Unit 08 Numerical Solutions of Nonlinear Equations

INTRODUCTION

Nonlinear equations, where variables appear with powers exceeding one or within trigonometric, exponential, or logarithmic functions, are ubiquitous in mathematics, science, and engineering. Since most such equations resist exact algebraic solutions, numerical techniques are essential for approximating their real roots. The Graphical Method offers a visual foundation by plotting functions and identifying x-axis intersections. More sophisticated numerical methods—the Bisection Method, Regula-Falsi Method, and Newton-Raphson Method—systematically refine approximations to achieve desired accuracy through iterative procedures grounded in function continuity and the intermediate value theorem. This chapter explores solution techniques for nonlinear equations and introduces the Trapezium Rule, a numerical integration method for estimating definite integrals when analytical integration is impractical. Beyond theoretical importance, these methods have substantial practical applications in determining chemical equilibrium concentrations, regulating cardiac rhythms, analyzing nonlinear electronic circuits with diodes and transistors, and solving cryptographic problems.

8.1 Basic Principles of Approximating Real Roots of Nonlinear Equations

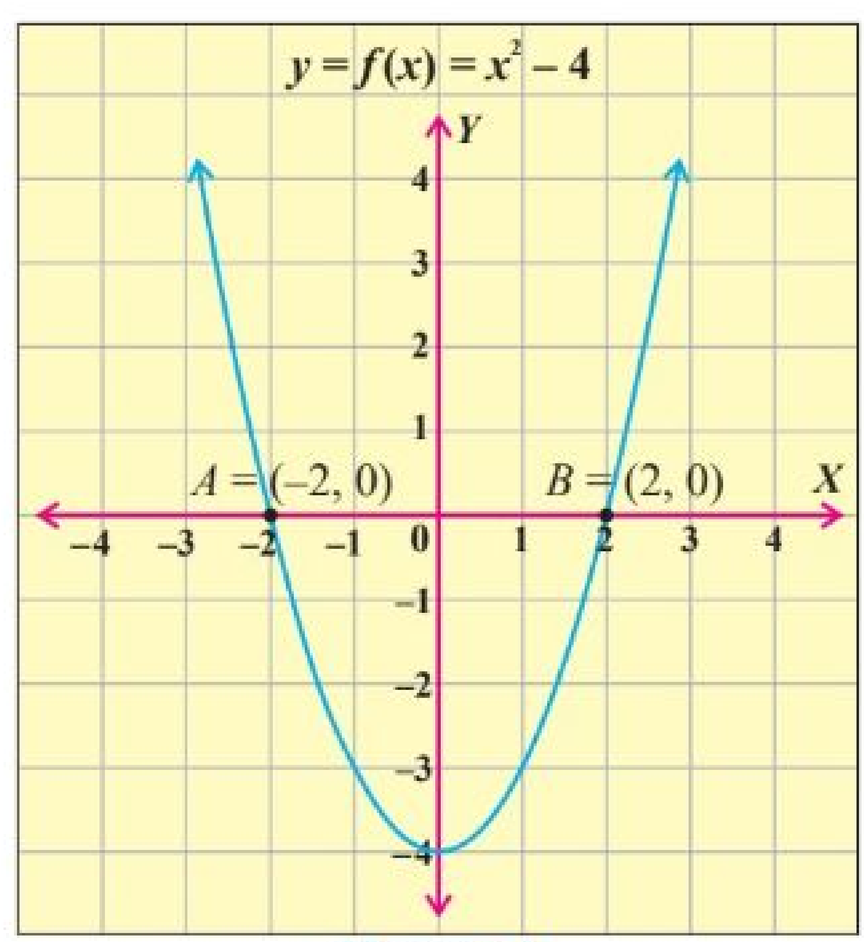

Figure 8.1

Before learning methods to find roots of nonlinear equations, it is important to understand some basic definitions. A **root** (or **zero**) of a function $y = f(x)$ is a value of x for which $y = 0$. In other words, a root is the solution of the equation $f(x) = 0$. Since the equation $y = 0$ represents the x-axis and a root is a common solution of the equations $y = f(x)$ and $y = 0$, the **geometric representation of a root** of the function $y = f(x)$ is the point on the Cartesian plane where the graph of the function intersects the x-axis. For example, if $f(x) = x^2 - 4$ then $f(x) = 0 \Rightarrow x^2 - 4 = 0$ $\Rightarrow x = 2$ or $x = -2$. So, 2 and –2 are roots of the function (see Figure 8.1). A **nonlinear equation** is an equation of the form, $y = f(x) =$

0, where f is not a linear function in the variable x, that is, $f(x)$ cannot be expressed in the form $ax + b$ for some real numbers a and b with $a \neq 0$. In such equations, the variable may appear with powers greater than one or inside trigonometric, exponential, logarithmic or other nonlinear functions. For example:

(i) $x^3 - 1 = 0$ (ii) $x^3 + 2x + 5 = 0$ (iii) $x - \cos x = 0$

Note that $x = 1$ is a root of the cubic function $y = x^3 - 1$ which can easily be found analytically. Consider another cubic function $y = x^3 + 2x + 5$. This equation has an exact root given by

$$x = \sqrt[3]{-\frac{5}{2} + \sqrt{\frac{707}{108}}} + \sqrt[3]{-\frac{5}{2} - \sqrt{\frac{707}{108}}}$$

which can be obtained analytically using algebraic methods but the expression is complicated and difficult to handle in practice. On the other hand, its approximate root (correct to two decimal places) $x \approx -1.33$ can easily be obtained using graphical or numerical methods. In fact, formulas for solving quadratic, cubic, and quartic equations do exist but no general algebraic formula is available for quintic or higher-degree polynomials. Therefore, numerical methods are essential for finding approximate roots in such cases. Before exploring these methods, we need to introduce a basic result known as the **Location of Root Theorem (LRT)**.

8.1.1 Location of Root Theorem (LRT)

Theorem 1: Let f be a real-valued function which is continuous on a closed interval $[a,b]$ If $f(a)f(b) < 0$ then f has at least one real root c in the open interval $]a,b[$

The geometric idea of the Location of Root Theorem (LRT) comes from the behavior of a continuous graph. If a function is continuous on an interval $[a,b]$, its graph can be drawn without lifting the pencil from the paper. Since $f(a)f(b) < 0$, the values of $f(a)$ and $f(b)$ have opposite signs. Without loss of generality, assume that $f(a) < 0$ and $f(b) > 0$. In this case the point $(a, f(a))$ lies below the x-axis and the point $(b, f(b))$ lies above the x-axis as shown in

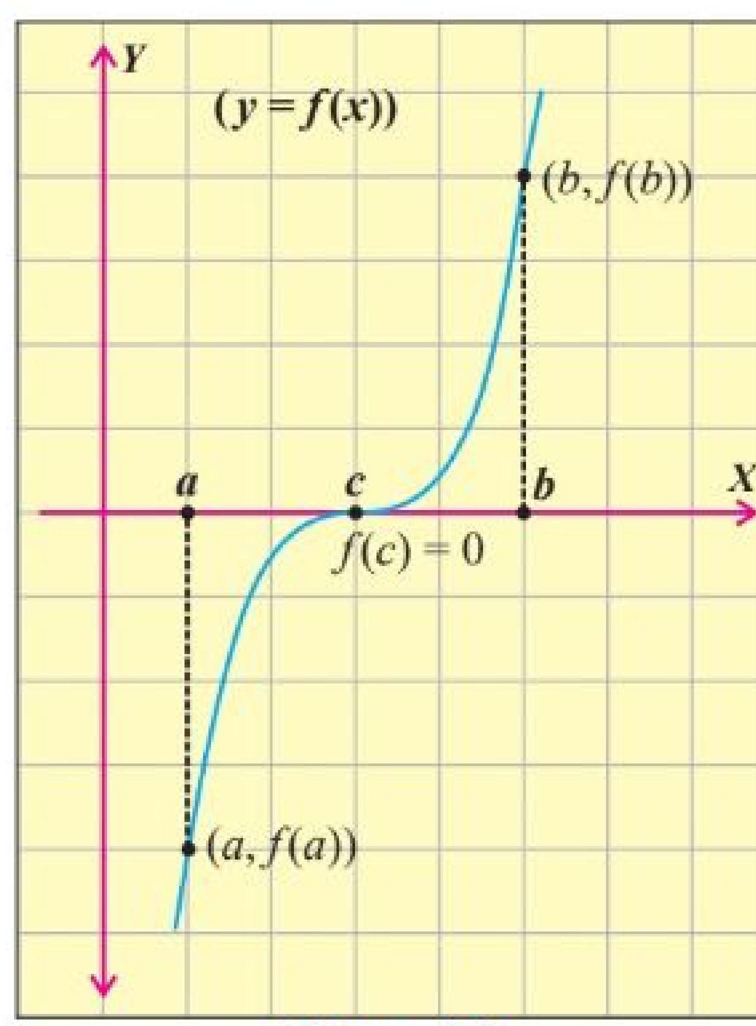

Figure 8.2

Figure 8.2. Since f is continuous on $[a, b]$, the graph can be drawn continuously

from the point $(a, f(a))$ to the point $(b, f(b))$ without any jump or break, in particular, at x-axis.

