UNIT 01 Complex Numbers

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Identify complex numbers, complex conjugate, absolute value or modulus of a complex number.
- Apply algebraic properties and perform basic operations on complex numbers.
- Demonstrate additive identity and multiplicative identity for the set of complex numbers.
- Find additive inverse and multiplicative inverse of a complex number z.
- Demonstrate the following properties of a complex number z.
  - $|z| = |-z| = |\overline{\overline{z}}| = |-\overline{z}|$
  - $\overline{\overline{z}} = z, z\overline{z} = |z|^2$
  - $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
  - $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}; \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$
- Apply the Geometric interpretation of a complex number, modulus of a complex number and algebraic operations of a complex number.
- Find real and imaginary parts of complex numbers of the type $(x + iy)^n$ and $\left(\dfrac{x_1 + iy_1}{x_2 + iy_2}\right)^n, x_2 + iy_2 \neq 0$ where $n = \pm 1$ and $n = \pm 2$.
- Solve the simultaneous linear equations with complex coefficients.

1.1 Complex Numbers

A complex number is a number expressed in the form $x + iy$, where x and y are real numbers and i is the imaginary unit, defined by $i^2 = -1$.

Example 1: Simplify the following:

(i) i^7 **(ii)** i^8 **(iii)** i^{17} **(iv)** i^{-25}

Solution:

(i) $i^7 = i^6 \times i = (i^2)^3 \times i = (-1)^3 \times i = -i$

(ii) $i^8 = (i^2)^4 = (-1)^4 = 1$

(iii) $i^{17} = i^{16} \times i = (i^2)^8 \times i = (-1)^8 \times i = i$

(iv) $i^{-25} = \dfrac{1}{i^{25}} = \dfrac{1}{i^{24} \times i} = \dfrac{1}{i^{(2)12} \times i} = \dfrac{1}{(-1)^{12} \times i} = \dfrac{1}{i} = \dfrac{1}{i} \times \dfrac{i}{i} = \dfrac{i}{i^2} = \dfrac{i}{-1} = -i$

> **Do you know?**
> Complex numbers are essential for many technologies like smartphone signal processing and MRI imaging.

Rectangular Form of a Complex Number

A complex number is of the form $x + iy$ (or $x + yi$), where x and y are real numbers, x is called the real part and y is called the imaginary part of the complex number.

(i) If $x = 0$, the complex number is said to be pure imaginary.

(ii) If $y = 0$, the complex number is said to be real.

(iii) It is customary to denote the standard rectangular form of a complex number $x + iy$ as z and we write $x = Re(z)$ and $y = Im(z)$.

For example, $Re(5 - 7i) = 5$ and $Im(5 - 7i) = -7$

Equality of Complex Numbers

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$. i.e. $x_1 = x_2$ and $y_1 = y_2$.

For example, if $\alpha + i\beta = -2 + 5i$, then $\alpha = -2, \beta = 5$.

Example 2: If $(2x - 1) + (y + 4)i = 5 + 7i$, then find the values of x and y.

Solution:

Given that $(2x-1) + (y + 4)i = 5 + 7i$

Since the two complex numbers are equal, their real and imaginary parts must be equal.

$$\therefore \quad 2x - 1 = 5 \quad , \quad y + 4 = 7$$
$$2x = 6 \quad , \quad y = 7 - 4$$
$$x = 3 \quad , \quad y = 3$$

Solved Exercise 1.1

1. Simplify the following:

(i) i^5

Sol. $i^5 = i^4 \times i = (i^2)^2 \times i = (-1)^2 \times i = 1 \times i = i$ as, $i^2 = -1$

(ii) i^{16}

Sol. $i^{16} = (i^2)^8 = (-1)^8 = 1$ as, $i^2 = -1$

(iii) $(-i)^{-19}$

Sol. $(-i)^{-19} = \dfrac{1}{(-i)^{19}} = \dfrac{1}{(-1 \times i)^{19}} = \dfrac{1}{(-1)^{19} \times i^{19}} = \dfrac{1}{-1 \times i^{18} \times i}$

$$\frac{1}{-1 \times (i^2)^9 \times i} = \frac{1}{-1 \times (-1)^9 \times i} = \frac{1}{-1 \times -1 \times i} = \frac{1}{1 \times i} \qquad \text{as, } i^2 = -1$$

$$\frac{1}{i} = \frac{1 \times i}{i \times i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

(iv) $27i^{-26}$

Sol. $27i^{-26} = \dfrac{27}{i^{26}} = \dfrac{27}{(i^2)^{13}} = \dfrac{27}{(-1)^{13}} = \dfrac{27}{-1} = -27$ as, $i^2 = -1$

(v) $i^{11} + i^5$

Sol. $i^{11} + i^5 = i^{10} \times i + i^4 \times i = (i^2)^5 \times i + (i^2)^2 \times i$

$= (-1)^5 \times i + (-1)^2 \times i = -1 \times i + 1 \times i = -i + i = 0$ as, $i^2 = -1$

(vi) $(i^4 + i^3 + i^2 + i)^2$

Sol. $(i^4 + i^3 + i^2 + i)^2 \quad = [(i^2)^2 + i^2 \times i + (-1) + i]^2$ as, $i^2 = -1$

$= [(-1)^2 + (-1) \times i - 1 + i]^2 = (1 - i - 1 + i)^2$

$= (1 - 1 - i + i)^2 = (0)^2 = 0$

(vii) $\left(\dfrac{i^8}{i^5}\right)^{-5}$

Sol. $\left(\dfrac{i^8}{i^5}\right)^{-5} = (i^{8-5})^{-5} = (i^3)^{-5} = (i^3)^{-5} = i^{-15}$

$= \dfrac{1}{i^{15}} = \dfrac{1}{i^{14} \times i} = \dfrac{1}{(i^2)^7 \times i} = \dfrac{1}{(-1)^7 \times i} = \dfrac{1}{-1 \times i}$ as, $i^2 = -1$

$= \dfrac{1}{-i} = \dfrac{1 \times i}{-i \times i} = \dfrac{i}{-i^2} = \dfrac{i}{-(-1)} = \dfrac{i}{1} = i$

(viii) $i^{13} \times i^{29}$

Sol. $i^{13} \times i^{29} \quad = i^{13+29} = i^{42} = (i^2)^{21} = (-1)^{21} = -1$

2. Write in terms of i.

(i) $2 + \sqrt{-4}$

Sol. $2 + \sqrt{-4} = 2 + \sqrt{4 \times (-1)} = 2 + \sqrt{4} \times \sqrt{-1} = 2 + 2i$ as, $\sqrt{-1} = i$

(ii) $3 - \sqrt{-7}$

Sol. $3 - \sqrt{-7} = 3 - \sqrt{7 \times (-1)} = 3 - \sqrt{7} \times \sqrt{-1} = 3 - \sqrt{7}i$ as, $\sqrt{-1} = i$

(iii) $\dfrac{2}{5} + \dfrac{\sqrt{-16}}{5}$

Sol. $\dfrac{2}{5} + \dfrac{\sqrt{-16}}{5} = \dfrac{2 + \sqrt{-16}}{5} = \dfrac{2 + \sqrt{16 \times (-1)}}{5}$

$= \dfrac{2 + \sqrt{16} \times \sqrt{-1}}{5} = \dfrac{2 + 4i}{5} = \dfrac{2}{5} + \dfrac{4i}{5}$ as, $\sqrt{-1} = i$

(iv) $\sqrt{2} - \sqrt{-3}$

Sol. $\sqrt{2} - \sqrt{-3} = \sqrt{2} - \sqrt{3 \times (-1)} = \sqrt{2} - \sqrt{3} \times \sqrt{-1} = \sqrt{2} - \sqrt{3}i$ as, $\sqrt{-1} = i$

3. Find the values of x and y.

(i) $(2x + 5) + (y - 3)i = 1 + 2i$

Sol. $(2x + 5) + (y - 3)i = 1 + 2i$

Since the two complex numbers are equal, their real and imaginary parts must be equal.

$2x + 5 = 1 \quad , \quad y - 3 = 2$

$2x = 1 - 5 \quad , \quad y = 2 + 3$

$2x = -4 \quad , \quad y = 5$

$x = -2$

(ii) $(3x+2)-(4-y)i = 5+3i$

Sol. $(3x+2)-(4-y)i = 5+3i$

Since the two complex numbers are equal, their real and imaginary parts must be equal.

$$3x+2=5 \quad , \quad -(4-y)=3$$
$$3x=5-2 \quad , \quad -4+y=3$$
$$3x=3 \quad , \quad y=3+4$$
$$x=\frac{3}{3} \quad , \quad y=7$$
$$x=1$$

(iii) $(2+i)x+(1-2i)y = 3+4i$

Sol. $(2+i)x+(1-2i)y = 3+4i$

$$2x+xi+y-2yi = 3+4i$$
$$2x+y+xi-2yi = 3+4i$$
$$(2x+y)+(x-2y)i = 3+4i$$

Since the two complex numbers are equal, their real and imaginary parts must be equal.

$$2x+y=3\ ,\ x-2y=4$$
$$2x+y=3 \quad \text{\_\_\_\_\_\_ (i)}$$
$$x-2y=4 \quad \text{\_\_\_\_\_\_ (ii)}$$

Multiply eq (i) by 2

$$4x+2y=6 \quad \text{\_\_\_\_\_\_ (iii)}$$

Add eq (ii) and eq (iii)

$$\begin{array}{l} x-\cancel{2y}=4 \\ 4x+\cancel{2y}=6 \\ \hline 5x \qquad =10 \end{array}$$

$$x=\frac{10}{5} \Rightarrow x=2$$

Put $x=2$ in eq. (i)

$$2(2)+y=3$$
$$4+y=3$$
$$y=3-4 \Rightarrow y=-1$$

So, $x=2,\ y=-1$

(iv) $(1-i)x+(2+i)y = 4-i$

Sol. $(1-i)x+(2+i)y = 4-i$

$$x-xi+2y+yi = 4-i$$
$$x+2y-xi+yi = 4-i$$
$$(x+2y)+(-x+y)i = 4-i$$

Since the two complex numbers are equal, their real and imaginary parts must be equal.

$$x+2y=4 \quad \text{\_\_\_ (i)} \qquad -x+y=-1 \quad \text{\_\_\_ (ii)}$$

By adding eq (i) and eq (ii), we get

$$\begin{array}{l} \cancel{x}+2y=4 \\ \cancel{-x}+\ y=-1 \\ \hline 3y \qquad =3 \end{array}$$

$$y=\frac{3}{3}$$
$$y=1$$

Put $y=1$ in eq. (i)

$$x+2(1)=4$$
$$x+2=4$$
$$x=2$$

So, $x=2,\ y=1$

(v) $(3x-1)+(2y-3)i = 8+7i$

Sol. $(3x-1)+(2y-3)i = 8+7i$

Since the two complex numbers are equal, their real and imaginary parts must be equal.

$$3x-1=8 \quad , \quad 2y-3=7$$
$$3x=8+1 \quad , \quad 2y=7+3$$
$$3x=9 \quad , \quad 2y=10$$
$$x=\frac{9}{3} \quad , \quad y=\frac{10}{2}$$
$$x=3 \quad , \quad y=5$$

1.2 Algebraic Operations on Complex Numbers

Scalar Multiplication of Complex Numbers

If $z=x+iy$ and $k\in R$, then we define $kz=(kx)+(ky)i$ or $(kx)+i(ky)$

The following diagram shows kz for $k=2,\ \frac{1}{2},\ -1$

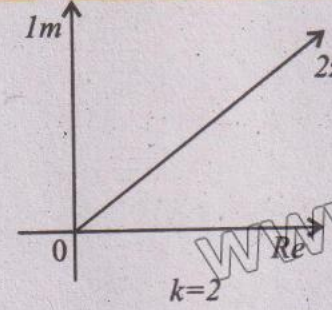


Figure 1

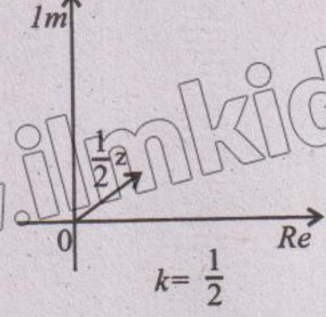


Figure 2

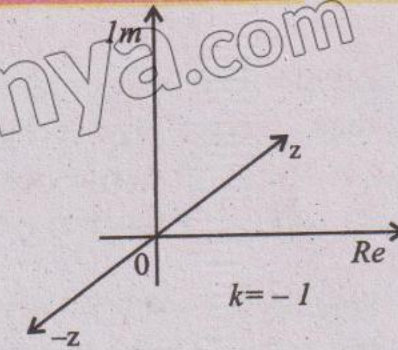
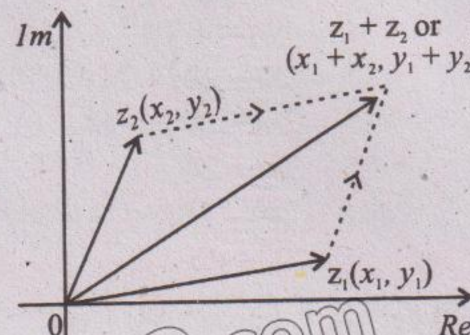


Figure 3

Addition of Two Complex Numbers



If $z_1 = x_1 + iy_1$ and $z_2 + iy_2$ where x_1, $x_2 + y_1$ and $y_2 \in$ R, then

$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$= (x_1 + x_2) + i\,(y_1 + y_2)$

When $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$, then by the parallelogram law of addition,

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

(i) If $z_1 = a + ib$, $z_2 = c + id$ where $a, b, c, d \in R$, then
$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i\,(b + d)$

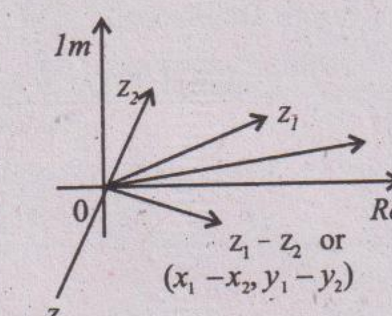
(ii) If $z_1 = a - ib$, $z_2 = c + id$, where $a, b, c, d \in R$, then
$z_1 + z_2 = (a - ib) + (c + id) = (a + c) + (-b + d)i$

Example 3: Add $(3 + 4i)$ and $(5 - 2i)$

Solution:

$(3 + 4i) + (5 - 2i)$
$= (3 + 5) + (4 - 2)i$
$= 8 + 2i$

Subtraction of Two Complex Numbers



If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where , $x_1, x_2, y_1, y_2 \in$ R, then $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i\,(y_1 - y_2)$

$z_1 - z_2 = z_1 + (-z_2)$
$= (x_1 + iy_1) + (-x_2 - iy_2)$
$= (x_1 - x_2) + i(y_1 - y_2)$

$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

Example 4: Simplify: (i) $(8 + 2i) - (5 - 6i)$ (ii) $(4 - 3i) - (2 - 5i)$

Solution:

(i) $(8 + 2i) - (5 - 6i)$
$= 8 + 2i - 5 + 6i$
$= (8 - 5) + (2i + 6i) = 3 + 8i$

(ii) $(4 - 3i) - (2 - 5i)$
$= 4 - 3i - 2 + 5i$
$= (4 - 2) + (-3 + 5)i = 2 + 2i$

Skilled Practice

If $z_1 - z_2 = 4 + 6i$ and $z_2 = 3 - 2i$, then find z_1.

Sol: Since $z_1 - z_2 = 4 + 6i$
$z_1 = 4 + 6i + z_2$
$z_1 = 4 + 6i + 3 - 2i$ as, $z_2 = 3 - 2i$
$z_1 = 7 + 4i$

Multiplication of Two Complex Numbers

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

then $\quad z_1z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$$= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2)$$

$$= x_1x_2 + x_1y_2i + y_1x_2i + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

Example 5

(i) Simplify: $(3 - 4i)(5 - 6i)$

(ii) If $z_1 = 2 + 3i$ and $z_2 = 4 + 7i$, then find z_1z_2.

