

UNIT 2

Quadratic Equations and Inequalities

Students' Learning Outcomes



After completing this unit, the students will be able to:

- ▶ Solve quadratic equations of the form $ax^2+bx+c=0$, $a(\neq 0)$, b , c are real numbers by factorization, quadratic formula, completing square and graphs.
- ▶ Find the intersecting point graphically when intersection occurs between
 - a linear function and coordinate axes.
 - two linear functions.
 - a linear and a quadratic function.
- ▶ Sketch and interpret the graphs of the quadratic function $y = ax^2+bx+c$, $a(\neq 0)$, b and c are real numbers.
- ▶ Establish relationship between roots and coefficients of quadratic equations.
- ▶ Form a quadratic equation when roots are given.
- ▶ Find discriminant of a given quadratic equation.
- ▶ Identify the nature of roots of a quadratic equation through discriminant.
- ▶ Solve quadratic inequalities in one unknown.
- ▶ Solve problems of "changing the subject of formula".
- ▶ Solve fractional equations that can be reduced to quadratic equations.
- ▶ Apply the concept of quadratic equations and quadratic inequalities, to real world problems (such as in physics, engineering and finance, i.e. calculating max and min heights in projectile motion, determining the maximum price on a company's budget, stability of population, growth of business, the relationship between hours worked and amount earned etc.).



INTRODUCTION

This unit explores into the study and applications of quadratic equations, a fundamental concept in algebra with wide-ranging uses in both academics and the real world. Students will learn to solve quadratic equations using factorization, the quadratic formula, completing the square and graphical methods. The unit also explores how to sketch and interpret the graphs of quadratic functions and examines the relationship between the roots and coefficients, as well as how to form equations

from given roots. Key concepts like the discriminant are used to determine the nature of the roots, whether they are real, repeated or complex. Students will also solve quadratic inequalities, tackle equations involving formulas and fractions and graphically identify points of intersection between linear and quadratic functions. Finally, the unit emphasizes practical applications, demonstrating how quadratic concepts are used in different fields.

Recall

An **algebraic expression** is a combination of constants and variables joined with the signs of arithmetic operations.

For example, 5 , $3x$, $2x-7y$, $3y^2-5y-2$ etc.

A **polynomial** in one variable ' x ' is an expression of the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where $a_0, a_1, \dots, a_n$ are real numbers and n is a non-negative integer.

If $a_n \neq 0$, then degree of polynomial is n .

For example, $5x^2 - x + 7$ is a polynomial of degree 2.

2.1 Quadratic Equation

A quadratic equation is a second degree polynomial equation in one variable.

For example, the following equations are quadratic equations:

(i) $2x^2 + 3x + 7 = 0$

(ii) $x^2 - 5x - 4 = 0$

(iii) $3y^2 - 6y + 3 = 0$

Note

If $a, b, c \in R$, $a \neq 0$ and $b \neq 0$, then equation $ax^2 + bx + c = 0$ is called the standard form of the quadratic equation in x .

If $a = 0$, the equation $ax^2 + bx + c = 0$ becomes $bx + c = 0$ which is a linear equation.

If $b = 0$, the equation $ax^2 + bx + c = 0$ becomes $ax^2 + c = 0$ which is pure quadratic equation.

2.1.1 Solution of Quadratic Equation of the Form

$ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$

a Factorization Method

The solution of a quadratic equation by factorization is based on the following result: If $(x + l)(x + m) = 0$, then $x + l = 0$ or $x + m = 0$, where $l, m \in R$. This is called zero product property.

Example 1 Solve by factorization method: $x^2 + \frac{7}{3}x = 2$

Solution

$$x^2 + \frac{7}{3}x = 2$$

$$3x^2 + 7x = 6$$

[multiplying both sides by 3]

$$3x^2 + 7x - 6 = 0$$

$$3x^2 + 9x - 2x - 6 = 0$$

$$3x(x + 3) - 2(x + 3) = 0$$

$$(x + 3)(3x - 2) = 0$$

Now if either of the factor $x + 3$ or $3x - 2$ is zero, then their product is zero. So,

$$x + 3 = 0 \text{ or } 3x - 2 = 0$$

$$x = -3 \text{ or } x = \frac{2}{3}$$

Hence, solution set is $\{-3, \frac{2}{3}\}$.

Example 2

Solve by factorization method: $\frac{2x-3}{2} = \frac{4x-6}{x}, x \neq 0$.

Solution

$$\frac{2x-3}{2} = \frac{4x-6}{x}$$

$$\frac{2x(2x-3)}{2} = \frac{(2x)(4x-6)}{x}$$

$$2x^2 - 3x = 8x - 12$$

$$2x^2 - 11x + 12 = 0$$

$$2x^2 - 8x - 3x + 12 = 0$$

$$2x(x-4) - 3(x-4) = 0$$

$$(2x-3)(x-4) = 0$$

$$2x-3=0 \text{ or } x-4=0$$

$$x = \frac{3}{2} \text{ or } x = 4$$

Hence, solution set is $\{\frac{3}{2}, 4\}$.

Do you know?

Every quadratic equation has at most two roots.

Note

An equation in which variable appears in the denominator of one or more terms is called a **fractional equation**.

(i) $\frac{2x}{5} - \frac{6}{x} = \frac{3}{7}, x \neq 0$

(ii) $\frac{x+3}{x-1} = \frac{2x-1}{x+5}, x \neq 1, -5$ are fractional equations.

