Chapter 6

SEQUENCES AND SERIES

SEQUENCE

A sequence is a function whose domain is a subset of the set of natural numbers. First term of the sequence is usually denoted by a_1, second by a_2, third by a_3 and similarly nth term of the sequence is denoted by a_n.

EXERCISE 6.1

Q.1 Write the first four terms of the following sequences, if

(i) $a_n = 2n - 3$ **(ii)** $a_n = (-1)^n n^2$

(iii) $a_n = (-1)^n (2n - 3)$ **(iv)** $a_n = 3n - 5$

(v) $a_n = \dfrac{n}{2n + 1}$ **(vi)** $a_n = \dfrac{1}{2^n}$

(vii) $a_n - a_{n-1} = n + 2,\ a_1 = 2$

(viii) $a_n = n\, a_{n-1},\ a_1 = 1$

(ix) $a_n = (n + 1)\, a_{n-1},\ a_1 = 1$

(x) $a_n = \dfrac{1}{a + (n - 1)\, d}$

Solution:

(i) $a_n = 2n - 3$

Given

$a_n = 2n - 3$

$n = 1, 2, 3, 4$

$n = 1 \Rightarrow a_1 = 2(1) - 3 = 2 - 3 = -1$

$n = 2 \Rightarrow a_2 = 2(2) - 3 = 4 - 3 = 1$

$n = 3 \Rightarrow a_3 = 2(3) - 3 = 6 - 3 = 3$

$n = 4 \Rightarrow a_4 = 2(4) - 3 = 8 - 3 = 5$

So the required terms are $-1, 1, 3, 5$.

(ii) $\mathbf{a_n = (-1)^n n^2}$

Given sequence

$a_n = (-1)^n n^2$

Put $n = 1, 2, 3, 4$

$n = 1 \Rightarrow a_1 = (-1)(1)^2 = -1$

$n = 2 \Rightarrow a^2 = (-1)^2 (2)^2 = 4$

$n = 3 \Rightarrow a_3 = (-1)^3 (3)^2 = (-1)(9) = -9$

$n = 4 \Rightarrow a_4 = (-1)^4 (4)^2 = (1)(16) = 16$

So required terms are $-1, 4, -9, 16$

(iii) $\mathbf{a_n = (-1)^n (2n - 3)}$

Given sequence

$a_n = (-1)^n (2n - 3)$

Put $n = 1, 2, 3, 4$

$n = 1 \Rightarrow a_1 = (-1)^1 (2(1) - 3) = -(2 - 3) = -(-1) = 1$

$n = 2 \Rightarrow a_2 = (-1)^2 (2(2) - 3) = 4 - 3 = 1$

$n = 3 \Rightarrow a_3 = (-1)^3 (2(3) - 3) = (-1)(6 - 3) = -3$

$n = 4 \Rightarrow a_4 = (-1)^4 (2(4) - 3) = 8 - 3 = 5$

So required terms are $1, 1, -3, 5$

(iv) $\mathbf{a_n = 3n - 5}$

Given sequence

$a_n = 3n - 5$

Put $n = 1, 2, 3, 4, 5$

$n = 1 \Rightarrow a_1 = 3(1) - 5 = 3 - 5 = -2$

$n = 2 \Rightarrow a_2 = 3(2) - 5 = 6 - 5 = 1$

$n = 3 \Rightarrow a_3 = 3(3) - 5 = 9 - 5 = 4$

$n = 4 \Rightarrow a_4 = 3(4) - 5 = 12 - 5 = 7$

so the required terms are $-2, 1, 4, 7$

(v) $\mathbf{a_n = \dfrac{n}{2n + 1}}$

Given sequence

$a_n = \dfrac{n}{2n + 1}$

Put $n = 1, 2, 3, 4$

$n = 1 \Rightarrow a_1 = \frac{1}{2(1)+1} = \frac{1}{2+1} = \frac{1}{3}$

$n = 2 \Rightarrow a_2 = \frac{2}{2(2)+1} = \frac{2}{4+1} = \frac{2}{5}$

$n = 3 \Rightarrow a_3 = \frac{3}{2(3)+1} \frac{3}{6+1} = \frac{3}{7}$

$n = 4 \Rightarrow a_4 = \frac{4}{2(4)+1} = \frac{4}{8+1} = \frac{4}{9}$

so the required terms are $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$

(vi) $\mathbf{a_n = \frac{1}{2^n}}$

Given sequence

$a_n = \frac{1}{2^n}$

Put $n = 1, 2, 3, 4$

$n = 1 \Rightarrow a_1 = \frac{1}{2^1} = \frac{1}{2}$

$n = 2 \Rightarrow a_2 = \frac{1}{2^2} = \frac{1}{4}$

$n = 3 \Rightarrow a_3 = \frac{1}{2^3} = \frac{1}{8}$

$n = 4 \Rightarrow a_4 = \frac{1}{2^4} = \frac{1}{16}$

so the required terms are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

(vii) $\mathbf{a_n - a_{n-1} = n + 2,\ a_1 = 2}$

Given

$a_n - a_{n-1} = n + 2, \qquad (1) \quad a_1 = 2$

Put $n = 2$ in (1)

$a_2 - a_1 + 2 + 2$

$a_2 - 2 = 4$

$a_2 = 4 + 2 = 6$

Put $n = 3$ in (1)

$a_3 - a_2 = 3 + 2$

$a_3 - 6 = 5$

$a_3 = 6 + 5 = 11$

Put $n = 4$ in (1)

$a_4 - a_3 = 4 + 2$

$a_4 - 11 = 6$

$a_4 = 6 + 11 = 17$

$\Rightarrow$ required terms are 2, 6, 11, 17

(viii) $\mathbf{a_n = n\, a_{n-1},\ a_1 = 1}$

Given

$a_n = n\, a_{n-1}$, (1) $a_1 = 1$

Put $n = 2$ in (1)

$a_2 = 2a_1 = 2(1) = 2$ ¤ $a_1 = 1$

Put $n = 3$ in (1)

$a_3 = 3a_2 = 3(2) = 6$

Put $n = 4$ in (1)

$a_4 = 4a_3$

$a_4 = 4(6) = 24$

$\Rightarrow$ required terms are 1, 2, 6, 24

(ix) $\mathbf{a_n = (n+1)\, a_{n-1},\ a_1 = 1}$

Given

$a_n = (n+1)\, a_{n-1}$ $a_1 = 1$ (1)

Put $n = 2$ in (1)

$a_2 = (2+1)\, a_1 = 3a_1 = 3(1) = 3$

Put $n = 3$ in (1)

$a_3 = (3+1)\, a_2 = 4a_2 = 4(3) = 12$

Put $n = 4$ in (1)

$a_4 = (4+1)\, a_3 = 5a_3 = 5(12) = 60$

$\Rightarrow$ required terms are 1, 3, 12, 60

(x) $\mathbf{a_n = \dfrac{1}{a + (n-1)\,d}}$

Given

$$a_n = \frac{1}{a + (n-1)\,d}$$

Put n = 1, 2, 3, 4

$$n = 1 \Rightarrow a_1 = \frac{1}{a + (1-1)\,d} = \frac{1}{a+0} = \frac{1}{a}$$

$$\text{Put } n = 2 \Rightarrow a_2 = \frac{1}{a + (2-1)\,d} = \frac{1}{a+d}$$

$$\text{Put } n = 3 \Rightarrow a_3 = \frac{1}{a + (3-1)\,d} = \frac{1}{a+2d}$$

$$\text{Put } n = 4 \Rightarrow a_4 = \frac{1}{a + (4-1)\,d} = \frac{1}{a+3d}$$

Q.2 Find the indicated terms of the following sequences:

(i) **2, 6, 11, 17, ... a_7** **(ii)** **1, 3, 12, 60, ... a_6**

(iii) $\mathbf{1, \frac{3}{2}, \frac{5}{4}, \frac{7}{8}, \ldots\ldots\ a_7}$ **(iv)** **1, 1, – 3, – 7, ... a_8**

(v) **1, – 3, 5, – 7, 9, – 11, ... a_8**

Solution:

Given

2, 6, 11, 17, ... a_7

from the given sequence, we have

$a_1 = 2$ $a_2 = 2 + 4 = 6$

$a_3 = 6 + 5 = 11$ $a_4 = 11 + 6 = 17$

$a_5 = 17 + 7 = 24$ $a_6 = 24 + 8 = 32$

$a_7 = 32 + 9 = 41$

$\Rightarrow$ required term is $a_7 = 41$

(ii) **1, 3, 12, 60, ... a_6**

Given

1, 3, 12, 60, ... a_6

from the given sequence, we have

$a_1 = 1$ $a_2 = 1 \times 3 = 3$

$a_3 = 3 \times 4 = 12$ $a_4 = 12 \times 5 = 60$

$a_5 = 60 \times 6 = 360$ $a_6 = 360 \times 7 = 2520$

so the required term $= a_6 = 2520$.

(iii) $\mathbf{1, \frac{3}{2}, \frac{5}{4}, \frac{7}{8}, \ldots\ldots a_7}$

Given

$1, \frac{3}{2}, \frac{5}{4}, \frac{7}{8}, \ldots\ldots a_7$

from the given sequence

$a_1 = 1$ $\qquad a_2 = \frac{1+2}{1(2)} = \frac{3}{2}$

$a_3 = \frac{3+2}{2(2)} = \frac{5}{4}$ $\qquad a_4 = \frac{5+2}{4(2)} = \frac{7}{8}$

$a_5 = \frac{7+2}{8(2)} = \frac{9}{16}$ $\qquad a_6 = \frac{9+2}{16(2)} = \frac{11}{32}$

$a_7 = \frac{11+2}{32(2)} = \frac{13}{64}$

$\Rightarrow$ required term $= a_7 = \frac{13}{64}$

(iv) $\mathbf{1, 1, -3, 5, -7, \ldots\ldots a_8}$

Given

$1, 1, -3, 5, -7, \ldots\ldots a_8$

from the given sequence

$a_1 = 1$ $\qquad a_2 = 1$

$a_3 = 1 - 4 = -3$ $\qquad a_4 = 1 + 4 = 5$

$a_5 = -3 - 4 = -7$ $\qquad a_6 = 5 + 4 = 9$

$a_7 = -7 - 4 = -11$ $\qquad a_8 = 9 + 4 = 13$

(v) $\mathbf{1, -3, 5, -7, 9, -11, \ldots\ldots a_8}$

Given

$1, -3, 5, -7, 9, -11, \ldots\ldots a_8$

From the given sequence

$a_1 = 1$ $\qquad a_2 = -3$

$a_3 = 5$ $\qquad a_4 = -7$

$a_5 = 9$ $\qquad a_6 = -11$

$a_7 = 13$ $\qquad a_8 = -15$

Hence required term $= a_8 = -15$

Q.3 Find the next two terms of the following sequences.

(i) 7, 9, 12, 16, ... **(ii) 1, 3, 7, 15, 31, ...**

(iii) – 1, 2, 12, 40, ... **(iv) 1, – 3, 5, – 7, 9, – 11, ...**

Solution:

(i) 7, 9, 12, 16, ...

Given sequence

7, 9, 12, 16, ...

$\Rightarrow$ $a_1 = 7$ $a_2 = 7 + 2 = 9$

$a_3 = 9 + 3 = 12$ $a_4 = 12 + 4 = 16$

$a_5 = 16 + 5 = 21$ $a_6 = 21 + 6 = 27$

so next two terms are 21, 27

(ii) 1, 3, 7, 15, 31, ...

Given sequence

1, 3, 7, 15, 31, ...

$\Rightarrow$ $a_1 = 1$ $a_2 = 1 + 2 = 3$

$a_3 = 3 + 4 = 7$ $a_4 = 7 + 8 = 15$

$a_5 = 15 + 16 = 31$ $a_6 = 31 + 32 = 63$

$a_7 = 63 + 64 = 127$

so next two terms are 63, 127.

(iii) – 1, 2, 12, 40, ...

Given sequence

– 1, 2, 12, 40, ...

$a_1 = -1 \times -1 = -1$

$a_2 = 1 \times 2 = 2$

$a_3 = 3 \times 4 = 12$

$a_4 = 5 \times 8 = 40$

$a_5 = 7 \times 16 = 112$

$a_6 = 9 \times 32 = 288$

$\Rightarrow$ the next two terms of the sequence are 112, 288

(iv) 1, – 3, 5, – 7, 9, – 11, ...

Given sequence

1, – 3, 5, – 7, 9, – 11, ...

$\Rightarrow$ $a_1 = 1$

$a_2 = -3$

$a_3 = 5$

$a_4 = -7$

$a_5 = 9$

$a_6 = -11$

$a_7 = 13$

$a_8 = -15$

$\Rightarrow$ next two terms are $12, -15$.

Arithmetic Progression (A. P)

A sequence $\{a_n\}$ is an Arithmetic sequence or Arithmetic progression (A.P) if $a_n - a_{n-1}$ is the same number for all $n \in N$ and $n > 1$. The difference of two consecutive terms of an A.P is called common difference and is usually denoted by d. $a_n = a_1 + (n-1)d$ is called the nth term or general term of the A.P.

General form of A.P $a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots$.

EXERCISE 6.2

Q.1 Write the first four terms of the following arithmetic sequence, if

(i) $a_1 = 5$ and other three consecutive terms are 23, 26, 29

(ii) $a_5 = 17$ and $a_9 = 37$

(iii) $a_7 = 7a_4$ and $a_{10} = 33$

Solution:

(i) $a_1 = 5$ and other three consecutive terms are 23, 26, 29

As the given sequence is arithmetic sequence so $d = 26 - 23 = 3$

and $a_1 = 5$ (given)

$\Rightarrow$ $a_2 = a_1 + d = 5 + 3 = 8$

$a_3 = a_2 + d = 8 + 3 = 11$

$a_4 = a_3 + d = 11 + 3 = 14$

$\Rightarrow$ first four terms of the sequence are 5, 8, 11, 14.

(ii) $a_5 = 17$ and $a_9 = 37$

As $a_5 = 17 \Rightarrow a_1 + 4d = 17$ ..... (1)

$a_9 = 37 \Rightarrow a_1 + 8d = 37$ ..... (2)

Subtracting (1) from (2), we get

$4d = 20$

$\Rightarrow \quad d = 5$

Put $d = 5$ in equation (1), we get

$a_1 + 4(5) = 17$

$a_1 + 20 = 17$

$\boxed{a_1 = -3}$

Now

$a_2 = a_1 + d = -3 + 5 = 2$

$a_3 = a_2 + d = 2 + 5 = 7$

$a_4 = a_3 + d = 7 + 5 = 12$

$\Rightarrow$ first four terms of the sequence are

$-3, 2, 7, 12, ...$

(iii) $3a_7 = 7a_4$ and $a_{10} = 33$

$3a_7 = 7a_4$

$\Rightarrow \quad 3(a_1 + 6d) = 7(a_1 + 3d)$

$3a_1 + 18d = 7a_1 + 21d$

$4a_1 + 3d = 0 \quad (1)$

and $\quad a_{10} = 33$

$\Rightarrow \quad a_1 + 9d = 33 \quad (2)$

from equation (1) $3d = -4a_1$

$9d = -12a_1$

Put $9d = -12a_1$ in equation (2), we get

$a_1 - 12a_1 = 33$

$-11a_1 = 33 \quad \Rightarrow \quad \boxed{a_1 = -3}$

Put $a_1 = -3$ in equation (1), we get

$4(-3) + 3d = 0$

$-12 + 3d = 0 \quad \Rightarrow \quad \boxed{d = 4}$

Now

$a_2 = a_1 + d = -3 + 4 = 1$

$a_3 = a_1 + 2d = -3 + 2(4) = -3 + 8 = 5$

$a_4 = a_1 + 3d = -3 + 3(4) = -3 + 12 = 9$

so first four terms of the sequence are $-3, 1, 5, 9, ...$

Q.2 If $a_{n-3} = 2n - 5$ find nth term of the sequence

Solution:

Given $a_{n-3} = 2n - 5$

Put $n = n + 3$, we get

$a_{n+3-3} = 2(n + 3) - 5$

$= 2n + 6 - 5$

$a_n = 2n + 1$

$\Rightarrow$ nth term of the sequence $= a_n = 2n + 1$

Q.3 If the 5th term of an A.P is 16 and 20th term is 46 what is the 12th term.

Solution:

Given

$a_5 = 16 \Rightarrow a_1 + 4d = 16$ ..... (1)

$a_{20} = 46 \Rightarrow a_1 + 19d = 46$ ..... (2)

subtracting equation (1) from equation (2), we get

$15\,d = 30$

$\boxed{d = 2}$

Put $d = 2$ in equation (1), we get

$a_1 + 4(2) = 16$

$a_1 + 8 = 16$

$a_1 = 16 - 8$

$\boxed{a_1 = 8}$

so required 12^{th} term $= a_{12} = a_1 + 11d$

$= 8 + 11(2)$

$\boxed{a_{12} = 30}$

Q.4 Find 13th term of the sequence x, 1, 2 – x, 3 – 2x, ...

Solution:

Given sequence

x, 1, 2 – x, 3 – 2x .....

$a_1 = x, \qquad d = 1 - x,$

so required 13th term will be

$a_{13} = a_1 + 12d = x + 12(1 - x) = x + 12 - 12x$

$a_{13} = 12 - 11x$

Q.5 Find 18th term of A.P if its 6th term is 19 and 9th term is 31.

(Gujranwala Board 2005)

Solution:

Given

$a_6 = 19 \Rightarrow a_1 + 5d = 19$ ..... (1)

$a_9 = 31 \Rightarrow a_1 + 8d = 31$ ..... (2)

subtracting equation (1) from equation (2)

$3d = 12$

$\boxed{d = 4}$

Put $d = 4$ in equation (1), we get

$a_1 + 5(4) = 19$

$\boxed{a_1 = -1}$

so 18th term of the A.P is

$a_{18} = a_1 + 17d$

$= -1 + 17(4)$

$= -1 + 68 = 67$

Q.6 Which term of A.P 5, 2, –1, ... is – 85? (Lahore Board 2010)

Solution:

Given sequence

5, 2, –1, ...

$a_1 = 5, \quad d = 2 - 5 = -3, \quad a_n = -85$

$n = ?$

As

$an = a_1 + (n-1)d$

Put values

$-85 = 5 + (n-1)(-3)$

$-85 - 5 = -3n + 3$

$-90 = -3n + 3$

$-90 - 3 = -3n$

$-93 = -3n \Rightarrow n = 31$

$\Rightarrow$ – 85 is the 31st term of the A.P

Q.7 Which term of the A.P. – 2, 4, 10, ... is 148?

Solution:

Given sequence

– 2, 4, 10, ...

$a = -2, \quad d = 4 - (-2) = 6 \quad a_n = 148 \quad n = ?$

As $a_n = a_1 + (n - 1)\, d$

Put values

$148 = -2 + (n - 1)(6)$

$148 = -2 + 6n - 6$

$148 = 6n - 8$

$148 + 8 = 6n$

$156 = 6n$

$n = 26$

$\Rightarrow$ 148 is 26^{th} term of the A.P.

Q.8 How many terms are there in the A.P. in which $a_1 = 11, \quad a_n = 68, \quad d = 3$.

Solution:

Given that

$a_1 = 11, \quad a_n = 68, \quad d = 3$

As

$a_n = a_1 + (n - 1)\, d$

$68 = 11 + (n - 1)(3)$

$68 = 11 + 3n - 3$

$68 = 8 + 3n$

$68 - 8 = 3n \Rightarrow \boxed{n = 20}$

$\Rightarrow$ 68 is the 20th term of the A.P.

Q.9 If nth term of the A.P. is 3n – 1, find A.P.

(Gujranwala Board 2007, Lahore Board 2007)

Solution:

Given nth term of the A.P is

$a_n = 3n - 1$

Put $n = 1, 2, 3, 4$

$n = 1 \Rightarrow a_1 = 3(1) - 1 = 3 - 1 = 2$

$n = 2 \Rightarrow a_2 = 3(2) - 1 = 5$

$n = 3 \Rightarrow a_3 = 3(3) - 1 = 8$

For $n = 4 \quad a_4 = 3(4) - 1 = 11$

so required A.P is 2, 5, 8, 11, ...

Q.10 Determine whether (i) – 19 (ii) 2 are the terms of the A.P. 17, 13, 9, ... or not.

Solution:

(i) Given

$a_1 = 17, \quad d = 13 - 17 = -4 \quad a_n = -19 \quad n = ?$

As $a_n = a_1 + (n-1)\,d$

$-19 = 17 + (n-1)(-4)$

$-19 = 17 - 4n + 4$

$-19 = 21 - 4n$

$4n = 21 + 19 = 40$

$\boxed{n = 10}$

– 19 is 10th term of the A.P.

(ii)

Here $a_1 = 17, \quad d = -4, \quad a_n = 2 \quad n = ?$

As $a_n = a_1 + (n-1)\,d$

$2 = 17 + (n-1)(-4)$

$2 = 17 - 4n + 4$

$2 = 21 - 4n = 19$

$4n = 21 - 2 = 19$

$n = \frac{19}{4}$ (which is not an integer)

$\Rightarrow$ 2 is not the term of given A.P.

Q.11 If *l*, m, n are the pth, qth, rth terms of an A.P. show that

(i) $l\,(q-r) + m\,(r-p) + n\,(p-q) = 0$

(ii) $p\,(m-n) + q\,(n-l) + r\,(l-m) = 0$

Solution:

(i) $l\,(q-r) + m\,(r-p) + n\,(p-q) = 0$

As $a_n = a_1 + (n-1)\,d$

$a_p = a_1 + (p-1)\,d$

$a_q = a_1 + (q-1)\,d$

$a_r = a_1 + (r-1)\,d$

it is given that $a_p = l, \quad a_q = m, \quad a_r = n$

$\Rightarrow$

$l = a_1 + (p-1)\,d$ (1)

$m = a_1 + (q-1)\,d$ (2)

$n = a_1 + (r-1)\,d$ (3)

To prove $l\,(q-r) + m\,(r-p) + n\,(p-q) = o$

Take L.H.S.

$l\,(q-r) + m\,(r-p) + n\,(p-q)$

Put values of l, m, n from equation (1), (2), (3) we get

$[a_1 + (p-1)d](q-r) + [a_1 + (q-1)d](r-p) + [a_1 + (r-1)d](p-q)$

$= (a_1 + pd - d)(q-r) + (a_1 + qd - d)(r-p) + (a_1 + rd - d)(p-q)$

$= a_1q - a_1r + pqd - pdr - dq + dr + a_1r - a_1p + qdr - qdp - dr + dp + a_1q$
$+ rdp - rdq - dp + dq$

$0 =$ R.H.S.

(ii) $p(m-n) + q(n-l) + r(l-m) = 0$

Now to prove

$p(m-n) + q(n-l) + r(l-m) = 0$

Take its L.H.S.

$p(m-n) + q(n-l) + r(l-m)$

Put values of l, m, n from (1), (2), (3)

$p[a_1 + qd - d - (a_1 + rd - d)] + q[a_1 + rd - d - (a_1 + pd - d)]$
$+ r[a_1 + pd - d - (a_1 + qd - d)]$

$= p[a_1 + qd - d - a_1 - rd + d)] + q[a_1 + rd - d - a_1 - pd + d)]$
$+ r[a_1 + pd - d - a_1 - qd + d)]$

$= p[qd - rd] + q[rd - pd] + r[pd - qd]$

$= pqd - prd + qrd - qpd + rpd - rqd$

$= 0 =$ R.H.S.

Hence proved.

Q.12 Find nth term of the sequence **(Lahore Board 2008)**

$\left(\frac{4}{3}\right)^2, \left(\frac{7}{3}\right)^2, \left(\frac{10}{3}\right)^2 \ldots\ldots$

Solution:

nth term of the sequence 4, 7, 10,….. is

$a_n = a_1 + (n-1)d$

$= 4 + (n-1)(3)$

$= 4 + 3n - 3$

$an = 3n + 1$

so the nth term of the given sequence is $\left(\frac{3n+1}{3}\right)^2$

Q.13 If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in A.P. Show that $b = \frac{2ac}{a+c}$. (Gujranwala Board 2003)

Solution:

As $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P

$\Rightarrow \quad \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$

$\frac{1}{b} + \frac{1}{b} = \frac{1}{c} + \frac{1}{a}$

$\frac{2}{b} = \frac{a+c}{ac} \quad \Rightarrow \quad \boxed{b = \frac{2ac}{a+c}}$

Q.14 If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in A.P. Show that common difference is $\frac{a-c}{2ac}$.

(Gujranwala Board 2006)

Solution:

As $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P

$\Rightarrow$

$d = \frac{1}{b} - \frac{1}{a} \qquad \ldots\ldots\ldots (1)$

and $d = \frac{1}{c} - \frac{1}{b} \qquad \ldots\ldots\ldots (2)$

Adding equation (1) and (2), we get

$2d = \frac{1}{b} - \frac{1}{a} + \frac{1}{c} - \frac{1}{b}$

$2d = \frac{-c+a}{ac} = \frac{a-c}{ac}$

$\boxed{d = \frac{a-c}{2ac}}$ Hence proved.

Arithmetic Mean (A.M)

A number A is said to be the A.M between the two numbers a and b if a, A, b are in A.P.

Middle term of three consecutive terms in A.P is the A.M between the extreme terms. If a, A, b are in A.P

then $A - a = b - a \Rightarrow A = \frac{a+b}{2}$

In general, we can say that a_n is the A.M between a_{n-1} and a_{n+1}

i.e.

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

EXERCISE 6.3

Q.1 Find A.M. between

(i) $3\sqrt{5}$ and $5\sqrt{5}$ (ii) $x - 3$ and $x + 5$

(iii) $1 - x + x^2$ and $1 + x + x^2$ (Gujranwala Board 2006, 2007)

Solution:

(i) $3\sqrt{5}$ and $5\sqrt{5}$

Let $a = 3\sqrt{5}$ $b = 5\sqrt{5}$

As $A.M. = \frac{a+b}{2}$

$$= \frac{3\sqrt{5} + 5\sqrt{5}}{2}$$

$$= \frac{8\sqrt{5}}{2} = 4\sqrt{5}$$

(ii) $x - 3$ and $x + 5$

Let $a = x - 3$, $b = x + 5$

As $A.M = \frac{a+b}{2}$

$$A.M = \frac{(x-3)+(x+5)}{2} = \frac{2x+2}{2} = \frac{2(x+1)}{2} = x + 1$$

(iii) $1 - x + x^2$ and $1 + x + x^2$

As $A.M = \frac{a+b}{2}$

$$= \frac{1 - x + x^2 + 1 + x + x^2}{2} = \frac{2 + 2x^2}{2} = \frac{2(1 + x^2)}{2} = 1 + x^2$$

Q.2 If 5, 8 are two A.M's between a and b, find a and b.

