# Chapter 12

# APPLICATION OF TRIGONOMETRY

## EXERCISE 12.1

**Q.1 Find the values of**

**(i)** $\sin 53^\circ 40'$

$= 0.8056$

**(ii)** $\cos 36^\circ 20'$

$= 0.8056$

**(iii)** $\tan 19^\circ 30'$

$= 0.3541$

**(iv)** $\cot 33^\circ 50'$

$= \dfrac{1}{\tan 33^\circ 50'} = 1.4919$

**(v)** $\cos 42^\circ 38'$

$= 0.7357$

**(vi)** $\tan 25^\circ 34'$

$= 0.4784$

**(vii)** $\sin 18^\circ 31'$

$= 0.3176$

**(viii)** $\cos 52^\circ 13'$

$= 0.6127$

**(ix)** $\cot 89^\circ 9'$

$= \dfrac{1}{\tan 89^\circ 9'} = 0.01483$

**Q.2 Find $\theta$ , if**

**(i) $\sin\theta = 0.5791$**

$\Rightarrow \quad \theta = \sin^{-1}(0.5791) = 35^{\circ}\,23'$

**(ii) $\cos\theta = 0.9316$**

$\theta = \cos^{-1}(0.9316) = 21^{\circ}\,18'$

**(iii) $\cos\theta = 0.5257$**

$\theta = \cos^{-1}(0.5257) = 58^{\circ}\,17'$

**(iv) $\tan\theta = 1.705$**

$\theta = \tan^{-1}(1.705) = 59^{\circ}\,36'$

**(v) $\tan\theta = 21.943$**

$\theta = \tan^{-1}(21.943) = 87^{\circ}\,23'$

**(vi) $\sin\theta = 0.5186$**

$\theta = \sin^{-1}(0.5186) = 31^{\circ}\,14'$

## EXERCISE 12.2

**Q.1 Find the unknown angles and sides of the following triangles.**

**(i) $\alpha = 45^{\circ}$, $\beta = 90^{\circ}$, $\gamma = m\angle B$**

**(ii) $\alpha = 60^{\circ}$, $\beta = 90^{\circ}$, $\gamma = m\angle c$**

**(iii) $\alpha = 90^{\circ}$, $\beta = m\angle B$, $\gamma = m\angle c$**

**Solution:**

**(i) $\alpha = 45^{\circ}$, $\beta = 90^{\circ}$, $\gamma = m\angle B$**

since $\alpha, \beta, \gamma$ are angles of triangle

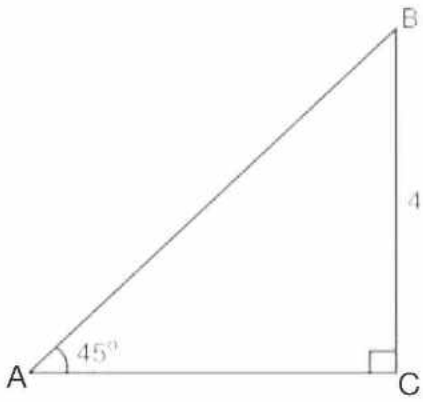
so $\alpha + \beta + \gamma = 180^{\circ}$

$45^{\circ} + 90^{\circ} + m < B = 180^{\circ}$

$m\angle B = 45^{\circ}$

$\sin 45^{\circ} = \dfrac{BC}{AB}$

$\sin 45^{\circ} = \dfrac{4}{AB}$

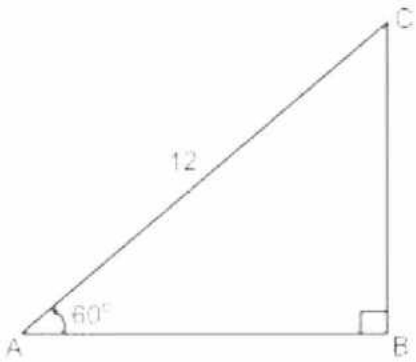


$$\frac{1}{\sqrt{2}} = \frac{4}{AB} \Rightarrow \boxed{AB = 4\sqrt{2}}$$

$$\cos 45^\circ = \frac{AC}{AB}$$

$$\frac{1}{\sqrt{2}} = \frac{AC}{4\sqrt{2}} \Rightarrow \boxed{AC = 4}$$

**(ii)** $\alpha = 60^\circ$, $\beta = 90^\circ$, $\gamma = m\angle c$




since $\alpha + \beta + \gamma = 180^\circ$

$60^\circ + 90^\circ + \gamma = 180^\circ$

$\gamma = 30^\circ$

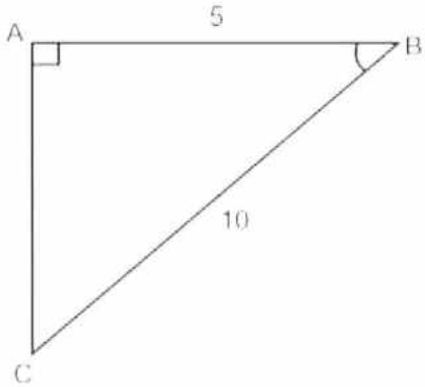
$$\sin 60^\circ = \frac{BC}{AC}$$

$$\frac{\sqrt{3}}{2} = \frac{BC}{12} \Rightarrow BC = \frac{\sqrt{3}}{2} \times 12 = 6\sqrt{3}$$

$$\cos 60^\circ = \frac{AB}{AC}$$

$$\frac{1}{2} = \frac{AB}{6\sqrt{3}} \Rightarrow AB = \frac{6\sqrt{3}}{2} = 3\sqrt{3}$$

**(iii)** $\alpha = 90^\circ$, $\beta = m\angle B$, $\gamma = m\angle c$




$$\cos\beta = \frac{AB}{BC}$$

$$\cos\beta = \frac{5}{10}$$

$$\beta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

$$\sin 60^\circ = \frac{AC}{BC}$$

$$\sin 60^\circ = \frac{AC}{10}$$

$$\boxed{AC = 8.66}$$

$\alpha + \beta + \gamma = 180^\circ$

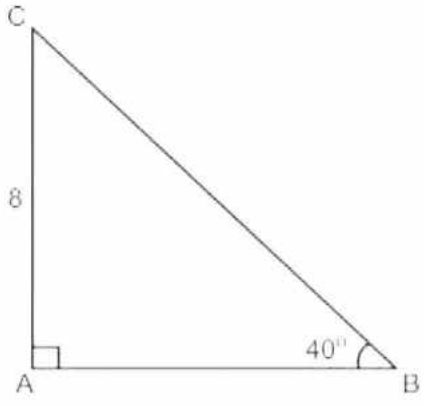
$90^\circ + 60^\circ + \gamma = 180^\circ$

$\boxed{\gamma = 30^o}$

**(iv)** $\alpha + \beta + \gamma = 180^o$

$90^o + 40^o + \gamma = 180^o$

$\boxed{\gamma = 50^o}$




$\sin 40^o = \dfrac{AC}{BC}$

$\sin 40^o = \dfrac{8}{BC}$

$\Rightarrow$ $\boxed{BC = 12.45}$

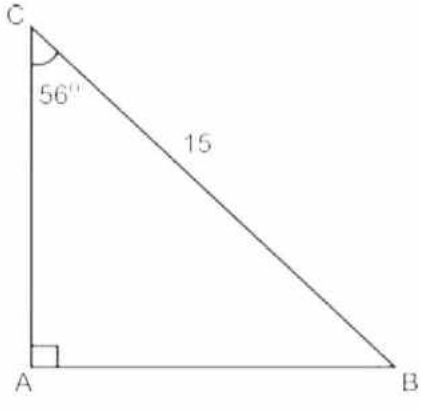
$\tan 40^o = \dfrac{AC}{AB} = \dfrac{8}{AB}$

$\boxed{AB = 9.53}$

**(v)** $\alpha + \beta + \gamma = 180^o$

$90^o + \beta + 56^o = 180^o$

$\boxed{\beta = 34^o}$




$\sin 56^o = \dfrac{AB}{BC}$

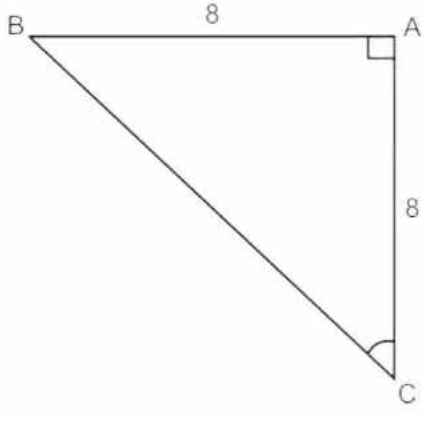
$\sin 56^o = \dfrac{AB}{15}$

$\Rightarrow$ $\boxed{AB = 12.44}$

$\cos 56^o = \dfrac{AC}{BC}$

$\cos 56^o = \dfrac{AC}{15}$

$\boxed{AC = 8.39}$




$\tan m\angle B = \dfrac{AC}{AB}$

$\tan m\angle B = \dfrac{8}{8}$

$m\angle B = \tan^{-1}(1)$

$\boxed{m \angle B = 45^o}$

$m \angle A + m \angle B + m \angle C = 180^o$

$45^o + 90^o + m \angle C = 180^o$

$\boxed{m \angle C = 45^o}$

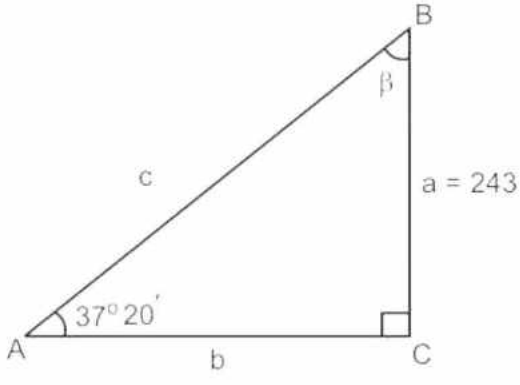
$\sin 45^o = \frac{AC}{AB}$

$\frac{1}{\sqrt{2}} = \frac{8}{BC}$

$\boxed{BC = 8\sqrt{2}}$

**Q.2 Solve the right triangle ABC, in which $\gamma = 90°$ $\alpha = 37^o\,20'$ , a = 243**

**Solution:**




Since $\alpha + \beta + \gamma = 180^o$

$37^o\,20' + \beta + 90^o = 180^o$

$\boxed{\beta = 52^o\,40'}$

$\sin 37^o\,20' = \frac{BC}{AB}$

$\sin 37^o\,20' = \frac{243}{c}$

$\Rightarrow$ $\boxed{c = 400}$

$\cos 37^o\,20' = \frac{b}{c}$

$\cos 37^o\,20' = \frac{b}{400}$

$\boxed{b = 318.04}$

**Q.3 $\gamma = 90^o$ , $\alpha = 62^o\,40'$ , b = 796**

**Solution:**

Since $\alpha + \beta + \gamma = 180^o$

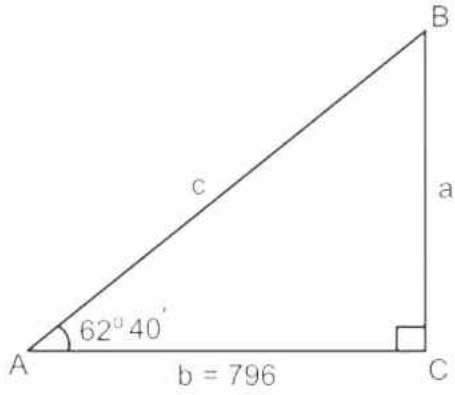
$62^o\,40' + \beta + 90^o = 180^o$

$\boxed{\beta = 27^o\,20'}$

$$\tan 62^{o}\,40' = \frac{a}{b}$$

$$\tan 62^{o}\,40' = \frac{a}{796}$$

$$\Rightarrow \quad \boxed{a = 1540}$$



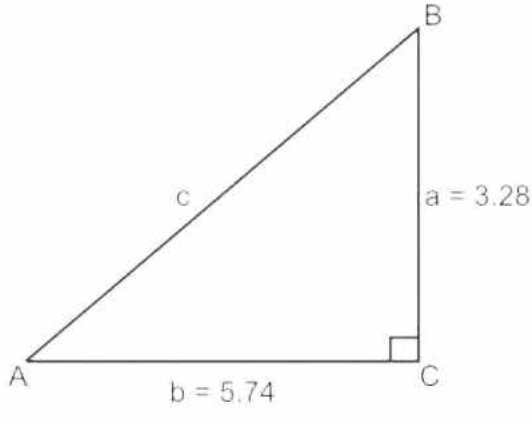

$$\sin 62^{o}\,40' = \frac{a}{c}$$

$$\sin 62^{o}\,40' = \frac{1540}{c}$$

$$\Rightarrow \quad \boxed{c = 1733.55}$$

**Q.4 $\gamma = 90^{o}$ , a = 3.28 , b = 5.74**

**Solution:**




$$\tan \alpha = \frac{a}{b}$$

$$\tan \alpha = \frac{3.28}{5.74}$$

$$\alpha = \tan^{-1}\left(\frac{3.28}{5.74}\right)$$

$$\Rightarrow \quad \boxed{\alpha = 29^{o}\,44'}$$

Since $\alpha + \beta + \gamma = 180^{o}$

$29^{o}\,44' + \beta + 90^{o} = 180^{o}$

$$\boxed{\beta = 60^{o}\,16'}$$

$$\sin 29^{o}\,44' = \frac{3.28}{c}$$

$$\boxed{c = 6.61}$$

**Q.5 $\gamma = 90^{o}$ , b = 68.4 , c = 96.2**

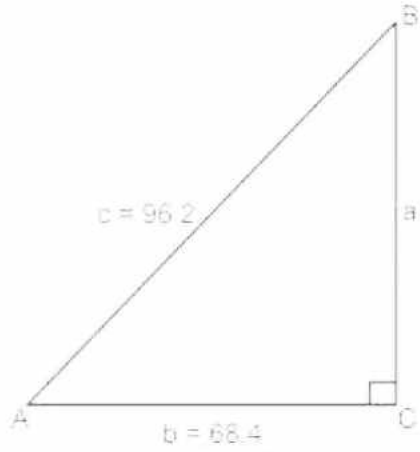
**Solution:**

$$\cos \alpha = \frac{b}{c}$$

$$\cos \alpha = \frac{68.4}{96.2}$$

$$\alpha = \cos^{-1}\left(\frac{68.4}{96.2}\right)$$

$\Rightarrow$ $\boxed{\alpha = 44^{o}\,40'}$




Since $\alpha + \beta + \gamma = 180^{o}$

$44^{o}\,40' + \beta + 90^{o} = 180^{o}$

$\boxed{\beta = 45^{o}\,20'}$

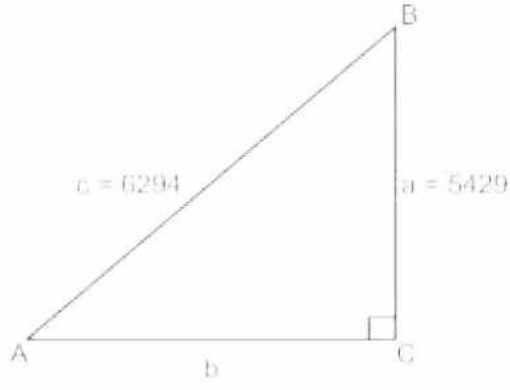
$$\sin\alpha = \frac{a}{c}$$

$$\sin 44^{o}\,40' = \frac{a}{96.2}$$

$\boxed{a = 67.6}$

**Q.6 $\gamma = 90^{o}$ , a = 5429 , c = 6294**

**Solution:**




$$\sin\alpha = \frac{a}{c}$$

$$\sin\alpha = \frac{5429}{6294}$$

$$\alpha = \sin^{-1}\left(\frac{5429}{6294}\right)$$

$\Rightarrow$ $\boxed{\alpha = 59^{o}\,36'}$

Since $\alpha + \beta + \gamma = 180^{o}$

$59^{o}\,36' + \beta + 90^{o} = 180^{o}$

$\boxed{\beta = 30^{o}\,24'}$

$$\cos\alpha = \frac{b}{c}$$

$$\cos(59^{o}\,36') = \frac{b}{6294}$$

$\boxed{b = 3184}$

**Q.7** $\gamma = 90^o$ , $\beta = 50^o\,10'$ , $c = 0.832$

**Solution:**

Since $\alpha + \beta + \gamma = 180^o$

$\alpha + 50^o\,10' + 90^o = 180^o$

$$\boxed{\alpha = 39^o\,50'}$$

$$\sin\alpha = \frac{a}{c}$$

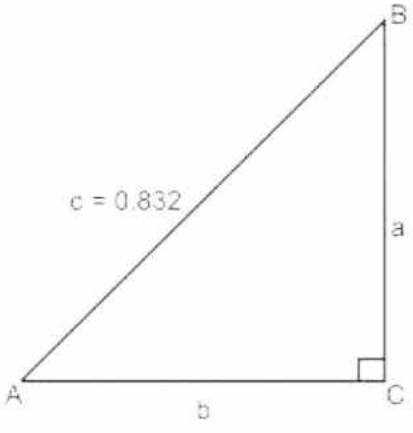
$$\sin 39^o\,50' = \frac{a}{0.832}$$

$\Rightarrow$ $$\boxed{a = 0.5329}$$

$$\cos\alpha = \frac{a}{b}$$

$$\cos 39^o\,50' = \frac{0.5329}{b}$$

$$\boxed{c = 0.6939}$$




## EXERCISE 12.3

**Q.1** **A vertical pole is 8m high and the length of its shadow is 6m. What is the angle of elevation of the sun at that moment?**

