Unit 06 Conics

INTRODUCTION

The Greek mathematicians **Apollonius** (260-200 B.C.) and **Pappus** (early fourth century A.D.) made significant contributions to the study of conic sections using Euclidean geometry. Here, we will employ the methods of **analytic geometry** to investigate their properties.

The theory of conics has important applications in various fields, including **space mechanics**, **oceanography**, and many other branches of science and technology.

Consider a circle, with its centre C, draw a line RS passes through c and perpendicular to the circle's plane. Choose a fixed point A on this line. Now, from A, draw every possible straight line that touches some point on the circle. These lines together form a surface known as a **right circular cone**. Each of these straight lines is called a **generator** (or a **ruling**) of the cone. The complete surface has two separate parts, called **nappes**, which meet only at point A. This meeting point is the **vertex** of the cone and the original line RS is the **axis** of come.

When you slice this cone with a plane, the intersection curve is called a **conic section** (or simply **conics**).

There are also special cases called **degenerate conics**. These happen when the slicing plane passes through the vertex A. Instead of getting a nice curve, the intersection might be just a single point, a single straight line, or a pair of intersecting lines.

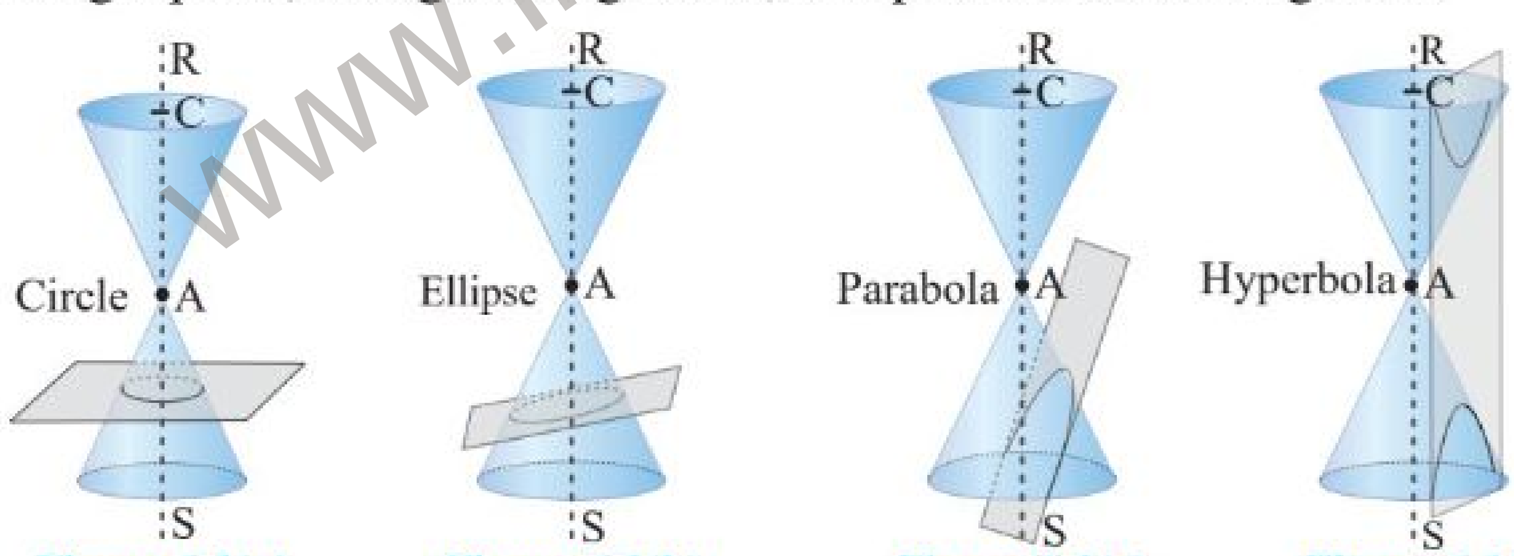


Figure 6.1(a) Figure 6.1(b) Figure 6.1(c) Figure 6.1(d)

Depending on the angle and position of the plane, you can get a circle, an ellipse, a parabola, or a hyperbola.

When this cone is intersected by a plane, different curves are obtained depending on the angle and position of the cutting plane:

- If the cutting plane is perpendicular to the axis, the section is a circle, see Figure 6.1(a).
- If the plane is slightly inclined and cuts only one nappe, the resulting curve is an ellipse, see Figure 6.1(b).

- If the plane is parallel to a generator and cuts only one nappe, the curve is a parabola, see Figure 6.1(c).
- If the plane is parallel to the axis and cuts both nappes, the curve is a hyperbola, see Figure 6.1(d).

We will being our study with the circle. Other conic sections will be discussed in subsequent section.

6.1 Circle

A circle is the set of all points in a plane that are at a fixed distance from a given fixed point. The fixed point is called the centre, and the fixed distance is called the radius of the circle.

Theorem 1: Equation of a circle with centre at the point (h, k) and radius r is $(x-h)^2+(y-k)^2=r^2$.

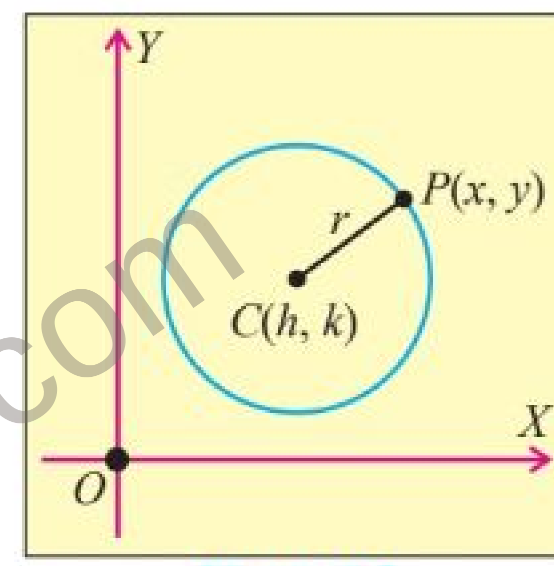


Figure 6.2

Proof: Let $C(h, k)$ be the centre of the circle and r be its radius. Let $P(x, y)$ be any point on the circle. Then by definition, the distance $|\overline{CP}|$ is equal to r. Using the distance formula between the points $P(x, y)$ and $C(h, k)$, we obtain

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

Squaring both sides gives

$$(x-h)^2+(y-k)^2=r^2 \quad \ldots(1)$$

which is the required equation of the circle.

Note:

(i) Equation (1) is known as the **standard form** of the equation of a circle.

(ii) If $r = 0$, equation (1) reduces to $(x-h)^2+(y-k)^2=0$. This represents a **point circle** centered at (h, k).

(iii) If $(h, k) = (0, 0)$, then equation (1) reduces to $x^2+y^2=r^2$. This equation represents a circle whose centre is at origin.

6.1.1 Parametric Equations of a Circle

Let $P(x, y)$ be any point on the circle $x^2+y^2=r^2$ and let the inclination of $\overline{OP}$ be θ, as shown in the Figure 6.3. Draw $\overline{PM}$ perpendicular to the x-axis. From right triangle OMP, we have

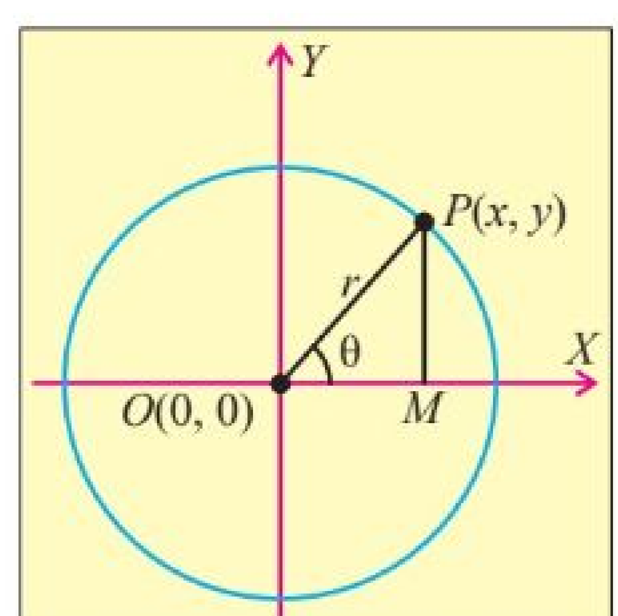


Figure 6.3

$$\cos\theta=\frac{|\overline{OM}|}{|\overline{OP}|}, \quad \sin\theta=\frac{|\overline{PM}|}{|\overline{OP}|}$$

Since $|\overline{OM}|=x$ and $|\overline{PM}|=y$, and $|\overline{OP}|=r$, it follows that

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}$$

Thus, the parametric equations of the circle are:

$$x = r\cos\theta,\ y = r\sin\theta$$

The point $P(r\cos\theta, r\sin\theta)$ lies on the circle $x^2 + y^2 = r^2$ for all values of θ.

Example 1 Write an equation of the circle with centre (–2, 1) and radius 5.

Solution: Using the standard form $(x-h)^2 + (y-k)^2 = r^2$ with $h = -2, k = 1$, and $r = 5$, we have

$$(x-(-2))^2 + (y-1)^2 = 5^2$$
$$(x+2)^2 + (y-1)^2 = 25$$

Hence, the required equation is $x^2 + y^2 + 4x - 2y - 20 = 0$.

Example 2 Prove that the locus of the point of intersection of the lines

$$x\cos\theta + y\sin\theta = a, \quad x\sin\theta - y\cos\theta = b$$

is a circle with centre at (0, 0) and radius $\sqrt{a^2+b^2}$, Where θ is a parameter.

Solution: The locus of the point of intersection of the given lines is obtained by eliminating θ between the two equations. To eliminate θ, we square both equations and add them:

$$(x\cos\theta + y\sin\theta)^2 + (x\sin\theta - y\cos\theta)^2 = a^2 + b^2$$

Expanding,

$$(x^2\cos^2\theta + 2xy\cos\theta\sin\theta + y^2\sin^2\theta) + (x^2\sin^2\theta - 2xy\sin\theta\cos\theta + y^2\cos^2\theta) = a^2 + b^2$$

Simplifying, $x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2$

Since $\cos^2\theta + \sin^2\theta = 1$, we obtain

$$x^2(1) + y^2(1) = a^2 + b^2 \quad \text{or} \quad x^2 + y^2 = \left(\sqrt{a^2+b^2}\right)^2$$

This is the equation of a circle with centre (0, 0) and radius $\sqrt{a^2+b^2}$.

6.1.2 General Form of an Equation of a Circle

Theorem 2: The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ …(1) where $g, f, c \in \mathbb{R}$ represents a circle.