As the graph moves from below the x- axis to above the x- axis, it must cross the x-axis at some point c in the interval [a, b]. At this point, $f(c)=0$. Which shows that c is a real root in the interval [a, b], Which shows that at least one root must exist. We call the interval $[a,b]$ an **interval of sign change for the function f**, because the function changes sign from positive to negative or vice versa.

Example 1 Show that the function $f(x)=x^4-200x-50$ has at least two real roots by finding its intervals of sign change.

Solution: The given function is $f(x)=x^4-200x-50$

Continuity:

Note that, $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x^4-200x-50)=f(c)\ \ \forall c\in\mathbb{R}$

$\Rightarrow f$ is continuous at every real number.

Interval of sign change:

At $x=-3$, $f(-3)=81+600-50=631>0$

At $x=-2$, $f(-2)=16+400-50=366>0$

Similar calculations are given in the table below.

x	–3	–2	–1	0	1	2	3	4	5	6
$f(x)$	631	366	151	–50	–249	–434	–569	–594	–425	46
Sign of $f(x)$	+	+	+	–	–	–	–	–	–	+

Note that, $f(-1)f(0)<0$. It means that, $[-1,0]$ is an interval of sign change for f. Similarly, $f(5)f(6)<0$. Thus, [5, 6] is another interval of sign change. We have shown two intervals of sign change for f. Since f is continuous on these intervals, it has at least two real roots.

8.2 Graphical Method for Approximating Real Roots

After learning the Location of Root Theorem (LRT), we can use a graphical method to approximate roots of nonlinear equations. This method provides a visual way to locate where a function crosses the x-axis.

The graphical method is mainly suitable for obtaining the root correct up to one or two decimal places and is used to provide an initial estimate for numerical methods such as the Bisection Method, Regula-Falsi Method, and Newton–Raphson Method.

Steps of Graphical Method:

1. Identify the interval of sign change where $f(a)\cdot f(b)<0$, if not provided.
2. Draw the graph of $y=f(x)$ on xy-plane within the interval of sign change only.
3. Observe where the graph crosses the x-axis.

Example 2 Find a root of the equation $x-\cos x=0$ graphically, approximate to two decimal places.

Solution:

Fixing Calculator: Fix the calculator to two decimal places.

Let $y=f(x)=x-\cos x$

Continuity:

Note that, $f(c)=\lim_{x\to c}(x-\cos x)\ \ \forall\, c\in\mathbb{R}$

$\Rightarrow$ f is continuous at every real number.

Identify Interval of Sign Change

Note that $f(0)=0-\cos 0=-1<0$ and $f(1)=1-\cos(1)=0.46>0$. Thus, [0,1] is the interval of sign change and hence by LRT, a root lies within this interval.

Draw the graph

Calculate some functional values on some points inside the interval [0, 1] as given in the following table.

x	f
0.1	–0.89
0.2	–0.78
0.3	–0.66
0.4	–0.52
0.5	–0.38
0.6	–0.23
0.7	–0.06
0.8	0.10
0.9	0.28

Draw the graph of $y=x-\cos x$ on the interval [0, 1] as shown in Figure 8.3.

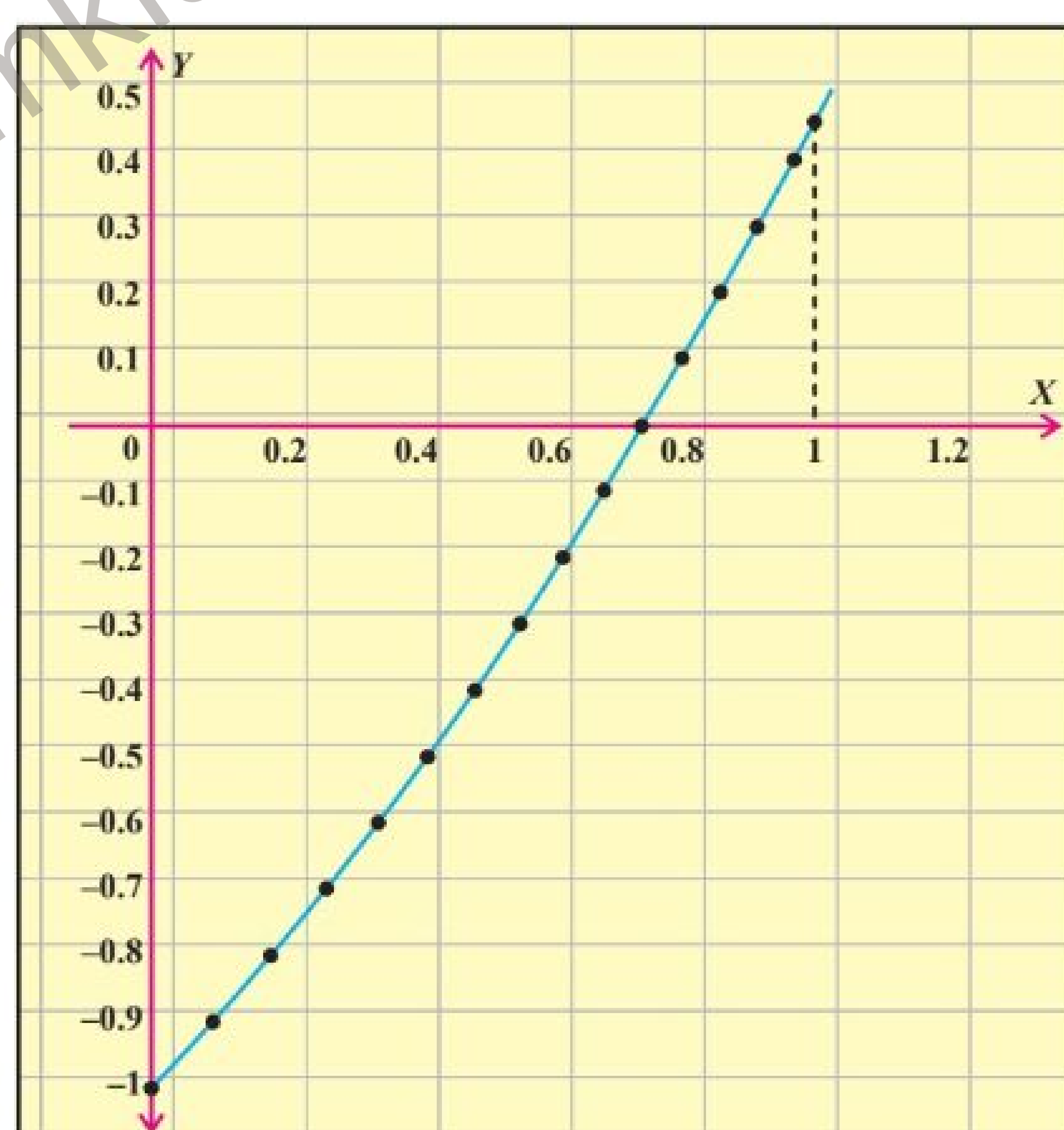

Figure 8.3

Read the root

It is clear from the figure, that the curve is crossing the x-axis at $x \approx 0.74$. Hence, the required root approximate to two decimal places is

$$\boxed{x \approx 0.74}$$

Note

While performing calculations with trigonometric equations, always keep your calculator in radian mode.