[Multiplying both sides by LCM of the denominators.
i.e., $2x$ to eliminate denominator]

[Transposing all terms to left hand side of '=' sign.]

b Solution of a Quadratic Equation by Completing Square Method**Example 3** Solve by completing square method: $3x^2 - 32 = -10x$ **Solution** $3x^2 - 32 = -10x$ **Step I** Simplifying equation so that the terms in x^2 and x are on one side of the equation and the term without x on the other side.

$$3x^2 + 10x = 32 \quad [\text{Transposing}]$$

Step II Make the coefficient of x^2 unity and positive by dividing throughout by the coefficient of x^2 .

$$x^2 + \frac{10}{3}x = \frac{32}{3}$$

Step III Add to each side of the equation the square of half the coefficient of x .

$$x^2 + \frac{10}{3}x + \left(\frac{5}{3}\right)^2 = \frac{32}{3} + \left(\frac{5}{3}\right)^2$$

$$\left(x + \frac{5}{3}\right)^2 = \frac{32}{3} + \frac{25}{9}$$

$$\left(x + \frac{5}{3}\right)^2 = \frac{96 + 25}{9} = \frac{121}{9}$$

Step IV Taking square root of both the side.

$$\therefore \pm \left(x + \frac{5}{3}\right) = \frac{11}{3}$$

$$x + \frac{5}{3} = \pm \frac{11}{3}$$

$$x + \frac{5}{3} = \frac{11}{3} \quad \text{or} \quad x + \frac{5}{3} = -\frac{11}{3}$$

$$x = \frac{11}{3} - \frac{5}{3} \quad \text{or} \quad x = -\frac{11}{3} - \frac{5}{3}$$

$$x = \frac{6}{3} = 2 \quad \text{or} \quad x = \frac{-16}{3} = -5\frac{1}{3}$$

Hence, solution set is $\left\{2, -5\frac{1}{3}\right\}$.**Skilled Practice!**

Solve by completing square method.

(i) $5x^2 - 18 = 2x$

(ii) $2x^2 - 7x = -20$

Challenge!The square of every real quantity is non-negative and therefore greater than zero. Thus, $(a - b)^2$ is non-negative i.e., $a^2 - 2ab + b^2 \geq 0$. Do you agree:

$$\frac{x+y}{2} > \sqrt{xy}$$

c Solution of a Quadratic Equation by Quadratic Formula

Derivation of Quadratic Formula by Completing Square Method

After suitable reduction and transposition every quadratic equation can be written in the form: $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

Now, $ax^2 + bx + c = 0$

We can rewrite the equation as:

Step I $ax^2 + bx = -c$ [Transposing]

Step II Dividing by a (the co-efficient of x^2), on both sides of the equation.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Step III We add square of half of the co-efficient of x i.e; $\left(\frac{b}{2a}\right)^2$ on both side of the equation.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Remember!

$ax^2 + bx + c = 0$, can equivalently be expressed as:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{since, } a \neq 0$$

Step IV Taking square root of both the sides.

$$\pm \left(x + \frac{b}{2a}\right) = \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

or $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ [Transposing]

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Remember!

Moving a term from one side of an equation to the other by changing its sign is called transposition

This is the quadratic formula.

Example 4 Solve by using quadratic formula: $3x^2 + 7x - 6 = 0$.

Solution $3x^2 + 7x - 6 = 0$ (given)
 $ax^2 + bx + c = 0$ (standard form)

Comparing the given equation with the standard form.

$$a = 3, b = 7, c = -6$$

Substituting the values of a, b, c in the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-6)}}{2(3)}$$

$$= \frac{-7 \pm \sqrt{49 + 72}}{6} = \frac{-7 \pm \sqrt{121}}{6}$$

$$= \frac{-7 \pm 11}{6}$$

$$= \frac{-7 + 11}{6}, \frac{-7 - 11}{6}$$

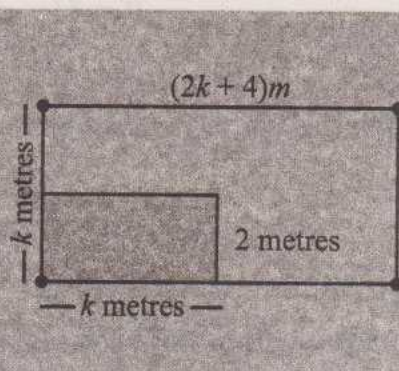
$$= \frac{4}{6}, \frac{-18}{6}$$

$$x = \frac{2}{3}, -3$$



Activity

A rectangular garden is in front of Kiran's house, whose dimensions are $(2k + 4)$ metres and k metres. A smaller rectangular portion of the garden of dimensions k metres and 2 metres is levelled. Find the area of unlevelled portion of the garden.



Hence, solution set is $\left\{\frac{2}{3}, -3\right\}$

d Solution of Quadratic Equation by Graphs

To solve the quadratic equation graphically means to find the points where the graph of quadratic equation crosses the x -axis. The x -values of these points are the solution of the quadratic equation. If the coefficient of x^2 in the quadratic equation is positive, then the graph will be $\cup$ shaped and if the coefficient of x^2 in quadratic equation is negative, then the graph will be $\cap$ shaped. To sketch the graph of quadratic function, we find its vertex by completing square method. $a(x - h)^2 + k$ is called the vertex form of quadratic function having vertex (h, k) and a is the coefficient of x^2 .

Do you know?

The graph of a quadratic function is called parabola.

Example 5 (i) Sketch and interpret the graph of quadratic function

$$y = 2x^2 - x - 10.$$

(ii) Solve the quadratic equation $2x^2 - x - 10 = 0$ graphically.

Solution

(i) Let us sketch the graph of $y = 2x^2 - x - 10$.

$$\begin{aligned} y &= 2\left(x^2 - \frac{1}{2}x - 5\right) = 2\left[x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 5\right] \\ &= 2\left[\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} - 5\right] \\ &= 2\left[\left(x - \frac{1}{4}\right)^2 - \frac{81}{16}\right] = 2\left(x - \frac{1}{4}\right)^2 - \frac{2(81)}{16} \\ &= 2\left(x - \frac{1}{4}\right)^2 - \frac{81}{8} = 2(x - 0.25)^2 - 10.13 \end{aligned}$$

Compare it with $y = a(x - h)^2 + k$. So, the vertex is $(0.25, -10.13)$. To sketch the graph of quadratic function we take some more points.

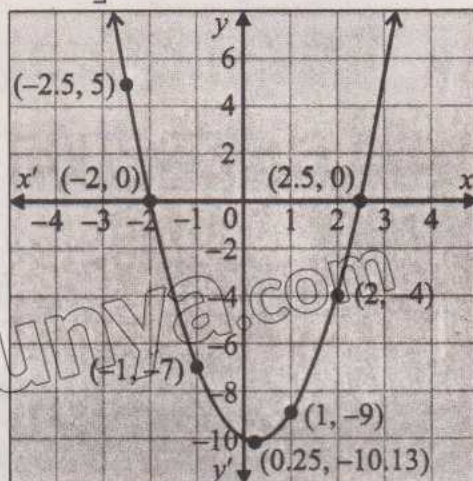


Figure 2.1

x	-2.5	-1	1	2
y	5	-7	-9	-4

By plotting the points and vertex, graph is sketched above. The graph of the given quadratic function $y = 2x^2 - x - 10$ opens upwards and it intersects the x -axis at $(-2, 0)$ and $(2.5, 0)$.