Proof: Equation (1) can be written as: $x^2 + 2gx + y^2 + 2fy = -c$

Completing the square for x and y, we add $g^2 + f^2$ to both sides:

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$$

This gives $(x+g)^2 + (y+f)^2 = \left(\sqrt{g^2+f^2-c}\right)^2$.

Alternatively, this can be written as $[x-(-g)]^2+[y-(-f)]^2=\left(\sqrt{g^2+f^2-c}\right)^2$, which is the general form of the equation of a circle with centre $(-g,-f)$ and radius $r=\sqrt{g^2+f^2-c}$, provided $g^2+f^2-c>0$.

Hence equation (1) represents a circle under the condition $g^2+f^2-c>0$.

Note:

(i) If $g^2+f^2-c=0$, equation (1) represents a point circle with centre $(-g,-f)$.

(ii) If $g^2+f^2-c<0$, equation (1) represents no real locus (an imaginary circle).

6.1.3 Working Rule to Find the Centre and Radius of a Circle in General Form

To find the centre and radius of a circle from its general equation, follow these steps:

Step 1: If necessary, make the coefficients of x^2 and y^2 equal to 1 by dividing the entire equation by the common coefficient of x^2 and y^2.

Step 2: Centre of the circle = $\left(-\dfrac{\text{coefficient of } x}{2},\ -\dfrac{\text{coefficient of } y}{2}\right)$

Step 3: Radius of the circle

$$=\sqrt{\left(\frac{\text{coefficient of } x}{2}\right)^2+\left(\frac{\text{coefficient of } y}{2}\right)^2-\text{constant term}}$$

Example 3 Equation $5x^2+5y^2+20x+25y+10=0$ represents a circle. Find its centre and radius.

Solution: The given equation can be written as:

$x^2+y^2+4x+5y+2=0$ ...(i) (by dividing both sides by 5).

(i) is the general form of the equation of a circle. Comparing with the standard general form *i.e.*

$x^2+y^2+2gx+2fy+c=0$ ...(ii)

we obtain $2g=4\Rightarrow g=2$, $2f=5\Rightarrow f=\dfrac{5}{2}$, $c=2$

Therefore, centre is $(-g,-f)=\left(-2,-\dfrac{5}{2}\right)$.

The radius is given by: Radius $=\sqrt{g^2+f^2-c}=\sqrt{(2)^2+\left(\dfrac{5}{2}\right)^2-2}=\sqrt{4+\dfrac{25}{4}-2}$

$$=\sqrt{2+\frac{25}{4}}=\sqrt{\frac{8+25}{4}}=\frac{\sqrt{33}}{2}$$

Thus, the given equation represents a circle with centre $\left(-2,-\dfrac{5}{2}\right)$ and radius $\dfrac{\sqrt{33}}{2}$.

Example 4 A point moves so that the sum of the squares of its distances from two points $A(1, 1)$ and $B(2, 2)$ is equal to 4. Prove that its locus is a circle.

Solution: Let the moving point be $P(x, y)$. Then the geometric condition is:

$$|\overline{PA}|^2 + |\overline{PB}|^2 = 4$$

Using the distance formula:

$$\left(\sqrt{(x-1)^2+(y-1)^2}\right)^2 + \left(\sqrt{(x-2)^2+(y-2)^2}\right)^2 = 4$$

$(x-1)^2+(y-1)^2+(x-2)^2+(y-2)^2 = 4$ (Simplifying)

$(x^2-2x+1)+(y^2-2y+1)+(x^2-4x+4)+(y^2-4y+4) = 4$ (Expanding)

$2x^2+2y^2-6x-6y+10 = 4$ (Collecting like terms)

$x^2+y^2-3x-3y+5 = 2$ (Dividing both sides by 2)

$x^2+y^2-3x-3y+3 = 0$ (Subtracting 2 from both sides)

It represents a circle with centre $\left(\frac{3}{2}, \frac{3}{2}\right)$ and $r = \frac{\sqrt{6}}{2}$

6.1.4 Equation of a Circle with Given End points of a Diameter

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of the diameter. Let $P(x, y)$ be any point on the circle.

From plane geometry, the angle subtended by a diameter at any point on the circle is a right angle. Therefore,

$$APB = 90°$$

Thus, the lines $\overline{AP}$ and $\overline{BP}$ are perpendicular to each other.

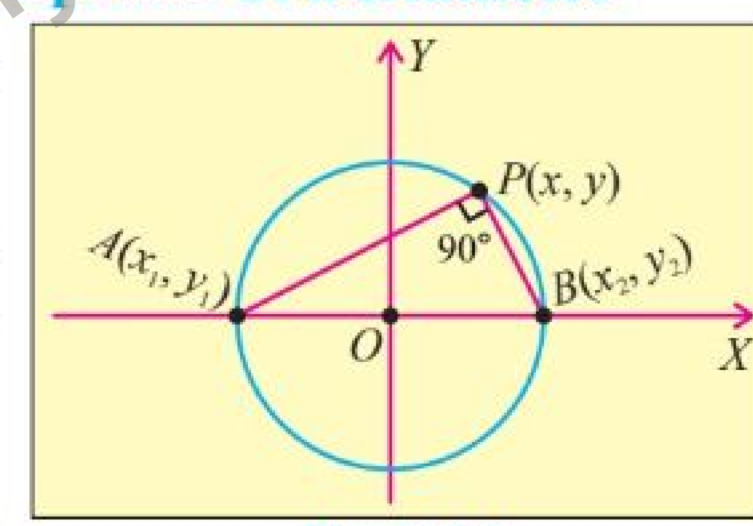


Figure 6.4

The slope of $\overline{AP}$ is $m_1 = \frac{y-y_1}{x-x_1}$ and the slope of $\overline{BP}$ is $m_2 = \frac{y-y_2}{x-x_2}$.

Using the condition for perpendicularity of two lines, $m_1 \cdot m_2 = -1$, we obtain:

$$\frac{y-y_1}{x-x_1} \cdot \frac{y-y_2}{x-x_2} = -1$$

Multiplying both sides by $(x-x_1)(x-x_2)$ gives:

$$(y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0.$$

This is the required equation of the circle.

Do You Know?
A circle has the maximum area for a given perimeter among all closed shapes.

6.1.5 Equations of Circles Determined by Given Conditions

The general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ contains three independent constants g, f and c. These can be determined if the equation satisfies three given conditions. We will discuss four different cases.

Case-1: A Circle Passing through Three Non-collinear Points

If three non-collinear points through which a circle passes are given, we can find the three independent constants g, f and c in the general equation.

Example 5 Find the equation of the circle passing through the points (–3, 3), (–1, 5) and (5, 1).

Solution: Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(A)$$

Since the circle passes through (–3, 3):

$$(A) \Rightarrow \quad (-3)^2 + (3)^2 + 2g(-3) + 2f(3) + c = 0$$
$$9 + 9 - 6g + 6f + c = 0$$
$$-6g + 6f + c + 18 = 0 \quad \ldots(1)$$

Since the circle passes through (–1, 5):

$$(A) \Rightarrow \quad (-1)^2 + (5)^2 + 2g(-1) + 2f(5) + c = 0$$
$$1 + 25 - 2g + 10f + c = 0$$
$$-2g + 10f + c + 26 = 0 \quad \ldots(2)$$

Since the circle passes through (5, 1):

$$(5)^2 + (1)^2 + 2g(5) + 2f(1) + c = 0$$
$$25 + 1 + 10g + 2f + c = 0$$
$$10g + 2f + c + 26 = 0 \quad \ldots(3)$$

Subtracting equation (2) from equation (1):

$$(-6g + 6f + c + 18) - (-2g + 10f + c + 26) = 0$$
$$-6g + 6f + c + 18 + 2g - 10f - c - 26 = 0$$
$$-4g - 4f - 8 = 0$$
$$g + f + 2 = 0 \quad \ldots(4)$$

Subtracting equation (3) from equation (2):

$$(-2g + 10f + c + 26) - (10g + 2f + c + 26) = 0$$
$$-2g + 10f + c + 26 - 10g - 2f - c - 26 = 0$$
$$-12g + 8f = 0$$
$$3g - 2f = 0 \quad \ldots(5)$$

From equation (4), $g = -f - 2$; Substituting into equation (5):

$$3(-f - 2) - 2f = 0$$

$$-3f - 6 - 2f = 0 \quad \Rightarrow \quad -5f = 6 \quad \Rightarrow \quad f = -\frac{6}{5}$$

Do You Know?

Infinitely many circles can pass through two points, but exactly one circle passes through three non-collinear points.

Equation of the circle in case of 3 non collinear points (x, y) (x_2, y_2) and (x_3, y_3) is given by

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Substituting $f=-\frac{6}{5}$ into equation (5):

$$3g-2\left(-\frac{6}{5}\right)=0 \quad \Rightarrow \quad 3g=-\frac{12}{5} \quad \Rightarrow \quad g=-\frac{4}{5}$$

Substituting $g=-\frac{4}{5}$ and $f=-\frac{6}{5}$ into equation (1):

$$-6\left(-\frac{4}{5}\right)+6\left(-\frac{6}{5}\right)+c+18=0 \quad \Rightarrow \quad \frac{24}{5}-\frac{36}{5}+c+18=0$$

$$\Rightarrow \quad -\frac{12}{5}+c+18=0 \quad \Rightarrow \quad c=\frac{12}{5}-18 \quad \Rightarrow \quad c=-\frac{78}{5}$$

Thus, the required equation of the circle is

$$x^2+y^2+2\left(-\frac{4}{5}\right)x+2\left(-\frac{6}{5}\right)y-\frac{78}{5}=0$$

Multiplying both sides by 5:

$$5x^2+5y^2-8x-12y-78=0$$

Case-2: A Circle Passing through Two Points and Having its Centre on a Given Line

Example 6 A circle with its centre on y-axis passes through the origin and the point (a, b). Find its equation.

Solution: Since the circle passes through the origin and has its centre on the y-axis, the centre is of the form $(0, r)$, where r is the radius (distance from centre to origin).

The circle also passes through (a, b). Therefore, the distance from the centre $(0, r)$ to (a, b) equals the radius r:

$$r=\sqrt{(a-0)^2+(b-r)^2}$$

Squaring both sides: $\quad r^2 = a^2+(b-r)^2$

$$r^2=a^2+b^2-2br+r^2 \quad \Rightarrow \quad r=\frac{a^2+b^2}{2b},\ b\neq 0$$

The equation of the circle with centre $(0, r)$ and radius r is:

$$(x-0)^2+(y-r)^2=r^2$$
$$x^2+y^2-2yr=0$$

$$x^2+y^2-2y\left(\frac{a^2+b^2}{2b}\right)=0 \quad \text{Substituting } r=\frac{a^2+b^2}{2b}:$$

Multiplying through by b:

$$b(x^2+y^2)-(a^2+b^2)y=0$$

Case 3: A Circle Passing through Two Points and having Tangent at One of These Points

Example 7 Find the equation of the circle passing through (1, 2) and touching the line $x - 2y + 4 = 0$ at (2, 3).