Solution:

(i) $(3 - 4i)(5 - 6i)$

$= 3(5 - 6i) - 4i(5 - 6i)$

$= 15 - 18i - 20i + 24i^2$

$= 15 - (18 + 20)i - 24$

$= 15 - 38i - 24$

$= (15 - 24) - 38i = -9 - 38i$

(ii) $z_1z_2 = (2 + 3i)(4 + 7i)$

Using

$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

$z_1z_2 = (2 \times 4 - 3 \times 7) + i(2 \times 7 + 3 \times 4)$

$= (8 - 21) + (14 + 12)i = -13 + 26i$

Division of Two Complex Numbers

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

Now, $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2}, z_2 \neq 0$

$$= \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{x_1x_2 - ix_1y_2 + iy_1x_2 - i^2y_1y_2}{(x_2)^2 - (iy_2)^2}$$

$$= \frac{x_1x_2 - i(x_1y_2 - y_1x_2) + y_1y_2}{x_2^2 + y_2^2} \qquad \text{as, } i^2 = -1$$

$$= \frac{x_1x_2 + y_1y_2 - i(x_1y_2 - y_1x_2)}{x_2^2 + y_2^2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}$$

Skilled Practice!

1. Simplify: $\left(\dfrac{1+2i}{3-i}\right)(2+i)$

Sol: $\left(\dfrac{1+2i}{3-i}\right)(2+i)$

$$= \frac{(1+2i)(2+i)}{3-i} = \frac{2+i+4i+2i^2}{3-i}$$

$$= \frac{2+5i+2(-1)}{3-i}$$

$$= \frac{2+5i-2}{3-i} = \frac{0+5i}{3-i}$$

$$= \frac{5i}{3-i} = \frac{5i}{3-i} \times \frac{3+i}{3+i}$$

$$= \frac{15i+5i^2}{9-(-1)}$$

$$= \frac{15i+5(-1)}{9+1} = \frac{-5+15i}{10}$$

$$= \frac{-5}{10} + \frac{15i}{10} = \frac{-1}{2} + \frac{3}{2}i$$

2. Find the complex number z:

if $\frac{z}{2+i} = 3 - i$

Sol: $\frac{z}{2+i} = 3 - i$

$z = (3 - i)(2 + i)$

$= 3(2 + i) - i(2 + i)$

$= 6 + 3i - 2i - i^2$

$= 6 + i - (-1) = 6 + 1 + i = 7 + i$

So, $z = 7 + i$

Example 6: Express $\frac{3+4i}{5-7i}$ in the form of $x + iy$ or $x + yi$.

Solution: $\frac{3+4i}{5-7i} = \frac{3+4i}{5-7i} \times \frac{5+7i}{5+7i} = \frac{15+21i+20i+28i^2}{(5)^2-(7i)^2}$

$= \frac{15-28+(21+20)i}{25+49}$ as $i^2 = -1$

$= \frac{-13+41i}{74} = \frac{-13}{74} + \frac{41}{74}i$

Properties of Complex Numbers

The complex numbers satisfy the following properties under Addition.	The complex numbers satisfy the following properties under multiplication.
(i) Closure Property For any two complex numbers z_1 and z_2, the sum $z_1 + z_2$ is also a complex number.	**(i) Closure Property** For any two complex numbers z_1 and z_2, the product z_1z_2 is also a complex number.
(ii) The Commutative Property For any two complex numbers z_1 and z_2, $z_1 + z_2 = z_2 + z_1$	**(ii) The Commutative Property** For any two complex numbers z_1 and z_2. $z_1z_2 = z_2 z_1$
(iii) The Associative Property For any three complex numbers z_1, z_2 and z_3, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$	**(iii) The Associative Property** For any three complex numbers z_1,z_2 and z_3, $(z_1z_2)z_3 = z_1(z_2z_3)$
(iv) The Additive Identity There exists a complex number $0 = 0 + 0i$ such that, for every complex number z, $z + 0 = 0 + z = z$ The complex number $0 = 0 + 0i$ is known as additive identity.	**(iv) The Multiplicative Identity** There exists a complex number $1 = 1 + 0i$ such that, for every complex number z, $z \times 1 = 1 \times z = z$ The complex number $1 = 1 + 0i$ is known as multiplicative identity.

(v) The Additive Inverse

For every complex number z there exists a complex number $-z$ such that,

$$z+(-z)=(-z)+z=0.$$

$-z$ is called the additive inverse of z.

(v) The Multiplicative Inverse

For any non-zero complex number z, there exists a complex number w such that,

$$zw=wz=1$$

w is called the multiplicative inverse of z and it is denoted by z^{-1}.

(vi) Distributive Property of Multiplication Over Addition

For any three complex numbers z_1, z_2, and z_3

$$z_1(z_2+z_3)=z_1z_2+z_1z_3 \text{ and } (z_1+z_2)z_3=z_1z_3+z_2z_3$$

Example 7: If $z=-7-4i$, then show that $z+0=z$.

Solution:

$$z+0=(-7-4i)+(0+0i)$$
$$=-7-4i+0+0i$$
$$=-7-4i=z$$

Shows $z+0=z$ (0 is the additive identity)

Example 8: Verify the multiplicative identity for $z=3-2i$.

Solution: $z\times 1=(3-2i)(1+0i)$
$$=3+0i-2i-0i^2$$
$$=3-2i=z$$
$$1\times z=(1+0i)(3-2i)$$
$$=3-2i+0i-0i^2$$
$$=3-2i=z$$

Hence, verified that $z\times 1=1\times z=z$

Example 9: Find the additive inverse of $z=7-10i$.

Solution: $z=7-10i$

Additive inverse of $z=-7+10i$

because $z+(-z)=7-10i+(-7+10i)$
$$=7-10i-7+10i=0$$

Example 10: Find multiplicative inverse of $4-3i$.

Solution: Let $z=4-3i$ (given)

Multiplicative inverse of $z=\dfrac{1}{4-3i}$

Then $z^{-1}=\dfrac{1}{4-3i}$

$$=\frac{4+3i}{(4-3i)(4+3i)}=\frac{4+3i}{(4)^2-(3i)^2}$$

$$z^{-1}=\frac{4+3i}{16+9}=\frac{4+3i}{25}=\frac{4}{25}+\frac{3i}{25}$$

Solved Exercise 1.2

1. Simplify and write in the form $a+bi$:

(i) $(2+5i)+(3-zi)$

Sol. $(2+5i)+(3-zi)=2+5i+3-zi$
$$=2+3+5i-zi$$
$$=5+(5-z)i$$

(ii) $(16-3i)+(9+2i)$

Sol. $(16-3i)+(9+2i)=16-3i+9+2i$
$$=16+9-3i+2i$$
$$=25-i$$

(iii) $(9-2i)-(7-3i)$

Sol. $(9-2i)-(7-3i)=9-2i-7+3i$
$$=9-7-2i+3i$$
$$=2+i$$

(iv) $(11+9i)-(9-7i)$

Sol. $(11+9i)-(9-7i)=11+9i-9+7i$
$$=11-9+9i+7i$$
$$=2+16i$$

(v) $(3+4i)(2-3i)$

Sol. $(3+4i)(2-3i) = 3(2-3i) + 4i(2-3i)$

$= 6 - 9i + 8i - 12i^2$

$= 6 - 9i + 8i - 12(-1)$

$= 6 - 9i + 8i + 12$

$= 6 + 12 - 9i + 8i$

$= 18 - i$

(vi) $(5-2i)(3-4i)$

Sol. $(5-2i)(3-4i) = 5(3-4i) - 2i(3-4i)$

$= 15 - 20i - 6i + 8i^2$

$= 15 - 20i - 6i + 8(-1)$

$= 15 - 20i - 6i - 8$

$= 15 - 8 - 20i - 6i$

$= 7 - 26i$

(vii) $(3-5i) \div (2-4i)$

Sol. $(3-5i) \div (2-4i) = \dfrac{3-5i}{2-4i}$

$$= \frac{3-5i}{2-4i} \times \frac{2+4i}{2+4i}$$

$$= \frac{3(2+4i) - 5i(2+4i)}{(2)^2 - (4i)^2}$$

$$= \frac{6+12i-10i-20i^2}{4-(16)i^2}$$

$$= \frac{6+12i-10i-20(-1)}{4-(16\times-1)} \quad \text{as, } i^2 = -1$$

$$= \frac{6+12i-10i+20}{4+16}$$

$$= \frac{6+20+12i-10i}{20}$$

$$= \frac{26+2i}{20}$$

$$= \frac{26}{20} + \frac{2i}{20} = \frac{13}{10} + \frac{i}{10}$$

(viii) $(5+2i) \div (6-3i)$

Sol. $(5+2i) \div (6-3i) = \dfrac{5+2i}{6-3i}$

$$= \frac{5+2i}{6-3i} \times \frac{6+3i}{6+3i}$$

$$\frac{5(6+3i) + 2i(6+3i)}{(6^2) - (3i)^2}$$

$$= \frac{30+15i+12i+6i^2}{36-(9i^2)}$$

$$= \frac{30+15i+12i+6(-1)}{36-(9\times-1)} \quad \text{as, } i^2 = -1$$

$$= \frac{30-6+15i+12i}{36+9}$$

$$= \frac{24+27i}{45}$$

$$= \frac{24}{45} + \frac{27i}{45}$$

$$= \frac{8}{15} + \frac{3i}{5}$$

2. Write additive inverse for each complex number:

(i) $3+2i$

Sol. Let $z = 3+2i$

Additive inverse of $z = -3-2i$

(ii) $4-3i$

Sol. Let $z = 4-3i$

Additive inverse of $z = -4+3i$

(iii) $5-7i$

Sol. Additive inverse of $z = -5+7i$

(iv) $-\dfrac{2}{3} + \dfrac{5}{4}i$

Sol. Additive inverse of $z = \dfrac{2}{3} - \dfrac{5}{4}i$

3. **Find multiplicative inverse for each complex number:**

(i) $4+5i$

Sol. Let $z = 4+5i$

The multiplicative inverse of $z = z^{-1}$

$$z^{-1} = \frac{1}{z} = \frac{1}{4+5i}$$

$$= \frac{1}{4+5i} \times \frac{4-5i}{4-5i}$$

$$= \frac{4-5i}{(4)^2-(5i)^2}$$

$$= \frac{4-5i}{16-25i^2}$$

$$= \frac{4-5i}{16-(25\times(-1))} \qquad \because i^2=-1$$

$$= \frac{4-5i}{16+25}$$

$$= \frac{4-5i}{41} = \frac{4}{41} - \frac{5i}{41}$$

(ii) $6+2i$

Sol. Let $z = 6+2i$

The multiplicative inverse of $z = z^{-1}$

$$z^{-1} = \frac{1}{z} = \frac{1}{6+2i}$$

$$= \frac{1}{6+2i} \times \frac{6-2i}{6-2i}$$

$$= \frac{6-2i}{(6)^2-(2i)^2} = \frac{6-2i}{36-4i^2}$$

$$= \frac{6-2i}{36-(4\times(-1))} \qquad \because i^2=-1$$

$$= \frac{6-2i}{36+4}$$

$$= \frac{6-2i}{40} = \frac{6}{40} - \frac{2i}{40}$$

$$= \frac{3}{20} - \frac{1}{20}i$$

(iii) $7-3i$

Sol. Let $z = 7-3i$

The multiplicative inverse of $z = z^{-1}$

$$z^{-1} = \frac{1}{z} = \frac{1}{7-3i}$$

$$= \frac{1}{7-3i} \times \frac{7+3i}{7+3i}$$

$$= \frac{7+3i}{(7)^2-(3)i^2}$$

$$= \frac{7-3i}{49-9i^2}$$

$$= \frac{7+3i}{49+9} \qquad \because i^2=-1$$

$$= \frac{7+3i}{58} = \frac{7}{58} + \frac{3i}{58}$$

(iv) $\sqrt{5}-4i$

Sol. Let $z = \sqrt{5}-4i$

The multiplicative inverse of $z = z^{-1}$

$$z^{-1} = \frac{1}{z} = \frac{1}{\sqrt{5}-4i}$$

$$= \frac{1}{\sqrt{5}-4i} \times \frac{\sqrt{5}+4i}{\sqrt{5}+4i}$$

$$= \frac{\sqrt{5}+4i}{(\sqrt{5})^2-(4i)^2}$$

$$= \frac{\sqrt{5}+4i}{5-16i^2}$$

$$= \frac{\sqrt{5}+4i}{5-(16\times(-1))} \qquad \because i^2=-1$$

$$= \frac{\sqrt{5}+4i}{5+16} = \frac{\sqrt{5}+4i}{21}$$

$$= \frac{\sqrt{5}}{21} + \frac{4}{21}i$$

4. If $z_1 = 2 + 5i$, $z_2 = 1 - 3i$ and $z_3 = 2 + i$, then verify that

(i) $z_1 + z_2 = z_2 + z_1$

Sol. L.H.S

$$
\begin{aligned}
z_1 + z_2 &= (2 + 5i) + (1 - 3i) \\
&= 2 + 5i + 1 - 3i \\
&= 2 + 1 + 5i - 3i \\
&= 3 + 2i \quad \_\_\_\_\_\_\_\_ (1)
\end{aligned}
$$

R.H.S

$$
\begin{aligned}
z_2 + z_1 &= (1 - 3i) + (2 + 5i) \\
&= 1 - 3i + 2 + 5i \\
&= 1 + 2 - 3i + 5i \\
&= 3 + 2i \quad \_\_\_\_\_\_\_\_ (2)
\end{aligned}
$$

From (1) and (2) we get

L.H.S = R.H.S

Hence, verified that $z_1 + z_2 = z_2 + z_1$

(ii) $z_1 z_2 = z_2 z_1$

Sol. L.H.S

$$
\begin{aligned}
z_1 . z_2 &= (2 + 5i).(1 - 3i) \\
&= 2(1 - 3i) + 5i(1 - 3i) \\
&= 2 - 6i + 5i - 15i^2 \\
&= 2 - 6i + 5i - 15(-1) \\
&= 2 - 6i + 5i + 15 \\
&= 2 + 15 - 6i + 5i \\
&= 17 - i \quad \_\_\_\_\_\_\_\_ (1)
\end{aligned}
$$

R.H.S

$$
\begin{aligned}
z_2 z_1 &= (1 - 3i)(2 + 5i) \\
&= 1(2 + 5i) - 3i(2 + 5i) \\
&= 2 + 5i - 6i - 15i^2 \\
&= 2 + 5i - 6i - 15(-1) \\
&= 2 + 5i - 6i + 15 \\
&= 2 + 15 + 5i - 6i \\
&= 17 - i \quad \_\_\_\_\_\_\_\_ (2)
\end{aligned}
$$

From (1) and (2) we get

L.H.S = R.H.S

Hence, verified that $z_1 z_2 = z_2 z_1$

(iii) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

Sol: L.H.S

$$
\begin{aligned}
(z_1 + z_2) + z_3 &= [(2 + 5i) + (1 - 3i)] + (2 + i) \\
&= [2 + 5i + 1 - 3i] + (2 + i) \\
&= (2 + 1 + 5i - 3i) + (2 + i) \\
&= (3 + 2i) + (2 + i) \\
&= 3 + 2i + 2 + i \\
&= 3 + 2 + 2i + i \\
&= 5 + 3i \quad \_\_\_\_\_\_\_\_ (1)
\end{aligned}
$$

R.H.S

$$
\begin{aligned}
z_1 + (z_2 + z_3) &= (2 + 5i) + [(1 - 3i)] + (2 + i)] \\
&= (2 + 5i) + [1 - 3i + 2 + i] \\
&= (2 + 5i) + (1 + 2 - 3i + i) \\
&= (2 + 5i) + (3 - 2i) \\
&= 2 + 5i + 3 - 2i \\
&= 2 + 3 + 5i - 2i \\
&= 5 + 3i \quad \_\_\_\_\_\_\_\_ (2)
\end{aligned}
$$

From (1) and (2) we get

L.H.S = R.H.S

Hence, verified that $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(iv) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Sol: L.H.S

$$
\begin{aligned}
(z_1 z_2) z_3 &= [(2 + 5i)(1 - 3i)](2 + i) \\
&= [2(1 - 3i) + 5i(1 - 3i)](2 + i) \\
&= (2 - 6i + 5i - 15i^2)(2 + i) \\
&= (2 - 6i + 5i - 15(-1))(2 + i) \\
&= (2 - 6i + 5i + 15)(2 + i) \\
&= (2 + 15 - 6i + 5i)(2 + i) \\
&= (17 - i)(2 + i) \\
&= 17(2 + i) - i(2 + i) \\
&= 34 + 17i - 2i - i^2 \\
&= 34 + 17i - 2i - (-1) \\
&= 34 + 17i - 2i + 1 \\
&= 34 + 1 + 17i - 2i \\
&= 35 + 15i \quad \_\_\_\_\_\_\_\_ (1)
\end{aligned}
$$

