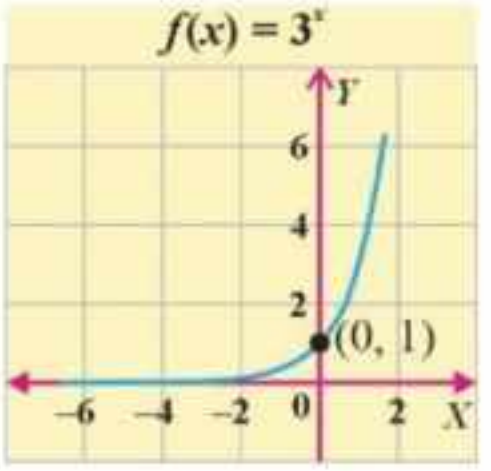
2. Transcendental Functions: $g(x) = \cot x$, $h(x) = e^{2x} + \sqrt{x^2 + 2x}$, $k(x) = \dfrac{e^x + e^{-x}}{2}$

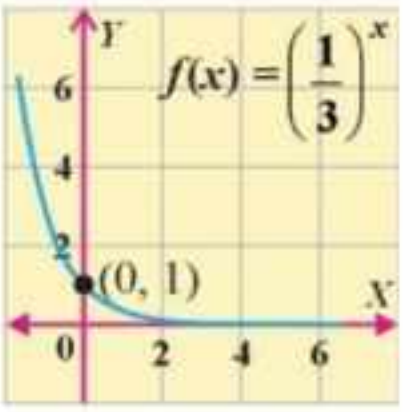
3. Fundamental Transcendental Functions: $g(x) = \sin x$, $h(x) = e^{2x}$, $r(x) = 15^x$

4. Non-Fundamental Transcendental Function: $k(x) = \cos x^2$

5. Dom $f =]-\infty, \infty[$, Range $f =]0, \infty[$. Graph is given in the following figure.

6. Dom $f =]-\infty, \infty]$, Range $f =]0, \infty[$. Graph is given in the following figure.





Exercise 1.3

1. (ii) $f^{-1}(x) = 4^x + 2$

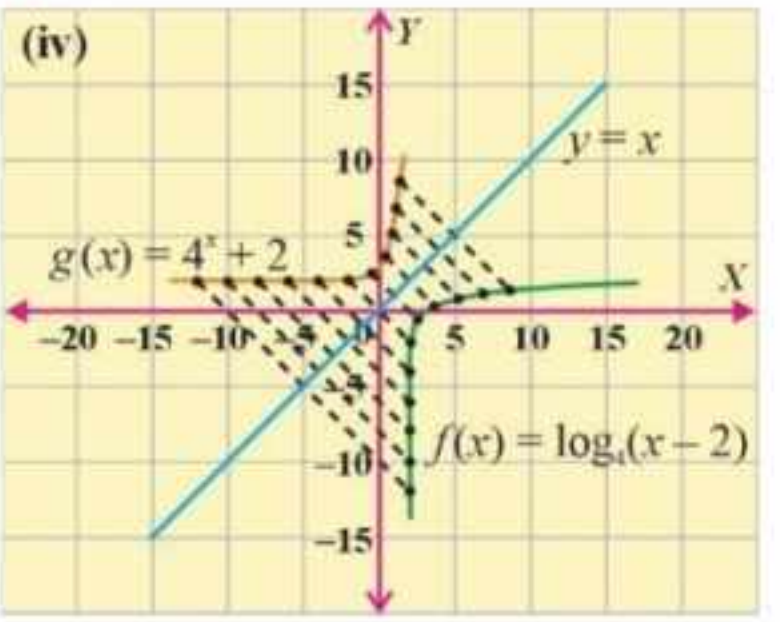
(iii) $D_f = R_{f^{-1}} =]2, \infty[, R_f = D_{f^{-1}} =]-\infty, \infty[$

(iii) Graphs of $f(x) = \log_4(x - 2)$ and $f^{-1}(x) = 4^x + 2$ are drawn in the figure.

Note that:

(a) any horizontal line crosses the curve of f only once. Which indicates that f is one to one

(b) both functions are symmetric about the line $y = x$.

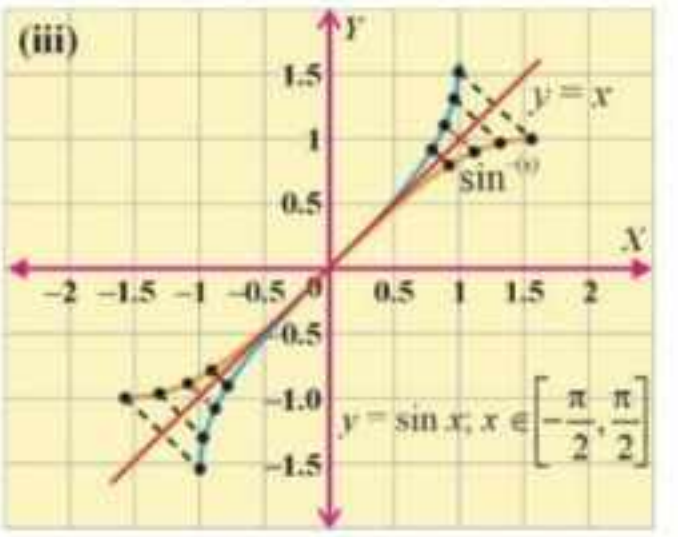


2. (i) $D_f = R_{f^{-1}} = [-1, 1], R_f = D_{f^{-1}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) Graphs of $f(x) = \sin^{-1}(x)$ and $f^{-1}(x) = \sin(x)$ are drawn in the figure, which clearly indicates that:

(a) any horizontal line crosses the curve of f only once. Which indicates that f is one to one.

