Chapter 10

TRIGONOMETRIC IDENTITIES

FUNDAMENTAL LAW OF TRIGONOMETRY

Let α, β any two angles (real number), then

$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta.$

DEDUCTIONS FROM FUNDAMENTAL LAW

$$\cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta$$

$$\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha$$

$$\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$$

DOUBLE ANGLE IDENTITIES

(i) $\sin 2\alpha = 2 \sin\alpha \cos\alpha$

(ii) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$ (Lahore Board 2005)

(iii) $$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

TRIPLE ANGLE IDENTITIES

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ (Gujranwala Board 2005)

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

HALF ANGLE IDENTITIES

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

SUM, DIFFERENCE & PRODUCT OF SINES & COSINES

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$$

and

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

$$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

where $P = \alpha+\beta$, $Q = \alpha-\beta$

EXERCISE 10.1

Q.1 Without using the table, find values of:

Solution:

(i) $\sin(-780°)$ (ii) $\cot(-855°)$ (iii) $\operatorname{cosec}(2040°)$

(iv) $\sec(-960°)$ (v) $\tan(1110°)$ (vi) $\sin(-300°)$

$$780^\circ = 8 \times 90^\circ + 60^\circ$$

$$-\sin(780^\circ) = -\sin(8 \times 90^\circ + 60^\circ)$$

$$= -\sin 60^\circ$$

$$= -\frac{\sqrt{3}}{2}$$

$$90^\circ \overline{)\ 780^\circ\ (8}$$
$$\underline{720^\circ}$$
$$60^\circ$$

(ii) −cot 855°

Solution:

$$855^\circ = 9 \times 90^\circ + 45^\circ$$

$$-\cot(855^\circ) = -\cot(9 \times 90^\circ + 45^\circ)$$

$$= -\cot 45^\circ$$

$$= -(-1)$$

$$= 1$$

$$90^\circ \overline{)\ 855^\circ\ (9}$$
$$\underline{810^\circ}$$
$$45^\circ$$

(iii) csc (2040°)

Solution:

$$2040^\circ = 22 \times 90^\circ + 60^\circ$$

Apply 'csc' both side.

$$\csc(2040^\circ) = \csc(22 \times 90^\circ + 60^\circ)$$

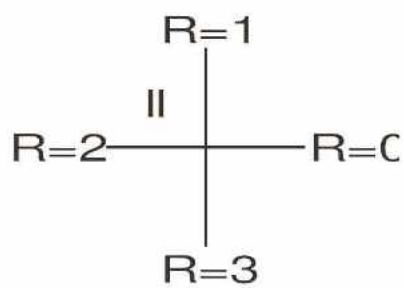
$$= -\csc 60^\circ$$

$$= -\frac{2}{\sqrt{3}}$$

$$90^\circ \overline{)\ 2040^\circ\ (22}$$
$$\underline{+1980^\circ}$$
$$60^\circ$$

For Quadrant

$$4 \overline{)\ 9\ (2}$$
$$\underline{8}$$
$$1$$



Note:

(i) When R = 0, then quad. I or IV

(ii) When R = 1, then quad. II or I

(iii) When R = 2, then quad. III or II

(iv) When R = 3, then quad. IV or III

(iv)

Solution:

$\sec(-960°) = \sec(960°)$

$960° = 10 \times 90° + 60°$

Apply 'sec' both sides.

$\sec f(780°) = \sec(10 \times 90° + 60°)$

$= -\sec 60°$

$= -2$

$90^\circ \overline{)\ 960^\circ\ (8}$

$\pm 900^\circ$

60°

$\therefore \quad \frac{960°}{90°} = 10.666$

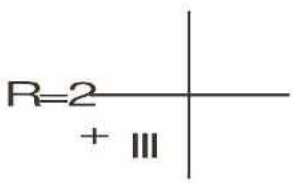
We take only 10.

For Quadrant

$4 \overline{)\ 10\ } 2$

8

$2 \rightarrow R$



(v) tan 1110°

Solution:

$1110° = 12 \times 90° + 30$

Apply 'tan' both sides.

$\tan(1110°) = \tan(12 \times 90° + 30°)$

$= \tan 30° = \frac{1}{\sqrt{3}}$

$90^\circ \overline{)\ 1110^\circ\ (12}$

$\pm 1080^\circ$

30°

$\therefore \quad \frac{1110°}{90°} = 120$

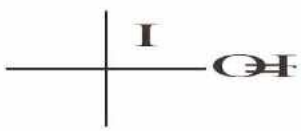
We take only 12.

For Quadrant

$4 \overline{)\ 12\ } 3$

12

$0 \rightarrow R$



(vi) $\sin(-300°) = -\sin 300°$

Solution:

$$300° = 3 \times 90° + 30°$$

Apply '−sin' both sides.

$$-\sin(300°) = -\sin(3 \times 90° + 30°)$$

$$= -\cos 30°$$

$\therefore$ Angle is in II quadrant cos is the in IV quadrant. So

$$= -\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} \quad \text{Ans.}$$

$90^\circ \overline{)\ 300^\circ\ (3}$
$\pm 270^\circ$
30°

Q.2 Express each of the following as a trigonometric function of an angle of positive degree measure of less than 45^o.

(i) $\sin 196^o$ (ii) $\cos 147^o$ (iii) $\sin 319^o$

(iv) $\cos 254^o$ (v) $\tan 294^o$ (vi) $\cos 728^o$

(vii) $\sin(-625^o)$ (viii) $\cos(-435^o)$ (ix) $\sin(150^o)$

Solution:

(i) $\sin 196^o$

$= \sin(180^o + 16^o)$

$= \sin 180^o \cos 16^o + \cos 180^o \sin 16^o$ $\quad \boxed{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$

$= 0 \times \cos 16^o + (-1)\sin 16^o$

$= -\sin 16^o$

Alternative Method:

$$\sin 196° = \sin(180° + 16°)$$

$$= \sin[2(90°) + 16°]$$

$$= -\sin 16°$$

(ii) cos 147°

$= \cos (180^\circ - 33^\circ)$ $\quad \boxed{\cos (\alpha - \beta) = \cos \beta \cos \beta + \sin \alpha \sin \beta}$

$= \cos 180^\circ \cos 33^\circ + \sin 180^\circ \sin 33^\circ$

$= -1 \times \cos 33^\circ + 0 \times \sin 33^\circ$

$= -\cos 33^\circ$

Alternative Method:

$\cos 147^\circ = \cos (180^\circ - 33^\circ)$

$= \cos [2(90^\circ) - 33^\circ]$

$= -\cos 33^\circ$

(iii) sin 319°

$= \sin (360^\circ - 41^\circ)$

$= \sin 360^\circ \cos 41^\circ - \sin 41^\circ \cos 360^\circ$

$= 0 \times \cos 41^\circ - \sin 41^\circ \times 1$

$= -\sin 41^\circ$

Alternative Method:

$\sin 319^\circ = \sin (360^\circ - 41^\circ)$

$= \sin [4(90^\circ) - 41^\circ]$

$= -\sin 41^\circ$

(iv) cos 254°

$\cos (270^\circ - 16^\circ)$

$= \cos 270^\circ \cos 16^\circ + \sin 270^\circ \sin 16^\circ$

$= 0 \times \cos 16^\circ + (-1) \times \sin 16^\circ$

$= -\sin 16^\circ$

Alternative Method:

$\cos 254^\circ = \cos (270^\circ - 16^\circ)$

$= \cos [3(90^\circ) - 16^\circ]$

$= -\sin 16^\circ$

(v) tan 294°

$\tan 294^\circ = \tan (270^\circ + 24^\circ)$

$= \tan [3(90^\circ) + 24^\circ]$

$= -\cot 24^\circ$

(vi) cos 728°

$\cos 728^\circ = \cos(720^\circ + 8^\circ)$

$= \cos [8(90^\circ) + 8^\circ]$

$= \cos 8^\circ$

(vii) **$\sin(-625^\circ)$**

$\sin(-625^\circ) = -\sin 625^\circ$

$= -\sin(630^\circ - 5^\circ)$

$= -\sin[7(90^\circ) - 5^\circ]$

$= -(-\cos 5^\circ)$

$= \cos 5^\circ$

(viii) **$\cos(-435^\circ)$**

$\cos(-435) = \cos 435^\circ$

$= \cos(450^\circ - 15^\circ)$

$= \cos[5(90^\circ) - 15^\circ]$

$= \sin 15^\circ$

(ix) **$\sin 150^\circ$**

$= \sin(180^\circ - 30^\circ)$

$= \sin 180^\circ \cos 30^\circ - \sin 30^\circ \cos 180^\circ$

$= 0 \times \cos 30^\circ - \sin 30^\circ (-1)$

$= \sin 30^\circ$

Alternative Method:

$\sin 150^\circ = \sin(180^\circ - 30^\circ)$

$= \sin[2(90^\circ) - 30^\circ]$

$= \sin 30^\circ$

Q.3 **Prove the following:**

(i) **$\sin(180^\circ + \alpha)\sin(90^\circ - \alpha) = -\sin\alpha\cos\alpha$**

(ii) **$\sin 780^\circ \sin 480^\circ + \cos 120^\circ \sin 30^\circ = \frac{1}{2}$**

(iii) **$\cos 306^\circ + \cos 234^\circ + \cos 162^\circ + \cos 18^\circ = 0$**

(iv) **$\cos 330^\circ \sin 600^\circ + \cos 120^\circ \sin 150^\circ = -1$**

Solution:

(i) **$\sin(180^\circ + \alpha)\sin(90^\circ - \alpha) = -\sin\alpha\cos\alpha$**

L.H.S. $= \sin(180^\circ + \alpha)\sin(90^\circ - \alpha)$

$= [\sin 180^\circ \cos\alpha + \cos 180^\circ \sin\alpha][\sin 90^\circ \cos\alpha - \cos 90^\circ \sin\alpha]$

$= [0 \times \cos\alpha + (-1)\sin\alpha][1 \times \cos\alpha - 0 \times \sin\alpha]$

$= (-\sin\alpha)(\cos\alpha)$

$= -\sin\alpha \cos\alpha$

$=$ R.H.S. Hence proved.

Alternative Method:

L.H.S $= \sin(180° + \alpha) \sin(90° - \alpha)$

$= \sin[2(90°) + \alpha] \sin(90° - \alpha)$

$= (-\sin\alpha)(\cos\alpha)$

$= -\sin\alpha \cos\alpha$

(ii) $\mathbf{\sin 780^o \sin 480^o + \cos 120^o \sin 30^o = \frac{1}{2}}$

L.H.S. $= \sin 780^o \sin 480^o + \cos 120^o \sin 30^o$ ..... (1)

$\text{Sin } 780^o = \sin(720° + 60°)$

$= \sin[8(90°) + 60°]$

$= \sin 60°$

$\sin 480^o = \sin(450° + 30°)$

$= \sin[5(90°) + 30°]$

$= \cos 30°$

$\cos 120° = \cos(180° - 60°)$

$= \cos[2(90°) - 60°]$

$= -\cos 60°$

L.H.S. $= \sin 780° \sin 480° + \cos 120° \sin 30°$

$= \sin 60° \cos 30° + (-\cos 60°) \sin 30°$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{3}{4} - \frac{1}{4}$

$= \frac{3-1}{4} = \frac{2}{4}$

$= \frac{1}{2} =$ R.H.S

(iii) $\mathbf{\cos 306^o + \cos 234^o + \cos 162^o + \cos 18^o = 0}$

L.H.S. $= \cos 306^o + \cos 234^o + \cos 162^o + \cos 18^o$

$= \cos(360^o - 54^o) + \cos(180^o + 54^o) + \cos(180^o - 18^o) + \cos 18^o$

$= \cos 54^o - \cos 54^o - \cos 18^o + \cos 18^o$ $\left(\begin{array}{c} \therefore \cos(2\pi - \theta) = \cos\theta \\ \cos(\pi + \theta) = -\cos\theta \\ \cos(\pi - \theta) = -\cos\theta \end{array}\right)$

$= 0$

$=$ R.H.S.

(iv) $\cos 330^\circ \sin 600^\circ + \cos 120^\circ \sin 150^\circ = -1$

L.H.S. $= \cos 330^\circ \sin 600^\circ + \cos 120^\circ \sin 150^\circ$

$= \cos (360^\circ - 30^\circ) \sin (540^\circ + 60^\circ) + \cos (180^\circ - 60^\circ) \sin (180^\circ - 30^\circ)$

$\cos[4(90°) - 30°] \sin[6(90°) + 60°] + \cos[2(90°) - 60°] \sin[2(90°) - 30°]$

$= \cos 30^\circ (-\sin 60^\circ) + (\cos 60^\circ) \sin 30^\circ$

$= \frac{-\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2}$

$= -\frac{3}{4} - \frac{1}{4} = \frac{-3-1}{4} = \frac{-4}{4} = -1 =$ R.H.S.

Q.4 Prove that

(i) $\dfrac{\sin^2 (\pi + \theta) \tan\left(\frac{3\pi}{2} + \theta\right)}{\cot^2\left(\frac{3\pi}{2} - \theta\right) \cos^2 (\pi - \theta) \operatorname{cosec} (2\pi - \theta)} = \cos \theta$

(ii) $\dfrac{\cos (90^\circ + \theta) \sec (-\theta) \tan (180^\circ - \theta)}{\sec (360^\circ - \theta) \sin (180^\circ + \theta) \cot (90^\circ - \theta)} = -1$

Solution:

(i) $\dfrac{\sin^2 (\pi + \theta) \tan\left(\frac{3\pi}{2} + \theta\right)}{\cot^2\left(\frac{3\pi}{2} - \theta\right) \cos^2 (\pi - \theta) \operatorname{cosec} (2\pi - \theta)} = \cos \theta$

L.H.S. $= \dfrac{\sin^2 (\pi + \theta) \tan\left(\frac{3\pi}{2} + \theta\right)}{\cot^2\left(\frac{3\pi}{2} - \theta\right) \cos^2 (\pi - \theta) \operatorname{cosec} (2\pi - \theta)}$

$= \dfrac{[\sin (\pi + \theta)]^2 \tan\left(\frac{3\pi}{2} + \theta\right)}{\left[\cot\left(\frac{3\pi}{2} - \theta\right)\right]^2 [\cos (\pi - \theta)]^2 \operatorname{cosec} (2\pi - \theta)}$

$= \dfrac{(-\sin \theta)^2 (-\cot \theta)}{(\tan \theta)^2 (-\cos \theta)^2 (-\operatorname{cosec} \theta)}$

$= \dfrac{\sin^2 \theta \left(\frac{-\cos \theta}{\sin \theta}\right)}{-\tan^2 \theta \cos^2 \theta \operatorname{cosec} \theta}$

$$= \frac{-\sin\theta\cos\theta}{-\frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta\cdot\frac{1}{\sin\theta}}$$

$$= \frac{\sin\theta\cos\theta}{\sin\theta}$$

$$= \cos\theta = \text{R.H.S.}$$

Hence proved.