R.H.S

$$\begin{aligned}
z_1(z_2z_3) &= (2+5i)\,[(1-3i)(2+i)] \\
&= (2+5i)\,[1(2+i)-3i(2+i)] \\
&= (2+5i)(2+i-6i-3i^2) \\
&= (2+5i)(2+i-6i-3(-1)) \\
&= (2+5i)(2+i-6i+3) \\
&= (2+5i)(2+3+i-6i) \\
&= (2+5i)(5-5i) \\
&= 2(5-5i)+5i(5-5i) \\
&= 10-10i+25i-25i^2 \\
&= 10-10i+25i-25(-1) \\
&= 10-10i+25i+25 \\
&= 10+25-10i+25i \\
&= 35+15i \qquad \text{______(2)}
\end{aligned}$$

From (1) and (2) we get

L.H.S = R.H.S

Hence, verified that $(z_1z_2)z_3 = z_1(z_2z_3)$

(v) $z_1 + (-z_1) = (-z_1) + z_1 = 0$

Sol:

$$\begin{aligned}
z_1 &= 2+5i \\
-z_1 &= -(2+5i) = -2-5i \\
z_1 + (-z_1) &= (2+5i)+(-2-5i) \\
&= 2+5i-2-5i \\
&= 2-2+5i-5i = 0+0 = 0 \\
(-z_1)+z_1 &= (-2-5i)+(2+5i) \\
&= -2-5i+2+5i \\
&= -2+2-5i+5i = 0+0 = 0
\end{aligned}$$

Hence, verified that $z_1 + (-z_1) = (-z_1) + z_1 = 0$

5. If $\dfrac{(1+i)^2}{2-i} = x+iy$ then find the values of x and y.

Sol:

$$\frac{(1+i)^2}{2-i} = x+iy$$

$$\frac{(1)^2+2(1)i+(i)^2}{2-i} = x+iy$$

$$\frac{1+2i+(-1)}{2-i} = x+iy$$

$$\frac{1+2i-1}{2-i} = x+iy$$

$$\frac{1-1+2i}{2-i} = x+iy$$

$$\frac{2i}{2-i} = x+iy$$

$$\frac{2i}{2-i}\times\frac{2+i}{2+i} = x+iy$$

$$\frac{4i+2i^2}{(2)^2-i^2} = x+iy$$

$$\frac{4i+2(-1)}{4-(-1)} = x+iy$$

$$\frac{4i-2}{4+1} = x+iy$$

$$\frac{-2+4i}{5} = x+iy$$

$$\frac{-2}{5}+\frac{4}{5}i = x+iy$$

By comparing real and imaginary parts of both sides

$$x = \frac{-2}{5},\ y = \frac{4}{5}$$

6. If $(2x + iy)(1 - i) = 4 + 2i$, then find the values of x and y.

Sol: $(2x + iy)(1 - i) = 4 + 2i$

$2x(1 - i) + iy(1 - i) = 4 + 2i$

$2x - 2xi + iy - i^2y = 4 + 2i$

$2x - 2xi + iy - (-1)y = 4 + 2i$

$2x - 2xi + iy + y = 4 + 2i$

$2x + y - 2xi + iy = 4 + 2i$

$2x + y + i(-2x + y) = 4 + 2i$

By comparing real and imaginary parts of both sides

$2x + y = 4$ – (i) , $-2x + y = 2$ – (ii)

Add eq (i) and eq (ii)

$$\begin{array}{r} \cancel{2x} + y = 4 \\ -\cancel{2x} + y = 2 \\ \hline 2y = 6 \end{array}$$

$$y = \frac{6}{2} = 3$$

Put y = 3 in eq (i)

$2x + 3 = 4$

$2x = 4 - 3 \Rightarrow 2x = 1$

$$x = \frac{1}{2}$$

So, $x = \frac{1}{2}$, $y = 3$

7. Find the values of a and b, if $(a + bi)(1 + 3i) = -8 + 11i$.

Sol: $(a + bi)(1 + 3i) = -8 + 11i$

$a(1 + 3i) + bi(1 + 3i) = -8 + 11i$

$a + 3ai + bi + 3bi^2 = -8 + 11i$

$a + 3ai + bi + 3b(-1) = -8 + 11i$

$a - 3b + 3ai + bi = -8 + 11i$

$(a - 3b) + (3a + b)i = -8 + 11i$

By comparing real and imaginary parts of both sides

$a - 3b = -8$ – (i),

$3a + b = 11$ _____ (ii)

Multiply eq. (ii) by 3

$9a + 3b = 33$ ________ (iii)

Add eq. (i) and eq. (iii)

$$\begin{array}{r} a - \cancel{3b} = -8 \\ 9a + \cancel{3b} = 33 \\ \hline 10a = 25 \end{array}$$

$$a = \frac{25}{10} \Rightarrow a = \frac{5}{2}$$

By putting $a = \frac{5}{2}$ in eq. (i)

$$\frac{5}{2} - 3b = -8$$

$$2\left(\frac{5}{2}\right) - 2(3b) = 2(-8)$$

$5 - 6b = -16$

$-6b = -16 - 5$

$-6b = -21 \Rightarrow 6b = 21$

$$b = \frac{21}{6}$$

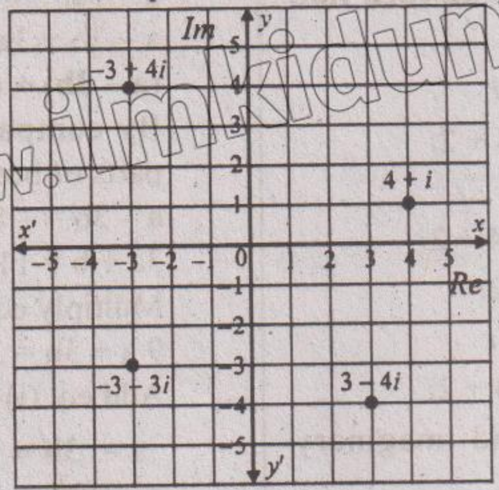
$$b = \frac{7}{2}$$

So, $a = \frac{5}{2}$, $b = \frac{7}{2}$

1.3 Complex or Argand Plane

A complex number z = a + ib is exclusively deremined by an ordered pair of real number (x,y). The numbers $2 - 5i$, 8 and $-7i$ are equivalent to $(2, -5)$, $(8, 0)$ and $0, -7)$ respectively. In this manner, a complex number $z = x + iy$ can be represented by the point (x,y) in the coordinate plane.

Here some complex numbers are plotted on the complex plane.



Conjugate of a Complex Number

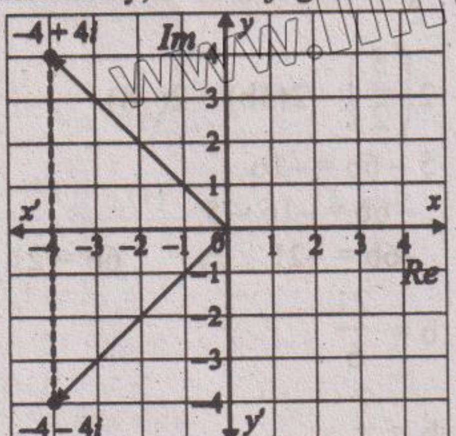
The conjugate of the complex number $x + iy$ is defined as the complex number $x - iy$.

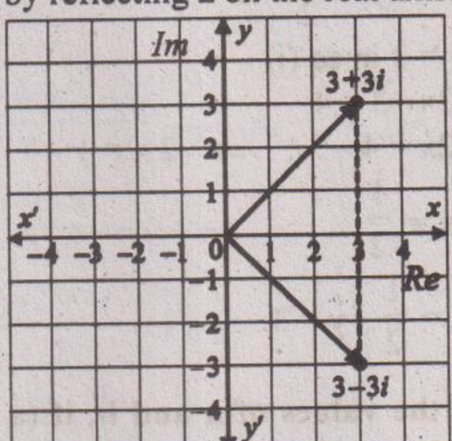
The complex conjugate of z is denoted by $\overline{z}$. To get the conjugate of the complex number z, simply change i by $-i$ in z. For instance $3 - 8i$ is the conjugate of $3 + 8i$. The product of a complex number with its conjugate is a real number.

For example, $(1 + 2i)(1 - 2i) = (1)^2 - (2i)^2 = 1 + 4 = 5$

Geometrical Representation of Conjugate of a Complex Number

Geometrically, the conjugate of z is obtained by reflecting z on the real axis.





Properties of Complex Conjugate:

For any two complex numbers z_1 and z_2, we have

Property 1: $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Proof: Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, where $x_1, y_1, x_2, y_2 \in R$

Now, $\overline{z_1 + z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)}$

$= \overline{(x_1 + x_2) + i(y_1 + y_2)}$

$= (x_1 + x_2) - i(y_1 + y_2) = x_1 + x_2 - iy_1 - iy_2$

$= (x_1 - iy_1) + (x_2 - iy_2) = \overline{z_1} + \overline{z_2}$

Example 11

If $z_1 = 4 + 3i$, $z_2 = 5 + 2i$, then prove that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

Sol. $z_1 + z_2 = 4 + 3i + 5 + 2i = 9 + 5i$

$\overline{z_1 + z_2} = 9 - 5i$ ________ (i)

$\overline{z_1} = 4 - 3i$

$\overline{z_2} = 5 - 2i$

$\overline{z_1} + \overline{z_2} = 4 - 3i + 5 - 2i$

$\overline{z_1} + \overline{z_2} = 9 - 5i$ ________ (ii)

Therefore, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ (From (i) and (ii))

Property 2 $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

Proof: Let $z_1 = (x_1 + iy_1)$

and $z_2 = (x_2 + iy_2)$ where $x_1, y_1, x_2, y_2 \in R$

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)$

Therefore: $\overline{z_1 z_2} = \overline{(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)}$

$= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + y_1 x_2)$ (i)

and $\overline{z_1}\,\overline{z_2} = \left(\overline{x_1 + iy_1}\right)\left(\overline{x_2 + iy_2}\right)$

$= (x_1 - iy_1)(x_2 - iy_2)$

$= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + y_1 x_2)$ (ii)

From (i) and (ii), we get

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

Example 12

If $z_1 = 3 + 4i, z_2 = 2 + 3i$, then prove that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

Sol. Now, $z_1 z_2 = (3 + 4i)(2 + 3i)$

$= 6 + 9i + 8i + 12i^2$

$= 6 + 17i - 12$

$= -6 + 17i$

$\overline{z_1 z_2} = -6 - 17i$ ________ (i)

and $\overline{z_1}\,\overline{z_2} = (3 - 4i)(2 - 3i)$

$= 6 - 9i - 8i + 12i^2$

$= 6 - 9i - 8i - 12$

$\overline{z_1}\,\overline{z_2} = -6 - 17i$ ________ (ii)

$\therefore$ $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ From (i) and (ii)

Property 3

$$\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2} \quad , \quad z_2 \neq 0$$

Proof: Let $z_1 = x_1 + iy_1$
and $z_2 = x_2 + iy_2$

$$\frac{z_1}{z_2}=\frac{x_1+iy_1}{x_2+iy_2}=\frac{x_1+iy_1}{x_2+iy_2}\times\frac{x_2-iy_2}{x_2-iy_2}$$

$$\frac{z_1}{z_2}=\frac{(x_1x_2+y_1y_2)-i(x_1y_2-y_1x_2)}{(x_2^2+y_2^2)}$$

$$\overline{\left(\frac{z_1}{z_2}\right)}=\frac{(x_1x_2+y_1y_2)-i(x_1y_2-y_1x_2)}{(x_2^2+y_2^2)} \quad \text{\_\_\_\_\_\_(i)}$$

Now,
$$\frac{\bar{z}_1}{\bar{z}_2}=\frac{x_1-iy_1}{x_2-iy_2}$$

$$=\frac{(x_1-iy_1)(x_2+iy_2)}{(x_2-iy_2)(x_2+iy_2)}$$

$$=\frac{x_1x_2+ix_1y_2-iy_1x_2-i^2y_1y_2}{x_2^2+y_2^2}$$

$$\therefore \quad \frac{\bar{z}_1}{\bar{z}_2}=\frac{(x_1x_2+y_1y_2)+i(x_1y_2-y_1x_2)}{x_2^2+y_2^2} \quad \text{\_\_\_\_\_\_(ii)}$$

$$\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2} \quad \text{(From (i) and (ii))}$$

Skilled Practice!

(i) Take any two complex numbers and prove that $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

Sol: Let $z_1 = (2+3i), z_2 = (4+5i)$

$\bar{z}_1 = 2-3i, \ \bar{z}_2 = 4-5i$

$z_1+z_2=(2+3i)+(4+5i)$

$=2+3i+4+5i$

$=6+8i$

$\overline{z_1+z_2}=\overline{6+8i}=6-8i$ ___ (i)

$\overline{z_1}+\overline{z_2}=(2-3i)+(4-5i)=2-3i+4-5i=6-8i$ _____ (ii)

Hence, Proved that $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$ (from (i) and (ii))

(ii) Take any two complex number and prove that $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

Sol. $z_1 z_2 = (2+3i)(4+5i)$

$= 8 + 10i + 12i + 15i^2$

$= 8 + 22i + 15 \times (-1)$

$= 8 + 22i - 15 = -7 + 22i$

$\overline{z_1 z_2} = \overline{-7 + 22} = -7 - 22i$ ____________(i)

$\overline{z_1}\,\overline{z_2} = (2 - 3i)(4 - 5i)$

$= 8 - 10i - 12i + 15i^2$

$= 8 - 22i + 15(-1)$

$= 8 - 15 - 22i = -7 - 22i$ ________(ii)

Hence proved that $\overline{z_1 z_2} = \overline{z_1}.\overline{z_2}$ (From (i) and (ii))

(iii) Take any two complex number and prove that = $\overline{\left(\dfrac{z_1}{z_2}\right)}\ \dfrac{\overline{z_1}}{\overline{z_2}}$, $(z_2 \neq 0)$

Sol. $\dfrac{z_1}{z_2} = \dfrac{2+3i}{4+5i}$

$= \dfrac{2+3i}{4+5i} \times \dfrac{4-5i}{4-5i}$

$= \dfrac{8-10i+12i-15i^2}{(4)^2-(5i)^2}$

$= \dfrac{8+2i-15\times(-1)}{(4)^2-(5i)^2}$

$= \dfrac{8+15+2i}{16+25} = \dfrac{23+2i}{41} = \dfrac{23}{41} + \dfrac{2}{41}i$

$\overline{\left(\dfrac{z_1}{z_2}\right)} = \overline{\dfrac{23}{41} + \dfrac{2}{41}i} = \dfrac{23}{41} - \dfrac{2}{41}i$ ____(i)

$\dfrac{\overline{z_1}}{\overline{z_2}} = \dfrac{2-3i}{4-5i}$

$= \dfrac{2-3i}{4-5i} \times \dfrac{4+5i}{4+5i}$

$= \dfrac{8+10i-12i-15i^2}{(4)^2-(5i)^2} = \dfrac{8-2i-15(-1)}{16-(25\times(-1))}$

$= \dfrac{8+15-2i}{16-(25\times(-1))} = \dfrac{23-2i}{16+25} = \dfrac{23-2i}{41}$

$= \dfrac{23}{41} - \dfrac{2}{41}i$ ____ ii

Hence proved that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ (From (i) and (ii))

Example 13

If $z_1 = 5 + 4i$, $z_2 = 3 + 2i$, then prove that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}$.