Solution: Let the equation of circle be:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(A)$$

Through (1, 2):

$$1^2 + 2^2 + 2g(1) + 2f(2) + c = 0$$
$$2g + 4f + c + 5 = 0 \quad \ldots(1)$$

Through (2, 3):

$$2^2 + 3^2 + 2g(2) + 2f(3) + c = 0$$
$$4g + 6f + c + 13 = 0 \quad \ldots(2)$$

Given line $x - 2y + 4 = 0$ has slope $\frac{1}{2}\cdot$ The normal through (2, 3) has slope –2.

$\Rightarrow$ equation of normal line is $y - 3 = -2(x - 2) \Rightarrow y - 3 = -2x + 4$

$\Rightarrow$ $2x + y - 7 = 0$ is the equation of the normal line.

This line being a normal through (2, 3) passes through the centre $(-g, -f)$ of the circle. Therefore:

$$2(-g) + (-f) - 7 = 0 \Rightarrow 2g + f = -7 \quad \ldots(3)$$

Subtracting (2) from (1):

$$[2g + 4f + c + 5] - [4g + 6f + c + 13] = 0$$
$$-2g - 2f - 8 = 0 \quad \Rightarrow \quad g + f = -4$$

$\Rightarrow$ $f = -4 - g \quad \ldots(4)$

Subtracting value of f in (3)

$$2g + (-4 - g) = -7 \quad \Rightarrow \quad 2g - 4 - g = -7 \quad \Rightarrow \quad \boxed{g = -3}$$

Then, from (4) $f = -4 - (-3) = -4 + 3 \quad \Rightarrow \quad \boxed{f = -1}$

Put values of f and g in (1)

$$2(-3) + 4(-1) + c + 5 = 0 \quad \Rightarrow \quad c = 5$$

Thus, the required substiting the values of f, gernd c is equation (A)

$x^2 + y^2 + 2(-3)x + 2 + (-1) + \Leftarrow 0$ equation of circle is: $x^2 + y^2 - 6x - 2y + 5 = 0.$

Case 4: A Circle Passing through Two Points and Touching a Given Line

Example 8 Find the equations of the circles passing through the points $A(3, 4)$ and $B(3, -4)$ and touching the line $x+2y+9=0$.

Solution: Let $O(h, k)$ be the centre of the required circle.

Since $\overline{OA}=\overline{OB} \Rightarrow |\overline{OA}|^2=|\overline{OB}|^2$, we have

$$(h-3)^2+(k-4)^2 = (h-3)^2+(k+4)^2$$

$$(k-4)^2 = (k+4)^2$$

$$k^2-8k+16=k^2+8k+16 \Rightarrow -8k=8k \Rightarrow 16k=0 \Rightarrow k=0$$

Thus, the centre is $O(h, 0)$. The radius is

$$r=|\overline{OA}|=\sqrt{(h-3)^2+(0-4)^2}=\sqrt{(h-3)^2+16}$$

The distance from the centre $(h, 0)$ to the line $x+2y+9=0$ is

$$\frac{|h+2(0)+9|}{\sqrt{1^2+2^2}}=\frac{|h+9|}{\sqrt{5}}$$

Since the circle touches the line. This distance equals to the radius:

$$\frac{|h+9|}{\sqrt{5}}=\sqrt{(h-3)^2+16}$$

Squaring both sides:

$$\frac{(h+9)^2}{5}=(h-3)^2+16$$

$$(h+9)^2=5[(h-3)^2+16]$$

$$h^2+18h+81=5(h^2-6h+9+16)$$

$$h^2+18h+81=5(h^2-6h+25)$$

$$h^2+18h+81=5h^2-30h+125$$

$$0=5h^2-30h+125-h^2-18h-81$$

$$0=4h^2-48h+44$$

$$h^2-12h+11=0$$

$$h^2-11h-h+11=0$$

$$h(h-11)-1(h-11)=0$$

$$\Rightarrow (h-11)(h-1)=0$$

$$\Rightarrow h=1, h=11$$

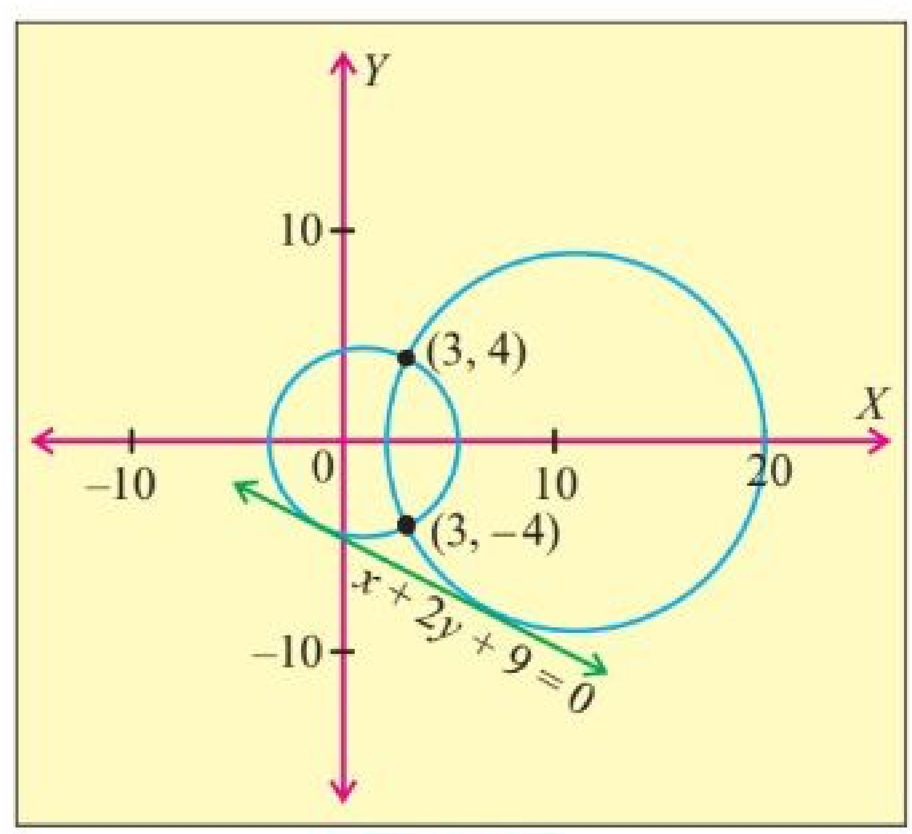


Figure 6.5

Thus, the centres of two circles are at (1, 0) and (11, 0)

Radius of circle with centre (1, 0):

$$r=\sqrt{(1-3)^2+(0-4)^2}=\sqrt{(-2)^2+16}=\sqrt{4+16}=\sqrt{20}$$

Radius of circle with centre (11, 0):

$$r=\sqrt{(11-3)^2+(0-4)^2}=\sqrt{(8)^2+16}=\sqrt{64+16}=\sqrt{80}$$

Equation of circle with centre at (1, 0) and radius $=\sqrt{20}$

$$(x-1)^2+(y-0)^2=(\sqrt{20})^2$$

$$(x-1)^2+y^2=20$$

Equation of circle with centre at (11, 0) and radius $=\sqrt{80}$

$$(x-11)^2+(y-0)^2=(\sqrt{80})^2$$

or $$(x-11)^2+y^2=80$$

Example 9 Find the equation of the circle which touches the y-axis at a distance of 3 units from the origin and cuts off an intercept of 8 units on the x-axis.

Solution: Let the point of tangency be (0, 3). Since the circle touches the y-axis at this point, the centre lies on the horizontal line through (0, 3), that is $y=3$, and its x-coordinate is $\pm r$ where r is radius.

Take the centre to the right of the y-axis:

Centre $C=(r, 3)$, radius r

Equation of circle:

$$(x-r)^2+(y-3)^2=r^2$$

$$(x^2-2rx+r^2)+(y^2-6y+9)=r^2$$

$$x^2+y^2-2rx-6y+9=0$$

For intercept on the x-axis, set $y=0$:

$$x^2-2rx+9=0$$

Let the roots be x_1, x_2. Then:

$$x_1+x_2=2r,\ x_1\cdot x_2=9$$

Intercept length $=|x_1-x_2|$ ( )

$$=\sqrt{(x_1+x_2)^2-4x_1x_2}=\sqrt{4r^2-36}$$

Given intercept = 8

$$\sqrt{4r^2-36}=8 \Rightarrow 4r^2-36=64$$

$$\Rightarrow 4r^2=100 \Rightarrow r^2=25 \Rightarrow r=5$$

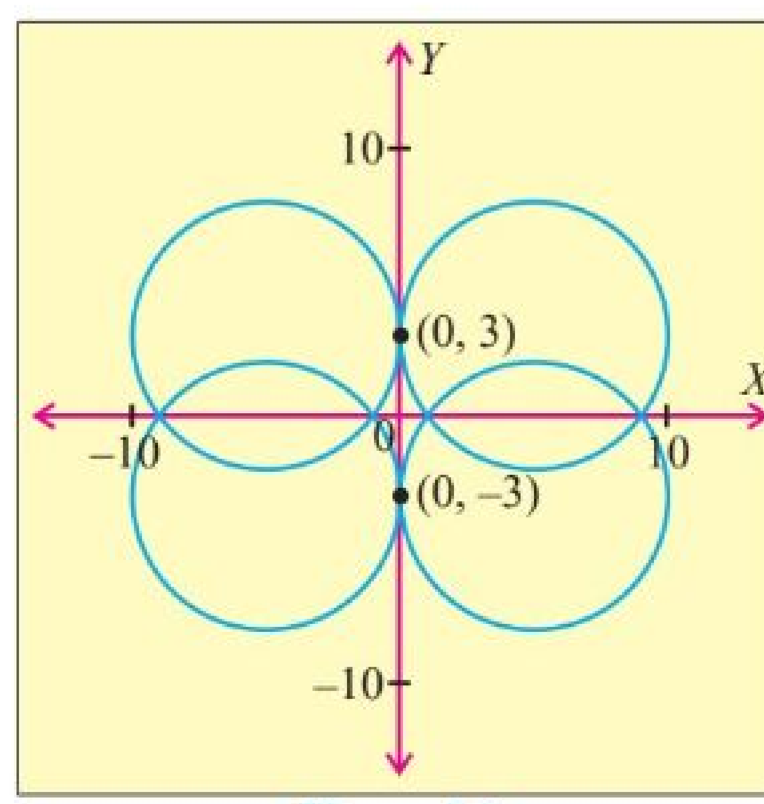


Figure 6.6

Thus, one circle with centre (5, 3) and radius 5 is:

$$(x-5)^2+(y-3)^2=25$$

By symmetry (tangency at (0, –3) and centre on left side of y-axis), we obtain four circles:

$$(x\pm 5)^2+(y\pm 3)^2=25 \text{ where } \pm \text{ signs are independent.}$$

EXERCISE 6.1

1. In each of the following, find an equation of the circle with
   (a) centre at (5, 7) and radius 2
   (b) centre at $(a\cos\theta, a\sin\theta)$ and radius a
   (c) ends of diameter (3, 4) and (2, –7).
2. Express each of the following, in the standard form and hence find its centre and radius:
   (a) $2x^2 + 2y^2 - 4x - 12y + 11 = 0$ (b) $x^2 + y^2 - 2ax + 2by - 2ab = 0$
3 Find an equation of circle passing through the points (2, 8), (5, –1) and (–3, 3), also find the centre and radius.
4. Find the equation of a circle which passes through the origin and cuts off intercepts 2 and 3 on the x-axis and y-axis respectively.
5. Find the equation of a circle passing through the points (1, 3), (2, –1) and having its centre on the line $x + y = 3$. Does the point (4, 1) lie on the circle?
6. Find the equation of the circle passing through the point (–3, – 4) and touching the line $3x - y - 1 = 0$ at the point (1, 2).
7. Find an equation of a circle of radius a lying in the second quadrant and tangent to both axes.
8. Find the equation of the circle which touches the x-axis at the point (5, 0) and passes through another point (3, 2).
9. Find the equation of the circle passing through the point of intersection of the lines $x - y - 5 = 0$, $2x + y - 4 = 0$ and it has its centre at (5, 0).
10. Find the equation of the circle touching the y-axis and also the straight line $3x - 4y = 0$ at the point (8, 6).
11. Find the equations of the circles passing through the points $A(3, 2)$ and $B(3, -2)$ and touching the line $x + 2y + 3 = 0$.
12. Find the equations of the circles of radius 5 that are tangent to the line $4x + 3y - 10 = 0$ at the point (1, 2).

6.2 Tangent and Normal to a Curve

If a line touches a curve at exactly one point, it is called a **tangent** to the curve, and the point is called the point of contact. In Figure 6.7, $\overleftrightarrow{LM}$ is a tangent to the curve at the point P.

The line perpendicular to the tangent at its point of contact is called the normal to the curve at that point. For example, in the Figure 6.7, $\overleftrightarrow{PQ} \perp \overleftrightarrow{LM}$ at the point P; thus, $\overleftrightarrow{PQ}$ is the normal to the curve at P.

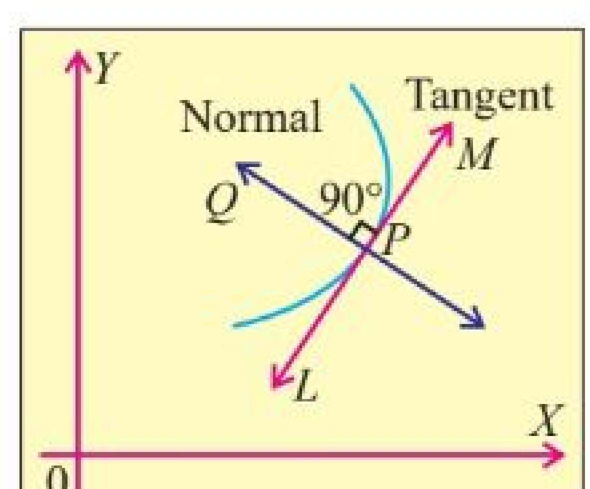


Figure 6.7

6.2.1 Equation of Tangent to a Circle

(a) Circle centered at origin: $x^2+y^2=r^2$

Let $P(x_1, y_1)$ be a point on the circle $x^2+y^2=r^2$ ...(i)

Differentiating both sides of the equation with respect of x, we have:

$$2x+2y\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{x}{y}$$

The slope of the tangent at $P(x_1, y_1)$ is the value of $\frac{dy}{dx}$ at this point:

$$m_T=\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}=-\frac{x_1}{y_1}$$

Therefore, the equation of the tangent at $P(x_1, y_1)$ is: $y-y_1=-\frac{x_1}{y_1}(x-x_1)$, $y_1\neq 0$

$$xx_1+yy_1=x_1^2+y_1^2 \qquad \text{...(ii)}$$

Since $P(x_1, y_1)$ lies on the circle $x^2+y^2=r^2$, we have

$$x_1^2+y_1^2=r^2 \qquad \text{...(iii)}$$

From equations (ii) and (iii), we obtain the required equati on of the tangent:

$$xx_1+yy_1=r^2$$

(b) General circle: $x^2+y^2+2gx+2fy+c=0$

Let $P(x_1, y_1)$ be a point on the circle

Differentiating both sides with respect to x:

$$2x+2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}+2f\frac{dy}{dx}=-2x-2g$$

$$2(y+f)\frac{dy}{dx}=-2(x+g) \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{x+g}{y+f}$$

The slope of the tangent at $P(x_1, y_1)$ is:

$$m_T=\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}=-\frac{x_1+g}{y_1+f}$$

Hence, the equation of the tangent is:

$$y-y_1=-\frac{x_1+g}{y_1+f}(x-x_1)$$

Multiplying both sides by (y_1+f):

$$(y-y_1)(y_1+f)=-(x_1+g)(x-x_1)$$

$$yy_1+fy-y_1^2-fy_1=-xx_1+x_1^2-gx+gx_1$$

$$xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1 \quad \text{(Rearranging)}$$

Adding both sides gx_1+fy_1:

$$xx_1+yy_1+gx+gx_1+fy+fy_1=x_1^2+y_1^2+2gx_1+2fy_1 \qquad \ldots(3)$$

Since $P(x_1,y_1)$ lies on the circle, we have:

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0$$

$$\Rightarrow \quad x_1^2+y_1^2+2gx_1+2fy_1=-c \qquad \ldots(4)$$

Substituting equation (4) into equation (3) and rearranging gives the required equation:

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

6.2.2 Rule for Writing the Equation of a Tangent to a Circle

We have proved above that the equation of the tangent to the circle

$x^2+y^2+2gx+2fy+c=0$ at the point (x_1,y_1) is:

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

This can be obtained from the equation of the circle by making the following substitutions:

(i) $x^2\to xx_1$ (ii) $y^2\to yy_1$

(iii) $x\to\dfrac{x+x_1}{2}$ (iv) $y\to\dfrac{y+y_1}{2}$

(v) The constant term c remains unchanged.

Note
The rule holds for all second-degree curves (conics) as well.

6.2.3 Equation of Normal to a Circle at P(x_1, y_1)

The tangent to the circle $x^2+y^2=r^2$ at $P(x_1,y_1)$ is $xx_1+yy_1=r^2$. The normal is

$$xy_1-yx_1=0 \quad \text{or} \quad \frac{x}{x_1}=\frac{y}{y_1}.$$

Example 10 Find the equation of the tangent and normal to the circle $x^2+y^2-4x+6y+12=0$ at the point on the circle whose ordinate is -2.

Solution: The equation of the circle is:

$$x^2+y^2-4x+6y+12=0$$

Here, $y=-2$, Substituting into the given equation of circle.

$$x^2+(-2)^2-4x+6(-2)+12=0 \;\Rightarrow\; x^2+4-4x-12+12=0$$

$$\Rightarrow\; x^2-4x+4=0 \;\Rightarrow\; (x-2)^2=0 \;\Rightarrow\; x=\pm2$$

Note
The centre of the circle, (0, 0), satisfies the equation of the normal, confirming that the normal to a circle always passes through its centre.

Thus, point of contact is $P(2, -2)$. The equation of the tangent at (x_1, y_1) is:

$$xx_1 + yy_1 - 2(x + x_1) + 3(y + y_1) + 12 = 0$$

Substituting $x_1 = 2$ and $y_1 = -2 \Rightarrow 2x + (-2)y - 2(x + 2) + 3(y - 2) + 12 = 0$

$$2x - 2y - 2x - 4 + 3y - 6 + 12 = 0 \Rightarrow y + 2 = 0$$

So, the equation of the tangent is $y = -2$ and the equation of the normal is $x = 2$.

6.2.4 Position of a Point with Respect to a Circle

Theorem 1: The point $P(x_1, y_1)$ lies outside, on, or inside the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \gtreqless 0$

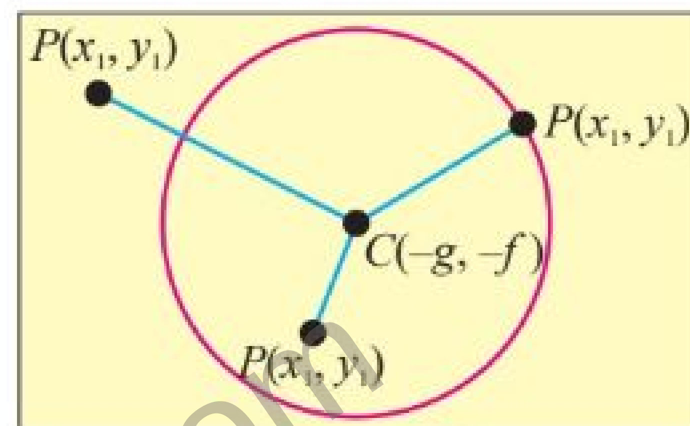


Figure 6.8

Proof: The centre of the circle is $C(-g, -f)$ and radius is $r = \sqrt{g^2 + f^2 - c}$. The point $P(x_1, y_1)$ lies outside, on, or inside the circle according as

$$|CP| > r,\ |CP| = r, \text{ or}$$

$CP < r$, that is $|CP| \gtreqless r$

that is, according as:

$$\sqrt{(x_1 + g)^2 + (y_1 + f)^2} \gtreqless \sqrt{g^2 + f^2 - c}$$

Squaring both sides (since all quantities are non-negative):

$$(x_1 + g)^2 + (y_1 + f)^2 \gtreqless g^2 + f^2 - c$$

$$x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 \gtreqless g^2 + f^2 - c$$

Cancelling g^2 and f^2:

$$x_1^2 + 2gx_1 + y_1^2 + 2fy_1 \gtreqless -c$$

Thus: $$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \gtreqless 0$$

Example 11 Determine whether the point $P(2, 3)$ lies outside, on, or inside the circle:

$$x^2 + y^2 + 2x - 3y - 10 = 0$$

Solution: Substituting $x = 2$ and $y = 3$ into the left-hand side of the circle's equation:

$$(2)^2 + (3)^2 + 2(2) - 3(3) - 10 = 4 + 9 + 4 - 9 - 10 = -2 < 0$$

Thus, the point $P(2, 3)$ lies inside the circle.