**Solution:**

Let the required angle be $= \theta$

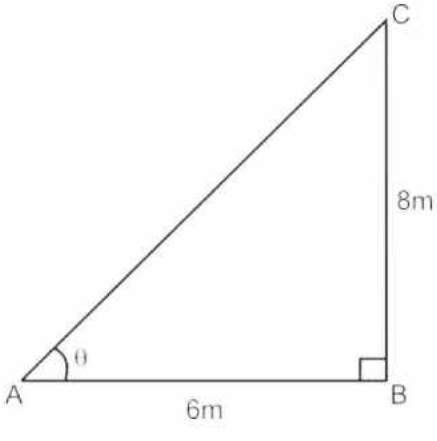
We know that

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan\theta = \frac{8}{6}$$

$$\theta = \tan^{-1}\left(\frac{8}{6}\right)$$

$$\theta = 53^o\,7'$$



**Q.2 A man 18dm tall observes that the angle of elevation of the top of a tree at a distance of 12m from him is $32^o$. What is the height of the tree?**

**Solution:**

Let the required height of the tree be = x + 18

we know that

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}}$$

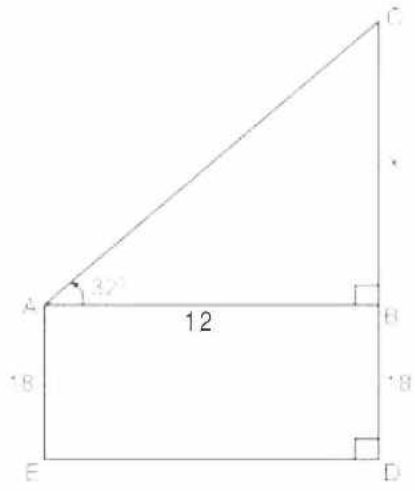
$$\tan 32^o = \frac{x}{12}$$

$$x = 7.50 \text{ m}$$

Required height = x + 18

= 7.50 + 1.8 m (18dm = 1.8m)

= 9.3m




**Q.3 At the top of a cliff 80m high, the angle of depression of a boat is $12^o$. How far is the boat from the cliff?**

**Solution:**

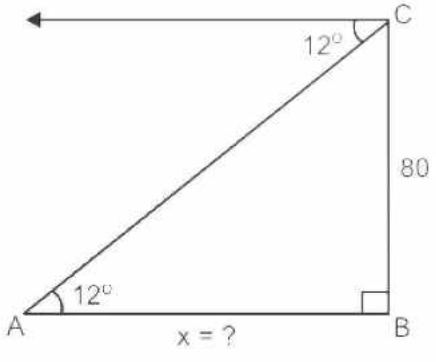
Let the boat from the cliff be = x

we know that

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 12^o = \frac{80}{x}$$

$$x = 376.37 \text{ m}$$




**Q.4 A ladder leaning against a vertical wall makes an angle of $24^o$ with the wall. Its foot is 5m from the wall. Find its length.**

**Solution:**

Let length of ladder be = $l$

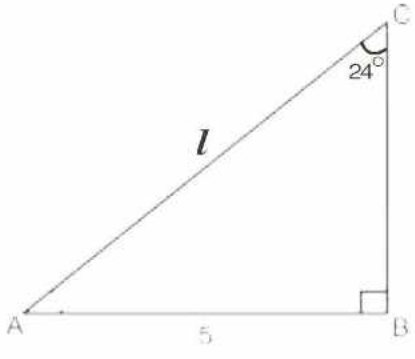
we know that

$$\sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}}$$

$$\sin 24^o = \frac{5}{l}$$

$$l = \frac{5}{\sin 24^\circ}$$

$$l = 12.30\text{m}$$



**Q.5 A kite is flying at a height of 67.2 m is attached to a fully stretched string inclined at an angle of 55° to the horizontal. Find the length of the string.**

**Solution:**

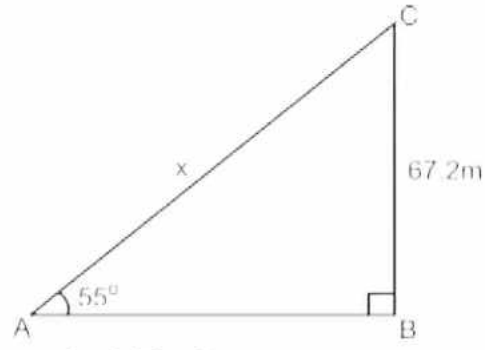
Let the string of the kite be = x

we know that

$$\sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}}$$

$$\sin 55^{o} = \frac{67.2}{x}$$

$$x = 82 \text{ m}$$




**Q.6 When the angle between the ground and the sun is 30°, flag pole casts a shadow of 40m long. Find the height of the top of flag. (Lahore Board 2008)**

**Solution:**

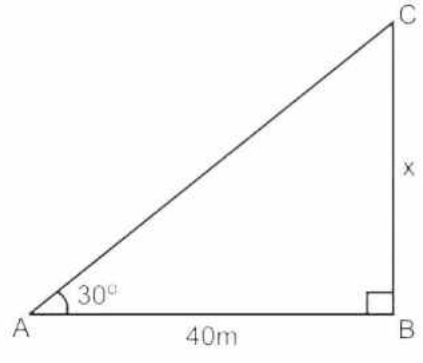
Let the height of the top of the flag be = x

we know that

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 30^{o} = \frac{x}{40}$$

$$x = 23.09 \text{ m}$$




**Q.7 A plane flying directly above a post 6000m away from an anti–aircraft observes the gun at an angle of depression of 27°. Find the height of the plane.**

**Solution:**

Let the height of plane be = x

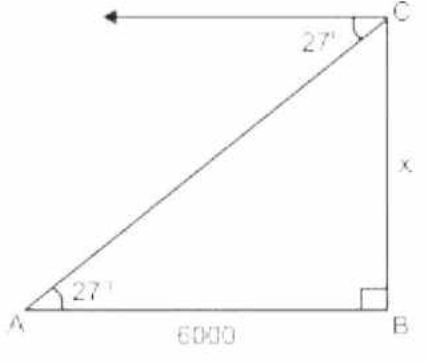
we know that

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan\theta = \frac{x}{6000}$$

$$\tan 27^{o} = \frac{x}{6000}$$

$$x = 3057.15 \text{ m}$$



**Q.8 A man on the top of a 100m high light house tower is in line with two ships on the same side of it, whose angle of depression from the man are $17^o$ and $19^o$ respectively. Find the distance between the ships.**

**Solution:**

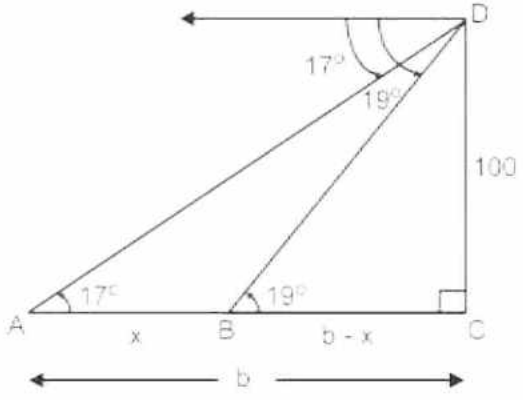
Let the distance between the ships be = x

First consider $\Delta DAC$

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 17^o = \frac{100}{b}$$

$$\Rightarrow \quad b = 327.08 \text{ m}$$




now we consider $\Delta DBC$

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 19^o = \frac{100}{b - x}$$

$$0.344 = \frac{100}{327 - x}$$

$$112.59 - 0.344x = 100$$

$$12.59 = 0.344x$$

$$\frac{12.59}{0.344} = x$$

$$x = 36.58 \text{ m}$$

**Q.9 P and Q are two points in line with a tree. If the distance between P and Q be 30m and the angle of elevation of the top of the tree at P and Q be $12^o$ and $15^o$ respectively. Find the height of tree.**

**Solution:**

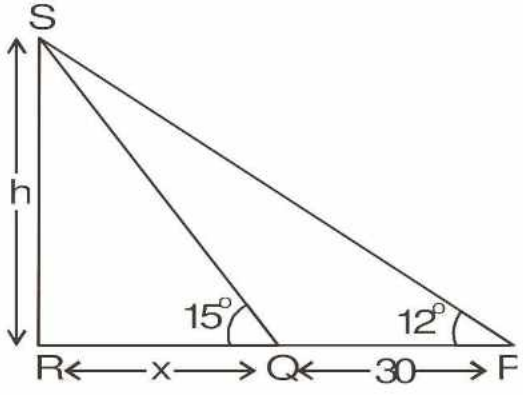
Let the height of the tree be = h

First consider $\Delta SQR$

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 15^o = \frac{h}{x}$$

$$\Rightarrow \quad \boxed{h = x \tan 15^\circ} \qquad \ldots\ldots\ldots (1)$$




Now we consider $\Delta SPR$

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}}$$

$$\tan 12 = \frac{h}{x + 30}$$

$\Rightarrow$ $\boxed{h = (x + 30)\tan 12}$ ..... (2)

Using eq. (1) and (2).

$x \tan 15° = x \tan 12° + 30 \tan 12°$

$x(\tan 15 - \tan 12°) = 30 \tan 12°$

$0.055x = 6.38$

$\boxed{x = 115.94}$ put in eq. (1)

$h = 115.94 \tan 15°$

$h = 31.066$ m

**Q.10 Two men are on the opposite sides of a 100m high tower. If the measure of the angles of the angles of elevation of the top of the tower are $18^o$ & $22^o$ respectively. Find the distance between them.**

**Solution:**

Let the distance between the two men $= x + y$

Consider $\Delta CAD$

$$\tan 18^o = \frac{100}{x}$$

$$x = \frac{100}{\tan 18^o}$$

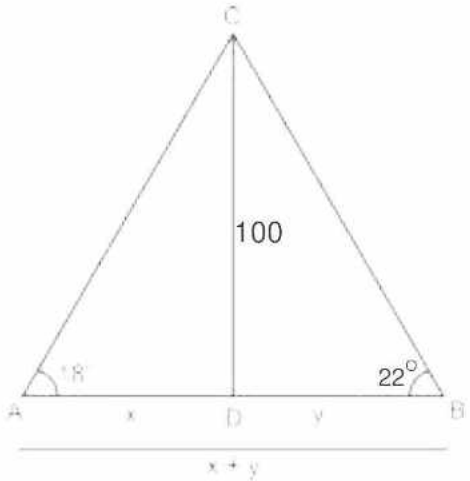
$x = 307.77$

Consider $\Delta BCD$

$$\tan 22^o = \frac{100}{y}$$

$$y = \frac{100}{\tan 22^o}$$

$y = 247.51$




Required distance between them $= x + y$

$= 307.77 + 247.51 = 555.28$ m

**Q.11 A man standing 60m away from a tower notices that the angle of elevation of the top and the bottom of a flag staff on the top of the tower are $64^o$ and $62^o$ respectively. Find the length of the flag staff.**

**Solution:**

Let the length of the flag staff be $= y$

First consider $\Delta BMA$

$$\tan 62^o = \frac{x}{60}$$

$x = 112.84$

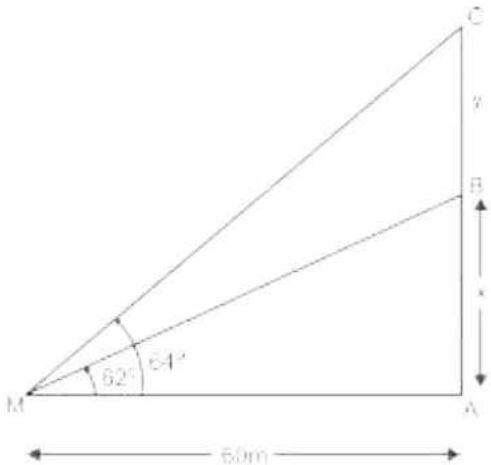
Now consider $\Delta CMA$

$$\tan 64^o = \frac{x + y}{60}$$

$60^o \times \tan 64^o = 112.84 + y$

$123 = x + 112.84$

$x = 10.2$ m



**Q.12 The angle of elevation of the top of 60m high tower from a point A, on the same level as the foot of the tower is $25^o$. Find the angle of elevation of the top of the tower from a point B, 20m nearer to A from the foot of the tower.**

**Solution:**

Let the required angle of elevation be $= \theta$

First consider $\Delta DAC$

$$\tan 25^o = \frac{60}{x + 20} \Rightarrow x + 20 = \frac{60}{\tan 25^\circ}$$

$$x = 128.67 - 20$$

$$x = 108.67$$

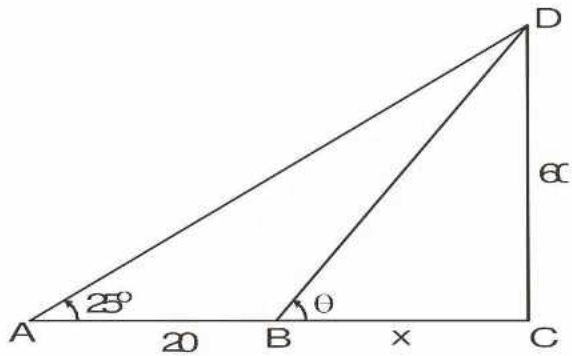
Now we consider $\Delta DBC$

$$\tan \theta = \frac{60}{x}$$

$$\tan \theta = \frac{60}{108.67}$$

$$\tan \theta = \frac{60}{108.67}$$

$$\theta = \tan^{-1}\left(\frac{60}{108.67}\right) = 28^o\ 54'$$




**Q.13 Two buildings A and B are 100m apart. The angle of elevation from the top of the building A to the top of the building B is $20^o$. the angle of elevation from the base of the building B to the top of the building A is $50^o$. Find the height of the building B.**