(ii) The graph intersects the x -axis at $x = 2.5$ and $x = -2$. Therefore, roots of the quadratic equation $2x^2 - x - 10 = 0$ are 2.5 and -2.

Hence, solution set is $\{2.5, -2\}$.

Remember!

For finding the solution of quadratic equation $ax^2 + bx + c = 0$ graphically, first of all we draw the graph of quadratic function $y = ax^2 + bx + c$.

EXERCISE

1. Write the following quadratic equations in standard form:

(i) $3x - 1 = 2x^2$

(ii) $2x(x + 1) = 4(2x + 3)$

(iii) $2x^2 - 4x = 4x + 7$

(iv) $4(3x - 2) = 9x^2$

(v) $2x + \frac{1}{x} = 5 - \frac{1}{x}, x \neq 0$

(vi) $\frac{6x+6}{20-x} = \frac{1}{x}, x \neq 0, 20$

2. Solve the following quadratic equations by factorization method:

(i) $x^2 - x - 6 = 0$

(ii) $x^2 + 3x - 28 = 0$

(iii) $6x^2 + 13x - 5 = 0$

(iv) $x^2 - \frac{3}{2}x = \frac{9}{2}$

(v) $\frac{3x-8}{x-2} = \frac{5x-2}{x+5}, x \neq 2, -5$

(vi) $\frac{1}{x-1} - \frac{1}{x+3} = \frac{1}{35}, x \neq 1, -3$

3. Solve the following quadratic equations by completing square method:

(i) $2x^2 + 5x + 2 = 0$

(ii) $x^2 + x = 42$

(iii) $12x^2 + 7x = 12$

(iv) $\frac{x+3}{2x-7} = \frac{2x-1}{x-3}, x \neq \frac{7}{2}, 3$

(v) $\frac{1}{1+x} - \frac{1}{3-x} = \frac{6}{35}, x \neq -1, 3$

(vi) $\frac{3x-1}{4x+7} = 1 - \frac{6}{x+7}, x \neq -\frac{7}{4}, -7$

4. Use quadratic formula to solve the following equations:

(i) $2x^2 - 5x + 3 = 0$

(ii) $2x^2 - 7x - 15 = 0$

(iii) $2x^2 + 7x - 15 = 0$

(iv) $x^2 + 11 = 7x$

(v) $\frac{x+4}{x-4} + \frac{x-2}{x-3} = 6\frac{1}{3}, x \neq 4, 3$

(vi) $\frac{3x-3}{x+1} = \frac{2x-1}{x-1}, x \neq -1, 1$

5. Solve the following quadratic equations graphically:

(i) $x^2 - 3x - 18 = 0$

(ii) $x^2 - 5x - 14 = 0$

(iii) $2x^2 + 13x + 6 = 0$

(iv) $4x^2 + 12x - 27 = 0$

2.2 Finding Intersection Point(s) Graphically

2.2.1 Finding Points of Intersection of a Linear Function and Coordinate Axes Graphically

Example 6 Find the points of intersection of $y = 2x + 4$ with coordinate axes graphically.

Solution $y = 2x + 4$ is an equation of a line. Now, we draw graph of $y = 2x + 4$.

Table of values

x	1	-4
y	6	-4

Remember!

To draw a straight line, it is essential to have at least two distinct points.

We plot points $(1, 6)$, $(-4, -4)$ and join these points.

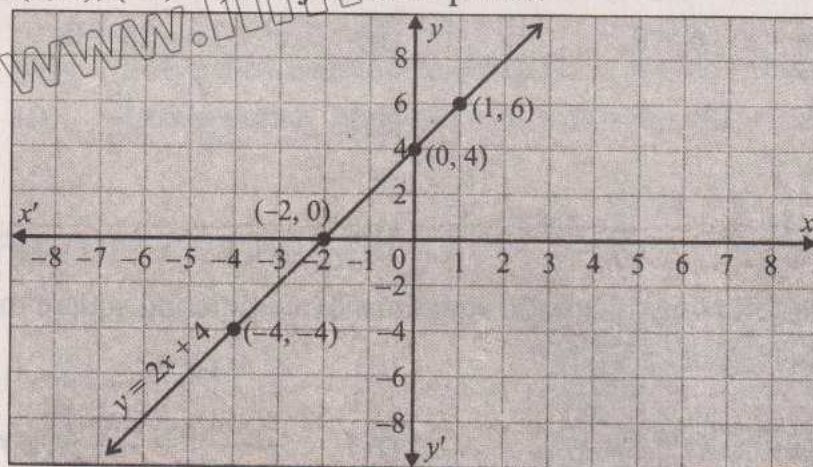


Figure 2.2

Point of intersection on x -axis is $(-2, 0)$ and the point of intersection on y -axis is $(0, 4)$.

2.2.2 Finding Point of Intersection of Two Linear Functions Graphically

Example 7 If $x + y = 6$ and $3x + y = 12$, then find their point of intersection graphically.

Solution $x + y = 6$ and $3x + y = 12$ are two linear equations. We draw graphs of both of these equations.

Table of values for $x + y = 6$

x	0	-4
y	6	10
Points are: $(0, 6)$, $(-4, 10)$		

Table of values for $3x + y = 12$

x	0	2
y	12	6
Points are: $(0, 12)$, $(2, 6)$		

Now, we plot points for both of these equations.

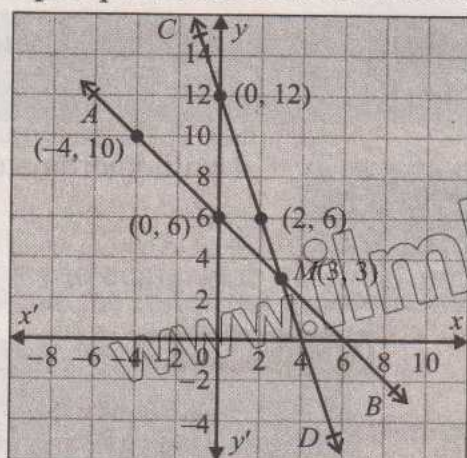


Figure 2.3

$\overleftrightarrow{AB}$ is the graph of $x + y = 6$

$\overleftrightarrow{CD}$ is the graph of $3x + y = 12$

Hence, the point of intersection is $M(3, 3)$.