Sol. $\dfrac{z_1}{z_2} = \dfrac{5+4i}{3+2i}$

$= \dfrac{(5+4i)(3-2i)}{(3+2i)(3-2i)} = \dfrac{15-10i+12i-8i^2}{(3)^2-(2i)^2}$

$= \dfrac{23+2i}{9+4}$

$$\frac{z_1}{z_2}=\frac{23+2i}{13}=\frac{23}{13}+\frac{2}{13}i$$

$$\overline{\left(\frac{z_1}{z_2}\right)}=\frac{23}{13}-\frac{2}{13}i \quad \text{\_\_\_\_ (i)}$$

$$\frac{\overline{z}_1}{\overline{z}_2}=\frac{5-4i}{3-2i}=\frac{(5-4i)(3+2i)}{(3-2i)(3+2i)}$$

$$=\frac{15+10i-12i-8i^2}{9+4}=\frac{15-2i+8}{9+4}$$

$$\frac{\overline{z}_1}{\overline{z}_2}=\frac{5-4i}{3-2i}=\frac{(5-4i)(3+2i)}{(3-2i)(3+2i)}$$

$$=$$

$$\frac{15+10i-12i-8i^2}{9+4}=\frac{15-2i+8}{9+4}$$

$$\frac{\overline{z}_1}{\overline{z}_2}=\frac{23-2i}{13}=\frac{23}{13}-\frac{2}{13}i \quad \text{\_\_\_\_ (ii)}$$

Therefore,

$$\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\overline{z}_1}{\overline{z}_2} \qquad \text{(From (i) and (ii))}$$

Property 4

$$\overline{\overline{z}}=z$$

Proof: Let $z=x+iy$

Then $\overline{z}=x-iy$

$\overline{\overline{z}}=x+iy=z$

Hence, proved that $\overline{\overline{z}}=z$

Example 14

If $z=7+3i$, then prove that $\overline{\overline{z}}=z$

Sol. $z=7+3i$

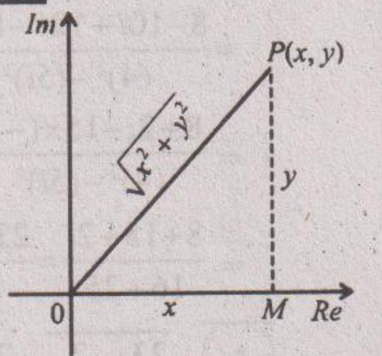
Then $\overline{z}=7-3i$

$\overline{\overline{z}}=7+3i=z$

Hence, proved that $\overline{\overline{z}}=z$

1.4 Modulus of a Complex Number

Just as the absolute value of a real number measures its distance from the origin on the real number line, the modulus of a complex number measures the distance from the origin in the complex plane.



Definition

If $z=x+iy$, then the modulus of z is denoted by $|z|$ and defined as $|z|=\sqrt{x^2+y^2}$

For example, (i) $|i|=|0+1i|=\sqrt{0^2+1^2}=1$

(ii) $|3-5i|=\sqrt{(3)^2+(-5)^2}=\sqrt{9+25}=\sqrt{34}$

Properties of Modulus of a Complex Number

Property 1

$$|z|=|\overline{z}|$$

Proof: Let $z=x+iy$

Then $|z|=\sqrt{x^2+y^2}$ ____ (i)

Now $\overline{z}=x-iy$

$|\overline{z}|=\sqrt{(x)^2+(-y)^2}=\sqrt{x^2+y^2}$ ____ (ii)

From (i) and (ii), we get

$|\overline{z}|=|z|$

Example 15

If $z = 5 + 4i$, show that $|z| = |\overline{z}|$

Sol. $z = 5 + 4i$

$$|z| = \sqrt{(5)^2+(4)^2}$$

$$= \sqrt{25+16} = \sqrt{41} \quad \_\_\_\text{(i)}$$

Now $\overline{z} = 5 - 4i$

$$|\overline{z}| = \sqrt{(5)^2+(-4)^2}$$

$$= \sqrt{25+16} = \sqrt{41} \quad \_\_\_\text{(ii)}$$

From (i) and (ii), we get

$$|z| = |\overline{z}|$$

Property 2: $|z| = |-z| = |\overline{\overline{z}}| = |-\overline{z}|$

Proof: Let $z = x + iy$

Then $|z| = \sqrt{x^2+y^2} \quad \_\_\_\text{(i)}$

$$|-z| = |-x-iy|$$

$$= \sqrt{(-x)^2+(-y)^2}$$

$$= \sqrt{x^2+y^2} \quad \_\_\_\text{(ii)}$$

$$\overline{z} = x - yi \Rightarrow \overline{\overline{z}} = x + yi$$

$$|\overline{\overline{z}}| = \sqrt{x^2+y^2} \quad \_\_\_\text{(iii)}$$

Here $\overline{z} = x - iy$ as $z = x + iy$

$$-\overline{z} = -x + iy$$

$$|-\overline{z}| = \sqrt{(-x)^2+(y)^2}$$

$$= \sqrt{x^2+y^2} \quad \_\_\_\text{(iv)}$$

From (i), (ii), (iii) and (iv), we have

$$|z| = |-z| = |\overline{\overline{z}}| = |-\overline{z}|$$

Property 3: $z\overline{z} = |z|^2$

Proof: Let $z = x + iy$

$$\therefore \overline{z} = x - iy$$

Now $z\overline{z} = (x + iy)(x - iy)$

$$= (x)^2 - (iy)^2$$

$$= x^2 - (i^2y^2)$$

$$= x^2 - (-y^2) \quad \text{as } i^2 = -1$$

$$= x^2 + y^2 = |z|^2 \quad \text{as } |z| = \sqrt{x^2+y^2}$$

Hence, $z\overline{z} = |z|^2$

Example 16 if $z = 5 + 3i$, show that $z\overline{z} = |z|^2$

Solution: $z = 5 + 3i$

$$\overline{z} = 5 - 3i$$

$$z\overline{z} = (5+3i)(5-3i)$$

$$= (5)^2 - (3i)^2$$

$$= (25) - (9i^2)$$

$$= 25 - (-9)$$

$$= 25 + 9 = 34 \quad \_\_\_\_\_ \text{(i)}$$

$$|z| = \sqrt{(5)^2+(3)^2}$$

$$|z|^2 = \left[\sqrt{(5)^2+(3)^2}\right]^2$$

$$= 25 + 9 = 34 \quad \_\_\_\_\_ \text{(ii)}$$

We get, $z\overline{z} = |z|^2$

From (i) and (ii)

Solved Exercise 1.3

1. Find the modulus of the following complex numbers:

(i) $4 + 3i$

Sol. $|4+3i| = \sqrt{(4)^2+(3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

(ii) $-5 - 4i$

Sol. $|-5-4i| = \sqrt{(-5)^2+(-4)^2} = \sqrt{25+16} = \sqrt{41}$

(iii) $\frac{3}{5}-\frac{4}{5}i$

Sol. $\left|\frac{3}{5}-\frac{4}{5}i\right|=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}=\sqrt{\frac{9}{25}+\frac{16}{25}}=\sqrt{\frac{25}{25}}=\sqrt{1}=1$

(iv) $-\sqrt{2}-\sqrt{3}i$

Sol. $|-\sqrt{2}-\sqrt{3}i|=\sqrt{\left(-\sqrt{2}\right)^2+\left(-\sqrt{3}\right)^2}=\sqrt{2+3}=\sqrt{5}$

2. If $z_1 = 2 + 7i$ and $z_2 = 4 - 3i$, then verify that:

(i) $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

Sol. L.H.S

$$\begin{aligned}
z_1+z_2 &= (2+7i)+(4-3i)\\
&= 2+7i+4-3i\\
&= 2+4+7i-3i\\
&= 6+4i
\end{aligned}$$

$$\overline{z+z_2}=\overline{6+4i}=6-4i$$

R.H.S

$$\begin{aligned}
z_1 &= 2+7i\\
\overline{z_1} &= 2-7i\\
z_2 &= 4-3i\\
\overline{z_2} &= 4+3i\\
\overline{z_1}+\overline{z_2} &= (2-7i)+(4+3i)\\
&= 2-7i+4+3i\\
&= 2+4-7i+3i\\
&= 6-4i
\end{aligned}$$

Hence, verified that $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

(ii) $\overline{z_1 z_2}=\overline{z_1}\,\overline{z_2}$

Sol. L.H.S

$$\begin{aligned}
z_1z_2 &= (2+7i)(4-3i)\\
&= 2(4-3i)+7i(4-3i)\\
&= 8-6i+28i-21i^2\\
&= 8-6i+28i-21(-1)\\
&= 8-6i+28i+21\\
&= 8+21-6i+28i\\
&= 29+22i
\end{aligned}$$

$$\overline{z_1 z_2}=\overline{29+22i}=29-22i$$

R.H.S

$$\begin{aligned}
z_1 &= 2+7i\\
\overline{z_1} &= 2-7i\\
z_2 &= 4-3i\\
\overline{z_2} &= 4+3i\\
\overline{z_1}\,\overline{z_2} &= (2-7i)(4+3i)\\
&= 8+6i-28i-21i^2\\
&= 8+6i-28i-21(-1)\\
&= 8+6i-28i+21\\
&= 8+21+6i-28i\\
&= 29-22i
\end{aligned}$$

Hence, verified that $\overline{z_1z_2}=\overline{z_1}\,\overline{z_2}$

(iii) $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\overline{z_1}}{\overline{z_2}}$

Sol. L.H.S

$$\frac{z_1}{z_2}=\frac{2+7i}{4-3i}=\frac{2+7i}{4-3i}\times\frac{4+3i}{4+3i}$$

$$=\frac{2(4+3i)+7i(4+3i)}{(4)^2-(3i)^2}=\frac{8+6i+28i+21i^2}{16-(9\times-1)}$$

$$= \frac{8+6i+28i+21(-1)}{16+9}$$
$$= \frac{8+6i+28i-21}{25}$$
$$= \frac{8+6i+28i-21}{25}$$
$$= \frac{8-21+6i+28i}{25}$$
$$= \frac{-13+34i}{25}$$
$$= \frac{-13}{25}+\frac{34i}{25}$$
$$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\frac{-13}{25}+\frac{34}{25}i} = -\frac{13}{25}-\frac{34}{25}i$$

R.H.S

$z_1 = 2+7i$

$\overline{z_1} = 2-7i$

$z_2 = 4-3i$

$\overline{z_2} = 4+3i$

$$\frac{\overline{z_1}}{\overline{z_2}} = \frac{2-7i}{4+3i}$$
$$= \frac{2-7i}{4+3i} \times \frac{4-3i}{4-3i} = \frac{2(4-3i)-7i(4-3i)}{(4)^2-(3i)^2}$$
$$= \frac{8-6i-28i+21i^2}{16-9i^2} = \frac{8-6i-28i+21i^2}{16-(9\times-1)}$$
$$= \frac{8-6i-28i+21i^2}{16-(9\times-1)} = \frac{8-6i-28i+21(-1)}{16+9}$$
$$= \frac{8-6i-28i-21}{25} = \frac{8-21-6i-28i}{25}$$
$$= \frac{-13-34i}{25} = \frac{-13}{25} - \frac{34}{25}i$$

Hence, verified that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

3. If $z = 5 - 2i$, then verify that

(i) $\overline{\overline{z}} = z$

Sol. $z = 5-2i$

$\overline{z} = 5+2i$

$\overline{\overline{z}} = 5-2i$

Hence, verified that $\overline{\overline{z}} = z$

(ii) $|z| = |\overline{z}|$

Sol. $z = 5-2i$

$\overline{z} = 5+2i$

$|z| = \sqrt{(5)^2+(2)^2} = \sqrt{25+4} = \sqrt{29}$ ____(i)

$|\overline{z}| = \sqrt{(5)^2+(2)^2} = \sqrt{25+4} = \sqrt{29}$ ____(ii)

Hence, verified that $|z| = |\overline{z}|$ (from (i) and (ii))

(iii) $|z| = |-z|$

Sol. $z = 5-2i$

$-\overline{z} = -5+2i$

$|z| = |5-2i| = \sqrt{(5)^2+(-2)^2} = \sqrt{25+4} = \sqrt{29}$ ____(i)

$|-z| = \sqrt{(-5)^2+(2)^2}$

$= \sqrt{25+4} = \sqrt{29}$ ____(ii)

Hence, verified that $|z| = |-z|$ (from (i) and (ii))

(iv) $z\bar{z} = |z|^2$

Sol. $z = 5 - 2i$

$\bar{z} = 5 + 2i$

$z\bar{z} = (5 - 2i)(5 + 2i)$

$= (5)^2 - (2i)^2$

$= 25 - 4i^2$

$= 25 - (4\times -1)$

$= 25 + 4$

$= 29$ ____(i)

$|z| = |5 - 2i| = \sqrt{(5)^2 + (2)^2}$

$= \sqrt{25+4} = \sqrt{29}$

$|z|^2 = \left(\sqrt{29}\right)^2$

$|z|^2 = 29$ ____(ii)

Hence, from (i) and (ii) verified that $z\bar{z} = |z|^2$

(v) $|z| = |-\bar{z}|$

Sol. $z = 5 - 2i$

$\bar{z} = 5 + 2i$

$-\bar{z} = -(5 + 2i) = -5 - 2i$

$|z| = |5 - 2i|$

$= \sqrt{(5)^2 + (2)^2}$

$= \sqrt{25+4} = \sqrt{29}$ ____(i)

$|-\bar{z}| = |-5 - 2i|$

$= \sqrt{(-5)^2 + (-2)^2}$ ____(ii)

$= \sqrt{25+4} = \sqrt{29}$

Hence, from (i) and (ii) verified that $|\bar{z}| = |-\bar{z}|$

4. If $z = 4 - 3i$, then verify that $|z| = |-z| = |\bar{\bar{z}}| = |-\bar{z}|$.

Sol. Given that $z = 4 - 3i$

$-z = -(4 - 3i) = -4 + 3i$

$-z = -4 + 3i$

$-\bar{z} = \overline{-4 + 3i} = -4 - 3i$

$-\bar{z} = -4 - 3i$

$|z| = |4 - 3i|$

$|z| = \sqrt{(4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25}$

$|z| = 5$ ____________(i)

$|-z| = |-4 + 3i|$

$|-z| = \sqrt{(-4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25}$

$|-z| = 5$ ____________(ii)

$-\bar{z} = -4 - 3i$

$|-\bar{z}| = |-4 - 3i|$

$= \sqrt{(-4)^2 + (-3)^2} = \sqrt{16+9} = 25$

$= 5$

$|-\bar{z}| = 5$ ____________(iii)

And $\bar{z} = \overline{4 - 3i}$

$\bar{z} = 4 + 3i$

$\bar{\bar{z}} = \overline{4 + 3i}$

$\bar{\bar{z}} = 4 - 3i$

$|\bar{\bar{z}}| = \sqrt{(4)^2 + (-3)^2}$

$|\bar{\bar{z}}| = \sqrt{16+9} = \sqrt{25}$

$|\bar{\bar{z}}| = 5$ ____________(iv)

Hence, from (i), (ii), (iii) and (iv) verified that $|z| = |-z| = |\bar{\bar{z}}| = |-\bar{z}|$

5. If $z_1 = 2 + 3i$, $z_2 = -1 + i$, evaluate:

(i) $\text{Re}(z_1z_2)$

Sol. $z_1z_2 = (2 + 3i)(-1 + i)$

$= 2(-1 + i) + 3i(-1 + i)$

$= -2 + 2i - 3i + 3i^2$

$= -2 + 2i - 3i + 3(-1)$

$= -2 - 3 + 2i - 3i = -5 - i$

$= -5 - i$

So $Re(z_1z_2) = -5$

(ii) $\text{Im}(z_1z_2)$

Sol. $z_1z_2 = (2+3i)(-1+i)$

$= 2(-1+i) + 3i(-1+i)$

$= -2 + 2i - 3i + 3i^2$

$= -2 + 2i - 3i + 3(-1)$

$= -2 - 3 + 2i - 3i$

$= -5 - i$

So $Im(z_1z_2) = -1$

1.5 Finding Real and imaginary Parts of Complex Number of $(x+iy)^n$ and $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$, where $x^2 + iy^2 \neq 0$ and $n = \pm 1$ and $n = \pm 2$

$(x+iy)^n$ when $n = \pm 1$

Let $z = (x+iy)^n$ be a complex number.