**Solution:**

Let the required height of building B be $= x + y$

We know that $\tan \theta = \frac{\text{perpendicular}}{\text{base}}$

$$\tan 20^o = \frac{y}{100}$$

$$y = 36.39$$

Now consider $\Delta ABC$

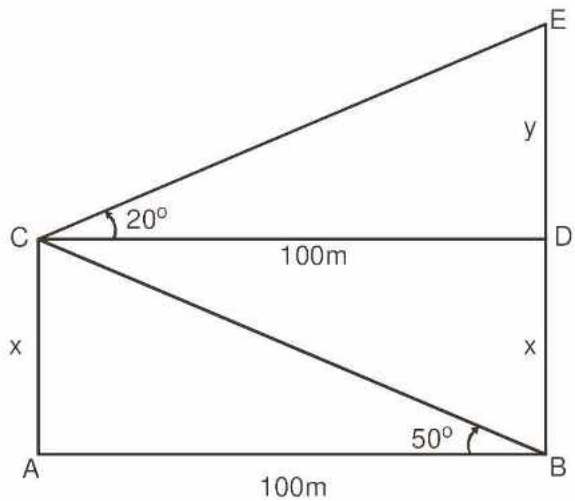
$$\tan 50^o = \frac{x}{100}$$

$$x = 119.2$$

required height of building B $= x + y$

$= 119.2 + 36.39$

$= 115.6$



**Q.14 A window washer is working in a hotel building at a distance of 20m from the building finds angle of elevation of the worker to be of $30^o$. The worker climbs up 12m and the observer moves 4m further away from the building. Find the new angle of elevation of the worker.**

**Solution:**

Let the required angle of elevation $= \theta$

First consider $\Delta ABC$

$$\tan 30^o = \frac{x}{20}$$

$\Rightarrow$ $x = 11.55m$

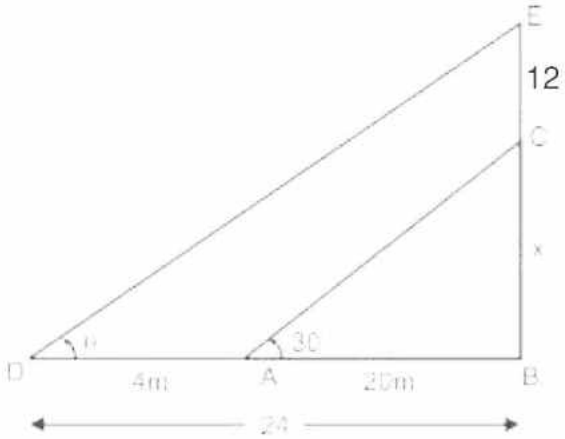
Now we consider $\Delta DBE$

$$\tan \theta = \frac{12 + x}{24}$$

$$\tan \theta = \frac{12 + 11.55}{24}$$

$\theta = \tan^{-1}(0.98125)$

$\theta = 44^o 27'$




**Q.15 A man standing on a bank of canal observes that the measure of the angle of elevation of a tree on the other side of the canal is $60^o$. On retreating 40m from the bank, he finds the measure of elevation of tree is 30o. find the height of the tree and width of the canal.**

**Solution:**

Let height of tree = h & width of canal = x

First consider $\Delta BAC$

$$\tan 30^o = \frac{h}{x + 40}$$

$h = (x + 40) \tan 30°$ ..... (1)

Now we consider $\Delta BDC$

$$\tan 60^o = \frac{h}{x} \Rightarrow h = x \tan 60° \quad \text{..... (2)}$$

From eq. (1) and (2).

$(x + 40) \tan 30° = x \tan 60°$

$x \tan 30° + 40 \tan 30° = x \tan 60°$

$x \tan 60° - x \tan 30° = 40 \tan 30°$

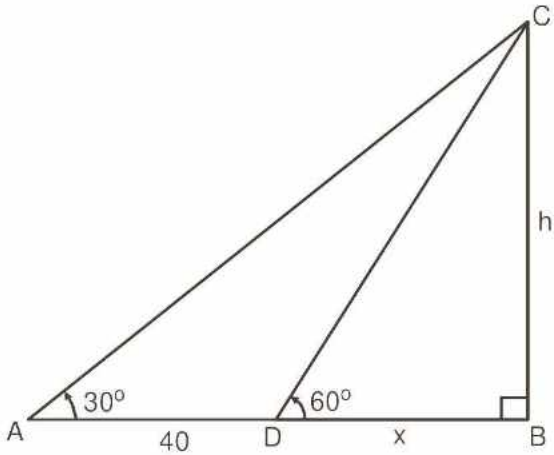
$x(\tan 60° - \tan 30°) = 40 \tan 30°$

$1.15x = 23.09$

$\boxed{x = 20}$ put in (2).

$h = 20 \tan 60°$

$\boxed{h = 34.64}$




**OBLIQUE TRIANGLE:**

A triangle, which is not right, is called an oblique triangle.

For the solution of oblique triangles we can use law of sines, law of cosines, law of tangents.

## 1. The Law of Cosine

In any triangle ABC, with usual notations

(i) $a^2 = b^2 + c^2 - 2bc \cos \alpha$

(ii) $b^2 = a^2 + c^2 - 2ac \cos \beta$

(iii) $c^2 = a^2 + b^2 - 2ab \cos \gamma$

## 2. Law of Sines

In any triangle ABC, with usual notations

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

## 3. The Law of Tangents

In any triangle ABC, with usual notations

(i) $$\frac{a-b}{a+b} = \frac{\tan \frac{\alpha - \beta}{2}}{\tan \frac{\alpha + \beta}{2}}$$

(ii) $$\frac{b-c}{b+c} = \frac{\tan \frac{\beta - \gamma}{2}}{\tan \frac{\beta + \gamma}{2}}$$

(iii) $$\frac{c-a}{c+a} = \frac{\tan \frac{\gamma - \alpha}{2}}{\tan \frac{\gamma + \alpha}{2}}$$

## 4. Half angle Formulas

In any triangle ABC, with usual notation

(i) $\sin \frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$

(ii) $\sin \frac{\beta}{2} = \sqrt{\frac{(S-a)(S-c)}{ac}}$

(iii) $\sin \frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{ab}}$

(i) $\cos \frac{\alpha}{2} \sqrt{\frac{S(S-a)}{bc}}$

(ii) $\cos \frac{\beta}{2} \sqrt{\frac{S(S-b)}{ac}}$

(iii) $\cos \frac{\gamma}{2} \sqrt{\frac{S(S-c)}{ab}}$

(i) $\tan \frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$

(ii) $\tan \frac{\beta}{2} = \sqrt{\frac{(S-a)(S-c)}{S(S-b)}}$

(iii) $\tan \frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$

# EXERCISE 12.4

**Q.1 Solve triangle ABC if**

**$\beta = 60^o$ , $\gamma = 15^o$ , $b = \sqrt{6}$** **(Gujranwala Board 2007)**

**Solution:**

Since $\alpha + \beta + \gamma = 180^o$

$\alpha + 60^o + 15^o = 180^o$

$\alpha = 180^o - 60^o - 15^o$

$\boxed{\alpha = 105}$

Now by law of sines

$$\frac{a}{\sin \alpha} = \frac{a}{\sin \beta}$$

$$\frac{a}{\sin 105^o} = \frac{\sqrt{6}}{\sin 60^o} \Rightarrow a = \frac{\sqrt{6}}{\sin 60^o} \times \sin 105^o$$

$\boxed{\alpha = 2.732}$

again

$$\frac{c}{\sin \gamma} = \frac{b}{\sin \beta}$$

$$\frac{c}{\sin 15^o} = \frac{\sqrt{6}}{\sin 60^o} \Rightarrow c = \frac{\sqrt{6}}{\sin 60^o} \times \sin 15^o$$

$\boxed{a = 0.7320}$

**Q.2 $\beta = 52^o$ , $\gamma = 89^o\,35'$ , $a = 89.35$**

**Solution:**

Since $\alpha + \beta + \gamma = 180^o$

$\alpha + 52^o + 89^o\,35' = 180^o$

$\alpha = 180^o - 52^o - 89^o\,35'$

$\boxed{\alpha = 38^o\,25'}$

Now by law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$\Rightarrow \quad \frac{89.35}{\sin 38^\circ 25'} = \frac{b}{\sin 52^\circ}$$

$$b = \frac{89.35}{\sin 38^\circ 25'} \times \sin 52^\circ$$

$$\boxed{b = 113.18}$$

again

$$\frac{c}{\sin \gamma} = \frac{a}{\sin \alpha}$$

$$\frac{c}{\sin 89^\circ 35'} = \frac{89.35}{\sin 38^\circ 25'}$$

$$c = \frac{89.35}{\sin 38^\circ 25'} \times \sin 89^\circ 35'$$

$$\boxed{c = 143.79}$$

**Q.3 $b = 125$ , $\gamma = 53^\circ$ , $\alpha = 47^\circ$**

**Solution:**

Since $\alpha + \beta + \gamma = 180^\circ$

$47^\circ + \beta + 35^\circ = 180^\circ$

$\beta = 180^\circ - 53^\circ - 47^\circ$

$$\boxed{\beta = 80^\circ}$$

Now by law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$\frac{a}{\sin 47^\circ} = \frac{125}{\sin 80^\circ}$$

$$a = \frac{125}{\sin 80^\circ} \times \sin 47^\circ$$

$$\boxed{a = 92.8}$$

and $$\frac{c}{\sin \gamma} = \frac{b}{\sin \beta}$$

$$\frac{c}{\sin 53^\circ} = \frac{125}{\sin 80^\circ}$$

$$c = \frac{125}{\sin 80^\circ} \times \sin 53^\circ$$

$$\boxed{c = 101}$$

**Q.4** $c = 16.1$ , $\alpha = 53^\circ 45'$ , $\gamma = 74^\circ 32'$

**Solution:**

Since $\alpha + \beta + \gamma = 180^\circ$

$42^\circ 45' + \beta + 74^\circ 32' = 180^\circ$

$\beta = 180^\circ - 42^\circ 45' - 74^\circ 32'$

$\boxed{\beta = 62^\circ 43'}$

Now by law of sines

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$$

$$\frac{a}{\sin 42^\circ 45'} = \frac{16.1}{\sin 74^\circ 32'}$$

$$a = \frac{16.1}{\sin 74^\circ 32'} \times \sin 42^\circ 45'$$

$\boxed{a = 11.3}$

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$\frac{b}{\sin 62^\circ 43'} = \frac{16.1}{\sin 74^\circ 32'}$$

$$b = \frac{16.1}{\sin 74^\circ 32'} \times \sin 62^\circ 43'$$

$\boxed{b = 14.8}$

**Q.5** $a = 53$ , $\beta = 88^\circ 36'$ , $\gamma = 31^\circ 54'$

**(Gujranwala Board 2006,Lahore Board 2007)**

**Solution:**

Since $\alpha + \beta + \gamma = 180^\circ$

$\alpha + 88^\circ 36' + 31^\circ 54' = 180^\circ$

$\alpha = 180^\circ - 88^\circ 36' - 31^\circ 54'$

$\boxed{\alpha = 59^\circ 30'}$

Now by law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$\frac{53}{\sin 59^\circ 30'} = \frac{b}{\sin 88^\circ 36'}$$

$$b = \frac{53}{\sin 59^\circ 30'} \times \sin 88^\circ 36'$$

$$\boxed{b = 61.49}$$

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$\frac{61.49}{\sin 88^\circ 36'} = \frac{c}{\sin 31^\circ 54'}$$

$$c = \frac{61.49}{\sin 88^\circ 36'} \times \sin 31^\circ 54'$$

$$\boxed{c = 32.5}$$

## EXERCISE 12.5

**Solve the triangle ABC, in which**

**Q.1 $b = 59$, $c = 34$, and $\alpha = 52^\circ$ (Gujranwala Board 2007)**

**Solution:**

Using law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$a^2 = (95)^2 + (34)^2 - 2\,(95)\,(34) \cos 52^\circ$$

$$= 9025 + 1156 - 3977$$

$$a^2 = 6204$$

$$\boxed{a = 78.76}$$

$$\because \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

Now $$\beta = \cos^{-1}\left[\frac{a^2 + c^2 - b^2}{2ac}\right]$$

$$\beta = \cos^{-1}\left[\frac{(78.76)^2 + (34)^2 - (95)^2}{2\,(78.76)\,(34)}\right]$$

$$\boxed{\beta = 71^\circ 53'}$$

so $$\gamma = 180^\circ - \beta - \alpha$$

$$= 180^\circ - 71^\circ 53' - 52^\circ$$

$$\boxed{\gamma = 56^\circ 7'}$$

$$\because \quad b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$2ac\cos\beta = a^2 + c^2 - b^2$$

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

**Q.2** $\mathbf{b = 12.5\,,\quad c = 23\,,\quad \&\quad \alpha = 38^{o}\,20'}$

**Solution:**

By law of cosines

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$= (12.5)^2 + (23)^2 - 2\,(12.5)\,(23)\cos 38^{o}\,20'$

$a^2 = 234.21$

$$\boxed{a = 15.3}$$

$$\because \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{Now} \quad \beta = \cos^{-1}\left[\frac{a^2 + c^2 - b^2}{2ac}\right]$$

$$\cos^{-1}\left[\frac{(15.3)^2 + (23)^2 - (12.5)^2}{2\,(15.3)\,(23)}\right]$$

$$\boxed{\beta = 30^{o}\,26'}$$

so $\gamma = 180^{o} - \alpha - \beta$

$= 180^{o} - 38^{o}\,20' - 30^{o}\,26'$

$$\boxed{\gamma = 111^{o}\,14'}$$

**Q.3** $\mathbf{a = \sqrt{3} - 1\,,\quad b = \sqrt{3} + 1\,,\quad \gamma = 60^{o}}$

**Solution:**

$a = \sqrt{3} - 1 = 0.7320$

$b = \sqrt{3} + 1 = 2.7320$

By law of cosines

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

$c^2 = (0.7320)^2 + (2.7320)^2 - 2\,(0.7320)\,(2.7320)\cos 60^{o}$

$c^2 = 6$

$$\boxed{c = \sqrt{6}}$$

$$\because \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\alpha = \cos^{-1}\left[\frac{b^2 + c^2 - a^2}{2bc}\right]$$

$$= \cos^{-1}\left[\frac{(2.7320)^2 + (6)^2 - (0.7320)^2}{2\,(2.7320)\,(\sqrt{6})}\right]$$

$$\boxed{\alpha = 15^{o}}$$

since $\alpha + \beta + \gamma = 180^{o}$

$\beta = 180^{o} - 15^{o} - 60^{o}$

$$\boxed{\beta = 105^{o}}$$

$$\because \quad a^2 = b^2 + c^2 - 2ac\cos\alpha$$

$$2ac\cos\alpha = b^2 + c^2 - a^2$$

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2ac}$$

**Q.4** $\alpha = 3$ , $c = 6$ , $\beta = 36^\circ 20'$

**Solution:**

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$= (3)^2 + (6)^2 - 2(3)(6)\cos 36^\circ 20'$$

$$b^2 = 16$$

$$\boxed{b = 4}$$

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\alpha = \cos^{-1}\left[\frac{b^2 + c^2 - a^2}{2bc}\right] \Rightarrow \cos^{-1}\left[\frac{16 + 36 - 9}{2(4)(6)}\right]$$

$$\boxed{\alpha = 26^\circ 23'}$$

$$\gamma = 180^\circ - \alpha - \beta$$

$$= 180^\circ - 26^\circ 23' - 36^\circ 20'$$

$$\boxed{\gamma = 117^\circ 17'}$$

**Q.5** $a = 7$ , $b = 3$ and $\gamma = 38^\circ 13'$

**Solution:**

By law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$c^2 = (7)^2 + (3)^2 - 2(7)(3)\cos 38^\circ 13'$$

$$c^2 = 25 \Rightarrow \boxed{c = 5}$$

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

Now $$\alpha = \cos^{-1}\left[\frac{b^2 + c^2 - a^2}{2bc}\right]$$

$$\alpha = \cos^{-1}\left[\frac{9 + 25 - 49}{2(3)(5)}\right]$$

$$\alpha = \cos{-1}\left(\frac{-1}{2}\right)$$

$$\boxed{\alpha = 120^\circ}$$

$$\beta = 180^\circ - \alpha - \gamma$$

$$= 180^\circ - 120^\circ - 38^\circ 13'$$

$$\boxed{\beta = 21^\circ 47'}$$

**Solve the following triangles, using first law of tangents, and then law of sines.**

**Q.6** $a = 36.21$ , $b = 42.09$ , $\gamma = 44^\circ 29'$

**Solution:**

We know that $\alpha + \beta + \gamma = 180^\circ$

$$\beta + \alpha = 180^\circ - \gamma$$

$$= 180^\circ - 44^\circ 29'$$

$$\beta + \alpha = 135^\circ 31' \qquad \ldots\ldots\ldots (1)$$

$$\frac{b-a}{b+a} = \frac{\tan\left(\frac{\beta-\alpha}{2}\right)}{\tan\left(\frac{\beta+\alpha}{2}\right)}$$

$$\frac{42.09 - 36.21}{42.09 + 36.21} = \frac{\tan\left(\frac{\beta-\alpha}{2}\right)}{\tan 67^\circ 45'}$$

$$\frac{5.88}{78.3} = \frac{\tan\left(\frac{\beta-\alpha}{2}\right)}{2.4443}$$

$$(0.0750)(2.4443) = \tan\left(\frac{\beta-\alpha}{2}\right)$$

$$\tan^{-1}(0.18355) = \frac{\beta-\alpha}{2}$$

$$2(10°24') = \beta - \alpha$$

$$\beta - \alpha = 20°48' \qquad \ldots\ldots\ldots (2)$$

Adding (1) to (2).