(Gujranwala Board 2005, 2007, Lahore Board 2004, 2011)

Solution:

As 5, 8 are A.M's between a and b

so

a, 5, 8, b are in A.P

$$\Rightarrow \quad 5 = \frac{a+8}{2}$$

$$\Rightarrow \quad 10 = a+8 \quad \Rightarrow \quad \boxed{a = 2}$$

and $\quad 8 = \frac{5+b}{2}$

$$16 = 5+b$$

$$\Rightarrow \quad \boxed{b = 11}$$

Q.3 Find 6 A.M's between 2 and 5. (Gujranwala Board 2003)

Solution:

Let $A_1, A_2, A_3, A_4, A_5, A_6$ are required A.M's between 2 and 5

then

$2, A_1, A_2, A_3, A_4, A_5, A_6, 5$ are in A.P

$a_1 = 2, \quad a_8 = 5$

$$\Rightarrow \quad a_1 + 7d = 5$$

$$2 + 7d = 5 \quad \Rightarrow \quad 7d = 5 - 2 = 3$$

$$d = \frac{3}{7}$$

Now

$$A_1 = a_2 = a_1 + d = 2 + \frac{3}{7} = \frac{14+3}{7} = \frac{17}{7}$$

$$A_2 = a_3 = a_1 + 2d = 2 + 2 \cdot \frac{3}{7} = 2 + \frac{6}{7} = \frac{14+6}{7} = \frac{20}{7}$$

$$A_3 = a_4 = a_1 + 3d = 2 + 3 \cdot \frac{3}{7} = 2 + \frac{9}{7} = \frac{14+9}{7} = \frac{23}{7}$$

$$A_4 = a_5 = a_1 + 4d = 2 + 4 \cdot \frac{3}{7} = 2 + \frac{12}{7} = \frac{14+12}{7} = \frac{26}{7}$$

$$A_5 = a_6 = a_1 + 5d = 2 + 5 \cdot \frac{3}{7} = 2 + \frac{15}{7} = \frac{14+15}{7} = \frac{29}{7}$$

$$A_6 = a_7 = a_1 + 6d = 2 + 6 \cdot \frac{3}{7} = 2 + \frac{18}{7} = \frac{14+18}{7} = \frac{32}{7}$$

Hence required.

$\Rightarrow$ $\frac{17}{7}, \frac{20}{7}, \frac{23}{7}, \frac{26}{7}, \frac{29}{7}, \frac{32}{7}$ are required A.Ms.

Q.4 Find four A.M's between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$.

Solution:

Let A_1, A_2, A_3, A_4 are required A.Ms

then

$\sqrt{2}, A_1, A_2, A_3, A_4, \frac{12}{\sqrt{2}}$ are in A.P

$a_1 = \sqrt{2}$,

$a_6 = \frac{12}{\sqrt{2}} \Rightarrow a_1 + 5d = \frac{12}{\sqrt{2}}$

$$\sqrt{2} + 5d = \frac{12}{\sqrt{2}}$$

$$5d = \frac{12}{\sqrt{2}} - \sqrt{2}$$

$$= \frac{12-2}{\sqrt{2}} = \frac{10}{\sqrt{2}}$$

$$d = \frac{10}{5\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$A_1 = a_2 = a_1 + d = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$A_2 = a_3 = a_1 + 2d = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$

$A_3 = a_4 = a_1 + 3d = \sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$

$A_4 = a_5 = a_1 + 4d = \sqrt{2} + 4\sqrt{2} = 5\sqrt{2}$

Are required A.Ms.

Q.5 Insert 7 A.M's between 4 and 8.

Solution:

Let $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ are required A.Ms

then

$4, A_1, A_2, A_3, A_4, A_5, A_6, A_7, 8$

Here

$\boxed{a_1 = 4}$ and $a_9 = 8$

$\Rightarrow a_1 + 8d = 8$

$4 + 8d = 8$

$4 + 8d = 8$

$$8d = 4$$

$$\boxed{d = \frac{1}{2}}$$

So,

$$A_1 = a_2 = a_1 + d = 4 + \frac{1}{2} = \frac{9}{2}$$

$$A_2 = a_3 = a_1 + 2d = 4 + 2 \cdot \frac{1}{2} = 5$$

$$A_3 = a_4 = a_1 + 3d = 4 + 3 \cdot \frac{1}{2} = 4 + \frac{3}{2} = \frac{11}{2}$$

$$A_4 = a_5 = a_1 + 4d = 4 + 4 \cdot \frac{1}{2} = 4 + 2 = 6$$

$$A_5 = a_6 = a_1 + 5d = 4 + 5 \cdot \frac{1}{2} = 4 + \frac{5}{2} = \frac{13}{2}$$

$$A_6 = a_7 = a_1 + 6d = 4 + 6 \cdot \frac{1}{2} = 4 + 3 = 7$$

$$A_7 = a_8 = a_1 + 7d = 4 + 7 \cdot \frac{1}{2} = 4 + \frac{7}{2} = \frac{15}{2}$$

Are required A.Ms.

Q.6 Find three A.Ms between 3 and 11.

Solution:

Let A_1, A_2, A_3 are required A.M's

then

$3, A_1, A_2, A_3, 11$ are in A.P

$\Rightarrow$ $a_1 = 3$ and $a_5 = 11$

$\Rightarrow a_1 + 4d = 11$

$\Rightarrow 3 + 4d = 11$

$4d = 8$

$$\boxed{d = 2}$$

So

$$A_1 = a_2 = a_1 + d = 3 + 2 = 5$$

$$A_2 = a_3 = a_1 + 2d = 3 + 2\,(2) = 7$$

$$A_3 = a_4 = a_1 + 3d = 3 + 3\,(2) = 9$$

are required A.Ms.

Q.7 Find n so that $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ may be A.M. between a and b.

Solution:

As we know that the A.M between a and b is

$$A = \frac{a+b}{2}$$

$$\frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \frac{a+b}{2}$$

$$\frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \frac{a+b}{2}$$

$\Rightarrow \quad 2(a^n + b^n) = (a+b)(a^{n-1} + b^{n-1})$

$\Rightarrow \quad 2a^n + 2b^n = a^n + a^{n-1}b + ab^{n-1} + b^n$

$\Rightarrow \quad 2a^n + 2b^n - a^n - b^n = a^{n-1}b + ab^{n-1}$

$\Rightarrow \quad a^n + b^n = a^{n-1}b + ab^{n-1}$

$\Rightarrow \quad a^n - a^{n-1}b = ab^{n-1} - b^n$

$\Rightarrow \quad a^{n-1}(a-b) = b^{n-1}(a-b)$

$\Rightarrow \quad a^{n-1} = b^{n-1}$

$\Rightarrow \quad \frac{a^{n-1}}{b^{n-1}} = 1$

$\Rightarrow \quad \left(\frac{a}{b}\right)^{n-1} = \left(\frac{a}{b}\right)^0$

$\Rightarrow \quad n - 1 = 0$

$\Rightarrow \quad \boxed{n = 1}$

Q.8 Show that sum of n A.Ms between a and b is equal to n times their A.M.

(Lahore Board 2010)

Solution:

Let $A_1, A_2, A_3, \ldots\ldots A_n$ are n arithmetic means between a and b \

then

a, $A_1, A_2, A_3, \ldots\ldots A_n$, b are in A.P

$\Rightarrow \quad a_1 = a$ and $a_{n+2} = b$

$\Rightarrow \quad a_1 + (n+1)d = b$

$$\Rightarrow \quad a + (n+1)\,d = b$$

$$(n+1)\,d = b - a$$

$$d = \frac{b-a}{n+1}$$

Now

$$A_1 = a_2 = a_1 + d = a + \frac{b-a}{n+1}$$

$$A_2 = a_3 = a_1 + 2d = a + 2 \cdot \frac{b-a}{n+1}$$

$$A_3 = a_4 = a_1 + 3d = a + 3 \cdot \frac{b-a}{n+1}$$

.

.

.

$$A_n = a_{n+1} = a_1 + nd = a + n \cdot \frac{b-a}{n+1}$$

$\therefore$ $A_1 + a_2 + A_3 + \ldots\ldots + A_n$

$$= a + \frac{b-a}{n+1} + a + 2 \cdot \frac{b-a}{n+1} + a + 3 \cdot \frac{b-a}{n+1} + \ldots\ldots + a + n \cdot \frac{b-a}{n+1}$$

$$= na + \frac{b-a}{n+1} + 2\frac{b-a}{n+1} + \ldots\ldots + n \cdot \frac{b-a}{n+1}$$

$$= na + \frac{b-a}{n+1}[1 + 2 + 3 + \ldots\ldots + n]$$

$$= na + \frac{b-a}{n+1}\,\frac{n(n+1)}{2}$$

$$= na + \frac{n(b-a)}{2} = \frac{2na + nb - na}{2}$$

$$= \frac{na + nb}{2} = n \cdot \frac{a+b}{2} = n\,A$$

Hence proved

Series

The sum of an indicated number of terms in a sequence is called a series.

For example,

2, 4, 6, 8, …….. is a sequence

$2 + 4 + 6 + 8 + \ldots\ldots$ is a series

The formula to find the sum of first n terms of an arithmetic series is

$$S_n = \frac{n}{2}(a_1 + a_n)$$

or $$S_n = \frac{n}{2}[2a_1 + (n-1)\,d]$$

EXERCISE 6.4

Q.1 Find the sum of all the integral multiples of 3 between 4 and 97.

Solution:

Integral multiples between 4 and 97 are

6, 9, 12, 15, ... 96

$a_1 = 6, \quad d = 9 - 6 = 3, \quad a_n = 96, \quad n = ?$

As $a_n = a_1 + (n-1)\,d$

$96 = 6 + (n-1)\,3$

$96 - 6 = 3n - 3$

$90 = 3n - 3$

$3n = 90 + 3$

$n = 31$

Now using formula

$$S_n = \frac{n}{2}\{2a_1 + (n-1)\,d\}$$

$$= \frac{31}{2}\{2(6) + (31-1)(3)\}$$

$$= \frac{31}{2}\{12 + 90\}$$

$$= \frac{31}{2}(102) = 1581$$

Q.2 Sum of the series

(i) $-3 + (-1) + 1 + 3 + 5 + \ldots\ldots\ldots a_{16}$ **(Lahore Board 2007)**

(ii) $\frac{3}{\sqrt{2}} + 2\sqrt{2} + \frac{5}{\sqrt{2}} + \ldots\ldots + a_{13}$

(iii) $1.11, + 1.41 + 1.71 + \ldots\ldots\ldots + a_{10}$

(iv) $-8 - \frac{7}{2} + 1 + \ldots\ldots\ldots + a_{11}$

(v) $(x - a) + (x + a) + (x + 3a) + \ldots\ldots\ldots$ **n terms**

(vi) $\frac{1}{1-\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1+\sqrt{x}} \ldots\ldots$ **n terms**

(vii) $\frac{1}{1+\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1-\sqrt{x}} + \ldots\ldots$ **n terms**

Solution:

(i) $-3 + (-1) + 1 + 3 + 5 + \ldots\ldots\ldots a_{16}$

$a_1 = -3, \quad d = -1 - (-3) = 2, \quad n = 16, \quad S_n = ?$

using

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$= \frac{16}{2}\{2(-3) + (16-1)(2)\}$$

$$= 8\{-6 + 30\}$$

$$= 192$$

(ii) $\frac{3}{\sqrt{2}} + 2\sqrt{2} + \frac{5}{\sqrt{2}} + \ldots\ldots + a_{13}$

$a_1 = \frac{3}{\sqrt{2}}, \quad d = 2\sqrt{2} - \frac{3}{\sqrt{2}} = \frac{4-3}{\sqrt{2}} = \frac{1}{\sqrt{2}}, \quad n = 13, \quad S_n = ?$

using

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$= \frac{13}{2}\left\{2\left(\frac{3}{\sqrt{2}}\right) + (13-1)\left(\frac{1}{\sqrt{2}}\right)\right\}$$

$$= \frac{13}{2}\left\{\frac{6}{\sqrt{2}} + \frac{12}{\sqrt{2}}\right\}$$

$$S_{13} = \frac{13}{2} \cdot \frac{18}{\sqrt{2}} = \frac{117}{\sqrt{2}}$$

(iii) $\mathbf{1.11, + 1.41 + 1.71 + ... + a_{10}}$

$a_1 = 1.11, \quad d = 1.41 - 1.11 = 0.30 \quad n = 10, \quad S_n = ?$

using

$$S_n = \frac{n}{2} \{2a + (n-1)\, d\}$$

$$= \frac{10}{2} \{2\,(1.11) + (10-1)\,(0.30)\}$$

$$S_{10} = 5\,\{2.22 + 2.70\}$$

$$= 24.6 \quad \text{Ans.}$$

(iv) $\mathbf{-8 - \frac{7}{2} + 1 + ... + a_{11}}$

Here $a_1 = -8 \quad d = 1 - \left(-3\frac{1}{2}\right) = 1 + \frac{7}{2} = \frac{9}{2}, \quad n = 11, \quad S_n = ?$

As $$S_n = \frac{n}{2} \{2a_1 + (n-1)\, d\}$$

$$= \frac{11}{2} \left\{2\,(-8) + (11-1)\left(\frac{9}{2}\right)\right\}$$

$$= \frac{11}{2} \{-16 + 45\}$$

$$= \frac{11}{2} \{29\} = 159.5$$

(v) $\mathbf{(x - a) + (x + a) + (x + 3a) + ... \text{ n terms}}$

$a_1 = x - a, \quad d = x + a - (x - a) = 2a, \quad n = n, \quad S_n = ?$

As $$S_n = \frac{n}{2} \{2a_1 + (n-1)\, d\}$$

$$= \frac{n}{2} \{2\,(x-a) + (n-1)\,(2a)\}$$

$$= n\,[(x-a) + (n-1)\, a]$$

$$= n\,[x - a + na - a]$$

$$= n\,[x + na - 2a]$$

$$= n\,(x + (n-2)\, a)$$

(vi) $\dfrac{1}{1-\sqrt{x}} + \dfrac{1}{1-x} + \dfrac{1}{1+\sqrt{x}}$ **..... n terms**

$a_1 = \dfrac{1}{1-\sqrt{x}}$, $d = \dfrac{1}{1-x} - \dfrac{1}{1-\sqrt{x}} = \dfrac{1-(1-\sqrt{x})}{1-x} = \dfrac{-\sqrt{x}}{1-x}$

$n = n$, $S_n = ?$ ¤ $1-x = (1+\sqrt{x})(1-\sqrt{x})$

As $S_n = \dfrac{n}{2}\{2a_1 + (n-1)d\}$

$= \dfrac{n}{2}\left\{\dfrac{2}{1-\sqrt{x}} + (n-1)\left(\dfrac{-\sqrt{x}}{1-x}\right)\right\}$

$= \dfrac{n}{2}\left[\dfrac{2}{1-\sqrt{x}} - \dfrac{(n-1)\sqrt{x}}{1-x}\right]$

$= \dfrac{n}{2}\left\{\dfrac{2(1+\sqrt{x}) - (n-1)\sqrt{x}}{1-x}\right\}$ ¤ $1-x = (1+\sqrt{x})(1-\sqrt{x})$

$= \dfrac{n}{2}\left\{\dfrac{2+2\sqrt{x}-n\sqrt{x}+\sqrt{x}}{1-x}\right\}$

$= \dfrac{n}{2}\left\{\dfrac{2+3\sqrt{x}-n\sqrt{x}}{1-x}\right\}$

$= \dfrac{n}{2}\left\{\dfrac{2+(3-n)\sqrt{x}}{1-x}\right\}$

(vii) $\dfrac{1}{1+\sqrt{x}} + \dfrac{1}{1-x} + \dfrac{1}{1-\sqrt{x}}$ **+ ..... n terms**

$a_1 = \dfrac{1}{1+\sqrt{x}}$, $d = \dfrac{1}{1-x} - \dfrac{1}{1+\sqrt{x}} = \dfrac{1-(1-\sqrt{x})}{1-x} = \dfrac{-\sqrt{x}}{1-x}$

$n = n$, $S_n = ?$

As $S_n = \dfrac{n}{2}\{2a + (n-1)d\}$

$= \dfrac{n}{2}\left\{2\dfrac{1}{1+\sqrt{x}} + (n-1)\dfrac{\sqrt{x}}{1-x}\right\}$

$= \dfrac{n}{2}\left\{\dfrac{2}{1+\sqrt{x}} + \dfrac{(n-1)\sqrt{x}}{1-x}\right\}$

$$= \frac{n}{2}\left\{\frac{2\left(1-\sqrt{x}\right)+(n-1)\sqrt{x}}{\left(1-\sqrt{x}\right)(1-x)}\right\}$$

$$= \frac{n}{2}\left\{\frac{2-2\sqrt{x}+n\sqrt{x}-\sqrt{x}}{1-x}\right\}$$

$$= \frac{n}{2}\left\{\frac{2-3\sqrt{x}+n\sqrt{x}}{1-x}\right\}$$

$$= \frac{n}{2}\left\{\frac{2+(n-3)\sqrt{x}}{1-x}\right\}$$

Q.3 How many terms of the series

(i) $-7+(-5)+(-3)+\ldots\ldots$ amount to 65

(ii) $-7+(-4)+(-1)+\ldots\ldots$ amount to 114

Solution:

(i) $-7+(-5)+(-3)+\ldots\ldots$

Here $a_1 = -7 \quad d = -5-(-7) = -5+7 = 2, \quad S_n = 65, \quad n = ?$

As $S_n = \frac{n}{2}\{2a_1+(n-1)d\}$

$$65 = \frac{n}{2}\{2(-7)+(n-1)(2)\}$$

$$65 = \frac{n}{2}\{-14+2n-2\}$$

$$65 = \frac{n}{2}(2n-16)$$

$$65 = n(n-8)$$

$$65 = n^2-8n \Rightarrow n^2-8n-65 = 0$$

$$n^2-13n+5n-65 = 0$$

$$n(n-13)+5(n-13) = 0$$

$$(n-13)(n+5) = 0$$

$\Rightarrow$ $n = 13$ or $n = -5$ (not possible)

$\Rightarrow$ $\boxed{n = 13}$

(ii) **$-7 + (-4) + (-1) +$ ..... amount to 114**

$a_1 = -7 \quad d = -4 - (-7) = -+7 = 3, \quad S_n = 114, \quad n = ?$

As $S_n = \frac{n}{2} \{2a_1 + (n-1)\, d\}$

$114 = \frac{n}{2} \{2(-7) + (n-1)(3)\}$

$114 = \frac{n}{2} [-14 + 3n - 3]$

$114 = \frac{n}{2} [3n - 17]$

$228 = 3n^2 - 17n$

$3n^2 - 17n - 228 = 0$

$3n^2 - 36n + 19n - 228 = 0$

$3n(n-12) - 19(n-12) = 0$

$(3n - 19)(n - 12) = 0$

$n = 12 \quad$ or $\quad n = \frac{-19}{3}$ (not possible)

$\Rightarrow$ $\boxed{n = 12}$

Q.4 Sum the series

(i) **$3 + 5 - 7 + 9 + 11 - 13 + 15 + 17 - 19 +$ ..... to 3n terms.**

(Lahore Board 2007, 2009)

(ii) **$1 + 4 - 7 + 10 + 13 - 16 + 19 + 22 - 25 +$ ..... to 3n terms**

Solution:

(i) **$3 + 5 - 7 + 9 + 11 - 13 + 15 + 17 - 19 +$ ..... + to 3n terms.**

It can be written as

$(3 + 5 - 7) + (9 + 11 - 13) + (15 + 17 - 19) +$ ..... n terms

$1 + 7 + 13 +$ ..... n terms

Now $a_1 = 1, \quad d = 7 - 1 = 6, \quad n = n, \quad S_n = ?$

As $S_n = \frac{n}{2} [2a_1 + (n-1)\, d]$

$= \frac{n}{2} [2(1) + (n-1)(6)]$

$S_n = \frac{n}{2} [2 + 6n - 6]$

$= \frac{n}{2}[6n-4]$

$= n(3n-2)$

(ii) $1+4-7+10+13-16+19+22-25+\ldots\ldots$ to 3n terms

$1+4-7+10+13-16+19+22-25+\ldots\ldots$ to 3n terms

It can be written as

$(1+4-7)+(10+13-16)+(19+22-25)+\ldots\ldots$ n terms

$-2+7+16+\ldots\ldots$ n terms

Now $a_1 = -2,\quad d = 7-(-2) = 9,\quad n = n,\quad S_n = ?$

As $S_n = \frac{n}{2}\{2a_1+(n-1)d\}$

$= \frac{n}{2}\{2(-2)+(n-1)(9)\}$

$= \frac{n}{2}\{-4+9n-9\}$

$= \frac{n}{2}[9n-13]$

Q.5 Find sum of 20 terms of the series whose rth term is 3r + 1

(Gujranwala Board 2007)

Solution:

Given that

$a_r = 3r+1$

Put $r = 1, 2, 3$

$r = 1 \Rightarrow a_1 = 3(1)+1 = 4$

$r = 2 \Rightarrow a_2 = 3(2)+1 = 7$

$r = 3 \Rightarrow a_3 = 3(3)+1 = 10$

$\Rightarrow$ Series will be

$4+7+10+\ldots\ldots$

$\Rightarrow$ $a_1 = 4,\quad d = 7-4 = 3,\quad n = 20,\quad S_n = ?$

As $S_n = \frac{n}{2}[2a_1+(n-1)d]$

$= \frac{20}{2}[2(4)+(20-1)(3)]$

$S_{20} = 10[8+57] = 650$

Q.6 If $S_n = n(2n-1)$ then find the series.

Solution:

Given

Put $n = 1, 2, 3, 4$

$n = 1 \Rightarrow S_1 = 1(2(1)-1) = 1$

$n = 2 \Rightarrow S_2 = 2(2(2)-1) = 2(4-1) = 6$

$n = 3 \Rightarrow S_3 = 3(2(3)-1) = 3(6-1) = 15$

$n = 4 \Rightarrow S_4 = 4(2(4)-1) = 4(8-1) = 28$

Now $a_1 = S_1 = 1$

$a_2 = S_2 - S_1 = 6 - 1 = 5$

$a_3 = S_3 - S_2 = 15 - 6 = 9$

$a_4 = S_4 - S_3 = 28 - 15 = 13$

so, the required series is

$1 + 5 + 9 + 13 + ...$

Q.7 The ratio of the sum of n terms of two series in A.P is $3n + 2 : n + 1$. Find the ratio of their 8th term. (Lahore Board 2004, 2006)

Solution:

Let two A.Ps with a, d and a′ , d′

$$\Rightarrow \quad \frac{\frac{n}{2}[2a + (n-1)d]}{\frac{n}{2}[2a' + (n-1)d']} = \frac{3n+2}{n+1}$$

$$\Rightarrow \quad \frac{2a + (n-1)d}{2a' + (n-1)d'} = \frac{3n+2}{n+1} \qquad \ldots\ldots\ldots (1)$$

To find the ratio of their 8^{th} terms

i.e. $$\frac{a_8}{a_8'} = \frac{a + 7d}{a' + 7d}$$

we will put $n = 15$ in equation (1) we get

$$\frac{2a + (15-1)d}{2a' + (15-1)d'} = \frac{3(15)+2}{15+1}$$

$$\Rightarrow \quad \frac{2a + 14d}{2a' + 14d'} = \frac{47}{16}$$

$$\frac{a + 7d}{a' + 7d'} = \frac{47}{16}$$

Q.8 If S_2, S_3, S_5 are sum of 2n, 3n, 5n terms of the A.P. Show that $S_5 = 5(S_3 - S_2)$ (Lahore Board 2011)

Solution:

As $S_n = \frac{2n}{2}[2a + (2n-1)d]$

It is given that

$S_2 = \frac{2n}{2}[2a + (2n-1)d]$ ..... (1)

$S_3 = \frac{3n}{2}[2a + (3n-1)d]$ ..... (2)

$S_5 = \frac{5n}{2}[2a + (5n-1)d]$ ..... (3)

Now from (1), (2)

$$S_3 - S_2 = \frac{3n}{2}[2a_1 + (3n-1)d] - \frac{2n}{2}[2a + (2n-1)d]$$

$$= \frac{n}{2}[6a + (9n-3)d] - \frac{n}{2}[4a + (4n-2)d]$$

$$= \frac{n}{2}[6a + 9nd - 3d] - \frac{n}{2}[4a + 4nd - 2d]$$

$$= \frac{n}{2}[6a + 9nd - 3d - 4a - 4nd + 2d]$$

$$= \frac{n}{2}[2a + 5nd - d]$$

$$= \frac{n}{2}[2a + (5n-1)d]$$

$$5(S_3 - S_2) = \frac{5n}{2}[2a + (5n-1)d]$$

$$= S_5 \text{ from equation (3)}$$

Hence proved.

Q.9 Obtain the sum of all the integers in the first 1000 integers which are neither divisible by 5 nor 2.

Solution:

The integers not divisible by 5 and 2

$\underline{1 + 3 + 7 + 9} + \underline{11 + 13 + 17 + 19} + \underline{21 + 23 + 27 + 29} + \ldots\ldots + \underline{991 + 993 + 997 + 999}$

$= 20 + 60 + 100 + \ldots\ldots + 3980$ which is A.P

$a_1 = 20, \quad d = 60 - 20 = 40, \quad a_n = 3980$

so using

$a_n = a_1 + (n - 1)\, d$

$3980 = 20 + (n - 1)\, 40$

$3980 = 20 + 40n - 40$

$3980 = 60 + 40n$

$3980 - 60 = 40n \quad \Rightarrow \quad n = 100$

$$S_n = \frac{n}{2}\,[2a_1 + (n - 1)\, d]$$

$$= \frac{100}{2}\,[2\,(20) + (100 - 1)\, 40]$$

$$= 50\,[40 + 3960]$$

$S_{100} = 50\,(4000) = 200000$

Q.10 S_8 and S_9 are sum of the first eight and nine terms of an A.P. Find S_9 if $50\, S_9 = 63\, S_8$ and $a_1 = 2$

Solution:

It is given that

$50\, S_9 = 63\, S_8$

$$50 \cdot \frac{9}{2}\,[2a_1 + (9 - 1)\, d] = 63 \cdot \frac{8}{2}\,[2a_1 + (8 - 1)\, d]$$

$25 \times 9\,[2\,(2) + 8d] = 63 \times 4\,[2\,(2) + 7d]$

$225\,(4 + 8d) = 252\,(4 + 7d)$

$900 + 1800d = 1008 + 1764d$

$900 - 1008 + 1800d - 1764d = 0$

$-108 + 36d = 0$

$36d = 108$

$$d = \frac{108}{36} \quad \Rightarrow \quad \boxed{d = 3}$$

So $S_9 = \frac{9}{2}\,[2a_1 + (9 - 1)\, d]$

$$= \frac{9}{2}\,[2\,(2) + 8\,(3)]$$

$$= \frac{9}{2}\,[4 + 24] = \frac{9}{2} \times 28 = 126$$

Q.12 The sum of S_9 and S_7 is 203 and $S_9 - S_7 = 49$ and S_7 and S_9 being sum of 7 or 9 terms of an A.P. respectively. Determine the series.