(b) both functions are symmetric about the line $y = x$.



3. The graph of f^{-1} is given in the figure.

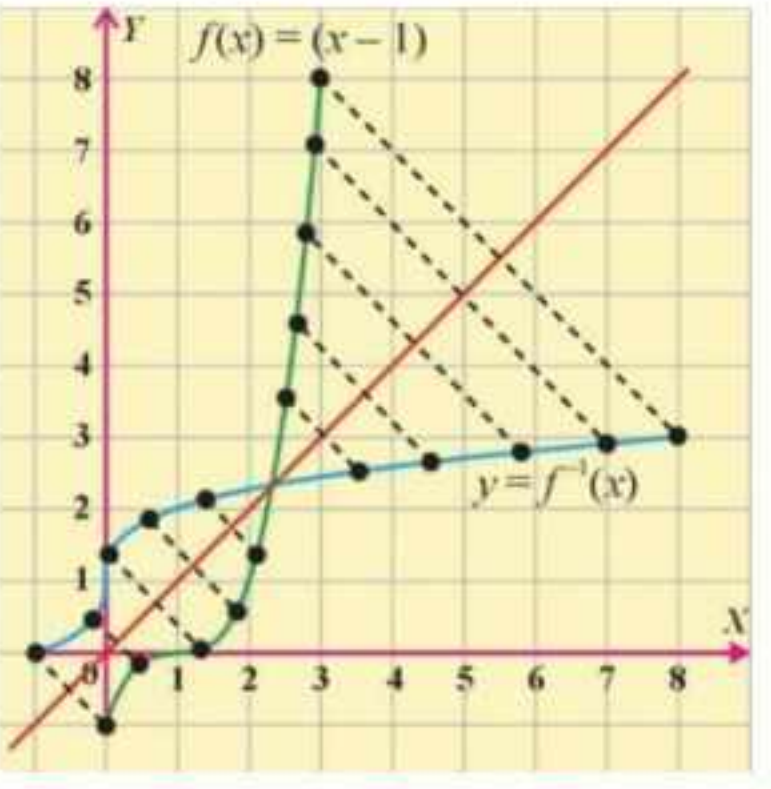
4. (i) $y = 2\sqrt{x}$ (ii) $y = \sqrt{x-3}$

(iii) $y = 2\sqrt{x-3}$

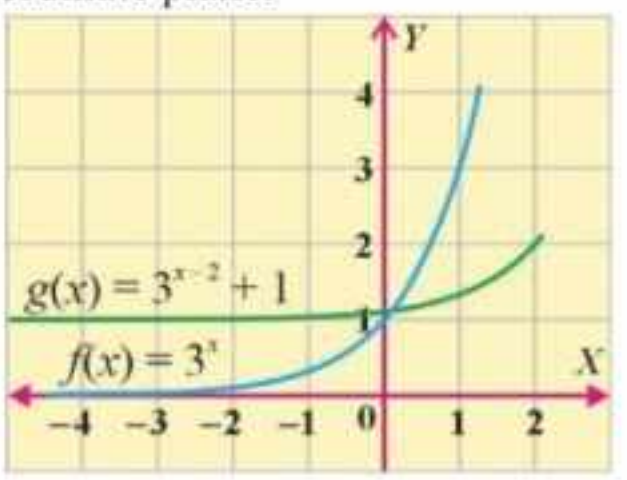
5. $y = \frac{2x+5}{2x+2}$

6. $y = \ln\sqrt{x-1} + 3$

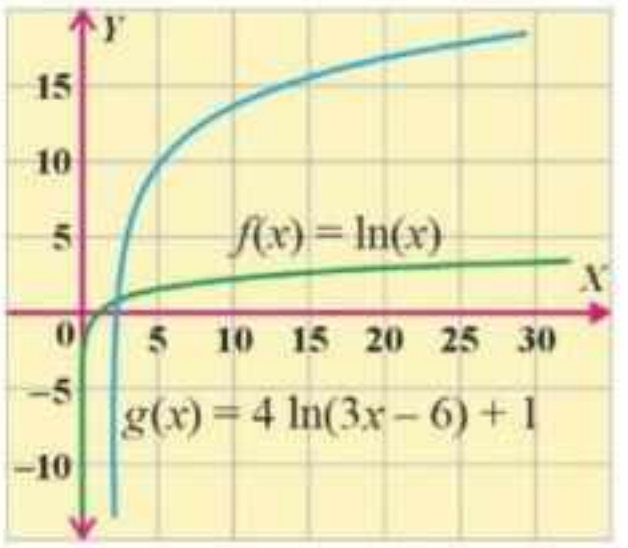
7. $y = \frac{1}{2}\sqrt{\frac{6x+3}{3x+5}} - 1$



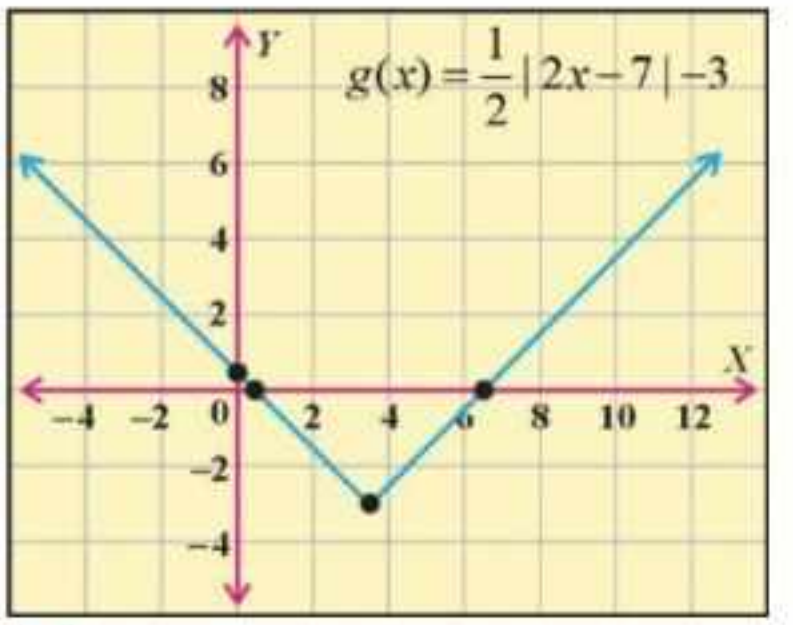
8. **Comparison:** From the figure, it is clear that the graph of $g(x)$ is obtained from $f(x)$ by shifting it 2 units to the right and 1 unit upward. Therefore, both graphs are identical in shape but differ in their position on the coordinate plane.