$$\beta + \alpha = 135^\circ 31'$$

adding $\beta - \alpha = 20^\circ 48'$

$$2\beta = 156^\circ 19'$$

$$\boxed{\beta = 78^\circ 10'}$$

Put in (2)

$$78^\circ 10' - \alpha = 20^\circ 48'$$

$$78^\circ 10' - 20^\circ 48' = \alpha$$

$$\boxed{\alpha = 57^\circ 22'}$$

We find 'c' using law of sines

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta}$$

$$c = \frac{42.09}{\sin 78^\circ 10'} \times \sin 44^\circ 29'$$

$$\boxed{c = 30.13}$$

**Q.7** $a = 93\ ,\ c = 101\ ,\ \beta = 80^\circ$

**Solution:**

We know that $\alpha + \beta + \gamma = 180^\circ$

$\alpha + \gamma = 180^\circ - \beta$

$\alpha + \gamma = 180^\circ - 80^\circ$

$\alpha + \gamma = 100^\circ$

By law of tangents.

$$\frac{c-a}{c+a} = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{\tan\left(\frac{\gamma+\alpha}{2}\right)}$$

$$\frac{101-93}{101+93} = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{\tan\left(\frac{\gamma+\alpha}{2}\right)}$$

$$\frac{8}{194} = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{\tan\left(\frac{100^\circ}{2}\right)}$$

$$0.04124 = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{1.1918}$$

$$\tan\left(\frac{\gamma-\alpha}{2}\right) = (0.04124)(1.1918)$$

$$\frac{\gamma-\alpha}{2} = \tan^{-1}(0.04915)$$

$$\gamma - \alpha = 2(2°48')$$

$$\gamma - \alpha = 5°37'$$

$\gamma - \alpha = 5^\circ 37'$

By adding.

$\alpha + \gamma = 100^\circ$

$-\alpha + \gamma = 5^\circ 37'$

$2\gamma = 105^\circ 37'$

$\boxed{\gamma = 52^\circ 49'}$ Put in (2)

$52^\circ 49' - \alpha = 5^\circ 37'$

$52^\circ 49' - 5^\circ 37' = \alpha$

$\boxed{\alpha = 47^\circ 11'}$

To find 'b' using law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$b = \frac{93}{\sin 47^{o}\ 11'} \times \sin 80^{o}$$

$$\boxed{b = 125}$$

**Q.8 $b = 14.8$ , $c = 16.1$ , $\alpha = 42^{o}\ 45'$**

**Solution:**

$$\because \quad \alpha + \beta + \gamma = 180°$$

$$42°45' + \beta + \gamma = 180°$$

$$\beta + \gamma = 180° - 42°45'$$

$$\beta + \gamma = 137°\ 15' \quad \ldots\ldots\ldots (1)$$

By law of tangents.

$$\frac{c - b}{c + b} = \frac{\tan\left(\frac{\gamma - \beta}{2}\right)}{\tan\left(\frac{\gamma + \beta}{2}\right)}$$

$$\frac{16.1 - 14.8}{16.1 + 14.8} = \frac{\tan\left(\frac{\gamma - \beta}{2}\right)}{\tan\left(\frac{137°15'}{2}\right)}$$

$$\frac{1.3}{30.9} = \frac{\tan\left(\frac{\gamma - \beta}{2}\right)}{2.555}$$

$$\tan\left(\frac{\gamma - \beta}{2}\right) = (0.0420)\ (2.555)$$

$$\frac{\gamma - \beta}{2} = \tan^{-1}(0.1075)$$

$$\gamma - \beta = 2(6°8')$$

$$\gamma - \beta = 12°16' \quad \ldots\ldots\ldots (2)$$

Adding (1) to (2).

$$\beta + \gamma = 137°15'$$

$$\gamma - \beta = 12°16'$$

---

$$2\gamma = 149°31'$$

$$\boxed{\gamma = 74°45'}$$

$$\boxed{\beta = 62°29'}$$

To find '$\alpha$' using law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$a = \frac{b}{\sin\beta} \times \sin\alpha$$

$$= \frac{14.8}{\sin 62^\circ 29'} \times \sin 42^\circ 45'$$

$$\boxed{a = 11.33}$$

**Q.9 a = 319 , b = 168 , $\gamma = 110^\circ 22'$**

**Solution:**

Since $\alpha + \beta + \gamma = 180^\circ$

$$\alpha + \beta = 180^\circ - \gamma$$

$$\alpha + \beta = 180^\circ - 110^\circ 22'$$

$$\alpha + \beta = 69^\circ 38' \qquad \ldots\ldots\ldots (1)$$

By law of tangent.

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{319-168}{319+168} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{69^\circ 38'}{2}\right)}$$

$$\frac{151}{487} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{0.695}$$

$$\tan\left(\frac{\alpha-\beta}{2}\right) = (0.3101)(0.695)$$

$$\frac{\alpha-\beta}{2} = \tan^{-1}(0.2155)$$

$$\alpha - \beta = 2(12^\circ 9')$$

$$\alpha - \beta = 24^\circ 20' \qquad \ldots\ldots\ldots (2)$$

Adding (1) and (2).

$\alpha + \beta = 69^\circ 38'$

$\underline{\alpha - \beta = 24^\circ 20'}$

$2\alpha = 93^\circ 58'$

$\boxed{\alpha = 46^\circ 58'}$ Put in (ii)

$\boxed{\beta = 22^\circ 39'}$

to find 'c' using law of sines

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$$

$$c = \frac{319}{\sin 46^\circ 60'} \times \sin 110^\circ 22'$$

$\boxed{c = 408.9}$

**Q.10 b = 61 , c = 32 , $\alpha = 59^\circ 30'$**

**Solution:**

since $\alpha + \beta + \gamma = 180^\circ$

$\beta + \gamma = 180^\circ - 59^\circ 30'$

$\beta + \gamma = 120^\circ 30'$ ..... (1)

By law of tangent.

$$\frac{b-c}{b+c} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{\beta+\gamma}{2}\right)}$$

$$\frac{61-32}{61+32} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{120^\circ 30'}{2}\right)}$$

$$\frac{29}{93} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan(60^\circ 15')}$$

$$\tan\left(\frac{\alpha-\beta}{2}\right) = (0.3118)(1.7496)$$

$$\frac{\beta-\gamma}{2} = \tan^{-1}(0.5455)$$

$$\beta - \gamma = 2(28^\circ 36')$$

$$\beta - \gamma = 57^\circ 14' \quad \text{..... (2)}$$

Adding eq. (1) and eq. (2).

$\beta + \gamma = 120^\circ 30'$

Adding $\beta - \gamma = 57^\circ 14'$

$2\beta = 177^\circ 44'$

$\boxed{\beta = 88^\circ 51'}$ Put in (3)

$\boxed{\gamma = 31^\circ 38'}$

To find 'a' we will use law of sines

$$\frac{a}{\sin \alpha} = \frac{B}{\sin \beta}$$

$$a = \frac{b}{\sin \beta} \times \sin \alpha$$

$$a = \frac{61}{\sin 88^\circ 52'} \times \sin 59^\circ 30'$$

$\boxed{a = 53}$

**Q.11 Measures of two sides of a triangle are in the ratio 3 : 2 and they include an angle of measure $57^\circ$. Find remaining two angles.**

**Solution:**

Let a and b the two sides of a triangle are in the ratio 3 : 2 and include angle is $\gamma = 57^\circ$.

Then $a = 3$, $b = 2$, and $\gamma = 57^\circ$

since $\alpha + \beta + \gamma = 180^\circ$

$\beta + \gamma = 180^\circ - \alpha$

$\alpha + \beta = 180^\circ - \gamma$

$\alpha + \beta = 123^\circ$ ..... (1)

By law of tangent.

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{3-2}{3+2} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{123^\circ}{2}\right)}$$

$$\frac{1}{5} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan(61°30')}$$

$$\tan\left(\frac{\alpha-\beta}{2}\right) = (0.2)\,(1.8418)$$

$$\frac{\alpha-\beta}{2} = \tan^{-1}(0.3684)$$

$$\alpha-\beta = 2(20°13')$$

$$\alpha-\beta = 24°\,27' \quad \ldots\ldots\ldots (1)$$

Adding eq. (1) and (2).

$\alpha + \beta = 123^o$

adding $\alpha - \beta = 40^o\,27'$

$2\alpha = 163^o\,27'$

$\boxed{\alpha = 81^o\,44'}$ Put in (2)

$\boxed{\beta = 41^o 16'}$

**Q.12 Two forces of 40N and 30N are represented by $\vec{AB}$ and $\vec{BC}$ which are inclined at an angle of $147^o\,25'$. Find $\vec{AC}$, the resultant of $\vec{AB}$ & $\vec{BC}$.**

**Solution:**

Given $\vec{AB} = 40$ N $\quad\therefore\quad c = 40$

$\vec{BC} = 30$ N $\quad\therefore\quad a = 30$

Now m $\angle$ ABC = 147o 25′

$\beta = 147^o\,25'$

By law of cosines

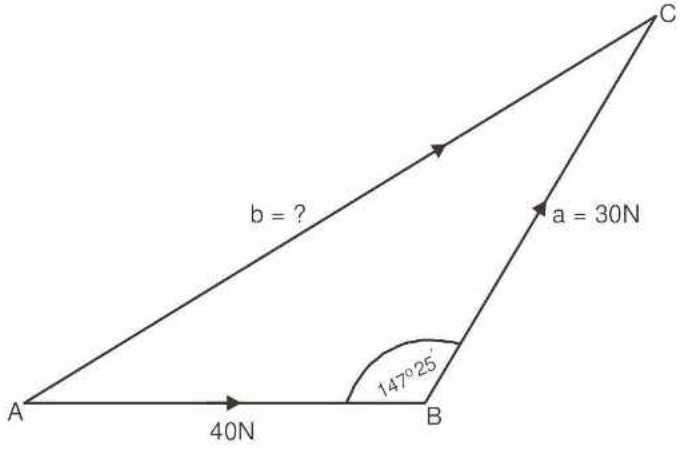
$b^2 = a^2 + c^2 - 2ac\cos\beta$

$= (30)^2 + (40)^2 - 2\,(30)\,(40)\cos 147^o\,25'$

$b^2 = 4522.262$

$b = 67.25$

$\vec{AC} = 67.25$ N



# EXERCISE 12.6

**Q.1 Solve the following triangles, in which**

**a = 7 , b = 7 , c = 9**

**Solution:**

$$S = \frac{a+b+c}{2} = \frac{7+7+9}{2} = \frac{23}{2} = 11.5$$

$S - a = 11.5 - 7 = 4.5$

$S - b = 11.5 - 7 = 4.5$

$S - c = 11.5 - 9 = 2.5$

Now by half angle formulas

$$\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{11.5(4.5)}{7 \times 9}}$$

$$\alpha = 2 \times \cos^{-1}(0.9063)$$

$$\boxed{\alpha = 50^\circ}$$

$$\cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\cos\frac{\beta}{2} = \sqrt{\frac{11.5(4.5)}{7 \times 9}}$$

$$\beta = 2 \times \cos^{-1}(0.9063)$$

$$\boxed{\beta = 50^\circ}$$

$$\cos\frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$\cos\frac{\gamma}{2} = \sqrt{\frac{11.5 \times 2.5}{49}}$$

$$\gamma = 2 \times \cos^{-1}(0.7659)$$

$$\boxed{\gamma = 80^\circ}$$

**Q.2 a = 32 , b = 40 , c = 66 (Lahore Board 2008)**

**Solution:**

$$S = \frac{a+b+c}{2} = \frac{32+40+66}{2} = 69$$

$S - a = 69 - 32 = 37$

$S - b = 69 - 40 = 29$

$S - c = 69 - 66 = 3$

Now by half angle formulas

$$\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\alpha = 2\cos^{-1}\sqrt{\frac{69 \times 37}{40 \times 66}}$$

$$\boxed{\alpha = 20^\circ\, 56'}$$

$$\cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\beta = 2\cos^{-1}\sqrt{\frac{69 \times 29}{32 \times 66}}$$

$$\boxed{\beta = 26^\circ\, 30'}$$

$$\cos\frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$\gamma = 2\cos^{-1}\sqrt{\frac{69 \times 3}{32 \times 40}}$$

$$\boxed{\gamma = 132^\circ\, 34'}$$

**Q.3 a = 28.3 , b = 31.7 , c = 42.8**

**Solution:**

$$S = \frac{a+b+c}{2} = \frac{28.3+31.7+42.8}{2} = 51.4$$

$S - a = 51.4 - 28.3 = 23.1$

$S - b = 51.4 - 31.7 = 19.7$

$S - c = 51.4 - 42.8 = 8.6$

Now by half angle formulas

$$\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}} \qquad \alpha = 2\cos^{-1}\sqrt{\frac{51.4 \times 23.1}{31.7 \times 42.8}}$$

$$\boxed{\alpha = 41^\circ 23'}$$

$$\cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}} \qquad \beta = 2\cos^{-1}\sqrt{\frac{51.4 \times 19.7}{28.3 \times 42.8}}$$

$$\boxed{\beta = 47^\circ 46'}$$

$$\cos\frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}} \qquad \gamma = 2\cos^{-1}\sqrt{\frac{51.4 \times 8.6}{28.3 \times 31.7}}$$

$$\boxed{\gamma = 90^\circ 51'}$$

**Q.4 a = 31.9 , b = 56.31 , c = 40.27**

**Solution:**

$$S = \frac{a+b+c}{2} = \frac{31.9+56.31+40.27}{2} = 64.24$$

$S - a = 64.24 - 31.9 = 32.34$

$S - b = 64.24 - 56.31 = 7.93$

$S - c = 64.24 - 40.27 = 23.97$

Now by half angle formulas

$$\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}} \qquad \alpha = 2\cos^{-1}\sqrt{\frac{64.24 \times 32.34}{56.31 \times 40.27}}$$

$$\boxed{\alpha = 33^\circ 39'}$$

$$\cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ab}} \qquad \beta = 2\cos^{-1}\sqrt{\frac{64.24 \times 7.93}{31.9 \times 56.31}}$$

$$\boxed{\beta = 115^\circ 38'}$$

$$\cos\frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}} \qquad \gamma = 2\cos^{-1}\sqrt{\frac{64.24 \times 23.97}{31.9 \times 56.31}}$$

$$\boxed{\gamma = 30^\circ 43'}$$

**Q.5 a = 4584 , b = 5140 , c = 3624**

**Solution:**

$$S = \frac{a+b+c}{2} = \frac{4584+5140+3624}{2} = 6674$$

$S - a = 6674 - 4584 = 2090$

$S - b = 6674 - 5140 = 1534$

$S - c = 6674 - 3624 = 3050$

Now by half angle formulas

$$\cos\frac{\alpha}{2} = \sqrt{\frac{S\,(S-a)}{bc}} \qquad \cos\frac{\beta}{2} = \sqrt{\frac{S\,(S-b)}{ac}} \qquad \cos\frac{\gamma}{2} = \sqrt{\frac{S\,(S-c)}{ab}}$$

$$\alpha = 2\cos^{-1}\sqrt{\frac{6674 \times 2090}{18627360}} \qquad \beta = 2\cos^{-1}\sqrt{\frac{6674 \times 1534}{16612416}} \qquad \gamma = 2\cos^{-1}\sqrt{\frac{6674 \times 3050}{23561760}}$$