6.2.5 Intersection of a Line and a Circle

Let the equation of the line be

$$y = mx + c \qquad \ldots(1)$$

and the equation of the circle be

$$x^2 + y^2 = a^2 \qquad \ldots(2)$$

The coordinates of the points of intersection are the simultaneous solutions of (1) and (2). Substituting the value of y from (1) into (2):

$$x^2 + (mx + c)^2 = a^2$$
$$x^2 + m^2x^2 + 2mxc + c^2 - a^2 = 0$$
$$x^2(1 + m^2) + 2mcx + (c^2 - a^2) = 0 \qquad \ldots(3)$$

Equation (3) is a quadratic in x and gives two values of x say x_1 and x_2. Corresponding to these values of x_1 and x_2, we get values of y as y_1 and y_2. Thus, the line $y = mx + c$ intersects the circle $x^2 + y^2 = a^2$ in at the most two points (x_1, y_1) and (x_2, y_2).

For the nature of the points of the intersection, we examine the discriminant, D, of (3).

$$D = (2mc)^2 - 4(1 + m^2)(c^2 - a^2) = 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2)$$
$$= 4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4m^2a^2 = -4c^2 + 4a^2 + 4m^2a^2$$
$$= 4a^2(1 + m^2) - 4c^2 = = 4[a^2(1 + m^2) - c^2]$$

Case-I: Two distinct real points

The two points of intersection of the line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ are real and distinct, if the discriminant > 0

that is, if $4[a^2(1 + m^2) - c^2] > 0 \quad \Rightarrow \quad a^2(1 + m^2) - c^2 > 0 \quad \Rightarrow \quad a^2(1 + m^2) > c^2$

Case-II: One real point (tangent)

The line $y = mx + c$ touch the circle $x^2 + y^2 = a^2$ at only one real point, that is, case of coincident points if the discriminant = 0

that is, if $4[a^2(1 + m^2) - c^2] = 0 \quad \Rightarrow \quad a^2(1 + m^2) - c^2 = 0 \quad \Rightarrow \quad a^2(1 + m^2) = c^2$

Case-III: No real intersection

The line $y = mx + c$ cuts the circle $x^2 + y^2 = a^2$ at two imaginary points (that is, does not cut the circle) if the discriminant < 0

that is, if $4\left[a^2(1+m^2) - c^2\right] < 0 \quad \Rightarrow \quad a^2(1+m^2) - c^2 < 0 \quad \Rightarrow \quad a^2(1+m^2) < c^2$

Corollary-1: Condition of Tangency of a Line to a Circle

From Case II, the line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ if:

$$|c| = a\sqrt{1+m^2} \quad \text{or} \quad c = \pm a\sqrt{1+m^2}$$

Therefore, the family of tangents to the circle is: $\boxed{y = mx \pm a\sqrt{1+m^2}}$

Note that $\dfrac{|c|}{\sqrt{1+m^2}}$ is the perpendicular distance from the centre (0, 0) to the line.

Hence, the condition for tangency is that the distance $\dfrac{|c|}{\sqrt{1+m^2}}$ equals the radius a.

Corollary 2: Coordinates of Point of Contact

For the tangent $y = mx + a\sqrt{1+m^2}$, the quadratic equation (3) has equal roots given by:

$$x = \frac{-2mc}{2(1+m^2)} = \frac{-mc}{1+m^2}$$

$$= \frac{-m}{1+m^2}\left(a\sqrt{1+m^2}\right) = \frac{-am}{\sqrt{1+m^2}}$$

Substituting into the equation $y = mx + a\sqrt{1+m^2}$

$$y = m\left(\frac{-am}{\sqrt{1+m^2}}\right) + a\sqrt{1+m^2} = \frac{-am^2}{\sqrt{1+m^2}} + a\sqrt{1+m^2} = \frac{-am^2 + a(1+m^2)}{\sqrt{1+m^2}} = \frac{a}{\sqrt{1+m^2}}$$

Thus, the point of contact is $\left(\frac{-am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$

Similarly, for the tangent $y = mx - a\sqrt{1+m^2}$, the point of contact $\left(\frac{am}{\sqrt{1+m^2}}, \frac{-a}{\sqrt{1+m^2}}\right)$.

6.2.6 Tangents from a Point to a Circle

Theorem 2: Two tangents can be drawn to a circle from any point $P(x_1, y_1)$. The tangents are **real and distinct**, **coincident**, or **imaginary** according as the point lies **outside**, **on**, or **inside** the circle.

Proof: Consider the circle $x^2 + y^2 = a^2$. The line $y = mx + a\sqrt{1+m^2}$ is a tangent for all m. If it passes through $P(x_1, y_1)$, then:

$$y_1 = mx_1 + a\sqrt{1+m^2} \quad \text{or} \quad y_1 - mx_1 = a\sqrt{1+m^2}$$

Squaring both sides:

$$(y_1 - mx_1)^2 = a^2(1+m^2) \Rightarrow y_1^2 - 2mx_1 y_1 + m^2 x_1^2 = a^2 + a^2m^2$$

or $\quad (x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 - a^2) = 0$

This is a quadratic in m. Its discriminant, D, is:

$$D = (-2x_1y_1)^2 - 4(x_1^2 - a^2)(y_1^2 - a^2)$$

$$= 4x_1^2y_1^2 - 4(x_1^2y_1^2 - a^2x_1^2 - a^2y_1^2 + a^4) = 4a^2(x_1^2 + y_1^2 - a^2)$$

The two values of m are:

- **Real and distinct** if $D > 0 \Rightarrow x_1^2 + y_1^2 > a^2$ (point lies outside the circle)
- **Real and coincident** if $D = 0 \Rightarrow x_1^2 + y_1^2 = a^2$ (point lies on the circle)
- **Imaginary** if $D < 0 \Rightarrow x_1^2 + y_1^2 < a^2$ (point lies inside the circle).

Example 12 Tangents are drawn from (4, 5) to the circle $x^2+y^2=29$. Find an equation of the line joining the points of contact (The line is called the chord of contact).

Solution: Let the points of contact of the two tangents be $P(x_1, y_1)$ and $Q(x_2, y_2)$.
The equation of the tangent at P is $xx_1+yy_1=29$ and at Q is $xx_2+yy_2=29$. Since both tangents pass through (4, 5): $4x_1+5y_1=29$ and $4x_2+5y_2=29$.
Thus, both $P(x_1, y_1)$ and $Q(x_2, y_2)$ satisfy the equation $4x+5y=29$. Hence, the equation of the chord of contact is: $4x+5y=29$.

Example 13 Find a combined equation of the pair of tangents drawn from (5, 0) to the circle $x^2+y^2=9$.

Solution: Let $P(h,k)$ be any point on either of the two tangents drawn from $A(5, 0)$ to the given circle. The equation of line AP is:

$$y-0=\frac{k-0}{h-5}(x-5) \Rightarrow y=\frac{k}{h-5}(x-5) \Rightarrow y(h-5)=k(x-5)$$

$$\Rightarrow kx-(h-5)y-5k=0$$

This line is tangent to the circle $x^2+y^2=9$. The perpendicular distance from the centre (0, 0) to the line must equal the radius 3: $\frac{|5k|}{\sqrt{k^2+(h-5)^2}}=3$

Squaring both sides: $25k^2=9[k^2+(h-5)^2] \Rightarrow 25k^2=9k^2+9(h-5)^2$

$$9(h-5)^2-16k^2=0$$

Since (h, k) is any point on the two tangents, the combined equation of the pair of tangents is:

$$9(x-5)^2-16y^2=0$$

Example 14 Find the equations of the pair of tangents drawn from $(4, -1)$ to the circle $x^2+y^2-2x+4y=0$. Also find the points of contact.

Solution: Given: $x^2+y^2-2x+4y=0 \quad \ldots(1)$

The circle has centre $(1, -2)$ and radius $\sqrt{(-1)^2+(2)^2}=\sqrt{5}$. The equation of a line through $(4, -1)$ with slope m is: $y+1=m(x-4) \Rightarrow mx-y-4m-1=0$
The perpendicular distance from the centre $(1, -2)$ to this line must equal the radius:

$$\frac{|m(1)-(-2)-4m-1|}{\sqrt{m^2+(-1)^2}}=\sqrt{5} \Rightarrow \frac{|m+2-4m-1|}{\sqrt{m^2+1}}=\sqrt{5} \Rightarrow \frac{|-3m+1|}{\sqrt{m^2+1}}=\sqrt{5}$$

Squaring both sides:

$$\Rightarrow (3m-1)^2=5(m^2+1) \Rightarrow 9m^2-6m+1=5m^2+5 \Rightarrow 4m^2-6m-4=0$$

$$\Rightarrow 2m^2-3m-2=0 \Rightarrow 2m^2-4m+m-2=0 \Rightarrow 2m(m-2)+1(m-2)=0$$

$$\Rightarrow (m-2)(2m+1)=0 \Rightarrow m=2,\ m=-\frac{1}{2}$$

Substituting these slopes into the line equation gives the tangents:

For $m=2$: $2x-y-8-1=0 \quad \Rightarrow \quad 2x-y-9=0$ ...(2)

$$\text{For } m=-\frac{1}{2}:\ -\frac{1}{2}x-y-4\left(-\frac{1}{2}\right)-1=0$$

$$\Rightarrow -\frac{1}{2}x-y+1=0 \Rightarrow x+2y-2=0 \qquad \text{...(3)}$$

Now we proceed to find the points of contact.

From (2) $y = 2x - 9$ putting in (1), we have

$$x^2 + (2x-9)^2 - 2x + 4(2x-9) = 0$$
$$x^2 + 4x^2 - 36x + 81 - 2x + 8x - 36 = 0$$
$$5x^2 - 30x + 45 = 0$$
$$x^2 - 6x + 9 = 0 \quad \Rightarrow \quad (x-3)^2 = 0 \quad \Rightarrow \quad x-3=0 \Rightarrow \quad x=3$$

Putting $x = 3$ in $y = 2x - 9$, we have $y = 2(3) - 9 = -3$

Therefore, one point of contact is $(3, -3)$

From (3) $y=\dfrac{2-x}{2}$ putting in (1), we have $x^2+\left(\dfrac{2-x}{2}\right)^2-2x+4\left(\dfrac{2-x}{2}\right)=0$

$$x^2+\frac{4-4x+x^2}{4}-2x+4-2x=0$$
$$x^2+\frac{x^2-4x+4}{4}-4x+4=0$$
$$4x^2 + x^2 - 4x + 4 - 16x + 16 = 0$$
$$4x^2 + x^2 - 4x + 4 - 16x + 16 = 0$$
$$5x^2 - 20x + 20 = 0$$
$$x^2 - 4x + 4 = 0$$
$$(x-2)^2 = 0 \quad \Rightarrow \quad x-2=0 \quad \Rightarrow \quad x=2$$

Putting $x=2$ in $y=\dfrac{2-x}{2}$, we have $y=\dfrac{2-2}{2}=0$

Therefore, other point of contact is $(2,0)$.