(ii) $\dfrac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)} = -1$

$$\text{L.H.S.} = \frac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)}$$

$$= \frac{-\sin\theta\sec\theta(-\tan\theta)}{\sec\theta(-\sin\theta)\tan\theta} \qquad \left(\begin{array}{c}\therefore \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta \\ \cos(-\theta) = \cos\theta \\ \tan(\pi-\theta) = -\tan\theta\end{array}\right)$$

$$= \frac{+1}{-1} = -1 = \text{R.H.S.}$$

Q.5 If α, β, γ are the angles of a triangle ABC, then prove that

(i) $\sin(\alpha+\beta) = \sin\gamma$ **(ii)** $\cos\left(\frac{\alpha+\beta}{2}\right) = \sin\frac{\gamma}{2}$

(iii) $\cos(\alpha+\beta) = -\cos\gamma$ **(iv)** $\tan(\alpha+\beta)+\tan\gamma = 0$

Solution:

(i) $\sin(\alpha+\beta) = \sin\gamma$

For a triangle ABC we know that

$\alpha+\beta+\gamma = 180^\circ$

$\alpha+\beta = 180^\circ-\gamma$

$\sin(\alpha+\beta) = \sin(180^\circ-\gamma)$

$= \sin\gamma \qquad (\therefore \sin(\pi-\theta) = \sin\theta)$

Hence proved.

(ii) $\mathbf{\cos\left(\frac{\alpha+\beta}{2}\right) = \sin\frac{\gamma}{2}}$

Since α, β, γ are angles of triangle ABC

so $\alpha + \beta + \gamma = 180^o$

$\alpha + \beta = 180^o - \gamma$

$\Rightarrow \quad \frac{\alpha+\beta}{2} = \frac{180^o - \gamma}{2} = 90^o - \frac{\gamma}{2}$

$\cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(90^o - \frac{\gamma}{2}\right)$

$\cos\left(\frac{\alpha+\beta}{2}\right) = \sin\frac{\gamma}{2}$ Hence proved.

(iii) $\mathbf{\cos(\alpha + \beta) = -\cos\gamma}$

since α, β, γ are angles of triangle

so $\alpha + \beta + \gamma = 180^o$

$\alpha + \beta = 180^o - \gamma$

$\cos(\alpha + \beta) = \cos(180^o - \gamma)$

$\cos(\alpha + \beta) = -\cos\gamma$ Hence proved.

(iv) $\mathbf{\tan(\alpha + \beta) + \tan\gamma = 0}$

since α, β, γ are angles of triangle so

$\alpha + \beta + \gamma = 180^o$

$\alpha + \beta = 180^o - \gamma$

taking tan on both sides

$\tan(\alpha + \beta) = \tan(180^o - \gamma) = -\tan\gamma$

$\tan(\alpha + \beta) + \tan\gamma = 0$

Hence proved.

EXERCISE 10.2

Q.1 Prove that

(i) $\sin(180 + \theta) = -\sin\theta$

(ii) $\cos(180^\circ + \theta) = -\cos\theta$

(iii) $\tan(270^\circ - \theta) = \cot\theta$

(iv) $\cos(\theta - 180^\circ) = -\cos\theta$

(v) $\cos(270^\circ + \theta) = \sin\theta$ **(Lahore Board 2009)**

(vi) $\sin(\theta + 270^\circ) = -\cos\theta$

(vii) $\tan(180^\circ + \theta) = \tan\theta$

(viii) $\cos(360^\circ - \theta) = \cos\theta$

Solution:

(i) $\sin(180 + \theta) = -\sin\theta$

$$\text{L.H.S} = \sin(180^\circ + \theta) = \sin 180^\circ \cos\theta + \cos 180^\circ \sin\theta$$
$$= 0 \times \cos\theta + (-1)\sin\theta$$
$$= -\sin\theta = \text{R.H.S.}$$

Hence proved.

(ii) $\cos(180^\circ + \theta) = -\cos\theta$

$$\text{L.H.S.} = \cos(180^\circ + \theta)$$
$$= \cos 180^\circ \cos\theta - \sin 180^\circ \sin\theta$$
$$= -1 \times \cos\theta - (0) \times \sin\theta$$
$$= -\cos\theta = \text{R.H.S.}$$

Hence proved.

(iii) $\tan(270^\circ - \theta) = \cot\theta$

$$\text{L.H.S.} = \tan(270^\circ - \theta)$$
$$= \frac{\sin(270^\circ - \theta)}{\cos(270^\circ - \theta)}$$
$$= \frac{\sin 270^\circ \cos\theta - \sin\theta \cos 270^\circ}{\cos 270^\circ \cos\theta + \sin 270^\circ \sin\theta}$$
$$= \frac{-\cos\theta}{-\sin\theta} = \cot\theta = \text{R.H.S.}$$

Hence proved.

(iv) $\cos(\theta - 180^\circ) = -\cos\theta$

L.H.S. $= \cos(\theta - 180^\circ)$

$= \cos\theta \cos 180^\circ + \sin\theta \sin 180^\circ$

$= -\cos\theta$ = R.H.S.

Hence proved.

(v) $\cos(270^\circ + \theta) = \sin\theta$ **(Lahore Board 2009)**

L.H.S. $= \cos(270^\circ + \theta)$

$= \cos 270^\circ \cos\theta - \sin 270^\circ \sin\theta$

$= 0 \times (\cos\theta) - (-1) \sin\theta$

$= \sin\theta$ = R.H.S.

Hence proved.

(vi) $\sin(\theta + 270^\circ) = -\cos\theta$

L.H.S. $= \sin(\theta + 270^\circ)$

$= \sin\theta \cos 270^\circ + \cos\theta \sin 270^\circ$

$= \sin\theta \times 0 + \cos\theta (-1)$

$= -\cos\theta$ = R.H.S.

Hence proved.

(vii) $\tan(180^\circ + \theta) = \tan\theta$

L.H.S. $= \tan(180^\circ + \theta)$

$$= \frac{\tan 180^\circ + \tan\theta}{1 - \tan 180^\circ \tan\theta}$$

$$= \frac{0 + \tan\theta}{1 - 0} = \frac{\tan\theta}{1} = \tan\theta = \text{R.H.S.}$$

Hence proved.

(viii) $\cos(360^\circ - \theta) = \cos\theta$

L.H.S. $= \cos(360^\circ - \theta)$

$= \cos 360^\circ \cos\theta + \sin 360^\circ \sin\theta$

$= 1 \times \cos\theta + 0 \times \sin\theta$

$= \cos\theta$ = R.H.S.

Hence proved.

(iv) $\cos(\theta - 180^\circ) = -\cos\theta$

L.H.S. $= \cos(\theta - 180^\circ)$

$= \cos\theta \cos 180^\circ + \sin\theta \sin 180^\circ$

$= -\cos\theta = \text{R.H.S.}$

Hence proved.

(v) $\cos(270^\circ + \theta) = \sin\theta$ **(Lahore Board 2009)**

L.H.S. $= \cos(270^\circ + \theta)$

$= \cos 270^\circ \cos\theta - \sin 270^\circ \sin\theta$

$= 0 \times (\cos\theta) - (-1) \sin\theta$

$= \sin\theta = \text{R.H.S.}$

Hence proved.

(vi) $\sin(\theta + 270^\circ) = -\cos\theta$

L.H.S. $= \sin(\theta + 270^\circ)$

$= \sin\theta \cos 270^\circ + \cos\theta \sin 270^\circ$

$= \sin\theta \times 0 + \cos\theta (-1)$

$= -\cos\theta = \text{R.H.S.}$

Hence proved.

(vii) $\tan(180^\circ + \theta) = \tan\theta$

L.H.S. $= \tan(180^\circ + \theta)$

$$= \frac{\tan 180^\circ + \tan\theta}{1 - \tan 180^\circ \tan\theta}$$

$$= \frac{0 + \tan\theta}{1 - 0} = \frac{\tan\theta}{1} = \tan\theta = \text{R.H.S.}$$

Hence proved.

(viii) $\cos(360^\circ - \theta) = \cos\theta$

L.H.S. $= \cos(360^\circ - \theta)$

$= \cos 360^\circ \cos\theta + \sin 360^\circ \sin\theta$

$= 1 \times \cos\theta + 0 \times \sin\theta$

$= \cos\theta = \text{R.H.S.}$

Hence proved.

(vi) $\tan 105^\circ$

$= \tan (60^\circ + 45^\circ)$

$= \dfrac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ \tan 45^\circ}$

$= \dfrac{\sqrt{3} + 1}{1 - \sqrt{3}}$ Ans.

Q.3 Prove that

(i) $\sin (45^\circ + \alpha) = \dfrac{1}{\sqrt{2}} (\sin \alpha + \cos \alpha)$

(ii) $\cos (\alpha + 45^\circ) = \dfrac{1}{\sqrt{2}} (\cos \alpha - \sin \alpha)$

Solution:

(i) $\sin (45^\circ + \alpha) = \dfrac{1}{\sqrt{2}} (\sin \alpha + \cos \alpha)$

L.H.S. $= \sin (45^\circ + \alpha) = \sin 45^\circ \cos \alpha + \cos 45^\circ \sin \alpha$

$= \dfrac{1}{\sqrt{2}} \cos \alpha + \dfrac{1}{\sqrt{2}} \sin \alpha$

$= \dfrac{1}{\sqrt{2}} (\cos \alpha + \sin \alpha) =$ R.H.S.

Hence proved.

(ii) $\cos (\alpha + 45^\circ) = \dfrac{1}{\sqrt{2}} (\cos \alpha - \sin \alpha)$

L.H.S. $= \cos (\alpha + 45^\circ)$

$= \cos \alpha \cos 45^\circ - \sin \alpha \sin 45^\circ$

$= \dfrac{1}{\sqrt{2}} \cos \alpha - \dfrac{1}{\sqrt{2}} \sin \alpha$

$= \dfrac{1}{\sqrt{2}} (\cos \alpha - \sin \alpha) =$ R.H.S. Hence proved.

Q.4 Prove that

(i) $\tan (45^\circ + A) \tan (45^\circ - A) = 1$

(ii) $\tan \left(\dfrac{\pi}{4} - \theta \right) + \tan \left(\dfrac{3\pi}{4} + \theta \right) = 0$

(iii) $\sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

(iv) $\dfrac{\sin\theta - \cos\theta \cdot \tan\frac{\theta}{2}}{\cos\theta + \sin\theta \cdot \tan\frac{\theta}{2}} = \tan\frac{\theta}{2}$

(v) $\dfrac{1 - \tan\theta\tan\phi}{1 + \tan\theta\tan\phi} = \dfrac{\cos(\theta + \phi)}{\cos(\theta - \phi)}$

Solution:

(i) $\tan(45^\circ + A)\tan(45^\circ - A) = 1$

L.H.S. $= \tan(45^\circ + A)\tan(45^\circ - A)$

$$= \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} \cdot \frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \tan A}$$

$$= \frac{1 + \tan A}{1 - \tan A} \cdot \frac{1 - \tan A}{1 + \tan A}$$

$= 1 =$ R.H.S. Hence proved.

(ii) $\tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right) = 0$

L.H.S. $= \tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right)$

$$= \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta} + \frac{\tan\frac{3\pi}{4} + \tan\theta}{1 - \tan\frac{3\pi}{4}\tan\theta}$$

$$= \frac{1 - \tan\theta}{1 + \tan\theta} + \frac{-1 + \tan\theta}{1 + \tan\theta}$$

$$= \frac{1 - \tan\theta - 1 + \tan\theta}{1 + \tan\theta}$$

$$= \frac{0}{1 + \tan\theta} = 0 = \text{R.H.S.}$$

Hence proved.

(iii) $\sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

L.H.S. $= \sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right)$

$$= \left[\sin\theta \cos\frac{\pi}{6} + \cos\theta \sin\frac{\pi}{6} \right] + \left[\cos\theta \cos\frac{\pi}{3} - \sin\theta \sin\frac{\pi}{3} \right]$$

$$= \left(\sin\theta \frac{\sqrt{3}}{2} + \cos\theta \frac{1}{2} \right) + \left(\cos\theta \frac{1}{2} - \sin\theta \frac{\sqrt{3}}{2} \right)$$

$$= \frac{\sqrt{3}}{2} \sin\theta + \frac{1}{2} \cos\theta + \frac{1}{2} \cos\theta - \frac{\sqrt{3}}{2} \sin\theta$$

$$= \cos\theta = \text{R.H.S.}$$

Hence proved.