$\boxed{\alpha = 60^\circ\,9'}$ $\boxed{\beta = 76^\circ\,34'}$ $\boxed{\gamma = 43^\circ\,17'}$

**Q.6 Find the smallest angle of the triangle ABC when**
**a = 37.34 , b = 3.24 , c = 35.06**

**Solution:**

By law of cosines

$b^2 = a^2 + c^2 - 2ac\cos\beta$

$2ac\cos\beta = a^2 + c^2 - b^2$

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\beta = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$

$$\beta = \cos^{-1}\left(\frac{(37.34)^2 + (35.06)^2 - (3.24)^2}{2\,(37.34)\,(35.06)}\right)$$

$\boxed{\beta = 3^\circ\,39'}$

**Q.7 find the measure of the greatest angle, if sides of triangle are 16, 20, 23.**

**(Lahore Board 2010)**

**Solution:**

$a = 16\ ,\quad b = 20\ ,\quad c = 33\ ,\quad \gamma = ?$

Now by law of cosines

$c^2 = a^2 + b^2 - 2ac\cos\gamma$

$2ac\cos\gamma = a^2 + b^2 - c^2$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ac}$$

$$\gamma = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

$$\gamma = \cos^{-1}\left(\frac{(16)^2 + (20)^2 - (33)^2}{2 \times 16 \times 20}\right)$$

$\gamma = 132^\circ\,35'$

**Q.8 The sides of a triangle are $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. Prove that the greatest angle of the triangle is $120^o$. (Gujranwala Board 2006)**

**Solution:**

$$a = x^2 + x + 1, \quad b = 2x + 1, \quad c = x^2 - 1, \quad \alpha = ?$$

$$S = \frac{a + b + c}{2} = \frac{x^2 + x + 1 + 2x + 1 + x^2 - 1}{2} = \frac{2x^2 + 3x + 1}{2}$$

$$S - a = \frac{2x^2 + 3x + 1}{2} - (x^2 + x + 1)$$

$$S - a = \frac{2x^2 + 3x + 1 - 2x^2 - 2x - 2}{2} = \frac{x - 1}{2}$$

By half angle formula

$$\cos \frac{\alpha}{2} = \sqrt{\frac{S(S - a)}{bc}}$$

$$\alpha = 2 \cos^{-1} \sqrt{\frac{\frac{2x^2 + 3x + 1}{2}\left(\frac{x - 1}{2}\right)}{(2x + 1)(x^2 - 1)}}$$

$$= 2 \cos^{-1} \sqrt{\frac{(2x^2 + 2x + x + 1)(x - 1)}{4(2x + 1)(x - 1)(x + 1)}}$$

$$= 2 \cos^{-1} \sqrt{\frac{[2x(x + 1) + (x + 1)](x - 1)}{4(2x + 1)(x - 1)(x + 1)}}$$

$$= 2 \cos^{-1} \sqrt{\frac{(2x + 1)(x + 1)(x - 1)}{4(2x + 1)(x - 1)(x + 1)}}$$

$$\alpha = 2 \cos^{-1}\left(\frac{1}{2}\right) = 2 \times 60^o = 120^o$$

Hence proved.

**Q.9 The measures of the sides of a triangular plot are 413, 214 and 375 meters. Find the measure of the corner angles of the plot.**

**Solution:**

$a = 413\ ,\ b = 214\ ,\ c = 375$

$$S = \frac{a+b+c}{2} = \frac{413+214+375}{2} = 501$$

$S - a = 501 - 413 = 88$

$S - b = 501 - 214 = 287$

$S - c = 501 - 375 = 126$

Now by half angle formulas

$$\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\alpha = 2\cos^{-1}\sqrt{\frac{501 \times 88}{214 \times 375}}$$

$$\boxed{\alpha = 84^\circ 20'}$$

$$\cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\beta = 2\cos^{-1}\sqrt{\frac{501 \times 287}{413 \times 375}}$$

$$\boxed{\beta = 31^\circ 2'}$$

$$\cos\frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$\gamma = 2\cos^{-1}\sqrt{\frac{501 \times 126}{413 \times 214}}$$

$$\boxed{\gamma = 64^\circ 38'}$$

**Q.10 Three villages A, B, C are connected by straight roads 6km, 9km, 13km. What angles these roads makes with each other.**

**Solution:**

Let, $a = 6\ ,\ b = 9\ ,\ c = 13$

$$S = \frac{a+b+c}{2} = \frac{6+9+13}{2} = 14$$

$S - a = 14 - 6 = 8$

$S - b = 14 - 9 = 5$

$S - c = 14 - 13 = 1$

Now by half angle formulas

$$\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\alpha = 2\cos^{-1}\sqrt{\frac{14 \times 8}{117 \times 375}}$$

$$\boxed{\alpha = 23^\circ 51'}$$

$$\cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\beta = 2\cos^{-1}\sqrt{\frac{14 \times 5}{78}}$$

$$\boxed{\beta = 37^\circ 21'}$$

$$\cos\frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$\gamma = 2\cos^{-1}\sqrt{\frac{14 \times 1}{54}}$$

$$\boxed{\gamma = 118^\circ 46'}$$

## Area of Triangle

**Case 1:** When two sides & included angle is given

$$\Delta = \frac{1}{2} bc \sin \alpha = \frac{1}{2} ab \sin \gamma = \frac{1}{2} ac \sin \beta$$

**Case II:** When one side & two angles are given

$$\text{Area of triangle} = \frac{a^2 \sin \beta \sin \gamma}{2 \sin \alpha} = \frac{b^2 \sin \alpha \sin \gamma}{2 \sin \beta} = \frac{c^2 \sin \alpha \sin \beta}{2 \sin \gamma}$$

**Case III:** When three sides are given

Area of triangle by Hero's formula is

$$\Delta = \sqrt{S\ (S-a)\ (S-b)\ (S-c)}$$

where a, b, c sides of triangle ABC

and $S = \frac{a+b+c}{2}$

## Exercise 12.7

**Q.1 Find the area of triangle ABC, given two sides and their included angle**

**(i)** $a = 200,\ b = 120,\ \gamma = 150^o$ **(Lahore Board 2006, Gujranwala Board 2007)**

**(ii)** $b = 37,\ C = 45°,\ \alpha = 30^o\ 50'$ **(Gujranwala Board 2005, Lahore Board 2008,2011)**

**(iii)** $a = 4.33,\ b = 9.25,\ \gamma = 56^o\ 44'$

**Solution:**

**(i)** $a = 200,\ b = 120,\ \gamma = 150^o$

$$\Delta = \frac{1}{2} ab \sin \gamma$$

$$= \frac{1}{2} (200) \times 120 \sin 150^o$$

$$= 12000 \times 0.50$$

$$= 6000 \text{ sq. units.}$$

**(ii)** $b = 37,\ C = 45°,\ \alpha = 30^o\ 50'$

$$\Delta = \frac{1}{2} bc \sin \alpha$$

$$= \frac{1}{2} (37) \times 45 \sin 30^o\ 50'$$

$$= 426.692 \text{ sq. units.}$$

**(iii)** $\mathbf{a = 4.33,\ b = 9.25,\ \gamma = 56^\circ 44'}$

$$\Delta = \frac{1}{2} ab \sin\gamma$$

$$= \frac{1}{2}(4.33) \times 9.25 \sin 56^\circ 44'$$

$$= 16.745 \text{ sq. units.}$$

**Q.2 Find area of triangle ABC, given one side and two angles.**

**(i)** $\mathbf{b = 25.4,\ \gamma = 36^\circ 41',\ \alpha = 45^\circ 17'}$

**(ii)** $\mathbf{c = 32,\ \alpha = 47^\circ 24',\ \beta = 70^\circ 16'}$

**(iii)** $\mathbf{a = 4.8,\ \gamma = 83^\circ 42',\ \gamma = 37^\circ 12'}$ **(Lahore Board 2010)**

**Solution:**

**(i)** $\mathbf{b = 25.4,\ \gamma = 36^\circ 41',\ \alpha = 45^\circ 17'}$

$\because\ \alpha + \beta + \gamma = 180^\circ$

$\beta = 180^\circ - 36^\circ 41' - 45^\circ 17'$

$\beta = 98^\circ 2'$

We know that area of triangle

$$\Delta = \frac{1}{2} b^2 \frac{\sin\alpha \times \sin\gamma}{\sin\beta}$$

$$= \frac{1}{2}(25.4)^2 \frac{\sin 45^\circ 17' \sin 36^\circ 41'}{\sin 98^\circ 2'}$$

$\Delta = 138.29$ sq. units.

**(ii)** $\mathbf{c = 32,\ \alpha = 47^\circ 24',\ \beta = 70^\circ 16'}$

$\because\ \alpha + \beta + \gamma = 180^\circ$

$\gamma = 180^\circ - \alpha - \beta$

$\gamma = 180^\circ - 47^\circ 24' - 70^\circ 16'$

$\gamma = 62^\circ 20'$

$$\text{Area of triangle} = \frac{1}{2} c^2 \frac{\sin\alpha \times \sin\beta}{\sin\gamma}$$

$$\Delta = \frac{1}{2}(32)^2 \frac{\sin 47^\circ 24' \sin 70^\circ 16'}{\sin 62^\circ 20'}$$

$\Delta = 400.57$ sq. units.

**(iii)** **$a = 4.8,\ \gamma = 83^o\ 42',\ \gamma = 37^o\ 12'$**

$\because\ \alpha + \beta + \gamma = 180°$

$\beta = 180^o - \alpha - \gamma$

$= 180^o - 83^o\ 42' - 37^o\ 12'$

$\beta = 59^o\ 6'$

$$\text{Area of triangle} = \frac{1}{2} a^2 \frac{\sin \beta \times \sin \gamma}{\sin \alpha}$$

$$\Delta = \frac{1}{2} (4.8)^2 \frac{\sin 59^o\ 6' \sin 37^o\ 12'}{\sin 83^o\ 42'}$$

$\Delta = 6.02$ sq. units.

**Q.3 Find the area of the triangle ABC, given three sides**

**(i)** **$a = 18,\ b = 24,\ c = 30$**

**(ii)** **$a = 524,\ b = 276,\ c = 315$**

**(iii)** **$a = 32.65,\ b = 42.81,\ c = 64.92$**

**Solution:**

**(i)** **$a = 18,\ b = 24,\ c = 30$**

$$S = \frac{a + b + c}{2} = \frac{18 + 24 + 30}{2} = 36$$

$S - a = 36 - 18 = 18$

$S - b = 36 - 24 = 12$

$S - c = 36 - 30 = 6$

$$\text{Required Area} = \sqrt{S\,(S - a)\,(S - b)\,(S - c)}$$

$$= \sqrt{36 \times 18 \times 12 \times 6} = 216 \text{ sq. units.}$$

**(ii)** **$a = 524,\ b = 276,\ c = 315$**

$$S = \frac{a + b + c}{2} = \frac{524 + 276 + 315}{2} = 557.5$$

$S - a = 557.5 - 524 = 33.5$

$S - b = 557.5 - 276 = 281.5$

$S - c = 557.5 - 315 = 242.5$

$$\text{Required Area} = \sqrt{S\,(S - a)\,(S - b)\,(S - c)}$$

$= \sqrt{557.5 \times 33.5 \times 281.5 \times 242.5} = 35705.89$ sq. units.

**(iii)** **a = 32.65, b = 42.81, c = 64.92**

$$S = \frac{a+b+c}{2} = \frac{32.65 + 42.81 + 64.92}{2} = 70.19$$

$S - a = 70.19 - 32.65 = 37.54$

$S - b = 70.19 - 42.81 = 27.38$

$S - c = 70.19 - 64.92 = 5.27$

Required Area $= \sqrt{S(S-a)(S-b)(S-c)}$

$= \sqrt{70.19 \times 37.54 \times 27.38 \times 5.27} = 616.60$ sq. units.

**Q.4 The area of triangle is 2437. If a = 79, c = 97. Find angle β.**

**(Lahore Board 2005).**

**Solution:**

$\Delta = 2437$ , $a = 79$ , $c = 97$ , $\beta = ?$

area of triangle $= \frac{1}{2} ac \sin\beta$

$$2437 = \frac{1}{2} \times 79 \times 97 \sin\beta$$

$$\beta = \sin^{-1} \frac{2437}{3831.5}$$

$\beta = 39^\circ 30'$

**Q.5 The area of triangle is 121.34. If $\alpha = 32^\circ 15'$ , $\beta = 65^\circ 37'$ find c and γ.**

**(Lahore Board 2004)**

**Solution:**

$\because \alpha + \beta + \gamma = 180^\circ$

$\gamma = 180^\circ - \alpha - \beta$

$= 180^\circ - 32^\circ 15' - 65^\circ 37'$

$\gamma = 82^\circ 8'$

$$\text{Area of triangle} = \frac{1}{2} c^2 \frac{\sin\alpha \times \sin\beta}{\sin\gamma}$$

$$121.34 = \frac{1}{2} c^2 \frac{\sin 32^\circ 15' \sin 65^\circ 37'}{\sin 82^\circ 8'}$$

$$c^2 = \frac{240.3962}{0.4860}$$

$c^2 = 494.64$

c = 22.24 sq. units.