EXERCISE 8.1

1. Find the real roots of the following functions correct to two decimal places using the graphical method, within the specified interval of sign change.
   (i) $f(x) = 3x + \sin(x) - e^x$ [0, 1] (ii) $f(x) = x^4 - x - 10$ [1, 2]
2. Find the real roots of the following functions correct to two decimal places using the graphical method, by finding some interval of sign change.
   (i) $f(x) = x^3 - x - 1$ (ii) $f(x) = xe^x - 2$ (iii) $f(x) = \ln(x) - 2$

8.3 Numerical Methods for Solving Nonlinear Equations

To find roots accurate to two or more decimal places, we need systematic numerical methods. These methods use the concept of intervals where the function changes sign to progressively narrow down the root. In this section, we will study three important numerical techniques—Bisection Method, Regula-Falsi Method, and Newton–Raphson Method.

8.3.1 General Rule of Accuracy, Tolerance and Calculator Setup

Before applying any numerical method for solving nonlinear equations, it is important to decide the desired accuracy of the root. This is done using a tolerance value, which determines how close the approximate root should be to the actual root to achieve the required number of decimal places.

Decide tolerance based on desired decimal places:

There are several methods to check the accuracy of a root in numerical methods, such as reducing the interval length and checking the function value. In this chapter, we will follow the function value criterion and consider the root accurate when $|f(x)|$ is sufficiently close to zero.

If n is the number of decimal places required for the root c of the function $f(x)$, then

$$|f(c)| < \frac{10^{-n}}{2}.$$

Examples:
- For 2 decimal places ($n = 2$): $|f(c)| < 0.005 \approx 0.01$
- For 3 decimal places ($n = 3$): $|f(c)| < 0.0005 \approx 0.001$

Calculator Setup

- To obtain a root correct to n decimal places, fix the calculator to at least $n + 1$ decimal places through all the intermediate calculations and round the final answer to the required accuracy.
- Use radian mode if trigonometric functions are involved.

8.3.2 Bisection Method

The Bisection Method is a simple and reliable numerical technique to find roots of nonlinear equations in one variable. It is based on the Location of Root Theorem (LRT): if a continuous function $f(x)$ changes sign over an interval $[a,b]$, that is, $f(a)f(b)<0$, then at least one root exists within that interval. The method systematically halves the interval to narrow down the location of the root until the desired accuracy is reached.

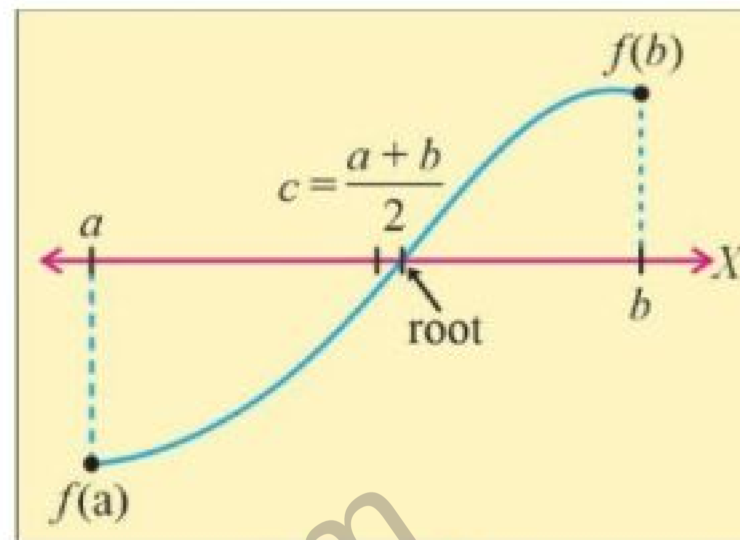

Figure 8.4

Steps of Bisection Method:

1. Check that continuity of f on $[a, b]$ and the condition $f(a)\ f(b) < 0$.
2. Compute the midpoint: $c = \dfrac{a+b}{2}$.
3. Check the function value $f(c)$:
   - If $|f(c)| \approx 0$, then c is the desired root.
   - Otherwise, determine the subinterval containing the root:
     - If $f(a) \cdot f(c) < 0$, the root lies in $[a, c]$. Set $b = c$.
     - If $f(c) \cdot f(b) < 0$, the root lies in $[c, b]$. Set $a = c$.
4. Repeat Steps 2–3 until $|f(c)|$ is near zero.

Note: the by section method guarantees that, after a finite number of steps, there must exists $c \in]a, b[$ such that $|f(c)| \approx 0$ up to the desired accuracy.

Example 3 Use bisection method to find a root of $f(x) = x^3 - 3x - 10$ within the interval $[2, 3]$ accurate up to two decimal places.

Solution: Check Initial Interval and Continuity:

Note that, $f(c) = \lim\limits_{x \to c}(x^3 - 3x - 10) \quad \forall\ c \in [2,3]$.

$\Rightarrow$ f is continuous on $[2, 3]$.

Clearly, f being a polynomial in x is continuous on $[2, 3]$.

$f(2) = 2^3 - 3(2) - 10 = -8 < 0, f(3) = 3^3 - 3(3) - 10 = 8 > 0$

Sign change occurs: $f(2)f(3) < 0 \Rightarrow$ root exists in $[2, 3]$.

Apply Bisection Method: (Intermediate values rounded to 4 decimal places)

Iteration 1: $a = 2,\ b = 3$

$c = \dfrac{2+3}{2} = 2.5$ $\qquad f(2.5) = 15.625 - 7.5 - 10 = -1.875 < 0$

Since $f(2.5) < 0$ and $f(3) = 8 > 0$, the root lies in $]2.5, 3[$. thus, consider $a = c$.

Iteration 2: $a = 2.5,\ b = 3$

$f(2.75) = 20.7969 - 8.25 - 10 = 2.5469 > 0$

Root lies in $]2.5, 2.75[$ and so, consider $b = c$.

Iteration 3:

$a = 2.5,\ b = 2.75 \qquad c = \dfrac{2.5+2.75}{2} = 2.625$

$f(2.625) = 18.087 - 7.875 - 10 = 0.212 > 0$

Root lies in $[2.5, 2.625]$ and so, set $b = c$.

> **Note**
> In the iteration table, the signs written inside the brackets along with the values of a, b and c means their functional values are negative or positive accordingly.

Continuing the same process, similar calculations are performed for the next iterations. The results are summarized in the following table.

Iteration	a	b	$c = \frac{a+b}{2}$	$f(c)$	New Interval
1	2 (–)	3 (+)	2.50 (–)	–1.875	$[2.5, 3]$
2	2.5 (–)	3 (+)	2.75 (+)	2.547	$[2.5, 2.75]$
3	2.5 (–)	2.75 (+)	2.625 (+)	0.212	$[2.5, 2.625]$
4	2.5 (–)	2.625 (+)	2.5625 (–)	–0.8611	$[2.5625, 2.625]$
5	2.5625 (–)	2.625 (+)	2.5938 (–)	–0.3308	$[2.5938, 2.625]$
6	2.5938 (–)	2.625 (+)	2.6094 (–)	–0.0609	$[2.6094, 2.625]$
7	2.6094 (–)	2.625 (+)	2.6172 (+)	0.0755	$[2.6094, 2.6172]$
8	2.6094 (–)	2.6172 (+)	2.6133 (+)	0.0072	Stop

Approximate Root: $|f(2.6133)| = 0.0072 \approx 0.01$

Final Root: $\boxed{x \approx 2.61 \text{ (correct to 2 decimal places)}}$

8.3.3 Regula-Falsi Method (False Position Method)

The Regula-Falsi Method (False Position) is a numerical method used to find a real root of a nonlinear equation $f(x) = 0$. Like bisection method, this method also based directly on the Location of Root Theorem (LRT). Suppose the function $y = f(x)$ is continuous on an interval $[a, b]$ and $f(a)f(b) < 0$, then there exists at least one root in $[a, b]$. Without loss of generality, we assume that $f(a) < 0$ and $f(b) > 0$. The idea of Regula-Falsi is to approximate the root by drawing a straight line through the points $(a, f(a))$ and $(b, f(b))$ (called **secant line**) and finding where this line crosses the x-axis. This intersection is used often as a better approximation of the root than the simple midpoint $c = \frac{a+b}{2}$ used in the Bisection Method (see Figure 8.5).