Solution:

Given that

$S_9 + S_7 = 203$ ..... (1)

$S_9 - S_7 = 49$ ..... (2)

Adding (1) and (2), we get

$2S_n = 252 \Rightarrow S_9 = 126$

Subtract (2) from (1), we get

$2S_7 = 154 \Rightarrow S_7 = 77$

$S_9 = 126$

$\Rightarrow \frac{9}{2}[2a_1 + (9-1)d] = 126$

$\Rightarrow \frac{9}{2}[2a + 8d] = 126$

$\Rightarrow 9(a + 4d) = 126$

$\Rightarrow a + 4d = 14$ ..... (3)

As $S_7 = 77$

$\Rightarrow \frac{7}{2}[2a_1 + 6d] = 77$

$\Rightarrow 7(a + 3d) = 77$

$\Rightarrow a + 3d = 11$ ..... (4)

Subtracting (4) from (3)

$d = 14 - 11 = 3$

Put $d = 3$ in (3)

$a + 4(3) = 14$

$a + 12 = 14 \Rightarrow a = 2$

so $a_1 = 2$

$a_2 = a_1 + d = 2 + 3 = 5$

$a_3 = a_1 + 2d = 2 + 2(3) = 8$

$\Rightarrow$ required series will be $2 + 5 + 8 + \ldots\ldots$

Q.13 S_7 and S_9 are the sums of the first seven and nine terms of an A.P. respectively if $\frac{S_9}{S_7} = \frac{18}{11}$ and $a_7 = 20$ find the series.

Solution:

Given that

$$\frac{S_9}{S_7} = \frac{18}{11}$$

$$\Rightarrow \quad \frac{\frac{9}{2}(2a_1 + 8d)}{\frac{7}{2}(2a_1 + 6d)} = \frac{18}{11}$$

$$\Rightarrow \quad \frac{9(a_1 + 4d)}{7(a_1 + 3d)} = \frac{18}{11}$$

$$\Rightarrow \quad 126(a_1 + 3d) = 99(a_1 + 4d)$$

$$\Rightarrow \quad 126a_1 + 378d = 99a_1 + 396d$$

$$\Rightarrow \quad 27a_1 - 18d = 0$$

$$\Rightarrow \quad 3a_1 - 2d = 0 \qquad \ldots\ldots\ldots (1)$$

Also given that

$$a_7 = 20 \quad \Rightarrow \quad a_1 + 6d = 20 \qquad \ldots\ldots\ldots (2)$$

Put $a_1 = 20 - 6d$ from (2) in (1)

$$3(20 - 6d) - 2d = 0$$

$$60 - 18d - 2d = 0$$

$$60 - 20d \quad \Rightarrow \quad \boxed{d = 3}$$

Put $d = 3$ in (1), we get

$$3a_1 - 2(3) = 0$$

$$3a_1 - 6 = 0 \quad \Rightarrow \quad 3a_1 = 6 \quad \Rightarrow \quad \boxed{a_1 = 2}$$

so $\quad a_2 = a_1 + d = 2 + 3 = 5$

$$a_3 = a_1 + 2d = 2 + 2(3) = 8$$

so the required series is

$$2 + 5 + 8 + \ldots\ldots\ldots$$

Q.14 The sum of three numbers in an A.P. is 24 and their product is 440. Find the numbers. (Lahore Board 2005, 2010)

Solution:

Let the required numbers are $a - d, a, a + d$

then as given

$$a - d + a + a + d = 24$$

$$\Rightarrow \quad 3a = 24 \quad \Rightarrow \quad \boxed{a = 8}$$

and $(a - 8)(a)(a + d) = 440$

$$\Rightarrow \quad a(a^2 - d^2) = 440$$

$$\Rightarrow \quad 8(64 - d^2) = 440 \qquad \because \ a = 8$$

$$\Rightarrow \quad 64 - d^2 = 55$$

$$\Rightarrow \quad 64 - 55 = d^2 \quad \Rightarrow \quad d^2 = 9 \quad \Rightarrow \quad d = \pm 3$$

when $a = 8$ and $d = 3$

then numbers are

$$a - d = 8 - 3 = 5$$

$$a = 8$$

$$a + d = 8 + 3 = 11$$

when $a = 8$ and $d = -3$

then numbers are

$$a - d = 8 + 3 = 11$$

$$a = 8$$

$$a + d = 8 - 3 = 5$$

so required numbers are

5, 8, 11 or 11, 8, 5.

Q.15 Find four numbers in A.P. whose sum is 32 and the sum of whose squares is 276.

Solution:

Let the required numbers are $a - 3d$, $a - d$, $a + d$, $a + 3d$ then it is given that

$$a - 3d + a - d + a + d + a + 3d = 32$$

$$4a = 32 \quad \Rightarrow \quad \boxed{a = 8}$$

Also given that

$(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 276$

$4a^2 = 20d^2 = 276$

$a^2 = 5d^2 = 69$

$5d^2 = 69 - a^2 = 69 - (8)^2 = 69 - 64$ ⌑ $a = 8$

$5d^2 = 5 \Rightarrow \boxed{d = \pm 1}$

when $a = 8, \quad d = 1$

$a - 3d = 8 - 3(1) = 5$

$a - d = 8 - 1 = 7$

$a + d = 8 + 1 = 9$

$a + 3d = 8 + 3(1) = 11$

when $a = 8, \ d = -1$

$a - 3d = 8 - 3(-1) = 11$

$a - d = 8 - (-1) = 9$

$a + d = 8 - 1 = 7$

$a + 3d = 8 + 3(-1) = 5$

so required numbers are

5, 7, 9, 11 or 11, 9, 7, 5

Q.16 Find the five numbers in A.P. whose sum is 25 and the sum of whose squares is 135. (Lahore Board 2003)

Solution:

Let the required numbers are $a - 2d, \ a - d, \ a, \ a + d, \ a + 2d$ then it is given that

$a - 2d + a - d + a + a + d + a + 2d = 25$

$5a = 25 \Rightarrow \boxed{a = 5}$

also it is given that

$(a-2d)^2 + (a-d)^2 + a^2 + (a+d)^2 + (a+2d)^2 = 135$

$a^2 + 4d^2 - 4da + a^2 + d^2 - 2ad + a^2 + a^2 + d^2 + 2ad + a^2 + 4d^2 + 4ad = 135$

$5a^2 + 10d^2 = 135$

$a^2 + 2d^2 = 27$

$2d^2 = 27 - a^2 = 27 - (5)^2 = 27 - 25 = 2$

$d^2 = 1 \Rightarrow \boxed{d = \pm 1}$

when $a = 5, \quad d = 1$

$a - 2d = 5 - 2(1) = 3$

$a - d = 5 - 1 = 4$

$a = 5$

$a + d = 5 + 1 = 6$

$a + 2d = 5 + 2(1) = 7$

when $a = 5,\ d = -1$

$a - 2d = 5 - 2(-1) = 7$

$a - d = 5 - (-1) = 6$

$a = 5$

$a + d = 5 + (-1) = 4$

$a + 2d = 5 + 2(-1) = 3$

so the required numbers are

3, 4, 5, 6, 7 or 7, 6, 5, 4, 3.

Q.17 The sum of the 6^{th} and 8^{th} terms of an A.P. is 40 and the product of 4^{th} and 7^{th} terms is 220. Find the A.P.

Solution:

Given that

$a_6 + a_8 = 40$

$\Rightarrow \quad a + 5d + a + 7d = 40$

$\Rightarrow \quad 2a + 12d = 40$

$\Rightarrow \quad a + 6d = 20 \qquad \ldots\ldots\ldots (1)$

Also it is given that

$a_4 \cdot a_7 = 220$

$\Rightarrow \quad (a + 3d)(a + 6d) = 220$

$\Rightarrow \quad (a + 3d)(20) = 220$ from (1)

$\Rightarrow \quad a + 3d = \frac{220}{20}$

$\Rightarrow \quad a + 3d = 11 \qquad \ldots\ldots\ldots (2)$

Subtracting (2) from (1), we get

$3d = 9 \quad \Rightarrow \quad \boxed{d = 3}$

Put $d = 3$ in (1), we get

$a + 6(3) = 20$

$a + 18 = 20 \Rightarrow \boxed{a = 2}$

$\Rightarrow \quad a_1 = 2$

$a_2 = a_1 + d = 2 + 3 = 5$

$a_3 = a_1 + 2d = 2 + 2(3) = 8$

$\Rightarrow$ required A.P. is 2, 5, 8, ...

Q.18 If a^2, b^2 and c^2 are in A.P. show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

Solution:

If a^2, b^2, c^2 are in A.P.

$\Rightarrow \quad b^2 - a^2 = c^2 - b^2$

To show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P. we will prove that

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

or $$\frac{b+c-(c+a)}{(c+a)(b+c)} = \frac{c+a-(a+b)}{(a+b)(c+a)}$$

or $$\frac{b+c-c-a}{b+c} = \frac{c+a-a-b}{a+b}$$

or $$\frac{b-a}{b+c} = \frac{c-b}{a+b}$$

or $(b-a)(b+a) = (c-b)(c+b)$

or $b^2 - c^2 = c^2 - b^2$ (given)

Hence $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

EXERCISE 6.5

Q.1 A man deposits in a bank Rs. 10 in the first month Rs. 15 in the second month, Rs. 20 in the third month and so on. Find how much he will have deposited in the bank by 9th months.

Solution:

Deposited amount is

10, 15, 20, ... which is A.P

Here $a_1 = 10, \quad d = 15 - 10 = 5, \quad n = 9$

We find S_9

As $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$= \frac{9}{2}[2(10) + (9-1)5]$

$= \frac{9}{2}[20 + 40]$

$= \frac{9}{2}\{60\} = 9(30) = \text{Rs. } 270$

Q.2 378 trees are planted in rows in the shape of an isosceles triangle. The number in the successive rows, decreasing by one from the base to the top. How many trees are there in the row which forms the base of the triangle.

Solution:

Let $a_1 = 1$ is the topmost row and a_n is the base row then sequence of trees is

1, 2, 3, ...

$\Rightarrow$ $1 + 2 + 3 + ...a_n = 378 \Rightarrow a_1 = 1, \quad d = 1$

$\Rightarrow$ $S_n = 378$

$\Rightarrow$ $\frac{n}{2}[2a_1 + (n-1)d] = 378$

$\Rightarrow$ $\frac{n}{2}[2(1) + (n-1)(1)] = 378$

$\Rightarrow$ $n[2 + n - 1] = 756$

$\Rightarrow$ $n(n+1) = 756$

$\Rightarrow$ $n^2 + n - 756 = 0$

$\Rightarrow$ $n^2 + 28n - 27n - 756 = 0$

$\Rightarrow$ $n(n+28) - 27(n+28) = 0$

$\Rightarrow$ $(n-27)(n+28) = 0$

$\Rightarrow$ $n = 27 \qquad n = -28$ (not possible)

$\Rightarrow$ we have to find a_{27}.

As $a_1 = 1, \quad a_2 = 2,$ and so on

$\Rightarrow$ $a_{27} = 27$

Q.3 A man borrows Rs. 1100 and agrees to repay with a total interest of Rs. 230 in 14 installments, each installment is less than the preceding by Rs. 10. What should be his first installment?

Solution:

Total amount to repay = Rs. 1100 + 230 = Rs. 1330

$d = -10$, $n = 14$, $S_n = 1330$, $a_1 = ?$

As

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$

$$1330 = \frac{14}{2}[2a_1 + (14-1)(-10)]$$

$$1330 = 7(2a_1 - 130)$$

$$\frac{1330}{7} = 2a_1 - 130$$

$$190 = 2a_1 - 130$$

$$190 + 130 = 2a_1$$

$$320 = 2a_1$$

$$\boxed{a_1 = 160}$$

$\Rightarrow$

First installment will be Rs. 160.

Q.4 A clock strikes once when its hour hand is at one, twice, when it is at two and so on. How many times a clock strikes in twelve hours. (Lahore Board 2007)

Solution:

The sequence is

1, ,2, 3, ..., 12 (which is an A.P)

$a_1 = 1$, $d = 1$, $n = 12$, $S_{12} = ?$

As $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$$= \frac{12}{2}[2(1) + (12-1)(1)]$$

$$= 6(2 + 11) = 6(13) = 78$$

Q.5 A student saves Rs. 12 at the end of the first week and goes on increasing his saving Rs. 4 weekly. After how many weeks will he be able to save Rs. 2100.

Solution:

The sequence is

12, 16, 20, ... (which is an A.P)

$a_1 = 12 \quad d = 16\ 12 = 4 \quad n = ? \quad S_n = 2100$

As

$$\Rightarrow \quad S_n = \frac{n}{2}[2a_1 + (n-1)\,d]$$

$$\Rightarrow \quad 2100 = \frac{n}{2}[2(12) + (n-1)\,4]$$

$$\Rightarrow \quad 2100 = \frac{n}{2}[24 + 4n - 4]$$

$$\Rightarrow \quad 2100 = \frac{n}{2}[4n + 20]$$

$$\Rightarrow \quad 2100 = n(2n + 10)$$

$$\Rightarrow \quad 2100 = 2n(n + 5)$$

$$\Rightarrow \quad 1050 = n(n + 5)$$

$$\Rightarrow \quad n^2 + 5n - 1050 = 0$$

$$\Rightarrow \quad n^2 + 35n - 30n - 1050 = 0$$

$$\Rightarrow \quad n(n + 35) - 30(n + 35) = 0$$

$$\Rightarrow \quad (n - 30)(n + 35) = 0$$

$$\Rightarrow \quad n = 30 \quad \text{or} \quad n = -35 \quad \text{(not possible)}$$

$\Rightarrow$ no. of weeks = 30 weeks.

Q.6 An object falling from rest, fall 9 meter during the first second, 27 meters during next second, 45 meters during the third second and so on.

(i) How far will it fall during the fifth second?

(ii) How far will it fall upto the fifth second?

Solution:

Given that

(i) The sequence is

9, 27, 45, ... (which is an A.P.)

$a_1 = 9,\ d = 18,\ n = 5,\ a_5 = ?$

As $a_n = a_1 + (n-1)\,d$

$a_5 = 9 + (5-1)\,(18) = 9 + 4\,(18) = 9 + 72 = 81\text{m}$

(ii) To find S_5

As $S_n = \frac{n}{2}\left[2a_1 + (n-1)\,d\right]$

$S_5 = \frac{5}{2}\left[2\,(9) + (5-1)\,(18)\right]$

$= \frac{5}{2}\left[18 + 4\,(18)\right] = \frac{5}{2}\left[90\right] = 225\text{m}$

Q.7 An investor earned Rs. 6000 for year 1980 and Rs. 12000 for the year 1990 on the same investment. If his earning have increased by the same amount each year, how much income he has received from the investment over the past eleven years?

Solution:

Earning in 1980 $= a_1 =$ Rs. 6000

Earning in 1990 $= a_{11} =$ Rs. 12000

As $a_{11} = a_1 + 10d$

$\Rightarrow$ $12000 = 6000 + 10d$

$12000 - 6000 = 10d$

$6000 = 10d$

$d = 600$

Now we find S_{11}

As $S_n = \frac{n}{2}\left[2a_1 + (n-1)\,d\right]$

$S_{11} = \frac{11}{2}\left[2\,(6000) + (11-1)\,(600)\right]$

$= \frac{11}{2}\left[12000 + 10\,(600)\right]$

$= \frac{11}{2}\left[18000\right] = 99000$

$\Rightarrow$ Total earning in 11 years = Rs. 99000

Q.8 The sum of interior angles of polygons having sides 3, 4, 5, ... etc. form an A.P. Find the sum of the interior angles for a 16 sided polygon.

Solution:

Sum of interior angles of 3–sided polygon (triangle) $= \pi$

Sum of interior angles of 4–sided polygon $= 2\pi$

Sum of interior angles of 16–sided polygon $= ?$

The sequence is $\pi, 2\pi, 3\pi, \ldots\ldots$ (which is an A.P.)

$a_1 = \pi, \quad d = 2\pi - \pi = \pi, \quad n = 14, \quad a_{14} = ?$

As $a_n = a_1 + (n-1)\,d$

$a_{14} = \pi + (14-1)(\pi) = \pi + 13\pi$

$a_{14} = 14\pi$

$\Rightarrow$ sum of interior angles of 14–sided polygon $= 14\pi$

Q.9 The prize money of Rs. 60,000 will be distributed among the eight teams according to their positions determined in the match–series. The award increases by the same amount for each higher position. If the last place team is given Rs. 4000, how much will be awarded to the first place team?

Solution:

The last place team is given $= a_8 =$ Rs. 4000

$S_8 = 60{,}000, \quad n = 8, \quad a_1 = ?$

As $S_n = \frac{n}{2}[a_1 + a_n]$

$S_8 = \frac{8}{2}[a_1 + a_8]$

$60{,}000 = 4\,[a_1 + 4000]$

$15000 = a_1 + 4000$

$15000 - 4000 = a_1$

$a_1 = 1100$

First place team will be awarded $=$ Rs. 1100

Q.10 An equilateral triangular base is filled by placing eight balls in the first row, 7 balls in the second row and so on with one ball in the last row and so on with one ball at the row. After this base layer, second layer is formed by placing 7 balls in its first row, 6 balls in its second row and so on with one ball in its last row. Continuing this process, a pyramid of balls is formed with one ball on top. How many balls are there in the pyramid?

Solution:

Balls in the 1^{st} layer $= 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$

Balls in the 2^{nd} layer $= 7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$

Balls in the 3^{rd} layer $= 6 + 5 + 4 + 3 + 2 + 1 = 21$

Balls in the 4^{th} layer $= 5 + 4 + 3 + 2 + 1 = 15$

Balls in the 5^{th} layer $= 4 + 3 + 2 + 1 = 10$

Balls in the 6^{th} layer $= 3 + 2 + 1 = 6$

Balls in the 7^{th} layer $= 2 + 1 = 3$

Balls in the 8^{th} layer $= 1 = 1$

Total Balls $= 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1 = 120$

GEOMETRIC PROGRESSION OR SEQUENCE (G.P):

A sequence $\{a_n\}$ is a geometric sequence or geometric progression if $\frac{a_n}{a_{n-1}}$ is the same non–zero number for all $n \in N$ and $n > 1$. The quotient $\frac{a_n}{a_{n-1}}$ is usually denoted by r and is called common ratio of the G.P.

General Term of G.P.

General term or nth term of G.P. is given by

$a_n = a_1 r^{n-1}$

EXERCISE 6.6

Q.1 Find the 5^{th} term of G.P. 3, 6, 12, ...

(Lahore Board 2006, Gujranwala Board 2007)

Solution:

Given sequence

3, 6, 12, ...

$a_1 = 3, \quad r = \frac{6}{3} = 2, \quad n = 5$

As $a_n = a_1 r^{n-1}$

$a_5 = (3)(2)^{5-1} = 3(2)^4 = 3(16) = 48$

Q.2 Find the 11^{th} term of the sequence $1 + i, 2, \frac{4}{1+i}$ (Lahore Board 2011)

Solution:

Given sequence

$1 + i, \ 2, \ \frac{4}{1+i}, \ldots\ldots$

$a_1 = 1 + i, \quad r = \frac{1}{1+i}, \quad n = 11$

As $a_n = a_1 r^{n-1}$

$$a_{11} = (1+i)\left(\frac{2}{1+i}\right)^{10}$$

$$= (1+i)\frac{2^{10}}{(1+i)^{10}} = \frac{2^{10}(1+i)}{\left[(1+i)^2\right]^5}$$

$$= \frac{2^{10}(1+i)}{(1+i^2+2i)^5} = \frac{2^{10}(1+i)}{(1-1+2i)^5}$$

$$= \frac{2^{10}(1+i)}{(2i)^5} = \frac{2^{10}(1+i)}{2^5 . i^5}$$

$$= 2^5 \frac{1+i}{i} = 2^5 \frac{1+i}{i} \times \frac{i}{i}$$

$$= 2^5 . \frac{(1+i)\,i}{i^2} = -32\,(i+i^2)$$

$$= -32\,(i-1) = 32\,(1-i)$$

Q.3 Find the 12^{th} term of $1 + i, 2i, -2 + 2i, \ldots\ldots$

Solution:

Given sequence

$1 + i,\ 2i,\ -2 + 2i, \ldots\ldots$

$a_1 = 1 + i, \quad r = \frac{2i}{1+i}, \quad n = 12, \quad a_{12} = ?$

As $a_n = a_1 r^{n-1}$

$$a_{12} = (1+i)\left(\frac{2i}{1+i}\right)^{11}$$

$$= \frac{(1+i)(2i)^{11}}{(1+i)^{11}}$$

$$= \frac{2^{11} . i^{11}}{(1+i)^{10}}$$

$$= \frac{2^{11} . (i^2)^5 . i}{\left[(1+i)^2\right]^5} = \frac{2^{11} . (-1)^5 . i}{(1+i^2+2i)^5}$$

$$= \frac{-2^{11} . i}{(1-1+2i)^5} = \frac{-i . 2^{11}}{2^5 . i^5}$$

$$= -\frac{2^6}{i^4} = -\frac{2^6}{(-1)^2} = -2^6 = -64$$

Q.4 Find the 11^{th} term of the sequence $1 + i, 2, 2(1 - i), \ldots\ldots$

(Lahore Board 2005)

Solution:

Given sequence

$1 + i, 2, 2(1 - i), \ldots\ldots$

$a_1 = 1 + i, \quad r = \frac{2}{1+i}, \quad n = 11$

As $a_n = a_1 r^{n-1}$

$$a_{11} = (1+i)\left(\frac{2}{1+i}\right)^{10}$$

$$= \frac{(1+i).2^{10}}{(1+i)^{10}} = \frac{2^{10}(1+i)}{[(1+i)^2]^5}$$

$$= \frac{2^{10}(1+i)}{(1+i^2+2i)^5} = \frac{2^{10}(1+i)}{(1-1+2i)^5}$$

$$= \frac{2^{10}(1+i)}{(2i)^5} = \frac{2^{10}(1+i)}{2^5.i^5} = \frac{2^5(1+i)}{(i^2)^2.i}$$

$$= \frac{3^2(1+i)}{(-1)^2.i} = \frac{32(1+i)}{i}.\frac{i}{i}$$

$$= \frac{32(i+i^2)}{i^2} = -32(i-1) = 32(1-i)$$

Q.5 If an automobile depreciates in values 5% every year, at the end of 4 years. What is the value of the automobile purchased for Rs. 12,000?

Solution:

$$5\% = \frac{5}{100} = 0.05$$

The value in first year = Rs. 12000

At the end of 1^{st} year = 12000 – 5% of 12000 = 12000 – (0.05) (12000) = 11400

At the end of 2^{nd} year = 11400 – 5% 11400 = 11400 – 570 = 10830

At the end of 3^{rd} year = 10830 – 5% 10830 = 10288.5

At the end of 4^{th} year = 10288.5 – 5% 10288.5 = 9774

So at the end of 4 year its value is = Rs. 9774

Q.6 Which term of the sequence is $x^2 - y^2, x + y, \frac{x+y}{x-y}, \ldots\ldots$ is $\frac{x+y}{(x-y)^9}$?

Solution:

Given sequence is

$$x^2 - y^2, x + y, \frac{x+y}{x-y}, \ldots\ldots$$

$$a_1 = x^2 - y^2, \quad r = \frac{x+y}{x^2-y^2} = \frac{1}{x-y}, \quad a_n = \frac{x+y}{(x-y)^9}, \quad n = ?$$

As $a_n = a_1 r^{n-1}$

$$\Rightarrow \quad \frac{x+y}{(x-y)^9} = (x^2 - y^2) \cdot \left(\frac{1}{x-y}\right)^{n-1}$$

$$\Rightarrow \quad \frac{x+y}{(x-y)^9} = \frac{(x+y)(x-y)}{(x-y)^{n-1}}$$

$$\Rightarrow \quad \frac{x+y}{(x-y)^9} = \frac{x+y}{(x-y)^{n-2}}$$

$$\Rightarrow \quad \frac{1}{(x-y)^9} = \frac{1}{(x-y)^{n-2}}$$

$$\Rightarrow \quad 9 = n - 2$$

$$\Rightarrow \quad \boxed{n = 11}$$

Q.7 If a, b, c, d are in G.P. Prove that (Gujranwala Board 2003)

(i) $a - b, b - c, c - d,$ are in G.P.

(ii) $a^2 - b^2, b^2 - c^2, c^2 - d^2,$ are in G.P.

(iii) $a^2 + b^2, b^2 + c^2, c^2 + d^2,$ are in G.P.

Solution:

Given that

a, b, c, d are in G.P.

$$\Rightarrow \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r \text{ (say)}$$

$$\Rightarrow \quad b = ar \qquad \ldots\ldots\ldots (1)$$

$$\Rightarrow \quad c = br = ar \cdot r = ar^2 \qquad \ldots\ldots\ldots (2)$$

$$\Rightarrow \quad d = cr = ar^2 \cdot r = ar^3 \qquad \ldots\ldots\ldots (3)$$

(i) To prove $a - b, b - c, c - d$ are in G.P. we will prove that

$$\frac{b-c}{a-b} = \frac{c-d}{b-c}$$

Take L.H.S. $= \dfrac{b-c}{a-b}$

$= \dfrac{ar - ar^2}{a - ar}$ from (1) and (2)

$= \dfrac{ar(1-r)}{a(1-r)} = r$

Now R.H.S. $= \dfrac{c-d}{b-c}$

$= \dfrac{ar^2 - 2r^3}{ar - ar^2}$ from (1), (2), (3)

$= \dfrac{ar^2(1-r)}{ar(1-r)} = r$

$\Rightarrow$ L.H.S. = R.H.S.

$\Rightarrow$ $a-b,\ b-c,\ c-d$ are in G.P.

(ii) To prove $a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G.P. we will prove that

$\dfrac{b^2-c^2}{a^2-b^2} = \dfrac{c^2-d^2}{b^2-c^2}$

Take L.H.S. $= \dfrac{b^2-c^2}{a^2-b^2}$

$= \dfrac{(ar)^2 - (ar^2)^2}{a^2 - (ar)^2}$ from (1) and (2)

$= \dfrac{a^2r^2 - a^2r^4}{a^2 - a^2r^2} = \dfrac{a^2r^2(1-r^2)}{a^2(1-r^2)} = r^2$

Now R.H.S. $= \dfrac{c^2-d^2}{b^2-c^2}$

$= \dfrac{(ar^2)^2 - (ar^3)^2}{(ar)^2 - (ar^2)^2}$ from (1), (2), (3)

$= \dfrac{a^2r^4 - a^2r^6}{a^2r^2 - a^2r^4} = \dfrac{a^2r^4(1-r^2)}{a^2r^2(1-r^2)} = r^2$

$\Rightarrow$ L.H.S. = R.H.S.

$\Rightarrow$ $a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G.P.