**Q.6 One side of a triangular garden is 30m. If its two corner angles are $22^{\circ}\frac{1}{2}$ and $112^{\circ}\frac{1}{2}$. Find the cost of planting the grass at the rate of Rs. 5 per square meter.**

**Solution:**

$a = 30$

$\beta = 22^{\circ}\frac{1}{2} = 22.5^{\circ} = 22^{\circ}\,30'$

$\gamma = 112^{\circ}\frac{1}{2} = 112^{\circ}30'$

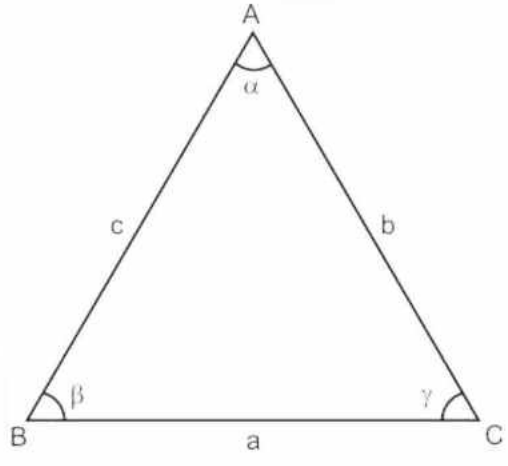
$\alpha = ?$

$\because\ \alpha + \beta + \gamma = 180^{\circ}$

$\alpha = 180^{\circ} - \beta - \gamma$

$= 180^{\circ} - 22^{\circ}\,30' - 112^{\circ}\,30'$

$\alpha = 45^{\circ}$




Area of triangle $= \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}$

$= \frac{1}{2}(30)^2\frac{\sin 22^{\circ}\,30'\sin 112^{\circ}\,30'}{\sin 45^{\circ}}$

$\Delta = 225$ sq. m

Grass planting @ Rs. 5/sq. m $= 225 \times 5$

$=$ Rs. 1125 Ans.

## EXERCISE 12.8

**Q.1 Show that**

**(i) $r = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$**

**(ii) $S = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$**

**Solution:**

**(i) $r = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$**

R.H.S. $= 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

$= 4\frac{abc}{4\Delta}\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{(S-a)(S-c)}{ac}}\sqrt{\frac{(S-a)(S-b)}{ab}}$

$$= \frac{a\,b\,c}{\Delta}\sqrt{\frac{(S-b)^2(S-c)^2(S-a)^2}{a^2b^2c^2}}$$

$$= \frac{a\,b\,c}{\Delta}\,\frac{(S-a)(S-b)(S-c)}{a\,b\,c}$$

$$= \frac{1}{\Delta}\,\frac{S(S-a)(S-b)(S-c)}{S} \quad \text{(multiply and dividing by S)}$$

$$= \frac{1}{\Delta} \times \frac{\Delta^2}{S}$$

$$= \frac{\Delta}{S} = r$$

= L.H.S. Hence proved.

**(ii)** $\mathbf{S = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$

$$\text{R.H.S.} = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\,\frac{a\,b\,c}{4\,\Delta}\sqrt{\frac{S(S-a)}{bc}}\sqrt{\frac{S(S-b)}{ac}}\sqrt{\frac{S(S-c)}{ab}}$$

$$= \frac{a\,b\,c}{\Delta}\sqrt{\frac{S^3(S-a)(S-b)(S-c)}{a^2b^2c^2}}$$

$$= \frac{S\,abc}{\Delta\,abc}\sqrt{S(S-a)(S-b)(S-c)}$$

$$= \frac{S}{\Delta} \times \Delta$$

= S = L.H.S.

Hence proved.

**Q.2** **Show that** $\mathbf{r = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2} = b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2} = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}}$

**Solution:**

$$\text{R.H.S.} = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}$$

$$= a\sqrt{\frac{(S-a)(S-c)}{ac}}\sqrt{\frac{(S-a)(S-b)}{ab}}\sqrt{\frac{bc}{S(S-a)}}$$

$$= a\sqrt{\frac{(S-a)(S-c)(S-a)(S-b)\,bc}{a^2\,bc\,S(S-a)}}$$

$$= \frac{a^2}{a}\sqrt{\frac{S\,(S-a)\,(S-b)\,(S-c)}{S^2}}$$

$$= \frac{\Delta}{S} = r = \text{L.H.S. Hence proved.}$$

Next R.H.S. $= b \sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2}$

$$= b\sqrt{\frac{(S-a)\,(S-b)}{ab}}\sqrt{\frac{(S-b)\,(S-c)}{bc}}\sqrt{\frac{ac}{S\,(S-b)}}$$

$$= b\sqrt{\frac{(S-a)\,(S-b)\,(S-b)\,(S-c)\,ac}{a\,b^2\,c\,S^2\,(S-b)}}$$

$$= \frac{b}{b}\sqrt{\frac{S\,(S-a)\,(S-b)\,(S-c)}{S^2}}$$

$$= \frac{\Delta}{S} = r = \text{L.H.S. Hence proved.}$$

Next R.H.S. $= c \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$

$$= c\sqrt{\frac{(S-b)\,(S-c)}{bc}}\sqrt{\frac{(S-a)\,(S-c)}{ac}}\sqrt{\frac{ab}{S\,(S-c)}}$$

$$= c\sqrt{\frac{(S-b)\,(S-c)\,(S-c)\,(S-a)\,ab}{ba\,c^2\,S\,(S-c)}}$$

$$= \frac{c}{c}\sqrt{\frac{S\,(S-a)\,(S-b)\,(S-c)}{S^2}}$$

$$= \frac{\sqrt{S\,(S-a)\,(S-b)\,(S-c)}}{S}$$

$$= \frac{\Delta}{S} = r = \text{L.H.S. Hence proved.}$$

**Q.3 Prove that**

**(i)** $\mathbf{r_1 = 4R \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$

**(ii)** $\mathbf{r_2 = 4R \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}}$

**(iii)** $\mathbf{r_3 = 4R \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}}$

**Solution:**

**(i)** $\mathbf{r_1 = 4R \sin\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}}$

R.H.S. $= 4R \sin\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

$$= 4\frac{a\,b\,c}{4\,\Delta}\sqrt{\frac{(S-b)\,(S-c)}{bc}}\sqrt{\frac{S\,(S-b)}{ac}}\sqrt{\frac{S\,(S-c)}{ab}}$$

$$= \frac{a\,b\,c}{\Delta}\sqrt{\frac{(S-b)^2\,(S-c)^2\,S^2}{a^2\,b^2\,c^2}}$$

$$= \frac{abc}{\Delta\,abc}\;\frac{S(S-b)(S-c)(S-a)}{S-a}$$

$$= \frac{\Delta^2}{\Delta\,(S-a)}$$

$$= \frac{\Delta}{S-a} = r_1 \quad \text{L.H.S.} \quad \text{Hence proved}$$

**(ii)** $\mathbf{r_2 = 4\,R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}}$

R.H.S. $= 4\,R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$

$$= 4\left(\frac{abc}{4\,\Delta}\right)\sqrt{\frac{S\,(S-a)}{bc}}\sqrt{\frac{(S-a)\,(S-c)}{a\,c}}\sqrt{\frac{S(S-c)}{ac}}$$

$$= \frac{a\,b\,c}{\Delta}\sqrt{\frac{S^2\,(S-a)^2\,(S-c)^2}{a^2\,b^2\,c^2}}$$

$$= \frac{a\,b\,c}{\Delta}\times\frac{S(S-a)\,(S-c)}{a\,b\,c}$$

$$= \frac{S\,(S-a)\,(S-b)\,(S-c)}{(S-b)\,\Delta}$$

$$= \frac{\Delta^2}{\Delta\,(S-b)}$$

$$= \frac{\Delta}{S-b} = r_2 = \text{L.H.S.} \quad \text{Hence proved}$$

Hence proved.

**(iii)** $\mathbf{r_3 = 4\,R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}}$

R.H.S. $= 4\,R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$

$$= 4\left(\frac{abc}{4\Delta}\right)\sqrt{\frac{S(S-a)}{bc}}\sqrt{\frac{S(S-b)}{ab}}\sqrt{\frac{(S-a)(S-b)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{S^2(S-a)^2(S-b)^2}{a^2b^2c^2}}$$

$$= \frac{abc}{\Delta}\frac{S(S-a)(S-b)(S-c)}{abc(S-c)}$$

$$= \frac{\Delta^2}{\Delta(S-c)}$$

$$= \frac{\Delta}{S-c} = r_3 = \text{L.H.S. Hence proved.}$$

**Q.4 Show that**

**(i)** $\mathbf{r_1 = S\tan\frac{\alpha}{2}}$

**(ii)** $\mathbf{r_2 = S\tan\frac{\beta}{2}}$ **(Lahore Board 2007)**

**(iii)** $\mathbf{r_3 = S\tan\frac{\gamma}{2}}$ **(Gujranwala Board 2005)**

**Solution:**

**(i)** $\mathbf{r_1 = S\tan\frac{\alpha}{2}}$

$$\text{R.H.S.} = S\tan\frac{\alpha}{2}$$

$$= S\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$$

$$= S\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-a)^2}}$$

$$= S\frac{\sqrt{S(S-a)(S-b)(S-c)}}{S(S-a)}$$

$$= \frac{\Delta}{S-a} = r_1 = \text{L.H.S. Hence proved}$$

**(ii)** $\mathbf{r_2 = S \tan \frac{\beta}{2}}$

R.H.S. $= S \tan \frac{\alpha}{2}$

$$= S\sqrt{\frac{(S-a)(S-c)}{S(S-b)}}$$

$$= S\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-b)^2}}$$

$$= S\frac{\sqrt{S(S-a)(S-b)(S-c)}}{S(S-b)}$$

$$= \frac{\Delta}{S-b} = r_2 = \text{L.H.S. Hence proved}$$

**(iii)** $\mathbf{r_3 = S \tan \frac{\gamma}{2}}$

R.H.S. $= S \tan \frac{\gamma}{2}$

$$= S\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

$$= S\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-c)^2}}$$

$$= S\frac{\sqrt{S(S-a)(S-b)(S-c)}}{S(S-c)}$$

$$= S \times \frac{\Delta}{S(S-b)} = r_3 = \text{L.H.S. Hence proved}$$

**Q.5 Prove that**

**(i)** $\mathbf{r_1 r_2 + r_2 r_3 + r_3 r_1 = S^2}$ **(Lahore Board 2009)**

**(ii)** $\mathbf{r\, r_1 r_2 r_3 = \Delta^2}$

**(iii)** $\mathbf{r_1 + r_2 + r_3 - r = 4R}$

**(iv)** $\mathbf{r_1 r_2 r_3 = rS^2}$

**Solution:**

**(i)** $\mathbf{r_1 r_2 + r_2 r_3 + r_3 r_1 = S^2}$

L.H.S. $= r_1 r_2 + r_2 r_3 + r_3 r_1$

$$= \frac{\Delta}{S-a} \times \frac{\Delta}{S-b} + \frac{\Delta}{S-b} \times \frac{\Delta}{S-c} + \frac{\Delta}{S-c} \times \frac{\Delta}{S-a}$$

$$= \frac{\Delta^2}{(S-a)(S-b)} + \frac{\Delta^2}{(S-b)(S-c)} + \frac{\Delta^2}{(S-c)(S-a)}$$

$$= \Delta^2\left[\frac{S-c+S-a+S-b}{(S-a)(S-b)(S-c)}\right]$$

$$= \Delta^2\left[\frac{3S-(a+b+c)}{(S-a)(S-b)(S-c)}\right]$$

$$= \Delta^2\left[\frac{3S-2S}{(S-a)(S-b)(S-c)}\right]$$

$$= \frac{\Delta^2 S \,.\, S}{S(S-a)(S-b)(S-c)}$$

$$= \frac{\Delta^2 S^2}{\Delta^2} = S^2 = \text{R.H.S. Hence proved}$$

**(ii)** $\mathbf{r\, r_1\, r_2\, r_3 = \Delta^2}$

L.H.S. $= r\, r_1\, r_2\, r_3$

$$= \frac{\Delta}{S} \times \frac{\Delta}{S-a} \times \frac{\Delta}{S-b} \times \frac{\Delta}{S-c}$$

$$= \frac{\Delta^4}{S(S-a)(S-b)(S-c)}$$

$$= \frac{\Delta^4}{\Delta^2} = \Delta^2 = \text{R.H.S. Hence proved}$$

**(iii)** $\mathbf{r_1 + r_2 + r_3 - r = 4R}$

L.H.S. $= r_1 + r_2 + r_3 - r$

$$= \frac{\Delta}{S-a} + \frac{\Delta}{S-b} + \frac{\Delta}{S-c} - \frac{\Delta}{S} = \Delta\left[\frac{1}{S-a} + \frac{1}{S-b} + \frac{1}{S-c} - \frac{1}{S}\right]$$

$$= \Delta\left[\frac{S-b+S-a}{(S-a)(S-b)} + \frac{S-(S-c)}{S(S-c)}\right]$$

$$= \Delta\left[\frac{2S-a-b}{(S-a)(S-b)} + \frac{S-S+c}{S(S-c)}\right]$$

$$= \Delta\left[\frac{2S-a-b-c+c}{(S-a)(S-b)} + \frac{c}{S(S-c)}\right]$$

$$= \Delta\left[\frac{2S-(a+b+c)+c}{(S-a)(S-b)} + \frac{c}{S(S-c)}\right]$$

$$= \Delta\left[\frac{2S-2S+c}{(S-a)(S-b)} + \frac{c}{S(S-c)}\right]$$

$$= \Delta.C\left[\frac{1}{(S-a)(S-b)} + \frac{1}{S(S-c)}\right]$$

$$= C\Delta\left[\frac{S(S-C)+(S-a)(S-b)}{S(S-a)(S-b)(S-c)}\right]$$

$$= C\Delta\left[\frac{S^2-CS+S^2-bS-aS+ab}{\Delta^2}\right]$$

$$= \frac{C}{\Delta}[2S^2 - S(a+b+C) + ab]$$

$$= \frac{C}{\Delta}[2S^2 - S(2S) + ab]$$

$$= \frac{C}{\Delta}[2S^2 - 2S^2 + ab]$$

$$= \frac{abc}{\Delta} = \frac{4abc}{4\Delta} = 4R$$

$$= \text{R.H.S Hence proved}$$

**(iv)** $\mathbf{r_1\, r_2\, r_3 = rS^2}$

L.H.S. $= r_1\, r_2\, r_3$

$$\frac{\Delta}{S-a} \times \frac{\Delta}{S-b} \times \frac{\Delta}{S-c} = \frac{\Delta^3}{(S-a)(S-b)(S-c)}$$

$$= \frac{S\,\Delta^3}{S(S-a)(S-b)(S-c)}$$

$$= \frac{S\,\Delta^3}{\Delta^2}$$

$$= S\,\Delta$$

$$= S\Delta \times \frac{S}{S}$$

$$= S^2 r$$

$$= \text{R.H.S.}$$

Hence proved.