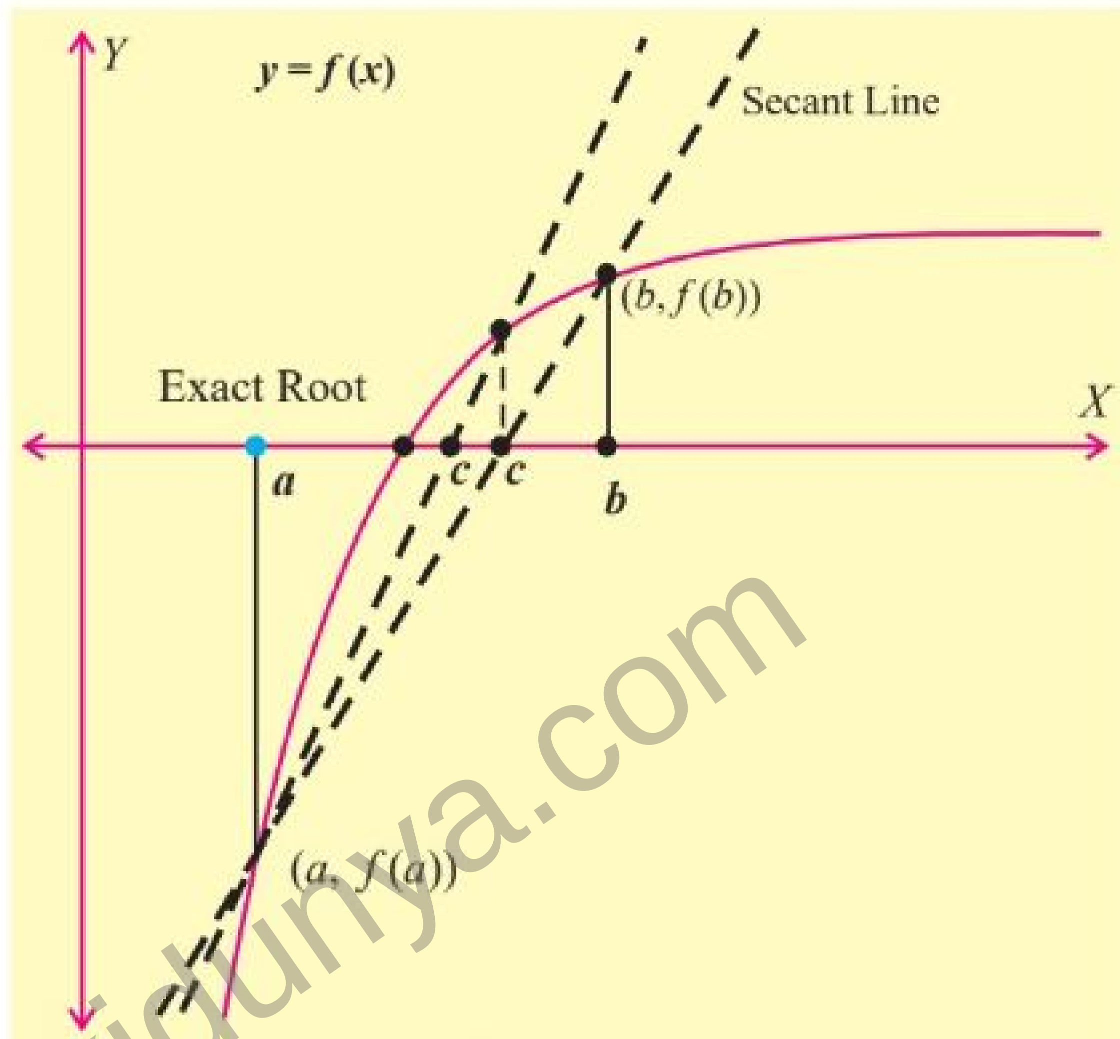

Figure 8.5

Slope of the secant line passing through the points $(a, f(a))$ and $(b, f(b))$ is $\frac{f(b)-f(a)}{b-a}$. Thus, the equation of secant line is given by:

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

For the x-intercepts of the secant line, we substitute $y = 0$ and $x = c$ in the above equation to get:

$$c = a - \frac{f(a)(b-a)}{f(b)-f(a)}$$

On simplifying:

$$c = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

Steps of the Regula-Falsi Method:

1. Check that continuity of f on $[a, b]$ and the condition $f(a)\,f(b) < 0$.

2. Compute the value of c using the Regula-Falsi formula:

$$c=\frac{af(b)-bf(a)}{f(b)-f(a)}$$

3. Check the function value $f(c)$:
   - If $|f(c)| \approx 0$, then c is the desired root.
   - Otherwise, determine the subinterval containing the root:
     - If $f(a)\cdot f(c)<0$, the root lies in $[a, c]$. Set $b=c$.
     - If $f(c)\cdot f(b)<0$, the root lies in $[c, b]$. Set $a=c$.
4. Repeat Steps 2–3 until $|f(c)|$ is less than the desired tolerance.

Note

The Regula-Falsi method guarantees that, after a finite number of steps, there exists some $c \in]a, b[$ such that $|f(c)| \approx 0$ upto the desired accuracy.

Example 4 Use Regula-Falsi method to find a root of $f(x) = x^3 - 3x - 10$ within the interval [2, 3] accurate up to two decimal places.

Solution: We set $a = 2$, $b = 3$ and apply the Regula-Falsi Formula:

$f(2) = -8 < 0$ and $f(3) = 8 > 0 \;\Rightarrow\; f$ has a root in [2, 3]

Continuity: Note that, $f(c)=\lim\limits_{x\to c}(x^3-3x-10)\;\forall\, c\in[2,3]$.

$\Rightarrow f$ is continuous on [2, 3].

Regula-Falsi Formula: $c=\dfrac{af(b)-bf(a)}{f(b)-f(a)}$

Iterations (Intermediate 4 decimals)

Iteration 1: $a = 2$, $b = 3$

$f(a) = f(2) = 2^3 - 3(2) - 10 = -8 < 0$

$f(b) = f(3) = 3^3 - 3(3) - 10 = 8 > 0$

$$c=\frac{af(b)-bf(a)}{f(b)-f(a)}=\frac{(2)(8)-(3)(-8)}{(8)-(-8)}=2.5$$

$f(c) = f(2.5) = (2.5)^3 - 3(2.5) - 10 = -1.8750 < 0$

Since $f(2.5)\,f(3) < 0$, root lies in the interval [2.5, 3].

Iteration 2: Now, $a = 2.5$, $b = 3$

$f(a)=f(2.5)=-1.8750<0,\;\; f(b)=f(3)=8>0$

$$c=\frac{af(b)-bf(a)}{f(b)-f(a)}=\frac{(2.5)(8)-(3)(-1.875)}{(8)-(-1.875)}=2.5949$$

$f(c)=f(2.5949)=-0.3119<0$

Since $f(2.5949)\,f(3)<0$, root lies in the interval [2.5949, 3].

Iteration 3: Now we take, $a = 2.5949,\ b = 3$

$$f(a) = f(2.5949) = -0.3119 < 0,\ \ f(b) = f(3) = 8 > 0$$

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{(2.5949)(8) - (3)(-0.3119)}{(8) - (-0.3119)} = 2.6101$$

$$f(c) = f(2.6101) = -0.0487 < 0$$

Since $f(2.6101)f(3) < 0$, root lies in the interval [2.6101, 3].

Iteration 4: Take, $a = 2.6101,\ b = 3$

$$f(a) = f(2.6101) = -0.0487 < 0,\ \ f(b) = f(3) = 8 > 0$$

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{(2.6101)(8) - (3)(-0.0487)}{(8) - (-0.0487)} = 2.6125$$

$$f(c) = f(2.6125) = -0.0068 < 0$$

Since $|f(2.6125)| = |-0.0068| = 0.0068 \approx 0.01 \Rightarrow 2.6125$ is a root.

