Chapter 5

PARTIAL FRACTIONS

We know that how to add two or more rational fractions into a single rational fraction. For example,

(i) $\dfrac{1}{x-1}+\dfrac{2}{x+2}=\dfrac{3x}{(x-1)(x+2)}$

(ii) $\dfrac{2}{x+1}+\dfrac{1}{(x+1)^2}+\dfrac{3}{x-2}=\dfrac{5x^2+5x-3}{(x+1)^2(x-2)}$

Here we shall learn how to reverse the order (i) and (ii)

Partial Fractions

To express a single rational function as a sum of two or more single rational functions which are called Partial Fractions.

Partial Fraction Resolution

Expressing a rational function as a sum of partial fractions is called partial fraction resolution.

Rational Fraction

The quotient of two polynomials $\dfrac{P(x)}{Q(x)}$ where $Q(x) \neq 0$, with no common factors is called a Rational Fraction. A rational fraction is of two types.

(1) Proper Rational Fraction

A rational fraction $\dfrac{P(x)}{Q(x)}$ is called a proper rational fraction if the degree of the polynomial $P(x)$ in the numerator is less than the degree of the polynomial $Q(x)$ in the denominator. For example,

$\dfrac{3}{x+1}$, $\dfrac{2x-5}{x^2+4}$ and $\dfrac{9x^2}{x^3-1}$ are proper rational fractions.

(2) Improper Rational Fraction

A rational fraction $\dfrac{P(x)}{Q(x)}$ is called an improper rational fraction if the degree of the polynomial $P(x)$ in the numerator is equal to or greater than the degree of the polynomial $Q(x)$ in the denominator. For example

$\frac{x}{2x-3}$, $\frac{(x-2)(x+1)}{(x-1)(x+4)}$ and $\frac{x^3-x^2+x+1}{x^2+5}$

are improper rational fractions.

EXERCISE 5.1

Resolve into Partial Fractions.

Q.1 $\frac{1}{x^2-1}$ **(Gujranwala Board 2007, Lahore Board 2011)**

Solution:

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)}$$

Let

$$\frac{1}{(x-1)(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} \qquad (1)$$

$$\frac{1}{(x+1)(x-1)} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

$$1 = A(x-1)+B(x+1) \qquad (2)$$

put $x = 1$ in equation (2), we get

$$1 = A(1-1)+B(1+1)$$

$$1 = A(0)+2B$$

$$1 = 2B \Rightarrow \boxed{B = \frac{1}{2}}$$

Put $x = -1$ in equation (2), we get

$$1 = A(-1-1)+B(-1+1)$$

$$1 = -2A+B(0)$$

$$-2A = 1 \Rightarrow \boxed{A = -\frac{1}{2}}$$

Put values of A and B in equation (1)

$$\frac{1}{(x+1)(x-1)} = \frac{-\frac{1}{2}}{(x+1)} + \frac{\frac{1}{2}}{(x-1)}$$

$$= \frac{-1}{2(x+1)} + \frac{1}{2(x-1)}$$ are required partial fractions.

Q.2 $\dfrac{x^2(x^2+1)}{(x+1)(x-1)}$ **(Lahore Board 2007, 2008)**

Solution:

$$\frac{x^2(x^2+1)}{(x+1)(x-1)} = \frac{x^4+x^2}{x^2-1}$$

Dividing

$$\begin{array}{r|l} & x^2+2 \\ x^2-1 & \sqrt{x^4+x^2} \\ & +x^4-x^2 \\ & -\quad + \\ \hline & 2x^2 \\ & 2x^2-2 \\ & -\quad + \\ \hline & 2 \end{array}$$

$\Rightarrow$

$$\frac{x^4+x^2}{x^2-1} = x^2+2+\frac{2}{x^2-1} \qquad (1)$$

Here

$$\frac{2}{x^2-1} = \frac{2}{(x+1)(x-1)} \qquad (2)$$

Take $\dfrac{2}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$

$$\frac{2}{(x+1)(x-1)} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

$$2 = A(x-1)+B(x+1) \qquad (3)$$

Put $x = 1$ in equation (3), we get

$2 = A(1-1)+B(1+1)$

$2 = A(0)+2B$

$2B = 2 \Rightarrow \boxed{B = 1}$

Put $x = -1$ in equation (3), we get

$2 = A(-1-1)+B(-1+1)$

$2 = -2A+(0)B$

$\Rightarrow \quad -2A = 2 \Rightarrow \boxed{A = -1}$

Put values of A, B in equation (2)

$$\frac{2}{(x+1)(x-1)} = \frac{-1}{x-1} + \frac{1}{(x-1)}$$

equation (1) becomes

$$\frac{x^4 + x^2}{x^2 - 1} = x^2 + 2 - \frac{1}{x+1} + \frac{1}{x-1}$$

Hence $x^2 + 2 - \frac{1}{x+1} + \frac{1}{x-1}$ are required partial fractions.

Q.3 $\dfrac{\mathbf{2x+1}}{\mathbf{(x-1)(x+2)(x+3)}}$

Solution:

Let

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3} \qquad (1)$$

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)}{(x-1)(x+2)(x+3)}$$

$$2x + 1 = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2) \qquad (2)$$

Put $x = 1$ in equation (2)

$$2(1) + 1 = A(1+2)(1+3) + B(1-1)(1+3) + C(1-1)(1+2)$$

$$3 = 12A + 0 + 0$$

$$12A = 3 \Rightarrow A = \frac{3}{12} \Rightarrow \boxed{A = \frac{1}{4}}$$

Put $x = -2$ in equation (2)

$$2(-2) + 1 = A(-2+2)(-2+3) + B(-2-1)(-2+3) + C(-2-1)(-2+2)$$

$$-4 + 1 = A(0) + B(-3)(1) + C(0)$$

$$-3 = -3B \Rightarrow \boxed{B = 1}$$

Put $x = -3$ in equation (2)

$$2(-3) + 1 = A(-3+2)(-3+3) + B(-3-1)(-3+3) + C(-3-1)(-3+2)$$

$$-6 + 1 = A(0) + B(0) + C(-4)(-1)$$

$$-5 = 0 + 0 + 4C$$

$$-5 = 4C \quad \Rightarrow \quad \boxed{C = \frac{-5}{4}}$$

Put these values in equation (1)

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{\frac{1}{4}}{x-1} + \frac{1}{x+2} - \frac{\frac{5}{4}}{x+3}$$

$$= \frac{1}{4(x-1)} + \frac{1}{x+2} - \frac{5}{4(x+3)}$$

are required partial fractions.

Q.4 $\dfrac{\mathbf{3x^2 - 4x - 5}}{\mathbf{(x-2)(x^2 + 7x + 10)}}$

Solution:

As $\dfrac{3x^2 - 4x - 5}{(x-2)(x^2+7x+10)} = \dfrac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)}$

Let

$$\frac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)} = \frac{A}{(x-2)} + \frac{B}{(x+2)} + \frac{C}{(x+5)} \qquad (1)$$

$$\frac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)} = \frac{A(x+2)(x+5) + B(x-2)(x+5) + C(x-2)(x+2)}{(x-2)(x+2)(x+5)}$$

$$3x^2 - 4x - 5 = A(x+2)(x+5) + B(x-2)(x+5) + C(x+2)(x-2) \qquad (2)$$

Put $x = 2$ in equation (2)

$$3(2)^2 - 4(2) - 5 = A(2+2)(2+5) + B(2-2)(2+5) + C(2-2)(2+2)$$

$$3(4) - 8 - 5 = A(4)(7) + 0 + 0$$

$$12 - 8 - 5 = 28A$$

$$4 - 5 = 28A \quad \Rightarrow \quad \boxed{A = -\frac{1}{28}}$$

Put $x = -2$ in equation (2), we get

$$3(-2)^2 - 4(-2) - 5 = A(-2+2)(-2+5) + B(-2-2)(-2+5) + C(-2-2)(-2+2)$$

$$3(4) + 8 - 5 = A(0) + B(-4)(3) + C(0)$$

$$12 + 8 - 5 = -12B$$

$$15 = -12\,B \quad \Rightarrow \quad B = -\frac{15}{12} \quad \Rightarrow \quad \boxed{B = -\frac{5}{4}}$$

Put $x = -5$ in equation (2), we get

$$3(-5)^2 - 4(-5) - 5 = A(-5+2)(-5+5) + B(-5-2)(-5+5) + C(-5-2)(-5+2)$$

$$3(25) + 20 - 5 = A(0) + B(0) + C(-7)(-3)$$

$$75 + 15 = 21C$$

$$90 = 21C \quad \Rightarrow \quad C = \frac{90}{21} \quad \Rightarrow \quad \boxed{C = \frac{30}{7}}$$

Put these values in equation (1)

$$\frac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)} = \frac{-\frac{1}{28}}{x-2} + \frac{-\frac{5}{4}}{x+2} + \frac{\frac{30}{7}}{x+5}$$

$$= \frac{1}{28(x-2)} - \frac{5}{4(x+2)} + \frac{30}{7(x+5)}$$

are required partial fractions.

Q.5 $\dfrac{1}{(x-1)(2x-1)(3x-1)}$

Solution:

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{A}{x-1} + \frac{B}{2x-1} + \frac{C}{3x-1} \qquad (1)$$

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{A(2x-1)(3x-1) + B(x-1)(3x-1) + C(x-1)(2x-1)}{(x-1)(2x-1)(3x-1)}$$

$$1 = A(2x-1)(3x-1) + B(x-1)(3x-1) + C(x-1)(2x-1) \qquad (2)$$

Put $x = 1$ in equation (2), we get

$$1 = A(2(1)-1)(3(1)-1) + B(1-1)(3(1)-1) + C(1-1)(2(1)-1)$$

$$1 = A(2-1)(3-1) + B(0) + C(0)$$

$$1 = A(1)(2) \quad \Rightarrow \quad \boxed{A = \frac{1}{2}}$$

Put $x = \frac{1}{2}$ in equation (2), we get

$$1 = A\left(2\left(\frac{1}{2}\right)-1\right)\left(3\left(\frac{1}{2}\right)-1\right)+B\left(\frac{1}{2}-1\right)\left(3\left(\frac{1}{2}\right)-1\right)$$
$$+C\left(\frac{1}{2}-1\right)\left(2\left(\frac{1}{2}\right)-1\right)$$

$$1 = A(1-1)\left(\frac{3}{2}-1\right)+B\left(-\frac{1}{2}\right)\left(\frac{3}{2}-1\right)+C\left(-\frac{1}{2}\right)(1-1)$$

$$1 = A(0)+B\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)+C(0)$$

$$1 = -\frac{1}{4}B \Rightarrow \boxed{B = -4}$$

Put $x = \frac{1}{3}$ in equation (2), we get

$$1 = A\left(2\left(\frac{1}{3}\right)-1\right)\left(3\left(\frac{1}{3}\right)-1\right)+B\left(\frac{1}{3}-1\right)\left(3\left(\frac{1}{3}\right)-1\right)$$
$$+C\left(\frac{1}{3}-1\right)\left(2\left(\frac{1}{3}\right)-1\right)$$

$$1 = A\left(\frac{2}{3}-1\right)(1-1)+B\left(\frac{-2}{3}\right)(1-1)+C\left(-\frac{2}{3}\right)\left(\frac{2}{3}-1\right)$$

$$1 = A(0)+B(0)+C\left(-\frac{2}{3}\right)\left(-\frac{1}{3}\right)$$

$$1 = \frac{2}{9}C \Rightarrow \boxed{C = \frac{9}{2}}$$

Put these values in equation (1), we get

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{\frac{1}{2}}{x-1}+\frac{-4}{2x-1}+\frac{\frac{9}{2}}{3x-1}$$
$$= \frac{1}{2(x-1)}-\frac{4}{(2x-1)}+\frac{9}{2(3x-1)}$$

are required partial fractions.

Q.6 $\dfrac{x}{(x-a)(x-b)(x-c)}$

Solution:

Let

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c} \quad \ldots\ldots\ldots (1)$$

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)}{(x-a)(x-b)(x-c)}$$

$$x = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b) \quad \ldots\ldots\ldots (2)$$

Put $x = a$ in equation (2), we get

$$a = A(a-b)(a-c) + B(a-a) + C(a-a)(a-b)$$

$$a = A(a-b)(a-c) + 0 + 0$$

$$a = A(a-b)(a-c)$$

$$\boxed{A = \frac{a}{(a-b)(a-c)}}$$

Put $x = b$ in equation (2), we get

$$b = A(b-b)(b-c) + B(b-a)(b-c) + C(b-a)(b-b)$$

$$b = 0 + B(b-a)(b-c) + 0$$

$$\boxed{B = \frac{b}{(b-a)(b-c)}}$$

Put $x = c$ in equation (2), we get

$$c = A(c-b)(c-c) + B(c-a)(c-c) + C(c-a)(c-b)$$

$$c = 0 + 0 + C(c-a)(c-b)$$

$$\boxed{C = \frac{c}{(c-a)(c-b)}}$$

put these values in equation (1), we get

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{a}{(a+b)(a-c)(x-a)} + \frac{b}{(b-a)(b-c)(x-b)} + \frac{c}{(c-a)(c-b)(x-c)}$$

are required partial fractions.