(iv) $$\mathbf{\frac{\sin\theta - \cos\theta \,.\, \tan\frac{\theta}{2}}{\cos\theta + \sin\theta \,.\, \tan\frac{\theta}{2}} = \tan\frac{\theta}{2}}$$

$$\text{R.H.S.} = \tan\frac{\theta}{2}$$

$$= \tan\left(\theta - \frac{\theta}{2} \right)$$

$$= \frac{\tan\theta - \tan\frac{\theta}{2}}{1 + \tan\theta \tan\frac{\theta}{2}} = \frac{\frac{\sin\theta}{\cos\theta} - \tan\frac{\theta}{2}}{1 + \frac{\sin\theta}{\cos\theta} \tan\frac{\theta}{2}}$$

$$= \frac{\sin\theta - \cos\theta \tan\frac{\theta}{2}}{\cos\theta + \sin\tan\frac{\theta}{2}} = \text{L.H.S.}$$

Hence proved.

Alternative Method:

$$\text{L.H.S} = \frac{\sin\theta - \cos\theta \tan\theta/2}{\cos\theta + \sin\theta \tan\theta/2}$$

$$= \frac{\sin\theta - \cos\theta \frac{\sin\theta/2}{\cos\theta/2}}{\cos\theta + \sin\theta \frac{\sin\theta/2}{\cos\theta/2}}$$

$$= \frac{\sin\theta \cos\theta/2 - \cos\theta \sin\theta/2}{\cos\theta \cos\theta/2 + \sin\theta \sin\theta/2}$$

$$= \frac{\sin(\theta - \theta/2)}{\cos(\theta - \theta/2)} = \frac{\sin\theta/2}{\cos\theta/2}$$

$$= \tan\theta/2$$

$$= \text{R.H.S}$$

(v) $$\mathbf{\frac{1 - \tan\theta \tan\phi}{1 + \tan\theta \tan\phi} = \frac{\cos(\theta + \phi)}{\cos(\theta - \phi)}}$$

$$\text{L.H.S.} = \frac{\cos(\theta + \phi)}{\cos(\theta - \phi)}$$

$$= \frac{\cos\theta \cos\phi - \sin\theta \sin\phi}{\cos\theta \cos\phi + \sin\theta \sin\phi}$$

Alternative Method:

$$\text{L.H.S} = \frac{1 - \tan\theta \tan\theta}{1 + \tan\theta \tan\theta}$$

Dividing up & down by $\cos\theta \cos\phi$

$$= \frac{\frac{\cos\theta\cos\phi}{\cos\phi\cos\theta} - \frac{\sin\theta\sin\phi}{\cos\theta\cos\phi}}{\frac{\cos\theta\cos\phi}{\cos\theta\cos\phi} + \frac{\sin\theta\sin\phi}{\cos\theta\cos\phi}}$$

$$= \frac{1 - \tan\theta\tan\phi}{1 + \tan\theta\tan\phi}$$

= L.H.S. Hence proved.

$$= \frac{1 - \frac{\sin\theta}{\cos\theta}\frac{\sin\theta}{\cos\theta}}{1 + \frac{\sin\theta}{\cos\theta}\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\frac{\cos\theta\cos\theta - \sin\theta\sin\theta}{\cos\theta\cos\theta}}{\frac{\cos\theta\cos\theta + \sin\theta\sin\theta}{\cos\theta\cos\theta}}$$

$$= \frac{\cos(\theta + \phi)}{\cos(\theta - \phi)}$$

= R.H.S

Q.5 Show that

$\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$

Solution:

L.H.S. $= \cos(\alpha + \beta)\cos(\alpha - \beta)$

$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= (\cos\alpha\cos\beta)^2 - (\sin\alpha\sin\beta)^2$

$= \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta$

$= \cos^2\alpha(1 - \sin^2\beta) - (1 - \cos^2\alpha)\sin^2\beta$

$= \cos^2\alpha - \cos^2\alpha\sin^2\beta - \sin^2\beta + \cos^2\alpha\sin^2\beta$

$= \cos^2\alpha - \sin^2\beta$ = R.H.S.

Again,

L.H.S. $= \cos(\alpha + \beta)\cos(\alpha - \beta)$

$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha in\beta)$

$= (\cos\alpha\cos\beta)^2 - (\sin\alpha\sin\beta)^2$

$= \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta$

$= (1 - \sin^2\alpha)\cos^2\beta - \sin^2\alpha(1 - \cos^2\beta)$

$= \cos^2\beta - \sin^2\alpha\cos^2\beta - \sin^2\alpha + \sin^2\alpha\cos^2\beta$

$= \cos^2\beta - \sin^2\alpha$ = R.H.S.

Q.6 Show that $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha$

Solution:

$$\text{L.H.S.} = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}$$

$$= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\sin\beta\cos\alpha}{\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$

$$= \frac{2\sin\alpha\cos\beta}{2\cos\alpha\cos\beta} = \tan\alpha = \text{R.H.S}$$

Hence proved.

Q.7 Show that

(i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

(ii) $\cot(\alpha-\beta) = \dfrac{\cos\alpha\cos\beta+1}{\cos\beta-\cos\alpha}$

(iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

Solution:

(i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

$$\text{L.H.S.} = \cot(\alpha+\beta) = \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}$$

$$= \frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$

Dividing up & down by $\sin\alpha\sin\beta$

$$= \frac{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}+\dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$= \frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$$

$$= \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

= R.H.S. Hence proved

Alternative Method:

$$\text{R.H.S} = \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

$$= \frac{\dfrac{\cos\alpha}{\sin\alpha}\dfrac{\cos\beta}{\sin\beta}-1}{\dfrac{\cos\alpha}{\sin\alpha}+\dfrac{\cos\beta}{\sin\beta}}$$

$$= \frac{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\cos\alpha\sin\beta+\sin\alpha\cos\beta}{\sin\alpha\sin\beta}}$$

$$= \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \cot(\alpha+\beta) = \text{L.H.S}$$

Q.6 Show that $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha$

Solution:

$$\text{L.H.S.} = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}$$

$$= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\sin\beta\cos\alpha}{\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$

$$= \frac{2\sin\alpha\cos\beta}{2\cos\alpha\cos\beta} = \tan\alpha = \text{R.H.S}$$

Hence proved.

Q.7 Show that

(i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

(ii) $\cot(\alpha-\beta) = \dfrac{\cos\alpha\cos\beta+1}{\cos\beta-\cos\alpha}$

(iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

Solution:

(i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

$$\text{L.H.S.} = \cot(\alpha+\beta) = \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}$$

$$= \frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$

Dividing up & down by $\sin\alpha\sin\beta$

$$= \frac{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}+\dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$= \frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$$

$$= \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

= R.H.S. Hence proved

Alternative Method:

$$\text{R.H.S} = \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

$$= \frac{\dfrac{\cos\alpha}{\sin\alpha}\dfrac{\cos\beta}{\sin\beta}-1}{\dfrac{\cos\alpha}{\sin\alpha}+\dfrac{\cos\beta}{\sin\beta}}$$

$$= \frac{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\cos\alpha\sin\beta+\sin\alpha\cos\beta}{\sin\alpha\sin\beta}}$$

$$= \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \cot(\alpha+\beta) = \text{L.H.S}$$

Q.6 Show that $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha$

Solution:

$$\text{L.H.S.} = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}$$

$$= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\sin\beta\cos\alpha}{\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$

$$= \frac{2\sin\alpha\cos\beta}{2\cos\alpha\cos\beta} = \tan\alpha = \text{R.H.S}$$

Hence proved.

Q.7 Show that

(i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

(ii) $\cot(\alpha-\beta) = \dfrac{\cos\alpha\cos\beta+1}{\cos\beta-\cos\alpha}$

(iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

Solution:

(i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

$$\text{L.H.S.} = \cot(\alpha+\beta) = \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}$$

$$= \frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$

Dividing up & down by $\sin\alpha\sin\beta$

$$= \frac{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}+\dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$= \frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$$

$$= \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

$$= \text{R.H.S. Hence proved}$$

Alternative Method:

$$\text{R.H.S} = \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

$$= \frac{\dfrac{\cos\alpha}{\sin\alpha}\dfrac{\cos\beta}{\sin\beta}-1}{\dfrac{\cos\alpha}{\sin\alpha}+\dfrac{\cos\beta}{\sin\beta}}$$

$$= \frac{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\cos\alpha\sin\beta+\sin\alpha\cos\beta}{\sin\alpha\sin\beta}}$$

$$= \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \cot(\alpha+\beta) = \text{L.H.S}$$

Q.9 If $\sin\alpha = \frac{4}{5}$ and $\sin\beta = \frac{12}{13}$ where $\frac{\pi}{2} < \alpha < \pi$ & $\frac{\pi}{2} < \beta < \pi$. Find

(i) $\sin(\alpha+\beta)$ **(ii)** $\cos(\alpha+\beta)$ **(iii)** $\tan(\alpha+\beta)$

(iv) $\sin(\alpha-\beta)$ **(v)** $\cos(\alpha-\beta)$ **(vi)** $\tan(\alpha-\beta)$

Solution:

(i) $\sin(\alpha+\beta)$

$\sin\alpha = \frac{4}{5}$ $(\frac{\pi}{2} < \alpha < \pi \Rightarrow \alpha$ is in II Quadrant)

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

$\cos\alpha = \frac{-3}{5}$ ($\because$ α is in II Quadrant)

$\sin\beta = \frac{12}{13}$

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}} = \pm\frac{5}{13}$$

$\cos\beta = \frac{-5}{13}$ (as β lie in II Quadrant).

we know $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{4}{5}\times\frac{-5}{13} + \frac{-3}{5}\times\frac{12}{13} = \frac{-4}{13} - \frac{36}{65} = \frac{-20-36}{65} = -\frac{56}{65}$$ Ans.

(ii) $\cos(\alpha+\beta)$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= -\frac{3}{5}\times\frac{-5}{13} - \frac{4}{5}\times\frac{12}{13} = \frac{15}{65} - \frac{48}{65} = \frac{15-48}{65} = \frac{-33}{65}$$ Ans.

(iii) $\tan(\alpha+\beta)$

$$= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\frac{-56}{65}}{\frac{-33}{65}} = \frac{56}{33}$$ Ans.

(iv) $\sin(\alpha-\beta)$

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \frac{4}{5}\times\frac{-5}{13} - \frac{-3}{5}\times\frac{12}{13} = \frac{-20}{65} + \frac{36}{65} = \frac{-20+36}{65} = \frac{16}{65}$$ Ans.

(v) $\cos(\alpha - \beta)$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \frac{-3}{5} \times \frac{-5}{13} + \frac{4}{5} \times \frac{12}{13} = \frac{15}{65} + \frac{48}{65}$

$\cos(\alpha - \beta) = \frac{63}{65}$ Ans.

(vi) $\tan(\alpha - \beta)$

$= \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\frac{16}{65}}{\frac{63}{65}}$

$\tan(\alpha - \beta) = \frac{16}{63}$ Ans.

Since $\sin(\alpha - \beta)$, $\cos(\alpha - \beta)$, $\tan(\alpha - \beta)$ are all +ve, so terminal side of $(\alpha - \beta)$ is in I Quadrant.

Now $\sin(\alpha + \beta)$ is – ve, $\cos(\alpha + \beta)$ is – ve and $\tan(\alpha + \beta)$ is +ve. Thus terminal side of $\alpha + \beta$ is in III Quadrant.

Q.10 Find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ given that

(i) **$\tan\alpha = \frac{3}{4}$, $\cos\beta = \frac{5}{13}$ and neither the terminal side of angle of measure α nor that of β in I quadrant.** **(Lahore Board 2005)**

(ii) **$\tan\alpha = -\frac{15}{18}$, $\sin\beta = -\frac{7}{25}$ and neither the terminal side of the angle of measure α nor β is in the IV quadrant.**

Solution:

(i) **$\tan\alpha = \frac{3}{4}$**

$\tan\alpha = \frac{3}{4} \Rightarrow \cot\alpha = \frac{4}{3}$

i.e. is + ve, so terminal side in the III Quadrant

we know that

$\sec\alpha = \sqrt{1 + \tan^2\alpha} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{16+9}{16}} = \sqrt{\frac{25}{16}} = \pm\frac{5}{4}$

$\sec\alpha = \frac{-5}{4} \Rightarrow \cos\alpha = \frac{-5}{4}$

$$\cos ec\ \alpha = \sqrt{1 + \cot^2 \alpha} = \sqrt{1 + \frac{16}{9}} = \sqrt{\frac{9+16}{9}} = \sqrt{\frac{25}{9}} = \pm\frac{5}{3}$$

$$\text{cosec}\ \alpha = \frac{-5}{3} \Rightarrow \sin\alpha = \frac{-3}{5}, \quad \sin\beta = \sqrt{1 - \frac{25}{169}} = \pm\frac{12}{13} \quad \sin\beta = -\frac{12}{13}$$

Now

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{-3}{5} \times \frac{5}{13} + \frac{-4}{5} \times \frac{-12}{13}$$

$$= \frac{-3}{13} + \frac{48}{65} = \frac{-15+48}{65}$$

$$\boxed{\sin(\alpha + \beta) = \frac{33}{65}} \quad \text{Ans.}$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{-4}{5} \times \frac{5}{13} - \frac{-3}{5} \times \frac{-12}{3}$$

$$= \frac{-4}{13} - \frac{36}{65} = \frac{-20-36}{65}$$

$$\boxed{\cos(\alpha + \beta) = \frac{-56}{65}} \quad \text{Ans.}$$

(ii) **$\tan\alpha = -\frac{15}{18}$, $\sin\beta = -\frac{7}{25}$ and neither the terminal side of the angle of measure α nor β is in the IV quadrant.**

$$\tan\alpha = \frac{-15}{8} \Rightarrow \cot\alpha = \frac{-8}{15}$$

since tan is – ve. Thus terminal side in the II Quadrant

$$\sec\alpha = \sqrt{1 + \tan^2\alpha} = \sqrt{1 + \frac{225}{64}} = \sqrt{\frac{64+225}{64}} = \sqrt{\frac{289}{64}} = \pm\frac{17}{8}$$

$$\sec\alpha = \frac{-17}{8} \Rightarrow \cos\alpha = \frac{-8}{17}$$

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - \frac{64}{289}} = \sqrt{\frac{289-64}{289}} = \sqrt{\frac{225}{289}} = \pm\frac{15}{17}$$

$$\sin\alpha = \frac{15}{17}$$

$$\text{cos ec } \alpha = \sqrt{1 + \cot^2 \alpha} = \sqrt{1 + \frac{16}{9}} = \sqrt{\frac{9+16}{9}} = \sqrt{\frac{25}{9}} = \pm\frac{5}{3}$$

$$\text{cosec } \alpha = \frac{-5}{3} \Rightarrow \sin \alpha = \frac{-3}{5} \quad , \quad \sin \beta = \sqrt{1 - \frac{25}{169}} = \pm\frac{12}{13} \quad \sin \beta = -\frac{12}{13}$$