2.2.3 Finding Points of Intersection of a Linear Function and a Quadratic Function Graphically

Example 8 If $y = 2 + 3x$ and $y = 2x^2 + 7x - 4$, then find their points of intersection graphically.

Solution We draw graphs of both of these equations.

Table of values for $y = 2 + 3x$

x	0	-2
y	2	-4
Points	(0, 2)	(-2, -4)

Now, $y = 2x^2 + 7x - 4$

$$= 2 \left(x^2 + \frac{7}{2}x - 2 \right) = 2 \left(x^2 + \frac{7}{2}x + \left(\frac{7}{4} \right)^2 - \left(\frac{7}{4} \right)^2 - 2 \right)$$

$$= 2 \left[\left(x + \frac{7}{4} \right)^2 - \frac{49}{16} - 2 \right] = 2 \left[\left(x + \frac{7}{4} \right)^2 - \frac{81}{16} \right] = 2 \left(x + \frac{7}{4} \right)^2 - 2 \left(\frac{81}{16} \right)$$

$$= 2(x + 1.75)^2 - 10.13$$

So, the vertex is $(-1.75, -10.13)$.

Table of values for $y = 2x^2 + 7x - 4$

x	-4	-3	0	1
y	0	-7	-4	5

We plot points of both equations and draw their graphs.

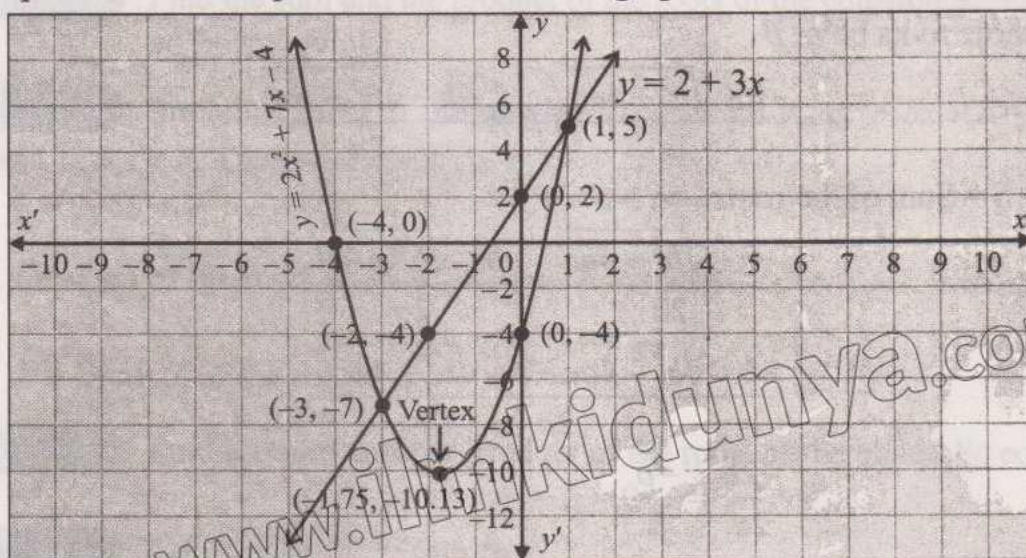


Figure 2.4

Points of intersection are : $(1, 5)$ and $(-3, -7)$

EXERCISE

2.2

- Find the points of intersection of the following linear equations with coordinate axes graphically:
 - $x + y = 8$
 - $x - y = 1$
 - $x - 2y = 1$
 - $x - 2y + 2 = 0$
 - $5x - 5y = 1$
- Solve the following system of linear equations graphically:
 - $x + y = 8$
 $3x - y = 4$
 - $x - y = 1$
 $x + 2y = 7$
 - $x - 2y = 1$
 $2x + y = 2$
 - $y = 2x + 2$
 $3x + 2y = 4$
 - $3y = 2x + 8$
 $x + y = 1$
- Solve the following equations graphically:
 - $y = 8x - 32$
 $y = x^2 - 6x + 8$
 - $y + x = 2$
 $y = 2x^2 + x - 10$

2.3

Relationship Between the Roots and Coefficients of a Quadratic Equation

The standard form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$ and $a, b, c \in R$. The roots of this equation are:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Let these roots be α, β

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Now $S = \text{Sum of the roots} = \alpha + \beta$

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a} = -\frac{b}{a}$$

$$S = \text{Sum of the roots} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\begin{aligned}
 P &= \text{Product of the roots} = \alpha\beta \\
 &= \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} \right) \\
 &= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{(2a)(2a)} = \frac{b^2 - (b^2 - 4ac)}{4a^2} \\
 &= \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}
 \end{aligned}$$

$$P = \text{Product of the roots} = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Important Note!

- (i) The standard form of a quadratic equation is $ax^2 + bx + c = 0$ where $a \neq 0$ and $a, b, c \in \mathbb{R}$.
- (ii) Sum of the roots $= -\frac{b}{a}$, Product of the roots $= \frac{c}{a}$.
- (iii) When the coefficient of x^2 in a quadratic equation is unity (one).
 - Sum of roots is equal to the coefficient of x with sign changed.
 - Product of the roots is equal to the constant term.
- (iv) If the constant term of $ax^2 + bx + c = 0$ is zero, then sum of its roots $= -\frac{b}{a}$ and products of the roots $= 0$.

Example 9 Find the sum and the product of the roots of the equation $3x^2 + 5x - 12 = 0$ without solving.

Solution Standard form of the quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ and $a, b, c \in \mathbb{R}$. The given equation is $3x^2 + 5x - 12 = 0$.

Here $a = 3$, $b = 5$, $c = -12$

$$S = \text{Sum of the roots} = -\frac{b}{a} = -\frac{5}{3}$$

$$P = \text{Product of the roots} = \frac{c}{a} = \frac{-12}{3} = -4$$

Example 10 Find the sum and the product of the roots of the quadratic equation $x^2 + 6x + 8 = 0$

Solution Coefficient of x^2 is unity.