Q.7 $\dfrac{6x^3 + 5x^2 - 7}{2x^2 - x - 1}$

Solution:

First dividing

$$2x^2 - x - 1 \;\overline{)\,6x^3 + 5x^2 - 7}$$

$$3x + 4$$

$$6x^3 - 3x^2 \quad -3x$$

$$- \quad + \quad +$$

$$8x^2 + 3x - 7$$

$$8x^2 - 4x - 4$$

$$- \quad + \quad +$$

$$7x - 3$$

$$\Rightarrow \quad \frac{6x^3 + 5x^2 - 7}{2x^2 - x - 1} = 3x + 4 + \frac{7x - 3}{2x^2 - x - 1} \qquad \ldots\ldots\ldots (1)$$

Take

$$\frac{7x - 3}{2x^2 - x - 1} = \frac{7x - 3}{(x - 1)(2x + 1)}$$

Let

$$\frac{7x - 3}{(x - 1)(2x + 1)} = \frac{A}{(x - 1)} + \frac{B}{(2x + 1)} \qquad \ldots\ldots\ldots (2)$$

$$\frac{7x - 3}{(x - 1)(2x + 1)} = \frac{A(2x + 1) + B(x - 1)}{(x - 1)(2x + 1)}$$

$$7x - 3 = A(2x + 1) + B(x - 1) \qquad \ldots\ldots\ldots (3)$$

Put $x = 1$ in equation (3), we get

$$7(1) - 3 = A(2(1) + 1) + B(1 - 1)$$

$$7 - 3 = A(2 + 1) + 0$$

$$\Rightarrow \quad 4 = 3A \quad \Rightarrow \quad \boxed{A = \frac{4}{3}}$$

Put $x = -\frac{1}{2}$ in equation (3), we get

$$7\left(-\frac{1}{2}\right) - 3 = A\left(2\left(-\frac{1}{2}\right) + 1\right) + B\left(-\frac{1}{2} - 1\right)$$

$$\frac{7}{2} - 3 = A(-1 + 1) + B\left(-\frac{3}{2}\right)$$

$$-\frac{13}{2} = 0 - \frac{3}{2}B$$

$$13 = 3b \Rightarrow \boxed{B = \frac{13}{3}}$$

Put these values in equation (2), we get

$$\frac{7x-3}{(x-1)(2x+1)} = \frac{\frac{4}{3}}{x-1} + \frac{\frac{13}{3}}{2x+1}$$

$$= \frac{4}{3(x-1)} + \frac{13}{3(2x+1)}$$

equation (1) becomes

$$\frac{6x^3+5x^2-7}{2x^2-x-1} = (3x+4) + \frac{4}{3(x-1)} + \frac{13}{3(2x+1)}$$

are required partial fractions.

Q.8 $\dfrac{\mathbf{2x^3+x^2-5x+3}}{\mathbf{2x^3+x^2-3x}}$

Solution:

First divide $2x^3+x^2-5x+3$ by $2x^3+x^2-3x$

i.e.

$$\begin{array}{r} 1 \\ 2x^3+x^2-3x \;\overline{)\,2x^3+x^2-5x+3} \\ 2x^3+x^2-3x \\ - \quad - \quad + \\ \hline -2x+3 \end{array}$$

$\Rightarrow$

$$\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x} = 1 + \frac{-2x+3}{2x^3+x^2-3x}$$

$$= 1 + \frac{-2x+3}{x(x-1)(2x+3)} \qquad \ldots\ldots\ldots (1)$$

Let

$$\frac{-2x+3}{x(x-1)(2x+3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{2x+3} \qquad \ldots\ldots\ldots (2)$$

$$\frac{-2x+3}{x(x-1)(2x+3)} = \frac{A(x-1)(2x+3) + Bx(2x+3) + Cx(x-1)}{x(x-1)(2x+3)}$$

$$-2x + 3 = A(x-1)(2x+3) + Bx(2x+3) + Cx(x-1) \quad \ldots\ldots\ldots (3)$$

Put $x = 0$ in equation (3), we get

$$-2(0) + 3 = A(0-1)\left(2(0)+3\right) + B(0)\left(2(0)+3\right) + C(0)(0-1$$

$$3 = A(-1)(3) + 0 + 0$$

$$3 = -3A \Rightarrow \boxed{A = -1}$$

Put $x = 1$ in equation (3), we get

$$-2(1) + 3 = A(1-1)\left(2(1)+3\right) + B(1)\left(2(1)+3\right) + C(1)(1-1)$$

$$-2 + 3 = A(0) + B(2+3) + C(0)$$

$$1 = 5B \Rightarrow \boxed{B = \frac{1}{5}}$$

Put $x = -\frac{3}{2}$ in equation (3), we get

$$-2\left(-\frac{3}{2}\right) + 3 = A\left(-\frac{3}{2} - 1\right)\left[2\left(-\frac{3}{2}\right) + 3\right] \; B\left(-\frac{3}{2}\right)\left[2\left(-\frac{3}{2}\right) + 3\right] + C\left(-\frac{3}{2}\right)\left(-\frac{3}{2} - 1\right)$$

$$3 + 3 = A(0) + B(0) + C\left(-\frac{3}{2}\right)\left(-\frac{3}{2} - 1\right)$$

$$6 = \frac{15}{4}C \quad \Rightarrow \quad C = \frac{6 \times 4}{15} = \frac{8}{5}$$

$$\Rightarrow \boxed{C = \frac{8}{5}}$$

Put these values in equation (2), we get

$$\frac{-2x+3}{x(x-1)(2x+3)} = \frac{-1}{x} + \frac{\frac{1}{5}}{x-1} + \frac{\frac{8}{5}}{(2x+3)}$$

$$= -\frac{1}{x} + \frac{1}{5(x-1)} + \frac{8}{5(2x+3)}$$

equation (1) becomes

$$\frac{2x^3 + x^2 - 5x + 3}{2x^3 + x^2 - 3x} = 1 - \frac{1}{x} + \frac{1}{5(x-1)} + \frac{8}{5(2x+3)}$$

are required Partial formula.

Q.9 $\dfrac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$

Solution:

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)} = \frac{x^3-9x^2+23x-15}{x^3-12x^2+44x-48}$$

By division

$$\begin{array}{r} 1 \\ x^3-12x^2+44x-48 \;\overline{)\; x^3-9x^2+23x-15} \\ x^3-12x^2+44x-48 \\ -\quad +\qquad -\quad + \\ \hline 3x^2-21x+33 \end{array}$$

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)} = 1+\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)}$$

$$= 1+\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} \qquad \ldots\ldots\ldots (1)$$

Now Let

$$\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} = \frac{A}{x-2}+\frac{B}{x-4}+\frac{C}{x-6} \qquad \ldots\ldots\ldots (2)$$

$$\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} = \frac{A(x-4)(x-6)+B(x-2)(x-6)+C(x-2)(x-4)}{(x-2)(x-4)(x-6)}$$

$$3x^2-21x+33 = A(x-4)(x-6)+B(x-2)(x-6)+C(x-2)(x-4) \ldots\ldots\ldots (3)$$

Put $x = 2$ in equation (3), we get

$$3(2)^2-21(2)+33 = A(2-4)(2-6)+B(0)+C(0)$$

$$3(4)-42+33 = A(-2)(-4)$$

$$12-42+33 = 8A$$

$$3 = 8A \quad \Rightarrow \quad \boxed{A = \frac{3}{8}}$$

Put $x = 4$ in equation (3), we get

$$3(4)^2 - 21(4) + 33 = A(0) + B(4-2)(4-6) + C(0)$$

$$3(16) - 84 + 33 = B(2)(-2)$$

$$48 - 84 + 33 = -4B$$

$$-3 = -4B \Rightarrow \boxed{B = \frac{3}{4}}$$

Put $x = 6$ in equation (3), we get

$$3(6)^2 - 21(6) + 33 = 0 + 0 + C(6-2)(6-4)$$

$$3(36) - 126 + 33 = C(4)(2)$$

$$108 - 126 + 33 = 8C$$

$$15 = 8C \Rightarrow \boxed{C = \frac{15}{8}}$$

Put these values in equation (2), we get

$$\frac{3x^2 - 21x + 33}{(x-2)(x-4)(x-6)} = \frac{\frac{3}{8}}{(x-2)} + \frac{\frac{3}{4}}{(x-4)} + \frac{\frac{15}{8}}{(x-6)}$$

$$= \frac{3}{8(x-2)} + \frac{3}{4(x-4)} + \frac{15}{8(x-6)}$$

$\Rightarrow$ equation (1) becomes

$$\frac{x^3 - 9x^2 + 23x - 15}{x^3 - 12x^2 + 44x - 48} = 1 + \frac{3}{8(x-2)} + \frac{3}{4(x-4)} + \frac{15}{8(x-6)}$$

are required Partial fractions.

Q.10 $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$ **(Lahore Board 2007)**

Solution:

Let

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{A}{(1-ax)} + \frac{B}{(1-bx)} + \frac{C}{(1-cx)} \quad \ldots\ldots\ldots (1)$$

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{A(1-bx)(1-cx) + B(1-ax)(1-cx) + C(1-ax)(1-bx)}{(1-ax)(1-bx)(1-cx)}$$

$$1 = A(1-bx)(1-cx) + B(1-ax)(1-cx) + C(1-ax)(1-bx) \quad \ldots\ldots\ldots (2)$$

Put $x = \frac{1}{a}$ in equation (2), we get

$$1 = A\left(1 - b \cdot \frac{1}{a}\right)\left(1 - c \cdot \frac{1}{a}\right) + B(0) + C(0)$$

$$1 = \left(1 - \frac{b}{a}\right)\left(1 - \frac{c}{a}\right)$$

$$1 = A\left(\frac{a-b}{a}\right)\left(\frac{a-c}{a}\right)$$

$$1 = \frac{A(a-b)(a-c)}{a^2} \Rightarrow \boxed{A = \frac{a^2}{(a-b)(a-c)}}$$

Put $x = \frac{1}{b}$ in equation (2), we get

$$1 = A(0) + B\left(1 - a \cdot \frac{1}{b}\right)\left(1 - c \cdot \frac{1}{b}\right)C(0)$$

$$1 = 0 + B\left(1 - a \cdot \frac{1}{b}\right)\left(1 - c \cdot \frac{1}{b}\right) + 0$$

$$1 = B\frac{(b-a)(b-c)}{b^2} \Rightarrow \boxed{B = \frac{b^2}{(b-a)(b-c)}}$$

Put $x = \frac{1}{c}$ in equation (2), we get

$$1 = 0 + 0 + C\left(1 - a \cdot \frac{1}{c}\right)\left(1 - b \cdot \frac{1}{c}\right)$$

$$1 = C\left(1 - \frac{a}{c}\right)\left(1 - \frac{b}{c}\right)$$

$$1 = C\left(\frac{c-a}{c}\right)\left(\frac{c-b}{c}\right)$$

$$1 = \frac{C(c-a)(c-b)}{c^2} \Rightarrow \boxed{C = \frac{c^2}{(c-a)(c-b)}}$$

Put these values in equation (1), we get

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{\frac{a^2}{(a-b)(a-c)}}{(1-ax)} + \frac{\frac{b^2}{(b-a)(b-c)}}{(1-bx)} + \frac{\frac{c^2}{(c-a)(c-b)}}{(1-cx)}$$

$$= \frac{a^2}{(a-b)(a-c)(1-ax)} + \frac{b^2}{(b-a)(b-c)(1-bx)} + \frac{c^2}{(c-a)(c-b)(1-bx)}$$

are required Partial fractions.