9. **Transformed function:** $g(x) = 4\ln(3x-6)+1$
Graphs of $f(x)$ and $g(x)$ are given in the figure.



10. Graph of $g(x)=\frac{1}{2}|2x-7|-3$ is given in the figure.



Answers to Exercise 1.4

1. **(i)** $S=\{4-2\sqrt{2},4+2\sqrt{2}\}$ **(ii)** $S=\left\{\frac{3+\sqrt{5}}{2}\right\}$ **(iii)** $S=\{1,3\}$

(iv) $S=\{1\}$ **(v)** $S=\left\{\frac{1}{2}\log_2\left(\frac{1+\sqrt{13}}{2}\right)\right\}$ **(vi)** $S=\left\{\log_2\left(\frac{5+\sqrt{21}}{2}\right),\log_2\left(\frac{5-\sqrt{21}}{2}\right)\right\}$

2. **(i)** $S=[0,1]$ **(ii)** $S=[2,\infty]$ **(iii)** $S=[3,5]$

(iv) $S=\left]\frac{1}{2},1\right[\cup\left]3,3+\sqrt{5}\right[$ **(v)** $S=\left[\frac{5+\sqrt{33}}{2},\infty\right[$ **(vi)** $S=\emptyset$

Answers to Exercise 1.5

1. 34.7 hours **2.** $k\approx 0.0511$ **3.** 27.73 hours **4.** $k\approx -0.0507$

5. 60 dB **6.** $I=10^{-4}\ \text{W/m}^2$ **7.** **(i)** $k\approx -0.13863$ **(ii)** $N=125$

8. $\frac{I_1}{I_2}=100$ **9.** $A=$ Rs. 4764.06 **10.** $t\approx 3$ years **11.** $A\approx$ Rs. 3664.2 **12.** $t\approx 8.42$ years

EXERCISE 3.1

1. **(i)** $y=20x^5+c$ **(ii)** $y=2t+6t^4-2t^5+c$ **2.** **(i)** $t+t^2+c$ **(ii)** u^3+2u^2+c

(iii) $\frac{100}{9}x^9+c$ **(iv)** $-\frac{2}{x}-\frac{3}{2x^2}+c$ **3.** **(i)** $\frac{5}{4}x^4+5x^2+20x+c$

(ii) $\frac{x^4}{4}+\frac{x^2}{2}+c$ (iii) $\frac{y^2}{2}+4\sqrt{y}-\frac{1}{y}+c$ (iv) $\frac{2}{3}x^{3/2}+4\sqrt{x}+c$

4. (i) $\frac{1}{140}\left(7x^2+8\right)^{10}+c$ (ii) $-\frac{1}{12}\left(x^3+2\right)^{-4}+c$ (iii) $-\frac{1}{3}\left(1+\frac{1}{x}\right)^3+c$

(iv) $\frac{1}{3}\left(1+\sqrt{x}\right)^6+c$ (v) $\frac{2}{21}\left(7x+8\right)^{3/2}+c$ (vi) $-\frac{1}{25}\left(7-x^5\right)^5+c$

(vii) $-\frac{1}{9}\left(t^3-3t^2+5\right)^{-3}+c$ **5.** (i) $\frac{2}{3}\sqrt{x+2}\,(x-1)+c$ (ii) $\sqrt{ax^2+2bx+c}+C$

(iii) $\frac{2}{3}(x+1)^{3/2}-\frac{2}{3}x^{3/2}+c$ (iv) $\frac{4}{9}\left(1+\sqrt{x}\right)^{9/2}-\frac{12}{7}\left(1+\sqrt{x}\right)^{7/2}+\frac{12}{5}\left(1+\sqrt{x}\right)^{5/2}-\frac{4}{3}\left(1+\sqrt{x}\right)^{3/2}+c$

(v) $-\frac{a+2bx}{2b^2(a+bx)^2}+c$ (vi) $\frac{2}{15}\left(x^3+1\right)^{5/2}-\frac{2}{9}\left(x^3+1\right)^{3/2}+c$

Exercise 3.2

1. (i) $\frac{23}{6}$ (ii) 89 **2.** (i) 13 (ii) 42 (ii) $\frac{71}{3}$ **3.** $\frac{100}{3}$ **4.** (i) 0 (ii) $\int_0^5 f(x)dx$

5. $\frac{1}{132}$ **6.** (i) $\frac{128}{15}$ (ii) $\frac{16}{15}$ (iii) 0 **7.** $\frac{8}{3}$ **8.** 25 **9.** 38 **13.** $\frac{a^3}{3}$

14. 18 **16.** (i) 1 (ii) $\frac{1}{2}\ln 2$ **17.** (i) $\frac{\sin t}{1+\sqrt{t}}$ (ii) $\frac{b}{1+b^2x^2}-\frac{a}{1+a^2x^2}$