6.2.7 Length of the Tangent from a Point to a Circle

Theorem 3: The length of the tangent drawn from a point $P(x_1, y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{x_1^{\,2}+y_1^{\,2}+2gx_1+2fy_1+c}$.

Proof: Let the centre of the circle be $C(-g, -f)$ and its radius be $r=\sqrt{g^2+f^2-c}$. Let PT be a tangent from P to the circle, touching it at T. Then, $CT \perp PT$.
In right triangle PTC:

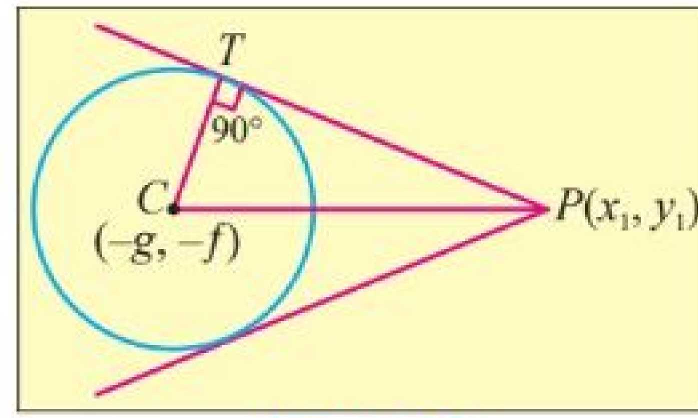


Figure 6.9

$$|PT|^2 = |PC|^2 - |CT|^2$$

Where, $$|PC|^2 = (x_1+g)^2 + (y_1+f)^2$$

$$|CT|^2 = r^2 = g^2 + f^2 - c$$

Therefore, $$|PT|^2 = (x_1+g)^2 + (y_1+f)^2 - (g^2+f^2-c)$$
$$= x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - g^2 - f^2 + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

Hence, $$|PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}.$$

6.2.8 Rule to Find the Length of the Tangent

(i) Write the equation of the circle with coefficients of x^2 and y^2 as 1.
(ii) Make the right-hand side of the equation zero.
(iii) Substitute the coordinates of the point into the left-hand side expression.
(iv) Take the square root of the result.

Example 15 Find the length of the tangent from point $P(2,-5)$ to the circle $5x^2+5y^2+10x+14y-15=0$.

Solution: Divide the equation by 5 to get the standard form:

$$x^2+y^2+2x+\frac{14}{5}y-3=0$$

The length of the tangent is:

$$\sqrt{(2)^2+(-5)^2+2(2)+\frac{14}{5}(-5)-3} = \sqrt{4+25+4-14-3} = \sqrt{16} = 4$$

Example 16 Length of a Chord of a Circle

Find the length of intercept cut off from the line $y=mx+c$ by the circle $x^2+y^2=a^2$.

Solution: The circle is $x^2+y^2=a^2$ and the line is $y=mx+c$.

Let the intersection points be $P(x_1, y_1)$ and $Q(x_2, y_2)$. Substituting the line into the circle:

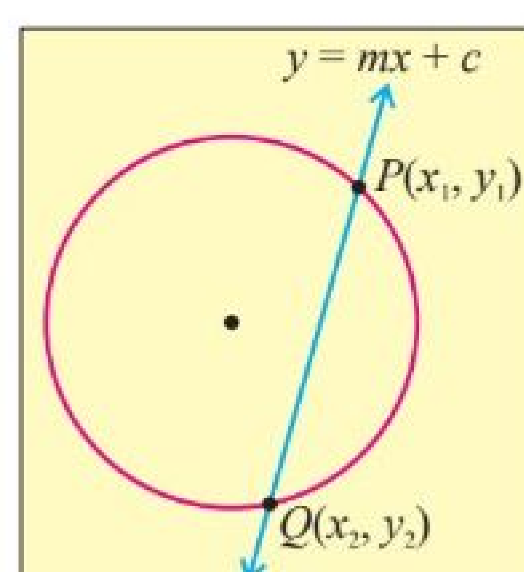


Figure 6.10

$$x^2+(mx+c)^2=a^2$$

$$x^2+m^2x^2+2mcx+c^2=a^2$$

$$(1+m^2)x^2+2mcx+(c^2-a^2)=0$$

The roots x_1 and x_2 satisfy:

$$x_1+x_2=-\frac{2mc}{1+m^2},\ x_1x_2=\frac{c^2-a^2}{1+m^2}$$

Since the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the line,

$$y_1=mx_1+c \text{ and } y_2=mx_2+c, \text{ so } y_1-y_2=m(x_1-x_2)$$

The length of the chord PQ is:

$$PQ=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\sqrt{(x_1-x_2)^2+m^2(x_1-x_2)^2}$$

$$=\sqrt{(1+m^2)(x_1-x_2)^2}=\sqrt{1+m^2}\ \sqrt{(x_1+x_2)^2-4x_1x_2}$$

$\because$ Substituting x_1-x_2 and x_1x_2

$$=\sqrt{1+m^2}\ \sqrt{\frac{4m^2c^2}{(1+m^2)^2}-\frac{4(c^2-a^2)}{1+m^2}}$$

$$=\sqrt{1+m^2}\ \sqrt{\frac{4m^2c^2-4(1+m^2)(c^2-a^2)}{(1+m^2)^2}}$$

$$=2\sqrt{1+m^2}\ \sqrt{\frac{m^2c^2-(c^2-a^2+m^2c^2-m^2a^2)}{(1+m^2)^2}}$$

$$=2\sqrt{1+m^2}\ \frac{\sqrt{m2c2-c2+a2-m2c2+m2a2}}{1+m2}$$

Hence, the length of the chord is: $PQ=\dfrac{2}{\sqrt{1+m2}}\sqrt{a2(1+m2)c2}$

> **Note**
> In numerical problems, it is often more convenient to find the points of intersection directly and then compute the distance between them.

EXERCISE 6.2

1. Find the equations of the tangent and normal to the circles
   (a) $x^2+y^2=8$ at $(2, 2)$ and at $(2\sqrt{2}\cos\theta,\ 2\sqrt{2}\sin\theta)$.
   (b) $x^2+y^2-6x-2y-7=0$ at $(4, 5)$
2. Show that the line $4x+3y-25=0$ is tangent to the circle $x^2+y^2=25$ and find the point of contact.
3. Find the equations of tangents to the circle $x^2+y^2=4$,
   (a) which are inclined at an angle of $45°$ to the axis of x.
   (b) which are parallel to the line $3x+4y+1=0$.
   (c) which are perpendicular to the line $6x-8y+3=0$.

4. Find the equations of the tangents to the circle $x^2+y^2-4x-2y-4=0$, which are parallel to the line $3x-4y+2=0$.
5. Find the equations of the tangents drawn from:
   (a) the point (–3, 7) to the circle $x^2+y^2=8$.
   (b) the point (12, –4) to the circle $x^2+y^2-4x-2y-20=0$.
   Also find the points of contact.
6. A circle has the parametric equations: $x=3+5\cos\theta, y=-1+5\sin\theta$
   (a) Find the Cartesian equation of the circle.
   (b) Find the centre and radius of the circle.
   (c) Find the parametric coordinates of the point P on the circle corresponding to $\theta=\frac{\pi}{2}$.
   (d) Find the equation of the tangent to the circle at point P.
   (e) Find the equation of the normal to the circle at point P.
   (f) Find the equation of the tangent to the circle at the point where $\theta=\pi$.

6.3 Parabola

A **Parabola** is the locus of a point which moves in a plane such that its distance from a fixed point, called the **focus**, is equal to its distance from a given fixed straight line, called the **directrix**.

6.3.1 Standard Equation of the Parabola

Let F be the focus and CD, the directrix of the parabola. Draw FZ perpendicular from the focus F to the directrix CD. Let V be the mid-point of FZ and let $FZ = 2a$. Take V as the origin, the x-axis along the line ZF and y-axis along the perpendicular to VF at V, as shown in the figure.

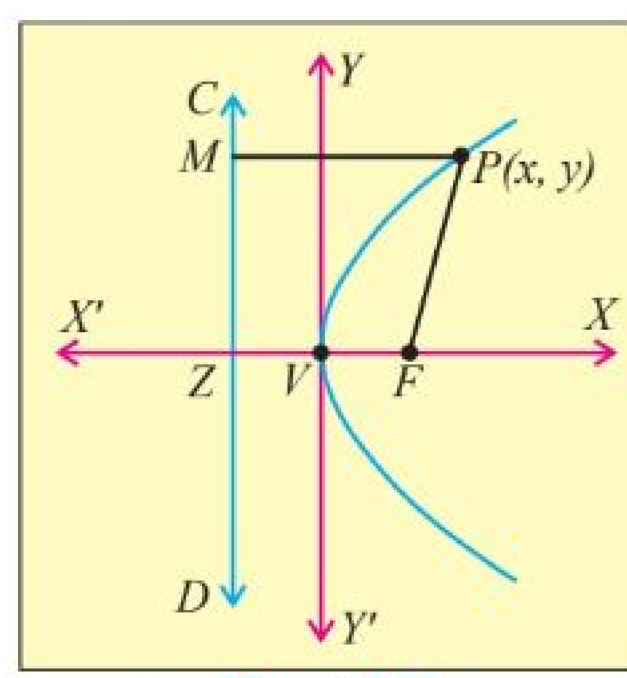


Figure 6.11

Since V is equidistant from the focus and the directrix, by definition, the point V lies on the parabola. The point V is called the vertex of the parabola.

Thus, the focus of the parabola is $F(a, 0)$ and the directrix is $x = -a$.

Let $P(x, y)$ be any point on the parabola and draw PM perpendicular to CD.

Therefore, $|PF|=\sqrt{(x-a)^2+(y-0)^2}=\sqrt{x^2-2ax+a^2+y^2}$

Distance of P from the directrix $x=-a$ is $PM=\frac{|x+a|}{\sqrt{1^2+0^2}}=|x+a|$

By the definition of a parabola, $|PF| = |PM|$

Therefore, $\sqrt{x^2-2ax+a^2+y^2}=|x+a|$

Squaring both sides, we get

$$x^2-2ax+a^2+y^2=(x+a)^2$$

$$x^2-2ax+a^2+y^2=x^2+2ax+a^2 \text{ which simplifies to } y^2=4ax$$

This is the standard equation of the parabola.