Q.11 $\dfrac{x^2 + a^2}{(x^2 + b^2)(x^2 + c^2)(x^2 + d^2)}$

Solution:

$$\frac{x^2 + a^2}{(x^2 + b^2)(x^2 + c^2)(x^2 + d^2)}$$

Let $x^2 = y$ and neglecting the square of each term

We have

$$\frac{x^2 + a^2}{(x^2 + b^2)(x^2 + c^2)(x^2 + d^2)} = \frac{y + a}{(y + b)(y + c)(y + d)}$$

Let

$$\frac{y + a}{(y + b)(y + c)(y + d)} = \frac{A}{(y + b)} + \frac{B}{(y + c)} + \frac{C}{(y + d)} \qquad \ldots\ldots\ldots (1)$$

$$\frac{y + a}{(y + b)(y + c)(y + d)} = \frac{A(y + c)(y + d) + B(y + b)(y + d) + C(y + b)(y + c)}{(y + b)(y + c)(y + d)}$$

$$y + a = A(y + c)(y + d) + B(y + b)(y + d) + C(y + b)(y + c) \qquad \ldots\ldots\ldots (2)$$

Put $y = -b$ in equation (2), we get

$$-b + a = A(-b + c)(-b + d) + 0 + 0$$

$$a - b = A(c - b)(d - b)$$

$$\boxed{A = \frac{(a - b)}{(c - b)(d - b)}}$$

Put $y = -c$ in equation (2), we get

$$-c + a = 0 + B(-c + b)(-c + d) + 0$$

$$a - c = B(b - c)(d - c)$$

$$\boxed{B = \frac{(a - c)}{(b - c)(d - c)}}$$

Put $y = -d$ in equation (2), we get

$$-d + a = 0 + 0 + C(-d + b)(-d + c)$$

$$a - d = C(b - d)(c - d)$$

$$\boxed{C = \frac{(a-d)}{(b-d)(c-d)}}$$

Put these values in equation (1), we get

$$\frac{y+a}{(y+b)(y+c)(y+d)} = \frac{(a-b)}{(c-b)(d-b)} \cdot \frac{1}{y+b} + \frac{(a-c)}{(b-c)(d-c)} \cdot \frac{1}{y+c} + \frac{(a-d)}{(b-b)(c-d)} \cdot \frac{1}{y+d}$$

$$\frac{a}{b+c+d} = \frac{a-b}{(c-b)(d-b)(y+b)} + \frac{a-c}{(b-c)(d-c)(y+c)} + \frac{a-d}{(b-d)(c-d)(y+d)}$$

Replacing the neglecting squares, we get

$$\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)} = \frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)(x^2+b^2)} + \frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)(x^2+c^2)} + \frac{a^2-d^2}{(b^2-d^2)(c^2-d^2)(x^2+d^2)}$$

are required Partial fractions.

EXERCISE 5.2

Resolve the following into Partial fraction.

Q.1 $\dfrac{2x^2-3x+4}{(x-1)^3}$

Solution:

Let

$$\frac{2x^2-3x+4}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \qquad \ldots\ldots\ldots (1)$$

$$\frac{2x^2-3x+4}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

$$2x^2-3x+4 = A(x-1)^2 + B(x-1) + C \qquad \ldots\ldots\ldots (2)$$

$$2x^2-3x+4 = A(x^2+1-2x) + Bx - B + C$$

$$2x^2-3x+4 = Ax^2 + A - 2Ax + Bx - B + C \qquad \ldots\ldots\ldots (3)$$

Put $x = 1$ in equation (2), we get

$$2(1)^2 - 3(1) + 4 = A(1-1)^2 + B(1-1) + C$$

$$2 - 3 + 4 = 0 + 0 + C$$

$$3 = C \quad \Rightarrow \quad \boxed{C = 3}$$

Equating coefficients of x^2, x in equation (3), we get

x^2 ; $\boxed{A = 2}$

x ; $-2A + B = -3$

$\Rightarrow -2(2) + B = -3$

$\Rightarrow -4 + B = -3$

$B = 4 - 3$

$\boxed{B = 1}$

Put values of A, B and C in equation (1)

$$\frac{2x^2 - 3x + 4}{(x-1)^3} = \frac{2}{(x-1)} + \frac{1}{(x-1)^2} + \frac{3}{(x-1)^3}$$

are required partial fractions.

Q.2 $\dfrac{\mathbf{5x^2 - 2x + 3}}{\mathbf{(x+2)^2}}$ **(Lahore Board 2005)**

Solution:

Let

$$\frac{5x^2 - 2x + 3}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} \quad \ldots\ldots\ldots (1)$$

$$\frac{5x^2 - 2x + 3}{(x+2)^2} = \frac{A(x+2)^2 + B(x+2) + C}{(x+2)^3}$$

$$5x^2 - 2x + 3 = A(x+2)^2 + B(x+2) + C \quad \ldots\ldots\ldots (2)$$

$$5x^2 - 2x + 3 = A(x^2 + 4 + 4x) + Bx + 2B + C$$

Put $x = -2$ in equation (2), we get

$$5(-2)^2 - 2(-2) + 3 = A(-2+2)^2 + B(-2+2) + C$$

$$5(4) + 4 + 3 = 0 + 0 + C$$

$20 + 7 = C \Rightarrow \boxed{C = 27}$

Equating coefficients of x^2, x, we have

x^2 ; $\boxed{A = 5}$

x ; $4A + B = -2$

$4(5) + B = -2$ ¤ $A = 5$

$20 + B = -2$

$B = -2 - 20$

$\boxed{B = -22}$

Put values of A, B and C in equation (1) we get

$$\frac{5x^2 - 2x + 3}{(x+2)^2} = \frac{5}{x+2} + \frac{-22}{(x+2)^2} + \frac{27}{(x+2)^3}$$

$$= \frac{5}{x+2} - \frac{22}{(x+2)^2} + \frac{27}{(x+2)^3}$$

are required partial fractions.

Q.3 $\dfrac{4x}{(x+1)^2(x-1)}$ **(Lahore Board 2009, Gujranwala Board 2003, 2005)**

Solution:

Let

$$\frac{4x}{(x+1)^2(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)} \quad \ldots\ldots\ldots (1)$$

$$\frac{4x}{(x+1)^2(x-1)} = \frac{A(x+1)(x-1) + B(x-1) + C(x+1)^2}{(x+1)^2(x-1)}$$

$$4x = A(x+1)(x-1) + B(x-1) + C(x+1)^2 \quad \ldots\ldots\ldots (2)$$

$$4x = A(x^2 - 1) + Bx - B + C(x^2 + 1 + 2x)$$

$$4x = Ax^2 - A + Bx - B + Cx^2 + C + 2Cx \quad \ldots\ldots\ldots (3)$$

Put $x = -1$ in equation (2), we get

$$4(-1) = A(0) + B(-1-1) + C(0)$$

$$-4 = -2B \Rightarrow \boxed{B = 2}$$

Put $x = 1$ in equation (2), we get

$$4(1) = A(0) + B(0) + C(1+1)^2$$

$$4 = C(2)^2 \Rightarrow 4 = 4C \Rightarrow \boxed{C = 1}$$

Equating coefficient of x^2 in equation (3) we get,

x^2 ; $A + C = 0$ ⌑ $C = 1$

$A + 1 = 0$

$\boxed{A = -1}$

Put values of A, B and C in equation (1), we have

$$\frac{4x}{(x+1)^2(x-1)} = \frac{-1}{(x+1)} + \frac{2}{(x+1)^2} + \frac{1}{(x-1)}$$

are required partial fractions.

Q.4 $\dfrac{9}{(x+2)^2(x-1)}$

Solution:

Let

$$\frac{9}{(x+2)^2(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x-1)} \qquad \ldots\ldots\ldots (1)$$

$$\frac{9}{(x+2)^2(x-1)} = \frac{A(x+2)(x-1) + B(x-1) + C(x+2)^2}{(x+2)^2(x-1)}$$

$$9 = A(x+2)(x-1) + B(x-1) + C(x+2)^2 \qquad \ldots\ldots\ldots (2)$$

$$9 = A(x^2 + x - 2) + Bx - B + C(x^2 + 4 + 4x)$$

$$9 = Ax^2 + Ax - 2A + Bx - B + Cx^2 + 4C + 4Cx \qquad \ldots\ldots\ldots (3)$$

Put $x = -2$ in equation (2), we get

$$9 = A(0) + B(-2-1) + C(0)$$

$$9 = -3B \Rightarrow \boxed{B = -3}$$

Put $x = 1$ in equation (2), we get

$$9 = A(0) + B(0) + C(1+2)^2$$

$$9 = 0 + 0 + C(3)^2$$

$$9 = 9C \Rightarrow \boxed{C = 1}$$

Equating co–efficient of x^2 in equation (3) we get

$$A + C = 0$$

$$A + 1 = 0 \qquad \because \quad C = 1$$

$$\boxed{A = -1}$$

Put values of A, B and C in equation (1) we get

$$\frac{9}{(x+2)^2(x-1)} = \frac{-1}{(x+2)} - \frac{3}{(x+2)^2} + \frac{1}{(x-1)}$$

are required partial fractions.

Q.5 $\dfrac{1}{(x-3)^2(x+1)}$ **(Lahore Board 2007)**

Solution:

Let

$$\frac{1}{(x-3)^2(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{(x+1)} \qquad \ldots\ldots\ldots (1)$$

$$\frac{1}{(x-3)^2(x+1)} = \frac{A(x-3)(x+1) + B(x+1) + C(x-3)^2}{(x-3)^2(x+1)}$$

$$1 = A(x-3)(x+1) + B(x+1) + C(x-3)^2 \qquad \ldots\ldots\ldots (2)$$

$$1 = A(x^2 - 2x - 3) + Bx + B + C(x^2 + 9 - 6x)$$

$$1 = Ax^2 - 2Ax - 3A + Bx + B + Cx^2 + 9C - 6Cx \qquad \ldots\ldots\ldots (3)$$

Put $x = 3$ in equation (2), we get

$$1 = A(0) + B(3+1) + C(0)$$

$$1 = 0 + 4B + 0$$

$$1 = 4B \Rightarrow \boxed{B = \frac{1}{4}}$$

Put $x = -1$ in equation (2), we get

$$1 = A(0) + B(0) + C(-1-3)^2$$

$$1 = 0 + 0 + C(-4)^2$$

$$1 = 16C \Rightarrow \boxed{C = \frac{1}{16}}$$

Equating coefficients of x^2 in equation (3) we get

$$A + C = 0$$

$$A + \frac{1}{16} = 0 \qquad \because \quad C = \frac{1}{16}$$

$$\boxed{A = -\frac{1}{16}}$$

Put values of A, B and C in equation (1), we get

$$\frac{1}{(x-3)^2(x+1)} = \frac{-\frac{1}{16}}{x-3} + \frac{\frac{1}{4}}{x-3} + \frac{\frac{1}{16}}{x-3}$$

$$= \frac{1}{16(x-3)} + \frac{1}{4(x-3)^2} + \frac{1}{16(x+1)}$$

Q.6 $\dfrac{x^2}{(x-2)(x-1)^2}$ **(Lahore Board 2011)**

Solution:

Let

$$\frac{x^2}{(x-2)(x-1)^2} = \frac{A}{(x-2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \quad \ldots\ldots\ldots (1)$$

$$\frac{x^2}{(x-2)(x-1)^2} = \frac{A(x-1)^2 + B(x-2)(x-1) + C(x-2)}{(x-2)(x-1)^2}$$

$$x^2 = A(x-1)^2 + B(x-2)(x-1) + C(x-2) \quad \ldots\ldots\ldots (2)$$

$$x^2 = A(x^2 + 1 - 2x) + B(x^2 - 3x - 2) + Cx - 2C$$

$$x^2 = Ax^2 + A - 2Ax + Bx^2 - 3Bx - 2B + Cx - 2C \quad \ldots\ldots\ldots (3)$$

Put $x = 2$ in equation (2), we get

$$(2)^2 = A(2-1)^2 + B(0) + C(0)$$

$$4 = A(3)^2$$

$$4 = 9A \Rightarrow \boxed{A = \frac{4}{9}}$$

Put $x = 1$ in equation (2), we get

$$(1)^2 = A(0) + B(0) + C(1-2)$$

$$1 = 0 + 0 + C(-1)$$

$$1 = -C \Rightarrow \boxed{C = -1}$$

Equating coefficients of x^2 in equation (3) we get

$$A + B = 1$$

$$\frac{4}{9} + B = 1$$

$$B = 1 - \frac{4}{9}$$

$$\boxed{B = \frac{5}{9}}$$

Put values of A, B and C in equation (1), we get

$$\frac{x^2}{(x-2)(x-1)^2} = \frac{\frac{4}{9}}{(x-2)} + \frac{\frac{5}{9}}{(x-1)} + \frac{-1}{(x-1)^2}$$

$$= \frac{4}{9(x-2)} + \frac{5}{9(x-1)} - \frac{1}{(x-1)^2}$$

Q.7 $\dfrac{1}{(x-1)^2(x+1)}$

Solution:

Let

$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)} \qquad \ldots\ldots\ldots (1)$$

$$\frac{1}{(x-1)^2(x+1)} = \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)}$$

$$1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \qquad \ldots\ldots\ldots (2)$$

$$1 = A(x^2-1) + Bx + B + C(x^2+1-2x)$$

$$1 = Ax^2 - A + Bx + B + Cx^2 + C - 2Cx \qquad \ldots\ldots\ldots (3)$$

Put $x = 1$ in equation (2), we get

$$1 = A(0) + 2B + 0$$

$$1 = 2B \quad \Rightarrow \quad \boxed{B = \frac{1}{2}}$$

Put $x = -1$ in equation (2), we get

$$1 = A(0) + B(0) + C(-1-1)^2$$

$$1 = 0 + 0 + C(-2)^2$$

$$1 = 4C \quad \Rightarrow \quad \boxed{C = \frac{1}{4}}$$

Equating coefficients of x^2 in equation (3), we get

$$A + C = 0$$

$$A + \frac{1}{4} = 0 \qquad \because \quad C = \frac{1}{4}$$

$$\boxed{A = \frac{1}{4}}$$

Put values of A, B and C in equation (1) we get

$$\frac{1}{(x-1)^2(x+1)} = \frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{2}}{(x-1)^2} + \frac{\frac{1}{4}}{x+1}$$

$$= \frac{-1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{1}{4(x+1)}$$

are required partial fraction.