Exercise 3.3

1. (i) $\ln\left|\csc x-\cot x\right|+c$ (ii) $-\cot x-x+c$ (iii) $2\sqrt{1+\sin^2 x}+c$

(iv) $x+\frac{\sin^2 x}{2}+c$ (v) $\sin x+\cos x+c$ (vi) $-\sqrt{2}\cos x+c$

(vii) $\tan\frac{x}{2}+c$ (viii) $\frac{\cos 6x}{12}-\frac{\cos 8x}{16}+c$ (ix) $\tan x-x+c$

(x) $2-\sqrt{2}$ (xi) $\frac{2}{3}$ (xii) $\tan x+\frac{\tan^3 x}{3}+c$ (xiii) $\frac{3x^2}{2}+\frac{3^x}{\ln 3}+\frac{x^4}{4}+c$

(xiv) $4\sqrt[4]{x}-4\ln\left(1+\sqrt[4]{x}\right)+c$ (xv) $2\sqrt{2+x}-2\ln\left(1+\sqrt{2+x}\right)+c$

(xvi) $\frac{x^{\ln 7+1}}{\ln 7+1}+c$ **2.** (i) $\ln\left|\ln\tan x\right|+c$ (ii) $-\ln\left|\sin x+\cos x\right|+c$

(iii) $\frac{1}{4}\ln\left(1+\sin^4 x\right)+c$ (iv) $\ln\left|\cos x+x\sin x\right|+c$ (v) $\frac{(\ln\cos x)^2}{2}+c$

(vi) $\frac{(\ln\sin x)^2}{2}+c$ (vii) $-\frac{1}{2\left(2+e^{2x}\right)}+c$ (viii) $\frac{\pi}{4}$ (ix) $\frac{\pi}{2}$

Exercise 3.4

1. (i) $\frac{1}{6}\ln\left|\frac{3+e^x}{3-e^x}\right|+c$ (ii) $\frac{1}{4}\ln\left|\frac{\sin x-2}{\sin x+2}\right|+c$ (iii) $\arctan\left(e^x\right)+c$

(iv) $\arcsin\left(\frac{\ln x}{2}\right)+c$ **(v)** $\frac{1}{30}\ln\left|\frac{3x-5}{3x+5}\right|+c$ **(vi)** $\frac{1}{2\sqrt{3}}\arctan\left(\frac{\sqrt{x^2-4}}{2}\right)+c$

(vii) $x-a\arctan\left(\frac{x}{a}\right)+c$ **(viii)** $\arctan(\ln x)+c$ **(ix)** $2-\frac{\pi}{2}$

(x) $\frac{x}{\sqrt{1+x^2}}+c$ **(xi)** $\frac{1}{3}\arctan\left(\frac{\sqrt{x^2-9}}{3}\right)+c$ **(xii)** $\arcsin x-\sqrt{1-x^2}+c$

(xiii) $\frac{\pi}{8}\ln 2$ **(xiv)** $-\frac{\sqrt{x^2+4}}{4x}+c$ **(xv)** $a\ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right|+\sqrt{a^2-x^2}+c$

Exercise 3.5

1. (i) $x\ln x-x+c$ **(ii)** $\frac{x^8\ln x}{8}-\frac{x^8}{64}+c$ **(iii)** $\frac{x\sin 2x}{2}+\frac{\cos 2x}{4}+c$ **(iv)** $-x\cot x+\ln|\sin x|+c$

(v) $\frac{xe^{2x}}{2}-\frac{e^{2x}}{4}+c$ **(vi)** $\frac{x^2e^{ax}}{a}-\frac{2xe^{ax}}{a^2}+\frac{2e^{ax}}{a^3}+c$ **(vii)** $x\tan x-\ln|\sec x|-\frac{x^2}{2}+c$

(viii) $\sin x\cdot\ln(\sin x)-\sin x+c$ **(ix)** $(e^x+1)\ln(1+e^x)-e^x+c$

(x) $\frac{(\ln x)^2\tan^{-1}(\ln x)}{2}-\frac{\ln x}{2}+\frac{\tan^{-1}(\ln x)}{2}+c$ **(xi)** $\frac{e^{5x}}{5}(x^5+1)+c$ **(xii)** $2\sqrt{x}e^{\sqrt{x}}+c$

(xiii) $\frac{e^{3x}}{10}(3\sin x-\cos x)-\frac{e^{3x}}{2}(\cos 3x+\sin 3x)+c$ **(xiv)** $x^2e^{x^2}+c$

(xv) $e^x\tan^{-1}(e^x)-\frac{1}{2}\ln(1+e^{2x})+c$ **(xvi)** $x\tan\frac{x}{2}-2\ln\left|\sec\frac{x}{2}\right|+c$

(xvii) $x\tan x+x\sec x-\ln|\sec x|-\ln|\sec x+\tan x|+c$

(xviii) $\frac{\sin x}{x}+c$ (xix) $2\sqrt{x}\tan^{-1}\sqrt{x}-\ln(1+x)+c$

(xx) $-2x\cos\sqrt{x}+4\sqrt{x}\sin\sqrt{x}+4\cos\sqrt{x}+c$ **(xxi)** $\frac{e^x}{1+x^2}+c$

(xxii) $\frac{\pi}{4}-\frac{1}{2}\ln 2$ **(xxiii)** $\frac{e^{\tan^{-1}x}(2x^2+2x+3)}{5(1+x^2)}+c$ **(xxiv)** $x\sin^{-1}x+\sqrt{1-x^2}+c$

(xxv) $-\frac{1}{2}\csc x\cot x+\frac{1}{2}\ln|\csc x-\cot x|+c$

Exercise 3.6

(i) $2\ln|x+1|+3\ln|x+2|+c$ **(ii)** $\ln|x+1|+2\ln|x+3|+c$

(iii) $\frac{x^2}{2}+2x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+c$ **(iv)** $x-7\ln|x-7|+\ln|x+1|+c$