Final Root: $\boxed{x \approx 2.61 \text{ (correct to 2 decimal places)}}$

8.3.4 Newton–Raphson Method

The Newton–Raphson Method is a numerical method used to find an approximate real root of a nonlinear equation $f(x) = 0$. This method works by drawing a tangent to the curve $y = f(x)$ at a point near the root and using the point where this tangent crosses the x-axis as a better approximation of the root. The initial guess chosen is based on a sign-change interval.

Since this method requires the function to be differentiable (having a slope of tangent), it usually reaches a good approximation faster than the Bisection or Regula-Falsi methods. The derivative of f should not be zero near the root. This method does not rely directly on the Location of Root Theorem (LRT).

Suppose x_0 is an initial guess for the root of $f(x) = 0$, where:

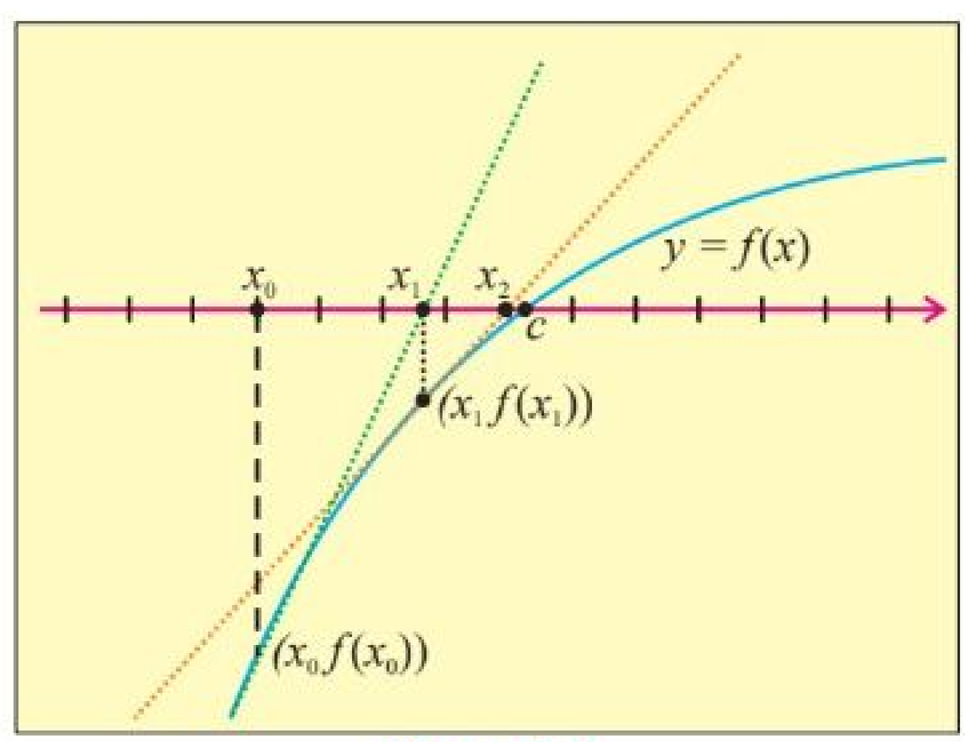

Figure 8.6

1. f is differentiable near x_0, and

2. $f'(x) \neq 0$ near x_0.

The slope of the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is given by $f'(x_0)$. Draw a tangent to the curve at this point (see Figure 8.6).

The equation of the tangent line is:

$$y-f(x_0)=f'(x_0)(x-x_0)$$

This tangent intersects the x-axis at $x=x_1$. Substituting $y=0$ and $x=x_1$ gives:

$$0-f(x_0)=f'(x_0)(x_1-x_0)$$

Solving for x_1:

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$

Similarly, if we draw a tangent at $x=x_1$, its x-intercept x_2 is:

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

Repeating this process produces a sequence of x-intercepts:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, n=0,1,2,\ldots, \quad f'(x_n)\neq 0$$

This is the Newton–Raphson formula, used to approximate the root of $f(x)=0$ iteratively.

From Figure 8.5, it is clear that each new approximation x_{n+1} is closer to the root, $x=c$, than the previous one x_n. If there exists some non-negative integer m such that $x_{m+1}-x_m$ is closed to zero, then x_m will be the desired root, i.e. $|f(x_m)|\approx 0$

Steps of Newton-Raphson Method:

1. **Prepare the equation:** Write the given equation in the form $f(x)=0$
   Choose an initial approximation x_0 from the interval where the function changes sign and ensure that:
   (i) $f(x)$ is differentiable within this interval. (ii) $f'(a)\neq 0$ near x_0.
2. **Set up the formula:** $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$, $n=0,1,2,\ldots$
   Use this to compute successive approximations.
3. **Check for convergence:** If $|f(x_{n+1})|\approx 0$, then stop.
4. **Conclude the root:** $x=x_{n+1}$ is the root correct to the desired accuracy.

Example 5 Use Newton-Raphson method to find a root of $f(x)=x^3+2x^2+10x-20$ correct to two decimal places.

Solution: Prepare the function, Given:

$$f(x)=x^3+2x^2+10x-20$$

Choose initial approximation:

Note that: $f(1) = -7 < 0$ and $f(2) = 16 > 0$.

- Note that, $f(c) = \lim_{x \to c}(x^3 + 2x^2 + 10x - 20) \quad \forall c \in [1, 2]$.

  $\Rightarrow$ is continuous on the interval of sign change [1, 2].

  $\Rightarrow$ Root lies in the interval [1, 2].

- Take initial guess: $x_0 = 1.500$.
- Calculator fixed to 3 decimal places (for accuracy).

Derivative: $f' = 3x^2 + 4x + 10$

- Note that, f' exists and $f'(x) \neq 0$ around $x_0 = 1.5$.
- $\Rightarrow$ Newton–Raphson method can be applied.

Newton–Raphson formula: $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$, $n = 0, 1, 2, \ldots$

Perform iterations: We have, $x_0 = 1.5$.

$f(x_0) = f(1.5) = (1.500)^3 + 2(1.500)^2 + 10(1.500) - 20 = 3.375 + 4.500 + 15 - 20 = 2.875 \neq 0$

Iteration 1: $f'(1.5) = 3(1.500)^2 + 4(1.500) + 10 = 6.750 + 6 + 10 = 22.750$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.500 - \frac{2.875}{22.750} = 1.374$$

$$f(x_1) = f(1.374) \approx (1.374)^3 + 2(1.374)^2 + 10(1.374) - 20 \approx 0.110 \neq 0$$

Iteration 2: $f'(1.374) \approx 3(1.374)^2 + 4(1.374) + 10 \approx 21.160$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.374 - \frac{0.110}{21.160} = 1.369$$

$$f(x_2) = f(1.369) \approx 1.369^3 + 2(1.369)^2 + 10(1.369) - 20 \approx 0.002$$

Stop iterations: $|f(1.369)| \approx 0.002 < 0.01$

Conclude the root: $x \approx 1.37$ (accurate to two decimal places)

EXERCISE 8.2

1. Find the real roots of the following functions correct to 3 decimal places using the bisection method, within the specified interval of sign change.

   (i) $f(x) = x^2 \sin(x) - 1 : [1, 2]$ (ii) $f(x) = 3x - \cos x - 1 : [0, 1]$

   (iii) $f(x) = x^4 - 2x - 10 : [1, 2]$ (iv) $f(x) = x^3 - 3x - 10 : [2, 3]$

2. Find the real roots of nonlinear equations correct to 2 decimal places using the Regula -Falsi method, by finding some interval of sign change if not specified.
   (i) $3x^3 - 9x^2 + 8 = 0$; [2, 3] (ii) $x^2 + 4\sin x = 0$; $[-2.2, -1]$
   (iii) $xe_x = 2$ (iv) $3x + \sin x = e^x$
3. Find the real roots of nonlinear equations using the Newton-Raphson method.
   (i) $x^3 + x^2 = 3\ (x + 1)$ near $x = 1$ accurate to 2 decimal places.
   (ii) $e^x = 4x$ near $x = 2$ accurate to 3 decimal places.
   (iii) $x^3 + 2x - 5 = 0$ accurate to 4 decimal places.
   (iv) $3x + \sin x = e^x$ accurate to 3 decimal places.
   (v) $\tan(x) = x$ around $x_0 = 4.3$ accurate to 4 decimal places.