Q.8 $\dfrac{x^2}{(x-1)^3(x+1)}$ **(Gujranwala Board 2003)**

Solution:

Let

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x+1)} \quad \ldots\ldots\ldots (1)$$

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3}{(x-1)^3(x+1)}$$

$$x^2 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3 \quad \ldots\ldots\ldots (2)$$

$$x^2 = A(x^3 - x^2 - x + 1) + Bx^2 - B + Cx + C + Dx^3 - 3Dx^2 + 3Dx - D$$

$$x^2 = Ax^3 - Ax^2 - Ax + A + Bx^2 - B + Cx + C + Dx^3 - 3Dx^2 + 3Dx - D \quad \ldots\ldots\ldots (3)$$

Put $x = 1$ in equation (2), we get

$$(1)^2 = A(0) + B(0) + C(1+1) + D(0)$$

$$1 = 0 + 0 + 2C + 0$$

$$2C = 1 \Rightarrow \boxed{C = \frac{1}{2}}$$

Put $x = -1$ in equation (2), we get

$$(-1)^2 = A(0) + B(0) + C(0) + D(-1-1)^3$$

$$1 = 0 + 0 + 0\,D(-2)^3$$

$$1 = -8D \Rightarrow \boxed{D = -\frac{1}{8}}$$

Equating coefficients of x^3, x^2 in equation (3), we get

x^3 ; $A + D = 0$

$A - \dfrac{1}{8} = 0$ ¤ $D = -\dfrac{1}{8}$

$$\boxed{A = \frac{1}{8}}$$

x^2 ; $-A + B - 3D = 1$

$-\dfrac{1}{8} + B - \left(-\dfrac{1}{8}\right) = 1$ ¤ $A = \dfrac{1}{8}$, $D = -\dfrac{1}{8}$

$$-\frac{1}{8} + B + \frac{3}{8} = 1$$

$$B = 1 + \frac{1}{8} - \frac{3}{8}$$

$$B = \frac{8 + 1 - 3}{8}$$

$$B = \frac{6}{8} \Rightarrow \boxed{B = \frac{3}{4}}$$

Put values of A, B, C and D in equation (1) we get

$$\frac{x^2}{(x-1)^3 (x+1)} = \frac{\frac{1}{8}}{(x-1)} + \frac{\frac{3}{4}}{(x-1)^2} + \frac{\frac{1}{2}}{(x-1)^2} + \frac{-\frac{1}{8}}{(x+1)}$$

$$= \frac{1}{8(x-1)} + \frac{3}{4(x-1)^2} + \frac{1}{2(x-1)^2} - \frac{1}{8(x+1)}$$

are required partial fractions.

Q.9 $\dfrac{x-1}{(x-2)(x+1)^3}$ **(Gujranwala Board 2006)**

Solution:

Let

$$\frac{x-1}{(x-2)(x+1)^3} = \frac{A}{(x-2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3} \quad \ldots\ldots\ldots (1)$$

$$\frac{x-1}{(x-2)(x+1)^3} = \frac{A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + D(x-2)}{(x-2)(x+1)^3}$$

$$(x-1) = A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + D(x-2) \quad \ldots\ldots\ldots (2)$$

$$x - 1 = A(x^3 + 1 + 3x^2 + 3x) + B(x^3 - 3x - 2) + C(x^2 - x + 2) + D(x-2)$$

$$x - 1 = Ax^3 + A + 3Ax^2 + 3Ax + Bx^3 - 3Bx - 2B + Cx^2 - Cx + 2C + Dx - 2D \quad \ldots\ldots (3)$$

Put $x = 2$ in equation (2), we get

$$2 - 1 = A(2+1)^3 + 0 + 0 + 0$$

$$1 = A(3)^3$$

$$1 = 27A \Rightarrow \boxed{A = \frac{1}{27}}$$

Put $x = -1$ in equation (2), we get

$$-1 - 1 = 0 + 0 + 0 + D(-1 - 2)$$

$-2 = D(-3)$

$$\boxed{D = \frac{2}{3}}$$

x^3 ; $A + B = 0$

$$\frac{1}{27} + B = 0 \qquad \text{¤} \quad A = \frac{1}{27}$$

$$\boxed{B = -\frac{1}{27}}$$

x^2 ; $3A + C = 0$

$$3 \cdot \frac{1}{27} + C = 0$$

$$\frac{1}{9} + C = 0 \Rightarrow \boxed{C = -\frac{1}{9}}$$

Put values of A, B, C and D in equation (1) we get

$$\frac{x-1}{(x-2)(x+1)^3} = \frac{\frac{1}{27}}{(x-2)} + \frac{-\frac{1}{27}}{(x+1)} + \frac{-\frac{1}{9}}{(x+1)^2} + \frac{\frac{2}{3}}{(x+1)^3}$$

$$= \frac{1}{27(x-2)} - \frac{1}{27(x+1)} - \frac{1}{9(x+1)^2} + \frac{2}{3(x+1)^3}$$

are required partial fractions.

Q.10 $\dfrac{4x^3}{(x^2-1)(x+1)^2}$ **(Gujranwala Board 2005, Lahore Board 2005)**

Solution:

$$\frac{4x^3}{(x^2-1)(x+1)^2} = \frac{4x^3}{(x+1)(x-1)(x+1)^2} = \frac{4x^3}{(x-1)(x+1)^3}$$

Let

$$\frac{4x^3}{(x-1)(x+1)^3} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3} \qquad \ldots\ldots\ldots (1)$$

$$\frac{4x^3}{(x-1)(x+1)^3} = \frac{A(x+1)^3 + B(x-1)(x+1)^2 + C(x-1)(x+1) + D(x-1)}{(x-1)(x+1)^3}$$

$$4x^3 = A(x+1)^3 + B(x-1)(x+1)^2 + C(x-1)(x+1) + D(x-1) \qquad \ldots\ldots\ldots (2)$$

$$4x^3 = A(x^3 + 1 + 3x^2 + 3x) + B(x^3 + x^2 - x - 1) + C(x^2 - 1) + D(x-1)$$

$$4x^3 = Ax^3 + A + 3Ax^2 + 3Ax + Bx^3 + Bx^2 - Bx - B + Cx^2 - C + Dx - D \qquad \ldots\ldots\ldots (3)$$

Put $x = 1$ in equation (2), we get

$4(1)^3 = A(1+1)^3 + 0 + 0 + 0$

$4 = (2)^3 A$

$4 = 8A \Rightarrow \boxed{A = \frac{1}{2}}$

Put $x = -1$ in equation (2), we get

$4(-1)^3 = 0 + 0 + 0 + D(-1-1)$

$-4 = -2D \Rightarrow \boxed{D = 2}$

Equating coefficients of x^3, x^2 in equation (3) we get

x^3 ; $A + B = 4$

$\frac{1}{2} + B = 4$ ∵ $A = \frac{1}{2}$

$B = 4 - \frac{1}{2}$

$\boxed{B = \frac{7}{2}}$

x^2 ; $3A + B + C = 0$

$3 \cdot \frac{1}{2} + \frac{7}{2} + C = 0$

$\frac{3}{2} + \frac{7}{2} + C = 0$

$\frac{10}{2} + c = 0$

$5 + C = 0 \Rightarrow \boxed{C = 5}$

Put values of A, B, C and D in equation (1), we get

$$\frac{4x^3}{(x-1)(x+1)^3} = \frac{\frac{1}{2}}{(x-1)} + \frac{\frac{7}{2}}{(x+1)} - \frac{5}{(x+1)^2} + \frac{2}{(x+1)^3}$$

$$= \frac{1}{2(x-1)} + \frac{7}{2(x+1)} - \frac{5}{(x+1)^2} + \frac{2}{(x+1)^3}$$

are required partial fractions.

Q.11 $\dfrac{2x+1}{(x+3)(x-1)(x+2)^2}$

Solution:

Let

$$\frac{2x+1}{(x+3)(x-1)(x+2)^2} = \frac{A}{(x+3)} + \frac{B}{(x-1)} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2} \quad \ldots\ldots\ldots (1)$$

$$\frac{2x+1}{(x+3)(x-1)(x+2)^2}$$

$$= \frac{A(x-1)(x+2)^2 + B(x+3)(x+2)^2 + C(x+3)(x-1)(x+2) + D(x+3)(x-1)}{(x+3)(x-1)(x+2)^2}$$

$$(2x+1) = A(x-1)(x+2)^2 + B(x+3)(x+2)^2 + C(x+3)(x-1)(x+2) + D(x+3)(x-1) \quad (2)$$

$$2x+1 = A(x^3+3x^2-4) + B(x^3+7x^2+16x+12) + C(x^3+4x^2+x-6) + D(x^2+2x-3)$$

$$2x+1 = Ax^3+3Ax^2-4A+Bx^3+7Bx^2+16Bx+12B+Cx^3+4Cx^2+Cx-6C + Dx^2+2Dx-3D \quad (3)$$

Put $x = -3$ in equation (2), we get

$$2(-3)+1 = A(-3-1)(-3+2)^2 + 0 + 0 + 0$$

$$-6+1 = A(-4)(-1)^2$$

$$-5 = -4A \Rightarrow \boxed{A = \frac{5}{4}}$$

Put $x = 1$ in equation (2), we get

$$2(1)+1 = 0 + B(1+3)(1+2)^2 + 0 + 0$$

$$2+1 = B(4)(3)^2$$

$$3 = B(4)(9)$$

$$3 = 36B \Rightarrow B = \frac{3}{36} \Rightarrow \boxed{B = \frac{1}{12}}$$

Put $x = -2$ in equation (2), we get

$$2(-2)+1 = 0 + 0 + 0 + D(-2+3)(-2-1)$$

$$-4+1 = D(1)(-3)$$

$$-3 = -3D \Rightarrow \boxed{D = 1}$$

Equating coefficients of x^3 in equation (3) we get

$$A + B + C = 0$$

$$\frac{5}{4} + \frac{1}{12} + C = 0$$

$$\frac{15+1}{12} + C = 0$$

$$\frac{16}{12} + C = 0$$

$$\frac{4}{3} + C = 0 \Rightarrow \boxed{C = -\frac{4}{3}}$$

Put values of A, B, C and D in equation (1), we get

$$\frac{2x+1}{(x+3)(x-1)(x+2)^2} = \frac{\frac{5}{4}}{(x+3)} + \frac{\frac{1}{12}}{(x-1)} + \frac{-\frac{4}{3}}{(x+2)} + \frac{1}{(x+2)^2}$$

$$= \frac{5}{4(x+3)} + \frac{1}{12(x-1)} - \frac{4}{3(x+2)} + \frac{1}{(x+2)^2}$$

are required partial fraction.

Q.12 $\dfrac{2x^4}{(x-3)(x+2)^2}$

Solution:

$$\frac{2x^4}{(x-3)(x+2)^2} = \frac{2x^4}{(x-3)(x^2+4+4x)} = \frac{2x^4}{x^3+x^2-8x-12}$$

$$\begin{array}{r} 2x-2 \\ x^3+x^2-8x-12 \;\overline{)\;2x^4} \\ 2x^4+2x^3-16x^2-24x \\ - \quad - \quad + \quad + \\ \hline -2x^3+16x^2+24x \\ -2x^3-2x^2+16x+24 \\ + \quad + \quad - \quad - \\ \hline 18x^2+8x-24 \end{array}$$

$$\frac{2x^4}{x^3+x^2-8x-12} = 2x-2+\frac{18x^2+8x-24}{x^3+x^2-8x-12}$$

$$= 2x-2+\frac{18x^2+8x-24}{(x-3)(x+2)^2} \quad \ldots\ldots\ldots (1)$$

Let $$\frac{18x^2+8x-24}{(x-3)(x+2)^2} = \frac{A}{(x-3)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} \quad \ldots\ldots\ldots (2)$$

$$\frac{18x^2+8x-24}{(x-3)(x+2)^2} = \frac{A(x+2)^2 + B(x-3)(x+2) + C(x-3)}{(x-3)(x+2)^2}$$

$18x^2 + 8x - 24 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$ ..... (3)

$18x^2 + 8x - 24 = A(x^2 + 4x + 4) + B(x^2 - x - 6) + C(x-3)$

$18x^2 + 8x - 24 = Ax^2 + 4Ax + 4A + Bx^2 - Bx - 6B + Cx - 3C$ ..... (4)

Put $x = 3$ in equation (3), we get

$18(3)^2 + 8(3) - 24 = A(3+2)^2 + 0 + 0$

$18(9) + 24 - 24 = A(5)^2$

$162 = 25A \Rightarrow \boxed{A = \frac{162}{25}}$

Put $x = -2$ in equation (3), we get

$18(-2)^2 + 8(-2) - 24 = 0 + 0 + C(-2-3)$

$-36 - 16 - 24 = C(-5)$

$32 = -5C \Rightarrow \boxed{B = -\frac{32}{5}}$

Equating coefficients of x^2 in equation (4), we get

$A + B = 18$

$B = 18 - A$

$B = 18 - \frac{162}{25} \Rightarrow \boxed{B = \frac{288}{25}}$

Put values of A, B, C in equation (2), we get

$$\frac{18x^2 + 8x - 24}{(x-3)(x+2)^2} = \frac{162}{25(x-3)} + \frac{288}{25(x+2)} - \frac{32}{5(x+2)^2}$$

equation (1) becomes

$$\frac{4x^3}{(x-3)(x+2)^2} = (2x-2) + \frac{162}{25(x-3)} + \frac{288}{25(x+2)} - \frac{32}{5(x+2)^2}$$

are required partial fractions.