(iii) To prove $a^2 + b^2,\ b^2 + c^2,\ c^2 + d^2$ are in G.P. we will prove that

$$\frac{b^2 + c^2}{a^2 + b^2} = \frac{c^2 + d^2}{b^2 + c^2}$$

Take L.H.S. $= \dfrac{b^2 + c^2}{a^2 + b^2}$

$$= \frac{(ar)^2 + (ar^2)^2}{a^2 + (ar)^2} \quad \text{from (1) and (2)}$$

$$= \frac{a^2r^2 + a^2 r^4}{a^2 + a^2 r^2)} = \frac{a^2 r^2 (1 + r^2)}{a^2 (1 + r^2)} = r^2$$

Now R.H.S. $= \dfrac{c^2 + d^2}{b^2 + c^2}$

$$= \frac{(ar^2)^2 + (ar^3)^2}{(ar)^2 + (ar^2)^2} \quad \text{from (1), (2) and (3)}$$

$$= \frac{a^2 r^4 + a^2 r^6}{a^2 r^2 + a^2 r^4} = \frac{a^2 r^4 (1 + r^2)}{a^2 r^2 (1 + r^2)} = r^2$$

$\Rightarrow$ L.H.S. = R.H.S.

$\Rightarrow$ $a^2 + b^2,\ b^2 + c^2,\ c^2 + d^2$ are in G.P.

Q.8 Show that the reciprocals of the terms of geometric sequence a_1, a_1r^2, a_1r^4 ..... form another geometric sequence.

Solution:

Given G.P. is $a_1, a_1 r^2, a_1 r^4, \ldots\ldots$

We will show

$$\frac{1}{a_1}, \frac{1}{a_1 r^2}, \frac{1}{a_1 r^4} \ldots\ldots \text{ is G.P.}$$

For this take $r = \dfrac{\dfrac{1}{a_1 r^2}}{\dfrac{1}{a_1}} = \dfrac{1}{a_1 r^2} \times a_1 = \dfrac{1}{r^2}$

and $r = \dfrac{\dfrac{1}{a_1 r^4}}{\dfrac{1}{a_1 r^2}} = \dfrac{1}{a_1 r^4} \times a_1 r^2 = \dfrac{1}{r^2}$

$\Rightarrow$ $\dfrac{1}{a_1}, \dfrac{1}{a_1 r^2}, \dfrac{1}{a_1 r^4}, \ldots\ldots$ is also G.P.

Q.9 Find the nth term of geometric sequence, if $\frac{a_5}{a_3} = \frac{4}{9}$ and $a_2 = \frac{4}{9}$

Solution:

Given that

$$\frac{a_5}{a_3} = \frac{4}{9}$$

$$\frac{a_1 r^4}{a_1 r^2} = \frac{4}{9} \quad \Rightarrow \quad r^2 = \frac{4}{9} \quad \Rightarrow \quad \boxed{r = \pm\frac{2}{3}}$$

Also given that

$$a_2 = \frac{4}{9} \quad \Rightarrow \quad a_1 r = \frac{4}{9} \quad \Rightarrow \quad a_1 = \frac{4}{9r}$$

when $r = \frac{2}{3} \quad \Rightarrow \quad a_1 = \frac{4}{9 \cdot \left(\frac{2}{3}\right)} = \frac{2}{3}$

when $r = -\frac{2}{3} \quad \Rightarrow \quad a_1 = \frac{4}{9\left(-\frac{2}{3}\right)} = -\frac{2}{3}$

so when $a_1 = \frac{2}{3}, \quad r = \frac{2}{3}$

then $a_n = a_1 r^{n-1} = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1} = \left(\frac{2}{3}\right)^n$

when $a_1 = -\frac{2}{3}, \quad r = -\frac{2}{3}$

then $a_n = a_1 r^{n-1} = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)^{n-1} = (-1)\left(\frac{2}{3}\right)(-1)^{n-1}\left(\frac{2}{3}\right)^{n-1} = (-1)^n\left(\frac{2}{3}\right)^n$

Q.10 Find three consecutive numbers in G.P whose sum is 26 and their product is 216.

Solution:

Let the required numbers are $\frac{a}{r}$, a, ar

then $\frac{a}{r} + a + ar = 26 \qquad \ldots\ldots\ldots (1)$

and $\frac{a}{r} \cdot a \cdot ar = 216$

$a^3 = 216 \Rightarrow \boxed{a = 6}$

Put $a = 6$ in equation (1), we get

$\frac{6}{r} + 6 + 6r = 26$

$\Rightarrow 6r^2 + 6r + 6 = 26r$

$\Rightarrow 6r^2 + 6r - 26r + 6 = 0$

$\Rightarrow 6r^2 - 20r + 6 = 0$

$\Rightarrow 3r^2 - 10r + 3 = 0$

$\Rightarrow 3r^2 - 9r - r + 3 = 0$

$\Rightarrow 3r(r - 3) - 1(r - 3) = 0$

$\Rightarrow (3r - 1)(r - 3) = 0$

$\Rightarrow 3r - 1 = 0$ or $6r - 2 = 0$

$\Rightarrow r = \frac{1}{3}$ or $r = 3$

when $a = 6$, $r = \frac{1}{3}$

$\frac{a}{r} = \frac{6}{\frac{1}{3}} = 18$

$a = 6$

$ar = 6 \cdot \frac{1}{3} = 2$

when $a = 6$, $r = 3$ then

$\frac{a}{r} = \frac{6}{3} = 2$

$a = 6$

$ar = 6(3) = 18$

$\Rightarrow$ required numbers are 2, 6, 18, or 18, 6, 2

Q.11 If the sum of four consecutive terms of G.P is 80 and A.M of second and fourth of them is 30. Find the term.

Solution:

Given that

$a + ar + ar^2 + ar^3 = 80$

or $a + ar^2 + ar + ar^3 = 80$ ..... (1)

also

$$\frac{ar + ar^3}{2} = 30$$

$\Rightarrow$ $ar + ar^3 = 60$ ..... (2)

Put (2) in (1), we get

$a + ar^2 + 60 = 80$

$a + ar^2 = 20$ ..... (3)

equation (2) can be written as

$r(a + ar^2) = 60$ ..... (4)

Put (3) in (4), we have

$r(20) = 60$

$\boxed{r = 3}$

Put $r = 3$ in equation (3), we get

$a + a(3)^2 = 20$

$a + 9a = 20$

$10a = 20 \Rightarrow \boxed{a = 2}$

$\Rightarrow$ $ar = 2(3) = 6$

$ar^2 = 2(3)^2 = 18$

$ar^3 = 2(3)^3 = 54$

$\Rightarrow$ required terms are 2, 6, 18, 54.

Q.12 If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in G.P. Show that common ratio is $\pm\sqrt{\frac{a}{c}}$

(Lahore Board 2009)

Solution:

Given that

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in G.P

$$\Rightarrow \quad r = \frac{\frac{1}{b}}{\frac{1}{a}} = \frac{1}{b} \cdot a = \frac{a}{b} \qquad \ldots\ldots\ldots (1)$$

and $$r = \frac{\frac{1}{c}}{\frac{1}{b}} = \frac{1}{c} \cdot b = \frac{b}{c} \qquad \ldots\ldots\ldots (2)$$

multiply equations (1) and (2), we get

$$r^2 = \frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c}$$

$$\Rightarrow \quad r^2 = \pm\sqrt{\frac{a}{c}}$$

Q.13 If the numbers 1, 4, and 3 are subtracted from three consecutive terms of an A.P, the resulting numbers are in G.P. Find the numbers, if their sum is 21.

Solution:

Let $a - d,\ a,\ a + d$ are in A.P.

then $a - d + a + a + d = 21$

$$\Rightarrow \quad 3a = 21 \quad \Rightarrow \quad \boxed{a = 7}$$

also given that

$a - d - 1,\ \ a - 4,\ \ a + d - 3$ are in G.P

$$\Rightarrow \quad \frac{a-4}{a-d-1} = \frac{a+d-3}{a-4}$$

Put $a = 7$

$$\Rightarrow \quad \frac{7-4}{7-d-1} = \frac{7+d-3}{7-4}$$

$$\Rightarrow \quad \frac{3}{6-d} = \frac{4+d}{3}$$

$$\Rightarrow \quad (6-d)(4+d) = 9$$

$$\Rightarrow \quad 24 + 6d - 4d - d^2 = 9$$

$$\Rightarrow \quad 24 + 2d - d^2 = 9$$

$$\Rightarrow \quad d^2 - 2d - 24 + 9 = 0$$

$$\Rightarrow \quad d^2 - 2d - 15 = 0$$

$\Rightarrow \quad d^2 - 5d + 3d - 15 = 0$

$\Rightarrow \quad d(d-5) + 3(d-5) = 0$

$\Rightarrow \quad (d+3)(d-5) = 0$

$\Rightarrow \quad d = -3 \quad \text{or} \quad d = 5$

when $a = 7, \ d = -3$

$a - d = 7 - (-3) = 10$

$a = 7$

$a + d = 7 + (-3) = 4$

when $a = 7, \ d = 5$

$a - d = 7 - 5 = 2$

$a = 7$

$a + d = 7 + 5 = 12$

so required numbers are 10, 7, 4 or 2, 7, 12

Q.14 If three consecutive numbers in A.P are increased by 1, 4, 15 respectively, the resulting numbers are in G.P. Find original numbers of if their sum is 6.

Solution:

Let $a - d, \ a, \ a + d$ be the consecutive terms in A.P.

then $a - d + a + a + d = 6$

$3a = 6 \quad \Rightarrow \quad \boxed{a = 2}$

also given that

$a - d + 1, \quad a + 4, \quad a + d + 15$ are in G.P

$$\Rightarrow \quad \frac{a+4}{a-d+1} = \frac{a+d+15}{a+4}$$

Put $a = 2$

$$\Rightarrow \quad \frac{2+4}{2-d+1} = \frac{2+d+15}{2+4}$$

$$\Rightarrow \quad \frac{6}{3-d} = \frac{d+17}{6}$$

$\Rightarrow \quad 36 = (3-d)(17+d)$

$\Rightarrow \quad 36 = 3d + 51 - d^2 - 17d$

$\Rightarrow \quad 36 = -d^2 - 14d + 51$

$\Rightarrow \quad d^2 + 14d - 51 + 36 = 0$

$\Rightarrow \quad d^2 + 14d - 15 = 0$

$\Rightarrow \quad d^2 + 15d - d - 15 = 0$

$\Rightarrow \quad d(d + 15) - 1(d + 15) = 0$

$\Rightarrow \quad (d - 1)(d + 15) = 0$

$\Rightarrow \quad d = 1 \quad \text{or} \quad d = -15$

when $a = 2, \quad d = 1$

$a - d = 2 - 1 = 1$

$a = 2$

$a + d = 2 + 1 = 3$

when $a = 2, \quad d = -15$

$a - d = 2 - (-15) = 2 + 15 = 17$

$a = 2$

$a + d = 2 + (-15) = 2 - 15 = -13$

so the required numbers are 1, 2, 3 or 17, 2, – 13

GEOMETRIC MEANS

A number G is said to be a geometric means (G.M) between two numbers a and b if a, G, b are in G.P. therefore

$$\frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab \Rightarrow G = \pm\sqrt{ab}$$

EXERCISE 6.7

Q.1 Find G.M. between

(i) – 2 and 8 **(Lahore Board 2007)**

(ii) – 2i and 8i **(Gujranwala Board 2007, Lahore Board 2008)**

Solution:

(i) – 2 and 8

Let $a = -2$ and $b = 8$

as G.M. $= \pm\sqrt{ab}$

$= \pm\sqrt{(-2i)(8i)} = \pm\sqrt{-16} = \pm\sqrt{-1}\sqrt{16} = \pm 4i$

(ii) – 2i and 8i

Let $a = -2i, \quad b = 8i$

as G.M. $= \pm\sqrt{ab}$

$= \pm\sqrt{(-2i)(8i)} = \pm\sqrt{-16i^2} = \pm\sqrt{-16(-1)} = \pm\sqrt{16} = \pm 4$

Q.2 Insert two G.M's between

(i) 1 and 8 **(Gujranwala Board 2007)**

(ii) 2 and 16 **(Lahore Board 2011)**

Solution:

(i) 1 and 8

Let the required G.Ms are G_1, G_2

$\Rightarrow$ $1, G_1, G_2, 8$ are in G.P.

Here

$a = 1, \quad a_4 = 8$

$\Rightarrow a_1 r^3 = 8$

$\Rightarrow (1) r^3 = 8 \quad \because r_1 = 1$

$\Rightarrow r^3 = 8 \Rightarrow r = 2$

$G_1 = a_2 = a_1 r = (1)(2) = 2$

$G_2 = a_3 = a_1 r^2 = (1)(2)^2 = 4$

$\Rightarrow$ 2, 4 are required G.M's between 1 and 8

(ii) 2 and 16

Let G_1, G_2 are required G.Ms

then $2, G_1, G_2, 16$ are in G.P.

Here $a_1 = 2$, and $a_4 = 16$

$\Rightarrow a_1 r^3 = 16$

$\Rightarrow 2r^3 = 16 \quad \because a_1 = 2$

$\Rightarrow r^3 = 8$

$\Rightarrow r = 2$

Now $G_1 = a_2 = a_1 r = (2)(2) = 4$

$G_2 = a_3 = a_1 r^2 = (2)(2)^2 = 8$

$\Rightarrow$ 4, 8 are required G.M's between 2 and 16

Q.3 Insert three G.M's between

(i) 1 and 16 **(Lahore Board 2004)**

(ii) 2 and 32

Solution:

(i) 1 and 16

Let the required G.Ms are G_1, G_2, G_3

then

$1, G_1, G_2, G_3, 16$ are in G.P

$a = 1, \quad a_5 = 16,$

$a_1 r^4 = 16$

$(1)\, r^4 = 16 \qquad \because\ a_1 = 1$

$r^4 = 16$

$r^2 = \pm 4$

when $r^2 = 4 \Rightarrow r = \pm\sqrt{4} = \pm 2$

when $r^2 = -4 \Rightarrow r = \pm\sqrt{-4} = \pm 2i$

when $r = 2$, $a_1 = 1$

$G_1 = a_2 = a_1 r = (1)(2) = 2$

$G_2 = a_3 = a_1 r^2 = (1)(2)^2 = 4$

$G_3 = a_4 = a_1 r^3 = (1)(2)^3 = 8$

when $r = -2$, $a_1 = 1$

$G_1 = a_2 = a_1 r = (1)(-2) = -2$

$G_2 = a_3 = a_1 r^2 = (1)(-2)^2 = 4$

$G_3 = a_4 = a_1 r^3 = (1)(-2)^3 = -8$

when $r = 2i$, $a_1 = 1$

$G_1 = a_2 = a_1 r = (1)(2i) = 2i$

$G_2 = a_3 = a_1 r^2 = (1)(2i)^2 = 4i^2 = -4$

$G_3 = a_4 = a_1 r^3 = (1)(2i)^3 = 8i^3 = 8\, i^2\, i = -8i$

when $r = -2i$, $a_1 = 1$

$G_1 = a_2 = a_1 r = (1)(-2i) = -2i$

$G_2 = a_3 = a_1 r^2 = (1)(-2i)^2 = 4i^2 = -4$

$G_3 = a_4 = a_1 r^3 = (1)(-2i)^3 = -8i^3 = -8\, i^2\, i = 8i$

(ii) 2 and 32

Let G_1, G_2, G_3 are required G.Ms,

then $2, G, G_2, G_3, 32$ are in G.P

Here $a_1 = 2, \quad a_5 = 32,$

$a_1 r^4 = 32$

$2 r^4 = 32 \qquad \because\ a_1 = 2$

$r^4 = 16$

$r^2 = \pm 4$

when $r^2 = 4 \Rightarrow r = \pm\sqrt{4} = \pm 2$

when $r^2 = -4 \Rightarrow r = \pm\sqrt{-4} = \pm 2i$

when $r = 2$, $a_1 = 2$

$G_1 = a_2 = a_1 r = (2)(2) = 4$

$G_2 = a_3 = a_1 r^2 = (2)(2)^2 = 4$

$G_3 = a_4 = a_1 r^3 = (2)(2)^3 = 16$

when $r = -2$, $a_1 = 2$

$G_1 = a_2 = a_1 r = (2)(-2) = -4$

$G_2 = a_3 = a_1 r^2 = (2)(-2)^2 = 8$

$G_3 = a_4 = a_1 r^3 = (2)(-2)^3 = -16$

when $r = 2i$, $a_1 = 2$

$G_1 = a_2 = a_1 r = (2)(2i) = 4i$

$G_2 = a_3 = a_1 r^2 = (2)(2i)^2 = 8i^2 = -8$

$G_3 = a_4 = a_1 r^3 = (2)(2i)^3 = 16i^3 = 16\, i^2\, i = -16i$

when $r = -2i$, $a_1 = 2$

$G_1 = a_2 = a_1 r = (2)(-2i) = -4i$

$G_2 = a_3 = a_1 r^2 = (2)(-2i)^2 = 8i^2 = -8$

$G_3 = a_4 = a_1 r^3 = (2)(-2i)^3 = -16i^3 = -16\, i^2\, i = 16$

Q.4 Insert four real G.M's between 3 and 96. (Gujranwala Board 2006)

Solution:

Let G_1, G_2, G_3, G_4, are G.M's

then $3, G_1, G_2, G_3, G_4, 96$ are in G.P.

Here $a_1 = 3$, $a_6 = 96$

$a_1 r^5 = 96$

$3 r^5 = 96$ ⌑ $a_1 = 3$

$r^5 = \frac{96}{3} = 32$

$\boxed{r = 2}$

Now $G_1 = a_2 = a_1 r = (3)(2) = 4$

$G_2 = a_3 = a_1 r^2 = (3)(2)^2 = 12$

$G_3 = a_4 = a_1 r^3 = (3)(2)^3 = 24$

$G_4 = a_5 = a_1 r^4 = (3)(2)^4 = 48$

$\Rightarrow$ 6, 12, 24, 48 are required G.Ms.

Q.5 If both x and y positive distinct real numbers, show that geometric mean between x and y is less than their arithmetic's mean.

Solution:

Let 'G' be the geometric mean between x and y and 'A' be the arithmetic mean between x and y.

$$G = \pm\sqrt{xy} \quad \text{and} \quad A = \frac{x+y}{2}$$

We will show that

$$G < A$$

i.e. $$\pm\sqrt{xy} < \frac{x+y}{2}$$

or $$\pm 2\sqrt{xy} < x+y$$

or $$0 < x+y \mp 2\sqrt{xy}$$

or $$x+y \mp 2\sqrt{xy} > 0$$

or $$\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2 \mp 2\sqrt{xy} > 0$$

or $$\left(\sqrt{x} \mp \sqrt{y}\right)^2 > 0$$

which is true as square is always positive.

Hence proved.

Q.6 For what value of n $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the positive geometric mean between a and b? (Gujranwala Board 2007, Lahore Board 2006)

Solution:

The positive G.M between a and b is given by

$$G.M. = \sqrt{ab} = a^{1/2} b^{1/2}$$

It is given that

$$\frac{a^n + b^n}{a^{n-1} + b^{n-1}} = a^{1/2} b^{1/2}$$

$$\Rightarrow \quad a^n + b^n = a^{1/2} b^{1/2} (a^{n-1} + b^{n-1})$$

$$\Rightarrow \quad a^n + b^n = a^{n-1/2} b^{1/2} + a^{1/2} b^{n-1/2}$$

$$\Rightarrow \quad a^n - a^{n-1/2} . b^{1/2} = a^{1/2} b^{n-1/2} - b^n$$

$$\Rightarrow \quad a^{n-1/2} (a^{1/2} - b^{1/2}) = b^{n-1/2} (a^{1/2} - b^{1/2})$$

$\Rightarrow \quad a^{n-1/2} = b^{n-1/2}$

$\Rightarrow \quad \frac{a^{n-1/2}}{b^{n-1/2}} = 1$

$\Rightarrow \quad \left(\frac{a}{b}\right)^{n-1/2} = \left(\frac{a}{b}\right)^{0}$

$\Rightarrow \quad n - \frac{1}{2} = 0 \quad \Rightarrow \quad \boxed{n = \frac{1}{2}}$

Q.7 A.M. of two positive integral numbers exceeds their (positive) G.M. by 2 and their sum is 20, find the numbers.

Solution:

Let a and b two positive integral numbers then their G.M and A.M is given by

$A.M. = \frac{a+b}{2}, \quad G.M = \sqrt{ab}$

By given condition

$\frac{a+b}{2} - 2 = \sqrt{ab} \qquad \ldots\ldots\ldots (1)$

also it is given that

$a + b = 20 \qquad \ldots\ldots\ldots (2)$

from equation (2)

$a = 20 - b \qquad \ldots\ldots\ldots (3)$

Put (3) in (1), we get

$\frac{20 - b + b}{2} - 2 = \sqrt{(20-b)\,b}$

$10 - 2 = \sqrt{20b - b^2}$

$8 = \sqrt{20b - b^2}$

squaring both sides we get

$64 = 20b - b^2$

$\Rightarrow \quad b^2 - 20b + 64 = 0$

$\Rightarrow \quad b^2 - 16b - 4b + 64 = 0$

$\Rightarrow \quad b(b-16) - 4(b-16) = 0$

$\Rightarrow \quad (b-4)(b-16) = 0$

$\Rightarrow$ $b = 4$ $\quad\quad$ $b = 16$

Put $b = 4$ in equation (3), we get

$a = 20 - 4 = 16$

Put $b = 16$ in equation (3), we get

$a = 20 - 16 = 4$

so the required numbers are

4, 16 or 16, 4

Q.8 The A.M. between two numbers is 5 and their (positive) G.M is 4. Find the numbers.

Solution:

Let a, b are required numbers

then by given conditions

$$\frac{a+b}{2} = 5$$

$\Rightarrow$ $a + b = 10$ $\quad\quad$ ..... (1)

and $\sqrt{ab} = 4$

$\Rightarrow$ $ab = 16$ $\quad\quad$ ..... (2)

from equation (1)

$a = 10 - b$ $\quad\quad$ ..... (3)

Put (3) in (2), we get

$(10 - b)\, b = 16$

$\Rightarrow$ $10b - b^2 = 16$

$\Rightarrow$ $b^2 - 10b + 16 = 0$

$\Rightarrow$ $b^2 - 8b - 2b + 16 = 0$

$\Rightarrow$ $b(b - 8) - 2(b - 8) = 0$

$\Rightarrow$ $(b - 2)(b - 8) = 0$

$\Rightarrow$ $b = 2$ or $b = 8$

Put $b = 2$ in equation (3), we get

$a = 10 - 2 = 8$

Put $b = 8$ in equation (3), we get

$a = 10 - 8 = 2$

so the required numbers are

2, 8 or 8, 2

sum of n terms of a geometric Series

The formulas to find the sum of n terms of a geometric series is given by

$$S_n = \frac{a_1(1-r^n)}{1-r} \quad \text{if} \quad |r| < 1$$

and

$$S_n = \frac{a_1(r^n-1)}{r-1} \quad \text{if} \quad |r| > 1$$

Infinite Geometric Series

The geometric series which has infinite number of terms is called infinite geometric series. For example,

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots\ldots + a_1r^{n-1} + \ldots\ldots$$

is an infinite geometric series.

The formula to find the sum of infinite terms of a geometric series is given by

$$S_\infty = \frac{a_1}{1-r} \quad \text{if} \quad |r| < 1$$

EXERCISE 6.8

Q.1 Find sum of first 15 terms of geometric sequence, $1, \frac{1}{3}, \frac{1}{9}, \ldots\ldots$

Solution:

Given sequence

$$1, \frac{1}{3}, \frac{1}{9}, \ldots\ldots$$

Here $a_1 = 1, \quad r = \frac{\frac{1}{3}}{1} = \frac{1}{3}, \quad n = 15$

As $S_n = \frac{a(1-r^n)}{1-r}$

$$S_n = \frac{1\left[1-\left(\frac{1}{3}\right)^{15}\right]}{1-\frac{1}{3}} = \frac{1-\left(\frac{1}{3}\right)^{15}}{\frac{2}{3}}$$

$$= \frac{3}{2}\left[1-\left(\frac{1}{3}\right)^{15}\right] = \frac{3}{2}\left[1-\frac{1}{14348907}\right]$$

$$= \frac{3}{2}\left[\frac{14348907-1}{14348907}\right] = \frac{3}{2}\left[\frac{14348906}{14348907}\right] = \frac{7174453}{4782969}$$

Q.2 Sum to n terms, the series

(i) .2 + .22 + .222 (ii) 3 + 33 + 333 + ... (Lahore Board 2010)

Solution:

(i) .2 + .22 + .222 + ...