8.4 Estimating the Value of a Definite Integral

In many practical problems, however, the objective is to evaluate a definite integral of a function over a given interval. Similar to nonlinear equations, definite integrals are not always easy to evaluate analytically, and in some cases exact solutions are impossible to obtain using standard integration techniques.
For example, the integrals

(i) $\int_0^1 e^{-x^2}\,dx$ and (ii) $\int_0^4 \sin(x^2)\,dx$

cannot be evaluated easily by elementary methods. In such situations, numerical methods are used to estimate the value of definite integrals with reasonable accuracy. One of the important numerical methods used for this purpose is the Trapezium Rule, which will be developed and applied in this section.

8.4.1 Trapezium (Trapezoidal) Rule

Consider a continuous function $f(x) \geq 0$ on $[a,b]$.

Let $h = \dfrac{b-a}{n}$ for some positive integer n. Divide the interval $[a,b]$ into n subintervals of equal length h as follows:

$$[a, a+h], [a+h, a+2h], [a+2h, a+3h], \ldots, [a+(n-1)h, a+nh=b]$$

Let $x_i = a + ih$; $i = 0,1,2,\ldots,n$. In this way the above subintervals can be written as follows:

$$[a = x_0,\ x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n = b].$$

The function values $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ at the end points of each subinterval represent the lengths of parallel sides of n trapeziums as shown in the

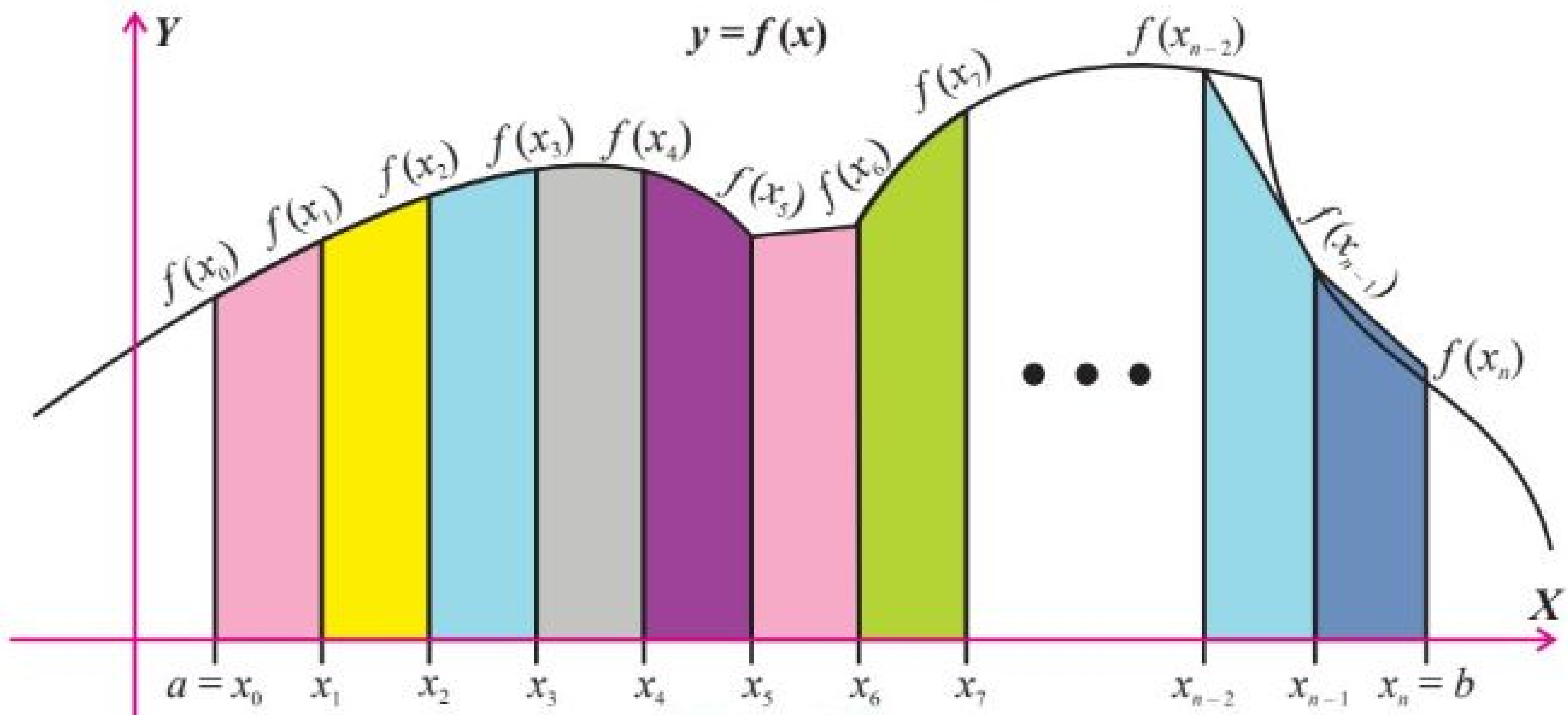

Figure 8.7

Figure 8.7. Length of the base of each trapezium is h, that is, the distance between the parallel sides of these trapeziums. Thus, the areas of these trapeziums are given by:

$$A_i = \frac{h}{2}[f(x_{i-1}) + f(x_i)]\ ;\ i = 1,2,\ldots,n$$

Since the value of the definite integral $\int_a^b f(x)\,dx$ is approximately equals to sum of areas of the trapeziums, we have:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} A_i = A_1 + A_2 + A_3 + \cdots + A_n$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \frac{h}{2}\left[f(x_2) + f(x_3)\right] + \ldots + \frac{h}{2}\left[f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Hence,

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Note:

1. From Figure 8.6, it is clear that the accuracy of the approximation improves as n is taken sufficiently large, because increasing the number of trapeziums reduces the gap between the exact area and the approximated area.
2. While applying the Trapezium Rule, the values obtained from the calculator should be rounded to at least four decimal places to maintain accuracy in the final result.
3. Although the rule is motivated using a positive function only to represent area, it is generally true for any continuous function on $[a,b]$ including those that

take negative values. In such cases, each trapezium approximates the signed area, and the sum provides a reliable estimate of the integral. The proof of this generalization is beyond the scope of the book.

Example 1 Use trapezium rule to approximate the definite integral $\int_0^1 e^{-x^2}dx$ by dividing the closed interval $[0,1]$ into 7 subintervals of equal length.

Solution: Here, $f(x) = e^{-x^2}$

Given: $a=0,\ b=1,\ n=7$

Continuity:

Note that, $f(c) = \lim\limits_{x\to c}(e^{-x^2})\quad \forall\, c\in[0,1]$.

$\Rightarrow\ f(x)=e^{-x^2}$ is continuous on $[0,1]$.

Step size: $h=\dfrac{b-a}{n}=\dfrac{1-0}{7}=\dfrac{1}{7}\cong 0.142$

Trapezium Rule: $\int_a^b f(x)dx \approx \dfrac{h}{2}\left[f(x_0)+2\left(f(x_1)+f(x_2)+\cdots+f(x_6)\right)+f(x_7)\right]$.

We have, $f(x)=e^{-x^2}$.