6.3.2 Elements of the Parabola

1. **Axis:** The straight line through the focus and perpendicular to the directrix is called axis of the parabola. In the figure, the axis is x-axis.
2. **Vertex:** The point where the axis meets the parabola is called the vertex. In the figure, V is the vertex with coordinates (0, 0).
3. **Chord and Focal Chord:** A line segment joining two distinct points on a parabola is called a chord. A chord passing through the focus is called a focal chord.
4. **Focal Distance:** The distance of a point on the parabola from the focus is called its focal distance. In the figure, $|FP|$ is the focal distance.
5. **Latus Rectum:** The focal chord perpendicular to the axis of the parabola is called the latus rectum. Its equation is $x = a$. It intersects the parabola $y^2 = 4ax$ at points where $y^2 = 4a^2$ that is, $y = \pm 2a$. Thus, the end points L and L' of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$.

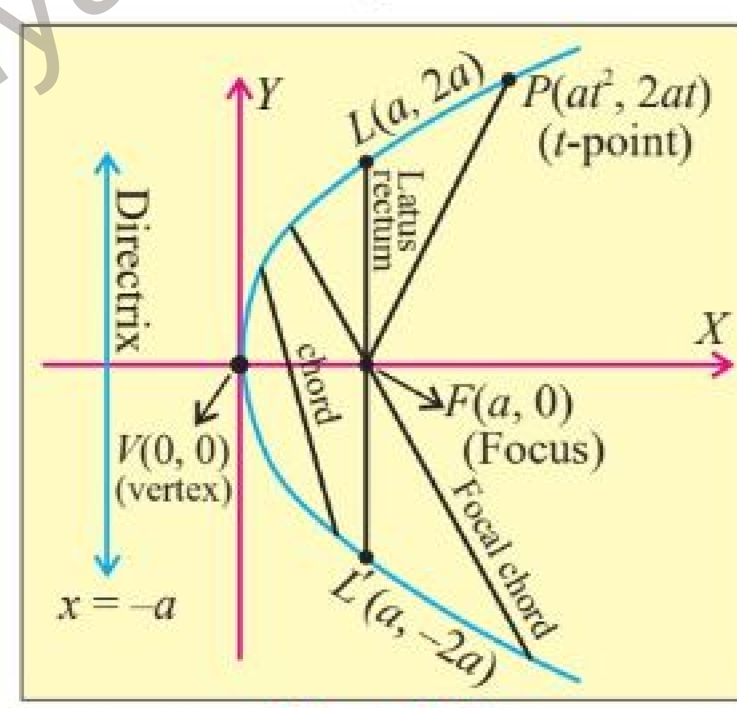


Figure 6.12

6. **Parametric Equations:** The equation $y^2 = 4ax$ can be expressed parametrically. Let $\dfrac{y}{2a}=t$ and $\dfrac{2x}{y}=t$.

   Then $y = 2at$ and $2x = yt = (2at)t = 2at^2$,

   So, $x=at^2$

   Hence, the parametric equations are

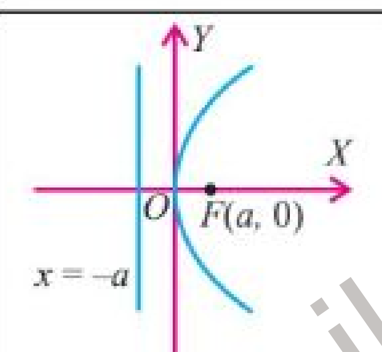
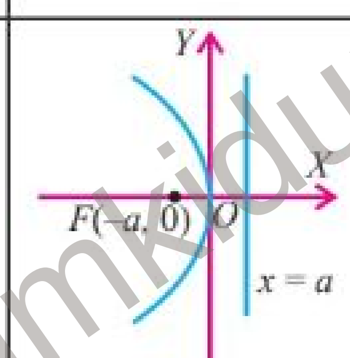
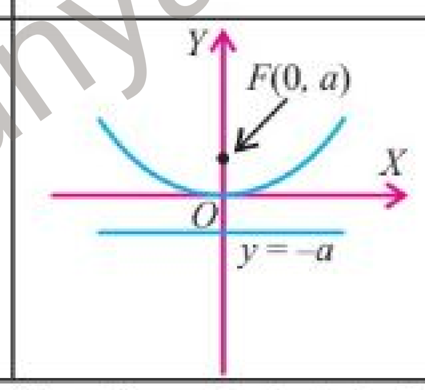
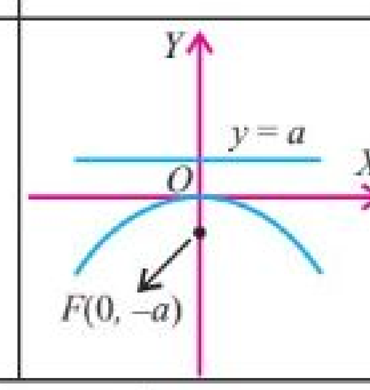
$$x=at^2,\ y=2at$$

   The point $(at^2, 2at)$ lies on the parabola for all real t and is often referred to as the t-point.

6.3.3 Other Standard Forms of the Parabola

By taking the directrix on the right side of the focus, or below or above the focus, we obtain three more standard forms.

The table below summarizes the key features of all four forms.

Parabola in standard form	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Vertex	$(0,\ 0)$	$(0,\ 0)$	$(0,\ 0)$	$(0,\ 0)$
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Equation of latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Sketch of parabola in standard form				

Example 17 Analyze the parabola $x^2 - 4x - 4y - 2 = 0$ and sketch its graph.

Solution: $x^2 - 4x - 4y - 2 = 0$ $\quad\ldots(1)$

Rewrite the equation as: $x^2 - 4x = 4y + 2$

Adding 4 on both sides, we have

$$x^2 - 4x + 4 = 4y + 6$$

$$(x-2)^2 = 4\left(y + \frac{3}{2}\right) \quad\ldots(2)$$

Let $x - 2 = X$ and $y + \frac{3}{2} = Y$. Then the equation (2) becomes $X^2 = 4Y$ $\quad\ldots(3)$

which is a parabola whose focus lies on $X = 0$ and whose length of latus rectum is equal to $4a = 4$, thus coordinates of the focus of (3) are:

$$X = 0, Y = 1, \text{ that is, } x - 2 = 0 \text{ and } y + \frac{3}{2} = 1 \quad \text{or} \quad x = 2,\ y = -\frac{1}{2}$$

Thus, coordinates of the focus of the parabola (1) are $\left(2, -\frac{1}{2}\right)$

Axis of (3) is $X = 0 \Rightarrow x - 2 = 0$ $x = 2$ is the axis of (1)

Vertex of (3) has coordinates

$$X = 0, Y = 0$$

that is, $x - 2 = 0,\ y + \frac{3}{2} = 0$

or $x = 2,\ y = -\frac{3}{2}$ are coordinates of the vertex of (1).

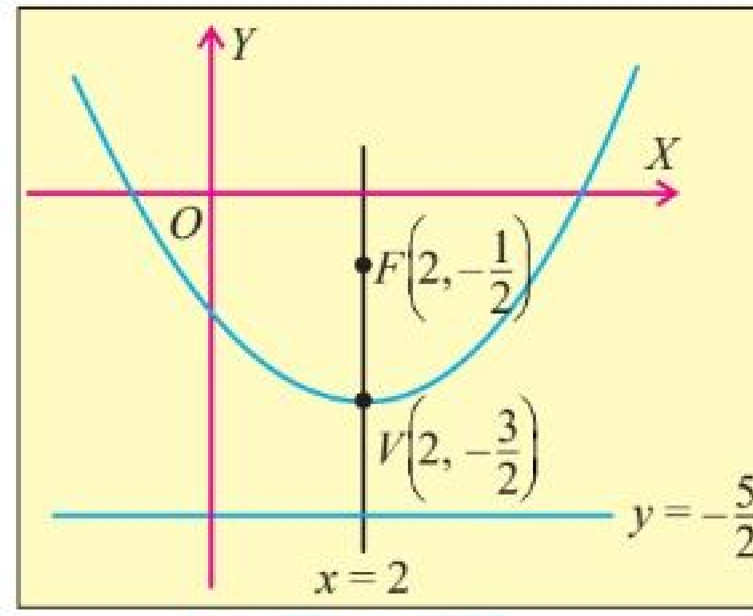


Figure 6.13

Equation of the directrix of (3) is:

$$Y = -1 \text{ that is, } y + \frac{3}{2} = -1 \text{ or } y = -\frac{5}{2} \text{ is equation of the directrix of (1).}$$

Length of the latus rectum of the parabola is 4.

The graph of (1) can easily be sketched and is as shown in the figure.

6.3.4 Tangents and Normals to a Parabola (Condition of the tangency to Parabola)

To find an equation of the tangent to a parabola, we first determine the slope of the tangent at the given point by calculating $\frac{dy}{dx}$ from the equation of the parabola. Using the point-slope form of a line, the equation of the tangent can then be written.

Since the normal to a curve at a point is perpendicular to the tangent at that point, its equation can be obtained readily from the equation of the tangent.

Let us consider the equation of a line

$$y = mx + c \qquad \ldots(1)$$

and the parabola

$$y^2 = 4ax \qquad \ldots(2)$$

To find their points of intersection, we solve equations (1) and (2) simultaneously. Substituting the value of y from (1) into (2), we obtain:

$$(mx + c)^2 = 4ax$$

$$m^2x^2 + 2mcx + c^2 = 4ax$$

$$m^2x^2 + 2mcx - 4ax + c^2 = 4ax$$

$$m^2x^2 + 2(mc - 2a)x + c^2 = 0 \qquad \ldots(3)$$

Line (1) is tangent to (3) if roofs of (3) are equal. This implies discriminant D, of (3) is zero.

$[2(mc-2a)]^2-4(m^2)(c^2)=0$

$4(mc-2a)^2-4m^2c^2=0$

Dividing both sides by 4, we get:

$(mc-2a)^2-m^2c^2=0$

$m^2c^2-4amc+4a^2-m^2c^2=0$

$-4amc+4a^2=0 \Rightarrow 4amc=4a^2 \Rightarrow mc=a \Rightarrow c=\frac{a}{m}, m\neq 0$

Thus, the required condition for the line $y=mx+c$ to be a tangent to the parabola $y^2=4ax$ is $c=\frac{a}{m}$.

Consequently, for any non-zero real number m, the line $y=mx+\frac{a}{m}$ is a tangent to the parabola $y^2=4ax$. Similarly, we can find condition of tangency for other parabolas

6.3.5 Similarly, we find condition of Tangency for other parabolas

Find the equations of the tangent and normal to the parabola $y^2=4ax$ at the point (x_1, y_1).