(v) $\frac{5}{3}\ln|4-3x|+\ln|2x+1|+c$ **(vi)** $-3\ln|x|+\ln|x-1|+\ln|x-2|+\ln|x-3|+c$

(vii) $\ln|x+2|+\frac{4}{x+2}-\frac{1}{2(x+2)^2}+c$ **(viii)** $e^{-x}+2\ln\left(\frac{e^x}{e^x+1}\right)+c$

(ix) $4\ln\left|\frac{x-2}{x+3}\right|-\frac{5}{x+3}+c$ **(x)** $\ln|x+1|+c$ **(xi)** $\ln|x-1|+2\arctan x+c$

(xii) $\frac{1}{6}\ln\left(x^2+1\right)+\frac{1}{3}\ln\left(x^2+4\right)+\frac{1}{3}\arctan x-\frac{1}{6}\arctan\left(\frac{x}{2}\right)+c$

(xiii) $\frac{3}{2}\ln|x-1|-\frac{1}{4}\ln\left(x^2+1\right)+\frac{x}{2\left(x^2+1\right)}+c$

(xiv) $\frac{1}{3}\ln|\sin x-1|-\frac{1}{6}\ln\left(\sin^2 x+\sin x+1\right)-\frac{\sqrt{3}}{3}\arctan\left(\frac{2\sin x+1}{\sqrt{3}}\right)+c$

Exercise 3.7

1. $\frac{1}{6}$ square units **2.** **(i)** Signed area $=\frac{2}{3}$, Total area $=4$

(ii) Signed area $=-6$, Total area $=\frac{46}{3}$ **3.** **(i)** $\frac{4}{3}$ **(ii)** $\frac{1}{3}$ **(iii)** $\frac{8}{5}$ **(iv)** $\frac{1}{3}$

4. 1 square unit **5.** $\frac{4}{3}$ square units **6.** 36 square units **7.** 1:1 **8.** **(a)** 9

(b) 9 **(c)** The function is nonnegative on $[0,3]$, so signed area equals total area.

9. $\frac{37}{12}$ square units **10.** 18 square units **11.** Consumer's Surplus = 27

Producer's Surplus = 36 **12.** $D(x)=24-2x$; $S(x)=12+x$ **13.** **(a)** $p_e=2\ln 2$

(b) $p=6-(3-\ln 2)x$ **14.** CS = Rs. 324 **15.** **(i)** $\frac{32\pi}{5}$ cubic units

(ii) $\frac{8\pi}{3}$ cubic units **(iii)** 4π cubic units **(iv)** $\frac{16\pi}{15}$ cubic units

Exercise 3.8

1. $I=2048$ units units, $I=\frac{1}{6}Mh^2$ **2.** **(a)** $v(t)=6t^2-36t+30$ m/s **(b)** -50 m

(c) 78 m **(d)** $v(t)=3t+\frac{2}{(t+2)^2}+\frac{9}{2}$ cm/s **(e)** $s(t)=\frac{3}{2}t^2+\frac{9}{2}t-\frac{2}{t+2}+3$ cm

3. **(a)** $N(t)=200\cdot 3^t$ **(b)** 16200 bacteria **4.** **(a)** $P(r)=25000+800t+30t^2$ **(b)** 36000 people

(c) 5650 people **(d)** $b=3$ $\int_2^4 f(x)dx=15$ **(e)** $\int_3^4 f(x)dx=1$

(f) $\int_3^4 f(x)dx=4$ **4.** **(a)** $T=\frac{\ln 4}{0.3}\approx 4.62$ hours **(b)** AUC = 30 mg·h/L

(c) One dose insufficient; 6 doses needed in 24 hours

4. **(a)** Total at $t=10$: $12e^{-2}+12e^{-0.4}$ **(b)** Student forgot the contribution of the second dose

(c) **(i)** $12e^{-3.2}+12e^{-1.6}$ **(ii)** $12e^{-3.2}+12e^{-1.6}+12$

(d) $C_{\text{total}}(t)=12e^{-0.2t}+12e^{-0.2(t-8)}++12e^{-0.2(t-16)}$

EXERCISE 4.1

1. **(i)** order 3, degree 1 **(ii)** order 2, degree 3 **(iii)** order 2, degree undefined

(iv) order 3, degree 2 **(vi)** order 2, degree undefined **2.** $\frac{dA}{dt}-rA=0$

3. $\frac{dT}{dt}+kT=0,\ k>0$. **4.** $\frac{dv}{dt}=-g-\frac{k}{m}v,\ k\geq 0,\ g$ is gravitational acceleration; $\frac{dv}{dt}+g=0$

5. $\frac{dN}{dt}=-kT,\ k>0$ **6.** $\frac{dx}{dt}=-\frac{k}{\pi r^2}\sqrt{x},\ k>0$ **7.** $\frac{dq}{dt}=-\frac{1}{RC}q$ **8.** $\frac{dI}{dt}=-\frac{R}{L}I$

EXERCISE 4.2

2. Separable: (*i*), (*ii*), (*v*), (*vi*) Homogeneous: (*iii*), (*iv*), (*vii*)

3. $\sin^{-1}x+\sin^{-1}x+C=0$ **4.** $e^y=e^x+C$ **5.** $2\tan\sqrt{y}=x+C$

6. $e^{-y}+2e^{\sqrt{x}}+C=0$ **7.** $y=\sin\left(x^2+C\right)$ **8.** $y=\ln\left|\frac{C\sqrt{1+x^2}}{1-x}\right|$

9. $y=x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$ **10.** $y=C(x+2)(2y-1)$ **11.** $\tan x\tan y=C$