Now

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$= \frac{-3}{5} \times \frac{5}{13} + \frac{-4}{5} \times \frac{-12}{13}$$

$$= \frac{-3}{13} + \frac{48}{65} = \frac{-15+48}{65}$$

$$\boxed{\sin (\alpha + \beta) = \frac{33}{65}} \quad \text{Ans.}$$

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \frac{-4}{5} \times \frac{5}{13} - \frac{-3}{5} \times \frac{-12}{3}$$

$$= \frac{-4}{13} - \frac{36}{65} = \frac{-20-36}{65}$$

$$\boxed{\cos (\alpha + \beta) = \frac{-56}{65}} \quad \text{Ans.}$$

(ii) **$\tan \alpha = -\frac{15}{18}$, $\sin \beta = -\frac{7}{25}$ and neither the terminal side of the angle of measure α nor β is in the IV quadrant.**

$$\tan \alpha = \frac{-15}{8} \Rightarrow \cot \alpha = \frac{-8}{15}$$

since tan is – ve. Thus terminal side in the II Quadrant

$$\sec \alpha = \sqrt{1 + \tan^2 \alpha} = \sqrt{1 + \frac{225}{64}} = \sqrt{\frac{64+225}{64}} = \sqrt{\frac{289}{64}} = \pm\frac{17}{8}$$

$$\sec \alpha = \frac{-17}{8} \Rightarrow \cos \alpha = \frac{-8}{17}$$

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \frac{64}{289}} = \sqrt{\frac{289-64}{289}} = \sqrt{\frac{225}{289}} = \pm\frac{15}{17}$$

$$\sin \alpha = \frac{15}{17}$$

Q.12 If α, β, γ are the angles of a triangle ABC, show that

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cdot\cot\frac{\gamma}{2}$$ **(Lahore Board 2004)**

Solution:

Since α, β, γ are the angles of a triangle ABC

$$\alpha + \beta + \gamma = 180^\circ$$

$$\frac{\alpha + \beta + \gamma}{2} = 90^\circ$$

$$\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90^\circ$$

$$\frac{\alpha}{2} + \frac{\beta}{2} = 90^\circ - \frac{\gamma}{2} \qquad \ldots\ldots\ldots (1)$$

taking tan on both sides

$$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \tan\left(90^\circ - \frac{\gamma}{2}\right)$$

$$\frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \cot\frac{\gamma}{2}$$

$$\frac{\frac{1}{\cot\frac{\alpha}{2}} + \frac{1}{\cot\frac{\beta}{2}}}{1 - \cot\frac{\alpha}{2}\cdot\cot\frac{\beta}{2}} = \cot\frac{\gamma}{2}$$

$$\frac{\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}}{\cot\frac{\alpha}{2}\cot\frac{\beta}{2} - 1} = \cot\frac{\gamma}{2}$$

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} = \cot\frac{\gamma}{2}\left(\cot\frac{\alpha}{2}\cot\frac{\beta}{2} - 1\right)$$

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2} - \cot\frac{\gamma}{2}$$

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

Hence proved.

Q.13 If $\alpha + \beta + \gamma = 180^o$, show that

$\cot\alpha \cot\beta + \cot\beta \cot\gamma + \cot\gamma \cot\alpha = 1$ **(Gujranwala Board 2005)**

Solution:

$\alpha + \beta + \gamma = 180^o$

$\alpha + \beta = 180^o - \gamma$

taking tan on both sides

$\tan(\alpha + \beta) = \tan(180^o - \gamma)$

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = -\tan\gamma$$

$$\frac{\frac{1}{\cot\alpha} + \frac{1}{\cot\beta}}{1 - \frac{1}{\cot\alpha}\frac{1}{\cot\beta}} = -\frac{1}{\cot\gamma}$$

$$\frac{\cot\alpha + \cot\beta}{\cot\alpha \cot\beta - 1} = \frac{-1}{\cot\gamma}$$

$\cot\gamma(\cot\alpha + \cot\beta) = -(\cot\alpha \cot\beta - 1)$

$\cot\alpha \cot\gamma + \cot\beta \cot\gamma = -\cot\alpha \cot\beta + 1$

$\cot\alpha \cot\gamma + \cot\beta \cot\gamma + \cot\alpha \cot\beta = 1$

Hence proved.

Q.14 Express the following in form $\gamma \sin(\theta + \phi)$ where terminal sides of the angles of measure θ and ϕ are in first quadrant

(i) **$12 \sin\theta + 5 \cos\theta$** **(ii)** **$3 \sin\theta - 4 \cos\theta$**

(iii) **$\sin\theta - \cos\theta$**

(iv) **$5 \sin\theta - 4 \cos\theta$** **(Gujranwala Board 2007)**

(v) **$\sin\theta + \cos\theta$** **(vi)** **$3 \sin\theta - 5 \cos\theta$**

Solution:

(i) **$12 \sin\theta + 5 \cos\theta$**

Since $r \sin(\theta + \phi) = r(\sin\theta \cos\phi + \cos\theta \sin\phi)$

$= r \sin\theta \cos\phi + r \cos\theta \sin\phi$

Let $r \cos\phi = 12$, $r \sin\phi = 5$

Squaring & adding

$r^2 \cos^2\phi + r^2 \sin^2\phi = 144 + 25$

$r^2 (\cos^2 \phi + \sin^2 \phi) = 169$

$r^2 = 169 \quad \Rightarrow \quad r = 13$

$\tan \phi = \dfrac{\sin \phi}{\cos \phi} = \dfrac{5}{12}$

$12 \sin\theta + 5 \cos\theta = \dfrac{13}{13}(12 \sin\theta + 5 \cos\theta)$

$= 13\left(\dfrac{12}{13}\sin\theta + \dfrac{5}{13}\cos\theta\right)$

$= r(\cos\phi \sin\theta + \sin\phi \cos\theta)$

$= r \sin(\theta + \phi)$

$= \boxed{\text{where } \phi = \tan^{-1}\dfrac{5}{12}}$ Ans.

(ii) 3 sin θ – 4 cos θ

$r \sin(\theta - \phi) = r \sin\theta \cos\phi - r \cos\theta \sin\phi$

Let $r \cos\phi = 3$ ..... (i)

$r \sin\phi = +4$ ..... (ii)

Taking square & adding

$r^2 \cos^2 \phi + r^2 \sin^2 \phi = (3)^2 + (+4)^2$

$r^2 (\cos\phi + \sin\phi)^2 = 9 + 16$

$r^2 = 25 \quad \Rightarrow \quad r = 5$

dividing (ii) by (i)

$\dfrac{r \sin\phi}{r \cos\phi} = \dfrac{+4}{3}$

$\tan\phi = \dfrac{4}{3}$

$3 \sin\theta - 4\cos\theta = \dfrac{5}{5}(3\sin\theta - 4\cos\theta)$

$= 5\left(\dfrac{3}{5}\sin\theta - \dfrac{4}{5}\cos\theta\right)$

$= 5(\cos\phi \sin\theta - \sin\phi \cos\theta)$

$= r \sin(\theta - \theta)$

(iii) **$\sin\theta - \cos\theta$**

$r\sin(\theta-\phi) = r\sin\theta\cos\phi - r\cos\theta\sin\phi$

Let $r\cos\phi = 1$ ..... (i)

$r\sin\phi = 1$ ..... (ii)

Taking square of (i) & (ii) & adding

$r^2(\cos^2\phi + \sin^2\phi) = 1 + 1$

$r^2 = 2$

$r = \sqrt{2}$

Dividing (ii) by (i) we have

$$\frac{r\sin\phi}{r\cos\phi} = 1$$

$\tan\phi = 1$

$\boxed{\sqrt{2}\ \sin(\theta-\phi)\ ,\ \tan\phi = 1}$ Ans.

(iv) **$5\sin\theta - 4\cos\theta$**

$r\sin(\theta-\phi) = r\sin\theta\cos\phi - r\sin\phi\cos\theta$

Let $r\cos\phi = 5$ ..... (i)

$r\sin\phi = 4$ ..... (ii)

Taking square & adding

$r^2(\cos^2\phi + \sin^2\phi) = (5)^2 + (4)^2$

$r^2 = 41$

$r = \sqrt{41}$

Dividing (ii) by (i)

$$\frac{r\sin\phi}{r\cos\phi} = \frac{4}{5}$$

$$\tan\phi = \frac{4}{5}$$

$$\boxed{\sqrt{41}\ \sin(\theta-\phi)\ ,\ \tan\phi = \frac{4}{5}}$$ Ans.

(v) **$\sin\theta + \cos\theta$** **(Gujranwala Board 2007)**

$r\sin(\theta+\phi) = r\sin\theta\cos\phi + r\cos\theta\sin\phi$

Let $r\cos\phi = 1$ ..... (i)

$r\sin\phi = 1$ ..... (ii)

Taking square of (i) & (ii) & adding

$r^2 (\cos^2 \phi + \sin^2 \phi) = 1 + 1$

$r^2 = 2 \Rightarrow r = \sqrt{2}$

Dividing (ii) by (i)

$\frac{r \sin \phi}{r \cos \phi} = 1$

$\tan \phi = 1$

$\boxed{r \sin (\theta + \phi) = \sqrt{2} \sin (\theta + \phi) \ , \ \tan \phi = 1}$ Ans.

(vi) $3 \sin \theta - 5 \cos \theta$

$r \sin (\theta - \phi) = r \sin \theta \cos \phi - r \cos \theta \sin \phi$

Let $r \cos \phi = 3$ ..... (i)

$r \sin \phi = 5$ ..... (ii)

Taking square & adding of (i) and (ii)

$r^2 (\cos^2 \phi + \sin^2 \phi) = (5)^2 + (3)^2$

$r^2 = 25 + 9$

$r^2 = 34 \Rightarrow r = \sqrt{34}$

$\tan \phi = \frac{5}{3}$ (dividing (ii) by (i)

$\boxed{r \sin (\theta - \phi) = \sqrt{34} \sin (\theta - \phi) \ , \ \tan \phi = \frac{5}{3}}$ Ans.

EXERCISE 10.3

Q.1 Find the values of $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$ when

(i) $\sin \alpha = \frac{12}{13}$ **(ii)** $\tan \alpha = \frac{12}{13}$

(iii) $\cos \alpha = \frac{3}{5}$ **where** $0 < \alpha < \frac{\pi}{2}$

Solution

(i) $\sin \alpha = \frac{12}{13} \quad 0 < \alpha < \frac{\pi}{2}$

$\sin \alpha = \frac{12}{13}$

$\cos \alpha = \sqrt{1 - \sin^2 \alpha} = \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{169 - 144}{169}} = \sqrt{\frac{25}{169}}$

$\cos \alpha = \frac{5}{13}$ (since α in I Quadrant)

(i) $\sin 2\alpha = 2 \sin \alpha \cos \alpha = 2 \times \frac{12}{13} \times \frac{5}{13}$

$$\boxed{\sin 2\alpha = \frac{120}{169}}$$ Ans.

(ii) $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

$= (\cos \alpha)^2 - (\sin \alpha)^2$

$= \frac{25}{169} - \frac{144}{169} = \frac{25 - 144}{169} = \frac{-119}{169}$ Ans.

$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$

$= \frac{120}{169} \div \frac{-119}{169} = \frac{120}{169} \times \frac{169}{-119}$

$$\boxed{\tan 2\alpha = \frac{-120}{119}}$$ Ans.

(ii) $\mathbf{\cos \alpha = \frac{3}{5}}$

$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{25 - 9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\sin \alpha = \frac{4}{5}$

$\sin 2\alpha = 2 \sin \alpha \cos \alpha$

$\sin 2\alpha = 2 \times \frac{4}{5} \times \frac{3}{5} = \frac{24}{25}$ Ans.

$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

$\cos 2\alpha = \frac{9}{25} - \frac{16}{25} = \frac{9 - 16}{25} = \frac{-7}{25}$ Ans.

$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{24}{25} \div \frac{-7}{25} = \frac{24}{25} \times \frac{-25}{7} = \frac{-24}{7}$ Ans.

Prove the following identities

Q.2 $\cot\alpha - \tan\alpha = 2\cot 2\alpha$.

Solution:

$$\text{R.H.S.} = 2\cot 2\alpha$$

$$= 2\frac{\cos 2\alpha}{\sin 2\alpha} = \frac{2(\cos^2\alpha - \sin^2\alpha)}{2\sin\alpha\cos\alpha}$$

$$= \frac{\cos^2\alpha}{\sin\alpha\cos\alpha} - \frac{\sin^2\alpha}{\sin\alpha\cos\alpha}$$

$$= \cot\alpha - \tan\alpha = \text{L.H.S.}$$

Hence proved.

Alternative Method:

$$\text{L.H.S} = \cot\alpha - \tan\alpha$$

$$= \frac{\cos\alpha}{\sin\alpha} - \frac{\sin\alpha}{\cos\alpha}$$

$$= \frac{\cos^2\alpha - \sin^2\alpha}{\sin\alpha\cos\alpha}$$

$$= \frac{2\cos 2\alpha}{2\sin\alpha\cos\alpha} = \frac{2\cos 2\alpha}{\sin 2\alpha}$$

$$= 2\cot 2\alpha$$

$$= \text{R.H.S}$$

Q.3 $\dfrac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha$ **(Lahore Board 2007, 2010)**

Solution:

$$\text{L.H.S.} = \frac{\sin 2\alpha}{1+\cos 2\alpha} = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha + \sin^2\alpha + \cos^2\alpha - \sin^2\alpha}$$

$$= \frac{2\sin\alpha\cos\alpha}{2\cos^2\alpha} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \text{R.H.S.}$$

Q.4 $\dfrac{1-\cos\alpha}{\sin\alpha} = \tan\dfrac{\alpha}{2}$.