**Q.6 Find R, r, $r_1$, $r_2$, $r_3$ if measures of the sides of triangle ΔABC are**
**(Gujranwala Board 2005, 2007) (Lahore Board, 2004, 2005, 2009)**

**(i)** $\mathbf{a = 13,\ b = 14,\ c = 15}$

**(ii)** $\mathbf{a = 34,\ b = 20,\ c = 42}$

**Solution:**

**(i)** $\mathbf{a = 13,\ b = 14,\ c = 15}$

$$S = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

$$= \sqrt{21(21-13)(21-14)(21-15)}$$

$$= \sqrt{21 \times 8 \times 7 \times 6} = 84$$

$$R = \frac{abc}{4\Delta} = \frac{13 \times 14 \times 15}{4(84)} = \frac{2730}{336} = 8.125$$

$$r = \frac{\Delta}{S} = \frac{84}{21} = 4$$

$$r_1 = \frac{\Delta}{S-a} = \frac{84}{8} = 10.5$$

$$r_2 = \frac{\Delta}{S-b} = \frac{84}{7} = 12$$

$$r_3 = \frac{\Delta}{S-c} = \frac{84}{6} = 14$$

**(ii)** $\mathbf{a = 34,\ b = 20,\ c = 42}$

$$S = \frac{a+b+c}{2} = \frac{34+20+42}{2} = \frac{96}{2} = 48$$

$$\Delta = \sqrt{S\,(S-a)\,(S-b)\,(S-c)}$$

$$= \sqrt{48\,(48-34)\,(48-20)\,(48-42)}$$

$$= \sqrt{48\,(14)\,(28)\,(6)} = 336$$

$$R = \frac{abc}{4\,\Delta} = \frac{34 \times 20 \times 42}{4 \times 336} = \frac{28560}{1344} = 21.25$$

$$r = \frac{\Delta}{S} = \frac{336}{48} = 7$$

$$r_1 = \frac{\Delta}{S-a} = \frac{336}{14} = 24$$

$$r_2 = \frac{\Delta}{S-b} = \frac{336}{28} = 12$$

$$r_3 = \frac{\Delta}{S-c} = \frac{336}{6} = 56$$

**Q.7 Prove that in an equilateral triangle**

**(i)** $\mathbf{r : R : r_1 = 1 : 2 : 3}$

**(ii)** $\mathbf{r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3}$

**Solution:**

**(i)** $\mathbf{r : R : r_1 = 1 : 2 : 3}$

We know that in an equilateral triangle

$$a = b = c$$

$$S = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$S-a = \frac{3a}{2} - a = \frac{3a-2a}{2} = \frac{a}{2}$$

$$S-b = S-a = \frac{a}{2}$$

$$S-c = S-a = \frac{a}{2}$$

$$\Delta = \sqrt{S\,(S-a)\,(S-a)\,(S-a)} = \sqrt{\frac{3a}{2}\times\frac{a}{2}\times\frac{a}{2}\times\frac{a}{2}} = \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3}\,a^2}{4}$$

$$r = \frac{\Delta}{S} = \frac{\sqrt{3}\,a^2}{4} \div \frac{3a}{2}$$

$$r = \frac{\sqrt{3}\,a^2}{4} \times \frac{2}{3a} = \frac{\sqrt{3}\,a}{6}$$

$$R = \frac{abc}{4\Delta} = \frac{a^3}{4\times\frac{\sqrt{3}\,a^2}{4}} = \frac{a^3}{\sqrt{3}\,a^2} = \frac{a}{\sqrt{3}}$$

$$r_1 = \frac{\Delta}{S-a} = \frac{\sqrt{3}\,a^2}{4} \div \frac{a}{2}$$

$$r_1 = \frac{\sqrt{3}a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3}a}{2}$$

L.H.S. $= r : R : r_1$

$$\frac{\sqrt{3}\,a}{6} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}\,a}{2}$$

Dividing throughout by $\frac{\sqrt{3}\,a}{6}$

$$\frac{\sqrt{3}\,a}{6}\times\frac{6}{\sqrt{3}\,a} : \frac{a}{\sqrt{3}}\times\frac{6}{\sqrt{3}\,a} : \frac{\sqrt{3}\,a}{2}\times\frac{6}{\sqrt{3}\,a}$$

$$= 1 : \frac{6}{3} : 3$$

$$= 1 : 2 : 3$$

= R.H.S.

Hence proved.

**(ii)** $\mathbf{r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3}$

We know that in an equilateral triangle

$a = b = c$

$$S = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$S-a = \frac{3a}{2} - a = \frac{3a-2a}{2} = \frac{a}{2}$$

$$S-b = \frac{3a}{2} - a = \frac{a}{2}$$

$$S-c = \frac{3a}{2} - a = \frac{a}{2}$$

$$\Delta = \sqrt{S\,(S-a)\,(S-b)\,(S-c)} = \sqrt{\frac{3a}{2}\times\frac{a}{2}\times\frac{a}{2}\times\frac{a}{2}} = \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3}\,a^2}{4}$$

$$r = \frac{\Delta}{S} = \frac{\sqrt{3}\,a^2}{4} \times \frac{2}{3a} = \frac{\sqrt{3}\,a}{6}$$

$$r_1 = \frac{\Delta}{S-a} = \frac{\sqrt{3}\,a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3}\,a}{2} \qquad R = \frac{a\,b\,c}{4\Delta} = \frac{a^3}{4 \times \frac{\sqrt{3}\,a^2}{4}} = \frac{a}{\sqrt{3}}$$

$$r_2 = \frac{\Delta}{S-b} = \frac{\sqrt{3}\,a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3}\,a}{2}$$

$$r_3 = \frac{\Delta}{S-c} = \frac{\sqrt{3}\,a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3}\,a}{2}$$

L.H.S. $= r : R : r_1 : r_2 : r_3$

$$= \frac{\sqrt{3}\,a}{6} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}\,a}{2} : \frac{\sqrt{3}\,a}{2} : \frac{\sqrt{3}\,a}{2}$$

Multiplying throughout by $\frac{6}{\sqrt{3}\,a}$

$$= \frac{\sqrt{3}\,a}{6} \times \frac{6}{\sqrt{3}\,a} : \frac{a}{\sqrt{3}} \times \frac{6}{\sqrt{3}\,a} : \frac{\sqrt{3}\,a}{2} \times \frac{6}{\sqrt{3}\,a} : \frac{\sqrt{3}\,a}{2} \times \frac{6}{\sqrt{3}\,a} : \frac{\sqrt{3}\,a}{2} \times \frac{6}{\sqrt{3}\,a}$$

$= 1 : 2 : 3 : 3 : 3$

$=$ R.H.S. Hence proved.

**Q.8 Prove that**

**(i)** $\Delta = r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$

**(ii)** $r = S \tan\frac{\alpha}{2} \tan\frac{\beta}{2} \tan\frac{\gamma}{2}$ **(Lahore Board 2010)**

**(iii)** $\Delta = 4\,R\,r \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

**Solution:**

**(i)** $\Delta = r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$

R.H.S. $= r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$

$$= \left(\frac{\Delta}{S}\right)^2 \sqrt{\frac{S\,(S-a)}{(S-b)\,(S-c)}} \sqrt{\frac{S\,(S-b)}{(S-a)\,(S-c)}} \sqrt{\frac{S\,(S-c)}{(S-a)\,(S-b)}}$$

$$= \frac{\Delta^2}{S^2}\sqrt{\frac{S^3(S-a)(S-b)(S-c)}{(S-b)^2(S-c)^2(S-a)^2}}$$

$$= \frac{\Delta^2}{S^2} S \sqrt{\frac{S(S-a)(S-b)(S-c)}{(S-a)^2(S-b)^2(S-c)^2}}$$

$$= \frac{\Delta^2}{S}\sqrt{\frac{S \cdot S}{S(S-a)(S-b)(S-c)}}$$

$$= \frac{\Delta^2}{S}\frac{S}{\sqrt{S(S-a)(S-b)(S-c)}} = \frac{\Delta^2}{\Delta} = \Delta = \text{L.H.S. Hence proved}$$

**(ii)** $\mathbf{r = S \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}}$

$$\text{R.H.S.} = S\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$$

$$= S\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}\sqrt{\frac{(S-a)(S-c)}{S(S-b)}}\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

$$= S\sqrt{\frac{(S-b)(S-c)(S-a)(S-c)(S-a)(S-b)}{S^3(S-a)(S-b)(S-c)}}$$

$$= \frac{S}{S}\sqrt{\frac{(S-a)^2(S-b)^2(S-c)^2}{S(S-a)(S-b)(S-c)}}$$

$$= \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2}}$$

$$= \frac{\sqrt{S(S-a)(S-b)(S-c)}}{S} = \frac{\Delta}{S} = r = \text{L.H.S. Hence proved}$$

Hence proved.

**(iii)** $\mathbf{\Delta = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$

$$\text{R.H.S.} = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\left(\frac{abc}{4\Delta}\right)\left(\frac{\Delta}{S}\right)\sqrt{\frac{S(S-a)}{bc}}\sqrt{\frac{S(S-b)}{ac}}\sqrt{\frac{S(S-c)}{ab}}$$

$$= \frac{abc}{S}\sqrt{\frac{S^3(S-a)(S-b)(S-c)}{a^2b^2c^2}}$$

$$= \frac{abc\,S}{S}\sqrt{\frac{S(S-a)(S-b)(S-c)}{a^2b^2c^2}}$$

$$= \frac{abc}{abc}\sqrt{S(S-a)(S-b)(S-c)} = \Delta = \text{L.H.S. Hence proved.}$$

**Q.9 Show that**

**(i)** $\dfrac{1}{2\,r\,R} = \dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ca}$

**(ii)** $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$

**Solution:**

**(i)** $\dfrac{1}{2\,r\,R} = \dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ca}$

$$\text{L.H.S.} = \frac{1}{2\,r\,R}$$

$$= \frac{1}{2\left(\frac{\Delta}{S}\right)\left(\frac{abc}{4\Delta}\right)} = \frac{2S}{abc} = \frac{a+b+c}{abc}$$

$$= \frac{a}{abc} + \frac{b}{abc} + \frac{c}{abc} = \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab} = \text{R.H.S. Hence proved}$$

**(ii)** $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$

$$\text{R.H.S.} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

$$= \frac{S-a}{\Delta} + \frac{S-b}{\Delta} + \frac{S-c}{\Delta}$$

$$= \frac{S-a+S-b+S-c}{\Delta} = \frac{3\,S-(a+b+c)}{\Delta}$$

$$= \frac{3S-2S}{\Delta} = \frac{S}{\Delta} = \frac{1}{r} = \text{L.H.S. Hence proved}$$

**Q.10 Prove that**

$$r = \frac{a \sin\frac{\beta}{2} \sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}} = \frac{b \sin\frac{\alpha}{2} \sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}} = \frac{c \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

**Solution:**

$$r = \frac{a \sin\frac{\beta}{2} \sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$$

$$\text{R.H.S.} = \frac{a \sin\frac{\beta}{2} \sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$$

$$= \frac{a\sqrt{\frac{(S-a)(S-c)}{ac}}\sqrt{\frac{(S-a)(S-b)}{ab}}}{\sqrt{\frac{S(S-a)}{bc}}}$$

$$= a\sqrt{\frac{\frac{(S-a)^2(S-c)(S-b)}{a^2bc}}{\frac{S(S-a)}{bc}}}$$

$$= a\sqrt{\frac{(S-a)^2(S-c)(S-b)}{a^2bc} \times \frac{abc}{S(S-a)}}$$

$$= a\sqrt{\frac{(S-a)(S-b)(S-c)S}{S^2a^2}}$$

$$= \frac{a}{a} \times \frac{\Delta}{S} = r = \text{L.H.S. Hence proved}$$

$$\mathbf{r = \frac{b \sin\frac{\alpha}{2} \sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}}$$

$$\text{R.H.S.} = \frac{b \sin\frac{\alpha}{2} \sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$$

$$= b\frac{\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{(S-a)(S-b)}{ab}}}{\sqrt{\frac{S(S-b)}{ac}}}$$

$$= b\sqrt{\frac{(S-b)^2(S-c)(S-a)}{ab^2c} \times \frac{ac}{S(S-b)}}$$

$$= b\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2b^2}}$$

$$= \frac{b}{b} \times \frac{\Delta}{S} = \frac{\Delta}{S} = r = \text{L.H.S. Hence proved}$$

$$\mathbf{r = \frac{c \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}}$$

$$\text{R.H.S.} = \frac{c \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

$$= \frac{c\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{(S-a)(S-c)}{ac}}}{\sqrt{\frac{S(S-c)}{ab}}}$$

$$= c\sqrt{\frac{(S-b)(S-c)^2(S-a)}{a\,b\,c^2} \times \frac{ba}{S(S-c)}}$$

$$= c\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2c^2}}$$

$$= \frac{\Delta}{S} = r = \text{L.H.S. Hence proved}$$

**Q.11 Prove that $abc(\sin\alpha + \sin\beta + \sin\gamma) = 4\Delta S$ (Lahore Board 2007, 2011)**

**Solution:**

$$\text{L.H.S.} = abc(\sin\alpha + \sin\beta + \sin\gamma)$$

$$= abc\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right) \qquad \because R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{\sin\gamma}$$

$$= abc\left(\frac{a+b+c}{2R}\right) = abc\left(\frac{2S}{2\left(\frac{abc}{4\Delta}\right)}\right)$$

$$= abc\left(\frac{4\Delta S}{abc}\right) = 4\Delta S = \text{R.H.S. Hence proved}$$

**Q.12 Prove that**

**(i) $(r_1 + r_2)\tan\frac{\gamma}{2} = C$ (Lahore Board 2006) (Gujranwala Board 2007)**

**(ii) $(r^3 - r)\cot\frac{\gamma}{2} = C$**

**Solution:**

**(i) $(r_1 + r_2)\tan\frac{\gamma}{2} = C$**

$$\text{L.H.S.} = (r_1 + r_2)\tan\frac{\gamma}{2}$$

$$= \left(\frac{\Delta}{S-a} + \frac{\Delta}{S-b}\right)\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

$$= \Delta\left(\frac{S-b+S-a}{(S-a)(S-b)}\right)\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-c)^2}}$$

$$= \Delta\left(\frac{2S - b - a}{(S-a)(S-b)}\right)\frac{\Delta}{S(S-c)}$$

$$= \frac{\Delta^2(a+b+c-b-a)}{S(S-a)(S-b)(S-c)} \qquad (\because\ 2S = a+b+c)$$

$$= \frac{\Delta^2 C}{\Delta^2} = C = \text{R.H.S. Hence proved.}$$

**(ii)** $\mathbf{(r^3 - r)\cot\frac{\gamma}{2} = C}$

$$\text{L.H.S.} = (r^3 - r)\cot\frac{\gamma}{2}$$

$$= \left(\frac{\Delta}{S-c} - \frac{\Delta}{S}\right)\sqrt{\frac{S(S-c)}{(S-a)(S-b)}}$$

$$= \Delta\left(\frac{1}{S-c} - \frac{1}{S}\right)\sqrt{\frac{S(S-a)(S-b)(S-c)}{(S-a)^2(S-b)^2}}$$

$$= \Delta\left(\frac{S - S + c}{S(S-c)}\right)\frac{\sqrt{S(S-a)(S-b)(S-c)}}{(S-a)(S-b)}$$

$$= \frac{\Delta C}{S(S-c)} \times \frac{\Delta}{(S-a)(S-b)}$$

$$= \frac{\Delta^2 C}{S(S-a)(S-b)(S-c)} = \frac{\Delta^2 C}{\Delta^2} = C = \text{R.H.S.}$$

Hence proved.