12. $2y=\ln\left|C(e^{2x}+e^{-2x})\right|$ **13.** $y\ln y=y+x\cos x-\sin x+C$ **14.** $\tan\left(\frac{y}{x}\right)=\ln|Cx|$

15. $\cos\left(\frac{y}{x}\right)=\ln|Cx|$ **16.** $x^3y=C(y+3x)$ **17.** $x^2+y^2=Cx^3$

18. $\tan^{-1}\left(\frac{y}{x}\right)=\ln|Cx|$ **19.** 19. $\frac{x}{x+y}=-\ln|Cx|$ **20.** $e^{\frac{y}{x}}=\ln|Cx|$

21. $2y^3-3xy^2+6x^2y=6x^3\ln\left|C(y+x)\right|$ **22.** $y=3+Cx^{\frac{1}{4}}$ **23.** $xy=C$

24. $y(y+1)=\frac{1}{2}x^2+C$ **25.** $\tan^{-1}y=\frac{x^2}{2}+\frac{\pi}{4}$ **26.** $s=4e^{1-t^2}$

27. $e^y(\sin x+\cos x)=1$ **28.** $y+\sqrt{x^2+y^2}=x^2$ **29.** $y=x+x\ln|x|$

EXERCISE 4.3

1. $v=t^2-7t+10;\ s=\frac{1}{3}t^3-\frac{7}{2}t^2+10t$ **2.** 863 (approximately) **3.** 619 (approximately)

4. **(a)** $k\approx 0.05776$ **(b)** 20 grams **(c)** 48 hours **5.** Rs. 1491.8 **6.** Rs. 15021

7. 7.32% **8.** 16.80°C **9.** 7.12 hours **10.** $I(t)=\frac{1}{5}\left(1-e^{-30t}\right)$

11. $I(t)=2e^{-\frac{1}{18}t}$ amps **12.** $I(t)=-\frac{1}{8}e^{-\frac{1}{800}t}$ amps **13.** **(i)** $x(t)=\left(3-\frac{t}{8\pi}\right)^2$

(ii) $V(t)=4\pi\left(3-\frac{t}{8\pi}\right)^2$ **(iii)** 75.4 minutes **14. (i)** velocity: $v(t)=1470-980t$

(ii) height: $h(t)=1470t-490t^2$ **(iii)** Maximum height: 1093 cms

15. (i) velocity: $v(t)=4.9\left(1-e^{-2t}\right)$ **(ii)** height: $h(t)=502.45-4.9t-2.45e^{-2t}$

Answers 5.1

2. $4x-y-1=0$ **3.** $5x+3y-18=0$ **4.** $6x+5y-16=0,\ 6x+y-24=0,\ y+2=0$

6. $x-2y-4=0,\ x-4=0,\ x+2y-4=0$

8. Medians: $14x-y-23=0,\ 11x+14y-23=0,\ 25x+13y-46=0$

Altitudes: $3x-4y+14=0,\ 4x-13y+64=0,\ 5x+y-22=0$

Perpendicular Bisectors: $6x-8y-25=0,\ 8x-26y-71=0,\ 5x+y-2=0$

10. $(1+3\lambda)x+(2-\lambda)y+(4\lambda-3)=0$ **(i)** $21x-14y+41=0$ **(ii)** $28x-21y+59=0$

11. $(2+\lambda)x+(-3+4\lambda)y+(5-7\lambda)=0$ **(i)** $22x-11y+17=0$ **(ii)** $11x-11y+20=0$

12. Not concurrent **14.** $m_1c_2-m_2c_1+m_2c_3-m_3c_2+m_3c_1-m_1c_3=0$

15. $k=4$ **16.** $a=-\frac{1}{3}$

Exercise 5.2

1. (i) 168.7°, acute angle: 11.3° **(ii)** 7.1° **2. (i)** 90° **(ii)** 45° **(iii)** 120°

3. (i) 18.4° **(ii)** 101.3°

(ii) 36.9°, 19.4°, 123.7°

4. $-2+\sqrt{3},\ -2-\sqrt{3}$ **5. (i)** 66.4°, 66.4°, 53.1°

(iv) 53.1°, 63.4°, 63.4° **6.** 126.9°, 74°, 106°, 53.1°

7. 112.6°, 67.4°, 112.6°, 67.4° **8.** 16.5 **9.** Collinear **10.** $\lambda=7$

Exercise 5.3

1. $(3x-4y)(2x+y)=0$, 79.7° **2.** $(2x-y)(x-3y)=0$, 45° **3.** $(5x+y)(x+y)=0$, 33.69°

4. $y=x(\tan\alpha\pm\sec\alpha)$, 90° **5.** $(7x-y)(x-y)=0$, 36.87° **6.** $k=2$

8. $2x^2+5xy-3y^2=0$ **9.** $x-y+100=0$, slope = 1 **10.** 1, 45°

11. $2x-y=0,\ x+3y=0$, 81.87° **12.** 0° **13.** $3x+y=0,\ x-3y=0$, 90°

EXERCISE 6.1

1. (a) $x^2+y^2-10x-14y+70=0$ **(b)** $x^2+y^2-2ax\cos\theta-2ay\sin\theta=0$

(c) $x^2+y^2-5x+3y-22=0$ **2. (i)** $(x-1)^2+(y-3)^2=\frac{9}{2}$; Center: $(1,3)$, Radius: $\frac{3\sqrt{2}}{2}$

(ii) $(x-a)^2+(y+b)^2=(a+b)^2$; Center: $(a,-b)$, Radius: $|a+b|$

3. $x^2+y^2-4x-6y-12=0$; Center: $(2,3)$, Radius: 5

4. $x^2+y^2-2x-3y=0$ **5.** $5x^2+5y^2-19x-11y+2=0$; Yes, (4, 1) lies on the circle.