Sum of the roots $= -6$ [Coefficient of x with sign changed]

Product of the roots $= 8$ [Constant term]

2.3.1 Form a Quadratic Equation When Roots are Given

The standard form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ and $a, b, c \in \mathbb{R}$.

Let α, β be its roots. We have established a relation between the roots and coefficients.

$$S = \text{Sum of the roots} = \alpha + \beta = -\frac{b}{a} \quad \dots \text{(i)}$$

$$P = \text{Product of the roots} = \alpha\beta = \frac{c}{a} \quad \dots \text{(ii)}$$

The given quadratic equation is $ax^2 + bx + c = 0$.

Dividing this equation by a , we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Writing this equation in terms of α, β

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad [\text{using (i) and (ii)}]$$

$$\text{or } x^2 - (\text{Sum of the roots})x + \text{Product of the roots} = 0$$

$$\text{or } x^2 - Sx + P = 0$$

Example 11 Form the equation whose roots are $\frac{2}{3}$ and $\frac{3}{4}$.

Solution **Method-I**

$$\text{Here } x = \frac{2}{3}, x = \frac{3}{4}$$

$$\therefore x - \frac{2}{3} = 0 \quad \text{or} \quad x - \frac{3}{4} = 0$$

Both of these statements are included in

$$\left(x - \frac{2}{3}\right)\left(x - \frac{3}{4}\right) = 0$$

$$x^2 - \frac{3}{4}x - \frac{2}{3}x + \frac{1}{2} = 0$$

$$x^2 - \frac{17}{12}x + \frac{1}{2} = 0$$

$12x^2 - 17x + 6 = 0$ is the required equation.

Method-II

Roots of the required equation are $\frac{2}{3}, \frac{3}{4}$

$$\begin{aligned} S = \text{Sum of the roots} &= \frac{2}{3} + \frac{3}{4} \\ &= \frac{8+9}{12} = \frac{17}{12} \quad \dots (i) \end{aligned}$$

$$P = \text{Product of the roots} = \frac{2}{3} \times \frac{3}{4} = \frac{1}{2} \quad \dots (ii)$$

Required equation is

$$x^2 - Sx + P = 0$$

Putting values of S and P in above equation

$$x^2 - \frac{17}{12}x + \frac{1}{2} = 0 \quad [\text{From (i) and (ii)}]$$

$$\text{So, } 12x^2 - 17x + 6 = 0$$

Example 12

If α, β are roots of the equation $x^2 - 7x + 10 = 0$, then form an equation whose roots are $2\alpha + 1$ and $2\beta + 1$.

Solution

As α, β are roots of $x^2 - 7x + 10 = 0$, therefore

$$\alpha + \beta = -\left(\frac{-7}{1}\right) = 7 \quad \dots (i), \quad \alpha\beta = \frac{10}{1} = 10 \quad \dots (ii)$$

$$S = \text{Sum of roots of the required equation} = 2\alpha + 1 + 2\beta + 1$$

$$= 2(\alpha + \beta) + 2$$

$$= 2(7) + 2 = 16$$

$$P = \text{Product of roots of the required equation} = (2\alpha + 1)(2\beta + 1)$$

$$= 4\alpha\beta + 2(\alpha + \beta) + 1$$

$$= 4(10) + 2(7) + 1 = 55$$

The required equation will be

$$x^2 - Sx + P = 0$$

$$x^2 - 16x + 55 = 0 \quad [\text{Putting values of } S \text{ and } P]$$

Example 13

Find the condition that the roots of $ax^2 + bx + c = 0$ may be equal in magnitude but opposite in sign, where $a \neq 0$.

Solution $ax^2 + bx + c = 0$ (given equation)

Let $\alpha, -\alpha$ be the roots of this equation

$$S = \text{Sum of the roots} = \alpha + (-\alpha) = -\frac{b}{a}$$

$$0 = -\frac{b}{a} \quad \text{where } a \neq 0$$

$\therefore b = 0$ is the required condition.

Challenge!

If one root of the quadratic equation is twice the other and their product is 8, find the equation and both roots.

EXERCISE 2.3

- Form a quadratic equation whose roots are given below:
 - $-4, 9$
 - $5, -7$
 - $\frac{-7}{5}, \frac{-6}{5}$
 - $\frac{-3}{2}, \frac{7}{2}$
 - $3 + \sqrt{5}, 3 - \sqrt{5}$
 - $-2 + \sqrt{3}, -2 - \sqrt{3}$
- Find the quadratic equation with roots exceeding by 2 than those of roots of $x^2 + 9x + 20 = 0$.
- Find the equation whose roots are double the roots of $x^2 - px + q = 0$.
- If α, β are the roots of the equation $x^2 + 2x + 4 = 0$, then find the equation whose roots are:
 - $\frac{1}{\alpha}, \frac{1}{\beta}$
 - $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$
 - $2\alpha - \frac{1}{\beta}, 2\beta - \frac{1}{\alpha}$
 - α^2, β^2
 - $2\alpha - 1, 2\beta - 1$
- Find the condition that roots of $ax^2 + bx + c = 0$ should be reciprocals of each other.
- Find the value of k , given that one root of $x^2 - (2k + 4)x + (7k + 1) = 0$ is 3.
- Find the value of m in the equation $2x^2 + 3x + m = 0$ when sum of its roots is equal to double the product of its roots.
- If α, β are the roots of $x^2 + ax + b = 0$ and α^2, β^2 are the roots of $x^2 + Ax + B = 0$, then prove that $A = 2b - a^2, B = b^2$.
- If α, β are the roots of $x^2 + px + q = 0$, then find the condition that
 - $\alpha = \beta$
 - $\alpha = \frac{1}{\beta}$

2.4 Discriminant of a Quadratic Equation

The quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$ and $a, b, c \in R$ has two roots.

Let these roots be α and β .

$$\text{Now, } \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Discriminant

The expression $b^2 - 4ac$, under the radical sign in quadratic formula is called the Discriminant (Disc.) of the equation, $ax^2 + bx + c = 0$. It determines the nature of character of the roots of the equation.

2.4.1 Nature of the Roots of a Quadratic Equation

Before proceeding to discuss the nature or character of the roots of a quadratic equation, it is important that the students should clearly distinguish among the terms rational, irrational, real and imaginary.