$S_n = .2 + .22 + .222 + \ldots\ldots$ n terms

$= 2\,[.1 + .11 + .111 + \ldots\ldots$ n terms$]$

$= \frac{2}{9}\,[.9 + .99 + .999 + \ldots\ldots$ n terms$]$

$= \frac{2}{9}\,[(1 - .1) + (1 - .01) + (1 - .001) + \ldots\ldots$ n terms$]$

$= \frac{2}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{100}\right) + \left(1 - \frac{1}{1000}\right) + \ldots..$ n terms$\right]$

$= \frac{2}{9}\left[(1 + 1 + \ldots..$ n terms $) - \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots..$ n terms$\right)\right]$

$= \frac{2}{9}\left[\,n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}}\,\right]$

$= \frac{2}{9}\left[\,n - \frac{1}{10}\,\frac{1 - \frac{1}{10^n}}{\frac{9}{10}}\,\right]$

$= \frac{2}{9}\left[\,n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$

(ii) 3 + 33 + 333 + ... (GujranwalaBoard 2007)

$Sn = 3 + 33 + 333 + \ldots\ldots$ n terms

$= 3\,[1 + 11 + 111 + \ldots\ldots$ n terms$]$

$= \frac{1}{3}\,[9 + 99 + 999 + \ldots\ldots$ n terms$]$

$= \frac{1}{3}\,[(10 - 1) + (100 - 1) + (1000 - 1) + \ldots\ldots$n terms$]$

$= \frac{1}{3}\,[(10 + 100 + 1000 + \ldots\ldots$ n terms$) - (1 + 1 + 1 + \ldots\ldots$ n times$)]$

$= \frac{1}{3}\left[\,\frac{10\,(10^n - 1)}{10 - 1} - n\,\right]$

$= \frac{1}{3}\left[\,\frac{10}{9}\,(10^n - 1) - n\,\right]$

Q.3 Sum to n terms of series

(i) $1 + (a + b) + (a^2+ab+b^2) + (a^3 + a^2b + ab^2 + b^3) + ...$ **(Lahore Board 2008)**

(ii) $r + (1 + k) r^2 + (1 + k + k^2) r^3 + ...$ **(Lahore Board 2003)**

Solution:

(i) $1 + (a + b) + (a^2+ ab + b^2) + (a^3 +a^2b+ab^2+b^3) + ...+ n$ **terms**

$S_n = 1 + (a + b) + (a^2+ ab + b^2) + (a^3 +a^2b+ab^2+b^3) + ...+ n$ terms

Multiply and dividing by $(a - b)$

$(a - b) S_n = a - b + (a - b)(a + b) + (a - b)(a^2+ ab + b^2) + (a - b)(a^3 +a^2b+ab^2+b^3)$

$+ .....+ n$ terms

$(a - b) S_n = a - b+(a^2 - b^2)+(a^3 - b^3)+(a^4 - b^4) +.....$ n terms

$(a - b) S_n = [(a + a^2 + a^3 + a^4 + ... n \text{ terms}) - (b + b^2 + b^3 + b^4 + ... n \text{ terms})$

$$\frac{a(a^n - 1)}{a - 1} - \frac{b(b^n - 1)}{b - 1}$$

$$S_n = \frac{1}{(a - b)}\left[\frac{a(a^n - 1)}{a - 1} - \frac{b(b^n - 1)}{b - 1}\right]$$

$$= \frac{1}{(a - b)}\left[\frac{a(b - 1)(a^n - 1) - b(a - 1)(b^n - 1)}{(a - 1)(b - 1)}\right]$$

$$= \frac{a(b - 1)(a^n - 1) - b(a - 1)(b^n - 1)}{(a - b)(a - 1)(b - 1)}$$

(ii) $r + (1 + k) r^2 + (1 + k + k^2) r^3 + ...$

$S_n = r + (1 + k) r^2 + (1 + k + k^2) r^3 + ...$ n terms

Multiply both sides by $(1 - k)$, we get

$(1 - k) S_n = r(1 - k) + (1 - k)(1 + k) r^2 + (1 - k)(1 + k + k^2) r^3 + ...$ n terms

$= r(1 - k) r+(1 - k^2) r^2+(1 - k^3) r^3 +.....$ n terms

$= r - rk + r^2 - r^2 k^2 + r^3 - r^3 k^3 + ...$ n terms

$= (r + r^2 + r^3 + ... n \text{ terms}) - (rk + r^2 k^2 + r^3 k^3 + ... n \text{ terms}]$

$$= \frac{r(r^n - 1)}{r - 1} - \frac{kr(k^n r^n - 1)}{kr - 1}$$

Q.4 Sum the series $2 + (1 - i) + \left(\frac{1}{i}\right) + ...$ **to 8 terms.**

Solution:

$$2 + (1 - i) + \frac{1}{i} + ...$$

$a = 2, \quad n = 8, \quad r = \frac{1-i}{2}, \quad S_8 = ?$

As $S_n = \frac{a(1-r^n)}{(1-r)}$

$$= \frac{2\left[1-\frac{(1-i)^8}{2}\right]}{1-\left(\frac{1-i}{2}\right)}$$

$$= \frac{2\left[1-\left(\frac{1-i}{2}\right)^8\right]}{\frac{1+i}{2}} \qquad \because \ (1-i)^2 = 1+i^2-2i = 1-1-2i = -2i$$

$$= \frac{4\left[1-\left(\frac{1-i}{2}\right)^8\right]}{1+i}$$

$$= \frac{4\left[1-\frac{((1-i)^2)^4}{2^8}\right]}{1+i}$$

$$= \frac{4\left[1-\frac{(-2i)^4}{2^8}\right]}{1+i}$$

$$= \frac{4\left[1-\frac{2^4 . i^4}{2^8}\right]}{1+i}$$

$$= \frac{4\left[1-\frac{+1}{2^4}\right]}{1+i}$$

$$= \frac{4\left[1-\frac{1}{16}\right]}{1+i}$$

$$= \frac{4\left(\frac{15}{16}\right)}{1+i} = \frac{15}{4(1+i)} \times \frac{1-i}{1-i} \qquad \because \ i^2 = -1$$

$$= \frac{15(1-i)}{4(1-i^2)} = \frac{15(1-i)}{4(1+1)} = \frac{15}{8}(1-i)$$

Q.5 Find the sums of the following infinite geometric series.

(i) $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + ...$ **(ii)** $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

(iii) $\frac{9}{4} + \frac{3}{2} + 1 + \frac{2}{3} + ...$ **(iv)** $2 + 1 + 0.5 + ...$

(v) $4 + 2\sqrt{2} + 2 + \sqrt{2} + 1 + ...$ **(vi)** $0.1 + 0.05 + 0.025 + ...$

Solution:

(i) $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + ...$

Here $a = \frac{1}{5}$, $r = \frac{\frac{1}{25}}{\frac{1}{5}} = \frac{1}{5}$, $S_\infty = ?$

As $S_\infty = \frac{a_1}{1-r}$

$= \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{\frac{1}{5}}{\frac{4}{5}} = \frac{1}{5} \times \frac{5}{4} = \frac{1}{4}$

(ii) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

Here $a_1 = \frac{1}{2}$, $r = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \times 2 = \frac{1}{2}$, $S_\infty = ?$

As $S_\infty = \frac{a_1}{1-r}$

$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$

(iii) $\frac{9}{4} + \frac{3}{2} + 1 + \frac{2}{3} + ...$

Here $a_1 = \frac{9}{4}$, $r = \frac{\frac{3}{2}}{\frac{9}{4}} = \frac{3}{2} \times \frac{4}{9} = \frac{3}{2}$, $S_\infty = ?$

As $S_\infty = \frac{a_1}{1-r} = \frac{\frac{9}{4}}{1-\frac{2}{3}} = \frac{\frac{9}{4}}{\frac{1}{3}} = \frac{9}{4} \times 3 = \frac{27}{4}$

(iv) **2 + 1 + 0.5 + ...**

Here $a_1 = 2$, $r = \frac{1}{2}$, $S_\infty = ?$

As $S_\infty = \frac{a_1}{1-r}$

$= \frac{2}{1-\frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4$

(v) $\mathbf{4 + 2\sqrt{2} + 2 + \sqrt{2} + 1 + ...}$

Here $a_1 = 4$, $r = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$, $S_\infty = ?$

As $S_\infty = \frac{a_1}{1-r}$

$= \frac{4}{1-\frac{1}{\sqrt{2}}} = \frac{4\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{4\sqrt{2}(\sqrt{2}+1)}{(\sqrt{2})^2-(1)^2}$

$= \frac{4(2+\sqrt{2})}{2-1} = 4(2+\sqrt{2})$

(vi) **0.1 + 0.05 + 0.025 + ...**

Here $a_1 = 0.1$, $r = \frac{0.05}{0.1} = 0.5$, $S_\infty = ?$

As $S_\infty = \frac{a_1}{1-r}$

$= \frac{0.1}{1-.05} = \frac{0.1}{0.5} = 0.2$

Q.6 Find vulgar fractions equivalent to the following recurring decimals.

(i) $\mathbf{1.\dot{3}\dot{4}}$ **(ii)** $\mathbf{0.\dot{7}}$

(iii) $\mathbf{0.\dot{2}\dot{5}\dot{9}}$ **(iv)** $\mathbf{1.\dot{5}\dot{3}}$

(v) $\mathbf{1.\dot{1}\dot{5}\dot{9}}$ **(vi)** $\mathbf{1.1\dot{4}\dot{7}}$

Solution:

(i) $\mathbf{1.\dot{3}\dot{4}}$

$1.\dot{3}\dot{4} = 1.3434343434 .....$

$= 1 + 0.34 + 0.0034 + 0.000034 + ...$

$= 1 + \frac{0.34}{1 - 0.01}$ $\because$ $a_1 = 0.34,$ $r = \frac{0.0034}{0.34} = 0.01$,

$= 1 + \frac{0.34}{1 - .01} = 1 + \frac{0.34}{.99} = 1 + \frac{34}{99} = \frac{133}{99}$

(ii) $\mathbf{0.\dot{7}}$

$0.\dot{7} = 0.777777 .....$

$= 0.7 + 0.07 + 0.007 + 0.0007 + ...$

$= \frac{0.7}{1 - .1} = \frac{0.7}{.9} = \frac{7}{9}$ $\because$ $a = 0.7,$ $r = \frac{0.07}{0.7} = 0.1$

(iii) $\mathbf{0.\dot{2}\dot{5}\dot{9}}$

$0.\dot{2}\dot{5}\dot{9} = 0.259259259259 .....$

$= 0.259 + 0.000259 + 0.000000259 + ...$

$= \frac{0.259}{1 - 0.001}$ $\because$ Here $a = 0.259,$ $r = \frac{0.000259}{0.259} = .001$

$= \frac{0.259}{.999} = \frac{259}{999}$

(iv) $\mathbf{1.\dot{5}\dot{3}}$

$1.\dot{5}\dot{3} = 1.53535353 .....$

$= 1 + 0.53 + 0.0053 + 0.000053 + ...$

$= 1 + \frac{0.53}{1 - 0.01}$ $\quad \because \quad a_1 = 0.53, \quad r = \frac{0.0053}{0.53} = 0.01, \quad n = \infty$

$= \frac{0.53}{0.99}$

$= 1 + \frac{53}{99} = \frac{152}{99}$

(v) $\mathbf{1.\dot{1}\dot{5}\dot{9}}$

$1.\dot{1}\dot{5}\dot{9} = 1.159159159 .....$

$= 0.159 + .000159 + .000000159 + ...$

$= \frac{0.159}{1 - 0.001}$ $\quad \because \quad a_1 = 0.159 \quad r = \frac{0.000159}{0.159} = 0.001$

$= \frac{0.159}{0.999} = \frac{159}{999}$

$= \frac{53}{333}$

(vi) $\mathbf{1.1\dot{4}\dot{7}}$

$1.1\dot{4}\dot{7} = 1 + .047 + .000047 + ...$

$= 1 + 0.1 + 0.047 + 0.000047 + ...$

$= 1 + 0.1 + \frac{0.047}{1 - 0.001}$

$= 1 + 0.1 + \frac{0.047}{0.99}$ $\quad \because \quad a_1 = .047 \quad r = \frac{0.000047}{0.0047} = 0.001$

$= 1 + 0.1 + \frac{47}{990}$

$= 1 + \frac{1}{10} + \frac{47}{990}$

$= 1 + \frac{99 + 47}{990}$

$= 1 + \frac{146}{9990} = \frac{1136}{990} = \frac{568}{495}$

Q.7 Find the sum to infinity of the series;

$r + (1 + k) r^2 + (1 + k + k^2) r^3 + ...$ r and k being proper functions.

Solution:

Given series

$r + (1 + k) r^2 + (1 + k + k^2) r^3 + ...$

$S = r + (1 + k) r^2 + (1 + k + k^2) r^3 + ...$to infinity

multiplying by $(1 - k)$, we get

$$(1 - k) S_n = r (1 - k) + (1 - k) (1 + k) r^2 + (1 - k) (1 + k + k^2) r^3 + ... \text{ to infinity}$$

$$= r (1 - k) r + (1 - k^2) r^2 + (1 - k^3) r^3 + ... \text{ to infinity}$$

$$= r - rk + r^2 - r^2 k^2 + r^3 - r^3 k^3 + ... \text{ to infinity}$$

$$= (r + r^2 + r^3 + ... \text{ n terms}) - (rk + r^2 k^2 + r^3 k^3 + ... \text{ n terms}$$

$$= \frac{r}{1 - r} = \frac{r k}{1 - r k}$$

$$= \frac{r (1 - kr) - rk (1 - r)}{(1 - r) (1 - rk)} = \frac{r - kr^2 - rk + r^2 k}{(1 - r) (1 - rk)}$$

$$(1 - k) S = \frac{r - rk}{(1 - r) (1 - rk)}$$

$$(1 - k) S = \frac{r (1 - k)}{(1 - r) (1 - rk)}$$

$$S = \frac{r}{(1 - r) (1 - rk)}$$

Q.8 If $y = \frac{x}{2} + \frac{1}{4} x^2 + \frac{1}{8} x^3 + ...$ and $0 < x < 2$ then prove that $x = \frac{2y}{1 + y}$

Solution:

$$y = \frac{x}{2} + \frac{1}{4} x^2 + \frac{1}{8} x^3 + ... \text{ to infinity}$$

Here $a_1 = \frac{x}{2}$, $r = \frac{\frac{x^2}{4}}{\frac{x}{2}} = \frac{x^2}{4} \cdot \frac{2}{x} = \frac{x}{2}$

so using $S_\infty = \frac{a_1}{1 - r}$

$$y = \frac{\frac{x}{2}}{1 - \frac{1}{2} x} = \frac{\frac{x}{2}}{\frac{2 - x}{2}} = \frac{x}{2 - x}$$

$$y = \frac{x}{2 - x}$$

$(2 - x)\, y = x$

$2y - xy = x$

$2y = x + xy$

$2y = x\,(1 + y)$

$x = \dfrac{2y}{y + 1}$

Q.9 If $y = \dfrac{2}{3}x + \dfrac{4}{9}x^2 + \dfrac{8}{27}x^3 + \ldots\ldots$ and if $0 < x < \dfrac{3}{2}$ then prove that $x = \dfrac{3y}{2\,(1 + y)}$

Solution:

Given series

$y = \dfrac{2}{3}x + \dfrac{4}{9}x^2 + \dfrac{8}{27}x^3 + \ldots\ldots$

Here $a_1 = \dfrac{2}{3}x$, $r = \dfrac{\frac{4}{9}x^2}{\frac{2}{3}x} = \dfrac{4x^2}{9} \times \dfrac{3}{2x} = \dfrac{2x}{3}$

using $S_\infty = \dfrac{a_1}{1 - r}$

$y = \dfrac{\frac{2}{3}x}{1 - \frac{2x}{3}} = \dfrac{\frac{2}{3}x}{\frac{3 - 2x}{3}} = \dfrac{2x}{3 - 2x}$

$y = \dfrac{2x}{3 - 2x}$

$y\,(3 - 2x) = 2x$

$3y - 2xy = 2x$

$3y = 2x\,(1 + y)$

$x = \dfrac{3y}{2\,(1 + y)}$

Hence proved.

Q.10 A ball is dropped from a height of 27 meters and it rebounds two third of the distance it falls. If it continues to fall in the same way what distance will it travel before coming to rest?

Solution:

Distance traveled in first fall = 27 meters

Height of first rebound $= 27 \times \frac{2}{3} = 12$ m

Height of second rebound $= 18 \times \frac{2}{3}$

Total distance $= 27 + 2\left[\frac{18}{1-\frac{2}{3}}\right]$ $\because$ $a_1 = 18$ $r = \frac{12}{18} = \frac{2}{3}$

$= 27 + 2 \times \frac{18}{\frac{1}{3}}$

$= 27 + 108 = 135$ meters

Q.11 What distance will a ball travel before coming to rest if it is dropped from a height of 75 meters and after each fall rebounds $\frac{2}{5}$ of the distance it fell?

Solution:

Distance of first fall $= 75$ meters

Height of first rebound $= 75 \times \frac{2}{5} = 30$ m

Height of second rebound $= 30 \times \frac{2}{5} = 12$ m

Height of third rebound $= 12 \times \frac{2}{5} = \frac{24}{5}$ m

Total distance $= 75 + 2\left[30 + 12 + \frac{24}{5} \text{ .... to infinity}\right]$

$= 75 + 2\left[\frac{30}{1-\frac{2}{5}}\right]$ $\because$ $a = 30$ $r = \frac{12}{30} = \frac{2}{5}$

$= 75 + 2\left[\frac{30}{\frac{3}{5}}\right]$

$= 75 + 2 \times \frac{150}{3} = 75 + 2\,(50) = 75 + 100 = 175$ m

Q.12 If $y = 1 + 2x + 4x^2 + 8x^3 + ...$

(i) Show that $x = \frac{y-1}{2y}$

(ii) Find the interval in which the series is convergent

Solution:

(i) $x = \frac{y-1}{2y}$

Given

$y = 1 + 2x + 4x^2 + 8x^3 + ... \infty$

Here $a_1 = 1$, $r = 2x$, using $S\infty = \frac{a_1}{1-r}$

$y = \frac{1}{1-2x} = y(1-2x) = 1$

$y - 2xy = 1$

$y - 1 = 2xy$

$x = \frac{y-1}{2y}$

(ii) $y = 1 + 2x + 4x^2 + 8x^3 + ...$

The given series will be convergent if

$|r| < 1 \qquad \because \ r = 2x$

$\Rightarrow \quad |2x| < 1$

$\Rightarrow \quad |x| < \frac{1}{2}$

$\Rightarrow \quad -\frac{1}{2} < x < \frac{1}{2}$

Q.13 If $y = 1 + \frac{x}{2} + \frac{x^2}{4} + ...$

(i) Show that $x = 2\left(\frac{y-1}{y}\right)$

(ii) Find the interval in which series is convergent.

Solution:

(i) $y = 1 + \frac{x}{2} + \frac{x^2}{4} + ...$

Given series

$y = 1 + \frac{x}{2} + \frac{x^2}{4} + ... \infty$

Here $a_1 = 1, \quad r = \frac{x}{2}, \quad$ using $S\infty = \frac{a_1}{1-r}$

$$y = \frac{1}{1-\frac{x}{2}} = \frac{1}{\frac{2-x}{2}}$$

$$y = \frac{2}{2-x}$$

$$y(2-x) = 2$$

$$2y - xy = 2$$

$$2y - 2 = xy$$

$$2(y-1) = xy$$

$$x = \frac{2(y-1)}{y}$$

(ii) $\mathbf{y = 1 + \frac{x}{2} + \frac{x^2}{4} + ...}$

The given series is convergent if

$|r| < 1$

Here $r = \frac{x}{2}$

$\Rightarrow \quad \left|\frac{x}{2}\right| < 1$

$\Rightarrow \quad |x| < 2 \quad \Rightarrow \quad -2 < x < 2$

Q.14 The sum of an infinite geometric series is 9 and sum of sequence of its terms is $\frac{81}{5}$. Find the series.

Solution:

Let the required geometric series is $a + ar + ar^2 + ar^3 + ... \infty$

By the given condition

$a + ar + ar^2 + ar^3 + ... \infty = 9$

or $\frac{a}{1-r} = 9$

or $a = 9(1-r) \qquad ..... (1)$

also it is given that

$$a^2 + a^2r^2 + a^2r^4 + a^2r^6 + ... \infty = \frac{81}{5}$$

or $\frac{a^2}{1-r^2} = \frac{81}{5}$

or $5a^2 = 81(1-r^2) \qquad ..... (2)$

Put equation (1) in equation (2), we get

$5\,[9\,(1-r)]^2 = 81\,(1-r^2)$

$5 \times 81\,(1-r)^2 = 81\,(1-r)\,(1+r)$

$5\,(1-r) = 1+r$

$5-5r = 1+r$

$5r+r-5+1 = 0$

$$6r-4 = 0 \Rightarrow r = \frac{4}{6} \Rightarrow \boxed{r = \frac{2}{3}}$$

Put $r = \frac{2}{3}$ in equation (1), we get

$$a = 9\left(1-\frac{2}{3}\right)$$

$$a = 9\left(\frac{1}{3}\right) = 3$$

so

the required series is

$a + a\,r + a\,r^2 + \ldots\ldots$

i.e. $3 + 3\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + \ldots..$

or $3 + 2 + \frac{4}{3} + \frac{8}{9}\ldots\ldots$

EXERCISE 6.9

Q.1 A man deposits in a bank Rs. 8 in the first year, Rs. 24 in the second year, Rs. 72 in the third year and so on. Find the amount he will have deposited in the bank by the fifth year.

Solution:

Deposited amount is given by

$8 + 24 + 72 + \ldots..$ which is a G.P

Here $a_1 = 8,\quad r = 3,\quad n = 5,\quad S_5 = ?$

As $$S_n = \frac{a_1\,(r^n-1)}{r-1}$$

$$S_5 = \frac{8\,(3^5-1)}{(3-1)}$$

$$= \frac{8\,(243-1)}{2} = \text{Rs. } 968$$

Q.2 A man borrows Rs. 32769 without interest and agrees to repay the loan installments, each installment being twice the proceeding one. Find amount of the last installment, if the amount of the first installment is Rs. 8.

Solution:

Let a_1 be the first installment

then $a_1 = 8$

$\Rightarrow$ $a_2 = 16$ and so on

$\Rightarrow$ $r = 2$, $S_n = 32760$

using $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$\Rightarrow$ $32760 = \dfrac{8(2^n - 1)}{2 - 1} = 8(2^n - 1)$

$\Rightarrow$ $\dfrac{32760}{8} = 2^n - 1$

$\Rightarrow$ $4095 + 1 = 2^n$

$\Rightarrow$ $4096 = 2^n$

$\Rightarrow$ $2^n = 2^{12}$

$\Rightarrow$ $\boxed{n = 12}$

Now we will find a_{12}

As $a_n = ar^{n-1}$

$a_{12} = (8)(2)^{12-1}$

$= 8(2)^{11}\ 8(2048) = 16384$

$a_{12} = $ Rs. 16384

is the amount of last installment

Q.3 The population of a certain village is 62500. What will be its population after 3 years if it increases at the rate of 4% annually?

Solution:

Given population = 62500

After first year = 62500 + 4% of 62500

= 62500 + (0.04) (62500) = 65000

After second year = 65000 + (0.04) (65000) = 67600

After third year = 67600 + (0.04) (67600) = 70304

So the population after 3rd year = 70304

Q.4 The enrollment of a famous school doubled after every eight year from 1970 to 1994. If the enrollment was 6000 in 1994, what was its enrollment in 1970?

Solution:

Let enrollment in 1970 = a

Then enrollment in 1978 = 2a

enrollment in 1986 = 4a

enrollment in 1994 = 8a

But it is given that

Enrolment in 1994 = 6000

$\Rightarrow$ $8a = 6000$

$$a = \frac{6000}{8}$$

$a = 750$

$\Rightarrow$ enrollment in 1970 = a = 750

Q.5 A singular cholera bacteria produces two bacteria in $\frac{1}{2}$ hour. If we start with a colony of bacteria, how many bacteria will we have in n hours?

Solution:

If at the start

Colony of bacteria = A

After every $\frac{1}{2}$ hour interval

number of bacteria are

A, 2A, 4A, 8A, 16A, ...

In one hour interval, number of bacterias are

A, 4A, 16A ..... which is a G.P

$a_1 = A, \quad r = \frac{4A}{A} = 4, \quad n = n + 1$

we will find a_{n+1} ¤ As A is present already

As $a_n = a_1 r^{n-1}$

$a_{n+1} = a_1 r^n$

$= A(4)^n = A4^n$

$= A\,2^{2n}$ is Answer

$\Rightarrow$ number of bacteria after n hours $= 4^n A$

Q.6 Joining mid points of the sides of an equilateral triangle, an equilateral triangle having half the perimeter of the original triangle is obtained. We form a sequence of rested equilateral triangles in the manner described with the original triangle having perimeter $\frac{3}{2}$. What will be total perimeter of all triangles formed in this way?

Solution:

Perimeter of original triangle $= \frac{3}{2}$

Perimeter of the second triangle $= \frac{\frac{3}{2}}{2} = \frac{3}{4}$

Perimeter of the third triangle $= \frac{\frac{3}{4}}{2} = \frac{3}{8}$

and so on.

We can write them as

$\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots\ldots$ which is a G.P.

we have to find the sum of this infinite G.P.

Here $a_1 = \frac{3}{2}, \quad r = \frac{\frac{3}{4}}{\frac{3}{2}} = \frac{1}{2},$

so using

$$S_\infty = \frac{a_1}{1-r}$$

$$= \frac{\frac{3}{2}}{1-\frac{1}{2}} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3$$

$\Rightarrow$ Total perimeter of all the triangles $= 3$.

Harmonic Progression: (H. P.)

A sequence of numbers is called a harmonic sequence or harmonic progression if the reciprocals of its terms are in arithmetic progression.

e.g. $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ are in H.P

since

1, 3, 5, 7 are in A.P.

nth term or general term of H.P is given by

$$\frac{1}{a_1 + (n-1)\,d}$$

Harmonic Mean

A number H is said to be the harmonic mean (H.M) between two numbers a and b if a, H, b are in H.P

Also $H = \frac{2ab}{a+b}$

EXERCISE 6.10

Q.1 Find the 9th term of the harmonic sequence

(i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ ..... **(ii)** $\frac{-1}{5}, \frac{-1}{3}, -1$ .....

Solution:

(i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ .....

Given sequence

$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ ..... which is H.P

3, 5, 7, ... is A.P

$a_1 = 3, \quad d = 5 - 3 = 2,$

As

$a_n = a + (n-1)\,d$

$a_9 = a_1 + 8d$

$= 3 + 8\,(2) = 3 + 16 = 19$

$\Rightarrow$ 9th term of A.P = 19

$\Rightarrow$ 9th term of H.P = $\frac{1}{19}$

(ii) $\frac{-1}{5}, \frac{-1}{3}, -1$ .....

Given sequence

$\frac{-1}{5}, \frac{-1}{3}, -1$ ..... which is H.P

$-5, -3, -1, \ldots\ldots$ is A.P

$a_1 = -5, \quad d = -3 - (-5) = 2, \quad n = 9, \quad a_9 = ?$

As

$a_n = a_1 + (n-1)\,d$

$a_9 = a_1 + 8d$

$= -5 + 8\,(2) = -5 + 16 = 11$

$\Rightarrow$

9^{th} term of H.P $= \frac{1}{11}$

Q.2 Find the 12th term of the following Harmonic sequence.

(i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8} \ldots\ldots$ **(ii)** $\frac{1}{3}, \frac{2}{9}, \frac{1}{6} \ldots\ldots$

Solution:

(i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8} \ldots\ldots$

Given sequence

$\frac{1}{2}, \frac{1}{5}, \frac{1}{8} \ldots\ldots$ which is H.P

$2, 5, 8, \ldots\ldots$ A.P

$a_1 = 2, \quad d = 5 - 2 = 3, \quad n = 12, \quad a_{12} = ?$

As

$a_n = a + (n-1)\,d$

$a_{12} = a_1 + 11d$

$= 2 + (12 - 1)$

$= 2 + 11\,(3) = 2 + 33 = 35$

$\Rightarrow$ 12th term of H.P $= \frac{1}{35}$

(ii) $\frac{1}{3}, \frac{2}{9}, \frac{1}{6} \ldots\ldots$

Given sequence

$\frac{1}{3}, \frac{2}{9}, \frac{1}{6} \ldots\ldots$ which is H.P

$3, \frac{9}{2}, 6 \ldots\ldots$ A.P

$a_1 = 3, \quad d = \frac{9}{2} - 3 = \frac{3}{2}, \quad n = 12, \quad a_{12} = ?$

As

$$a_n = a + (n-1)\,d$$

$$a_{12} = a_1 + 11d$$

$$= 3 + 11\left(\frac{3}{2}\right) = 3 + \frac{33}{2} = \frac{39}{2}$$

so

the 12th term of H.P $= \frac{2}{39}$

Q.3 Insert five harmonic means between the following given numbers.