(i) For $n = 1$

We have $z = (x+iy)^1$

$= x + iy$

$\text{Re}(z) = x, \text{Im}(z) = y$

(ii) For $n = -1$

We have $z = (x+iy)^{-1}$

$$= \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

$$\therefore \text{Re}(z) = \frac{x}{x^2+y^2}, \text{Im}(z) = \frac{-y}{x^2+y^2}$$

Skilled Practice!

Find real and imaginary parts of $(x+iy)^n$ for $n = \pm 2$

Sol. $= (x+iy)^2$

$= x^2 + 2xyi + (yi)^2$

$= x^2 + 2xyi + (y^2 \times -1)$

$= x^2 + 2xyi - y^2$

$= x^2 - y^2 + 2xyi$

So Real part $= x^2 - y^2$ and Imaginary part $= 2xy$

Now $(x+yi)^{-2}$

$$= \left(\frac{1}{x+yi}\right)^2$$

$$= \left(\frac{1}{x+yi} \times \frac{x-yi}{x-yi}\right)^2$$

$$= \left(\frac{x-yi}{(x)^2-(yi)^2}\right)^2 = \left(\frac{x-yi}{x^2+y^2}\right)^2 = \frac{(x-yi)^2}{(x^2+y^2)^2}$$

$$= \frac{x^2 - 2xyi + y^2 i^2}{\left(x^2+y^2\right)^2} = \frac{x^2 - 2xyi - y^2}{\left(x^2+y^2\right)^2} = \frac{\left(x^2 - y^2\right) - 2xyi}{\left(x^2+y^2\right)^2}$$

$$= \frac{x^2 - y^2}{\left(x^2+y^2\right)^2} - \frac{2xyi}{\left(x^2+y^2\right)^2}$$

So, Real part $= \dfrac{x^2 - y^2}{\left(x^2+y^2\right)^2}$, imaginary part $= \dfrac{-2xy}{\left(x^2+y^2\right)^2}$

Example 17: Find real and imaginary parts of $(4 + 3i)^{-1}$.

Sol. Let $z = (4+3i)^{-1}$

$$= \frac{1}{(4+3i)}$$

$$= \frac{1(4-3i)}{(4+3i)(4-3i)} = \frac{4-3i}{16-9i^2}$$

$$= \frac{4-3i}{16+9} = \frac{4-3i}{25} = \frac{4}{25} - \frac{3}{25}i$$

$\therefore$ Re(z) $= \dfrac{4}{25}$, Im(z) $= -\dfrac{3}{25}$

Example 18: Find real and imaginary parts of $z = (4 + 3i)^{-1}$.

Sol. $z = (4+3i)^{-1}$

$$= \frac{1}{(4+3i)^2}$$

$$= \frac{1}{(4)^2 + (3i)^2 + 2(4)(3i)} = \frac{1}{16-9+24i}$$

$$= \frac{1}{7+24i} = \frac{(7-24i)}{(7+24i)(7-24i)}$$

$$= \frac{7-24i}{49-576i^2} = \frac{7-24i}{49-576(-1)}$$

$$= \frac{7-24i}{49+576} = \frac{7-24i}{625}$$

$$= \frac{7}{625} - \frac{24}{625}i$$

$\therefore$ Re(z) $= \dfrac{7}{625}$, Im(z) $= -\dfrac{24}{625}$

Do you know?

$$\operatorname{Re}(z)=\frac{z+\bar{z}}{2},$$

$$\operatorname{Im}(z)=\frac{z-\bar{z}}{2i}$$

Example 19: Find real and imaginary parts of $z=\left(\frac{4+3i}{3+2i}\right)^{-1}$.

Sol. $z=\left(\frac{4+3i}{3+2i}\right)^{-1}$

$$=\frac{3+2i}{4+3i}=\frac{(3+2i)(4-3i)}{(4+3i)(4-3i)}$$

$$=\frac{12-9i+8i-6i^2}{16-9i^2}=\frac{12-9i+8i-6(-1)}{16-9(-1)}=\frac{12-i+6}{16+9}$$

$$=\frac{18-i}{25}=\frac{18}{25}-\frac{1}{25}i$$

$\therefore \operatorname{Re}(z)=\frac{18}{25}, \operatorname{Im}(z)=-\frac{1}{25}$

Example 20: Find real and imaginary parts of $z=\left(\frac{1+i}{1-i}\right)^{-2}$.

Sol. $z=\left(\frac{1+i}{1-i}\right)^{-2}$

$$=\left(\frac{1-i}{1+i}\right)^{2}=\left(\frac{1-i}{1+i}\times\frac{1-i}{1-i}\right)^{2}=\left(\frac{1-i-i+i^2}{1^2-i^2}\right)^{2}$$

$$=\left(\frac{1-2i-1}{1+1}\right)^{2}=\left(\frac{-2i}{2}\right)^{2}=(-i)^2=-1$$

$$=-1+0i$$

$\therefore \operatorname{Re}(z)=-1, \operatorname{Im}(z)=0$

1.6 Solution of Simultaneous Linear Equations with Complex Coefficients

More than one equation which are to be satisfied by the same values of the variables involved are called simultaneous equations or a system of equations.

Example 21: Solve the given simultaneous linear equations with complex coefficients for z and w:

$$5z-(3+i)w=7-i$$

$$(2-i)z+2iw=4$$

Sol. $5z - (3+i)w = 7 - i$ ________ (i)

$(2-i)z + 2iw = 4$ ________ (ii)

Step I From equation (i), we solve for w in terms of z.

$$5z - (3+i)w = 7 - i$$

$$(3+i)w = 5z - 7 + i$$

$$w = \frac{5z - 7 + i}{3+i} \quad \text{———— (iii)}$$

Step II Put the expression of w in equation (ii)

$$(2-i)z + 2i\left(\frac{5z-7+i}{3+i}\right) = 4$$

$$(2-i)(3+i)z + 10iz - 14i + 2i^2 = 4(3+i)$$

$$(6 + 2i - 3i - i^2)z + 10iz - 14i + 2i^2 = 12 + 4i$$

$$(6 - i + 1)z + 10iz - 14i - 2 = 12 + 4i$$

$$(7 - i)z + 10iz - 14i - 2 = 12 + 4i$$

$$7z - iz + 10iz - 14i - 2 = 12 + 4i$$

$$7z + 9iz = 12 + 4i + 14i + 2$$

$$7z + 9iz = 14 + 18i$$

$$z(7 + 9i) = 2(7 + 9i)$$

$$z = \frac{2(7+9i)}{7+9i}$$

$$z = 2$$

Step III Put z = 2 in (iii), we get

$$w = \frac{5(2) - 7 + i}{3+i} = \frac{10 - 7 + i}{3+i}$$

$$w = \frac{3+i}{3+i} = 1 \Rightarrow w = 1$$

Hence, z = 2, w = 1

Solved Exercise 1.4

Q.1: Find the real and imaginary parts of the following complex numbers:

(i) $(8-3i)^2$

Sol. $(8-3i)^2$

$$= (8)^2 - 2(8)(3i) + (3i)^2$$

$$= 64 - 48i + (9(-1))$$

$$= 64 - 9 - 48i = 55 - 48i$$

$\therefore$ Real part = 55, imaginary part = –48

(ii) $(5+3i)^{-1}$

Sol. $(5+3i)^{-1}$

$$=\frac{1}{(5+3i)}$$

$$=\frac{1}{5+3i}\times\frac{5-3i}{5-3i}$$

$$=\frac{5-3i}{(5)^2-(3i)^2}==\frac{5-3i}{25-9i^2}$$

$$=\frac{5-3i}{25-(9\times-1)}=\frac{5-3i}{25+9}$$

$$=\frac{5-3i}{34}=\frac{5}{34}-\frac{3i}{34}$$

$\therefore$ Real part $=\frac{5}{34}$, Imaginary part $=\frac{-3}{34}$

(iii) $(4-5i)^{-1}$

Sol. $(4-5i)^{-1}$

$$=\frac{1}{4-5i}$$

$$=\frac{1}{4-5i}\times\frac{4+5i}{4+5i}=\frac{4+5i}{(4)^2-(5i)^2}$$

$$=\frac{4+5i}{16-(25\times-1)}=\frac{4+5i}{16+25}$$

$$=\frac{4+5i}{41}=\frac{4}{41}+\frac{5i}{41}$$

$\therefore$ Real part $=\frac{4}{41}$, Imaginary part $=\frac{5}{41}$

(iv) $(4-3i)^{-2}$

Sol. $(4-3i)^{-2}$

$$=\left(\frac{1}{4-3i}\right)^2$$

$$=\left(\frac{1}{4-3i}\times\frac{4+3i}{4+3i}\right)^2$$

$$=\left(\frac{4+3i}{(4)^2-(3i)^2}\right)^2$$

$$=\left(\frac{4+3i}{16-9i^2}\right)^2$$

$$=\left(\frac{4+3i}{16-(9\times-1)}\right)^2$$

$$=\left(\frac{4+3i}{16+9}\right)^2=\left(\frac{4+3i}{25}\right)^2$$

$$=\frac{(4)^2+2(4)(3i)+(3i)^2}{(25)^2}$$

$$=\frac{16+24i+9i^2}{625}$$

$$=\frac{16+24i+(9\times-1)}{625}$$

$$=\frac{16+24i-9}{625}$$

$$=\frac{7+24i}{625}=\frac{7}{625}+\frac{24i}{625}$$

$\therefore$ Real part $=\frac{7}{625}$, Imaginary part $=\frac{24}{625}$

(v) $\left(\frac{3+2i}{4+3i}\right)^{-1}$

Sol. $\left(\frac{3+2i}{4+3i}\right)^{-1}=\left(\frac{4+3i}{3+2i}\right)^{+1}$

$$=\frac{4+3i}{3+2i}\times\frac{3-2i}{3-2i}$$

$$=\frac{4(3-2i)+3i(3-2i)}{(3)^2-(2i)^2}$$

$$=\frac{12-8i+9i-6i^2}{9-4i^2}$$

$$=\frac{12-8i+9i-6i^2}{9-(4\times-1)}$$

$$=\frac{12-8i+9i-6(-1)}{9+4}$$

$$=\frac{12-8i+9i+6}{13}$$

$$=\frac{18+i}{13}=\frac{18}{13}+\frac{i}{13}$$

$\therefore$ Real part $=\frac{18}{13}$, Imaginary part $=\frac{1}{13}$

(vi) $\left(\frac{2-i}{2+i}\right)^{-2}$

Sol. $\left(\frac{2-i}{2+i}\right)^{-2}=\left(\frac{2+i}{2-i}\right)^{2}$

$$=\left(\frac{2+i}{2-i}\times\frac{2+i}{2+i}\right)^{2}$$

$$=\left(\frac{2(2+i)+i(2+i)}{(2)^2-(i)^2}\right)^{2}$$

$$=\left(\frac{4+2i+2i+i^2}{4-(-1)}\right)^{2}$$

$$=\left(\frac{4+2i+2i+(-1)}{4+1}\right)^{2}$$

$$=\left(\frac{4-1+2i+2i}{5}\right)^{2}$$

$$=\left(\frac{3+4i}{5}\right)^{2}$$

$$=\frac{(3)^2+2(3)(4i)+(4i)^2}{25}$$

$$\frac{9+24i+16i^2}{25}$$

$$=\frac{9+24i+(16\times-1)}{25}$$

$$=\frac{9+24i-16}{25}$$

$$=\frac{-7+24i}{25}=\frac{-7}{25}+\frac{24}{25}i$$

$\therefore$ Real part $=\frac{-7}{25}$, Imaginary part $=\frac{24}{25}$

(vii) $\left(\frac{1-2i}{1+i}\right)^{2}$

Sol. $\left(\frac{1-2i}{1+i}\right)^{2}$

$$=\left(\frac{1-2i}{1+i}\times\frac{1-i}{1-i}\right)^{2}$$

$$=\left(\frac{1(1-i)-2i(1-i)}{(1)^2-(i)^2}\right)$$

$$=\left(\frac{1-i-2i+2i^2}{1-(-1)}\right)^{2}$$

$$=\left(\frac{1-i-2i+2(-1)}{1+1}\right)^{2}$$

$$=\left(\frac{1-i-2i-2}{2}\right)^{2}$$

$$=\left(\frac{-1-3i}{2}\right)^{2}$$

$$=\frac{(-1-3i)(-1-3i)}{2\times2}$$

$$=\frac{-1(-1-3i)-3i(-1-3i)}{4}$$

$$=\frac{1+3i+3i+9i^2}{4}$$

$$=\frac{1+6i+(9\times-1)}{4}=\frac{1+6i-9}{4}$$

$$=\frac{-8+6i}{4}=\frac{-8}{4}+\frac{6i}{4}$$

$$=-2+\frac{3i}{2}$$

$\therefore$ Real part $=-2$, Imaginary part $=\frac{3}{2}$

Q.2: Solve the following simultaneous linear equations with complex coefficients for w and z:

(i) $3z+(2+i)w=11-i$

$(2-i)z-w=-1+i$

Sol. $3z+(2+i)w=11-i$ ____ (i)

$(2-i)z-w=-1+i$ ____ (ii)

From (i) solve for w in term of z

$3z+(2+i)w=11-i$

$(2+i)w=11-i-3z$

$w=\dfrac{11-i-3z}{2+i}$ ____ (iii)

Put the value of $w=\dfrac{11-i-3z}{2+i}$ from (iii) into (ii)

$(2-i)z-\dfrac{11-i-3z}{2+i}=-1+i$

$\dfrac{(2+i)(2-i)z-(11-i-3z)}{2+i}=-1+i$

$\left[(4)^2-(i)^2\right]z-11+i+3z$

$=(-1+i)(2+i)$

$(4-(-1))z-11+i+3z$

$=-2-i+2i+i^2$

$(4+1)z-11+i+3z$

$=-2+i+(-1)$

$5z-11+i+3z=-3+i$

$5z+3z+i-i=-3+11$

$8z=8$

$z=\dfrac{8}{8}$

$z=1$ ____ (iv)

From (iv) put z = 1 into (iii)

$w=\dfrac{11-i-3(1)}{2+i}$

$=\dfrac{11-i-3}{2+i}$

$=\dfrac{8-i}{2+i}$

$=\dfrac{8-i}{2+i}\times\dfrac{2-i}{2-i}$

$=\dfrac{8(2-i)-i(2-i)}{(2)^2-i^2}$

$=\dfrac{16-8i-2i+i^2}{4-(-1)}$

$=\dfrac{16-10i+(-1)}{4+1}=\dfrac{16-1-10i}{5}$

$=\dfrac{15-10i}{5}$

$=\dfrac{5(3-2i)}{5}$

$=3\div 2i$

Hence, $z=1,\ w=3-2i$

(ii) $2z+(3+i)w=9-i$

$-iz-iw=-1+i$

Sol. $2z+(3+i)w=9-i$ ____ (i)

$-iz-iw=-1+i$ ____ (ii)

From (ii) solve for z in terms of w

$-iz=-1+i+iw$

$iz=1-i-iw$

$z=\dfrac{1-i-i^2w}{i}$

$z=\dfrac{1-i-iw}{i}\times\dfrac{i}{i}$

$z=\dfrac{i-i^2-i^2w}{i^2}$

$z=\dfrac{i-(-1)-(-1)w}{-1}$

$z=\dfrac{i+1+w}{-1}$

$z=-i-1-w$ ____ (iii)

Put $z=-i-1-w$ in (i)

$2z+(3+i)w=9-i$

$2(-i-1-w)+3w+wi=9-i$

$-2i-2-2w+3w+wi=9-i$

$-2i+i+w+wi=9+2$

$-i + w + wi = 11$
$w + wi - i = 11$
$w + wi = 11 + i$
$w(1 + i) = 11 + i$
$$w = \frac{11+i}{1+i}$$
$$w = \frac{11+i}{1+i} \times \frac{1-i}{1-i}$$
$$w = \frac{11-11i+i-i^2}{(1)^2 - i^2}$$
$$w = \frac{11-10i-(-1)}{1-(-1)}$$
$$w = \frac{11+1-10i}{1+1}$$
$$w = \frac{12-10i}{2}$$
$$w = \frac{2(6-5i)}{2} = 6-5i$$
$w = 6 - 5i$

Put $w = 6 - 5i$ in (iii)

$z = -i - 1 - w$
$z = -i - 1 - (6 - 5i)$
$z = -i - 1 - 6 + 5i$
$z = -7 - i + 5i$
$z = -7 + 4i$