Solution:

$$\text{L.H.S.} = \frac{1-\cos\alpha}{\sin\alpha}$$

$$= \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2} = \text{R.H.S.}$$

Hence proved.

Q.5 $\dfrac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \sec 2\alpha - \tan 2\alpha.$

Solution:

$$\text{R.H.S.} = \sec 2\alpha - \tan 2\alpha$$

$$= \frac{1}{\cos 2\alpha} - \frac{\sin 2\alpha}{\cos 2\alpha}$$

$$= \frac{1 - \sin 2\alpha}{\cos 2\alpha} = \frac{1 - (2\sin\alpha\cos\alpha)}{\cos^2\alpha - \sin^2\alpha}$$

$$= \frac{\cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha}$$

$$= \frac{(\cos\alpha - \sin\alpha)^2}{(\cos\alpha - \sin\alpha)(\cos\alpha + \sin\alpha)}$$

$$= \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \text{L.H.S. Hence proved.}$$

Q.6 $\sqrt{\dfrac{1 + \sin\alpha}{1 - \sin\alpha}} = \dfrac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}}$

Solution:

$$\text{L.H.S.} = \sqrt{\frac{1 + \sin\alpha}{1 - \sin\alpha}}$$

$\because \sin 2\alpha = 2\sin\alpha\cos\alpha$

$\sin\alpha = 2\sin\alpha/2\cos\alpha/2$

$\because \sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} = 1$

$$= \sqrt{\frac{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}}$$

$$= \sqrt{\frac{\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)^2}{\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)^2}}$$

$$= \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}} \text{ R.H.S. Hence proved.}$$

Q.7 $\dfrac{\text{cosec } \theta + 2 \text{ cosec } 2\theta}{\sec \theta} = \cot \dfrac{\theta}{2}$

Solution:

$$\text{L.H.S.} = \frac{\text{cosec } \theta + 2 \text{ cosec } 2\theta}{\sec \theta}$$

$$= \left(\frac{1}{\sin \theta} + 2 \frac{1}{\sin 2\theta} \right) \cos \theta$$

$$= \left(\frac{1}{\sin \theta} + \frac{2}{2 \sin \cos \theta} \right) \cos \theta$$

$$= \left(\frac{\cos \theta + 1}{\sin \theta \cos \theta} \right) \cos \theta$$

$$= \frac{2 \cos^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \cot \frac{\theta}{2} = \text{R.H.S.}$$

Hence proved.

Q.8 $1 + \tan \alpha \tan 2\alpha = \sec 2\alpha$. **(Gujranwala Board 2006, Lahore Board 2006)**

Solution:

$$\text{L.H.S.} = 1 + \tan \alpha \, . \, \tan 2\alpha$$

$$= 1 + \frac{\sin \alpha}{\cos \alpha} \frac{\sin 2\alpha}{\cos 2\alpha}$$

$$= \frac{\cos 2\alpha \cos \alpha + \sin 2\alpha \sin \alpha}{\cos \alpha \cos 2\alpha}$$

$$= \frac{\cos(2\alpha - \alpha)}{\cos \alpha \cos 2\alpha}$$

$$= \frac{\cos \alpha}{\cos \alpha \cos 2\alpha}$$

$$= \frac{1}{\cos 2\alpha}$$

$$= \sec 2\alpha$$

$$= \text{R.H.S}$$

Hence proved.

Q.9 **Prove that** $\dfrac{2\sin\theta\sin 2\theta}{\cos\theta+\cos 3\theta} = \tan 2\theta\tan\theta$

(Gujranwala Board 2007, Lahore Board 2006)

Solution:

$$\text{L.H.S.} = \frac{2\sin\theta\sin2\theta}{\cos\theta+\cos3\theta} = \frac{2\sin\theta\sin2\theta}{\cos\theta+4\cos^3\theta-3\cos\theta}$$

$$= \frac{2\sin\theta\sin2\theta}{4\cos^3\theta-2\cos\theta}$$

$$= \frac{2\sin\theta\sin2\theta}{2\cos\theta(2\cos^2\theta-1)}$$

$$= \frac{2\sin\theta\sin2\theta}{2\cos\theta\cos2\theta} = \tan\theta\tan2\theta$$

$$= \text{R.H.S}$$

Q.10 $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2.$

Solution:

$$\text{L.H.S.} = \frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin(3\theta-\theta)}{\sin\theta\cos\theta}$$

$$= \frac{\sin 2\theta}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{R.H.S.}$$

Hence proved.

Q.11 $\dfrac{\cos 3\theta}{\cos\theta} + \dfrac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta.$

Solution:

$$\text{L.H.S.} = \frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} =$$

$$= \frac{\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{\cos\theta\sin\theta}$$

$$= \frac{\sin(3\theta+\theta)}{\cos\theta\sin\theta} = \frac{\sin 4\theta}{\cos\theta\sin\theta} = \frac{2\sin 2\theta\cos 2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\,.\,2\sin\theta\cos\theta\cos 2\theta}{\sin\theta\cos\theta} \qquad (\because\ \sin 2\theta = 2\sin\theta\cos\theta)$$

$$= 4\cos 2\theta = \text{R.H.S.}$$

Hence proved.

Q.12 $\dfrac{\tan\frac{\theta}{2}+\cot\frac{\theta}{2}}{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}} = \sec\theta.$

Solution:

$$\text{L.H.S.} = \frac{\tan\frac{\theta}{2}+\cot\frac{\theta}{2}}{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}}$$

$$= \frac{\frac{\sin\theta/2}{\cos\theta/2}+\frac{\cos\theta/2}{\sin\theta/2}}{\frac{\cos\theta/2}{\sin\theta/2}-\frac{\sin\theta/2}{\cos\theta/2}}$$

$$= \frac{\frac{\sin^2\theta/2+\cos^2\theta/2}{\sin\theta/2\cos\theta/2}}{\frac{\cos^2\theta/2-\sin^2\theta/2}{\sin\theta/2\cos\theta/2}}$$

$$= \frac{1}{\cos^2\theta/2-\sin^2\theta/2} = \frac{1}{\cos\theta}$$

$$= \sec\theta$$

$$= \text{R.H.S}$$

Q.13 $\dfrac{\sin 3\theta}{\cos\theta}+\dfrac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$ **(Gujranwala Board 2006)**

Solution:

$$\text{L.H.S.} = \frac{\sin 3\theta}{\cos\theta}+\frac{\cos 3\theta}{\sin\theta}$$

$$= \frac{\sin 3\theta\sin\theta+\cos 3\theta\cos\theta}{\cos\theta\sin\theta}$$

$$= \frac{\cos(3\theta-\theta)}{\cos\theta\sin\theta} = \frac{\cos 2\theta}{\cos\theta\sin\theta}$$

multiplying & dividing by 2

$$= 2\frac{\cos 2\theta}{2\sin\theta\cos\theta}$$

$$= 2\frac{\cos 2\theta}{\sin 2\theta} = 2\cot 2\theta = \text{R.H.S.}$$

Hence proved.

Q.14 Reduce $\sin^4\theta$ to an expression involving only function of multiples of θ, raised to first power. (Lahore Board 2004, 2010)

Solution:

$$\text{Sin}^4\theta = (\sin^2\theta)^2 \qquad \left(\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$= \left(\frac{1-\cos 2\theta}{2}\right)^2 \qquad \left(\because \cos^2 2\theta = \frac{1+\cos 4\theta}{2}\right)$$

$$= \frac{1-2\cos 2\theta+\cos^2 2\theta}{4}$$

$$= \frac{1}{4}\left[1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right]$$

$$= \frac{1}{4}\left[\frac{2+4\cos 2\theta+1+\cos 4\theta}{2}\right]$$

$$= \frac{1}{4}\left[\frac{3-4\cos 2\theta+\cos 4\theta}{2}\right]$$

$$= \frac{3-4\cos 2\theta+\cos 4\theta}{8}$$

Q.15 Find values of $\sin\theta$, $\cos\theta$ when θ is

(i) 18^o (ii) 36^o (iii) 54^o (iv) 72^o (v) 140^o

Hence prove that $\cos 36^o \cos 72^o \cos 108^o \cos 144^o = \frac{1}{16}$

Solution:

(i) $\theta = 18^o$

$5\theta = 5 \times 18^o = 90^o$

Now $3\theta + 2\theta = 90^o \Rightarrow 3\theta = 90^o - 2\theta$

taking cos on both sides

$\cos 3\theta = \cos(90^o - 2\theta) = \cos 90^o \cos 2\theta + \sin 90^o \sin 2\theta$

$\cos 3\theta = \sin 2\theta$

$4\cos^3\theta - 3\cos\theta = 2\sin\theta\cos\theta$

$\cos\theta[4\cos^2\theta - 3] = 2\sin\theta\cos\theta$

$4(1-\sin^2\theta) - 3 - 2\sin\theta = 0$

$4 - 4\sin^2\theta - 3 - 2\sin\theta = 0$

$4\sin^2\theta + 2\sin\theta - 1 = 0$

$a = 4,\ b = 2,\ c = -1$ By Quadratic formula

$$\sin\theta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4\,(4)\,(-1)}}{2\,(4)} = \frac{-2 \pm \sqrt{4 + 16}}{8}$$

$$\sin\theta = \frac{-2 \pm \sqrt{20}}{8}$$

$$\sin\theta = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

As 18^o is in I quadrant so

$$\sin 18^\circ = \frac{\sqrt{5} - 1}{4}$$

$$\cos\theta = \sqrt{1 - \sin^2\theta}$$

$$\cos 18^o = \sqrt{\frac{10 + 2\sqrt{5}}{16}} = \sqrt{\frac{10 + 2\sqrt{5}}{16}} = \cos 18^o$$

(ii) $\theta = 36^o$

$$5 \times \theta = 5 \times 36^o$$

$$5\theta = 180^o$$

$$3\theta + 2\theta = 180^o \Rightarrow 3\theta = 180^o - 2\theta$$

$$\sin 3\theta = \sin(180^o - 2\theta)$$

$$3\sin\theta - 4\sin^3\theta = \sin 2\theta$$

$$\sin\theta\,(3 - 4\sin^2\theta) = 2\sin\theta\cos\theta$$

$$3 - 4\sin^2\theta - 2\cos\theta = 0$$

$$3 - 4\,(1 - \cos^2\theta) - 2\cos\theta = 0$$

$$3 - 4 + 4\cos^2\theta - 2\cos\theta = 0$$

$$4\cos^2\theta - 2\cos\theta - 1 = 0$$

$a = 4$, $b = -2$, $c = -1$ By Quadratic formula

$$\cos\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\cos\theta = \frac{2 \pm \sqrt{4 - 4\,(4)\,(-1)}}{2\,(4)}$$

$$= \frac{2 \pm \sqrt{4 + 16}}{8} = \frac{2 \pm \sqrt{20}}{8} = \frac{2 \pm 2\sqrt{5}}{8}$$

$$\cos 36^\circ = \frac{1+\sqrt{5}}{4}$$

As 36° is in I quadrant so

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{1+\sqrt{5}}{4}\right)^2}$$

$$\sin\theta = \sqrt{\frac{10-2\sqrt{5}}{16}}$$

$$\boxed{\sin 36^\circ = \sqrt{\frac{10-2\sqrt{5}}{16}}}$$

(iii) $\sin 54^\circ$

$$\sin 54^\circ = \sin(90^\circ - 36^\circ)$$

$$= \cos 36^\circ \quad \left(\therefore \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta\right)$$

$$\boxed{\sin 54^\circ = \frac{\sqrt{5}+1}{4}} \quad \text{Ans.}$$

$\cos 54^\circ$

$$= \cos(90^\circ - 36^\circ) = \sin 36^\circ \quad \left(\therefore \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta\right)$$

$$\boxed{\cos 54^\circ = \sqrt{\frac{10-2\sqrt{5}}{16}}}$$

(iv) 72°

$$\sin 72^\circ = \sin(90^\circ - 18^\circ) = \cos 18^\circ \quad \left(\therefore \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta\right)$$

$$= \cos 18^\circ$$

$$\boxed{\sin 72^\circ = \sqrt{\frac{10+2\sqrt{5}}{16}}}$$

$$\cos 72^\circ = \cos(90^\circ - 18^\circ) = \sin 18^\circ$$

$$\boxed{\cos 72^\circ = \frac{\sqrt{5}-1}{4}}$$

(v) $\sin 144^\circ$

$$\sin 144^\circ = \sin((180^\circ - 36^\circ) \quad (\therefore \sin(\pi-\theta) = \sin\theta)$$

$= \sin 36^\circ$

$$\boxed{\sin 144^\circ = \sqrt{\frac{10 - 2\sqrt{5}}{16}}} \quad \text{Ans.}$$

$\cos 144^\circ = \cos (180^\circ - 36^\circ) = -\cos 36^\circ$

$$\boxed{\cos 144^\circ = -\left(\frac{1+\sqrt{5}}{4}\right)} \quad \text{Ans.}$$

Next $\cos 36^\circ . \cos 72^\circ . \cos 108^\circ . \cos 144^\circ = \frac{1}{16}$

L.H.S. $= \cos 36^\circ \cos 72^\circ \cos (180^\circ - 72^\circ) \cos 144$

$= \cos 36^\circ \cos 72^\circ (-\cos 72^\circ) \cos 144^\circ$

$$= \left(\frac{1+\sqrt{5}}{4}\right)\left(\frac{\sqrt{5}-1}{4}\right)\left(-\left(\frac{\sqrt{5}-1}{4}\right)\right)\left(\frac{1+\sqrt{5}}{4}\right)$$

$$= \left(\frac{1+\sqrt{5}}{4}\right)^2 \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= \left[\frac{(1+\sqrt{5})(\sqrt{5}-1)}{16}\right]^2$$

$$= \left[\frac{(\sqrt{5})^2 - (1)^2}{16}\right]^2 = \left(\frac{5-1}{16}\right)^2$$

$$= \left(\frac{4}{16}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16} = \text{R.H.S.}$$

Hence proved.