(i) $-\frac{2}{5}$ **and** $\frac{2}{13}$ **(ii)** $\frac{1}{4}$ **and** $\frac{1}{24}$

Solution:

(i) $-\frac{2}{5}$ **and** $\frac{2}{13}$

Let A_1, A_2, A_3, A_4, A_5 are arithmetic means between $-\frac{2}{5}$ and $\frac{2}{13}$

$\Rightarrow$ $-\frac{5}{2}, A_1, A_2, A_3, A_4, A_5, \frac{13}{2}$ are in A.P

$\Rightarrow$ $a_1 = -\frac{5}{2}, \quad a_7 = \frac{13}{2}$

$\Rightarrow$ $a_1 + 6d = \frac{13}{2}$ ¤ $a_1 = -\frac{5}{2}$

$$-\frac{5}{2} + 6d = \frac{13}{2}$$

$\Rightarrow$ $6d = \frac{13}{2} + \frac{5}{2}$

$$6d = \frac{18}{2} = 9 \Rightarrow d = \frac{3}{2}$$

$\Rightarrow$ $A_1 = a_2 = a_1 + d = \frac{-5}{2} + \frac{3}{2} = \frac{-2}{2} = -1 \Rightarrow H_1 = -1$

$$A_2 = a_3 = a_1 + 2d = \frac{-5}{2} + 2\left(\frac{3}{2}\right) = \frac{-5}{2} + 3 = -\frac{1}{2} \Rightarrow H_2 = -2$$

$$A_3 = a_4 = a_1 + 3d = \frac{-5}{2} + 3\left(\frac{3}{2}\right) = \frac{-5}{2} + \frac{9}{2} = 2 \Rightarrow H_3 = \frac{1}{2}$$

$$A_4 = a_5 = a_1 + 4d = \frac{-5}{2} + 4\left(\frac{3}{2}\right) = \frac{-5}{2} + 6 = -\frac{7}{2} \Rightarrow H_4 = \frac{2}{7}$$

$$A_5 = a_6 = a_1 + 5d = \frac{-5}{2} + 5\left(\frac{3}{2}\right) = \frac{-5}{2} + \frac{15}{2} = 5 \Rightarrow H_5 = \frac{1}{5}$$

$\Rightarrow$ Required H.Ms are $-1, -2, \frac{1}{2}, \frac{2}{7}$ and $\frac{1}{5}$

(ii) $\frac{1}{4}$ **and** $\frac{1}{24}$

Let A_1, A_2, A_3, A_4, A_5 are A.Ms between 4 and 24

$\Rightarrow$ 4, A_1, A_2, A_3, A_4, A_5, 24 are in A.P

$a_1 = 4, \quad a_7 = 24,$

$a_1 + 6d = 24$

$4 + 6d = 24$

$6d = 24 - 4$

$\Rightarrow \quad 6d = 20 \Rightarrow d = \frac{20}{6} = \frac{10}{3}$

$$A_1 = a_2 = a_1 + d = 4 + \frac{10}{3} = \frac{22}{3} \Rightarrow H_1 = \frac{3}{22}$$

$$A_2 = a_3 = a_1 + 2d = 4 + 2\left(\frac{10}{3}\right) = 4 + \frac{20}{3} = \frac{32}{3} \Rightarrow H_2 = \frac{3}{32}$$

$$A_3 = a_4 = a_1 + 3d = 4 + 3\left(\frac{10}{3}\right) = 14 \Rightarrow H_3 = \frac{1}{14}$$

$$A_4 = a_5 = a_1 + 4d = 4 + 4\left(\frac{10}{3}\right) = 4 + \frac{40}{3} = \frac{52}{3} \Rightarrow H_4 = \frac{3}{52}$$

$$A_5 = a_6 = a_1 + 5d = 4 + 5\left(\frac{10}{3}\right) = 4 + \frac{50}{3} = \frac{62}{3} \Rightarrow H_5 = \frac{3}{62}$$

so the required five H.Ms are

$\frac{3}{22}, \frac{3}{32}, \frac{1}{14}, \frac{3}{52}$, and $\frac{3}{62}$

Q.4 Insert four H.Ms between the given numbers:

(i) $\frac{1}{3}$ **and** $\frac{1}{23}$ **(ii)** $\frac{7}{3}$ **and** $\frac{7}{11}$

(iii) 4 and 20

Solution:

(i) $\frac{1}{3}$ **and** $\frac{1}{23}$

Let A_1, A_2, A_3, A_4 are four A.M's between 3 and 23.

$\Rightarrow$ 3, A_1, A_2, A_3, A_4, 23 are A.P.

$\boxed{a_1 = 3}$, $\quad a_6 = 23$

$a_1 + 5d = 23$

$3 + 5d = 23$

$5d = 20$

$\boxed{d = 4}$

$A_1 = a_2 = a_1 + d = 3 + 4 = 7 \Rightarrow H_1 = \frac{1}{7}$

$A_2 = a_3 = a_1 + 2d = 3 + 2(4) = 11 \Rightarrow H_2 = \frac{1}{11}$

$A_3 = a_4 = a_1 + 3d = 3 + 3(4) = 15 \Rightarrow H_3 = \frac{1}{15}$

$A_4 = a_5 = a_1 + 4d = 3 + 4(4) = 19 \Rightarrow H_4 = \frac{1}{19}$

so required 4 H.Ms are $\frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \frac{1}{19}$

(ii) $\frac{7}{3}$ **and** $\frac{7}{11}$

Let A_1, A_2, A_3, A_4 are four A.Ms between $\frac{7}{3}$ and $\frac{11}{7}$

$\Rightarrow$ $\frac{3}{7}, A_1, A_2, A_3, A_4, \frac{11}{7}$ are in A.P

$a_1 = \frac{3}{7}$ and $a_6 = \frac{11}{7}$

$a_1 + 5d = \frac{11}{7}$

$\frac{3}{7} + 5d = \frac{11}{7}$

$5d = \frac{11}{7} - \frac{3}{7} = \frac{8}{7}$

$d = \frac{8}{35}$

$A_1 = a_2 = a_1 + d = \frac{3}{7} + \frac{8}{35} = \frac{23}{35} \Rightarrow H_1 = \frac{35}{23}$

$A_2 = a_3 = a_1 + 2d = \frac{3}{7} + 2\left(\frac{8}{35}\right) = \frac{3}{7} + \frac{16}{35} = \frac{31}{35} \Rightarrow H_2 = \frac{35}{31}$

$$A_3 = a_4 = a_1 + 3d = \frac{3}{7} + 3\left(\frac{8}{35}\right) = \frac{3}{7} + \frac{24}{35} = \frac{39}{35} \Rightarrow H_3 = \frac{35}{39}$$

$$A_4 = a_5 = a_1 + 4d = \frac{3}{7} + 4\left(\frac{8}{35}\right) = \frac{3}{7} + \frac{32}{35} = \frac{47}{35} \Rightarrow H_4 = \frac{35}{47}$$

so the required 4 H.Ms are

$$\frac{35}{23}, \frac{35}{31}, \frac{35}{39}, \frac{35}{47}$$

(iii) 4 and 20

Let A_1, A_2, A_3, A_4 are four A.Ms between $\frac{1}{4}$ and $\frac{1}{2}$

$\Rightarrow$ $\frac{1}{4}, A_1, A_2, A_3, A_4, \frac{1}{20}$ are in A.P

$a_1 = \frac{1}{4}$ and $a_6 = \frac{1}{20}$

$$a_1 + 5d = \frac{1}{20}$$

$$\frac{1}{4} + 5d = \frac{1}{20}$$

$$5d = \frac{1}{20} - \frac{1}{4} = -\frac{4}{20} = -\frac{2}{10}$$

$$d = -\frac{2}{50}$$

$\Rightarrow$ $A_1 = a_2 = a_1 + d = \frac{1}{4} - \frac{2}{50} = \frac{21}{100} \Rightarrow H_1 = \frac{100}{21}$

$$A_2 = a_3 = a_1 + 2d = \frac{1}{4} + 2\left(-\frac{2}{50}\right) = \frac{1}{4} - \frac{4}{50} = \frac{17}{100} = \Rightarrow H_2 = \frac{100}{17}$$

$$A_3 = a_4 = a_1 + 3d = \frac{1}{4} + 3\left(-\frac{2}{50}\right) = \frac{1}{4} - \frac{6}{50} = \frac{13}{100} \Rightarrow H_3 = \frac{100}{3}$$

$$A_4 = a_5 = a_1 + 4d = \frac{1}{4} + 4\left(-\frac{2}{50}\right) = \frac{1}{4} - \frac{8}{50} = \frac{9}{100} \Rightarrow H_4 = \frac{100}{9}$$

so the required four H.Ms are

$$\frac{100}{21}, \frac{100}{17}, \frac{100}{13}, \frac{100}{9}$$

Q.5 If the 7^{th} and 10^{th} terms of an H.P are $\frac{1}{3}$ and $\frac{5}{21}$ respectively, find its 14^{th} term.

Solution:

7th term of H.P $= \frac{1}{3}$

7th term of A.P $= 3 = a_7$

10th term of H.P $= \frac{5}{21}$

10th term of A.P $= \frac{21}{5} = a_{10}$

$\Rightarrow$ $a_1 + 6d = 3$ ..... (1)

$a_1 + 9d = \frac{21}{5}$ ..... (2)

subtracting (1) from (2), we get

$3d = \frac{21}{5} - 3 = \frac{6}{5}$

$$\boxed{d = \frac{2}{5}}$$

Put $d = \frac{2}{5}$ in (1), we get

$a_1 + 6 . \frac{2}{5} = 3$

$a_1 + \frac{12}{5} = 3$

$a_1 = 3 - \frac{12}{5} = \frac{15 - 12}{5}$

$$\boxed{a_1 = \frac{3}{5}}$$

As in A.P

$a_{14} = a_1 + 13d$

$= \frac{3}{5} + 13\left(\frac{2}{5}\right)$

$= \frac{3}{5} + \frac{26}{5} = \frac{29}{5}$

$\Rightarrow$ 14^{th} term of the given H.P $= \frac{5}{29}$

Q.6 If first term of an H.P. is $-\frac{1}{3}$ and fifth term is $\frac{1}{5}$. Find its 9th term.

Solution:

1^{st} term of H.P $= -\frac{1}{3}$

1^{st} term of A.P $= -3 = a_1$

5^{th} term of H.P $= \frac{1}{5}$

5^{th} term of A.P $= 5 = a_5$

As $a_5 = a_1 + 4d$

$5 = -3 + 4d$

$5 + 3 = 4d \Rightarrow 8 = 4d = 8 \Rightarrow \boxed{d = 2}$

so 9^{th} term of A.P

$a_9 = a_1 + 8d$

$= -3 + 8\,(2)$

$= 13$

and 9th term of H.P $= \frac{1}{13}$

Q.7 If 5 is the HM between 2 and b. Find b. (Lahore Board 2007, 2010)

Solution:

If H is H.M between a and b

then

$$\text{H.M.} = \frac{2ab}{a+b}$$

but here 5 is the H.M between 2 and b.

$$\Rightarrow \quad 5 = \frac{2\,(2)\,b}{2+b}$$

$$\Rightarrow \quad 5 = \frac{4b}{2+b}$$

$$\Rightarrow \quad 5\,(2+b) = 4b$$

$$\Rightarrow \quad 10 + 5b - 4b = 0$$

$$\Rightarrow \quad 10 + b = 0 \Rightarrow \boxed{b = -10}$$

Q.8 If the numbers $\frac{1}{k}, \frac{1}{2k+1}, \frac{1}{4k-1}$ are in harmonic sequence. Find k.

(Lahore Board 2003, Gujranwala Board 2007)

Solution:

As $\frac{1}{k}, \frac{1}{2k+1}, \frac{1}{4k-1}$ are in H.P

$\Rightarrow$ $k, 2k+1, 4k-1$ are in A.P

$\Rightarrow$ $2k+1-k = 4k-1-(2k+1)$ common difference should same

$\Rightarrow$ $k+1 = 4k-1-2k-1$

$\Rightarrow$ $k+1 = 2k-2$ (B)

$\Rightarrow$ $2k-2-k-1 = 0$

$\Rightarrow$ $k-3 = 0$

$\Rightarrow$ $\boxed{k = 3}$

Q.9 Find n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be HM between a and b.

(Lahore Board 2003)

Solution:

As harmonic mean between a and b is given by

$$H = \frac{2ab}{a+b}$$

therefore

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{2ab}{a+b}$$

$\Rightarrow$ $(a+b)(a^{n+1}+b^{n+1}) = 2ab(a^n+b^n)$

$\Rightarrow$ $a^{n+2}+ab^{n+1}+ba^{n+1}+b^{n+2} = 2a^{n+1}b+2ab^{n+1}$

$\Rightarrow$ $a^{n+2}+b^{n+2} = a^{n+1}b+ab^{n+1}$

$\Rightarrow$ $a^{n+2}-a^{n+1}b = ab^{n+1}-b^{n+2}$

$\Rightarrow$ $a^{n+1}(a-b) = b^{n+1}(a-b)$

$\Rightarrow$ $a^{n+1} = b^{n+1}$

$\Rightarrow$ $\frac{a^{n+1}}{b^{n+1}} = 1$

$\Rightarrow$ $\left(\frac{a}{b}\right)^{n+1} = \left(\frac{a}{b}\right)^0$

$\Rightarrow$ $n+1 = 0$

$\Rightarrow$ $\boxed{n = -1}$

Q.10 a^2, b^2, c^2 are in A.P. Show that $b + c, c + a, a + b$ are in H.P.

(Gujranwala Board 2004)

Solution:

Given that a^2, b^2, c^2 in A.P.

$\Rightarrow \quad b^2 - a^2 = c^2 - b^2 \qquad \ldots\ldots\ldots (1)$

To show that $b + c,\ c + a,$ are in H.P we have to show that

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P

For this we will show that

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

or $$\frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$$

$$\frac{b-a}{b+c} = \frac{c-b}{a+b}$$

$$(b-a)(b+a) = (c-b)(c+b)$$

$$b^2 - a^2 = c^2 - b^2$$

which is true from (1)

$\Rightarrow \quad \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P

$\Rightarrow \quad b + c,\ c + a,\ a + b$ are in H.P

Q.11 The sum of the first and fifth terms of the harmonic sequence is $\frac{4}{7}$, if the first term is $\frac{1}{2}$, find the sequence.

Solution:

Given that

$$\frac{1}{a_1} + \frac{1}{a_5} = \frac{4}{7}$$

or $$\frac{1}{a_1} + \frac{1}{a_1 + 4d} = \frac{4}{7} \qquad \ldots\ldots\ldots (1)$$

also $$\frac{1}{a_1} + \frac{1}{2 + 4d} = \frac{4}{7}$$

Put this value of a1 in equation (1), we get

$$\frac{1}{2} + \frac{1}{2 + 4d} = \frac{4}{7}$$

$$\Rightarrow \quad \frac{1}{2+4d} = \frac{4}{7} - \frac{1}{2} = \frac{8-7}{14}$$

$$\Rightarrow \quad \frac{1}{2+4d} = \frac{1}{14} \Rightarrow 2+4d = 14$$

$$4d = 12$$

$$\boxed{d = 3}$$

so required H.P is

$$\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \ldots\ldots$$

$$\Rightarrow \quad \frac{1}{2}, \frac{1}{2+3}, \frac{1}{2+2(3)}, \ldots\ldots$$

$$\Rightarrow \quad \frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots\ldots$$

Q.12 If, A, G and H are arithmetic geometric and harmonic means between a and b respectively. Show that $G^2 = A\,H$.

Solution:

As A, G, H are arithmetic, geometric and harmonic means between a and b.

$$\Rightarrow \quad A = \frac{a+b}{2}, \; G = \pm\sqrt{ab}, \; H = \frac{2ab}{a+b}$$

and $$A\,H = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = G^2$$

$$\Rightarrow \quad A\,H = G^2 \text{ or } G^2 = AH.$$

Hence proved.

Q.13 Find A, G and H and show that $G^2 = A.H.$ if

(i) $a = -2, \; b = -6$ **(ii) $a = 2i, \; b = 4i$**

(iii) $a = 9, \; b = 4$

Solution:

(i) $a = -2, \; b = -6$

$$A.M. = \frac{a+b}{2} = \frac{-2+-6}{2} = -\frac{8}{2} = -4$$

$$G = \sqrt{ab} = \pm\sqrt{(-2)(-6)} = \pm\sqrt{12}$$

$$H.M. = \frac{2ab}{a+b} = \frac{2(-2)(-6)}{-2+(-6)} = \frac{24}{-8} = -3$$

Now $$AH = (-4)(-3) = 12 = G^2$$

$$\Rightarrow \quad G^2 = A\,H$$

(ii) **$a = 2i,\ b = 4i$**

$$A = \frac{a+b}{2} = \frac{2i+4i}{2} = \frac{6i}{2} = 3i$$

$$G = \pm\sqrt{ab} = \pm\sqrt{(2i)(4i)} = \pm\sqrt{8i^2} = \pm\sqrt{-8}$$

$$H = \frac{2ab}{a+b} = \frac{2(2i)(4i)}{2i+4i} = \frac{16i^2}{61} = \frac{8}{3}i$$

Now $A.H = 3i \times \frac{8}{3}i = 8i^2 = -8 = G^2$

$\Rightarrow$ $G^2 = AH$

(iii) **$a = 9,\ b = 4$**

$$A = \frac{a+b}{2} = \frac{9+4}{2} = \frac{13}{2}$$

$$G = \pm\sqrt{ab} = \pm\sqrt{(9)(4)} = \pm\sqrt{36} = \pm 6$$

$$H = \frac{2ab}{a+b} = \frac{2(9)(4)}{9+4} = \frac{72}{13}$$

Now $AH = \frac{13}{2} \cdot \frac{72}{13} = 36 = G^2$

$\Rightarrow$ $G^2 = AH.$

Q.14 Find A. G, H and verify that $A < G < H$ $(G > 0)$ if

(i) **$a = 2,\ b = 8$** **(ii)** **$a = -\frac{2}{5},\ b = -\frac{8}{5}$**

(Lahore Board 2008)

Solution:

(i) **$a = 2,\ b = 8$**

As $A = \frac{a+b}{2} = \frac{2+8}{2} = \frac{10}{2} = 5$

$$G = \pm\sqrt{ab} = \pm\sqrt{(2)(8)} = \pm\sqrt{16} = \pm 4$$

$\Rightarrow$ $G = 4$ ¤ $G > 0$

$$H = \frac{2ab}{a+b} = \frac{2(2)(8)}{2+8} = \frac{32}{10} = \frac{16}{5}$$

Here $5 > 4 < \frac{16}{5}$ $\Rightarrow$ $A > G > H$

(ii) $\mathbf{a = \frac{-2}{5}, \quad \frac{-8}{5}}$

As $A = \frac{a+b}{2} = \frac{\frac{2}{5}+\frac{8}{5}}{2} = \frac{\frac{2+8}{5}}{2} = \frac{\frac{10}{5}}{2} = \frac{2}{2} = 1$

$G = \sqrt{ab} = \sqrt{\left(\frac{2}{5}\right)\left(\frac{8}{5}\right)} = \sqrt{\frac{16}{25}} = \frac{4}{5} \quad \because G > 0$

$G = \frac{4}{5}$

$H = \frac{2ab}{a+b} = \frac{2\left(\frac{2}{5}\right)\left(\frac{8}{5}\right)}{\frac{2}{5}+\frac{8}{5}} = \frac{\frac{32}{25}}{\frac{10}{5}} = \frac{32}{25} \cdot \frac{5}{10} = \frac{16}{25}$

Here $+1 > \frac{4}{5} > \frac{16}{25} \Rightarrow A > G > H.$

Q.15 Find A, G, H and verify that $A < G < H$ $(G < 0)$ if

(i) $\mathbf{a = -2, \ b = -8}$ **(Lahore Board 2009)**

(ii) $\mathbf{a = \frac{-2}{5}, \ b = \frac{-8}{5}}$

Solution:

(i) $\mathbf{a = -2, \ b = -8}$

As $A = \frac{a+b}{2} = \frac{-2-8}{2} = \frac{-10}{2} = -5$

$G = -\sqrt{ab} = -\sqrt{(-2)(-8)} = -\sqrt{16} = -4 \quad \because G < 0$

$H = \frac{2ab}{a+b} = \frac{2(-2)(-8)}{-2-8} = \frac{32}{-10} = -\frac{16}{5}$

Here $-5 < -4 < -\frac{16}{5}$

$\Rightarrow \quad A < G < H.$

(ii) $\mathbf{a = \frac{-2}{5}, \ b = \frac{-8}{5}}$

As $A = \frac{a+b}{2} = \frac{-\frac{2}{5}-\frac{8}{5}}{2} = \frac{\frac{-2-8}{5}}{2} = \frac{\frac{-10}{5}}{2} = \frac{-2}{2} = 1$

$$G = \sqrt{ab} = \sqrt{\left(\frac{-2}{5}\right)\left(\frac{-8}{5}\right)} = -\sqrt{\frac{16}{25}} = -\frac{4}{5} \quad \because\ G > 0$$

$$H = \frac{2ab}{a+b} = \frac{2\left(-\frac{2}{5}\right)\left(-\frac{8}{5}\right)}{-\frac{2}{5}-\frac{8}{5}} = \frac{\frac{32}{25}}{\frac{-10}{5}} = -\frac{32}{25} \times \frac{5}{10} = \frac{-16}{25}$$

Here $-1 < -\frac{4}{5} < -\frac{16}{25} \Rightarrow A < G < H.$

Q.16 If the H.M. and A. M. between two numbers are 4 and $\frac{9}{2}$ respectively. Find the numbers. (Lahore Board 2003, 2007)

Solution:

Let a and b are required numbers

then by given conditions

$$\frac{a+b}{2} = \frac{9}{2} \Rightarrow a + b = 9 \quad \ldots\ldots\ldots (1)$$

and $$\frac{2ab}{a+b} = 4$$

$$2ab = 4(a+b)$$

$$2ab = 4a + 4b \quad \ldots\ldots\ldots (2)$$

from (1) $a = 9 - b$ ..... (3)

Put $a = 9 - b$ in (2), we get

$\Rightarrow$ $2(9-b)b = 4(9-b) + 4b$

$\Rightarrow$ $18b - 2b^2 = 36 - 4b + 4b$

$\Rightarrow$ $18b - 2b^2 = 36$

$\Rightarrow$ $9b - b^2 = 18$

$\Rightarrow$ $b^2 - 9b + 18 = 0$

$\Rightarrow$ $b^2 - 6b - 3b + 18 = 0$

$\Rightarrow$ $b(b-6) - 3(b-6) = 0$

$\Rightarrow$ $(b-6)(b-3) = 0$

$\Rightarrow$ $b = 6$ or $b = 3$

Put $b = 6$ in equation (3), we get

$a = 9 - 6 = 3$

Put $b = 3$ in equation (3), we get

$a = 9 - 3 = 6$

so the required numbers are 3, 6, or 6, 3

Q.17 If the (positive) G.M. and H.M. between two numbers 4 and $\frac{16}{5}$. Find numbers.

Solution:

Let a and b are required numbers

then by given conditions

$\sqrt{ab} = 4 \Rightarrow ab = 16 \quad \because G > 0 \quad \ldots\ldots\ldots (1)$

and $\frac{2ab}{a+b} = \frac{16}{5}$

$10ab = 16(a+b) \quad \ldots\ldots\ldots (2)$

From equation (1)

$a = \frac{16}{b} \quad \ldots\ldots\ldots (3)$

Put (3) in (2), we get

$10 \frac{16}{b} b = 16\left(\frac{16}{b} + b\right)$

$10 = \frac{16}{b} + b$

Multiplying by b, we get

$\Rightarrow \quad 10b = 16 + b^2$

$\Rightarrow \quad b^2 - 10b + 16 = 0$

$\Rightarrow \quad b^2 - 8b - 2b + 16 = 0$

$\Rightarrow \quad b(b-8) - 2(b-8) = 0$

$\Rightarrow \quad (b-2)(b-8) = 0$

$\Rightarrow \quad b = 2 \quad \text{or} \quad b = 8$

Put $b = 2$ in equation (3), we get

$a = \frac{16}{2} = 8$

Put $b = 8$ in equation (3), we get

$a = \frac{16}{8} = 2$

So the required numbers are 2, 8 or 8, 2.

Q.18 **If the numbers $\frac{1}{2}, \frac{4}{21}$ and $\frac{1}{36}$ are subtracting from three consecutive terms of a G.P., the resulting numbers are in H.P. Find the numbers if their product is $\frac{1}{27}$.**

Solution:

Let $\frac{a}{r}$, a, ar are required term of G.P. then by given condition

$$\frac{a}{r} a \cdot ar = \frac{1}{27} \Rightarrow a^3 = \frac{1}{27} \Rightarrow \boxed{a = \frac{1}{3}}$$

and $\frac{a}{r} - \frac{1}{2}, a - \frac{4}{21}, ar - \frac{1}{36}$ are in H.P.

Put $a = \frac{1}{3}$

$$\frac{1}{3r} - \frac{1}{2}, \frac{1}{3} - \frac{4}{21}, \frac{r}{3} - \frac{1}{36} \text{ are in H.P.}$$

$$\frac{2-3r}{6r}, \frac{3}{21}, \frac{12r-1}{36} \text{ are in H.P.}$$

$$\frac{6r}{2-3r}, 7, \frac{36}{12r-1} \text{ are in A.P.}$$

$$\Rightarrow 7 = \frac{\frac{6r}{2-3r} + \frac{36}{12r-1}}{2}$$

$$\Rightarrow 14 = \frac{6r}{2-3r} + \frac{36}{12r-1}$$

$$\Rightarrow 7 = \frac{3r}{2-3r} + \frac{18}{12r-1}$$

$$\Rightarrow 7 = \frac{36r^2 - 3r + 36 - 54r}{(2-3r)(12r-1)}$$

$$\Rightarrow 7 = \frac{36r^2 - 57r + 36}{(2-3r)(12r-1)}$$

$$\Rightarrow 7(2-3r)(12r-1) = 36r^2 - 57r + 36$$

$$\Rightarrow 7(-36r^2 + 27r - 2) = 36r^2 - 57r + 36$$

$$\Rightarrow -252r^2 + 189r - 14 = 36r^2 - 57r + 36$$

$$\Rightarrow 288r^2 - 246r + 50 = 0$$

$$\Rightarrow 2(144r^2 - 123r + 25) = 0$$

$$\Rightarrow 144r^2 - 123r + 25 = 0$$

$$\Rightarrow 144r^2 - 48r - 75r + 25 = 0$$

$\Rightarrow \quad 48r(3r-1) - 25(3r-1) = 0$

$\Rightarrow \quad (3r-1)(48r-25) = 0$

$\Rightarrow \quad r = \frac{1}{3} \quad \text{or} \quad r = \frac{25}{48}$

When $a = \frac{1}{3}$, $r = \frac{1}{3}$ then

$$\frac{a}{r} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1, \quad a = \frac{1}{3}, \quad ar = \frac{1}{3}\,\frac{1}{3} = \frac{1}{9}$$

When $a = \frac{1}{3}$, $r = \frac{25}{48}$

$$\frac{a}{r} = \frac{\frac{1}{3}}{\frac{25}{48}} = \frac{16}{25}, \quad a = \frac{1}{3}, \quad ar = \frac{1}{3}\,\frac{25}{48} = \frac{25}{144}$$

So the required numbers are

$$1, \frac{1}{3}, \frac{1}{9} \quad \text{or} \quad \frac{16}{25}, \frac{1}{3}, \frac{25}{144}$$

FORMULAE FOR THE SUMS

$$\sum_{k=1}^{n} 1 = n$$

$$1 + 2 + 3 + \ldots\ldots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

EXERCISE 6.11

Sum the following series upto n terms.