Hence, $z = -7 + 4i$, $w = 6 - 5i$

(iii) $\mathbf{z - 4w = 3i}$
$\mathbf{2z + 3w = 11 - 5i}$

Sol. $z - 4w = 3i$ ____ (i)
$2z + 3w = 11 - 5i$ ____ (ii)

From (i) solve for w in terms of z

$z - 4w = 3i$
$-4w = 3i - z$
$4w = -3i + z$
$$w = \frac{-3i+z}{4} \quad \text{\_\_\_\_ (iii)}$$

Put $w = \dfrac{-3i+z}{4}$ in (ii)

$$2z + 3\left(\frac{-3i+z}{4}\right) = 11-5i$$
$$\frac{4(2z)+3(-3i+z)}{4} = 11-5i$$
$8z - 9i + 3z = 4(11 - 5i)$
$8z - 9i + 3z = 44 - 20i$
$11z = 44 - 20i + 9i$
$11z = 44 - 11i$
$$z = \frac{44-11i}{11}$$
$$z = \frac{11(4-i)}{11}$$
$z = 4 - i$

Put $z = 4 - i$ in (iii)

$$w = \frac{-3i+4-i}{4}$$
$$w = \frac{-4i+4}{4}$$
$$w = \frac{\cancel{4}(-i+1)}{\cancel{4}}$$
$w = -i + 1$

Hence, $z = 4 - i$, $w = 1 - i$

(iv) $\mathbf{z + w = 3i}$
$\mathbf{2z + 3w = 2}$

Sol. $z + w = 3i$ ________ (i)
$2z + 3w = 2$ ________ (ii)

From (i) Solve for w in terms of z

$z + w = 3i$
$w = 3i - z$ ________ (iii)

put $w = 3i - z$ into (ii)

$2z + 3(3i - z) = 2$
$2z + 9i - 3z = 2$
$2z - 3z = 2 - 9i$
$-z = 2 - 9i$
$z = -2 + 9i$

put $z = -2 + 9i$ in (iii)

$w = 3i - (-2 + 9i)$
$w = 3i + 2 - 9i$
$w = 2 + 3i - 9i$
$w = 2 - 6i$

Hence, $z = -2 + 9i$, $w = 2 - 6i$

(v) $\mathbf{2z + (3 + i)\,w = 1}$
$\mathbf{-z - (1 - i)\,w = 2}$

Sol. $2z + (3 + i)\,w = 1$ ____ (i)
$-z - (1 - i)\,w = 2$ ____ (ii)

From (ii) solve z in terms of w

$-z - (1 - i)\,w = 2$
$-z = 2 + (1 - i)\,w$
$z = -2 - (1 - i)\,w$
$z = -2 - w + wi$ ____ (iii)

Put $z = -2 - w + wi$ in (i)

$2z + (3 + i)\,w = 1$
$2(-2 - w + wi) + 3w + wi = 1$
$-4 - 2w + 2wi + 3w + wi = 1$
$-2w + 3w + 2wi + wi = 1 + 4$
$w + 3wi = 5$
$w(1 + 3i) = 5$

$$w = \frac{5}{1+3i}$$

$$w = \frac{5}{1+3i} \times \frac{1-3i}{1-3i}$$

$$w = \frac{5-15i}{(1)^2 - (3i)^2}$$

$$w = \frac{5-15i}{1-9i^2}$$

$$w = \frac{5-15i}{1-(9\times -1)}$$

$$w = \frac{5-15i}{1+9}$$

$$w = \frac{5-15i}{10}$$

$$w = \frac{5}{10} - \frac{15}{10}i$$

$$w = \frac{1}{2} - \frac{3}{2}i$$

Put $w = \frac{1}{2} - \frac{3}{2}i$ in (iii)

$$z = -2 - w + wi$$

$$z = -2 - \left(\frac{1}{2} - \frac{3}{2}i\right) + \left(\frac{1}{2} - \frac{3}{2}i\right)i$$

$$z = -2 - \frac{1}{2} + \frac{3}{2}i + \frac{1}{2}i - \frac{3}{2}i^2$$

$$z = -2 - \frac{1}{2} + \frac{3}{2}i + \frac{1}{2}i - \frac{3}{2}(-1)$$

$$z = -2 - \frac{1}{2} + \frac{3}{2} + \frac{3}{2}i + \frac{1}{2}i$$

$$z = \frac{-4-1+3}{2} + \frac{3i+i}{2}$$

$$z = \frac{-2}{2} + \frac{4i}{2}$$

$$z = -1 + 2i$$

Hence, $z = -1 + 2i$, $w = \frac{1}{2} - \frac{3}{2}i$

Solved Review Exercise 1

1. Four possible answers are given for the following questions. Choose the correct answer:

(i) $i^2 + i^4 =$

(a) -1 (b) 0 (c) 1 (d) 2

(ii) Real part of $(2 - 3i)(2 + 3i)$ is:

(a) -3 (b) 1 (c) 4 (d) 13

(iii) Imaginary part of $(2-i)(2+i)$ is:

(a) 0 (b) 1 (c) 7 (d) 9

(iv) $x + iy$ will be pure imaginary number, when:

(a) $y = 0$ (b) $x = 0$ (c) $i = 0$ (d) $x = 0, y = 0$

(v) What is additive inverse of $5 - 2i$?

(a) $5 + 2i$ (b) $-5 - 2i$ (c) $5 - 2i$ (d) $-5 + 2i$

(vi) What is multiplicative inverse of $z = 1 + i$?

(a) $1 - i$ (b) i (c) $\frac{1}{2} - \frac{1}{2}i$ (d) $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

(vii) If $z = 4 - 3i$, then $z\bar{z} =$

(a) 3 (b) 9 (c) 16 (d) 25

(viii) Conjugate of $9 - 4i$ is:

(a) $-9 - 4i$ (b) $9 + 4i$ (c) $9 + 9i$ (d) $4 - 9i$

(ix) If $z = 4 + 4i$, then $z + \bar{z} =$

(a) 8 (b) $8 + 8i$ (c) $8i$ (d) 0

(x) If $z = 5 + 4i$, then $|z| =$

(a) 9 (b) 25 (c) 41 (d) $\sqrt{41}$

ANSWERS:

(i) 0	(ii) 13	(iii) 0	(iv) $x = 0$	(v) $-5 + 2i$
(vi) $\frac{1}{2} - \frac{1}{2}i$	(vii) 25	(viii) $9 + 4i$	(ix) 8	(x) $\sqrt{41}$

2. (i) Is "0" a complex number? Explain.

Ans. Yes, every real number is a complex number with an imaginary part of zero. It can be written as $0 + 0i$.

(ii) What is the result of multiplying a complex number by its conjugate?

Ans. The result is always a non-negative real number.

(iii) State the condition for two complex numbers to be equal.

Ans. Two complex numbers are equal if and only if their real numbers are equal and their imaginary parts are also equal.

3. Simplify:

(i) i^{37}

Sol. $i^{37} = i^{36}.\, i = (i^2)^{18}.i = (-1)^{18}.i = 1.i = i$

(ii) $i^{13} \times i^{11}$

Sol. $i^{13} \times i^{11} = i^{13+11} = i^{24} = (i^2)^{12} = (-1)^{12} = 1$

(iii) $(-i)^{-9}$

Sol. $(-i)^{-9} = \frac{1}{(-i)^9} = \frac{1}{(-1 \times i)^9} = \frac{1}{(-1)^9 \times i^9} = \frac{1}{-1 \times i^8 \times i}$

$$= \frac{1}{-1 \times (i^2)^4 \times i} = \frac{1}{-1 \times (-1)^4 \times i} = \frac{1}{-1 \times 1 \times i} = \frac{1}{-1 \times i} = \frac{1}{-i}$$

$$\frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = \frac{i}{-(-1)} = \frac{i}{1} = i$$

(iv) $(3-4i)(5-6i)$

Sol. $(3-4i)(5-6i)$

$= 3(5-6i) - 4i(5-6i)$

$= 15 - 18i - 20i + 24i^2$

$= 15 - 24 - 38i$

$= -9 - 38i$

(v) $(3+4i) \div (5-7i)$

Sol. $(3+4i) \div (5-7i)$

$= \dfrac{3+4i}{5-7i}$

$= \dfrac{3+4i}{5-7i} \times \dfrac{5+7i}{5+7i} = \dfrac{3(5+7i)+4i(5+7i)}{(5)^2-(7i)^2}$

$= \dfrac{15+21i+20i+28i^2}{25-(49\times -1)} = \dfrac{15+41i+28(-1)}{25+49}$

$= \dfrac{15-28+41i}{74} = \dfrac{-13+41i}{74} = -\dfrac{13}{74} + \dfrac{41i}{74}$

4. Find additive and multiplicative inverse of $z = 8 + 9i$.

Sol: Given that $z = 8 + 9i$

Additive Inverse $= -(8+9i) = -8 - 9i$

Multiplicative Inverse $= z^{-1} = \dfrac{1}{8+9i}$

$= \dfrac{1}{8+9i} \times \dfrac{8-9i}{8-9i}$

$= \dfrac{8-9i}{(8)^2-(9i)^2} = \dfrac{8-9i}{64-(81\times -1)}$

$= \dfrac{8-9i}{64+81} = \dfrac{8-9i}{145} = \dfrac{8}{145} - \dfrac{9i}{145}$

5. If $z_1 = 3 + 4i$ and $z_2 = 2 + 3i$, then verify that

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Sol: $z_1 = 3 + 4i$

$\overline{z_1} = 3 - 4i$

$z_2 = 2 + 3i$

$\overline{z_2} = 2 - 3i$

L.H.S

$z_1 + z_2 = (3+4i) + (2+3i)$

$= 3 + 4i + 2 + 3i$

$= 3 + 2 + 4i + 3i = 5 + 7i$

$\overline{z_1 + z_2} = \overline{5+7i} = 5 - 7i$

R.H.S

$\overline{z}_1 + \overline{z}_2 = (3-4i) + (2-3i)$

$= 3 - 4i + 2 - 3i$

$= 3 + 2 - 4i - 3i = 5 - 7i$

Hence, verified that $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$

(ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

Sol: $z_1 = 3 + 4i$

$\overline{z_1} = 3 - 4i$

$z_2 = 2 + 3i$

$\overline{z_2} = 2 - 3i$

L.H.S	R.H.S
$z_1z_2 = (3+4i)(2+3i)$	$\overline{z}_1\overline{z}_2 = (3-4i)(2-3i)$
$= 3(2+3i) + 4i(2+3i)$	$= 3(2-3i) - 4i(2-3i)$
$= 6 + 9i + 8i + 12i^2$	$= 6 - 9i - 8i + 12i^2$
$= 6 + 17i + 12(-1)$	$= 6 - 17i + 12(-1)$
$= 6 - 12 + 17i$	$= 6 - 12 - 17i$
$= -6 + 17i$	$= -6 - 17i$
$\overline{z_1z_2} = \overline{-6+17i} = -6 - 17i$	Hence, verified that $\overline{z_1z_2} = \overline{z}_1\overline{z}_2$

(iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$

Sol. $z_1 = 3 + 4i$, $\overline{z_2} = 2 + 3i$

$\overline{z}_1 = 3 - 4i$, $\overline{z_2} = 2 - 3i$

L.H.S

$$\frac{z_1}{z_2} = \frac{3+4i}{2+3i}$$

$$= \frac{3+4i}{2+3i} \times \frac{2-3i}{2-3i}$$

$$= \frac{3(2-3i)+4i(2-3i)}{(2)^2-(3i)^2}$$

$$= \frac{6-9i+8i-12i^2}{4-(9\times(-1))}$$

$$= \frac{6-i-(12\times(-1))}{4-(-9)} = \frac{6-i+12}{4+9}$$

$$= \frac{18-i}{13} = \frac{18}{13} - \frac{i}{13}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\frac{18}{13} - \frac{i}{13}} = \frac{18}{13} + \frac{i}{13}$$

R.H.S

$$\frac{\overline{z}_1}{\overline{z}_2} = \frac{3-4i}{2-3i}$$

$$= \frac{3-4i}{2-3i} \times \frac{2+3i}{2+3i}$$

$$= \frac{3(2+3i)-4i(2+3i)}{(2)^2-(3i)^2}$$

$$= \frac{6+9i-8i-12i^2}{4-(9\times(-1))}$$

$$= \frac{6+i-12(-1)}{4-(-9)} = \frac{6+i+12}{4+9}$$

$$= \frac{18+i}{13} = \frac{18}{13} + \frac{i}{13}$$

Hence, verified that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$

(iv) $|z_1| = |\overline{-z_1}|$

Sol. $z_1 = 3 + 4i$

$\overline{z_1} = 3 - 4i$

$-\overline{z_1} = -(3 - 4i) = -3 + 4i$

L.H.S $= |z_1| = |3 + 4i| = \sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

R.H.S $= |-\overline{z}_1| = |-3 + 4i| = \sqrt{(-3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

Hence, v erified that $|z_1| = |\overline{z}_1|$

(v) $\overline{\overline{z_2}} = z_2$

Sol. $z_2 = 2 + 3i$

$$\overline{z_2} = \overline{2+3i} = 2 - 3i$$

$$\overline{\overline{z_2}} = \overline{2-3i} = 2 + 3i$$

Hence, proved that $\overline{\overline{z_2}} = z_2$

(vi) $z_1\overline{z_1} = |z_1|^2$

Sol. $z_1 = 3 + 4i$

$$\overline{z_1} = \overline{3+4i} = 3 - 4i$$

L.H.S

$$\begin{aligned} z_1\overline{z_1} &= (3+4i)(3-4i) \\ &= (3)^2 - (4i)^2 \\ &= 9 - 16i^2 \\ &= 9 - (16 \times (-1)) \\ &= 9 + 16 = 25 \end{aligned}$$

R.H.S

$$\begin{aligned} |z_1| &= |3+4i| = \sqrt{(3)^2 + (4)^2} \\ &= \sqrt{9+16} = \sqrt{25} = 5 \\ |z_1|^2 &= (5)^2 = 25 \end{aligned}$$

Hence, verified that $z_1\overline{z_1} = |z_1|^2$

6. If $z_1 = 5 + 4i$, $z_2 = 3 + 2i$, then find

(i) $z_1 z_2$

Sol. $z_1 = 5 + 4i$, $z_2 = 3 + 2i$

$$\begin{aligned} z_1z_2 &= (5+4i)(3+3i) \\ &= 5(3+2i) + 4i(3+2i) \\ &= 15 + 10i + 12i + 8i^2 \\ &= 15 + 22i + 8(-1) \\ &= 15 + 22i - 8 \\ &= 7 + 22i \end{aligned}$$

(ii) $\dfrac{z_1}{z_2}$

Sol. $z_1 = 5 + 4i$, $z_2 = 3 + 2i$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{5+4i}{3+2i} = \frac{5+4i}{3+2i} \times \frac{3-2i}{3-2i} \\ &= \frac{5(3-2i)+4i(3-2i)}{(3)^2-(2i)^2} = \frac{15-10i+12i-8i^2}{9-(4(-1))} \\ &= \frac{15+2i-8(-1)}{9-(-4)} = \frac{15+2i+8}{9+4} \\ &= \frac{23+2i}{13} = \frac{23}{13} + \frac{2i}{13} \end{aligned}$$

(iii) $\overline{z_1 z_2}$

Sol. $z_1 = 5 + 4i$, $z_2 = 3 + 2i$

$$\begin{aligned} \overline{z_1} &= 5 - 4i, \quad \overline{z_2} = 3 - 2i \\ \overline{z_1 z_2} &= (5-4i)(3-2i) \\ &= 5(3-2i) - 4i(3-2i) \\ &= 15 - 10i - 12i + 8i^2 \\ &= 15 - 22i + 8(-1) \\ &= 15 - 22i - 8 \\ &= 7 - 22i \end{aligned}$$

(iv) $|z_1z_2|$

Sol. $z_1 = 5 + 4i$, $z_2 = 3 + 2i$

$$\begin{aligned} z_1z_2 &= (5+4i)(3+2i) \\ &= 5(3+2i) + 4i(3+2i) \\ &= 15 + 10i + 12i + 8i^2 \\ &= 15 + 22i + 8(-1) \\ &= 15 + 22i - 8 \\ &= 7 + 22i \\ |z_1z_2| &= |7+22i| = \sqrt{(7)^2 + (22)^2} \\ &= \sqrt{49+484} = \sqrt{533} \end{aligned}$$