Chapter 12

# APPLICATION OF TRIGONOMETRY

## MULTIPLE CHOICE QUESTIONS

(1) The value of R =

(a) $\frac{abc}{S-a}$ (b) $\frac{abc}{4\Delta}$

(c) $\frac{4abc}{\Delta}$ (d) $\frac{abc}{\Delta}$

**[Lahore Board 2005]**

(2) In the law of tangents $\frac{\tan\frac{\alpha-p}{2}}{\tan\frac{\alpha+p}{2}}$ =

(a) $\frac{a-b}{a+b}$ (b) $\frac{a-p}{a+p}$

(c) $\frac{a+p}{a-p}$ (d) $\frac{a+b}{a-b}$

**[Lahore Board 2005]**

(3) Radius of the inscribed circle is:

(a) $r = \frac{\Delta}{S}$ (b) $r = \frac{abc}{4\Delta}$

(c) $r = \frac{S}{\Delta}$ (d) $r = \frac{S-a}{\Delta}$

**[Lahore Board 2005]**

(4) Hero's formula is used to calculate:

(a) Area of $\Delta$ (b) Sides of $\Delta$

(c) Angles of $\Delta$ (d) None of these

**[Lahore Board 2005]**

(5) If three sides of a triangle are given then area of triangle =

(a) $S(S-a)(S-b)(S-c)$ (b) $(S-a)(S-b)(S-c)$

(c) $\sqrt{(S-a)(S-b)(S-c)}$ (d) $\sqrt{S(S-a)(S-b)(S-c)}$

**[Lahore Board 2005]**

(6) $\dfrac{c+a}{c-a} =$

(a) $\dfrac{\tan\frac{\gamma+\alpha}{2}}{\tan\frac{\gamma-\alpha}{2}}$ (b) $\dfrac{\tan\frac{\alpha+\gamma}{2}}{\tan\frac{\alpha-\gamma}{2}}$

(c) $\dfrac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$ (d) None of these

**[Lahore Board 2005]**

(7) For any triangle ABC $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$ is called ———

(a) Cosine law (b) Sine law

(c) Law of tangent (d) Hero's formula

**[Gujranwala Board 2005]**

(8) By Hero's formula $\Delta$ =

(a) $S(S-a)(S-b)(S-c)$ (b) $(S-a)(S-b)(S-c)$

(c) $\sqrt{S(S-a)(S-b)(S-c)}$ (d) None of these

**[Gujranwala Board 2005]**

(9) $\cos\dfrac{\gamma}{2} =$

(a) $\sqrt{\dfrac{S(S-c)}{ab}}$ (b) $\sqrt{\dfrac{S(S-b)}{ca}}$

(c) $\sqrt{\dfrac{S(S-a)}{ab}}$ (d) None of these

**[Gujranwala Board 2005]**

(10) The in-radius r is equal to:

(a) $S\Delta$ (b) $\frac{1}{S\Delta}$

(c) $\frac{S}{\Delta}$ (d) $\frac{\Delta}{S}$

**[Gujranwala Board 2005]**

**[Lahore Board 2008-10]**

(11) Circle down inside a triangle touching its three sides is called ———

(a) Circum-circle (b) In-circle

(c) Ex-circle (d) None of these

**[Gujranwala Board 2005]**

(12) $\Delta$ =

(a) $\frac{abc}{4R}$ (b) rS

(c) $r_2(S - b)$ (d) All of these

**[Gujranwala Board 2005]**

(13) A circle passing through the vertices of a triangle is called:

(a) Circum-circle (b) In-circle

(c) Escribed circle (d) None of these

**[Gujranwala Board 2006-08]**

(14) With usual notation, the value of a – b + c is:

(a) S + b (b) S – b

(c) 2S – b (d) 2(S – b)

**[Gujranwala Board 2006]**

(15) Area of circumcircle is:

(a) $\frac{abc}{4R}$ (b) abC/R

(c) rS (d) $\frac{1}{2}$ bc sin$\alpha$

**[Gujranwala Board 2006]**

(16) Radius of escribed circle opposite to the vertex A is:

(a) $\frac{\Delta}{a}$ (b) $\frac{\Delta}{S}$

(c) $\frac{S-a}{\Delta}$ (d) $\frac{\Delta}{S-a}$

**[Lahore Board 2006, Lahore Board 2015]**

**[Gujranwala Board 2010]**

(17) $S(S-a)(S-b)(S-c) =$

(a) $\Delta$ (b) $\Delta^2$

(c) $\Delta^3$ (d) None of these

**[Lahore Board 2006]**

(18) In case of escribed circle, $\frac{\Delta}{S-C}$ is equal to:

(a) $r_1$ (b) $r_2$

(c) $r_3$ (d) None of these

**[Lahore Board 2006]**

(19) The radius of the circle passes through three vertices of a triangle is called:

(a) Circum-radius (b) In-radius

(c) Ex-radius (d) None of these

**[Lahore Board 2006]**

(20) $\frac{S-a}{\Delta} =$

(a) r (b) $r_1$

(c) $\frac{1}{r_1}$ (d) None of these

**[Lahore Board 2006]**

(21) The circum radius R is given by:

(a) $\frac{a+b+c}{4\Delta}$ (b) $\frac{a+b+c}{\Delta}$

(c) $\frac{abc}{\Delta}$ (d) $\frac{abc}{4\Delta}$

**[Lahore Board 2007]**

**[Gujranwala Board 2007]**

(22) If in a right angled triangle base = 30.8 , hypoteneuse = 37.2 then $\cos\alpha$ will be:

(a) 0.7032 (b) 0.8280

(c) 0.7513 (d) 0.8655

**[Lahore Board 2007]**

(23) The greatest angle is opposite to:

(a) Smallest side (b) Greatest side

(c) Same side (d) Right side

**[Lahore Board 2007]**

(24) In a right triangle no angle is greater than:

(a) $90°$ (b) $80°$

(c) $60°$ (d) $45°$

**[Lahore Board 2008]**

(25) If a, b, c have their usual meanings then $\frac{c^2 + a^2 - b^2}{2ac}$ is equal to:

(a) $\cos\alpha$ (b) $\cos\beta$

(c) $\cos\gamma$ (d) $\sin\beta$

**[Gujranwala Board 2009]**

(26) If $\Delta ABC$ is right angled then the law of cosines reduces to:

(a) Law of sines (b) Law of cosines

(c) Law of tangents (d) Pythagoras theorem

**[Lahore Board 2009]**

(27) With usual notations for triangle R equals:

(a) $\frac{b}{2\sin\alpha}$ (b) $\frac{a}{2\sin\beta}$

(c) $\frac{c}{2\sin\gamma}$ (d) $\frac{\Delta}{5}$

**[Lahore Board 2009]**

**[Lahore Board 2010]**

(28) The value of R = ————

(a) $\frac{\Delta}{S}$ (b) $\frac{a}{\sin\alpha}$

(c) $\frac{b}{\sin\beta}$ (d) $\frac{c}{2\sin\gamma}$

**[Lahore Board 2011]**

(29) In any triangle ABC with usual notation $\frac{a^2 + b^2 - c^2}{2ab} =$

(a) $\cos\alpha$ (b) $\cos\beta$

(c) $\cos\gamma$ (d) $\sin\gamma$

**[Lahore Board 2012]**

(30) In any triangle ABC, $a = 4$ , $b = 10$ , $\gamma = 30°$ then area of triangle is:

(a) 5 (b) 40

(c) 10 (d) 20

**[Lahore Board 2012]**

(31) Circle passing through three vertices of a triangle is called:

(a) Circumcircle (b) In-circle

(c) E-circle (d) Semicircle

**[Lahore Board 2014, Lahore Board 2013]**

(32) Radius of E-circle, $r_1$ is equal to:

(a) $\frac{\Delta}{S - a}$ (b) $\frac{\Delta}{\sqrt{S - a}}$

(c) $\frac{\Delta}{S - C}$ (d) $\frac{\Delta}{S}$

**[Lahore Board 2013]**

(33) $r_1 =$

(a) $\frac{\Delta}{S - b}$ (b) $\frac{\Delta}{S - a}$

(c) $\frac{\Delta}{S - C}$ (d) $\frac{S - a}{\Delta}$

**[Lahore Board 2013, Lahore Board 2015]**

(34) If a, b, c are sides of a triangle then $\frac{C^2 + a^2 - b^2}{2aC} =$

(a) $\cos\alpha$ (b) $\cos\beta$

(c) $\cos\gamma$ (d) $\sin\beta$

**[Lahore Board 2013]**

(35) The angle made by an object at a lower position is called:

(a) Angle of elevation (b) Angle of depression

(c) Right angle (d) None of these

**[Lahore Board 2014]**

(36) By law of cosines $\cos\alpha$ = ————

(a) $\frac{b^2 - c^2 - a^2}{2bc}$ (b) $\frac{a^2 + b^2 - c^2}{2bc}$

(c) $\frac{c^2 + a^2 - b^2}{2ca}$ (d) $\frac{b^2 + c^2 - a^2}{2bc}$

(37) A triangle can be constructed if:

(a) One side is given (b) 2 sides are given

(c) Three sides are given (d) Three angle are given

(38) If $\alpha = 35°17'$ , $\beta = 45°13'$ then $\gamma$ = ?

(a) $99°$ (b) $99°30'$

(c) $99°35'$ (d) $99°40'$

(39) If $b = 3$ , $c = 5$ and $\alpha = 120°$ then a =

(a) 5 (b) 6

(c) 7 (d) 8

(40) In escribed circles $r_3$ =

(a) $\frac{\Delta}{S}$ (b) $\frac{\Delta}{S - a}$

(c) $\frac{\Delta}{S - b}$ (d) $\frac{\Delta}{S - c}$

(41) If the measures of the sides of a triangle ABC are 17, 10, 21 then $r_2$ =

(a) 12 (b) 6

(c) 84 (d) 8

(42) In a triangle ABC, with usual notations 2R =

(a) $\frac{b}{\sin\beta}$ (b) $\frac{c}{\sin\gamma}$

(c) Both 'a' and 'b' (d) None of these

(43) The point of intersection of the right bisectors of the sides of a triangle is called:

(a) Circum-centre (b) In-centre

(c) Ex-centre (d) None of these

(44) The point of intersection of the bisectors of angles of the triangle is called:

(a) Circum-centre (b) In-centre

(c) Ex-centre (d) None of these

(45) In $\Delta ABC$, with usual notations $(S - a) \tan\frac{\alpha}{2}$ =

(a) r (b) $r_1$

(c) $r_2$ (d) $r_3$

(46) In an equilateral triangle $r_1 : R : r$ =

(a) 1 : 2 : 3 (b) 1 : 2 : 2

(c) 1 : 1 : 2 (d) 3 : 2 : 1

(47) In a triangle ABC, with usual notations if $a = 7$ , $b = 3$ , $c = 5$ , $\alpha$ =

(a) 90° (b) 120°

(c) 150° (d) 180°

(48) In a triangle ABC, with usual notations if $\alpha = 57°21'$ , $a = 36.21$ , $\beta = 78°10'$ , $b$ =

(a) 40 (b) 41.9

(c) 42.09 (d) 43.5

(49) If $\sin\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$ then $\frac{\alpha}{2}$ is:

(a) An acute angle (b) An obtuse angle

(c) A right angle (d) None of these

(50) $2\sin^2\frac{\alpha}{2}$ =

(a) $1 + \cos\alpha$ (b) $\cos^2\alpha - \sin^2\alpha$

(c) $1 - \cos\alpha$ (d) $\sin^2\alpha - \cos^2\theta$

(51) A triangle which is not right angle is called:

(a) Equilateral triangle (b) Oblique triangle

(c) Isosceles triangle (d) Acute triangle

(52) Sum of opposite angles of a cyclic quadrilateral =

(a) 90° (b) 180°

(c) 270° (d) 360°

(53) In-radius of a circle is given by:

(a) $R = \frac{a}{2\sin\alpha}$ (b) $r = \frac{\Delta}{S}$

(c) $r_1 = \frac{\Delta}{S-a}$ (d) None of these

# Answers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| *(1)* | b | *(2)* | b | *(3)* | a | *(4)* | a | *(5)* | d |
| *(6)* | a | *(7)* | b | *(8)* | c | *(9)* | a | *(10)* | d |
| *(11)* | b | *(12)* | d | *(13)* | a | *(14)* | d | *(15)* | a |
| *(16)* | d | *(17)* | b | *(18)* | c | *(19)* | a | *(20)* | c |
| *(21)* | d | *(22)* | b | *(23)* | b | *(24)* | a | *(25)* | b |
| *(26)* | d | *(27)* | c | *(28)* | d | *(29)* | c | *(30)* | c |
| *(31)* | a | *(32)* | a | *(33)* | b | *(34)* | b | *(35)* | b |
| *(36)* | d | *(37)* | d | *(38)* | b | *(39)* | c | *(40)* | d |
| *(41)* | b | *(42)* | c | *(43)* | a | *(44)* | b | *(45)* | a |
| *(46)* | d | *(47)* | b | *(48)* | c | *(49)* | a | *(50)* | c |
| *(51)* | b | *(52)* | b | *(53)* | b | | | | |

# SOLUTION

**Q.14** $a - b + c = a + b + c - b - b$

$= a + b + c - 2b$

$= 2S - 2b$

$= 2(S - b)$

**Q.15** As radius of circum-circle $= R = \frac{abc}{4\Delta}$

$\Rightarrow \quad \Delta = \frac{abc}{4R}$

**Q.20** As,

$r_1 = \frac{\Delta}{S - a}$

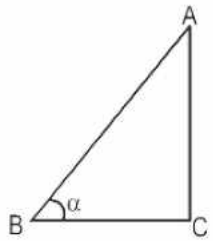
$\Rightarrow \quad \frac{1}{r_1} = \frac{S - a}{\Delta}$

**Q.22** As,

$\cos\alpha = \frac{\text{base}}{\text{hypoteneuse}}$

$= \frac{30.8}{37.2}$

$= 0.8279$




**Q.32** As

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$2bc\cos\alpha = b^2 + c^2 - a^2$

$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$

**Q.35** As,

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$= (3)^2 + (5)^2 - 2(3)(5)\cos(120°)$

$= 9 + 25 - 30\left(-\frac{1}{2}\right)$

$= 9 + 25 + 5$

$= 49$

$a = 7$

**Q.37** As, $r_2 = \dfrac{\Delta}{S-b}$

And, $S = \dfrac{a+b+c}{2} = \dfrac{17+10+21}{2} = \dfrac{48}{2} = 24$

$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$

$= \sqrt{24(7)(14)(3)} = 84$

So, $r_2 = \dfrac{\Delta}{S-b} = \dfrac{84}{24-10} = \dfrac{84}{14} = 6$

**Q.41** $(S-a)\tan\dfrac{\alpha}{2} = (S-a)\sqrt{\dfrac{(S-b)(S-c)}{S(S-a)}}$

$= \sqrt{\dfrac{(S-a)(S-b)(S-c)}{S}}$

$= \sqrt{\dfrac{S(S-a)(S-b)(S-c)}{S^2}}$

$= \dfrac{\Delta}{S} = r$

**Q.42** As triangle is equilateral.

$\Rightarrow \quad a = b = c$

$\Rightarrow \quad S = \dfrac{a+a+a}{2} = \dfrac{3a}{2}$

$\Delta = \sqrt{\dfrac{3a}{2}\left(\dfrac{3a}{2}-a\right)\left(\dfrac{3a}{2}-a\right)\left(\dfrac{3a}{2}-a\right)}$

$= \dfrac{\sqrt{3}\,a^2}{4}$

$r_1 = \dfrac{\Delta}{S-a} = \dfrac{\frac{\sqrt{3}\,a^2}{4}}{\frac{3a}{2}-a} = \dfrac{\sqrt{3}\,a^2}{4}\times\dfrac{2}{a} = \dfrac{\sqrt{3}}{2}a$

$R = \dfrac{a.a.a}{4\Delta} = \dfrac{a^3}{4\left(\frac{\sqrt{3}\,a^2}{4}\right)} = \dfrac{a}{\sqrt{3}}$

$r = \dfrac{\Delta}{S} = \dfrac{\sqrt{3}\,a^2/4}{3a/2} = \dfrac{a}{2\sqrt{3}}$

$$r_1 : R : r = \frac{\sqrt{3}\,a}{2} : \frac{a}{\sqrt{3}} : \frac{a}{2\sqrt{3}}$$

$$= \frac{\sqrt{3}\,.\,\sqrt{3}\,a}{2\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{a}{2\sqrt{3}}$$

$$= \frac{3}{2} : 1 : \frac{1}{2}$$

Multiplying by 2.

$$= 3 : 2 : 1$$

**Q.43** As,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(3)^2 + (5)^2 - (7)^2}{2(3)(5)}$$

$$= \frac{9 + 25 - 49}{30}$$

$$= \frac{-15}{30} = -\frac{1}{2}$$

$$\alpha = \cos^{-1}\left(-\frac{1}{2}\right) = 120°$$

**Q.44** As,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\Rightarrow \quad b = \frac{a\sin\beta}{\sin\alpha} = \frac{(36.21)\sin 78°10'}{\sin 57°21'} = \frac{(36.21)(0.9788)}{0.8420}$$

$$= 42.09$$