Compute function values (round to 4 decimal places):

$x_0=0,\qquad f(0)=1$

$x_1=0+\dfrac{1}{7}=\dfrac{1}{7},\ f\left(\dfrac{1}{7}\right)=e^{-1/49}=0.9798\quad x_2=\dfrac{1}{7}+\dfrac{1}{7}=\dfrac{2}{7},\qquad f\left(\dfrac{2}{7}\right)=e^{-4/49}=0.9216$

$x_3=\dfrac{2}{7}+\dfrac{1}{7}=\dfrac{3}{7},\ f\left(\dfrac{3}{7}\right)=e^{-9/49}=0.8322\quad x_4=\dfrac{3}{7}+\dfrac{1}{7}=\dfrac{4}{7},\ f\left(\dfrac{4}{7}\right)=e^{-16/49}=0.7214$

$x_5=\dfrac{4}{7}+\dfrac{1}{7}=\dfrac{5}{7},\qquad f\left(\dfrac{5}{7}\right)=e^{-25/49}=0.6004\quad x_6=\dfrac{5}{7}+\dfrac{1}{7}=\dfrac{6}{7},\ f\left(\dfrac{6}{7}\right)=e^{-36/49}=0.4797$

$x_7=\dfrac{6}{7}+\dfrac{1}{7}=1,\qquad f(1)=0.3679$

Apply Trapezium Rule:

$$\int_0^1 e^{-x^2}dx \approx \frac{1}{14}\left[1+2(0.9798+0.9216+0.8322+0.7214+0.6004+0.4797)+0.3679\right]$$

$$=\frac{1}{14}\left[1+2(4.5351)+0.3679\right]=0.7456$$

Hence, $\boxed{\int_0^1 e^{-x^2}dx \approx 0.7456}$

Example 2 Use Trapezium Rule to approximate the definite integral

$$\int_0^6 \frac{x}{1+x^2}\,dx$$

by subdividing the closed interval $[0,6]$ each of length 1.

Also compare the answer by solving the integral using the Fundamental Theorem of Calculus.

Solution: Given: $a=0,\ b=6,\ h=1$

Number of subintervals:

$$n=\frac{b-a}{h}=\frac{6-0}{1}=6$$

Points: $x_0=0, x_1=1, x_2=2, x_3=3, x_4=4, x_5=5, x_6=6$

Function: $f(x)=\dfrac{x}{1+x^2}$

Continuity: Note that, $f(c)=\lim\limits_{x\to c}\left(\dfrac{x}{1+x^2}\right)\ \forall\, c\in[0,6]$.

$\Rightarrow f(x)=\dfrac{x}{1+x^2}$ is continuous on $[0,6]$.

Trapezium Rule Formula:

$$\int_0^6 f(x)\,dx \approx \frac{h}{2}\left[f(x_0)+2\left(f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)\right)+f(x_6)\right]$$

Compute function values (4 decimal places):

$f(0)=0/(1+0)=0.0000$

$f(1)=1/(1+1)=0.5000$

$f(2)=2/(1+4)=0.4000$

$f(3)=3/(1+9)=3/10=0.3000$

$f(4)=4/(1+16)=4/17\approx 0.2353$

$f(5)=5/(1+25)=5/26\approx 0.1923$

$f(6)=6/(1+36)=6/37\approx 0.1622$

Apply Trapezium Rule:

$$=\int_0^6 \frac{x}{1+x^2}\,dx \approx \frac{1}{2}\left[0+2(0.5+0.4+0.3+0.2353+0.1923)+0.1622\right]$$

$$=0.5\left[2(1.6276)+0.1622\right]$$

$$=0.5(3.4174)=1.708\,7$$

Trapezium Rule Answer: $\boxed{\int_0^6 \frac{x}{1+x^2}dx \approx 1.7087}$

Exact Value using Fundamental Theorem of Calculus:

We have:

$$\int_0^6 \frac{x}{1+x^2}dx = \frac{1}{2}\int_0^6 \frac{2x}{1+x^2}dx = \frac{1}{2}ln\left|1+x^2\right|_0^6 \qquad \left(\because \int \frac{\frac{du}{dx}}{u}dx = ln|u|+c\right)$$

$$= \frac{1}{2}\ln(1+36) - \frac{1}{2}\ln(1+0)$$

(By Fundamental Theorem of Calculus)

$$= \frac{1}{2}\ln(37) = 0.5\times 3.6109 \approx 1.8055$$

Comparison:

$\boxed{\int_0^6 \frac{x}{1+x^2}dx \approx 1.7087}$ (By Trapezium Rule) $\boxed{\int_0^6 \frac{x}{1+x^2}dx \approx 1.8055}$ (Exact Value)

Note: The difference between the approximate value and the exact value of the definite integral is $|1.708 - 1.806| = 0.098$ which can be decreased by increasing the number of subintervals.

EXERCISE 8.3

Use Trapezoidal Rule to approximate the value of the following definite integrals.

1. $\int_0^2 \cos(x)dx$ taking $n=4$
2. $\int_0^1 e^x dx$ taking $n=7$
3. $\int_1^7 \ln(x)dx$ taking $n=6$
4. $\int_0^2 x^3 dx$ taking $n=8$
5. $\int_0^4 \sqrt{x}dx$ taking $n=4$
6. $\int_{-1}^1 e^{-x^2} dx$ taking $h=0.4$
7. $\int_{-1}^1 \left(x^4-x+1\right)dx$ taking $h=0.2$
8. $\int_0^{\pi} \sin(x^2)dx$ taking $h=\frac{\pi}{6}$
9. Show by the trapezium rule that the approximate area bounded by the curve $y=\sqrt{8-2x-x^2}$ and the $x-$axis using $n=8$ is 13.5 square units. Calculate the exact area by using the formula for finding the area of semi-circle.
10. Setting $h=0.5$, use the trapezium rule to find the approximate area bounded by the curve $y=e^{x^2}$ from the line $\text{x}=0$ to the line $\text{x}=4$.

8.5 Real-Life Applications of Nonlinear Equations

In real life, many physical, biological, chemical, and technological problems lead to approximate real solutions to nonlinear equations. In such situations, solving equations by algebraic methods becomes complicated, and sometimes no exact formula exists. Therefore, numerical methods provide a practical and systematic way to obtain approximate solutions with desired accuracy.

For example, in chemical reactions, scientists need to determine equilibrium concentrations of substances, which often leads to nonlinear equations. In biology and medicine, the regulation of heart beats and nerve signals involves nonlinear mathematical models. Similarly, in electronic circuits, components such as diodes and transistors follow exponential laws, producing nonlinear equations that must be solved numerically. In cryptography, nonlinear equations play an important role in designing secure communication systems and encryption algorithms.

8.5.1 Chemical Reactions

In many chemical reactions, equilibrium concentration or reaction rate leads to nonlinear equations. These equations are often difficult to solve analytically, so numerical methods are used to find approximate values.

Example 1 In a chemical reaction, the concentration x (in moles per liter) of a product at equilibrium satisfies the relation $xe^x = 1$.
Find the equilibrium concentration correct to two decimal places using Newton-Raphson Method.

Solution: Define the function: $f(x) = xe^x - 1$

Derivative: $f'(x) = e^x + xe^x$ exists at every real number x.

Newton-Raphson formula: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{x_n e^{x_n} - 1}{e^{x_n} + x_n e^{x_n}}$$

$$f(0) = -1$$
$$f(1) = i.e^1 - 1 = 1.718$$

So, root lies between 0 and 1

Initial guess:

Note that $f(0.5) = 0.5e^{0.5} - 1 = -0.1756$, which is closed to zero.

Also f is differentiable near $x_0 = 0.5$ and hence continuous as well. Thus, we take $x_0 = 0.5$ as the initial guess and Newton-Raphson Formula is applicable.

First iteration:

$f(0.5) = 0.5 \cdot e^{0.5} - 1 = -0.1757, \qquad f'(0.5) = e^{0.5} + 0.5.e^{0.5} = 2.4730$

$x_1 = 0.5 - \dfrac{-0.1757}{2.4730} = 0.5711, \quad f(x_1) = f(0.5711) = 0.5711 \cdot e^{0.5711} - 1 \approx 0.0107$

Second iteration:

$f'(x_1) = f'(0.5711) = e^{0.5711} + 0.5711 \cdot e^{0.5711} \approx 2.7813$

$x_2 = 0.5711 - \dfrac{0.0107}{2.7813} \approx 0.5673$

$f(x_2) = f(0.5673) = 0.5673 \cdot e^{0.5673} - 1 \approx 1.0004 - 1 \approx 0.0004 \approx 0$

That is, $|f(0.5673)| \approx 0$

$$\boxed{x \approx 0.57 \text{ moles per liter}}$$

8.5.2 Regulation of Heartbeats

The human heartbeat is regulated by electrical signals that control the contraction and relaxation of the heart muscles. The variation of these signals over time can often be modeled by nonlinear equations.