7. Find real and imaginary parts of $z = (2+7i)^{-1}$

Sol. $z = (2+7i)^{-1}$

$$= \frac{1}{2+7i}$$

$$= \frac{1}{2+7i} \times \frac{2-7i}{2-7i}$$

$$= \frac{2-7i}{(2)^2-(7i)^2} = \frac{2-7i}{4-49i^2} = \frac{2-7i}{4-(49\times -1)}$$

$$= \frac{2-7i}{4+49} = \frac{2-7i}{53} = \frac{2}{53} - \frac{7}{53}i$$

so, real part $= \frac{2}{53}$, imaginary part $= -\frac{7}{53}$

8. Solve the given simultaneous linear equations with complex coefficients for z and w:

$iz + (2-i)w = 4+i$

$iz + (3+i)w = 3+3i$

Sol. $iz + (2-i)w = 4+i$ ______ (i)

$iz + (3+i)w = 3+3i$ ______ (ii)

subtract (i) From (ii)

$$\cancel{iz} + (3+i)w = 3+3i$$
$$\underset{-}{\cancel{iz}} \pm (2+i)w = \underset{-}{4} \pm i$$

$$[(3+i)-(2-i)]w = -1+2i$$

$$(3+i-2+i)w = -1+2i$$

$$(1+2i)w = -1+2i$$

$$w = \frac{-1+2i}{1+2i}$$

$$w = \frac{-1+2i}{1+2i} \times \frac{1-2i}{1-2i} = \frac{-1(1-2i)+2i(1-2i)}{(1)^2-(2i)^2}$$

$$= \frac{-1+2i+2i-4i^2}{1-(4\times -1)} = \frac{-1+4i-4(-1)}{1+4}$$

$$= \frac{-1+4i+4}{5} = \frac{3+4i}{5}$$

$$w = \frac{3}{5} + \frac{4}{5}i$$

put $w = \frac{3}{5} + \frac{4}{5}i$ into (i)

$$iz+(2-i)\left(\frac{3}{5}+\frac{4}{5}i\right) = 4+i$$

$$iz+2\left(\frac{3}{5}+\frac{4}{5}i\right)-i\left(\frac{3}{5}+\frac{4}{5}i\right) = 4+i$$

$$iz+\frac{6}{5}+\frac{8}{5}i-\frac{3}{5}i-\frac{4}{5}i^2 = 4+i$$

$$iz+\frac{6}{5}-\frac{4}{5}(-1)+\left(\frac{8}{5}-\frac{3}{5}\right)i = 4+i$$

$$iz+\frac{6}{5}+\frac{4}{5}+\left(\frac{5}{5}\right)i = 4+i$$

$$iz+\frac{10}{5}+i = 4+i$$

$$iz+2+i = 4+i$$

$$iz = 4-2+i-i=2 \quad \Rightarrow \quad iz=2$$

$$z = \frac{2}{i}$$

$$z = \frac{2}{i}\times\frac{i}{i}=\frac{2i}{i^2}$$

$$z = \frac{2i}{-1}=-2i$$

so, $z=-2i$, $w=\frac{3}{5}+\frac{4}{5}i$

9. Solve $(3-4i)(a+bi)=1+0i$ and find the values of a and b.

Sol. $(3-4i)(a+bi)=1+0i$

$$3(a+bi)-4i(a+bi) = 1+0i$$

$$3a+3bi-4ai-4bi^2 = 1+0i$$

$$3a+3bi-4ai-4b(-1) = 1+0i$$

$$3a+4b+3bi-4ai = 1+0i$$

$$(3a+4b)+(3b-4a)i = 1+0i$$

By compairing real parts with real parts and imaginary parts with imaginary parts of both sides.

$3a+4b=1$ _____ (i), $\quad -4a+3b=0$ _____ (ii)

Multiply (i) by 4 and (ii) by 3

$12a+16b=4$ ____ (iii), $\quad -12a+9b=0$ _____ (iv)

Add (iii) and (iv)

$$\begin{array}{r} \cancel{12a}+16b=4 \\ -\cancel{12a}+9b=0 \\ \hline +25b=4 \end{array}$$

$$b = \frac{4}{25}$$

Put $b = \frac{4}{25}$ in (i)

$$3a + 4\left(\frac{4}{25}\right) = 1$$

$$3a + \frac{16}{25} = 1$$

$$25 \times 3a + 25 \times \frac{16}{25} = 25 \times 1$$

$$75a + 16 = 25$$

$$75a = 25 - 16 = 9$$

$$a = \frac{\cancel{9}^{3}}{_{25}\cancel{75}} = \frac{3}{25}$$

$$a = \frac{3}{25}$$

so, $a = \frac{3}{25}$, $b = \frac{4}{25}$

10. Solve the equation for x and y: $(3 - 2i)(x + yi) = 2(x - 2yi) + 2i - 1$

Sol. $(3 - 2i)(x + yi) = 2(x - 2yi) + 2i - 1$

$$3x + 3yi - 2xi - 2yi^2 = 2x - 4yi + 2i - 1$$

$$3x - 2y(-1) - 2xi + 3yi = 2x - 1 - 4yi + 2i$$

$$3x + 2y + (-2x + 3y)i = (2x - 1) + (-4y + 2)i$$

By comparing real parts with real parts and imaginary parts with imaginary parts of both sides

$$3x + 2y = 2x - 1 \quad , \quad -2x + 3y = -4y + 2$$

$$3x - 2x + 2y = -1 \quad , \quad -2x + 3y + 4y = 2$$

$$x + 2y = -1 \_\_\_ \text{(i)} \quad , \quad -2x + 7y = 2 \_\_\_ \text{(ii)}$$

Multiply (i) by 2

$$2x + 4y = -2 \_\_\_\_ \text{(iii)}$$

Add (ii) and (iii)

$$\cancel{-2x} + 7y = 2$$

$$\cancel{2x} + 4y = -2$$

$$11y = 0$$

$$y = \frac{0}{11} = 0 \quad \Rightarrow \quad y = 0$$

Put y = 0 in (i)

$$x + 2(0) = -1$$

$$x + 0 = -1$$

$$x = -1$$

So, $x = -1$, $y = 0$

Objective Type Questions

Additional MCQ's

Multiple Choice Questions prepared in the light of new Examination Techniques
(Knowledge, Understanding, Application, Analytical & Conceptual)

1.1 Complex Numbers

○ Encircle the correct option.

1. **$x^2 + 1 = 0$ has real solution.**
(a) one (b) two (c) no (d) three
2. **$i =$ ______**
(a) $\sqrt{-1}$ (b) ± 1 (c) $0i$ (d)
3. **$i^7 =$ ______**
(a) i (b) 1 (c) -1 (d) $-i$
4. **$i^{-25} =$ ______**
(a) i (b) $-i$ (c) 1 (d) -1
5. **The imaginary unit is represented by:**
(a) α (b) i (c) β (d) ω
6. **The idea of complex numbers first emerged in:**
(a) 15th century (b) 16th century (c) 17th century (d) 18th century

1.1.1 Rectangular Form of a Complex Number

7. **Which of the following is a complex number?**
(a) 7 (b) $\sqrt{7}$ (c) $x + yi$ (d) $x^2 - 1$
8. **The real part of $-3 + 4yi$ is:**
(a) 4 (b) y (c) i (d) -3
9. **$(-i)^4 =$ ______**
(a) 1 (b) i^3 (c) -1 (d) 4
10. **If $y = 0$, the complex number is said to be:**
(a) real (b) imaginary (c) even (d) prime
11. **The imaginary part of $(5x - 7i)$ is:**
(a) 5 (b) -7 (c) x (d) i
12. **The real part of $(8 + 2i) - (4 - 3i)$ is:**
(a) 8 (b) 4 (c) $2i$ (d) $4 - i$
13. **The imaginary part of $(7 - 4i)$ is:**
(a) 2 (b) 16 (c) 3 (d) -4

1.1.2 Equality of Complex Numbers

14. **If $x + yi = 5 - 9i$ the $y =$ ______**
(a) 9 (b) -9 (c) $x + 5$ (d) yi
15. **If $2a + (b - 1)i = 10 + 4i$ then $a =$ ______**
(a) 5 (b) 10 (c) $2a + 10$ (d) $2a - 10$

16. **If $(3x + 2) + 4i = 14 - yi$ then x and y will be:**
(a) $x = 4, y = 3$ (b) $x = -4, y = 4$ (c) $x = 3, y = 4$ (d) $x = 4, y = -4$

17. **If $-5x + yi = -10$ then y = ________**
(a) 10 (b) -5 (c) 0 (d) -2

1.2 Algebric Operations on Complex Numbers

18. **If $z = 3 + 5i$ then $3z$ = ________**
(a) $9 + 15i$ (b) $6 + 8i$ (c) $3 - 5i$ (d) $9 + 25i^2$

19. **If $z = x + yi$ then $-5z$ = ________**
(a) $5z = -5x + yi$ (b) $-5x - 5yi$ (c) $x - 5yi$ (d) $5x + 5yi$

20. **By adding $(4 + 5i)$ and $(-6 + 7i)$, we get:**
(a) $-6 + 12i$ (b) $-10 + 12i$ (c) $10 + 12i$ (d) $-2 + 12i$

21. **$(4 - 3i) - (2 - 5i)$ = ________**
(a) $2 + 2i$ (b) $2 - 3i$ (c) $6 - 5i$ (d) $-2 - 2i$

22. **$(5 + 2i) + (3 - 7i)$ = ________**
(a) $2 - 5i$ (b) $15 - 9i$ (c) $8 - 5i$ (d) $8 + 9i$

23. **$(4 + i)(4 - i)$ = ________**
(a) 20 (b) 17 (c) 15 (d) 16

24. **$\dfrac{1+i}{1-i}$ in the form of $a + bi$ is:**
(a) $1 + i$ (b) $-i$ (c) -1 (d) i

1.2.6 Properties of Complex Numbers

25. **The complex numbers has the clouser property and:**
(a) Division (b) Subtraction
(c) Only addition (d) Addition and multiplication

26. **Which is the associative property of complex numbers?**
(a) $z_1(z_2z_3) = (z_1z_2)z_3$ (b) $z_1 + z_2 = z_2 + z_1$
(c) $z + 0 = z$ (d) $z - z = 0$

27. **Which is the additive identity of complex numbers?**
(a) $(1, 1)$ (b) $(0, 0i)$ (c) $(1, 0i)$ (d) $(1, i)$

28. **The multiplicative identity of complex numbers is:**
(a) $(1 + 0i)$ (b) $(0, 0i)$ (c) $(1, 1i)$ (d) $(-i, i)$

29. **The additive inverse of $(7 - 2i)$ is:**
(a) $(-2 + 7i)$ (b) $(-7, -2i)$ (c) $(0, 0)$ (d) $(-7 + 2i)$

30. **If z and w are multiplicative inverse of each other then zw = ________**
(a) 0 (b) 1 (c) i (d) $-i$

31. **For any two complex numbers z_1 and z_2:**
(a) $z_1z_2 = z_1 + z_2$ (b) $z_1 + z_2 = \dfrac{x_1}{x_2}$ (c) $z_1 + z_2 = z_2 + z_1$ (d) $z_1 - z_2 = \dfrac{z_1}{z_2}$

1.3 Complex or Argand Plane

32. **The number $(2 - 7i)$ is equivalent to:**
(a) $(2, -7)$ (b) $(2 + 7)$ (c) $(-2 + 7i)$ (d) $(2 - 7, i)$

33. **$z = -7i$ can be represented as:**
(a) $(7 + -i)$ (b) $(1 - 7i)$ (c) $+7i$ (d) $(0, -7)$

34. **The complex plane is named in honour of:**
(a) Jean Argand (b) Leonhard Euler (c) Neurten (d) Cavl

35. **Which order pair represents the complex number $z = 8$?**
(a) $(0, 8)$ (b) $(8, 8)$ (c) $(8, 0)$ (d) $(8, i)$

1.3.1 Conjugate of a Complex Number

36. **The conjugate of a complex number $a + bi$ is expressed as:**
(a) $-a - bi$ (b) $a - bi$ (c) $-a + bi$ (d) $a + bi^2$

37. **What is the conjugate of $-4 + 4i$?**
(a) $4 - 4i$ (b) $-4 + 4i$ (c) $-8i$ (d) $-4 - 4i$

38. **The product of a complex number with its conjugate is a:**
(a) real number (b) complex number (c) even number (d) odd number

39. **$(1 + 2i)(1 - 2i) =$ ___________**
(a) 3 (b) $3i$ (c) 5 (d) $5i$

40. **The conjugate of $-11i$ is:**
(a) $11i$ (b) -11 (c) $-11i$ (d) $11i^2$

1.3.3 Properties of Complex Conjugate

41. **For any two complex numbers z_1 and z_2 we have $\overline{z_1 + z_2} =$ _________**
(a) $\overline{z_1 z_2}$ (b) $\overline{z_1 - z_2}$ (c) $\overline{z_1} + \overline{z_2}$ (d) $z_1 z_2$

42. **For $z_1 = 4 + 3i$, $z_2 = 5 - 2i$ we have $\overline{z_1 + z_2} =$ ________**
(a) $9 - i$ (b) $9 + i$ (c) $5 + 5i$ (d) $4 - 5i$

43. **If $z = -3 + 7i$ then $\bar{\bar{z}}$**
(a) $-3 - 7i$ (b) $3 - 7i$ (c) $-3 + 7i$ (d) $7 - 3i$

44. **If $z = 3 - 7i$ then $z.\bar{z} =$ _____**
(a) $3 - 21i$ (b) $57i$ (c) $58i$ (d) $21 - 9i$

45. **If z is a complex number then $z.\bar{z} =$ _________**
(a) $|z|^2$ (b) $z + \bar{z}$ (c) $\bar{\bar{z}}.\bar{z}$ (d) $z - 2\bar{z}$

46. **If $z = 8 - 3i$ then $|z| =$ _________**
(a) 53 (b) 63 (c) 73 (d) 83

1.4.1 Properties of Modulus of a Complex Number

47. **If $z = 13 + 7i$, then $|\bar{\bar{z}}| =$ _________**
(a) $13 - 7i$ (b) 169 (c) 812 (d) 218

48. If $z = -\sqrt{3} - \sqrt{5}i$ then $|z| =$ ________

(a) 8 (b) $\sqrt{34}$ (c) $\sqrt{3}+\sqrt{5}$ (d) $\sqrt{8}\,i$

1.5 Finding Real and imaginary Parts of Complex Number of $(x + iy)^n$ and $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^n$, where $x^2 + iy_2 \neq 0$ and $n = \pm 1$ and $n = \pm 2$

49. $(4 + 3i)^{-2} =$ ____________

(a) $(4 + 3i)^2$ (b) $(-4 - 3)^{-2}$ (c) $\sqrt{(4)^2 + (3)^2}$ (d) $\frac{1}{(4-3i)^2}$

50. If z is a complex number then $\text{Re}(z) =$ ________

(a) $\frac{z-\bar{z}}{2i}$ (b) $\frac{z+\bar{\bar{z}}}{2}$ (c) $\frac{z+\bar{z}}{2}$ (d) $z.\bar{z}$

51. If z is a complex number the $\text{Im}(z) =$ ________

(a) $\frac{z-\bar{z}}{2i}$ (b) $\frac{z-\bar{z}}{2}$ (c) $\frac{z+\bar{z}}{2}$ (d) $\frac{z+\bar{z}}{2i}$

52. $\left(\frac{5z-3i}{3-i}\right)^{-2} =$ ____________

(a) $\left(\frac{5z+3i}{3-i}\right)^{-2}$ (b) $\left(\frac{5z-3i}{3-i}\right)^{2}$ (c) $\left(\frac{3-i}{5z-3i}\right)^{2}$ (d) $\left(\frac{3+i}{5z-3i}\right)^{-2}$

ANSWERS:

1. no **2.** $\sqrt{-1}$ **3.** $-i$ **4.** $-i$ **5.** i **6.** 16th century
7. $x + yi$ **8.** -3 **9.** 1 **10.** real **11.** -7 **12.** 4 **13.** -4
14. -9 **15.** 5 **16.** $x = 4, y = -4$ **17.** 0 **18.** $9 + 15i$
19. $-5x - 5yi$ **20.** $-2 + 12i$ **21.** $2 + 2i$ **22.** $8 - 5i$ **23.** 17
24. i **25.** addition and multiplication **26.** $z_1(z_2z_3) = (z_1z_2)z_3$ **27.** $(0, 0i)$
28. $(1+0i)$ **29.** $(-7+2i)$ **30.** 1 **31.** $z_1 + z_2 = z_2 + z_1$ **32.** $(2, -7)$ **33.** $(0, -7)$
34. Jean Argand **35.** (8, 0) **36.** $a - bi$ **37.** $-4 - 4i$ **38.** real number
39. 5 **40.** $11i$ **41.** $\overline{z_1} + \overline{z_2}$ **42.** $9 - i$ **43.** $-3 + 7i$ **44.** $57i$ **45.** $|z|^2$
46. 73 **47.** 218 **48.** 8 **49.** $\frac{1}{(4-3i)^2}$ **50.** $\frac{z+\bar{z}}{2}$
51. $\frac{z-\bar{z}}{2i}$ **52.** $\left(\frac{3-i}{5z-3i}\right)^2$

Additional Short Question

Short Answer Questions prepared in the light of new Examination Techniques
(Knowledge, Understanding, Application, Analytical & Conceptual)

1.1 Complex Numbers

Give short answers.