Q.1 $1 \times 1 + 2 \times 4 + 3 \times 7 + \ldots\ldots$

Solution:

a_n of $1, 2, 3, \ldots\ldots$ is n ¤ $a_n = a_1 + (n-1)d$

and a_n of $1, 4, 7, \ldots\ldots$ is $1 + (n-1)(+3) = 3n - 2$ so nth term of the given series is

$T_n = n(3n-2) = 3n^2 - 2n \quad \Rightarrow \quad T_k = 3k^2 - 2k$

Let S_n is the required sum then

$$S_n = T_1 + T_2 + \ldots\ldots + T_n$$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (3k^2 - 2k)$$

$$= \sum_{k=1}^{n} 3k^2 - \sum_{k=1}^{n} 2k$$

$$= 3\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k$$

$$= 3\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1)}{2} - \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}[2n+1-2]$$

$$= \frac{n(n+1)}{2}[2n-1]$$

$$= \frac{n(n+1)(2n-1)}{2}$$

$$= \frac{n}{2}(2n^2 - n + 2n - 1)$$

$$= \frac{n}{2}(2n^2 + n - 1)$$

Q.2 $1 \times 3 + 3 \times 6 + 5 \times 9 + \ldots\ldots$

Solution:

a_n of 1, 3, 5, ... is $1 + (n-1)(2) = 2n - 1$

a_n of 3, 6, 9, ... is $3 + (n-1)(3) = 3n$

so nth term of the given series is

$T_n = (2n-1)(3n) = 6n^2 - 3n \quad \Rightarrow \quad T_k = 6k^2 - 3k$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (6k^2 - 3k)$$

$$= 6 \sum_{k=1}^{n} k^2 - 3 \sum_{k=1}^{n} k$$

$$= 6 \frac{n(n+1)(2n+1)}{6} - 3 \frac{n(n+1)}{2}$$

$$= \frac{2n(n+1)(2n+1)}{2} - \frac{3n(n+1)}{2}$$

$$= \frac{n(n+1)}{2} [4n + 2 - 3] = \frac{n(n+1)(4n-1)}{2}$$

Q.3 $1 \times 4 + 2 \times 7 + 3 \times 10 + \ldots\ldots\ldots$

Solution:

a_n of $1, 2, 3, \ldots\ldots$ is n

a_n of $4, 7, 10, \ldots\ldots$ is $4 + (n-1)(3) = 3n + 1$

nth term of the given series is

$T_n = n(3n+1) = 3n^2 + n$

and $T_k = k(3k+1) = 3k^2 + k$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (3k^2 + k)$$

$$= \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} k$$

$$= 3 \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$= 3\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}[2n+1+1]$$

$$= \frac{n(n+1)(2n+2)}{2} = \frac{2n(n+1)(n+1)}{2} = n(n+1)^2$$

Q.4 $3 \times 5 + 5 \times 9 + 7 \times 13 + ...$

Solution:

a_n of 3, 5, 7, ... is $3 + (n-1)(2) = 2n + 1$

a_n of 5, 9, 13, ... is $5 + (n-1)(4) = 4n + 1$

so nth term of the given series is

$T_n = (2n+1)(4n+1) = 8n^2 + 6n + 1$

and $T_k = 8k^2 + 6k + 1$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + ... + T_n$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (8k^2 + 6k + 1)$$

$$= \sum_{k=1}^{n} 8k^2 + \sum_{k=1}^{n} 6k + \sum_{k=1}^{n} 1$$

$$= 8\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 8\frac{n(n+1)(2n+1)}{6} + 6\frac{n(n+1)}{2} + n$$

$$= \frac{4n(n+1)(2n+1)}{3} + 3n(n+1) + n$$

$$= \frac{4n(n+1)(2n+1)}{3} + 3n^2 + 3n + n$$

$$= \frac{n(4n+4)(2n+1)}{3} + 3n^2 + 4n$$

$$= n\left[\frac{(4n+4)(2n+1) + 9n + 12}{3}\right]$$

$$= \frac{n}{3}[8n^2 + 12n + 4 + 9n + 12]$$

$$= \frac{n}{3}[8n^2 + 21n + 16]$$

Q.5 $1^2 + 3^2 + 5^2 + ...$ (Lahore Board 2006)

Solution:

a_n of 1, 3, 5, ... is $1 + (n-1)(2) = 2n - 1$

nth term of the given series is

$T_n = (2n-1)^2 = 4n^2 - 4n + 1$

and $T_k = 4k^2 - 4k + 1$

Let S_n be the required sum

$\Rightarrow$ $S_n = T_1 + T_2 + T_3 + ... + T_n$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (4k^2 - 4k + 1)$$

$$= \sum_{k=1}^{n} 4k^2 - \sum_{k=1}^{n} 4k + \sum_{k=1}^{n} 1$$

$$= 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 4\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n$$

$$= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n$$

$$= n\left[\frac{2n(n+1)(2n+1)}{3} - 2(n+1) + 1\right]$$

$$= n\frac{2(n+1)(2n+1) - 6(n+1) + 3}{3}$$

$$= \frac{n}{3}[4n^2 + 6n + 2 - 6n - 6 + 3]$$

$$= \frac{n}{3}[4n^2 - 1]$$

Q.6 $2^2 + 5^2 + 8^2 + ...$

Solution:

a_n of 2, 5, 8, ... is $2 + (n - 1)(3) = 3n - 1$

nth term of the given series is

$T_n = (3n - 1)^2 = 9n^2 - 6n + 1$

and $T_k = 9k^2 - 6k + 1$

Let S_n be the required sum

$$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + ... + T_n$$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} 9k^2 - 6k + 1$$

$$= \sum_{k=1}^{n} 9k^2 - \sum_{k=1}^{n} 6k + \sum_{k=1}^{n} 1$$

$$= 9 \sum_{k=1}^{n} k^2 - 6 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= \frac{9n(n+1)(2n+1)}{6} - 6\frac{n(n+1)}{2} + n$$

$$= \frac{3n(n+1)(2n+1)}{2} - 3n(n+1) + n$$

$$= \frac{n}{2}[3(n+1)(2n+1) - 6(n+1) + 2]$$

$$= \frac{n}{2}[6n^2 + 9n + 3 - 6n - 6 + 2]$$

$$= \frac{n}{2}[6n^2 + 3n - 1]$$

Q.7 $2 \times 1^2 + 4 \times 2^2 + 6 \times 3^2 + ...$

Solution:

a_n of 2, 4, 6, ... is $2 + (n - 1)(2) = 2n$

a_n of 1, 2, 3, ... is n

nth term of the given series is

$T_n = 2n\,(n)^2 = 2n^3$

and $T_k = 2k^3$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} 2k^3$$

$$= 2\sum_{k=1}^{n} k^3 = 2\left[\frac{n\,(n+1)}{2}\right]^2$$

$$= 2\,\frac{n^2\,(n+1)^2}{4} = \frac{n^2\,(n+1)^2}{2}$$

Q.8 $\mathbf{3 \times 2^2 + 5 \times 3^2 + 7 \times 4^2 + ...}$

Solution:

a_n of 3, 5, 7, ... is $3 + (n-1)\,(2) = 2n + 1$

a_n of 2, 3, 4, ... is $n + 1$

nth term of the given series is

$T_n = (2n+1)\,(n+1)^2 = 2n^3 + 5n^2 + 4n + 1$

and $T_k = 2k^3 + 5k^2 + 4k + 1$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (2k^3 + 5k^2 + 4k + 1)$$

$$= \sum_{k=1}^{n} 2k^3 + \sum_{k=1}^{n} 5k^2 + \sum_{k=1}^{n} 4k + \sum_{k=1}^{n} 1$$

$$= 2\sum_{k=1}^{n} k^3 + 5\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 2\left[\frac{n(n+1)}{2}\right]^2 + 5\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} + n$$

$$= \frac{n^2(n+1)^2}{2} + \frac{5n(n+1)(2n+1)}{6} + 2n(n+1) + n$$

$$= \frac{n^2(n^2+2n+1)}{2} + \frac{5n(2n^2+3n+1)}{6} + 2n(n+1) + n$$

$$= \frac{n}{6}\left[3n(n^2+2n+1) + 5(2n^2+3n+1) + 12(n+1) + 6\right]$$

$$= \frac{n}{6}\left[3n^3 + 6n^2 + 3n + 10n^2 + 15n + 5 + 12n + 12 + 6\right]$$

$$= \frac{n}{6}\left[3n^3 + 16n^2 + 30n + 23\right]$$

Q.9 $2 \times 4 \times 7 + 3 \times 6 \times 10 + 4 \times 8 \times 13 + \ldots\ldots$

Solution:

a_n of $2, 3, 4, \ldots\ldots$ is $n + 1$

a_n of $4, 6, 8, \ldots\ldots$ is $4 + (n-1)\,2 = 2n + 2$

a_n of $7, 10, 13, \ldots\ldots$ is $7 + (n-1)\,(3) = 3n + 4$

nth term of the given series is

$T_n = (n+1)(2n+2)(3n+4) = 6n^3 + 20n^2 + 22n + 8$

and $T_k = 6k^3 + 20k^2 + 22k + 8$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (6k^3 + 20k^2 + 22k + 8)$$

$$= \sum_{k=1}^{n} 6k^3 + \sum_{k=1}^{n} 20k^2 + \sum_{k=1}^{n} 22k + \sum_{k=1}^{n} 8$$

$$= 6\sum_{k=1}^{n} k^3 + 20\sum_{k=1}^{n} k^2 + 22\sum_{k=1}^{n} k + \sum_{k=1}^{n} 8$$

$$= 6\frac{n^2(n+1)^2}{4} + 20\frac{n(n+1)(2n+1)}{6} + 22\frac{n(n+1)}{2} + 8n$$

$$= \frac{3n^2(n^2+1+2n)}{2} + \frac{10n(n+1)(2n+1)}{3} + 11n(n+1) + 8n$$

$$= n\left[\frac{3n(n^2+1+2n)}{2} + \frac{10(2n^2+3n+1)}{3} + 11(n+1) + 8\right]$$

$$= n\left[\frac{9n(n^2+1+2n) + 20(2n^2+3n+1) + 66(n+1) + 48}{6}\right]$$

$$= \frac{n}{6}\left[9n^3 + 9n + 18n^2 + 40n^2 + 60n + 20 + 66n + 66 + 48\right]$$

$$= \frac{n}{6}\left[9n^3 + 58n^2 + 135n + 134\right]$$

Q.10 $1 \times 4 \times 6 + 4 \times 7 \times 10 + 7 \times 10 \times 14 + ...$

Solution:

a_n of $1, 4, 7, ...$ is $1 + (n-1)^3 = 3n - 2$

a_n of $4, 7, 10, ...$ is $4 + (n-1)(3) = 3n + 1$

a_n of $6, 10, 14, ...$ is $6 + (n-1)(4) = 4n + 2$

nth term of the given series is

$T_n = (3n-2)(3n+1)(4n+2) = 36n^3 + 6n^2 - 14n - 4$

and $T_k = 36k^3 + 6k^2 - 14k - 4$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + ... + T_n$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (36k^3 + 6k^2 - 14k - 4)$$

$$= 36\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 - 14\sum_{k=1}^{n} k - 4\sum_{k=1}^{n} k$$

$$= 36\frac{n^2(n+1)^2}{4} + 6\frac{n(n+1)(2n+1)}{6} - 14\frac{n(n+1)}{2} - 4n$$

$$= 9n^2(n^2+1+2n) + n(2n^2+3n+1) - 7n(n+1) - 4n$$

$= n\left[9n^3 + 9n + 18n^2 + 2n^2 + 3n + 1 - 7n - 7 - 4\right]$

$= n\left[9n^3 + 20n^2 + 5n - 10\right]$

Q.11 $1 + (1 + 2) + (1 + 2 + 3) + \ldots\ldots\ldots$

Solution:

nth term of the given series is

$$T_n = 1 + 2 + 3 + \ldots\ldots n = \frac{n(n+1)}{2} = \frac{1}{2}(n^2 + n)$$

and $T_k = \frac{1}{2}(k^2 + k)$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} \frac{1}{2}(k^2 + k)$$

$$= \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{1}{2}\sum_{k=1}^{n} k$$

$$= \frac{1}{2}\,\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\,\frac{n(n+1)}{2}$$

$$= \frac{1}{2}\,\frac{n(n+1)}{2}\left[\frac{2n+1}{3} + 1\right]$$

$$= \frac{n(n+1)}{4}\left[\frac{2n+1+3}{3}\right]$$

$$= \frac{n(n+1)}{4}\left(\frac{2n+4}{3}\right) = \frac{n(n+1)(n+2)}{6}$$

Q.12 $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots\ldots$

Solution:

nth term of the given series is

$$T_n = 1^2 + 2^2 + 3^2 + \ldots\ldots n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}(2n^3 + 3n^2 + n)$$

and $T_k = \frac{1}{2}(2k^3 + 3k^2 + k)$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} \frac{1}{6}(2k^3 + 3k^2 + k)$$

$$= \frac{1}{6}\left[2\sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right]$$

$$= \frac{1}{6}\left[2\frac{n^2(n+1)^2}{4} + 3\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{1}{6}\left[\frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{12}\left[n(n+1) + (2n+1) + 1\right]$$

$$= \frac{n(n+1)}{12}\left[n^2 + n + 2n + 1 + 1\right]$$

$$= \frac{n(n+1)}{12}\left[n^2 + 3n + 2\right]$$

$$= \frac{n(n+1)}{12}\left[n^2 + 2n + n + 2\right]$$

$$= \frac{n(n+1)}{12}\left[(n+1)(n+2)\right]$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

Q.13 $2 + (2 + 5) + (2 + 5 + 8) + \ldots\ldots$

Solution:

nth term of the given series is

$T_n = 2 + 5 + 8 + \ldots\ldots$

Using $S_n = \left[2a_1 + (n-1)d\right]$ ¤ $a_1 = 2,\ d = 3$

$$T_n = \frac{n}{2}\left[2(2) + (n-1)(3)\right]$$

$$= \frac{n}{2}[3n + 1] = \frac{1}{2}(3n^2 + n)$$

Let S_n be the required sum

$$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$$

$$= \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{2}(3k^2 + k)$$

$$= \frac{1}{2}\left[3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k \right]$$

$$= \frac{1}{2}\left[2\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right]$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2} \right]$$

$$= \frac{1}{2}\,\frac{n(n+1)}{2}[2n + 1 + 1]$$

$$= \frac{1}{4}\,n(n+1)(2n+2)$$

$$= \frac{n(n+1)(n+1)}{2} = \frac{n(n+1)^2}{2}$$

Q.14 Sum the series.

(i) $\mathbf{1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots (2n-1)^2 - (2n)^2}$

(ii) $\mathbf{1^2 - 3^2 + 5^2 - 7^2 + \ldots\ldots (4n-3)^2 - (4n-1)^2}$

(iii) $\mathbf{\dfrac{1^2}{1} + \dfrac{1^2+2^2}{2} + \dfrac{1^2+2^2+3^2}{3} + \ldots\ldots}$ **+ to n terms**

Solution:

(i) $\mathbf{1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots (2n-1)^2 - (2n)^2}$

As $T_n = (2n-1)^2 - (2n)^2 = 4n^2 - 4n + 1 - 4n^2$

$$= -4n + 1$$

$$T_k = -4k + 1$$

Let S_n be the required sum

$$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$$

$$= \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (-4k + 1)$$

$$= -4\sum_{k=1}^{n} + \sum_{k=1}^{n} 1$$

$$= -4\,\frac{n(n+1)}{2} + n = -2n(n+1) + n$$

$$= -2n^2 - 2n + n = -2n^2 - n = -n(2n+1)$$

(ii) $\mathbf{1^2 - 3^2 + 5^2 - 7^2 + \ldots\ldots (4n-3)^2 - (4n-1)^2}$

$T_n = (4n-3)^2 - (4n-1)^2 = 16n^2 - 24n + 9 - 16n^2 + 8n - 1$

$= -16n + 8$

$T_k = -16k + 8$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (-16k + 8)$$

$$= -16 \sum_{k=1}^{n} k + 8 \sum_{k=1}^{n} 1$$

$$= -16 \frac{n(n+1)}{2} + 8n$$

$$= -8n(n+1) + 8n = -8n^2 - 8n + 8n = -8n^2$$

(iii) $\mathbf{\dfrac{1^2}{1} + \dfrac{1^2 + 2^2}{2} + \dfrac{1^2 + 2^2 + 3^2}{3} + \ldots\ldots \text{ to } n \text{ terms}}$

$$T_n = \frac{1}{n}[1^2 + 2^2 + 3^2 + \ldots\ldots + n^2]$$

$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{1}{6}(2n^2 + 3n + 1)$$

$$T_k = \frac{1}{6}(2k^2 + 3k + 1)$$

Let S_n be the required sum

$$S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$$

$$= \sum_{k=1}^{n} T_k = \frac{1}{6} \sum_{k=1}^{n} (2k^2 + 3k + 1)$$

$$= \frac{1}{6}\left[2 \sum_{k=1}^{n} k^2 + 3 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right]$$

$$= \frac{1}{6}\left[2 \frac{n(n+1)(2n+1)}{6} + 3 \frac{n(n+1)}{2} + n\right]$$

$$= \frac{1}{6}\left[\frac{2n(n+1)(2n+1) + 9n(n+1) + 6n}{6}\right]$$

$$= \frac{n}{36}\left[2(2n^2 + 3n + 1) + 9(n+1) + 6\right]$$

$$= \frac{n}{36}[4n^2 + 15n + 17]$$

Q.15 Find the sum to n terms of the series whose nth terms are given

(i) $3n^2 + n + 1$ **(ii) $n^2 + 4n + 1$**

Solution:

(i) $3n^2 + n + 1$

$T_n = 3n^2 + n + 1$

$T_k = 3k^2 + k + 1$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (3k^2 + k + 1)$$

$$= 3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 3\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$$

$$= \frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2} + \frac{2n}{2}$$

$$= \frac{n}{2}\left[(n+1)(2n+1) + n + 1 + 2\right]$$

$$= \frac{n}{2}\left[2n^2 + n + 2n + 1 + n + 3\right]$$

$$= \frac{n}{2}\left[2n^2 + 4n + 4\right] = n(n^2 + 2n + 2)$$

(ii) $n^2 + 4n + 1$

As $T_n = n^2 + 4n + 1$

$T_k = k^2 + 4k + 1$

Let S_n be the required sum

$\Rightarrow \quad S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (k^2 + 4k + 1)$$

$$= \sum_{k=1}^{n} k^2 + 4 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} + n$$

$$= \frac{n(n+1)(2n+1)}{6} + 2n(n+1) + n$$

$$= \frac{n}{6}\left[(n+1)(2n+1) + 12(n+1) + 6\right]$$

$$= \frac{n}{6}\left[2n^2 + 3n + 1 + 12n + 12 + 6\right]$$

$$= \frac{n}{6}\left[2n^2 + 15n + 19\right]$$

Q.16 Given nth terms of the series. Find the sum to 2n terms.

(i) $3n^2 + 2n + 1$ **(ii) $n^3 + 2n + 3$**

Solution:

(i) $3n^2 + 2n + 1$

As $T_n = 3n^2 + 2n + 1$

$T_k = 3k^2 + 2k + 1$

Let S_n be the sum of n terms

Then $S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (3k^2 + 2k + 1)$$

$$= 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 3\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} + n$$

$$= \frac{n(n+1)(2n+1)}{2} + \frac{2n(n+1)}{2} + n$$

$$= \frac{n}{2}\left[(n+1)(2n+1) + 2(n+1) + 2\right]$$

$$= \frac{n}{2}\left[2n^2 + 3n + 1 + 2n + 2 + 2\right]$$

$$= \frac{n}{2}\left[2n^2 + 5n + 5\right] \qquad \ldots\ldots\ldots (1)$$

To find sum up to $2n$ terms

$\Rightarrow$ put $n = 2n$ in (1)

$$S_{2n} = \frac{2n}{2}\left[2(2n)^2 + 5(2n) + 5\right]$$

$$= n\left[8n^2 + 10n + 5\right]$$

(ii) $n^3 + 2n + 3$

As $T_n = n^3 + 2n + 3$

$T_k = k^3 + 2k + 3$

Let S_n be the sum of n terms

$\Rightarrow$ $S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (k^3 + 2k + 3)$$

$$= \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k + 3\sum_{k=1}^{n} 1$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + 2\frac{n(n+1)}{2} + 3n$$

$$= \frac{n^2(n+1)^2}{4} + n(n+1) + 3n$$

$$= \frac{n}{4}\left[n(n^2 + 1 + 2n) + 4(n+1) + 12\right]$$

$$= \frac{n}{4}\left[n^3 + n + 2n^2 + 4n + 4 + 12\right]$$

$$= \frac{n}{4}\left[n^3 + 2n^2 + 5n + 16\right] \qquad \ldots\ldots\ldots (1)$$

To find sum of $2n$ terms of the given series

Put $n = 2n$ in (1), we get

$$S_{2n} = \frac{2n}{4}\left[(2n)^3 + 2(2n)^2 + 5(2n) + 16\right]$$

$$= \frac{n}{2}\left[8n^3 + 8n^2 + 10n + 16\right]$$

$$= n(4n^3 + 4n^2 + 5n + 8)$$

CHAPTER 6

SEQUENCES AND SERIES

MULTIPLE CHOICE QUESTIONS

(1) If a_1 and d are first term and common difference respectively of A.P then nth term of A.P:

(a) $a_n = a_1 + (n-1)d$ (b) $a_n = a_1 + (n+1)d$

(c) $a_n = a_1 + (n-2)d$ (d) $a_n = \frac{a_1}{1+(n-1)d}$

[Lahore Board 2005]

(2) $\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \ldots\ldots\ldots$ is a/an:

(a) Arithmetic progression (b) Geometric progression

(c) Harmonic progression (d) None of these

[Lahore Board 2005]

(3) The 5th term of the G.P 3, 6, 12, ... is:

(a) 15 (b) 48

(c) 2 (d) 3

[Lahore Board 2005]

(4) If $a_n = (-1)^{n-1}(n-7)$ then $a_3 =$

(a) –3 (b) –4

(c) –5 (d) None of these

[Lahore Board 2005]

(5) A geometric mean between 'a' and 'b' is:

(a) $\frac{a+b}{2}$ (b) $\frac{2}{a+b}$

(c) $\pm\sqrt{ab}$ (d) $\frac{2ab}{a+b}$

[Gujranwala Board 2005]

(6) The formula for the sum of 'n' terms of an A.P is:

(a) $\frac{n}{2}\{2a_1 + (n + a)d\}$ (b) $\frac{n}{2}\{a_1 + (n - 1)d\}$

(c) $\frac{n}{2}\{2a_1 + (n - 1)d\}$ (d) $\frac{n}{2}\{a_1 - (n - 1)d\}$

[Gujranwala Board 2005]

(7) $3 + 6 + 9 + \ldots\ldots 3n =$

(a) $\frac{3n(n + 1)}{2}$ (b) $\frac{3n(n - 1)}{2}$

(c) $\frac{2n(n + 1)}{3}$ (d) $\frac{2n(n - 1)}{3}$

(8) The series $3 + 33 + 333 + \ldots\ldots$ is:

(a) Arithmetic series (b) Geometric series

(c) Harmonic series (d) None of these

[Gujranwala Board 2005]

(9) The nth term of an A.P is $\frac{1}{2}(3 - n)$ the first three terms are:

(a) 3, 2, 1 (b) 1, 2, 3

(c) 1, 2, 1 (d) $1, \frac{1}{2}, 0$

[Gujranwala Board 2006]

(10) The geometric means between 2 and 4 are:

(a) $\pm 8\sqrt{2}$ (b) $\pm 4\sqrt{2}$

(c) $\pm 2\sqrt{2}$ (d) $\pm 3\sqrt{2}$

[Gujranwala Board 2006]

(11) Infinite geometric series is convergent if:

(a) $|r| < 1$ (b) $|r| > 1$

(c) $|r| = 1$ (d) All of these

[Gujranwala Board 2006]

(12) The A.M between $\frac{1}{a}$ and $\frac{1}{b}$ is:

(a) $\frac{a + b}{ab}$ (b) $\frac{a + b}{2ab}$

(c) $\frac{2ab}{a + b}$ (d) None of these

[Gujranwala Board 2006]

(13) $A \times H =$ _____

(a) ab (b) $2ab$

(c) $a + b$ (d) None of these

[Gujranwala Board 2006]

(14) If a, A, b are in A.P then 2A is:

(a) $a - b$ (b) $\frac{a+b}{2}$

(c) $a + b$ (d) $b - a$

[Lahore Board 2006]

(15) $\frac{n^2(n+1)^2}{4} =$ _____

(a) $1^3 + 2^3 + 3^3 + \ldots\ldots n^3$ (b) $1^2 + 2^2 + 3^2 + \ldots\ldots n^2$

(c) $1 + 2 + 3 + \ldots\ldots n$ (d) None of these

[Lahore Board 2006]

(16) nth term of G.P is:

(a) ar^n (b) ar^{n-1}

(c) $\frac{a}{r^n}$ (d) $\frac{r^n}{a}$

[Lahore Board 2014, Lahore Board 2006]

(17) A.M between 4 and 8 is:

(a) 4 (b) 5

(c) 6 (d) 7

[Lahore Board 2006]

(18) Geometric mean between 2a and 2b is:

(a) $\pm\sqrt{ab}$ (b) $\frac{4ab}{a+b}$

(c) $\pm 2\sqrt{ab}$ (d) $\frac{a+b}{4ab}$

[Gujranwala Board 2007]

(19) The sum of an infinite geometric series with common ratio 'r' exists only if:

(a) $-1 < r < 1$ (b) $-1 \leq r \leq 1$

(c) $r < -1, r > 1$ (d) $r = \pm 1$

[Gujranwala Board 2007]

(20) $\sum_{k=1}^{n} k^2 =$ ————

(a) $\frac{n(n+1)(n+2)}{2}$ (b) $\frac{n(n+1)(2n+1)}{6}$

(c) $\frac{(2n+1)(n+2)n}{6}$ (d) $\left\{\frac{n(n+1)}{2}\right\}^2$

[Gujranwala Board 2007]

(21) If $a_{n-3} = 2n - 5$, its nth term is:

(a) $2n + 1$ (b) $2n + 3$

(c) $2n - 2$ (d) $2n - 8$

[Gujranwala Board 2007]

(22) Geometric mean between $-2i$ and $8i$ is:

(a) $\pm 4i$ (b) ± 4

(c) $\pm 2i$ (d) $\pm 6i$

[Gujranwala Board 2007]

(23) Harmonic mean between 3 and 7 is:

(a) $\frac{5}{21}$ (b) $\frac{21}{5}$

(c) 5 (d) $\sqrt{21}$

[Gujranwala Board 2007]

(24) Arithmetic mean between $\sqrt{2}$ and $3\sqrt{2}$ is:

(a) $\sqrt{2}$ (b) $2\sqrt{2}$

(c) 6 (d) 3

[Lahore Board 2014, Lahore Board 2007]