EXERCISE 5.3

Resolve the following into partial fractions.

Q.1 $\dfrac{9x-7}{(x^2+1)(x+3)}$ **(Lahore Board 2004, 2010)**

Solution:

Let

$$\frac{9x-7}{(x^2+1)(x+3)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x+3)} \quad \text{..... (1)}$$

$$\frac{9x-7}{(x^2+1)(x+3)} = \frac{(Ax+B)(x+3) + C(x^2+1)}{(x^2+1)(x+3)}$$

$9x - 7 = (Ax+B)(x+3) + C(x^2+1)$ ..... (2)

$9x - 7 = Ax^2 + 3Ax + Bx + 3B + Cx^2 + C$ ..... (3)

Put $x = -3$ in equation (2), we get

$9(-3) - 7 = 0 + C[(-3)^2 + 1]$

$-27 - 7 = 0 + C(9 + 1)$

$-34 = 10C \Rightarrow \boxed{C = \frac{-17}{5}}$

Equating coefficients of x^2, x in equation (3) we get

x^2 ; $A + C = 0$

$A - \frac{17}{5} = 0$

$\Rightarrow \boxed{A = \frac{17}{5}}$

x ; $3A + B = 9$

$3\left(\frac{17}{5}\right) + B = 9$

$\frac{51}{5} + B = 9$

$B = 9 - \frac{51}{5}$

$= \frac{45 - 51}{5}$

$\boxed{B = \frac{-6}{5}}$

Put values of A, B, C in equation (1) we get

$$\frac{9x - 7}{(x^2 + 1)(x + 3)} = \frac{\frac{17}{5}x - \frac{6}{5}}{x^2 + 1} - \frac{\frac{17}{5}}{x + 3}$$

$$= \frac{17x - 6}{5(x^2 + 1)} - \frac{17}{5(x + 3)}$$

are required partial fractions.

Q.2 $\dfrac{1}{(x^2 + 1)(x + 1)}$ **(Lahore Board 2003, 2006)**

Solution:

Let

$$\frac{1}{(x^2 + 1)(x + 1)} = \frac{Ax + B}{(x^2 + 1)} + \frac{C}{(x + 1)} \quad \text{..... (1)}$$

$$\frac{1}{(x^2 + 1)(x + 1)} = \frac{(Ax + B)(x + 1) + C(x^2 + 1)}{(x^2 + 1)(x + 1)}$$

$1 = (Ax + B)(x + 1) + C(x^2 + 1)$ ..... (2)

$1 = Ax^2 + Ax + Bx + B + Cx^2 + C$ ..... (3)

Put $x = -1$ in equation (2), we get

$1 = 0 + C\left((-1)^2 + 1\right)$

$1 = C(1 + 1)$

$1 = 2C \Rightarrow \boxed{C = \frac{1}{2}}$

Equating coefficients of x^2 and x in equation (3) we get

$x^2 \quad ; \quad A + C = 0$

$A = -C$

$\boxed{A = -\frac{1}{2}}$

Equating coefficient of x

$x \quad ; \quad A + B = 0$

$B = -A$

$\boxed{B = \frac{1}{2}}$

Put values of A, B and C in equation (1) we get

$$\frac{1}{(x^2+1)(x+1)} = \frac{-\frac{1}{2}x + \frac{1}{2}}{(x^2+1)} + \frac{\frac{1}{2}}{(x+1)}$$

$$= \frac{-x+1}{2(x^2+1)} + \frac{1}{2(x+1)}$$

are required partial fractions.

Q.3 $\dfrac{\mathbf{3x + 7}}{\mathbf{(x^2 + 4)(x + 3)}}$

Solution:

Let

$$\frac{3x+7}{(x^2+4)(x+3)} = \frac{Ax+B}{(x^2+4)} + \frac{C}{(x+3)} \qquad \ldots\ldots\ldots (1)$$

$$\frac{3x+7}{(x^2+4)(x+3)} = \frac{(Ax+B)(x+3) + C(x^2+4)}{(x^2+4)(x+3)}$$

$3x + 7 = (Ax + B)(x + 3) + C(x^2 + 4) \qquad \ldots\ldots\ldots (2)$

$3x + 7 = Ax^2 + 3Ax + Bx + 3B + Cx^2 + 4C \qquad \ldots\ldots\ldots (3)$

Put $x = -3$ in equation (2), we get

$3(-3) + 7 = 0 + C\left((-3)^2 + 4\right)$

$-9 + 7 = C(9 + 4)$

$$-2 = 13C \Rightarrow \boxed{C = -\frac{2}{13}}$$

Equating coefficients of x^2 and x in equation (3) we get

x^2 ; $A + C = 0$

$A = -C$

$$\boxed{A = \frac{2}{13}}$$

x ; $3A + B = 3$

$B = 3 - 3A$

$$= 3 - 3 \cdot \frac{2}{13}$$

$$= 3 - \frac{6}{13}$$

$$= \frac{39 - 6}{13}$$

$$\boxed{B = \frac{33}{13}}$$

Put values of A, B and C in equation (1) we get

$$\frac{3x + 7}{(x^2 + 4)(x + 3)} = \frac{\frac{2}{13}x + \frac{33}{13}}{x^2 + 4} + \frac{-\frac{2}{13}}{x + 3}$$

$$= \frac{2x + 33}{13(x^2 + 4)} - \frac{2}{13(x + 3)}$$

are required partial fractions.

Q.4 $\dfrac{\mathbf{x^2 + 15}}{\mathbf{(x^2 + 2x + 5)(x - 1)}}$

Solution:

Let

$$\frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)} = \frac{Ax + B}{(x^2 + 2x + 5)} + \frac{C}{(x - 1)} \quad \ldots\ldots\ldots (1)$$

$$\frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)} = \frac{(Ax + B)(x - 1) + C(x^2 + 2x + 5)}{(x^2 + 2x + 5)(x - 1)}$$

$$x^2 + 15 = (Ax + B)(x - 1) + C(x^2 + 2x + 5) \quad \ldots\ldots\ldots (2)$$

$$x^2 + 15 = Ax^2 - Ax + Bx - B + Cx^2 + 2Cx + 5C \quad \ldots\ldots\ldots (3)$$

Put $x = 1$ in equation (2), we get

$(1)^2 + 15 = 0 + C\left((1)^2 + 2(1) + 5\right)$

$1 + 15 = C(1 + 2 + 5)$

$16 = 8C \Rightarrow \boxed{C = 2}$

Equating coefficients of x^2, and x in equation (3), we get

x^2 ; $A + C = 1$

$A = 1 - C$

$A = 1 - 2$

$\boxed{A = 1}$

x ; $-A + B + 2C = 0$

$B = A - 2C$

$B = -1 - 2(2) = -1 - 4$

$\boxed{B = -5}$

Put values of A, B and C in equation (1) we get

$$\frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)} = \frac{-x - 5}{x^2 + 2x + 5} + \frac{2}{(x - 1)}$$

are required partial fraction.

Q.5 $\dfrac{\mathbf{x^2}}{\mathbf{(x^2 + 4)(x + 2)}}$

Solution:

Let

$$\frac{x^2}{(x^2 + 4)(x + 2)} = \frac{Ax + B}{(x^2 + 4)} + \frac{C}{(x + 2)} \qquad \ldots\ldots\ldots (1)$$

$$\frac{x^2}{(x^2 + 4)(x + 2)} = \frac{(Ax + B)(x + 2) + C(x^2 + 4)}{(x^2 + 4)(x + 2)}$$

$x^2 = (Ax + B)(x + 2) + C(x^2 + 4)$ ..... (2)

$x^2 = Ax^2 + 2Ax + Bx + 2B + Cx^2 + 4C$ ..... (3)

Put $x = -2$ in equation (2), we get

$(-2)^2 = 0 + C\left((-2)2 + 4\right)$

$4 = 0 + C(4 + 4)$

$4 = 8C \Rightarrow \boxed{C = \frac{1}{2}}$

Equating coefficients of x^2 and x in equation (3), we get

$x^2 \quad ; \quad A + C = 1$

$$A = 1 - C = 1 - \frac{1}{2}$$

$$\boxed{A = \frac{1}{2}}$$

$x \quad ; \quad 2A + B = 0$

$$2\left(\frac{1}{2}\right) + B = 0$$

$$1 + B = 0$$

$$\boxed{B = -1}$$

Put values of A, B and C in equation (1) we get

$$\frac{x^2}{(x^2+4)(x+2)} = \frac{\frac{1}{2}x - 1}{x^2+4} + \frac{\frac{1}{2}}{x+2}$$

$$= \frac{x-2}{2(x^2+4)} + \frac{1}{2(x+2)}$$

are required partial fraction.

Q.6 $\dfrac{\mathbf{x^2+1}}{\mathbf{x^3+1}}$ **(Lahore Board 2003)**

Solution:

Let

$$\frac{x^2+1}{x^3+1} = \frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2-x+1)} \quad \ldots\ldots\ldots (1)$$

$$\Rightarrow \quad \frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{A(x^2-x+1) + (Bx+C)(x+1)}{(x+1)(x^2-x+1)}$$

$$x^2 + 1 = A(x^2 - x + 1) + (Bx + C)(x + 1) \quad \ldots\ldots\ldots (2)$$

$$x^2 + 1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C \quad \ldots\ldots\ldots (3)$$

Put $x = -1$ in equation (2), we get

$$(-1)^2 + 1 = A\left((-1)2 - (-1) + 1\right)$$

$1 + 1 = A(1 + 1 + 1)$

$$2 = 3A \Rightarrow \boxed{A = \frac{2}{3}}$$

Equating coefficients of x2 and x in equation (3), we get

x^2 ; $A + B = 1$

$$\frac{2}{3} + B = 1$$

$$B = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\boxed{B = \frac{1}{3}}$$

x ; $-A + B + C = 0$

$$-\frac{2}{3} + \frac{1}{3} + C = 0$$

$$-\frac{1}{3} + C = 0$$

$$\boxed{C = \frac{1}{3}}$$

Put values of A, B and C in equation (1), we get

$$\frac{x^2 + 1}{x^3 + 1} = \frac{\frac{2}{3}}{x + 1} + \frac{\frac{1}{3}x + \frac{1}{3}}{x^2 - x + 1}$$

$$= \frac{2}{3(x + 1)} + \frac{x + 1}{3(x^2 - x + 1)}$$

are required partial fraction.

Q.7 $\dfrac{\mathbf{x^2 + 2x + 2}}{\mathbf{(x^2 + 3)(x + 1)(x - 1)}}$

Solution:

Let

$$\frac{x^2 + 2x + 2}{(x^2 + 3)(x + 1)(x - 1)} = \frac{Ax + B}{(x^2 + 3)} + \frac{C}{(x + 1)} + \frac{D}{(x - 1)} \quad \ldots\ldots\ldots (1)$$

$$\frac{x^2 + 2x + 2}{(x^2 + 3)(x + 1)(x - 1)} = \frac{(Ax + B)(x + 1)(x - 1) + C(x^2 + 3)(x - 1) + D(x^2 + 3)(x + 1)}{(x^2 + 3)(x + 1)(x - 1)}$$

$x^2 + 2x + 2 = (Ax + B)(x + 1)(x - 1) + C(x^2 + 3)(x - 1) + D(x^2 + 3)(x + 1)$ ..... (2)

$x^2 + 2x + 2 = Ax^3 - Ax + Bx^2 - B + Cx^3 - Cx^2 + 3Cx - 3C + Dx^3 + Dx^2 + 3Dx + 3D$

Put $x = -1$ in equation (2), we get

$(-1)^2 + 2(-1) + 2 = 0 + C((-1)^2 + 3)(-1 - 1) + 0$

$1 - 2 + 2 = C(1 + 3)(-2)$

$1 = -8C \Rightarrow \boxed{C = -\frac{1}{8}}$

Put $x = 1$ in equation (2), we get

$(1)^2 + 2(1) + 2 = 0 + 0 + D((1)^2 + 3)(1 + 1)$

$1 + 2 + 2 = D(1 + 3)(2)$

$5 = 8D \Rightarrow \boxed{D = \frac{5}{8}}$

Equating coefficients of x^3 and x^2 in equation (3), we get

x^3 ; $A + C + D = 0$

$A = -C - D$

$= \frac{1}{8} - \frac{5}{8} = \frac{-4}{8}$

$\boxed{A = -\frac{1}{2}}$

x^2 ; $B + D - C = 1$

$B = 1 - D + C = 1 - \frac{5}{8} - \frac{1}{8}$

$\boxed{B = \frac{1}{4}}$

Put values of A, B, C, D in equation (1), we get

$$\frac{x^2 + 2x + 2}{(x^2 + 3)(x + 1)(x - 1)} = \frac{-\frac{1}{2}x + \frac{1}{4}}{(x^2 + 3)} + \frac{-\frac{1}{8}}{(x + 1)} + \frac{\frac{5}{8}}{(x - 1)}$$

$$= \frac{1 - 2x}{4(x^2 + 3)} - \frac{1}{8(x + 1)} + \frac{5}{8(x - 1)}$$

are required partial fraction.