1. Define a complex number.

Ans. A complex number is a number expressed in the form $x + iy$. Where x and y are real numbers and i is the imaginary unit.

2. Why does the equation $x^2 + 1 = 0$ has no "real" solution?

Ans. By solving for x, we get $x^2 = -1$. Since the square of any real number is always positive, there is no real number that equals -1 when squared.

3. How is imaginary unit i(iota) defined?

Ans. The imaginary unit i is defined as the square root of -1. It is expressed as $i = \sqrt{-1}$

4. Simplify i^{27}.

Sol. $i^{27} = i^{26} \times i = (i^2)^{13} \times i = (-1)^{13} \times i = -1 \times i = -i$

5. Simplify $(-i)^{17}$

Sol. $(-i)^{17} = (-1 \times i)^{17} = (-1)^{17}(i)^{17} = -1(i)^{16} \times i = -1(i^2)^8 \times i$

$= -1(-1)^8 \times i = -1(1) \times i = -i$

1.1.1 Rectangular Form of a Complex Number

6. What is a "pure imaginary" number?

Ans. A complex number $z = x + yi$ is said to be pure imaginary number if its real part is zero. $(x = 0)$

7. What is a "real" complex number?

Ans. A complex number is said to be real if its imaginary part is zero. $(y = 0)$

8. Identify the real and imaginary part of $z = 7 - 11i$.

Sol. Given that $z = 7 - 11i$

The real part Re(z) is 7.

The imaginary part Im(z) is -11.

1.1.2 Equality of Complex Numbers

9. Under what condition are two complex numbers equal?

Ans. Two complex numbers are equal if and only if their real parts are equal $(x_1 = x_2)$ and their imaginary parts are equal $(y_1 = y_2)$.

10. If $a + bi = 9 - 13i$, find the values of a and b.

Sol. Given that $a + bi = 9 - 13i$

By the principle of equality

Equating the real parts, Equating the imaginary parts

$a = 9 \quad , \quad b = -13$

11. Find the value of x and y if $(3x + 3) + (5 - y)i = 7 + 3i$.

Sol. Given that $(3x + 3) + (5 - y)i = 7 + 3i$

Since the to complex numbers are equal, their real and imaginary parts are also equal.

$3x + 3 = 7 \quad , \quad 5 - y = 3$

$3x = 7 - 3 \quad , \quad 5 - 3 = y$

$3x = 4 \quad , \quad 2 = y \Rightarrow y = 2$

$x = \frac{4}{3}$

So, $x = \frac{4}{3}, y = 2$

1.2 Algebraic Operations on Complex Numbers

12. If $z = 1 - 2yi$ then find $-3z$.

Sol. $z = 1 - 2yi$

$-3z = -3(1 - 2yi)$

$= -3 + 6yi$

13. Simplify: $(17 - 7i) + (19 - 7i)$

Sol. $(17 - 7i) + (19 - 7i)$

$= 17 - 7i + 19 - 7i$

$= 17 + 19 - 7i - 7i$

$= 36 - 14i$

14. Simplify: $(21 - 12i) - (-11 + 17i)$

Sol. $(21 - 12i) - (-11 + 17i)$

$= 21 - 12i + 11 - 17i$

$= 21 + 11 - 12i - 17i$

$= 32 - 29i$

15. Simplify $(5 - 3i)(4 + 7i)$

Sol. $(5 - 3i)(4 + 7i)$

$= 5(4 + 7i) - 3i(4 + 7i)$

$= 20 + 35i - 12i - 21i^2$

$= 20 + 23i - 21(-1)$

$= 20 + 23i + 21$

$= 41 + 23i$

16. If $z_1 + z_2 = 19 - 11i$ and $z_1 = 3 - 7i$ then find z_2.

Sol. $z_1 + z_2 = 19 - 11i$

$z_2 = 19 - 11i - z_1$

$z_2 = 19 - 11i - (3 - 7i) \quad \because z_1 = 3 - 7i$

$z_2 = 19 - 11i - 3 + 7i$

$z_2 = 19 - 3 - 11i + 7i$

$z_2 = 16 - 4i$

So, $z_2 = 16 - 4i$

17. Express $(2 + \sqrt{3}i)(-7 + 2i)$ in the form of $a + bi$.

Sol. $(2 + \sqrt{3}i)(-7 + 2i)$

$= 2(-7 + 2i) + \sqrt{3}i(-7 + 2i)$

$= -14 + 4i - 7\sqrt{3}i + 2\sqrt{3}i^2$

$= -14 + (4 - 7\sqrt{3})i + 2\sqrt{3}(-1)$

$= -14 + (4 - 7\sqrt{3})i - 2\sqrt{3}$

$= (-14 - 2\sqrt{3}) + (4 - 7\sqrt{3})i$

18. Find the complex number z if $\frac{z}{3 - 5i} = 2 + i$

Sol. $\frac{z}{3 - 5i} = 2 + i$

$z = (2 + i)(3 - 5i)$

$z = 2(3 - 5i) + i(3 - 5i)$

$z = 6 - 10i + 3i - 5i^2$

$z = 6 - 7i - 5(-1) = 6 + 5 - 7i$

$z = 11 - 7i$

So, $z = 11 - 7i$

19. Simplify $(7-3i) \div (6+2i)$ and write in the form of $a + bi$.

Sol. $(7-3i) \div (6+2i)$

$$= \frac{7-3i}{6+2i}$$

$$= \frac{7-3i}{6+2i} \times \frac{6-2i}{6-2i}$$

$$= \frac{7(6-2i)-3i(6-2i)}{(6)^2-(2i)^2}$$

$$= \frac{42-14i-18i+6i^2}{36-(4(-1))}$$

$$= \frac{42-32i+6(-1)}{36-(-4)}$$

$$= \frac{42-32i-6}{36+4}$$

$$= \frac{36-32i}{40} = \frac{36}{40} - \frac{32}{40}i$$

$$= \frac{9}{10} - \frac{4}{5}i$$

1.2.6 Properties of Complex Numbers

20. What is the Additive identity of a complex number?

Ans. The additive identity of a complex number is $0 = 0 + 0i$. When this is added to any complex number z, the result remains z.

21. If $z_1 = 2 + 3i$ and $z_2 = 4 + i$, show that the commutative property of addition holds.

Sol. $z_1 = 2 + 3i$, $z_2 = 4 + i$

$$z_1 + z_2 = (2+3i) + (4+i)$$
$$= 2 + 4 + 3i + i = 6 + 4i$$
$$z_2 + z_1 = (4+i) + (2+3i)$$
$$= 4 + 2 + i + 3i = 6 + 4i$$

Since both result are same

$$z_1 + z_2 = z_2 + z_1$$

22. Write the Associative property for multiplication of complex numbers.

Ans. For any three complex numbers z_1, z_2 and z_3

$$(z_1z_2)z_3 = z_1(z_2z_3)$$

23. What is the multiplicative identity in the set of complex numbers?

Ans. The multiplicative identity in the complex numbers is $1 = 1 + 0i$.

24. Write the additive inverse of $z = 9 - 7i$

Sol. $z = 9 - 7i$

$$-z = -(9-7i)$$
$$-z = -9 + 7i$$

So, the additive inverse of $9 - 7i$ is $-9 + 7i$.

25. Find the multiplicative inverse of $z = \sqrt{3} - 5i$.

Sol. $z = \sqrt{3} - 5i$

$$z^{-1} = \frac{1}{\sqrt{3}-5i}$$

$$z^{-1} = \frac{1}{\sqrt{3}-5i} \times \frac{\sqrt{3}+5i}{\sqrt{3}+5i}$$

$$= \frac{\sqrt{3}+5i}{(\sqrt{3})^2-(5i)^2} = \frac{\sqrt{3}+5i}{3-25i^2}$$

$$= \frac{\sqrt{3}+5i}{3-(25\times -1)}$$

$$= \frac{\sqrt{3}+5i}{3+25} = \frac{\sqrt{3}+5i}{28}$$

$$= \frac{\sqrt{3}}{28} + \frac{5}{28}i$$

26. If $z_1 = 2 + 5i$, $z_2 = 7 - 3i$ then verify that $z_1 + z_2 = z_2 + z_1$

Sol. $z_1 = 2 + 5i$, $z_2 = 7 - 3i$

L.H.S

$z_1 + z_2 = (2 + 5i) + (7 - 3i)$
$= 2 + 5i + 7 - 3i$
$= 2 + 7 + 5i - 3i$
$= 9 + 2i$

R.H.S

$z_2 + z_1 = (7 - 3i) + (2 + 5i)$
$= 7 - 3i + 2 + 5i$
$= 7 + 2 - 3i + 5i$
$= 9 + 2i$

Hence verified that $z_1 + z_2 = z_2 + z_1$

1.3 Complex or Argand Plane

27. What is an Argand Plane?

Ans. An Argand plane is a coordinate plane where the horizontal axis(x-axis) represent the real part and the vertical axis(y-axis) represent the imaginary part of a complex number.

28. Represent the complex number $-7i$ as an ordered pair.

Ans. Since there is no real part, the real value is 0. Therefore, the order pair is $(0, -7)$

29. Represent the ordered pair $(-7, -3)$ as a complex number.

Ans. The complex number of $(-7, -3)$ is $-7 - 3i$.

1.3.1 Conjugate of a Complex Number

30. Define the conjugate of a complex number.

Ans. The conjugate of a complex number $x + yi$ is defined as the complex number $x - yi$.

31. What is the simple rule to find the conjugate of any complex number z.

Ans. To get the conjugate of the complex number z, simply change the i by $-i$ in z. For example, $3 - 5i$ is the conjugate of $3 + 5i$.

32. What is the conjugate of $-4 + 4i$.

Sol. The conjugate of $-4 + 4i$ is $-4 - 4i$.

1.3.3 Properties of Complex Conjugate

33. If $z = 19 - 7i$ then verify that $\bar{\bar{z}} = z$

Sol. $z = 19 - 7i$

$\bar{z} = \overline{19 - 7i}$

$\bar{z} = 19 + 7i$

$\bar{\bar{z}} = \overline{19 + 7i}$

$\bar{\bar{z}} = 19 - 7i = z$

So, verified that $\bar{\bar{z}} = z$

34. If $z_1 = 4 + 3i$ and $z_2 = -5 - 7i$ then verified that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Sol. $z_1 = 4 + 3i$, $z_2 = -5 - 7i$

$\overline{z_1} = 4 - 3i$, $\overline{z_2} = -5 + 7i$

L.H.S

$z_1 + z_2 = (4 + 3i) + (-5 - 7i)$

$= 4 + 3i - 5 - 7i = -1 - 4i$

$\overline{z_1 + z_2} = \overline{-1-4i} = -1 + 4i$

R.H.S

$\overline{z_1} + \overline{z_2} = (4 - 3i) + (-5 + 7i)$

$= 4 - 3i - 5 + 7i = -1 + 4i$

So, verified that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

35. If $z_1 = 11 - 4i$, $z_2 = -5 + 4i$ then find $\overline{z_1 z_2}$.

Sol. $z_1 z_2 = (11 - 4i)(-5 + 4i)$

$= 11(-5 + 4i) - 4i(-5 + 4i)$

$= -55 + 44i + 20i - 16i^2$

$= -55 + 64i - 16 \times (-1)$

$= -55 + 64i + 16 = -39 + 64i$

$\overline{z_1 z_2} = \overline{-39 + 64i}$

$= -39 - 64i$

1.4.1 Properties of Modulus of a Complex Number

36. If $z = -3 + \sqrt{5}i$, show that $|z| = |\overline{z}|$

Sol. $z = -3 + \sqrt{5}i$

$\overline{z} = -3 - \sqrt{5}i$

$|z| = |-3 + \sqrt{5}i| = \sqrt{(-3)^2 + (+\sqrt{5})^2} = \sqrt{9+5} = \sqrt{14}$ ....(i)

$|\overline{z}| = |-3 - \sqrt{5}i| = \sqrt{(-3)^2 + (-\sqrt{5})^2} = \sqrt{9+5} = \sqrt{14}$ ....(ii)

Hence verified that $|z| = |\overline{z}|$ (from (i), (ii))

37. If $z = 7 - 13i$, show that $z\overline{z} = |z|^2$.

Sol. $z = 7 - 13i$

$\overline{z} = 7 + 13i$

$z\overline{z} = (7 - 13i)(7 + 13i)$

$= (7)^2 - (13i)^2$

$= 49 - (169 \times -1)$

$= 49 + 169 = 218$ ....(i)

$|z| = |7 - 13i| = \sqrt{(7)^2 + (-13)^2}$

$= \sqrt{49 + 169} = \sqrt{218}$

$|z|^2 = (\sqrt{218})^2$

$|z|^2 = 218$ ....(ii)

Therefore $z\overline{z} = |z|^2$ from (i) and (ii)

38. Find the modulus of $-7 - \sqrt{11}i$.

Sol. Let $z = -\sqrt{7} - \sqrt{11}i$

$|z| = |-\sqrt{7} - \sqrt{11}i|$

$= \sqrt{(-\sqrt{7})^2 + (\sqrt{11})^2}$

$= \sqrt{7+11} = \sqrt{18}$

1.5 Finding Real and imaginary Parts of Complex Number of $(x + iy)^n$ and $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^n$, where $x^2 + iy^2 \neq 0$ and $n = \pm 1$ and $n = \pm 2$

39. Find the real and imaginary parts of $(3 - 5i)^{-1}$

Sol. Let $z = (3 - 5i)^{-1}$

$$= \frac{1}{(3-5i)}$$

$$= \frac{1}{3-5i} \times \frac{3+5i}{3+5i}$$

$$= \frac{3+5i}{(3)^2 + (5i)^2} = \frac{3+5i}{9-(25\times -1)} = \frac{3+5i}{9+25}$$

$$= \frac{3+5i}{34} = \frac{3}{34} + \frac{5}{34}i$$

$$\text{Re}(z) = \frac{3}{34}, \ \text{Im}(z) = \frac{5}{34}$$

40. Find the real and imaginary parts of $(1 - i)^{-2}$

Sol. Let $z = (1 - i)^{-2}$

$$= \left(\frac{1}{(1-i)}\right)^2$$

$$= \left(\frac{1}{1-i} \times \frac{1+i}{1+i}\right)^2$$

$$= \left(\frac{1+i}{(1)^2 - (i)^2}\right)^2$$

$$= \frac{(1+i)^2}{[1-(-1)]^2} = \frac{1+2i+i^2}{(1+1)^2} = \frac{1+2i+(-1)}{(2)^2}$$

$$= \frac{1+2i-1}{4} = \frac{2i}{4} = \frac{1}{2}i = 0 + \frac{1}{2}i \text{ as } z = x + iy$$

So, $\text{Re}(z) = 0, \ \text{Im}(z) = \frac{1}{2}$

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