6. $3x^2+3y^2+72x-38y-11=0$ **7.** $x^2+y^2+2ax-2ay+a^2=0$

8. $x^2+y^2-10x-4y+25=0$ **9.** $x^2+y^2-10x+17=0$

10. $x^2+y^2-40x+20y+100=0$; $x^2+y^2-10x-20y+100=0$

11. $x^2+y^2-4x-1=0$, $x^2+y^2-14x+29=0$

12. $x^2+y^2-10x-10y+25=0$, $x^2+y^2+6x+2y-15=0$

EXERCISE 6.2

1. **(a)** At (2, 2): Tangent: $x+y=4$ Normal: $x-y=0$ At $\left(2\sqrt{2}\cos\theta, 2\sqrt{2}\sin\theta\right)$: Tangent: $x\cos\theta+y\sin\theta=2\sqrt{2}$ Normal: $x\sin\theta-y\cos\theta=0$

(b) Tangent: $x+4y-24=0$, Normal: $4x-y-11=0$

2. Condition of tangency: $c^2=a^2\left(1+m^2\right)$ Point of contact: $\left(-\frac{am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$ or $\left(\frac{am}{\sqrt{1+m^2}}, -\frac{a}{\sqrt{1+m^2}}\right)$ depending on the sign of c.

3. $(4,3)$ **4.** **(a)** $x-y\pm2\sqrt{2}=0$ **(b)** $3x+4y\pm10=0$ (c) $4x+3y\pm10=0$

5. $3x-4y+13=0$ and $3x-4y-17=0$

6. **(a)** Tangent: $x+y-4=0$, $\left(\text{Point of contact:}(2,2)\right)$

(b) Tangent: $41x+y+116=0$, $\left(\text{Point of contact:}\left(-\frac{82}{29}, -\frac{2}{29}\right)\right)$

(c) Tangent: $y+4=0$, $\left(\text{Point of contact:}(2,-4)\right)$

$4x+3y-36=0$, $\left(\text{Point of contact:}(6,4)\right)$

7. **(a)** $(x-3)^2+(y+1)^2=25$ **(b)** Centre: $(3,-1)$, Radius: 5 **(c)** P: $(3,4)$

(d) $y=4$ **(e)** $x=3$ **(f)** $x=-2$

EXERCISE 6.3

1. **(i)** Vertex: $(0,0)$ Focus: $(3,0)$ Directrix: $x=-3$ **(ii)** Vertex: $(0,1)$ Focus: $(0,3)$ Directrix: $y=-1$ **(iii)** Vertex: $(3,0)$ Focus: $(1,0)$ Directrix: $x=5$

(iv) Vertex: $(2,0)$ Focus: $(2,2)$ Directrix: $y=-2$ **2.** **(i)** $(y-2)^2=-16(x-1)$

(ii) $(x-3)^2=12(y-2)$ **(iii)** $(y-3)^2=8(x-6)$ **(iv)** $4y^2=5x+6$

3. **(i)** $x^2=-2dy+d^2$ **(ii)** $y^2=-2dx+d^2$ where d is the directed distance from origin to directrix

6. $y=-\frac{5}{4}$ **7.** $a=2, b=-12, c=16$ **8.** Tangent: $ty=x+at^2$, Normal: $y=-tx+2at+at^3$

9. Tangent: $2x-y-7=0$, Normal: $x+2y-11=0$

10. Tangent: $3x-y+1=0$, Point of tangency: $\left(\frac{1}{3}, 2\right)$

EXERCISE 6.4

1. **(i)** $\frac{x^2}{16}+\frac{y^2}{41}=1$ **(ii)** $\frac{(x+2)^2}{9}+\frac{(y-1)^2}{16}=1$ **(iii)** $\frac{x^2}{36}+\frac{y^2}{20}=1$

(iv) $\frac{(x+3)^2}{25}+\frac{(y-1)^2}{9}=1$ **2.** **(i)** Centre: $(0,0)$, Foci: $\left(\pm\sqrt{5},0\right)$, Eccentricity: $e=\frac{\sqrt{5}}{3}$, Vertices: $(\pm3,0)$, Directrices: $x=\pm\frac{9\sqrt{5}}{5}$

(ii) Centre: $(-3,2)$, Foci: $\left(-3\pm\sqrt{5},2\right)$, Eccentricity: $e=\frac{\sqrt{5}}{3}$, Vertices: $(0,2)$ and $(-6,2)$

Directrices: $x=2\pm\frac{9}{\sqrt{5}}$ 3. $\frac{x^2}{25}+\frac{y^2}{16}=1$ 4. $\frac{x^2}{36}+\frac{y^2}{20}=1$

5. $\frac{(x-5)^2}{25}+\frac{(y-3)^2}{9}=1$ **6.** $e=\frac{3}{5}$ **7.** $\frac{(x+2)^2}{20}+\frac{(y-3)^2}{16}=1$

8. 4 units **9.** 8 by 6 **10.** $\frac{32}{5}$ and $\frac{18}{5}$ 11. Minimum distance = 2

12. $11+2\sqrt{10}$ **13.** $\frac{6\sqrt{2}-\sqrt{26}}{2}$ **14.** $k=\pm\sqrt{37}$ **15.** The line is a secant.