Q.8 $\dfrac{1}{(x-1)^2(x^2+2)}$ **(Gujranwala Board 2006)**

Solution:

Let

$$\frac{1}{(x-1)^2(x^2+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(x^2+2)} \qquad \ldots\ldots\ldots (1)$$

$$\frac{1}{(x-1)^2(x^2+2)} = \frac{A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2}{(x-1)^2(x^2+2)}$$

$$1 = A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2 \quad \ldots\ldots\ldots (2)$$

$$1 = A(x^3 - x^2 + 2x - 2) + B(x^2+2) + (Cx+D)(x^2+1-2x)$$

$$1 = Ax^3 - Ax^2 + 2Ax - 2A + Bx^2 + 2B + Cx^3 + Cx - 2Cx^2 + Dx^2 + D - 2Dx \quad \ldots\ldots (3)$$

Put $x = 1$ in equation (2), we ge

$$1 = 0 + B\left((1)^2 + 2\right) + 0$$

$$1 = B(1+2)$$

$$1 = 3B \Rightarrow \boxed{B = \frac{1}{3}}$$

Equating coefficients of x^3, x^2, and x in equation (3), we get

$$1 = 0 + B\left((1)^2 + 2\right) + 0$$

$$1 = B(1+2)$$

$$1 = 3B \Rightarrow \boxed{B = \frac{1}{3}}$$

Equating coefficients of x^3, x^2, and x in equation (3), we get

x^3 ; $A + C = 0$ ..... (i)

x^2 ; $-A + B - 2C + D = 0$ ¤ $B = \dfrac{1}{3}$

$$-A - 2C + D = -\frac{1}{3} \qquad \ldots\ldots\ldots (ii)$$

x ; $2A + C - 2D = 0$ ..... (iii)

from (i) $A = -C$

Put in (ii) and (iii).

$$-(-C) - 2C + D = \frac{1}{3}$$

$$-C + D = \frac{1}{3} \qquad \ldots\ldots\ldots \text{(iv)}$$

and $\quad 2(-C) + C - 2D = 0$

$$-2C + C - 2D = 0$$

$$-C - 2D = 0 \qquad \ldots\ldots\ldots \text{(v)}$$

Subtracting (iv) from (v)

$$-C - 2D = 0$$

$$-C + D = -\frac{1}{3}$$

$$+ \quad - \quad +$$

$$-3D = 0 + \frac{1}{3}$$

$$-3D = \frac{1}{3} \Rightarrow \boxed{D = -\frac{1}{9}}$$

Put $D = -\frac{1}{9}$ in equation (v)

$$-C - 2\left(-\frac{1}{9}\right) = 0$$

$$-C + \frac{2}{9} = \Rightarrow \boxed{C = \frac{2}{9}}$$

Put this value in equation (i) we get

$$\boxed{A = -C = \frac{-2}{9}}$$

Putting values of A, B, C and D in equation (1) we get

$$\frac{1}{(x-1)^2(x^2+2)} = \frac{-\frac{2}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} + \frac{\frac{2}{9}x - \frac{1}{9}}{(x^2+2)}$$

$$= \frac{-2}{9(x-1)} + \frac{1}{3(x-1)^2} + \frac{2x-1}{9(x^2+2)}$$

are required partial fraction.

Q.9 $\dfrac{x^4}{1-x^4}$

Solution:

$$\frac{x^4}{1-x^4} = \frac{-x^4}{x^4-1}$$

By division

$$\begin{array}{r} -1 \\ x^4-1 \;\overline{)\,-x^4} \\ -x^4+1 \\ + \quad - \\ \hline -1 \end{array}$$

$$\Rightarrow \quad \frac{-x^4}{x^4-1} = -1 - \frac{1}{x^4-1} \qquad \ldots\ldots\ldots (1)$$

$$\frac{1}{x^4-1} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(x^2+1)}$$

Let

$$\frac{1}{(x+1)(x-1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \qquad \ldots\ldots\ldots (2)$$

$$\frac{1}{(x+1)(x-1)(x^2+1)} = \frac{A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x-1)}{(x+1)(x-1)(x^2+1)}$$

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x-1) \qquad \ldots\ldots\ldots (3)$$

$$1 = A(x^3 - x^2 + x - 1) + B(x^3 + x^2 + x + 1) + (Cx+D)(x^2-1)$$

$$1 = Ax^3 - Ax^2 + Ax - A + Bx^3 + Bx^2 + Bx + B + Cx^3 - Cx - Cx + Dx^2 - D \qquad \ldots\ldots\ldots (4)$$

Put $x = 1$ in equation (3), we get

$$1 = A(-1-1)\left((-1)^2+1\right) + 0 + 0$$

$$1 = A(-2)(1+1)$$

$$1 = 4A \quad \Rightarrow \quad \boxed{A = -\frac{1}{4}}$$

Put $x = -1$ in equation (3), we get

$$1 = 0 + B(1+1)\left((1)^2+1\right) + 0$$

$1 = B(2)(1+1)$

$1 = 4B \Rightarrow \boxed{B = \frac{1}{4}}$

Equating coefficients of x^3 and x^2 in equation (4) we get

x^3 ; $A + B + C = 0$

$-\frac{1}{4} + \frac{1}{4} + C = 0$

$\Rightarrow \boxed{C = 0}$

x^2 ; $-A + B + D = 0$

$\frac{1}{4} + \frac{1}{4} + D = 0$

$\boxed{D = -\frac{1}{2}}$

Put values of A, B, C, D in equation (2) we get

$$\frac{1}{(1-x)(1+x)(1+x^2)} = \frac{-\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1} + \frac{(0)x - \frac{1}{2}}{x^2+1}$$

$$= \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)}$$

Equation (1) becomes

$$\frac{-x^4}{x^4-1} = -1 - \left[\frac{-1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)}\right]$$

$$= -1 + \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{2(x^2+1)}$$

are required partial fractions.

Q.10 $\dfrac{x^2 - 2x + 3}{x^4 + x^2 + 1}$

Solution:

$$\frac{x^2-2x+3}{x^4+x^2+1} = \frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)}$$

Let

$$\frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)} = \frac{Ax+B}{(x^2+x+1)} + \frac{Cx+D}{(x^2-x+1)} \quad \ldots\ldots\ldots (1)$$

$$\frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)} = \frac{(Ax+B)(x^2-x+1)(Cx+D)(x^2+x+1)}{(x^2+x+1)(x^2-x+1)}$$

$$x^2 - 2x + 3 = (Ax+B)(x^2+x+1)(Cx+D)(x^2-x+1)$$

$$x^2 - 2x + 3 = Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + Cx^3 + Cx^2 + Cx + Dx^2 + Dx + D \quad \ldots\ldots\ldots (2)$$

Equating coefficients of x^3, x^2, x, x and constant term in equation (2), we get

x^3 ; $A + C = 0$ (i)

x^2 ; $-A + B + C + D = 1$ (ii)

x ; $A - B + C + D = -2$ (iii)

cons ; $B + D = 3$ (iv)

Put $A + C = 0$ in equation (iii), we get

$-B + D = -2$ (v)

adding equation (iv) and (v), we get

$2D = 1 \Rightarrow \boxed{D = \frac{1}{2}}$

Put $D = \frac{1}{2}$ in equation (iv), we get

$B + \frac{1}{2} = 3$

$B = 3 - \frac{1}{2} = \frac{6-1}{2}$

$\boxed{B = \frac{5}{2}}$

Put $B + D = 3$ in equation (ii), we get

$-A + C + 3 = 1$

$-A + C = -2$ (vi)

adding equation (vi) and (i)

$2C = -2$

$\boxed{C = -1}$

Put $C = -1$ in equation (i), we get

$A - 1 = 0$

$\boxed{A = 1}$

Put values of A, B, C, and D in equation (1), we get

$$\frac{x^2 - 2x + 3}{(x^2 - x + 1)(x^2 + x + 1)} = \frac{x + \frac{5}{2}}{x^2 + x + 1} + \frac{-x + \frac{1}{2}}{x^2 - x + 1}$$

$$= \frac{(2x + 5)}{2(x^2 + x + 1)} - \frac{2x - 1}{2(x^2 - x + 1)}$$

are required partial fraction.

EXERCISE 5.4

Resolve the following partial fractions.

Q.1 $\dfrac{x^3 + 2x + 2}{(x^2 + x + 1)^2}$

Solution:

Let

$$\frac{x^3 + 2x + 2}{(x^2 + x + 1)^2} = \frac{(Ax + B)}{(x^2 + x + 1)} + \frac{(Cx + D)}{(x^2 + x + 1)^2} \qquad \ldots\ldots\ldots (1)$$

$$\frac{x^3 + 2x + 2}{(x^2 + x + 1)^2} = \frac{(Ax + B)(x^2 + x + 1) + (Cx + D)}{(x^2 + x + 1)^2}$$

$$x^3 + 2x + 2 = (Ax + B)(x^2 + x + 1) + (Cx + D)$$

$$x^3 + 2x + 2 = Ax^3 + Ax^2 + Ax + Bx^2 + Bx + B + Cx + D \ldots\ldots\ldots (2)$$

Equating coefficients of x^3, x^2, x and constant term in equation (2), we get

x^3 ; $\boxed{A = 1}$

x^2 ; $A + B = 0$

$1 + B = 0 \Rightarrow \boxed{B = -1}$

x ; $A + B + C = 2$

$1 - 1 + C = 2 \Rightarrow \boxed{C = 2}$

Cons ; $B + D = 2$

$-1 + D = 2 \Rightarrow \boxed{D = 3}$

Put values of A, B, C, D in equation (1) we get

$$\frac{x^3 + 2x + 2}{(x^2 + x + 1)^2} = \frac{x - 1}{(x^2 + x + 1)} + \frac{2x + 3}{(x^2 + x + 1)^2}$$

are required partial fraction.

Q.2 $\dfrac{x^2}{(x^2 + 1)^2 (x - 1)}$

Solution:

Let

$$\frac{x^2}{(x - 1)(x^2 + 1)^2} = \frac{A}{(x - 1)} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} \qquad (1)$$

$$\frac{x^2}{(x-1)(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1)}{(x-1)(x^2+1)^2}$$

$$x^2 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1) \quad \ldots\ldots\ldots (2)$$

$$x^2 = A(x^4+1+2x^2) + (Bx+C)(x^3+x-x^2-1) + (Dx+E)(x-1)$$

$$x^2 = Ax^4 + A + 2Ax^2 + Bx^4 + Bx^2 - Bx^3 - Bx + Cx^3 + Cx - Cx^2 - C$$
$$+ Dx^2 - Dx + Ex - E \quad \ldots\ldots\ldots (3)$$

Put $x = 1$ in equation (2), we get

$$(1)^2 = A((1)^2+1)^2 + 0 + 0$$

$$1 = A(1+1)^2$$

$$1 = A(2)^2 \Rightarrow = \boxed{A = \frac{1}{4}}$$

Equating coefficients of x^4, x^3, x^2 and constant term in equation (3), we get

x^4 ; $A + B = 0$

$$B = -A$$

$$\boxed{B = -\frac{1}{4}}$$

x^3 ; $-B + C = 0$

$$C = B$$

$$\boxed{C = -\frac{1}{4}}$$

x^2 ; $2A + B - C + D = 1$

$$D = 1 - 2A - B + C$$

$$= 1 - 2\left(\frac{1}{4}\right) + \frac{1}{4} - \frac{1}{4}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$\boxed{D = \frac{1}{2}}$$

x ; $-B + C - D + E = 0$

$$\frac{1}{4} - \frac{1}{4} - \frac{1}{2} + E = 0$$

$$\boxed{E = \frac{1}{2}}$$

Put values of A, B, C, D and E in equation (1) we get

$$\frac{x^2}{(x^2+1)^2(x-1)} = +\frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}x - \frac{1}{4}}{x^2+1} + \frac{\frac{1}{2}x + \frac{1}{2}}{(x^2+1)^2}$$

$$= \frac{1}{4(x-1)} - \frac{x+1}{4(x^2+1)} + \frac{x+1}{2(x^2+1)^2}$$

are required partial fraction.