Now, $\sqrt{25}$ or 5, $4\frac{1}{2}$, $\frac{5}{6}$ are rational and real. $\sqrt{7}$, $\sqrt{5}$, $\sqrt{3}$ are irrational and real.

$\sqrt{-3}$, $\sqrt{-7}$, $\sqrt{-4}$ are imaginary or unreal.

As we know that the two roots of $ax^2 + bx + c = 0$, where $a \neq 0$ and $a, b, c \in R$

$$\text{are } \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Then we have the following results:

- (i) If $b^2 - 4ac > 0$ and is a perfect square, then the roots are rational (real) and unequal.
- (ii) If $b^2 - 4ac > 0$ and is not a perfect square, then the roots are irrational (real) and unequal.
- (iii) If $b^2 - 4ac = 0$, then the roots are real and equal.
- (iv) If $b^2 - 4ac < 0$, then the roots are imaginary (complex conjugates).

By applying these tests, the nature of the roots of any quadratic equation may be determined without solving the equation.

Example 14 Examine the nature of roots of the following quadratic equations:

(i) $3x^2 - 8x - 2 = 0$

(ii) $4x^2 + 4x + 1 = 0$

(iii) $2x^2 + 3x + 4 = 0$

(iv) $x^2 + 4x + 3 = 0$

Solution(i) Here, $a = 3$, $b = -8$, $c = -2$

$$\begin{aligned}\text{Disc.} &= b^2 - 4ac \\ &= (-8)^2 - 4(3)(-2) \\ &= 64 + 24 = 88 > 0\end{aligned}$$

As the discriminant is positive but not a perfect square. Therefore, the roots are irrational and unequal.

(iii) Here, $a = 2$, $b = 3$, $c = 4$

$$\begin{aligned}\text{Disc.} &= b^2 - 4ac \\ &= (3)^2 - 4(2)(4) \\ &= 9 - 32 \\ &= -23 < 0\end{aligned}$$

Which is negative. Therefore, the roots are imaginary.

Note

If the roots of $ax^2 + bx + c = 0$ are rational, then $b^2 - 4ac$ must be a perfect square.

Example 15

Find the value of k if the equation $(k+1)x^2 + (2k-1)x + (k-1) = 0$ has equal roots.

SolutionHere $a = k+1$, $b = 2k-1$, $c = k-1$

$$\text{Disc.} = b^2 - 4ac = 0 \quad (\text{roots are equal})$$

$$(2k-1)^2 - 4(k+1)(k-1) = 0$$

$$(4k^2 - 4k + 1) - 4(k^2 - 1) = 0$$

$$4k^2 - 4k + 1 - 4k^2 + 4 = 0$$

$$-4k + 5 = 0$$

$$4k = 5$$

$$k = \frac{5}{4}$$

EXERCISE**2.4**

1. Examine the nature of roots of the following quadratic equations:

(i) $3x^2 - 9x - 2 = 0$

(ii) $x^2 + 6x + 9 = 0$

(iii) $2x^2 + 4x + 5 = 0$

(iv) $7x^2 - 6x - 1 = 0$

(v) $5x^2 - 2x + 10 = 0$

(vi) $x^2 - 8x + 16 = 0$

2. For what values of x , the roots of $3x^2 + x + 9t = 0$ are real and unequal?
3. If the quadratic equation $16x^2 + 7px + 49 = 0$ has equal roots, then find the values of p .
4. If the quadratic equation $4u^2 + 8u + q = 0$ has unequal and real roots, find the possible values for q .
5. Find the value for m , if the quadratic equation $mx^2 - 8x + 1 = 0$ has real and equal roots.

2.5 Solution of Quadratic Inequalities in One Unknown

Inequality In Mathematics an inequality is a relation which makes a un-equal comparison between two numbers or other mathematical expressions. We have learned how to use the inequality symbols $<$ (less than), $>$ (greater than), $\leq$ (less than or equal to) and $\geq$ (greater than or equal to).

Here we learn how to solve quadratic inequalities in one unknown.

There are following four steps to solve quadratic inequality:

- | | |
|---------------------------------------|---------------------------|
| (i) Solve the corresponding equation. | (ii) Determine intervals. |
| (iii) Test points in each interval. | (iv) Conclusion. |

Example 16 Solve the inequality $x^2 - 5x + 6 > 0$.

Solution $x^2 - 5x + 6 > 0$

Step I Solve the corresponding or associated equation.

Set the quadratic expression equal to zero (0) to find the roots.

$$x^2 - 5x + 6 = 0$$

$$x^2 - 3x - 2x + 6 = 0$$

$$(x - 2)(x - 3) = 0 \quad [\text{Factorize}]$$

Now, $x - 2 = 0$ or $x - 3 = 0$

So, the solutions are $x = 2$ and $x = 3$.

Here, critical points (2, 0) and (3, 0).

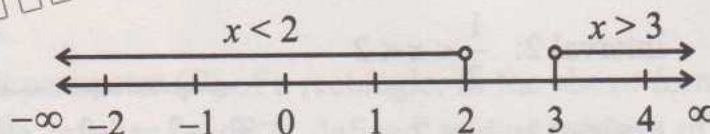
Step II Determine intervals

The critical points divide the number line into three intervals.

Interval 1: $x < 2$

Interval 2: $2 < x < 3$

Interval 3: $x > 3$



Step III Test points in each interval.

Choose a test point in each interval and substitute it into the inequality.

For $x = 1$: $(1 - 2)(1 - 3) = 2 > 0$.

So, this interval satisfies the inequality.

For $x = 2.5$: $(2.5 - 2)(2.5 - 3) = -0.25 < 0$.

So, this interval does not satisfy the inequality.

For $x = 4$: $(4 - 2)(4 - 3) = 2 > 0$.

So, this interval satisfies the inequality.

Step IV Conclusion:

The solution of this inequality $x^2 - 5x + 6 > 0$

is $x \in (-\infty, 2) \cup (3, \infty)$

Example 17 Solve the inequality $2x^2 - 5x + 2 \leq 0$.