EXERCISE 10.4

Q.1 Express the following product as sums and differences

(i) $2 \sin 3\theta \cos \theta$ **(Lahore Board 2006)**

(ii) $2 \cos 5\theta \sin 3\theta$

(iii) $\sin 5\theta \cos 2\theta$ **(Gujranwala Board 2004)**

(iv) $2 \sin 7\theta \sin 2\theta$

(v) $\cos (x + y) \sin (x - y)$

(vi) $\cos (2x + 30) \cos (2x - 30)$

(vii) $\sin 12^\circ \sin 46^\circ$

(viii) $\sin (x + 45^\circ) \sin (x - 45^\circ)$

Solutions:

(i) **$2 \sin 3\theta \cos\theta$**

$= \sin(3\theta + \theta) + \sin(3\theta - \theta)$ $\quad (\because\ 2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta))$

$= \sin 4\theta + \sin 2\theta$ Ans.

(ii) **$2 \cos 5\theta \sin 3\theta$**

$= \sin(5\theta + 3\theta) - \sin(5\theta - 3\theta)$ $\quad (\because\ 2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta))$

$= \sin 8\theta - \sin 2\theta$ Ans.

(iii) **$\sin 5\theta \cos 2\theta$** **(Gujranwala Board 2004)**

multiple & divide by 2

$= \frac{1}{2}[2 \sin 5\theta \cos 2\theta]$

$= \frac{1}{2}[\sin(5\theta + 2\theta) + \sin(5\theta - 2\theta)]$

$= \frac{1}{2}[\sin 7\theta + \sin 3\theta]$ Ans.

(iv) **$2 \sin 7\theta \sin 2\theta$**

$= -[-2 \sin 7\theta \sin 2\theta]$

$= -\cos[(7\theta + 2\theta) - \cos(7\theta - 2\theta)]$ $\quad (\because\ -2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

$= -[\cos 9\theta - \cos 5\theta]$ Ans.

(v) **$\cos(x + y) \sin(x - y)$**

multiply & dividing by 2

$= \frac{1}{2}[2 \cos(x + y) \sin(x - y)]$ $\quad (\because\ 2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta))$

$= \frac{1}{2}[\sin(x + y + x - y) - \sin(x + y - x + y]$

$= \frac{1}{2}[\sin(2x) - \sin 2y]$ Ans.

(vi) **$\cos(2x + 30^\circ) \cos(2x - 30^\circ)$**

multiply & divide by 2

$= \frac{1}{2}[2 \cos(2x + 30^\circ) \cos(2x - 30^\circ)]$ $\quad (\because\ 2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta))$

$= \frac{1}{2}[\cos(2x + 30^\circ + 2x - 30^\circ) + \cos(2x + 30^\circ - 2x + 30^\circ)]$

$= \frac{1}{2}[\cos 4x + \cos 60^\circ]$ Ans.

(vii) $\sin 12^\circ \sin 46^\circ$

multiply & divide by -2

$= \frac{-1}{2}[-2 \sin 12^\circ \sin 46^\circ)$ $\quad (\because\ -2\sin\alpha \sin\beta = \cos(\alpha + \beta) - \cos(\alpha - \beta))$

$= \frac{-1}{2}[\cos(12^\circ + 46^\circ) - \cos(12^\circ - 46^\circ)]$

$= \frac{-1}{2}[\cos 58^\circ - \cos(-34^\circ)]$

$= \frac{-1}{2}[\cos 58^\circ - \cos 34^\circ]$ Ans. $\quad (\because\ \cos(-\theta) = \cos\theta)$

(viii) $\sin(x + 45^\circ) \sin(x - 45^\circ)$

multiply & dividing by -2

$= \frac{-1}{2}[-2 \sin(x + 45^\circ) \sin(x - 45^\circ)]$ $\quad (\because\ -2\sin\alpha \sin\beta = \cos(\alpha + \beta) - \cos(\alpha - \beta))$

$= \frac{-1}{2}[\cos(x + 45^\circ + x - 45^\circ) - \cos(x + 45^\circ - x + 45^\circ)]$

$= -\frac{1}{2}[\cos 2x - \cos 90^\circ]$

Q.2 Express the following sums and differences as product

(i) $\sin 5\theta + \sin 3\theta$ **(Lahore Board 2006,2007)**

(ii) $\sin 8\theta - \sin 4\theta$

(iii) $\cos 6\theta + \cos 3\theta$

(iv) $\cos 7\theta - \cos\theta$ **(Lahore Board 2009)**

(v) $\cos 12 + \cos 48$ **(Lahore Board 2010)**

(vi) $\sin(x + 30) + \sin(x - 30)$

Solution:

(i) $\sin 5\theta + \sin 3\theta$

$= 2 \sin \frac{5\theta + 3\theta}{2} \cos \frac{5\theta - 3\theta}{2}$ $\quad \left(\therefore\ \sin P + \sin Q = 2 \sin \frac{P + Q}{2} \cos \frac{P - Q}{2}\right)$

$= 2 \sin 4\theta \cos\theta$ Ans.

(ii) $\sin 8\theta - \sin 4\theta$

$= 2\cos\frac{8\theta + 4\theta}{2}\sin\frac{8\theta - 4\theta}{2}$ $\quad\because \left(\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}\right)$

$= 2\cos 6\theta \sin 2\theta$ Ans.

(iii) $\cos 6\theta + \cos 3\theta$

$= 2\cos\frac{6\theta + 3\theta}{2}\cos\frac{6\theta - 3\theta}{2}$ $\quad\because \left(\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}\right)$

$= 2\cos\frac{9\theta}{2}\cos\frac{3\theta}{2}$ Ans.

(iv) $\cos 7\theta - \cos\theta$ **(Lahore Board 2009)**

$= -2\sin\frac{7\theta + \theta}{2} . \sin\frac{7\theta - \theta}{2}$ $\quad\because \left(\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}\right)$

$= -2\sin 4\theta \sin 3\theta$ Ans.

(v) $\cos 12^\circ + \cos 48^\circ$ **(Lahore Board 2010)**

$= 2\cos\frac{12^\circ + 48^\circ}{2}\cos\frac{12^\circ - 48^\circ}{2}$

$= 2\cos\frac{60^\circ}{2}\cos\frac{-36^\circ}{2}$ $\quad(\because \cos(-\theta) = \cos\theta)$

$= 2\cos 30^\circ \cos 18^\circ$ Ans.

(vi) $\sin(x + 30^\circ) + \sin(x - 30^\circ)$

$= 2\sin\frac{x + 30^\circ + x - 30^\circ}{2}\cos\frac{x + 30^\circ - x + 30^\circ}{2}$

$= 2\sin\frac{2x}{2}\cos\frac{60^\circ}{2}$

$= 2\sin x \cos 30^\circ$ Ans.

Q.3 Prove the following identities

(i) $\dfrac{\sin 3x - \sin x}{\cos x - \cos 3x} = \cot 2x$

(ii) $\dfrac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} = \tan 5x$

(iii) $\dfrac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \tan\left(\dfrac{\alpha - \beta}{2}\right)\cot\left(\dfrac{\alpha + \beta}{2}\right)$ **(Lahore Board 2007)**

Solution:

(i) $\dfrac{\sin 3x - \sin x}{\cos x - \cos 3x} = \cot 2x$

$$\text{L.H.S.} = \frac{\sin 3x - \sin x}{-(\cos 3x - \cos x)} \qquad \left(\because \sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)\right)$$

$$= \frac{2\cos\dfrac{3x+x}{2}\sin\dfrac{3x-x}{2}}{-\left(-2\sin\dfrac{3x+x}{2}\sin\dfrac{3x-x}{2}\right)} \left(\because \cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)\right)$$

$$= \frac{2\cos 2x \sin x}{2\sin 2x \sin x} = \frac{\cos 2x}{\sin 2x} = \cot 2x = \text{R.H.S.}$$

Hence proved.

(ii) $\dfrac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} = \tan 5x$

$$\text{L.H.S.} = \frac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} \qquad \left(\because \sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)\right)$$

$$= \frac{2\sin\dfrac{8x+2x}{2}\cos\dfrac{8x-2x}{2}}{2\cos\dfrac{8x+2x}{2}\cos\dfrac{8x-2x}{2}} \left(\because \cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)\right)$$

$$= \frac{2\sin 5x \cos 3x}{2\cos 5x \cos 3x} = \frac{\sin 5x}{\cos 5x} = \tan 5x = \text{R.H.S.}$$

Hence proved.

(iii) $\dfrac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \tan\dfrac{\alpha-\beta}{2}\cot\dfrac{\alpha+\beta}{2}$ **(Lahore Board 2007)**

$$\text{L.H.S.} = \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} \qquad \left(\because \sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)\right)$$

$$= \frac{2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}}{2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}} \left(\because \sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)\right)$$

$$= \cot\frac{\alpha+\beta}{2} \cdot \tan\frac{\alpha-\beta}{2} = \text{R.H.S.}$$

Hence the proof.

Q.4 Prove that

(i) $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$ **(Lahore Board 2008)**

(ii) $\sin\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{2}\cos 2\theta$ **(Lahore Board 2005)**

(iii) $\frac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta$

Solution:

(i) $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$

L.H.S. $= \cos 20^\circ + \cos 100^\circ + \cos 140^\circ$

$= [\cos 140^\circ + \cos 20^\circ] + \cos 100^\circ \quad \left(\because \cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)\right)$

$= 2\cos\frac{140^\circ + 20^\circ}{2}\cos\frac{140^\circ - 20^\circ}{2} + \cos 100^\circ$

$= 2\cos 80^\circ \cos 60^\circ + \cos 100^\circ$

$= 2\cos 80^\circ \frac{1}{2} + \cos 100^\circ$

$= \cos 80^\circ + \cos 100^\circ$

$= 2\cos\frac{80^\circ + 100^\circ}{2}\cos\frac{80^\circ - 100^\circ}{2}$

$= 2\cos 90^\circ \cos(-10^\circ)$

$= 2 \times 0 \times \cos 10^\circ$

$= 0 =$ R.H.S.

Hence proved.

(ii) $\sin\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{2}\cos 2\theta$

L.H.S. $= \sin\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} + \theta\right)$ $\quad \left(\because \begin{array}{l} \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{array}\right)$

$= \left(\sin\frac{\pi}{4}\cos\theta - \cos\frac{\pi}{4}\sin\theta\right)\left(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right)$

$= \left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)$

$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$

$= \frac{1}{2}(\cos^2\theta - \sin^2\theta)$

$= \frac{1}{2}\cos 2\theta =$ R.H.S. Hence proved.

(iii) $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta$

L.H.S. $= \dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta}$ $\left(\because \begin{array}{l} \sin P + \sin Q = 2\sin\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right) \\ \cos P + \cos Q = 2\cos\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right) \end{array} \right)$

$= \dfrac{[\sin 7\theta + \sin\theta] + [\sin 5\theta + \sin 3\theta]}{[\cos 7\theta + \cos\theta] + [\cos 5\theta + \cos 3\theta]}$

$= \dfrac{2\sin\dfrac{(7\theta+\theta)}{2}\cos\dfrac{7\theta-\theta}{2} + 2\sin\dfrac{5\theta+3\theta}{2}\cos\dfrac{5\theta-3\theta}{2}}{2\cos\dfrac{7\theta+\theta}{2}\cos\dfrac{7\theta-\theta}{2} + 2\cos\dfrac{5\theta+3\theta}{2}\cos\dfrac{5\theta-3\theta}{2}}$

$= \dfrac{2\sin 4\theta\cos 3\theta + 2\sin 4\theta\cos\theta}{2\cos 4\theta\cos 3\theta + 2\cos 4\theta\cos\theta}$

$= \dfrac{2\sin 4\theta(\cos 3\theta + \cos\theta)}{2\cos 4\theta(\cos 3\theta + \cos\theta)}$

$= \dfrac{\sin 4\theta}{\cos 4\theta} = \tan 4\theta$ = R.H.S.

Hence proved.

Q.5 Prove that

(i) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \dfrac{1}{16}$

(Gujranawala Board 2004, Lahore Board 2008)

(ii) $\sin\dfrac{\pi}{9}\sin\dfrac{2\pi}{9}\sin\dfrac{\pi}{3}\sin\dfrac{4\pi}{9} = \dfrac{3}{16}$

(iii) $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \dfrac{1}{16}$ **(Lahore Board 2011)**

Solution:

(i) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \dfrac{1}{16}$

L.H.S. $= \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

$= \dfrac{1}{2}\cos 40^\circ \cos 20^\circ \cos 80^\circ$

$= \dfrac{1}{2}[\cos 40^\circ \cos 20^\circ]\cos 80^\circ$ $(\because\ 2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta))$

$$= \frac{1}{2}\left[\frac{2}{2}\cos 40^\circ \cos 20^\circ\right]\cos 80^\circ$$

$$= \frac{1}{2} \times \frac{1}{2}\left[\cos(40^\circ + 20^\circ) + \cos(40^\circ - 20^\circ)\right]\cos 80^\circ$$

$$= \frac{1}{4}\left[\cos 60^\circ + \cos 20^\circ\right]\cos 80^\circ$$

$$= \frac{1}{4}\left[\frac{1}{2} + \cos 20^\circ\right]\cos 80^\circ$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80^\circ + \cos 80^\circ \cos 20^\circ\right] \quad (\because\ 2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta))$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80^\circ + \frac{2}{2}\cos 80^\circ \cos 20^\circ\right]$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80^\circ + \frac{1}{2}\left\{\cos(80^\circ + 20^\circ) + \cos(80^\circ - 20^\circ)\right\}\right]$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80^\circ + \frac{1}{2}\cos 100^\circ + \frac{1}{2}\cos 60^\circ\right]$$

$$= \frac{1}{8}\left[\cos 80^\circ + \cos 100^\circ + \cos 60^\circ\right]$$

$$= \frac{1}{8}\left[\cos 80^\circ + \cos(180^\circ - 80^\circ) + \cos 60^\circ\right] \quad (\because\ \cos(\pi - \theta) = -\cos\theta)$$

$$= \frac{1}{8}\left[\cos 80^\circ - \cos 80^\circ + \cos 60^\circ\right]$$

$$= \frac{1}{8}\left[\cos 60^\circ\right]$$

$$= \frac{1}{8}\left[\frac{1}{2}\right] = \frac{1}{16} = \text{R.H.S.}$$

Hence proved.