16. $\frac{x^2}{16}+\frac{y^2}{9}=1$

EXERCISE 6.5

1. (i) $\frac{y^2}{64}-\frac{x^2}{36}=1$ (ii) $\frac{y^2}{4}-\frac{x^2}{60}=1$ (iii) $\frac{y^2}{50}-\frac{x^2}{50}=1$

2.(i) Centre: (0,0); Vertices: (±6,0); Foci: (±10,0); Eccentricity: $e=\frac{5}{3}$; Directrices: $x=\pm\frac{18}{5}$ (ii)

Centre: $(-3,2)$, Vertices: $\left(-3\pm\sqrt{5},2\right)$; Foci: $(0,2)$ and $(-6,2)$; Eccentricity: $e=\frac{3}{\sqrt{5}}$ Directrices:

$x=-3\pm\frac{5}{3}$

(iii) Centre: (0,0); Vertices: (0,±5); Foci: (0,± $\sqrt{41}$); Eccentricity: $e=\frac{\sqrt{41}}{5}$;

Directrices: $y=\pm\frac{25}{\sqrt{41}}$ 3. $\frac{y^2}{49}-\frac{x^2}{25}=1$ 4. $\frac{(x-1)^2}{16}-\frac{(y-3)^2}{4}=1$

5. $\frac{(y+1)^2}{16}-\frac{(x-2)^2}{36}=1$ 6. $\frac{(x+1)^2}{12}-\frac{(y-4)^2}{24}=1$

7. $\frac{(y-4)^2}{16}-\frac{(x-2)^2}{3}=1$ 8. $\frac{x^2}{16}-\frac{y^2}{4}=1$

9. Tangent: $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$, Normal: $ax\tan\theta+by\sec\theta=(a^2+b^2)\sec\theta\tan\theta$

10. $\frac{9}{2}$ 11. $e=\sqrt{2}$ 12. $e=\frac{\sqrt{34}}{4}$ 13. $\frac{x^2}{4}-\frac{y^2}{12}=1$ 14. $e=\sqrt{\frac{3}{2}}$

15. $5y=10x-14$ and $5y=10x+14$ 16. $e=\frac{4}{3}$ 17. 6

19. $y=2x-5$ and $y=-3x+10$ 20. $a^2l^2-b^2m^2=1$

EXERCISE 6.6

1. 75 million km **2.** $y=\frac{4h}{\ell^2}x^2$ **3.** 12.75 m **4.** **(a)** $20\sqrt{5}$ m **(b)** $40\sqrt{5}$ m

5. **(a)** 8 m **(b)** 18 m **6.** (a) $\frac{x^2}{144}+\frac{y^2}{64}=1$ **(b)** $2\sqrt{15}$ m **7.** **(a)** 6.36×10^{-10} m

(b) 5.31×10^{-23} s **(c)** $r_{min}=1.384\times10^{-10}$ m and $r_{max}=2.756\times10^{-10}$ m

8. 126.65 miles **9.** **(a)** 36000 km **(b)** 6000 km **(c)** $\frac{1}{6}$ **10.** 31.75 meters

11. $\frac{5\sqrt{39}}{8}$ meters

Answers 8.1

1. **(i)** 0.36 **(ii)** 1.86 **(i)** 1.32 **(ii)** 0.85 **(iii)** 7.39

Answers 8.2

2. **(i)** 1.068 **(ii)** 0.605 **(iii)** 1.929 **(iv)** 2.613 **3.** **(i)** 2.61 **(ii)** −1.93 **(iii)** 0.85
(iv) 0.36 **4.** **(i)** 1.73 **(ii)** 2.153 **(iii)** 2.0946 **(iv)** 0.361 **(v)** 4.4934

Answers 8.3

1. 0.8903 **2.** 1.7212 **3.** 7.5522 **4.** 4.0625 **5.** 5.1463
6. 1.4704 **7.** 2.9411 **8.** 0.6179 **9.** Exact Area = 14.137 **10.** 2.3203×10^6

Answers 8.4

1. 0.12 moles per liter. **2.** 4.88 mL/s **3.** 0.539 coulombs **4.** 65.311 unit/seconds
5. 0.524 radian **6.** 20.488 volts

Answers 9.1

Inverse Trigonometric function Exercise 1 Answers

1. **(i)** $\frac{\pi}{3}$ **(ii)** $\frac{3\pi}{4}$ **(iii)** $\frac{-\pi}{3}$ **(iv)** $\frac{\pi}{6}$ **(v)** $\frac{-\pi}{4}$

(vi) $\frac{-\pi}{4}$ **(vii)** $\frac{\pi}{3}$ **(viii)** $\frac{3\pi}{4}$ **(ix)** $\frac{\pi}{2}$ **(x)** 0

(xi) $\frac{\pi}{3}$ **2.** **(i)** $\left[\frac{2}{5},\frac{4}{5}\right]$ **(ii)** [3, 4] **(iii)** $-\sqrt{2}\le x\le\sqrt{2}$

(iv) $\left[\frac{1}{e},e\right]$ **(v)** [−2, 2] **(vi)** $(-\infty,-3]\cup[-1,\infty)$ **(vii)** $\left(-\infty,-\frac{1}{3}\right]\cup[1,\infty)$

(viii) $\left[\frac{1}{e},e\right]$ **4.** **(i)** $\frac{3}{\sqrt{13}}$ **(ii)** $\frac{\sqrt{50}}{7}$ **(iii)** $\frac{1}{7}$ **(vi)** $\frac{1}{\sqrt{3}}$

(v) $\sqrt{\frac{11+4\sqrt{6}}{5}}$ **(vi)** $\sqrt{10}$

Answers 10

Solution of Trigonometric Equation Exercise 2 Answers

1. $x=-2.97, 0.41$ and 2.33 radians **2.** $x=1.16$ radians **3.** $x=0.65$ radians
4. $x=-1.66$ radians **5.** $x=-2.04, 0.82$ radians **6.** $x=-0.89, 0, 0.89$ radians

7. $x=0.40, 1.57, 2.74$ radians **8.** $x=1.89$ radians **9.** $\theta\approx53.13^\circ$

10. **(a)** $h=8.96$m **(b)** $\theta\approx1.15^\circ$ **11.** $h=11.55$ m, $\theta\approx49.1^\circ$