Example 2 The electrical activity of the heart is represented by a variable x, which measures the signal strength of the heartbeat. At steady-state, the signal satisfies the nonlinear equation:

$$x^3 - x - 1 = 0$$

Find the real steady-state value of the heartbeat signal near $x = 1$ using the Bisection Method, correct to two decimal places.

Solution: Identify the function: $\quad f(x) = x^3 - x - 1$

Continuity: Note that, $f(c) = \lim\limits_{x \to c}(x^3 - x - 1), \quad \forall c \in \mathbb{R}$

$\Rightarrow f$ is continuous at every real number.

Interval of sign change:

$f(1) = 1^3 - 1 - 1 = -1 < 0 \qquad f(2) = 8 - 2 - 1 = 5 > 0$

$\Rightarrow [1,2]$ is an interval of sign change

$\Rightarrow f$ has a root in [1, 2].

Iterations Table: (Intermediate 4 decimal places)

Iteration	a	b	$c=\frac{a+b}{2}$	$f(c)$	New Interval
1	1 (–)	2 (+)	1.5 (+)	0.875	$[1,1.5]$
2	1 (–)	1.5 (+)	1.25 (–)	–0.2969	$[1.25,1.5]$
3	1.25 (–)	1.5 (+)	1.375 (+)	0.2246	$[1.25,1.375]$
4	1.25 (–)	1.375 (+)	1.3125 (–)	–0.0515	$[1.3125,1.375]$
5	1.3125 (–)	1.375 (+)	1.3438 (+)	0.0828	[1.3125, 1.3438]
6	1.3125 (–)	1.3438 (+)	1.3282 (+)	0.0149	[1.3125, 1.3282]
7	1.3125 (–)	1.3282 (+)	1.3204 (–)	–0.0183	[1.3204, 1.3282]
8	1.3204 (–)	1.3282 (+)	1.3243 (–)	–0.0018	Stop

Note that, $|f(1.3243)| = |-0.0018| < 0.01$

Final Answer: $\boxed{x \approx 1.32}$ (correct to two decimal places)

Thus, the steady-state value of the heartbeat signal is approximately 1.32 units.

8.5.3 Electronic Circuits

Electronic components such as diodes and transistors often produce nonlinear voltage-current relationships. In circuits, the voltage or current at certain points may satisfy nonlinear equations.

Example 3 Diode/AC Circuit

In a circuit, the voltage V across a component satisfies the equation:

$$\sin V - 0.5V = 0$$

Find a non-zero real value of V correct to two decimal places using Regula-Falsi method.

Note: V represents the voltage in volts.

Solution: Define Function: $f(V) = \sin V - 0.5V$

Select Interval: $f(1) = \sin 1 - 0.5 \approx 0.3415 > 0$

$f(2) = \sin 2 - 1 \approx -0.0907 < 0$

$\Rightarrow [1, 2]$ is an interval of sign change.

Continuity: Note that, $f(c) = \lim_{V \to c} (\sin V - 0.5V) \quad \forall c \in [1, 2]$.

$\Rightarrow$ f is continuous on the interval of sign change [1, 2].

$\Rightarrow$ Regula-Falsi Method is applicable with the interval [1, 2].

Regula-Falsi Iterations: $V_c = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

Iterations Table: (4 decimal places)

Iteration	a	b	$V_c = \dfrac{a f(b) - bf(a)}{f(b) - f(a)}$	$f(V_c)$	New Interval
1	1 (+)	2 (–)	1.7901 (+)	0.0810	[1.7901, 2]
2	1.7901 (+)	2 (–)	1.8891 (+)	0.0052	[1.8891, 2]
3	1.8891 (+)	2 (–)	1.8951 (+)	0.0003	Stop

Checking: $|f(1.8951)| = 0.0003 < 0.01$

Final Answer: $\boxed{V \approx 1.90 \text{ volts}}$ (two decimal places)

8.5.4 Cryptography

Cryptographic algorithms sometimes involve high-degree nonlinear equations to generate secure keys or pseudo-random numbers.

Example 4 Key Parameter

In a cryptosystem, a key parameter x satisfies the equation:

$$x^5 - 10x + 5 = 0$$

Find a real value of x correct to two decimal places using the Regula-Falsi method.

Note: x represents a component of the encryption key; its approximate value is crucial for secure key generation.

Solution: Define Function: $f(x) = x^5 - 10x + 5$

Select Interval: $f(1) = 1 - 10 + 5 = -4 < 0,\ f(2) = 32 - 20 + 5 = 17 > 0$

Continuity: Note that, $f(c) = \lim\limits_{x \to c}(x^5 - 10x + 5)\quad \forall\, c \in [1, 2]$.

f is continuous on the interval of sign change [1, 2].

Regula-Falsi Iteration Table: (4 decimal places)

Iteration	a	b	$x_c = \dfrac{a f(b) - bf(a)}{f(b) - f(a)}$	$f(x_c)$	New Interval
1	1.0 (–)	2.0 (+)	1.1905 (–)	–4.5136	[1.1905, 2]
2	1.1905 (–)	2.0 (+)	1.3603 (–)	–3.9453	[1.3603, 2]
3	1.3603 (–)	2.0 (+)	1.4808 (–)	–2.6880	[1.4808, 2]
4	1.4808 (–)	2.0 (+)	1.5523 (–)	–1.5098	[1.5523, 2]
5	1.5523 (–)	2.0 (+)	1.5888 (–)	–0.7641	[1.5888, 2]
6	1.5888 (–)	2.0 (+)	1.6065 (–)	–0.3645	[1.6065, 2]
7	1.6065 (–)	2.0 (+)	1.6148 (–)	–0.1682	[1.6148, 2]
8	1.6148 (–)	2.0 (+)	1.6186 (–)	–0.0764	[1.6186, 2]
9	1.6186 (–)	2.0 (+)	1.6203 (–)	–0.0350	[1.6203, 2]
10	1.6203 (–)	2.0 (+)	1.6581 (+)	+0.9520	[1.6203, 1.6581]
11	1.6203 (–)	1.6581 (+)	1.6216 (–)	–0.0031	Stop

Checking: $|f(1.6216)| = 0.0031 < 0.01$

Final Answer: $\boxed{x \approx 1.62}$

EXERCISE 8.4

1. In a certain chemical reaction, the concentration of a substance at equilibrium is governed by the relation $x^2+0.2x-0.04=0$, where x is the equilibrium concentration (in moles per liter). Use bisection method to find the positive equilibrium concentration of the substance accurate to two decimal places.
2. The flow rate of blood in an artery is modeled by the nonlinear equation $Q=k\left(P-\frac{Q^2}{R}\right)$, where Q is the flow rate (mL/s), P = 100 mmHg is the blood pressure, k = 0.05, and R = 10. Determine the possible real values of the flow rate Q accurate to two decimal places.
3. In a nonlinear LC circuit, the equilibrium charge q on the capacitor satisfies $q^3-4q+2=0$. Find the equilibrium values of charge stored on the capacitor lying in the interval [0, 1] using Regula-Falsi method accurate to three decimal places.
4. In a certain cryptographic system, the decoding speed v (in units per second) is related to the security delay function $d(v)=\frac{v^2}{20+v}$, where $d(v)$ represents the decoding delay in milliseconds. For secure transmission, the system requires a decoding delay of 50 milliseconds. Use Newton–Raphson Method with initial guess $v_0=35$ to determine the decoding speed v that satisfies $d(v)=50$, correct to three decimal places.
5. In a secure communication system, the phase shift θ (in radians) of an encrypted signal is modeled by the nonlinear function
$$f(\theta)=\sin\theta-0.5.$$
For proper synchronization of the encrypted signal, the phase shift must satisfy $f(\theta)=0$. Use the Newton–Raphson Method with initial guess $\theta_0=0.5$ to determine the required phase shift θ, correct to three decimal places.
6. In a nonlinear electric circuit, the voltage x (in volts) across a component is modeled by
$$f(x)=-0.1x^2+2x+1.$$
Use the Newton–Raphson Method with initial guess $x_0=15$ to estimate the operating voltage at which the circuit reaches equilibrium. The equilibrium voltage is the voltage at which the net change in the circuit becomes zero, i.e., the circuit reaches a stable operating point.