(ii) $\mathbf{\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9} = \frac{3}{16}}$ **(Lahore Board 2011)**

$$\text{L.H.S.} = \sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9}$$

$$= \sin\frac{\pi}{9}\sin\frac{2\pi}{9}\,\frac{\sqrt{3}}{2}\sin\frac{4\pi}{9}$$

$$= \frac{\sqrt{3}}{2}\sin 20^\circ \sin 40^\circ \sin 80^\circ$$

$$= \frac{\sqrt{3}}{2}\left[\sin 40^\circ \sin 20^\circ\right]\sin 80^\circ \quad (\because\ -2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta))$$

$$= \frac{\sqrt{3}}{2}\left[\frac{-2}{-2}\sin40^\circ \sin20^\circ\right]\sin80^\circ$$

$$= \frac{-\sqrt{3}}{2}\left[\frac{1}{2}\{\cos(40^\circ + 20^\circ) - \cos(40^\circ - 20^\circ)\}\sin 80^\circ\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\{\cos 60^\circ - \cos 20^\circ\}\sin 80^\circ\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\left\{\frac{1}{2} - \cos20^\circ\right\}\sin80^\circ\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80^\circ - \sin80^\circ\cos20^\circ\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80^\circ - \frac{2}{2}\sin80^\circ\cos20^\circ\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80^\circ - \frac{1}{2}\sin(80^\circ + 20^\circ) + \sin(80^\circ - 20^\circ)\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80^\circ - \frac{1}{2}\sin100^\circ - \frac{1}{2}\sin60^\circ\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80^\circ - \frac{1}{2}\sin(180^\circ - 80^\circ) - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80^\circ - \frac{1}{2}\sin 80^\circ - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{-\sqrt{3}}{4}\left(-\frac{\sqrt{3}}{4}\right)$$

$$= \frac{3}{16} = \text{R.H.S}$$

Hence proved.

(iii) $\mathbf{\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}}$

L.H.S. $= \sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$

$$= \frac{1}{2}\sin 10^\circ \sin 50^\circ \sin 70^\circ$$

$$= \frac{-1}{2 \times 2}\left[(-2\sin 10^\circ \sin 50^\circ)\cdot \sin 70^\circ\right]$$

$$= \frac{-1}{4}\left[\{\cos(10^\circ + 50^\circ) - \cos(10^\circ - 50^\circ)\}\sin 70^\circ\right]$$

$$= \frac{-1}{4}\left[(\cos 60^\circ - \cos 40^\circ)\sin 70^\circ\right]$$

$$= \frac{-1}{8}\left[\, 2\cos 60^\circ \sin 70^\circ - 2\cos 40^\circ \sin 70^\circ \,\right] \quad \because\ 2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

$$= \frac{-1}{8}\left[\, 2\,\frac{1}{2}\sin 70^\circ - \{\, \sin(40^\circ + 70^\circ) - \sin(40^\circ - 70^\circ) \,\} \,\right]$$

$$= \frac{-1}{8}\left[\, \sin 70^\circ - \{\, \sin 110^\circ - \sin(-30^\circ) \,\} \,\right]$$

$$= \frac{-1}{8}\left[\, \sin 70^\circ - \{\, \sin 110^\circ + \sin 30^\circ \} \,\right]$$

$$= \frac{-1}{8}\left[\, \sin(180^\circ - 110^\circ) - \sin 110^\circ - \sin 30^\circ \,\right]$$

$$= \frac{-1}{8}\left[\, \sin 110^\circ - \sin 110^\circ - \sin 30^\circ \,\right]$$

$$= \frac{-1}{8}\left[\, -\frac{1}{2} \,\right]$$

$$= \frac{1}{16} = \text{R.H.S.}$$

Hence proved.

Chapter 10

TRIGONOMETRIC IDENTITIES

MULTIPLE CHOICE QUESTIONS

(1) $\sin 2\alpha$ =

(a) $\cos^2\alpha - \sin^2\alpha$ (b) $2\sin^2\alpha + 1$

(c) $2\sin\alpha \cos\alpha$ (d) $\sin\alpha \cdot \cos\alpha$

[Lahore Board 2005-08, Lahore Board 2015]

(2) $\cos 3\alpha$ =

(a) $4\cos^3\alpha - 3\cos\alpha$ (b) $3\cos\alpha - 4\cos^3$

(c) $3\cos\alpha + 4\cos^3\alpha$ (d) None of these

[Lahore Board 2005]

(3) $\sec x$ =

(a) $\sec(x + 2\pi)$ (b) $\sec(x + \pi)$

(c) $\cos x$ (d) $\sin x$

[Lahore Board 2005]

(4) $\sin P + \sin Q$ =

(a) $2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}$ (b) $-2 \sin \frac{P+Q}{2} \cdot \sin \frac{P-Q}{2}$

(c) $2 \cos \frac{P+Q}{2} \cdot \cos \frac{P-Q}{2}$ (d) None of these

[Lahore Board 2005]

(5) $\cos(\alpha - \beta)$ =

(a) $\cos\alpha \cos\beta + \sin\alpha \sin\beta$ (b) $\cos\alpha \cos\beta - \sin\alpha \sin\beta$

(c) $\sin\alpha \cos\beta + \cos\alpha \sin\beta$ (d) $\sin\alpha \cos\beta - \cos\alpha \sin\beta$

[Gujranwala Board 2005]

(6) $\operatorname{cosec}(\pi - \alpha) =$

(a) $\sin\alpha$ (b) $\cos\alpha$

(c) $\cot\alpha$ (d) $\operatorname{cosec}\alpha$

[Gujranwala Board 2005]

(7) $1 + \cos 2\alpha =$

(a) $\cos^2\alpha$ (b) $2\cos^2\alpha$

(c) $\sin^2\alpha$ (d) $2\sin\alpha$

[Gujranwala Board 2005]

(8) $\cos\left(\frac{\pi}{2} - \beta\right) =$

(a) $\cos\beta$ (b) $\cos\frac{\pi}{2}$

(c) $\sin\beta$ (d) $-\sin\beta$

[Gujranwala Board 2005]

(9) $2\sin\alpha\cos\beta =$

(a) $\sin(\alpha + \beta) + \cos(\alpha - \beta)$ (b) $\sin(\alpha + \beta) + \sin(\alpha - \beta)$

(c) $\sin(\alpha + \beta) - \sin(\alpha - \beta)$ (d) None of these

[Gujranwala Board 2006]

(10) $\sec\left(\frac{3\pi}{2} - \theta\right) =$

(a) $\operatorname{cosec}\theta$ (b) $-\operatorname{cosec}\theta$

(c) $-\sec\theta$ (d) None of these

[Gujranwala Board 2006]

(11) $\pm\sqrt{\frac{1 + \cos\alpha}{2}} =$

(a) $\cos\frac{\alpha}{2}$ (b) $\sin\frac{\alpha}{2}$

(c) $\tan\frac{\alpha}{2}$ (d) None of these

[Gujranwala Board 2006]

(12) $\sin 3\alpha =$

(a) $3\sin\alpha + 4\sin^3\alpha$ (b) $4\sin\alpha + 3\sin^3\alpha$

(c) $3\sin\alpha - 4\sin^3\alpha$ (d) $3\sin\alpha + 4\sin^3\alpha$

[Lahore Board 2006]

[Gujranwala Board 2006]

(13) If $r\cos\theta = 3$, $r\sin\theta = 4$ then r is:

(a) 25 (b) -5

(c) 5 (d) -25

[Lahore Board 2006]

(14) $\tan(180° + \alpha)$ is equal to:

(a) $\tan\alpha$ (b) $-\tan\alpha$

(c) $\cot\alpha$ (d) $-\cot\alpha$

[Lahore Board 2006]

(15) $\cos2\theta$ is equal to:

(a) $\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$ (b) $\dfrac{1 - \tan^2\theta}{1 - \tan^2\theta}$

(c) $\dfrac{2\tan\theta}{1 + \tan^2\theta}$ (d) $\dfrac{1 + \tan^2\theta}{2\tan\theta}$

[Gujranwala Board 2007]

(16) $\sin\theta$ =

(a) $2\sin\dfrac{\theta}{2}$ (b) $\sin\dfrac{\theta}{2} \cdot \cos\dfrac{\theta}{2}$

(c) $2\cos^2\dfrac{\theta}{2}$ (d) $2\sin\dfrac{\theta}{2} \cdot \cos\dfrac{\theta}{2}$

[Lahore Board 2007]

(17) co-ratio of cosine is:

(a) sec (b) sine

(c) cosec (d) cos

[Lahore Board 2008]

(18) $\tan2\alpha$ equals:

(a) $\dfrac{\tan\alpha}{1 - \tan^2\alpha}$ (b) $\dfrac{\tan2\alpha}{1 - \tan^2\alpha}$

(c) $\dfrac{2\tan\alpha}{1 - \tan^2\alpha}$ (d) $\dfrac{2\tan\alpha}{1 + \tan^2\alpha}$

[Gujranwala Board 2008]

(19) $\tan(270° + \theta)$ is equal to:

(a) $\cot\theta$ (b) $\tan\theta$

(c) $-\cot\theta$ (d) $-\tan\theta$

[Lahore Board 2014, Gujranwala Board 2009]

(13) If $r\cos\theta = 3$, $r\sin\theta = 4$ then r is:

(a) 25 (b) -5

(c) 5 (d) -25

[Lahore Board 2006]

(14) $\tan(180° + \alpha)$ is equal to:

(a) $\tan\alpha$ (b) $-\tan\alpha$

(c) $\cot\alpha$ (d) $-\cot\alpha$

[Lahore Board 2006]

(15) $\cos2\theta$ is equal to:

(a) $\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$ (b) $\dfrac{1 - \tan^2\theta}{1 - \tan^2\theta}$

(c) $\dfrac{2\tan\theta}{1 + \tan^2\theta}$ (d) $\dfrac{1 + \tan^2\theta}{2\tan\theta}$

[Gujranwala Board 2007]

(16) $\sin\theta$ =

(a) $2\sin\dfrac{\theta}{2}$ (b) $\sin\dfrac{\theta}{2} \cdot \cos\dfrac{\theta}{2}$

(c) $2\cos^2\dfrac{\theta}{2}$ (d) $2\sin\dfrac{\theta}{2} \cdot \cos\dfrac{\theta}{2}$

[Lahore Board 2007]

(17) co-ratio of cosine is:

(a) sec (b) sine

(c) cosec (d) cos

[Lahore Board 2008]

(18) $\tan2\alpha$ equals:

(a) $\dfrac{\tan\alpha}{1 - \tan^2\alpha}$ (b) $\dfrac{\tan2\alpha}{1 - \tan^2\alpha}$

(c) $\dfrac{2\tan\alpha}{1 - \tan^2\alpha}$ (d) $\dfrac{2\tan\alpha}{1 + \tan^2\alpha}$

[Gujranwala Board 2008]

(19) $\tan(270° + \theta)$ is equal to:

(a) $\cot\theta$ (b) $\tan\theta$

(c) $-\cot\theta$ (d) $-\tan\theta$

[Lahore Board 2014, Gujranwala Board 2009]

(20) $\tan\frac{\alpha}{2}$ is equal to:

(a) $\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$ (b) $\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$

(c) $\pm\sqrt{\frac{1-\cos\alpha}{2}}$ (d) $\sqrt{\frac{1+\cos\alpha}{2}}$

[Gujranwala Board 2009]

(21) $\cos\left(\frac{\pi}{2}+\theta\right)$ equals:

(a) $\cos\theta$ (b) $-\sin\theta$

(c) $\sin\theta$ (d) $-\cos\theta$

[Lahore Board 2009]

(22) $2\sin12°\sin46°$ equals:

(a) $\cos34° + \cos58°$ (b) $\sin34° - \sin58°$

(c) $\sin34° + \sin58°$ (d) $\cos34° - \cos58°$

[Lahore Board 2009]

(23) $\sin\left(\frac{\pi}{2}-\theta\right)$ equals:

(a) $\cos\theta$ (b) $\sin\theta$

(c) $-\cos\theta$ (d) $-\sin\theta$

[Lahore Board 2009]

(24) $2\sin\alpha\cos\beta$ equals:

(a) $\sin(\alpha+\beta) - \sin(\alpha-\beta)$ (b) $\cos(\alpha+\beta) + \cos(\alpha-\beta)$

(c) $\sin(\alpha+\beta) + \sin(\alpha-\beta)$ (d) $\cos(\alpha+\beta) - \cos(\alpha-\beta)$

[Lahore Board 2009]

(25) $\sin\left(\frac{3\pi}{2}+\theta\right) =$

(a) $\cos\theta$ (b) $-\cos\theta$

(c) $\sin\theta$ (d) $-\sin\theta$

[Lahore Board 2010]

(26) $2\cos5\theta \cdot \sin3\theta =$

(a) $\sin8\theta - \sin2\theta$ (b) $\sin8\theta + \sin2\theta$

(c) $\cos8\theta + \cos2\theta$ (d) $\sin4\theta - \sin\theta$

[Lahore Board 2010]