(25) Harmonic mean between a, b is:

(a) $\frac{ab}{a+b}$ (b) $\frac{a+b}{ab}$

(c) $\frac{2ab}{a+b}$ (d) $\frac{a-b}{ab}$

[Lahore Board 2007-08-09, Lahore Board 2015]

(26) If in an A.P $a_n = \frac{n}{2n+1}$ then a_4 is equal to:

(a) 4 (b) 3

(c) $\frac{4}{3}$ (d) $\frac{4}{9}$

[Lahore Board 2007]

(27) Geometric mean between 2 and 8 is:

(a) ± 2 (b) ± 8

(c) ± 4 (d) ± 16

[Lahore Board 2007]

(28) $\sum_{k=1}^{n} k =$ ——————

(a) k (b) $\frac{n(n+1)}{2}$

(c) $\frac{k(k-1)}{2}$ (d) k^n

[Lahore Board 2007, Gujranwala Board 2008]

(29) Common difference of the A.P 17, 13, 9, ... equals:

(a) 4 (b) –4

(c) 30 (d) 15

[Lahore Board 2008]

(30) No term in G.P is:

(a) 3 (b) 2

(c) 1 (d) 0

[Lahore Board 2008-09]

(31) If nth term of the A.P is $3n - 1$ then first term equals:

(a) –1 (b) 4

(c) 2 (d) –4

[Lahore Board 2008]

(32) The sequence 3, 6, 12, ... is:

(a) A.P (b) G.P

(c) H.P (d) Finite

[Lahore Board 2008]

(33) $1 + 2 + 3 + 4 + \ldots\ldots + n$ equals:

(a) $\frac{n(n-1)}{2}$ (b) $\frac{n(n+1)}{2}$

(c) $\frac{n+1}{2}$ (d) $\frac{n(n+1)^2}{2}$

(34) If $a_n = (n+1)a_{n-1}$, $a_1 = 1$ then a_2 is:

(a) -3 (b) 3

(c) -2 (d) 2

[Gujranwala Board 2008]

(35) Common difference of harmonic sequence $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\ldots$ is:

(a) $\frac{-2}{15}$ (b) $\frac{2}{15}$

(c) 2 (d) $\frac{1}{2}$

[Gujranwala Board 2008]

(36) The geometric mean between 1 and 16 is:

(a) -4 (b) 4

(c) ± 4 (d) $\pm\frac{1}{4}$

[Gujranwala Board 2008, Lahore Board 2015]

(37) A.M between a and b is:

(a) $\frac{a-b}{\sqrt{2}}$ (b) $\frac{a+b}{2}$

(c) $\sqrt{ab}$ (d) $a+b$

[Gujranwala Board 2008]

(38) In G.P 3, 6, 12, ... common ratio 'r' equals:

(a) 3 (b) 6

(c) 2 (d) $\frac{1}{2}$

[Gujranwala Board 2008]

(39) An infinite arithmetic series consisting of non-zero terms is:

(a) Convergent

(b) Divergent

(c) Neither convergent nor divergent

(d) Oscillatory

[Lahore Board 2009]

(40) The product of n geometric means between a and b equals ———— where A, G, H have their usual meanings:

(a) A^n (b) H^n

(c) G^n (d) nG

[Lahore Board 2009]

(41) If $a_{n-2} = 3n - 11$, then 5th term is:

(a) 4 (b) 7

(c) 10 (d) 13

[Lahore Board 2010]

(42) Let $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ be H.M between a and b then n =

(a) 0 (b) 1

(c) $\frac{1}{2}$ (d) -1

[Lahore Board 2010]

(43) The next term of the sequence 7, 9, 12, ... is:

(a) 16 (b) 15

(c) 14 (d) 18

[Gujranwala Board 2010]

(44) G.M between $2i$ and $8i$ is:

(a) $\pm 4i$ (b) ± 4

(c) 4 (d) -4

[Gujranwala Board 2010, Lahore Board 2015]

(45) Sum of the series $1^3 + 2^3 + 3^3 + ...$ up to n terms =

(a) $\frac{n(n+1)}{2}$ (b) $\left[\frac{n(n+1)}{2}\right]^2$

(c) $\frac{n(n-1)(2n+1)}{6}$ (d) $\frac{n(n+1)}{3}$

[Lahore Board 2010-11]

(46) A.M between $x - 3$ and $x + 5$ =

(a) $x + 1$ (b) $x - 1$

(c) $x - 3$ (d) $x + 5$

[Gujranwala Board 2011]

(47) If a, A, b are in A.P then 2A =

(a) $\frac{a+b}{2}$ (b) $a+b$

(c) $a-b$ (d) $\frac{a-b}{2}$

[Lahore Board 2012]

(48) 20th term of $1+3+5+\ldots\ldots$ is:

(a) 38 (b) 39

(c) 40 (d) 41

[Lahore Board 2012]

(49) If $a = 2i$, $b = 4i$, then $G =$ ———

(a) $\pm 2\sqrt{2}\,i$ (b) $\sqrt{6}\,i$

(c) $\pm 2i$ (d) $4i$

[Lahore Board 2012]

(50) The 3rd term of sequence $a_n = (-1)^n (n-7)$ is:

(a) 8 (b) 4

(c) –4 (d) –8

[Lahore Board 2013]

(51) The series $1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\ldots\ldots$ is:

(a) Arithmetic (b) Harmonic

(c) Convergent (d) Divergent

[Lahore Board 2013]

(52) a_7 of the sequence 2, 6, 11, 17, ... is:

(a) 41 (b) 51

(c) 32 (d) 40

[Lahore Board 2013]

(53) The A.M. between $x-3$ and $x+5$ is:

(a) $x+1$ (b) $x-1$

(c) $x-3$ (d) $x+5$

[Lahore Board 2013]

(54) If the 5th term of an A.P is 13 and 17th term if 49, then a_n = ?

(a) $a_n = 2+(n-1)3$ (b) $a_n = 1+(n-1)3$

(c) $a_n = 3+(n-1)4$ (d) None of these

(55) If $a_{n-2} = 3n-11$ then nth term of the sequence is:

(a) $a_n = 3n-5$ (b) $a_n = 3n-4$

(c) $a_n = 2n-5$ (d) $a_n = 2n-4$

(56) If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P then b = ———

(a) $\frac{ac}{a+c}$ (b) $\frac{a+c}{ac}$

(c) $\frac{2ac}{a+c}$ (d) None of these

(57) Sum of n terms of an arithmetic series is:

(a) $S_n = \frac{n}{2}(2a_1 + a_n)$ (b) $S_n = \frac{n}{2}(2a_1 + 2a_n)$

(c) $S_n = \frac{n}{2}(a_1 + a_n)$ (d) None of these

(58) $(x - a) + (x + a) + (x + 3a) + \ldots\ldots$ is a/an:

(a) Arithmetic series (b) Geometric series

(c) Harmonic series (d) None of these

(59) If $S_n = n(2n - 1)$, then the series is:

(a) $1 + 5 + 9 + \ldots\ldots$ (b) $1 + 4 + 8 + \ldots\ldots$

(c) $1 + 3 + 5 + \ldots\ldots$ (d) All of these

(60) 'O' cannot be a term of:

(a) A.P (b) H.P

(c) G.P (d) Both (b) and (c)

(61) $1 + i, 2, 2(1 - i), \ldots\ldots$ is a/an:

(a) A.P (b) G.P

(c) H.P (d) None of these

(62) If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in G.P then r = ?

(a) $\pm\sqrt{\frac{c}{a}}$ (b) $\pm\frac{c}{a}$

(c) $\pm\frac{a}{c}$ (d) $\pm\sqrt{\frac{a}{c}}$

(63) The formula to find the sum of 'n' terms of the geometric series is

$S_n = \frac{a_1(r^n - 1)}{r - 1}$ if

(a) $|r| < 1$ (b) $|r| > 1$

(c) $|r| = 1$ (d) None

(64) The series $1 + 2 + 4 + 8 + \ldots\ldots$ is:

(a) Convergent (b) Divergent

(c) Oscillatory (d) All of these

(65) $2.\dot{2}\dot{3} = ?$

(a) $\frac{220}{99}$ (b) $\frac{221}{89}$

(c) $\frac{220}{199}$ (d) $\frac{221}{99}$

(66) If a, b are two distinct real numbers and $a > 0$, $b > 0$, $G > 0$ then:

(a) $A > G > H$ (b) $A < G < H$

(c) $A = G = H$ (d) $G > A > H$

(67) If a, b are two distinct real numbers and $a < 0$, $b < 0$, $G < 0$ then:

(a) $A > G > H$ (b) $A < G < H$

(c) $A = G = H$ (d) $G > A > H$

(68) $\sum_{k=1}^{n} [(k^m - (k-1)^m] =$ ————

(a) m^n (b) n^m

(c) m^m (d) n^n

(69) $\sum_{k=1}^{n} k^3 =$ ————

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(2n+1)}{2}$

(c) $\frac{n(n^2+1+2n)}{4}$ (d) $\frac{n^4+2n^3+n^2}{4}$

(70) If $a_n + a_{n+1} = 2n$ and $a_3 = 2$ then $a_4 = ?$

(a) 4 (b) –4

(c) 2 (d) –2

(71) Sum of 'n' terms of sequence $\{3^{n-1}\}$ is:

(a) $3^{n-1} - 1$ (b) $\frac{3^{n-1}-1}{2}$

(c) $2(3^{n-1} - 1)$ (d) $\frac{3^n - 1}{2}$

(72) Infinite arithmetic series $\sum_{n=1}^{\infty} (n+2) =$ ————

(a) Convergent (b) Oscillatory

(c) Divergent (d) None of these

(73) The formula to find the sum of infinite terms of arithmetic series is:

(a) $S_\infty = \frac{a_1}{1-r}$ (b) $S_\infty = \frac{a_1(1-r^n)}{1-r}$

(c) $S_\infty = \frac{a_1(r^n-1)}{r-1}$ (d) None of these

(74) Fourth partial sum of the sequence $\{n^2\}$ is:

(a) $1 + 2 + 3 + 4$ (b) $1 + 2 + 4 + 8$

(c) $1 + 4 + 9 + 16$ (d) None of these

(75) If 4, G, 16 are in G.P then common ratio r =

(a) 1 or -1 (b) 2 or -2

(c) 4 or -4 (d) 3 or -3

(76) Interval of convergence for the series $\frac{x}{2} + \frac{}{14} x^2 + \frac{1}{8} x^3 + \ldots\ldots$ is

(a) $[-2, 2]$ (b) $[-1, 1]$

(c) $]-2, 2[$ (d) Both (a) and (c)

(77) If $a, b \in C$ then G^2 = ———

(a) $A \times G$ (b) A^2

(c) H^2 (d) $A \times H$

(78) $\sum_{k=1}^{n} k^2$ = ———

(a) $\frac{n(2n^2 + 3n + 1)}{6}$ (b) $\frac{n(n + 1)(2n + 1)}{6}$

(c) $\frac{(n + 1)(2n + 1)}{3}$ (d) Both (a) and (b)

(79) A.M between $(x - 2)^2$ and $(x + 2)^2$ is the sum of squares of:

(a) x and 1 (b) x and 2

(c) x^2 and 2^2 (d) None of these

(80) nth term of H.P $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \ldots\ldots$ is:

(a) $\frac{1}{3n + 1}$ (b) $\frac{1}{4n - 2}$

(c) $\frac{1}{3n - 2}$ (d) $\frac{1}{3n}$

(81) If $S_n = 2^{n-1}$ then the series will be:

(a) $1 + 2 + 4 + \ldots\ldots$ (b) $1 + 1 + 2 + 4 + \ldots\ldots$

(c) $1 + 2 + 3 + \ldots\ldots$ (d) None of these

(82) The series $1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots$ converges to:

(a) 0 (b) 1

(c) 2 (d) ∞

(83) If $r = -1$ then the sum of n terms of geometric series is:

(a) a_1 (b) 0

(c) $2a_1$ (d) Either (a) or (b)

(84) If $an = (-1)^{n+1}$ find 26th term.

(a) -1 (b) 1

(c) 26 (d) -26

(85) In geometric sequence nth term equal to:

(a) $a_1 + (n-1)d$ (b) $\frac{n}{2}(2a_1 + (n-1)d)$

(c) $a_1 r^{n-1}$ (d) $\frac{a_1}{1-r}$

(86) $5, \frac{5}{2}, \frac{5}{4}, \ldots\ldots\ldots$ is in:

(a) Sequence (b) A.P

(c) G.P (d) H.P

(87) If $a_n - a_{n-1} = n + 1$ and $a_4 = 14$ then $a_5 =$ ————

(a) a (b) 14

(c) 20 (d) 5

Answers

(1)	a	*(2)*	c	*(3)*	b	*(4)*	b	*(5)*	c
(6)	c	*(7)*	a	*(8)*	d	*(9)*	d	*(10)*	c
(11)	a	*(12)*	b	*(13)*	a	*(14)*	c	*(15)*	a
(16)	b	*(17)*	c	*(18)*	c	*(19)*	a	*(20)*	b
(21)	a	*(22)*	b	*(23)*	b	*(24)*	b	*(25)*	c
(26)	d	*(27)*	c	*(28)*	b	*(29)*	b	*(30)*	d
(31)	c	*(32)*	b	*(33)*	b	*(34)*	b	*(35)*	c
(36)	c	*(37)*	b	*(38)*	c	*(39)*	b	*(40)*	c
(41)	c	*(42)*	d	*(43)*	a	*(44)*	a	*(45)*	b
(46)	a	*(47)*	b	*(48)*	b	*(49)*	a	*(50)*	b
(51)	c	*(52)*	a	*(53)*	a	*(54)*	b	*(55)*	a
(56)	c	*(57)*	c	*(58)*	a	*(59)*	a	*(60)*	d
(61)	b	*(62)*	d	*(63)*	b	*(64)*	b	*(65)*	d
(66)	a	*(67)*	b	*(68)*	b	*(69)*	d	*(70)*	a
(71)	d	*(72)*	c	*(73)*	d	*(74)*	c	*(75)*	b
(76)	c	*(77)*	d	*(78)*	d	*(79)*	b	*(80)*	d
(81)	b	*(82)*	a	*(83)*	d	*(84)*	a	*(85)*	c
(86)	c	*(87)*	c						

SOLUTION

Q.2 As 2, 7, 12, ... are in A.P because $d = 7 - 2 = 5$ and $d = 12 - 7 = 5$

$\Rightarrow$ $\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \ldots\ldots$ are in H.P

Q.3 As 3, 6, 12, ... is G.P

Where $r = \frac{6}{3} = 2$, $a_1 = 3$

As, $a_5 = a_1 r^4 = 3(2)^4 = 3(16) = 48$

Q.4 $a_n = (-1)^{n-1}(n - 7)$

Put, $n = 3$

$a_3 = (-1)^{3-1}(3 - 7)$

$= (-1)^2(-4) = -4$

Q.7 $3 + 6 + 9 + \ldots\ldots + 3n$

$= 3[1 + 2 + 3 + \ldots\ldots n]$

$= 3\left[\sum_{k=1}^{n} k\right] = 3\left[\frac{n(n+1)}{2}\right] = \frac{3n(n+1)}{2}$

Q.8 $3 + 33 + 333 + \ldots\ldots$

As,

$\frac{33}{3} = 11$ and $\frac{333}{33} = 10.14.2011$

$\Rightarrow$ Ratio between every two consecutive terms is not same so it is not geometric series. Similarly not an arithmetic and harmonic series.

Q.9 $a_n = \frac{1}{2}(3 - n)$

$a_1 = \frac{1}{2}(3 - 1) = \frac{1}{2}(2) = 1$

$a_2 = \frac{1}{2}(3 - 2) = \frac{1}{2}(1) = \frac{1}{2}$

$a_3 = \frac{1}{2}(3 - 3) = \frac{1}{2}(0) = 0$

Q.10 Let, $a = 2$, $b = 4$

$$G.M = \pm\sqrt{ab}$$

$$= \pm\sqrt{2 \times 4} = \pm\sqrt{8} = \pm 2\sqrt{2}$$

Q.12 As A.M between a and b is given as:

$$A.M = \frac{a+b}{2}$$

$$\text{A.M between } \frac{1}{a} \text{ and } \frac{1}{b} = \frac{\frac{1}{a}+\frac{1}{b}}{2}$$

$$= \frac{\frac{b+a}{ab}}{2} = \frac{a+b}{ab} \times \frac{1}{2}$$

Q.13 As, $A \times H = G^2$

$\Rightarrow$ $A \times H = (\pm\sqrt{ab})^2$

$\Rightarrow$ $A \times H = ab$

Q.14 If, a, A, b are in A.P.

$\Rightarrow$ $A = \frac{a+b}{2}$

$$2A = 2 \cdot \frac{a+b}{2} = a+b$$

Q.15 As,

$$1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3 = \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$$

Q.17

$$A.M = \frac{a+b}{2}$$

$$= \frac{4+8}{2} = \frac{12}{2} = 6$$

Q.18 G.M between a and b $= \pm\sqrt{ab}$

G.M between 2a and 2b $= \pm\sqrt{2a \cdot 2b}$

$= \pm\sqrt{4ab}$

$= \pm 2\sqrt{ab}$

Q.19 The sum of an infinite geometric series with common ratio r exists only.

If $|r| < 1$ or $-1 < r < 1$

Q.21 $a_{n-3} = 2n - 5$

Put, $n = n + 3$

$a_n = 2(n + 3) - 5$

$= 2n + 6 - 5 = 2n + 1$

Q.22 Geometric mean between a and b $= \pm\sqrt{ab}$

Geometric mean between $-2i$ and $8i$ $= \pm\sqrt{-2i \cdot 8i}$

$= \pm\sqrt{-16i^2}$

$= \pm\sqrt{16} = \pm 4$

Q.23 H.M between a and b $= \dfrac{2ab}{a + b}$

H.M between 3 and 7 $= \dfrac{2(3)(7)}{3 + 7}$

$= \dfrac{2(21)}{10} = \dfrac{21}{5}$

Q.24 A.M between a and b $= \dfrac{a + b}{2}$

A.M between $\sqrt{2}$ and $3\sqrt{2}$ $= \dfrac{\sqrt{2} + 3\sqrt{2}}{2}$

$= \dfrac{4\sqrt{2}}{2} = 2\sqrt{2}$

Q.26 $a_n = \dfrac{n}{2n + 1}$

$a_4 = \dfrac{4}{2(4) + 1} = \dfrac{4}{8 + 1} = \dfrac{4}{9}$

Q.27 G.M $= \pm\sqrt{ab}$

$= \pm\sqrt{2 \times 8}$

$= \pm\sqrt{16} = \pm 4$

$\Rightarrow$ 4 and -4 are G.M between 2 and 8.

Q.29 17, 13, 9, ...

$d = 13 - 17 = -4$

Q.30 If '0' is the term of G.P we cannot find common ratio r.

Q.31 $a_n = 3n - 1$

$a_1 = 3(1) - 1$

$= 3 - 1 = 2$

Q.32 3, 6, 12, ...

$$r = \frac{6}{3} = 2 \quad , \quad r = \frac{12}{6} = 2$$

Ratio 'r' is same.

$\Rightarrow$ Given sequence is G.P.

Q.33 $1 + 2 + 3 + \ldots\ldots\ldots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Q.34 $a_n = (n+1)a_{n-1} \quad , \quad a_1 = 1$

$a_2 = (2+1)a_1$

$= 3(1) = 3$

Q.36 $G.M = \pm\sqrt{1 \times 16} = \pm\sqrt{16} = \pm 4$

Q.38 3, 6, 12, ...

$$r = \frac{6}{3} = 2$$

Q.40 As,

$$G_1, G_2, G_3, \ldots\ldots\ldots G_n = a^n \left(\frac{b}{a}\right)^{\frac{1+2+3+\ldots\ldots n}{n}} = a^n \left(\frac{b}{a}\right)^{n/2}$$

$$(G_1, G_2, G_3, \ldots\ldots\ldots G_n)^{1/n} = a\left(\frac{b}{a}\right)^{1/2} = (ab)^{1/2} = G$$

$\Rightarrow$ $G_1, G_2, G_3, \ldots\ldots\ldots G_n = G^n$

Q.50 Given $a_5 = 13 \Rightarrow a_1 + 4d = 13$

$a_{17} = 49 \Rightarrow a_1 + 16d = 49$

$-12d = -36$

$\boxed{d = 3}$

Put, $d = 3$ in $a_1 + 4d = 13$

$a_1 + 4(3) = 13$

$a_1 + 12 = 13 \Rightarrow \boxed{a_1 = 1}$

$\Rightarrow$ $a_n = a_1 + (n-1)d$

$= 1 + (n-1)(3)$

Q.51 $a_{n-2} = 3n - 11$

Put, $n = n + 2$

$a_n = 3(n+2) - 11 = 3n + 6 - 11 = 3n - 5$

Q.52 As, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$$\Rightarrow \quad \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\frac{1}{b} + \frac{1}{b} = \frac{1}{c} + \frac{1}{a}$$

$$2 \cdot \frac{1}{b} = \frac{a+c}{ac}$$

$$\frac{1}{b} = \frac{a+c}{2ac} \Rightarrow b = \frac{2ac}{a+c}$$

Q.54 $(x - a) + (x + a) + (x + 3a) + \ldots\ldots\ldots$

As, $x + a - (x - a) = x + a - x + a = 2a$

$x + 3a - (x + a) = x + 3a - x - a = 2a$

$\Rightarrow$ Common difference is same

$\Rightarrow$ Given series is arithmetic series

Q.55 $S_n = n(2n - 1)$

$S_1 = 1(2 - 1) = 1$

$S_2 = 2(4 - 1) = 6$

$S_3 = 3(6 - 1) = 15$

Now, $a_1 = S_1 = 1$

$a_2 = S_2 - S_1 = 6 - 1 = 5$

$a_3 = S_3 - S_2 = 15 - 6 = 9$

$\Rightarrow$ Series will be $1 + 5 + 9 + \ldots\ldots\ldots$

Q.57 $1 + i, 2, 2(1 - i), \ldots\ldots\ldots$

As, $\frac{2}{1+i} = \frac{2}{1+i} \cdot \frac{1-i}{1-i} = \frac{2(1-i)}{(1)^2 - (i)^2} = \frac{2(1-i)}{2} = 1 - i$

$$\frac{2(1-i)}{2} = 1 - i$$

$\Rightarrow$ Common ratio is same.

$\Rightarrow$ Given sequence is geometric sequence.

Q.58 $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in G.P.

$$\Rightarrow \quad r = \frac{1/b}{1/a} = \frac{a}{b} \qquad \ldots\ldots\ldots (1)$$

and $$r = \frac{1/c}{1/b} = \frac{b}{c} \qquad \ldots\ldots\ldots (2)$$

Multiplying (1) and (2).

$$r^2 = \frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c}$$

$$r = \pm\sqrt{\frac{a}{c}}$$

Q.61 $2.\dot{2}\dot{3} = 2.23232323\ldots\ldots$

$= 2 + \{.23 + 0.0023 + 0.000023 + \ldots\ldots\}$

$= 2 + \left[\dfrac{.23}{1 - \dfrac{1}{100}}\right]$

$= 2 + \dfrac{100 \times .23}{99} = 2 + \dfrac{23}{99}$

$= \dfrac{198 + 23}{99} = \dfrac{221}{99}$

Q.64 $\sum_{k=1}^{n} [(k^m - (k-1)^m)]$

$= (1^m - 0^m) + (2^m - 1^m) + (3^m - 2^m) + \ldots\ldots + [(n-1)^m - (n-2)^m] + [n^m - (n-1)^m] = n^m$

Q.65 As,

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4} = \frac{n^2(n^2+2n+1)}{4} = \frac{n^4+2n^3+n^2}{4}$$

Q.66 $a_n + a_{n+1} = 2n \quad , \quad a_3 = 2$

$n = 3 \Rightarrow a_3 + a_4 = 2(3)$

$2 + a_4 = 6 \Rightarrow \boxed{a_4 = 4}$

Q.67 $a_n = 3^{n-1}$

$a_1 = 1 \quad , \quad a_2 = 3^{2-1} = 3$

$a_3 = 3^2 = 9 \quad , \quad a_4 = 27$

$\Rightarrow$ Sequence is 1, 3, 9, 27, ...

Sum of 'n' terms of sequence $= \dfrac{a(r^n - 1)}{r - 1}$

$= \dfrac{1(3^n - 1)}{3 - 1} = \dfrac{3^n - 1}{2}$

Q.71 As 4, G, 16, are in G.P

$\Rightarrow G = \pm\sqrt{4.16} = \pm\sqrt{64} = \pm 8$

$\Rightarrow$ Sequence is 4, 8, 16 or 4, –8, 16 if 4, 8, 16 then r = 2 if 4, –8, 16 then r = –2.

Q.72 $\dfrac{x}{2} + \dfrac{1}{4}x^2 + \dfrac{1}{8}x^3 + \ldots\ldots$ is convergent if $|r| < 1$ or $\left|\dfrac{x}{2}\right| < 1$ or $-1 < \dfrac{x}{2} < 1$

or $-2 < x < 2$

or $x \in (-2, 2)$

or $x \in \,]-2, 2[$

Q.74 As $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$= \frac{n(2n^2+n+2n+1)}{6} = \frac{n(2n^2+3n+1)}{6}$$

Q.75 $$A.M = \frac{(x-2)^2+(x+2)^2}{2}$$

$$= \frac{x^2+4-4x+x^2+4+4x}{2} = \frac{2x^2+8}{2} = x^2+4$$

Which is sum of square of x and 2.

Q.76 As 3, 6, 9, ... are in A.P.

$a_n = 3+(n-1)(3) = 3+3n-3 = 3n$

nth term of A.P $= 3n$

nth term of H.P $= \frac{1}{3n}$

Q.77 $S_n = 2^{n-1}$

$n = 1 \Rightarrow S_1 = 2^{1-1} = 2^0 = 1$

$n = 2 \Rightarrow S_2 = 2^{2-1} = 2^1 = 2$

$n = 3 \Rightarrow S_3 = 2^{3-1} = 2^2 = 4$

$a_1 = S_1 = 1$, $a_2 = S_2 - S_1 = 2-1 = 1$

$a_3 = S_3 - S_2 = 4-2 = 2$ ..... Series will be $1+1+2+$.....

Q.79 If $r = -1$ then

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1(1-(-1)^n)}{2}$$

i.e. $S_n = a_1$ if 'n' is positive odd integer

$S_n = 0$ if 'n' is positive even integer

Q.84 $a_n = (-1)^{n+1}$ $a_{26} = ?$

$a_{26} = (-1)^{26+1} = (-1)^{27} = -1$

Q.85 $a_n = a_1 r^{n-1}$

Q.86 $5, \frac{5}{2}, \frac{5}{4}$ ..... is in G.P

Because, $\frac{5}{2} \div 5 = \frac{5}{4} \div \frac{5}{2} = \frac{1}{2}$

Q.87 Put, $n = 5$

$a_5 - a_4 = 6$

$a_5 = 6+14$

$a_5 = 20$