Q.3 $\dfrac{\mathbf{2x-5}}{\mathbf{(x^2+2)^2(x-2)}}$

Solution:

Let

$$\frac{2x-5}{(x-2)(x^2+2)^2} = \frac{A}{(x-2)} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2} \quad \ldots\ldots (1)$$

$$\frac{2x-5}{(x-2)(x^2+2)^2} = \frac{A(x^2+2)^2 + (Bx+C)(x-2)(x^2+1) + (Dx+E)(x-2)}{(x-2)(x^2+2)^2}$$

$$2x-5 = A(x^2+2)^2 + (Bx+C)(x-2)(x^2+2) + (Dx+E)(x-2) \quad \ldots\ldots (2)$$

$$2x-5 = A(x^4+4+4x^2) + (Bx+C)(x^3-2x^2+2x-4) + (Dx+E)(x-2)$$

$$2x-5 = Ax^4 + 4A + 4Ax^2 + Bx^4 - 2Bx^3 + 2Bx^2 - 4Bx + Cx^3 - 2Cx^2 + 2Cx - 4C$$
$$+ Dx^2 - 2Dx + Ex - 2E \quad \ldots\ldots (3)$$

Put $x = 2$ in equation (2), we get

$$2(2) - 5 = A\left((2)^2 + 2\right)^2 + 0 + 0$$

$$4 - 5 = A(4+2)^2$$

$$-1 = A(6)^2 \Rightarrow \boxed{A = -\frac{1}{36}}$$

Equating coefficients of x^4, x^3, x^2 and x in equation (3), we get

x^4 ; $A + B = 0$

$$B = -A = -\left(-\frac{1}{36}\right)$$

$$\boxed{B = \frac{1}{36}}$$

x^3 ; $-2B + C = 0$

$$-2\left(-\frac{1}{36}\right) + C = 0$$

$$\boxed{C = \frac{1}{18}}$$

x^2 ; $4A + 2B - 2C + D = 0$

$$4\left(-\frac{1}{36}\right) + 2\left(-\frac{1}{36}\right) - 2\left(-\frac{1}{18}\right) + D = 0$$

$$-\frac{1}{9} + \frac{1}{18} - \frac{1}{9} + D = 0$$

$$D = \frac{1}{9} + \frac{1}{9} - \frac{1}{18}$$

$$= \frac{2 + 2 - 1}{18} = \frac{3}{18}$$

$$\boxed{D = \frac{1}{6}}$$

x ; $-4B + 2C - 2D + E = 0$

$$E = 2 + 4B - 2C + 2D$$

$$= 2 + 4\left(\frac{1}{36}\right) - \left(\frac{1}{18}\right) + 2\left(\frac{1}{6}\right)$$

$$= 2 + \frac{1}{9} - \frac{1}{9} + \frac{1}{3}$$

$$= 2 + \frac{1}{3} = \frac{6 + 1}{3}$$

$$\boxed{E = \frac{7}{3}}$$

Put values of A, B, C, D, E in equation (1) we get

$$\frac{2x - 5}{(x^2 + 2)^2 (x - 2)} = \frac{-\frac{1}{36}}{(x - 2)} + \frac{\frac{1}{36}x + \frac{1}{18}}{x^2 + 2} + \frac{\frac{1}{6}x + \frac{7}{3}}{(x^2 + 2)^2}$$

$$= \frac{-1}{36\,(x - 2)} \frac{x + 2}{36\,(x^2 + 2)} + \frac{x + 14}{6\,(x^2 + 2)^2}$$

Q.4 $\dfrac{8x^2}{(x^2+1)^2(1-x^2)}$

Solution:

As

$$\frac{8x^2}{(1-x^2)(x^2+1)^2} = \frac{8x^2}{(1-x)(1+x)(x^2+1)^2}$$

Let

$$\frac{8x^2}{(1-x)(1+x)(x^2+1)^2} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} \quad \ldots\ldots\ldots (1)$$

$$\frac{8x^2}{(1-x)(1+x)(x^2+1)^2}$$

$$= \frac{A(1+x)(x^2+1)^2 + B(1-x)(x^2+1)^2 + (Cx+D)(x^2+1)(1-x)(1+x) + (Ex+F)(1-x)(1+x)}{(1-x)(1+x)(x^2+1)^2}$$

$$8x^2 = A(1+x)(x^2+1)^2 + B(1-x)(x^2+1)^2 + (Cx+D)(x^2+1)(1-x)(1+x) + (Ex+F)(1-x)(1+x) \quad \ldots\ldots\ldots (2)$$

$$8x^2 = A(x^5+x^4+2x^3+2x^2+x) + B(-x^5+x^4-2x^3+2x^2-x) + C(-x^5+x) + D(-x^4+1) + (Ex+F)(1-x^2)$$

$$8x^2 = Ax^5 + Ax^4 + 2Ax^3 + 2Ax^2 + Ax - Bx^5 + Bx^4 - 2Bx^3 + 2Bx^2 - Bx - Cx^5 + Cx - Dx^4 + D + Ex - Ex^3 + F - Fx^2 \quad \ldots\ldots\ldots (3)$$

Put $x = 1$ in equation (2), we get

$$8(1)^2 = A(1+1)\left((1)^2+1\right)^2$$

$$8 = A(2)(1+1)^2$$

$$8 = A(2)(4) \Rightarrow \boxed{A = 1}$$

Put $x = -1$ in equation (2), we get

$$8(-1)^2 = 0 + B(+1+1)\left((-1)^2+1\right)^2$$

$$8 = B(+2)(1+1)^2$$

$$8 = B(8)$$

$$\boxed{B = 1}$$

Equating coefficients of x^5, x^4, x^3 in equation (3) we get

x^5 ; $A - B - C = 0$

$C = A - B = 1 - 1 = 0$

$\boxed{C = 0}$

x^4 ; $A + B - D = 0$

$D = A + B = 1 + 1$

$\boxed{D = 2}$

x^3 ; $2A - 2B - E = 0$

$E = 2A - 2B$

$= 2(1) - 2(1)$

$\boxed{E = 0}$

Put values of A, B, C, D and E in equation (1) we get

$$\frac{8x^2}{(1-x)(1+x)(x^2+1)^2} = \frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{x^2+1} - \frac{4}{(x^2+1)^2}$$

are required partial fractions.

Q.5 $\dfrac{\mathbf{4x^4 + 3x^3 + 6x^2 + 5x}}{\mathbf{(x-1)(x^2+x+1)^2}}$

Solution:

Let

$$\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{A}{(x-1)} + \frac{Bx + C}{(x^2+x+1)} + \frac{Dx + E}{(x^2+x+1)^2} \quad \ldots\ldots\ldots (1)$$

$$\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{A(x^2+x+1)^2 + (Bx+C)(x-1)(x^2+x+1) + (Dx+E)(x-1)}{(x-1)(x^2+x+1)^2}$$

$$4x^4 + 3x^3 + 6x^2 + 5x = A(x^2+x+1)^2 + (Bx+C)(x-1)(x^2+x+1) + (Dx+E)(x-1)$$

$\ldots\ldots\ldots (2)$

$$4x^4 + 3x^3 + 6x^2 + 5x = Ax^4 + 2Ax^3 + 3Ax^2 + 2Ax + A + Bx^4 - Bx + Cx^3 - C + Dx^2 - Dx + Ex - E$$

$\ldots\ldots\ldots (3)$

Put $x = 1$ in equation (2), we get

$$4(1)^4 + 3(1)^3 + 6(1)^2 + 5(1) = A\left((1)^2 + 1 + 1\right)^2 + 0 + 0$$

$$4 + 3 + 6 + 5 = A(1 + 1 + 1)^2$$

$18 = 9A \Rightarrow \boxed{A = 2}$

x^4 ; $A + B = 4$

$B = 4 - A = 4 - 2$

$\boxed{B = 2}$

x^3 ; $2A + C = 3$

$C = 3 - 2A = 3 - 2(2) = 3 - 4$

$\boxed{C = -1}$

x^2 ; $3A + D = 6$

$D = 6 - 3A = 6 - 3(2) = 6 - 6 = 0$

$\boxed{D = 0}$

x ; $2A - B - D + E = 5$

$E = 5 - 2A + B + D$

$= 5 - 2(2) + 2 + 0$

$\boxed{E = 3}$

Put values of A, B, D, D and E in equation (1)

$$\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{2}{(x-1)} + \frac{2x-1}{(x^2+x+1)} + \frac{3}{(x^2+x+1)^2}$$

are required partial fractions.

Q.6 $\dfrac{\mathbf{2x^4 - 3x^3 - 4x}}{\mathbf{(x^2+2)^2 (x+1)^2}}$

Solution:

Let

$$\frac{2x^4 - 3x^3 - 4x}{(x+1)^2 (x^2+2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}$$

$$\frac{2x^4 - 3x^3 - 4x}{(x+1)^2 (x^2+2)^2}$$

$$= \frac{A(x+1)(x+2)^2 + B(x^2+2)^2 + (Cx+D)(x+1)^2 (x+2) + (Ex+F)(x+1)^2}{(x+1)^2 (x^2+2)^2}$$

$$2x^4 - 3x^3 - 4x = A(x+1)(x+2)^2 + B(x^2+2)^2 + (Cx+D)(x+1)^2 (x+2) + (Ex+F)(x+1)^2 \quad \ldots\ldots (2)$$

$$2x^4 - 3x^3 - 4x = A(x^5 + x^4 + 4x^3 + 4x^2 + 4x + 4) + B(x^4 + 4 + 4x^2)$$
$$+ C(x^5 + 2x^4 + 3x^3 + 4x^2 + 2x)\ D(x^4+2x^3+3x^2+4x+2) + E(x^3+2x^2+x)$$
$$+ F(x^2 + 2x + 1)$$

$$2x^4 - 3x^3 - 4x = Ax^5 + Ax^4 + 4Ax^3 + 4Ax^2 + 4Ax + 4A + Bx^4 + 4B + 4Bx^2$$
$$+ Cx^5 + 2Cx^4 + 3Cx^3 + 4Cx^2 + 2Cx + Dx^4 + 2Dx^3 + 3Dx^2 + 4Dx$$
$$+ 2D + Ex^3 + 2Ex^2 + Ex + Fx^2 + 2Fx + F \qquad (3)$$

Put $x = -1$ in equation (2), we get

$$2(-1)^4 - 3(-1)^3 - 4(-1) = 0 + B((-1)2 + 2)^2 + 0 + 0$$

$$2 + 3 + 4 = B(1 + 2)^2$$

$$9 = B(3)^2$$

$$9 = 9B \Rightarrow \boxed{B = 1}$$

Equating coefficients of x^5, x^4, x^3, x^2, x and constant in equation (3), we get

x^5 ; $A + C = 0$ (i)

x^4 ; $A + B + 2C + D = 2$ (ii)

x^3 ; $4A + 3C + 2D + E = -3$ (iii)

x^2 ; $4A + 4B + 4C + 3D + 2E + F = 0$ (iv)

x ; $4A + 2C + 4D + E + 2F = -4$ (v)

cons ; $4A + 4B + 2D + F = 0$ (vi)

from (i)

$C = -A$

from (ii)

$D = 2 - A - B - 2C = 2 - A - 1 + 2A = 1 + A$ (vii)

from (iii)

$E = -3 - 4A - 3C - 2D = -3 - 4A + 3A - 2 - 2A = -3A - 5$ (viii)

Subtracting (vi) from (iv), we get

$4C + D + 2E = 0$

$\Rightarrow \quad -4A + 1 + A + 2(-3A - 5) = 0$

$-9A = 9$

$\Rightarrow \boxed{A = -1}$

Put this value of A in (i), (vii), (viii) we get

(i) $\Rightarrow$ $C = -(-1) = 1 \Rightarrow \boxed{C = -1}$

(vii) $\Rightarrow$ $D = 1 + (-1) = 0 \Rightarrow \boxed{D = 0}$

(viii) $\Rightarrow$ $E = -3(-1) - 5 = 3 - 5 = -2$

$\Rightarrow \boxed{E = -2}$

from equation (vi)

$4(-1) + 4(1) + 2(0) + F = 0$

$-4 + 4 + 0 + F = 0$

$\boxed{F = 0}$

Put values of A, B, C, D, E and F in equation (1), we get

$$\frac{2x^4 - 3x^3 - 4x}{(x+1)^2 (x^2+2)^2} = \frac{-1}{x+1} + \frac{1}{(x+1)^2} + \frac{x}{x^2+2} - \frac{2x}{(x^2+2)^2}$$

are required partial fractions.