(27) $\cos\frac{\alpha}{2}$ is equal to:

(a) $\frac{1+\cos\alpha}{2}$ (b) $\frac{1-\cos\alpha}{2}$

(c) $\frac{1+\sin\alpha}{2}$ (d) $\pm\sqrt{\frac{1+\cos\alpha}{2}}$

[Gujranwala Board 2010]

(28) $\tan 2\alpha$ = _____

(a) $\frac{2\tan\alpha}{1-\tan^2\alpha}$ (b) $\frac{2\tan\alpha}{1+\tan^2\alpha}$

(c) $\frac{\tan\alpha}{1-\tan^2\alpha}$ (d) $\frac{\tan\alpha}{1+\tan^2\alpha}$

[Lahore Board 2012]

(29) $\cos(\pi-\theta)$ =

(a) $\sin\theta$ (b) $-\sin\theta$

(c) $\cos\theta$ (d) $-\cos\theta$

[Lahore Board 2012]

(30) $\cos(\pi-\theta)$ =

(a) $\sin\theta$ (b) $-\sin\theta$

(c) $\cos\theta$ (d) $-\cos\theta$

[Lahore Board 2013]

(31) $\sin(-300°)$ =

(a) $-\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{2}{\sqrt{3}}$ (d) 0

[Lahore Board 2013]

(32) Distance between the points A(3, 8) and B(5, 6) is:

(a) $\sqrt{2}$ (b) $2\sqrt{2}$

(c) $\sqrt{3}$ (d) $3\sqrt{3}$

(33) Fundamental law of trigonometry is _____ where α, β are any two angles:

(a) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(b) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

(c) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

(d) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(34) $\cos\frac{\pi}{12}$ =

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}-1}{\sqrt{2}}$

(c) $\frac{\sqrt{3}+1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

(35) $\sec(-300°)$ =

(a) 1 (b) -1

(c) 2 (d) -2

(36) If α, β, γ are angles of triangle then $\cos(\alpha+\beta)$ = ————

(a) $\cos\gamma$ (b) $-\cos\gamma$

(c) $\sin\gamma$ (d) $\sin\frac{\gamma}{2}$

(37) $\sin(\alpha+\beta) \cdot \sin(\alpha-\beta)$ =

(a) $\sin^2\alpha - \sin^2\beta$ (b) $\sin^2\alpha + \sin^2\beta$

(c) $\cos^2\alpha - \cos^2\beta$ (d) $\cos^2\alpha + \cos^2\beta$

(38) $\frac{\cos11° + \sin11°}{\cos11° - \sin11°}$ =

(a) $\tan 65°$ (b) $\tan 54°$

(c) $\tan 56°$ (d) $\tan 37°$

(39) $\frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A}$ =

(a) $\sin A$ (b) $\cos A$

(c) $\cot A$ (d) $\tan A$

(40) $2\sin7\theta\cos3\theta$ =

(a) $\sin10\theta + \sin4\theta$ (b) $\sin10\theta - \sin4\theta$

(c) $\cos10\theta + \cos4\theta$ (d) $\cos10\theta - \cos4\theta$

(41) $\sin5x + \sin7x$ =

(a) $2\sin3x\cos x$ (b) $2\sin x\cos x$

(c) $2\sin x\cos6x$ (d) $2\sin6x\cos x$

(42) $\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)$ =

(a) $\frac{1}{2}\cos2\theta$ (b) $\frac{1}{2}\sin2\theta$

(c) $\frac{1}{2}\sin\theta\cos\theta$ (d) $2\sin\theta \cdot \cos2\theta$

(43) If $x > y > 0$ then point P(x, y) lies in quadrant:

(a) I (b) II (c) III (d) IV

(44) $\tan\left(-\alpha + \frac{\pi}{2} - \beta\right) =$

(a) $\tan(\alpha + \beta)$ (b) $\tan(\alpha - \beta)$

(c) $\cot(\alpha + \beta)$ (d) $\cot(\alpha - \beta)$

(45) If $\cot\alpha + \cot\theta = 0$ then $\alpha = ?$

(a) $\frac{\pi}{2} - \theta$ (b) $\pi - \theta$

(c) $\pi + \theta$ (d) None of these

(46) If $\frac{2\cos\theta(1 - \cos^2\theta)}{\sin 2\theta} > 0$ and $\frac{\sec^2\theta - 1}{\tan^2\theta \cdot \cot\theta} < 0$ then 'θ' lies in the quadrant:

(a) I (b) II

(c) III (d) IV

(47) $\sec\theta \neq$

(a) $\frac{\sqrt{1 + \cot^2\theta}}{\cot\theta}$ (b) $\frac{1}{\sqrt{1 - \sin^2\theta}}$

(c) $\sqrt{\operatorname{cosec}^2\theta - 1}$ (d) $\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta - 1}}$

(48) $\tan\frac{5\alpha}{2} =$

(a) $\pm\sqrt{\frac{1 - \cos 5\alpha}{1 + \cos 5\alpha}}$ (b) $\pm\sqrt{\frac{1 - \cos\frac{5\alpha}{2}}{1 + \cos\frac{5\alpha}{2}}}$

(c) $\pm\sqrt{\frac{1 + \cos 5\alpha}{1 - \cos 5\alpha}}$ (d) None of these

(49) $\cos\alpha - 2\cos\frac{\alpha + \beta}{2} \cdot \cos\frac{\alpha - \beta}{2} + \cos\beta =$

(a) $\sin\alpha\cos\beta$ (b) 0

(c) $\sin(\alpha - \beta) + \cos\alpha$ (d) $\cos\alpha \cdot \cos\beta$

(50) The value of $\cos 315°$ is:

(a) 0 (b) 1

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{2}}$

(51) $\cos 2\alpha =$ ————

(a) $2\cos^2\alpha + 1$ (b) $2\cos^2\alpha - 1$

(c) $2\sin^2\alpha + 1$ (d) $2\sin^2 d - 1$

Answers

(1)	c	*(2)*	a	*(3)*	a	*(4)*	a	*(5)*	a
(6)	d	*(7)*	b	*(8)*	c	*(9)*	b	*(10)*	b
(11)	a	*(12)*	c	*(13)*	c	*(14)*	a	*(15)*	a
(16)	d	*(17)*	b	*(18)*	c	*(19)*	c	*(20)*	a
(21)	b	*(22)*	d	*(23)*	a	*(24)*	c	*(25)*	b
(26)	a	*(27)*	d	*(28)*	a	*(29)*	d	*(30)*	d
(31)	b	*(32)*	b	*(33)*	b	*(34)*	d	*(35)*	c
(36)	d	*(37)*	c	*(38)*	c	*(39)*	d	*(40)*	a
(41)	d	*(42)*	a	*(43)*	a	*(44)*	c	*(45)*	b
(46)	b	*(47)*	c	*(48)*	a	*(49)*	b	*(50)*	d
(51)	b								

SOLUTION

Q.6 $\operatorname{cosec}(\pi - \alpha)$

$$= \frac{1}{\sin(\pi - \alpha)} = \frac{1}{\sin\pi\cos\alpha - \cos\pi\sin\alpha}$$

$$= \frac{1}{0 - (-1)\sin\alpha}$$

$$= \frac{1}{\sin\alpha} = \operatorname{cosec}\alpha$$

Q.7 As, $\cos 2\alpha = 2\cos^2\alpha - 1$

$1 + \cos 2\alpha = 2\cos^2\alpha$

Q.8 $\cos\left(\frac{\pi}{2} - \beta\right)$

$= \cos\frac{\pi}{2}\cos\beta + \sin\frac{\pi}{2}.\sin\beta$

$= 0 + (1)\sin\beta$

$= \sin\beta$

Q.13 $r\cos\theta = 3$, $r\sin\theta = 4$

$r^2\cos^2\theta = 9$ ..... (1) $r^2\sin^2\theta = 16$ ..... (2)

Adding (1) and (2).

$r^2\cos^2\theta + r^2\sin^2\theta = 9 + 16$

$r^2(\cos^2\theta + \sin^2\theta) = 25$

$r^2 = 25$

$r = 5$

Q.14 $\tan(180° + \alpha)$

$= \tan(\pi + \alpha)$

$= \tan\alpha$

Q.15 $\cos 2\theta$

$= \cos^2\theta - \sin^2\theta$

$= \frac{\cos^2\theta - \sin^2\theta}{1}$

SOLUTION

Q.6 $\operatorname{cosec}(\pi - \alpha)$

$$= \frac{1}{\sin(\pi - \alpha)} = \frac{1}{\sin\pi\cos\alpha - \cos\pi\sin\alpha}$$

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Q.15 $\cos 2\theta$

$= \cos^2\theta - \sin^2\theta$

$= \frac{\cos^2\theta - \sin^2\theta}{1}$

Q.32 $\cos\frac{\pi}{12}$

$$\frac{\pi}{12} = \frac{\pi}{12}\times\frac{180}{\pi} = 15^\circ$$

$$\cos\frac{\pi}{12} = \cos 15^\circ$$

$$= \cos(45^\circ - 30^\circ)$$

$$= \cos 45^\circ \cdot \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Q.33 $\sec(-300^\circ)$

$$= \frac{1}{\cos(-300^\circ)}$$

$$= \frac{1}{\cos(300^\circ)}$$

$$= \frac{1}{\cos(360^\circ - 60^\circ)}$$

$$= \frac{1}{\cos\left(2\pi - \frac{\pi}{3}\right)}$$

$$= \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$$

Q.34 As, $\alpha + \beta + \gamma = 180^\circ$

$$\alpha + \beta = 180^\circ - \gamma$$

$\Rightarrow$ $\cos(\alpha + \beta)$

$$= \cos(180^\circ - \gamma)$$

$$= -\cos\gamma$$

Q.35 $\sin(\alpha+\beta)\,.\,\sin(\alpha-\beta)$

$= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$= (\sin\alpha\cos\beta)^2 - (\cos\alpha\sin\beta)^2$

$= \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta$

$= \sin^2\alpha(1-\sin^2\beta) - (1-\sin^2\alpha)\sin^2\beta$

$= \sin^2\alpha - \sin^2\alpha\sin^2\beta - \sin^2\beta + \sin^2\alpha\sin^2\beta$

$= \sin^2\alpha - \sin^2\beta$

Q.36 As,

$$\tan 56^\circ = \tan(45^\circ+11^\circ) = \frac{\sin(45^\circ+11^\circ)}{\cos(45^\circ+11^\circ)}$$

$$= \frac{\sin45^\circ\cos11^\circ + \cos45^\circ\sin11^\circ}{\cos45^\circ\cos11^\circ - \sin45^\circ\sin11^\circ}$$

$$= \frac{\frac{1}{\sqrt{2}}\,.\,\cos11^\circ + \frac{1}{\sqrt{2}}\sin11^\circ}{\frac{1}{\sqrt{2}}\cos11^\circ - \frac{1}{\sqrt{2}}\sin11^\circ}$$

$$= \frac{\cos11^\circ + \sin11^\circ}{\cos11^\circ - \sin11^\circ}$$

Q.37 $\dfrac{\sin A + \sin2A}{1+\cos A+\cos2A}$

$$= \frac{\sin A + \sin2A}{1+\cos A + 2\cos^2A - 1}$$

$$= \frac{\sin A + 2\sin A\cos A}{\cos A + 2\cos^2A}$$

$$= \frac{\sin A(1+2\cos A)}{\cos A(1+2\cos A)}$$

$$= \frac{\sin A}{\cos A} = \tan A$$

Q.38 As,

$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

$2\sin70\cos30 = \sin(70+30) + \sin(70-30)$

$= \sin100 + \sin40$

Q.39 As,

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cdot\cos\frac{P-Q}{2}$$

$$\Rightarrow \quad \sin 5x + \sin 7x = 2\sin\frac{5x+7x}{2}\cos\frac{5x-7x}{2}$$

$$= 2\sin\frac{12x}{2}\cdot\cos\frac{-2x}{2}$$

$$= 2\sin 6x\cdot\cos x$$

Q.40 As,

$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta)-\cos(\alpha-\beta)$$

$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$$

$$\Rightarrow \quad \sin\left(\frac{\pi}{4}-\theta\right).\sin\left(\frac{\pi}{4}+\theta\right) = -\frac{1}{2}\left[\cos\left(\frac{\pi}{4}-\theta+\frac{\pi}{4}+\theta\right)-\cos\left(\frac{\pi}{4}-\theta-\frac{\pi}{4}-\theta\right)\right]$$

$$= -\frac{1}{2}\left[\cos\left(2\frac{\pi}{4}\right)-\cos(-2\theta)\right]$$

$$= -\frac{1}{2}\left[\cos\frac{\pi}{2}-\cos 2\theta\right]$$

$$= -\frac{1}{2}[0-\cos 2\theta]$$

$$= -\frac{1}{2}\cos 2\theta$$

Q.42 $\tan\left(-\alpha+\frac{\pi}{2}-\beta\right)$

$= \tan\left(\frac{\pi}{2}-(\alpha+\beta)\right)$

$= \cot(\alpha+\beta)$

Q.43 As, $\cot\alpha + \cot\theta = 0$

$\Rightarrow \quad \cot\alpha = -\cot\theta$

Which is possible only if $\alpha = \pi - \theta$

Q.44

$$\frac{2\cos\theta(1-\cos^2\theta)}{\sin 2\theta} > 0$$

Q.39 As,

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cdot\cos\frac{P-Q}{2}$$

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$$= -\frac{1}{2}\left[\cos\left(2\frac{\pi}{4}\right)-\cos(-2\theta)\right]$$

$$= -\frac{1}{2}\left[\cos\frac{\pi}{2}-\cos 2\theta\right]$$

$$= -\frac{1}{2}[0-\cos 2\theta]$$

$$= -\frac{1}{2}\cos 2\theta$$

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Q.44

$$\frac{2\cos\theta(1-\cos^2\theta)}{\sin 2\theta} > 0$$