Solution $2x^2 - 5x + 2 \leq 0$

Step I Solve the corresponding or associated equation.

$$\begin{aligned} 2x^2 - 5x + 2 &= 0 \\ 2x^2 - 4x - x + 2 &= 0 \\ 2x(x - 2) - 1(x - 2) &= 0 \\ (2x - 1)(x - 2) &= 0 \end{aligned}$$

Now, $2x - 1 = 0$ or $x - 2 = 0$

So, the solutions are $x = \frac{1}{2}$ and $x = 2$

Here, critical points are $\left(\frac{1}{2}, 0\right)$ and $(2, 0)$

Note

For solution, we use open interval (a, b) when equation has $<$ or $>$ sign and closed interval $[a, b]$ when equation has $\leq$ or $\geq$ sign.

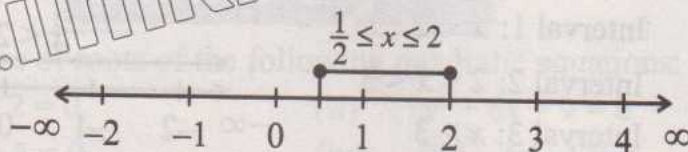
Step II Determine Intervals

The critical points $\left(\frac{1}{2}, 0\right)$, $(2, 0)$ divides the number line into three intervals.

Interval 1: $x < \frac{1}{2}$

Interval 2: $\frac{1}{2} < x < 2$

Interval 3: $x > 2$



Step III Test points in each interval.

For $x = 0$: $(2(0) - 1)(0 - 2) = (-1)(-2) = 2 > 0$,

So, this interval does not satisfy the inequality.

For $x = 1$: $(2(1) - 1)(1 - 2) = (1)(-1) = -1 < 0$,

So, this interval satisfies the inequality.

For $x = 3$: $(2(3) - 1)(3 - 2) = (6 - 1)(1) = 5 > 0$,

So, this interval does not satisfy the inequality.

Step IV Conclusion:

The solution of the inequality is: $x \in \left[\frac{1}{2}, 2\right]$.

EXERCISE 2.5

1. Solve the following inequalities:

(i) $x^2 + 3x - 4 > 0$

(ii) $2x^2 - 8x + 6 > 0$

(iii) $x^2 + x - 6 < 0$

(iv) $x^2 - 6x + 9 < 0$

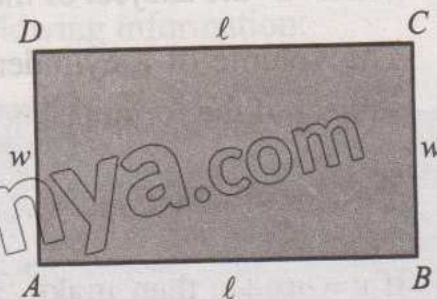
(v) $4x^2 - 16x + 15 \leq 0$

(vi) $-x^2 + 3x - 2 \geq 0$

2.6 Changing the Subject of Formula

A formula expresses the precise relationship between various measurable quantities in a problem or situation in an equation form.

The figure on the right is of a rectangle. Its length is ℓ units and width is w units. Let P be its perimeter.



Now,
$$P = \ell + w + \ell + w$$

$$= 2\ell + 2w$$

$$P = 2(\ell + w) \text{ units}$$

We have constructed a formula for perimeter (P) of a rectangle. In the above formula $P = 2(\ell + w)$, the letter P on the left-hand-side is defined as the **subject** of the formula, while the letters on the right-hand-side are called terms of the formula.

Perimeter (P) of square of length (ℓ) is $P = 4\ell$.

We change the subject P and rewrite this formula as $\ell = \frac{1}{4}P$, ℓ has become subject of the formula. If value of P is given, then we can directly find the value of ℓ .

Example 18 The area of a circle is $A = \pi r^2$. Make r the subject of the formula.

Solution

$$A = \pi r^2$$

or $\pi r^2 = A$

$$r^2 = \frac{A}{\pi}$$

$$\therefore r = \sqrt{\frac{A}{\pi}} \quad [r \text{ is always positive}]$$

EXERCISE 2.6

1. Make F the subject of the formula, $C^\circ = \frac{5}{9}(F^\circ - 32)$.
2. The formula for finding simple interest is $I = PRT$.
 - (a) Make P the subject of the formula.
 - (b) Make T the subject of the formula.
3. Make ' a ' the subject of the formula $S = 2a + (n - 1)d$.
4. The volume of a cylinder is given by the formula, $V = \pi r^2 h$. Make ' h ' the subject of the formula.
5. The area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$, make h the subject of the formula.
6. If $y = mx + c$, then make ' x ' the subject of this equation.
7. Perimeter (P) of a rectangle is $P = 2(\ell + w)$, make ℓ as the subject of this formula.
8. The equation of a parabola is $y^2 = 4ax$, make ' x ' as a subject of this equation.

9. If $P = S - C$, where S is selling price and C is cost price. Make S as subject of the equation.
10. Volume of the cone is $V = \frac{1}{3}\pi r^2 h$, make ' h ' as subject of this formula.

2.7 Applications of Quadratic Equation in Real World

Example 19 A ball is thrown upward with an initial velocity of 20 ms^{-1} . Calculate the maximum height it reaches above ground level. Calculate time of return to the ground.

Solution Maximum height

At its highest point, the ball's velocity is zero. So, we have the following quantities:

$$\text{Initial velocity} = v_i = 20\text{ ms}^{-1}$$

$$\text{Final velocity} = v_f = 0\text{ ms}^{-1}$$

$$\text{Gravitational acceleration} = g = -10\text{ ms}^{-2}$$

$$\text{Distance (displacement)} = s = ?$$

The relevant equation of motion is $v_f^2 = v_i^2 + 2gs$

Substituting values of v_i , v_f and g , we get

$$0^2 = (20)^2 + 2(-10)s$$

$$0 = 400 - 20s$$

$$\text{so, } s = \frac{400}{20} = 20\text{ m}$$

Maximum Time of Return: When the ball returns to the point from which it was thrown, its displacement (s) is zero. We have the following information:

$$\text{Displacement} = s = 0$$

$$\text{Initial velocity} = v_i = 20\text{ ms}^{-1}$$

$$\text{Gravitational acceleration} = g = -10\text{ ms}^{-2}$$

$$\text{Time} = t = ?$$

The relevant equation of motion:

$$s = v_i t + \frac{1}{2}gt^2$$

Putting the values of s , v_i and g

Note

The equations of motion describe the relationship between displacement, velocity, time and acceleration for objects moving in a straight line with uniform acceleration.

$$\text{First equation : } v_f = v_i + at$$

$$\text{Second equation : } s = v_i t + \frac{1}{2}at^2$$

$$\text{Third equation : } v_f^2 = v_i^2 + 2as$$

$$0 = 20t + \frac{1}{2}(-10)t^2$$

$$0 = 20t - 5t^2$$

$$0 = t[20 - 5t]$$

Now $t = 0$ sec or $20 - 5t = 0 \Rightarrow t = \frac{20}{5} = 4$ sec

- (i) $t = 0$ sec. means, the ball had zero displacement at the time it was thrown.
 (ii) $t = 4$ sec. means, the ball returned to the ground after 4 sec, it is the return time or the required time.