CHAPTER 5

PARTIAL FRACTIONS

MULTIPLE CHOICE QUESTIONS

(1) The function of the form $f(x) = \frac{p(x)}{q(x)}$ where $q(x) \neq 0$ and $p(x)$, $q(x)$ are polynomials is called:

(a) Identity (b) Equation
(c) Fraction (d) Algebraic relation

[Lahore Board 2005]

(2) Partial fractions of $\frac{1}{x^2 - 1}$ is equivalent to:

(a) $\frac{Ax + B}{x^2 - 1}$ (b) $\frac{A}{x + 1} + \frac{B}{x - 1}$

(c) $\frac{A}{x + 1}$ (d) $\frac{B}{x - 1}$ **[Lahore Board 2005-07-08]**

(3) The fraction $\frac{x^2 + 7x + 3}{x + 1}$ is:

(a) Improper (b) Proper
(c) Equivalent (d) None of these

[Gujranwala Board 2005]

(4) Partial fractions of $\frac{1}{x^3 + 1}$ will be of the form:

(a) $\frac{A}{x - 1} + \frac{B}{x^2 - x + 1}$ (b) $\frac{A}{x + 1} + \frac{B}{x^2 - x + 1}$

(c) $\frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1}$ (d) $\frac{A}{x - 1} + \frac{B}{x^2 + 1}$

[Gujranwala Board 2005]

(5) A fraction in which the degree of numerator is greater than or equal to the degree of denominator is called:

(a) A proper fraction (b) An improper fraction
(c) An equation (d) An algebraic relation

[Gujranwala Board 2006]

(6) $\frac{x^3+1}{(x-1)(x+2)}$ is:

(a) Proper fraction (b) Improper fraction
(c) Identity (d) None of these

[Gujranwala Board 2006]

(7) Partial fraction of $\frac{7x^2+25}{(x+3)(x+4)}$ will be of the form:

(a) $\frac{A}{x+3}+\frac{B}{x+4}$ (b) $1+\frac{A}{x+3}+\frac{B}{x+4}$

(c) $7+\frac{A}{x+3}+\frac{B}{x+4}$ (d) None of these

[Lahore Board 2014, Lahore Board 2006]

(8) The fraction $\frac{2x^2+5}{x-3}$ is:

(a) Proper (b) Improper
(c) Irrational (d) None of these

[Lahore Board 2006]

(9) Partial fraction of $\frac{1}{(x+1)(x^2-1)}$ will be of the form:

(a) $\frac{A}{x+1}+\frac{Bx+C}{x^2-1}$ (b) $\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x+1}$

(c) $\frac{A}{x-1}+\frac{B}{(x+1)^2}+\frac{C}{x+1}$ (d) $\frac{A}{x-1}+\frac{Bx+C}{(x+1)^2}$

[Gujranwala Board 2007]

(10) Partial fraction of $\frac{x^2+1}{(x+1)(x-1)}$ are of the form:

(a) $\frac{A}{x+1}+\frac{B}{x-1}$ (b) $1+\frac{A}{x+1}+\frac{B}{x-1}$

(c) $1+\frac{A}{x+1}+\frac{Bx+C}{x-1}$ (d) $\frac{Ax+B}{x+1}+\frac{C}{x-1}$

[Gujranwala Board 2007]

(11) Types of rational fractions are:

(a) Three (b) Two
(c) Four (d) Infinite

[Lahore Board 2007]

(12) $\frac{1}{(x^2+1)(x+1)}$ equals:

(a) $\frac{A}{x^2+1}+\frac{Bx+C}{x+1}$ (b) $\frac{A}{x^2+1}+\frac{B}{x+1}$

(c) $\frac{Ax+B}{x^2+1}$ (d) $\frac{Ax+B}{x^2+1}+\frac{C}{x+1}$

[Lahore Board 2008]

(13) $\frac{x^4}{1-x^4}$ is:

(a) Proper fraction (b) Improper fraction

(c) Decimal (d) Equation

[Lahore Board 2008]

(14) Partial fractions of $\frac{1}{x^3-1}$ are of the form:

(a) $\frac{A}{x-1}+\frac{B}{x^2+x+1}$ (b) $\frac{A}{x+1}+\frac{B}{x^2-x+1}$

(c) $\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$ (d) $\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$

[Gujranwala Board 2008]

(15) Partial fraction of $\frac{(x-a)(x-b)}{(x-c)(x-d)}$ will be of the form:

(a) $\frac{A}{x-c}+\frac{B}{x-d}$ (b) $1+\frac{A}{x-a}+\frac{B}{x-b}$

(c) $\frac{A}{x-a}+\frac{B}{x-b}$ (d) $1+\frac{B}{x-c}+\frac{A}{x-d}$

[Lahore Board 2010]

(16) Partial fraction of $\frac{x}{(x-1)(x-2)}$ will be of the form:

(a) $\frac{A}{x-1}+\frac{B}{x-2}$ (b) $\frac{1}{x-1}$

(c) $\frac{1}{x+2}$ (d) $1+\frac{A}{x-1}+\frac{B}{x+2}$

[Gujranwala Board 2011]

(17) Conditional equation $3x-1=0$ is true only if:

(a) $x=3$ (b) $x=-3$

(c) $x=\frac{1}{3}$ (d) $x=-\frac{1}{3}$

[Lahore Board 2012]

(18) Partial fractions of $\frac{1}{(x+1)(x^2-1)}$ will be of the form:

(a) $\frac{a}{x+1}+\frac{Bx+C}{x^2-1}$ (b) $\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x+1}$

(c) $\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1}$ (d) $\frac{A}{x-1}+\frac{Bx+C}{(x+1)^2}$

[Lahore Board 2012]

(19) The function of the form $\frac{p(x)}{q(x)}$, $q(x) \neq 0$ is called:

(a) Identity (b) Equation

(c) Fraction (d) Formula

[Lahore Board 2013]

(20) Partial fractions of $\frac{2}{x^2-1}$ are:

(a) $\frac{1}{x-1}+\frac{1}{x+1}$ (b) $\frac{1}{x-1}-\frac{1}{x+1}$

(c) $\frac{1}{2(x+1)}-\frac{1}{2(x-1)}$ (d) $\frac{1}{2(x-1)}-\frac{1}{2(x+1)}$

[Lahore Board 2013]

(21) A conditional equation is also called:

(a) Identity (b) Equation

(c) Fraction (d) None of these

(22) An equation which holds good for all values of the variable is called:

(a) Identity (b) Equation

(c) Conditional equation (d) All of these

(23) Expressing a rational function as a sum of partial fractions is called:

(a) Fraction (b) Partial fraction resolution

(c) Equation (d) Identity

(24) $(x+3)(x+4) = x^2+7x+12$ is a/an:

(a) Fraction (b) Conditional equation

(c) Identity (d) All of these

(25) $\frac{3x^2-1}{x-2} =$ _____

(a) $3x-6+\frac{A}{x-2}$ (b) $\frac{Ax+B}{x-2}$

(c) $3x+6+\frac{Ax+B}{x^2-1}$ (d) $3x+6+\frac{11}{x-2}$

(26) Partial fraction of $\frac{x^2 - 10x + 13}{(x-1)(x^2 - 5x + 6)}$ will be of the form:

(a) $\frac{A}{x-1} + \frac{B}{x^2 + 5x + 6}$ (b) $1 + \frac{A}{x-1} + \frac{Bx + C}{x^2 - 5x + 6}$

(c) $\frac{A}{x-1} + \frac{Bx + C}{x^2 + 5x + 6}$ (d) $\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

(27) $\frac{4x^2 + 8x}{x^4 + 2x^2 + 9} =$ _____

(a) $\frac{4x^2 + 8x}{(x^2 + 2x + 3)(x^2 + 2x + 3)}$ (b) $\frac{4x^2 + 8x}{(x^2 + 2x - 3)(x^2 + 2x + 3)}$

(c) $\frac{4x^2 + 8x}{(x^2 - 2x - 3)(x^2 + 2x + 3)}$ (d) $\frac{4x^2 + 8x}{(x^2 + 2x + 3)(x^2 - 2x + 3)}$

(28) Partial fraction of $\frac{x - b}{x^3 - b^3}$ where $x \neq b$ is of the form:

(a) $\frac{A}{x-b} + \frac{Bx + C}{x^2 + bx + b^2}$ (b) $\frac{A}{x-b} + \frac{B}{x^2 + bx + a^2}$

(c) $\frac{B}{x^2 + bx + b^2}$ (d) None of these

(29) Partial fraction of $\frac{3x - 4}{(x-1)^2}$ is of the form:

(a) $\frac{A}{(x-1)^2}$ (b) $\frac{A}{x-1} + \frac{Bx + C}{(x-1)^2}$

(c) $\frac{A}{x-1} + \frac{Bx + C}{(x-1)^2}$ (d) None of these

(30) $\cos^2\theta + \sin^2\theta = 1$ is a/an:

(a) Identity (b) Conditional equation

(c) Fraction (d) None of these

(31) An equation in which two algebraic expressions are equal for particular values of the variable is called:

(a) Identity (b) Conditional equation

(c) Fraction (d) None of these

(32) The quotient of two polynomials $\frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$ with no common factors is called a/an:

(a) Partial fraction resolution (b) Identity

(c) Rational fraction (d) None of these

(33) Partial fractions of $\frac{1}{x(x+1)}$ are of the form:

(a) $\frac{1}{x-1} + \frac{1}{x+1}$ (b) $\frac{1}{x-1} - \frac{1}{x+1}$

(c) $\frac{1}{x} + \frac{1}{x+1}$ (d) $\frac{1}{x} - \frac{1}{x-1}$

[Lahore Board 2014]

Answers

(1)	c	*(2)*	b	*(3)*	a	*(4)*	c	*(5)*	b
(6)	b	*(7)*	c	*(8)*	b	*(9)*	c	*(10)*	b
(11)	b	*(12)*	d	*(13)*	b	*(14)*	c	*(15)*	d
(16)	a	*(17)*	c	*(18)*	c	*(19)*	c	*(20)*	b
(21)	b	*(22)*	a	*(23)*	b	*(24)*	c	*(25)*	d
(26)	d	*(27)*	d	*(28)*	d	*(29)*	d	*(30)*	a
(31)	b	*(32)*	c	*(33)*	d				

SOLUTION

Q.2 As

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1}+\frac{B}{x-1}$$

Q.3 As degree of numerator is greater than the degree of denominator so it is an improper fraction.

Q.4 As,

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$$

So let,

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

Q.6 $$\frac{x^3+1}{(x-1)(x+2)} = \frac{x^3+1}{x^2+x-2}$$

As degree of the polynomial in numerator is greater than the degree of the polynomial in the denominator so it is improper fraction.

Q.7 $$\frac{7x^2+25}{(x+3)(x+4)} = \frac{7x^2+25}{x^2+7x+12}$$

$$\begin{array}{r} 7 \\ x^2+7x+12\overline{)\,7x^2+25} \\ -\,7x^2+84+49x \\ \hline -49x-59 \end{array}$$

$$\Rightarrow \quad \frac{7x^2+25}{(x+3)(x+4)} = 7+\frac{-49x-59}{(x+3)(x+4)}$$

$$= 7+\frac{A}{x+3}+\frac{B}{x+4}$$

Q.9 $$\frac{1}{(x+1)(x^2-1)} = \frac{1}{(x+1)(x+1)(x-1)} = \frac{1}{(x-1)(x+1)^2}$$

We Let,

$$\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1}+\frac{B}{(x+1)^2}+\frac{C}{x+1}$$

Q.10 $\dfrac{x^2+1}{(x+1)(x-1)} = \dfrac{x^2+1}{x^2-1}$

$$x^2-1 \overline{)\, x^2+1 \,(\, 1}$$
$$x^2-1$$
$$- \quad +$$
$$2$$

$$\Rightarrow \quad \frac{x^2+1}{x^2-1} = 1+\frac{2}{x^2-1} = 1+\frac{2}{(x+1)(x-1)}$$

Its P.F will be of the form.

$$1+\frac{A}{x+1}+\frac{B}{x-1}$$

Q.14 As,

$$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)}$$

So, P.F of $\dfrac{1}{x^3-1}$ are of the form $\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+1}$

Q.24 As,

$$\frac{x^2-10x+13}{(x-1)(x^2-5x+6)} = \frac{x^2-10x+13}{(x-1)(x-2)(x-3)}$$

$$\Rightarrow \quad \text{P.F of } \frac{x^2-10x+13}{(x-1)(x-2)(x-3)} \text{ are of the form } \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

Q.25 As,

$$x^4+2x^2+9 = (x^2+2x+3)(x^2-2x+3)$$

$$\Rightarrow \quad \frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)}$$

Q.26 As,

$$\frac{x-b}{x^3-b^3} = \frac{x-b}{(x-b)(x^2+bx+b^2)} = \frac{1}{x^2+bx+b^2}$$

Q.28 As it is true for all values of 'θ' so it is an identity.