Example 20 A company models its profit P (rupees in thousands) by the equation $P(x) = -2x^2 + 80x - 500$, where x is the price per item (in rupees). Find the price that gives the maximum profit and calculate the maximum profit.

Solution $P(x) = -2x^2 + 80x - 500$

Since $-2 < 0$, the parabola opens downward and vertex gives the maximum point.

$$x = \frac{-b}{2a} = \frac{-80}{2(-2)} = \frac{-80}{-4} = 20$$

Hence, the price per item that gives maximum profit is Rs. 20.

$$\begin{aligned} P(20) &= -2(20)^2 + 80(20) - 500 \\ &= -2(400) + 1600 - 500 \\ &= 300 \text{ (in thousands)} \end{aligned}$$

Hence, maximum profit is Rs. 300,000.

Note

We can also find vertex by using $x = \frac{-b}{2a}$, $y = f\left(\frac{-b}{2a}\right)$

Example 21 A ball is thrown upwards from the ground. Its height after t seconds is modeled by:

$$h(t) = -5t^2 + 20t \text{ (in metres)}$$

For what time interval is the ball at least 15 metres above the ground?

Solution

Here, $-5t^2 + 20t \geq 15$

$$-5t^2 + 20t - 15 \geq 0$$

$$5t^2 - 20t + 15 \leq 0$$

By solving the quadratic equation $5t^2 - 20t + 15 = 0$,

$$\begin{aligned} t &= \frac{-(-20) \pm \sqrt{(-20)^2 - 4(5)(15)}}{2(5)} \\ &= \frac{20 \pm \sqrt{400 - 300}}{10} \\ &= \frac{20 \pm \sqrt{100}}{10} \\ &= \frac{20 \pm 10}{10} \\ &= \frac{20-10}{10}, \frac{20+10}{10} \\ &= 1, 3 \end{aligned}$$

So, the ball is at least 15 metres high when $1 \leq t \leq 3$ seconds.

EXERCISE

2.7

1. A town's population is modeled by $P(t) = -2t^2 + 40t + 800$, where t is years since 2020. Find the years when the population will be at least 1000.
2. A company models its profit P in thousands of rupees by the equation: $P(x) = -5x^2 + 150x - 1000$, where x is the price per item in rupees. Find the price that gives maximum profit.
3. A toy car rolls down an incline and covers a distance given by the equation $d = t^2 - 0.5t$ metres, where t is the time in seconds. Find the time when the car has travelled a distance 12.5 metres.
4. A ball's height (in metres) after t seconds is $h(t) = -4t^2 + 24t$. For what time interval is the ball at least 20m above the ground?
5. A ball is thrown upward with an initial velocity of 40 ms^{-1} . Calculate the maximum height it reaches above ground level.
6. A freelancer's earnings follow the model $E(h) = -2h^2 + 40h$, where E is earning in rupees and h is hours worked per week. What is the maximum number of hours he should work to maximize earnings?

REVIEW EXERCISE

2

1. Four possible answers are given for the following questions. Choose the correct answer:

- (i) The type of the equation $2x^2 - x + 1 = 0$ is:
(a) Quadratic (b) linear (c) third degree (d) Pure quadratic
- (ii) What is the discriminant of $x^2 + 5x - 5 = 0$?
(a) -20 (b) 20 (c) 25 (d) 45
- (iii) The solution set of $3x^2 - 9 = 0$ is:
(a) $\{3\}$ (b) $\{\pm 3\}$ (c) $\{\pm\sqrt{3}\}$ (d) $\{\sqrt{3}\}$
- (iv) Sum of the roots of $3x^2 + 5x - 12 = 0$ is:
(a) $\frac{5}{3}$ (b) $-\frac{5}{3}$ (c) $\frac{12}{3}$ (d) $\frac{3}{5}$
- (v) Product of the roots of $3x^2 + 5x - 12 = 0$ is:
(a) -4 (b) 3 (c) 4 (d) 5
- (vi) What are the roots of $(x - 3)(x + 3) = 0$?
(a) 3, -3 (b) 3, 3 (c) -3, -3 (d) 9, 0
- (vii) 3 and 2 are the roots of:
(a) $x^2 + 5x + 6 = 0$ (b) $x^2 + 6x + 5 = 0$
(c) $x^2 - 5x + 6 = 0$ (d) $x^2 + 6x - 5 = 0$
- (viii) If $b^2 - 4ac > 0$ and is a perfect square, then the roots of $ax^2 + bx + c = 0$ are:
(a) equal (b) unequal (c) imaginary (d) irrational
- (ix) If $b^2 - 4ac = 0$, then the roots of $ax^2 + bx + c = 0$ are:
(a) unequal (b) irrational (c) imaginary (d) equal
- (x) Subject "c" of $x - 2c = b$ is:
(a) $x + b$ (b) $b - x$ (c) $\frac{x - b}{2}$ (d) $\frac{b - x}{2}$

2. Solve the following quadratic equations by factorization method, by completing square method and by quadratic formula:

(i) $8x^2 = x + 7$

(ii) $2x^2 - x - 10 = 0$

3. Form a quadratic equation whose roots are; $6, \frac{3}{2}$.

4. Examine the nature of the roots of the following equations:

(i) $15x^2 + 11x + 2 = 0$

(ii) $x^2 - x - 1 = 0$

5. If a ball is thrown upward with a velocity v , the maximum height it reaches can be determined by using a formula $h = \frac{v^2}{2g}$. Rearrange the formula to make v the subject.

6. If the equation $x^2 + 2(1 + k)x + k^2 = 0$ has equal roots, then find the value of k .