Solution: Differentiating $y^2=4ax$ with respect to x, we obtain

$$2y\frac{dy}{dx}=4a \quad \Rightarrow \quad \frac{dy}{dx}=\frac{2a}{y}$$

Thus, the slope of the tangent at (x_1, y_1) is $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}=\frac{2a}{y_1}$

Using the point-slope form, the equation of the tangent at (x_1, y_1) is

$$y-y_1=\frac{2a}{y_1}(x-x_1).$$

Multiplying both sides by y_1, we get

$$yy_1-y_1^2=2ax-2ax_1.$$

Rearranging terms,

$$yy_1-2ax=y_1^2-2ax_1$$

Since the point (x_1, y_1) lies on the parabola, it satisfies $y_1^2=4ax_1$. Substituting this value, we have

$$yy_1-2ax=4ax_1-2ax_1=2ax_1.$$

Bringing all terms to one side,

$$yy_1-2ax-2ax_1=0.$$

Which simplifies to $yy_1 = 2a(x + x_1)$.

This is the required equation of the tangent.

The slope of the normal is the negative reciprocal of the slope of the tangent. Hence,

$$\text{Slope of the normal} = -\frac{y_1}{2a}.$$

Using the point-slope form, the equation of the normal is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

EXERCISE 6.3

1. Find the focus, vertex and directrix of the following parabolas also sketch the graph:
   (i) $y^2 = 12x$ (ii) $x^2 = 8(y - 1)$
   (iii) $y^2 = -8(x - 3)$ (v) $x^2 - 4x - 8y + 4 = 0$
2. Write the equation of the parabola with given elements.
   (i) Focus (–3, 2); directrix $x = 5$ (ii) Focus (3, 5), vertex (3, 2)
   (iii) Directrix $x = 4$ vertex (6, 3) (iv) Axis $y = 0$ through (2, 2) and (6, –3)
3. Find the equation of the parabola having its focus at the origin and directrix is parallel to: (i) x-axis (ii) y-axis.
4. Show that the an equation of parabola with focus at ($a\cos\theta$, $a\sin\theta$) and directrix $x\cos\theta + y\sin\theta + a = 0$ is $(x\sin\theta - y\cos\theta)^2 = 4a(x\cos\theta + y\sin\theta)^2$.
5. Prove that in parabola a circle described on latus rectum as diameter touches the directrix.
6. A parabola is defined by the equation $y = x^2 - 4x + 3$. Find the coordinates of its vertex, focus, and the equation of its directrix.
7. The parabola $y = ax^2 + bx + c$ has a vertex at (3, –2) and passes through the point (1, 6). Find the values of a, b, and c
8. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.
9. Find the equations of the tangent and normal to the parabola
   $$x^2 - 6x - 2y + 11 = 0 \text{ at the point } (5, 3).$$
10. Find the equation of the tangent to the parabola $y^2 = 12x$ which is parallel to the line $3x - y + 4 = 0$. Also find the point of tangency.

6.4 Ellipse

An **ellipse** is the set of all points in a plane for which the sum of distances from two fixed points, called the **foci** (plural of focus), is constant (see Figure 14). Kepler's first law states that the orbits of the planets in the solar system are elliptical.

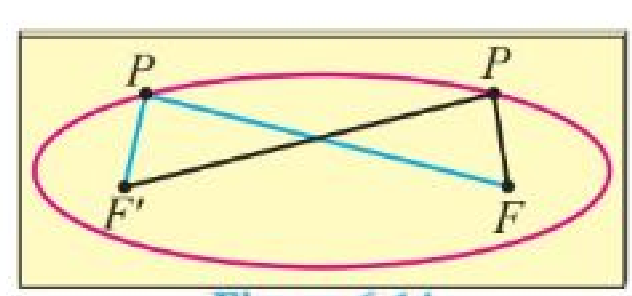


Figure 6.14

Derivation of the Standard Equation

To obtain the simplest equation for an ellipse, we place the foci on the x-axis at the points $F'(-c, 0)$ and $F(c, 0)$, so that the origin is the midpoint of the segment joining them (see Figure 17). Let the constant sum of distances from any point $P(x, y)$ on the ellipse to the foci be $2a$, where $a > 0$.

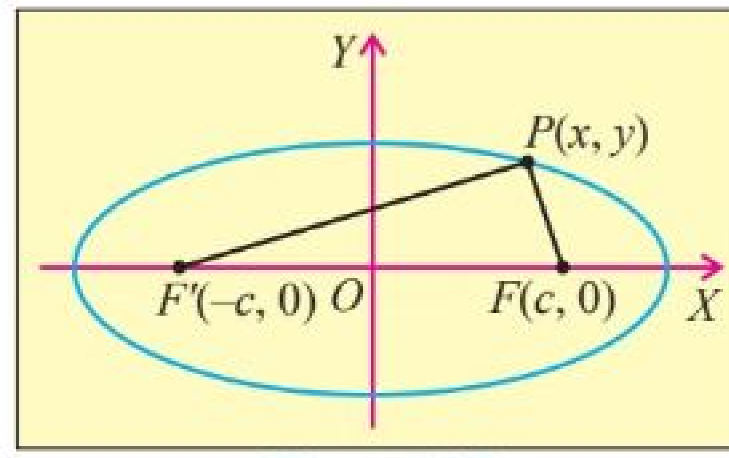


Figure 6.15

Thus, for any point $P(x, y)$ on the ellipse,

$$|PF'| + |PF| = 2a$$

Using the distance formula,

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

To simplify, isolate one radical and square both sides:

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

Squaring, $$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

Expanding the squares,

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

Cancelling the common terms x^2, c^2, and y^2 from both sides,

$$2cx = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} - 2cx$$

Bring the terms involving cx together:

$$4cx = 4a^2 - 4a\sqrt{(x-c)^2 + y^2}$$

Divide by 4 and isolate the remaining radical:

$$a\sqrt{(x-c)^2 + y^2} = a^2 - cx$$

Now we square both sides again:

$$a^2\left[(x-c)^2 + y^2\right] = a^4 - 2a^2cx + c^2x^2$$

Do You Know?

The word "focus" means "fireplace" in Latin.

Expanding the left side:

$$a^2(x^2 - 2cx + c^2 + y^2) = a^4 - 2a^2cx + c^2x^2$$

$$a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 = a^4 - 2a^2cx + c^2x^2$$

$$a^2x^2 + a^2c^2 + a^2y^2 = a^4 + c^2x^2$$

Rearranging, $$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

From triangle $PF'F$ in Figure 17, we note that $|PF'| + |PF| > |F'F|$, that is, $2a > 2c$, so $a > c$ and thus $a^2 - c^2 > 0$. For convenience, we define a new constant b such that $b^2 = a^2 - c^2$

Substituting, the equation of the ellipse becomes

$$b^2x^2 + a^2y^2 = a^2b^2$$

Finally, dividing both sides by a^2b^2 gives the standard form of the equation of an ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Since $b^2 = a^2 - c^2$ and $a > c$, it follows that $a > b$.

6.4.1 Key Features of the Ellipse

- **Foci**: The foci are at $(\pm c,\ 0)$, where $c^2 = a^2 - b^2$.
- **Centre**: The midpoint C of the segment joining the foci is called the **centre**. For the equation above, the centre is at the origin (0, 0).
- **Vertices**: Setting $y = 0$ gives $x = \pm a$. The points $A'(-a,\ 0)$ and $A(a,\ 0)$ are the **vertices**. The line segment $A'A$ of length $2a$ is the **major axis**.
- **Co-vertices**: Setting $x = 0$ gives $y = \pm b$. The points $B'(0,\ -b)$ and $B(0,\ b)$ are called the **co-vertices**. The line segment $B'B$ of length $2b$ is the **minor axis**. Since $a > b$, the major axis is longer than the minor axis.
- **Symmetry**: The equation is unchanged if x is replaced by $-x$ and y by $-y$; hence the ellipse is symmetric about the x-axis, y-axis, and the origin.
- **Eccentricity:** The **eccentricity** e of an ellipse is defined as $e = \frac{c}{a}\cdot$

  Since $a > c > 0$, we have $0 < e < 1$.
- **Directrices:** For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a > b$, there are two directrices given by the vertical lines $x = \pm\frac{a}{e}\cdot$
- **Latus Rectum:** Each of the focal chords perpendicular to the major axis is called a **latus rectum** (plural: latera recta). The length of each latus rectum is $\frac{2b^2}{a}\cdot$
- **Parametric Equations:** For all real θ, the point $(a\cos\theta,\ b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ Thus, the **parametric equations** of the ellipse are: $x = a\cos\theta, y = b\sin\theta.$
- **Foci on the y-axis:** If the foci are on the y-axis at $(0, \pm c)$, the roles of a and b are interchanged, and the standard equation becomes $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, $a > b$: with vertices $(0,\ \pm a)$, and covertices $(\pm b,\ 0)$.

- **Relation between *a*, *b* and *e*:**
  Since $b^2 = a^2 - c^2$, setting $c = ae \Rightarrow b^2 = a^2 - a^2e^2 = a^2(1-e^2)$.
- **Circle as a Special Case:** If the foci coincide, then $c = 0$ and $a = b$, so the ellipse becomes a circle with radius $r = a = b$, and has equation: $x^2 + y^2 = a^2$. Clearly for circle $e = 0$.
- **Second definition of ellipse:**
  The ellipse can also be defined as, if a point P moves such that the ratio of its distance from a fixed point F (the focus) to its distance from a fixed straight line (the directrix) is equal to a constant, e (less than 1), the locus of P is an ellipse of eccentricity e.

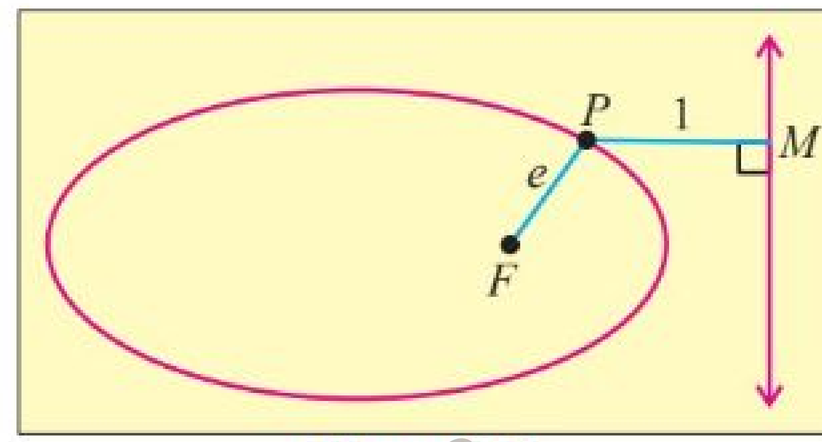


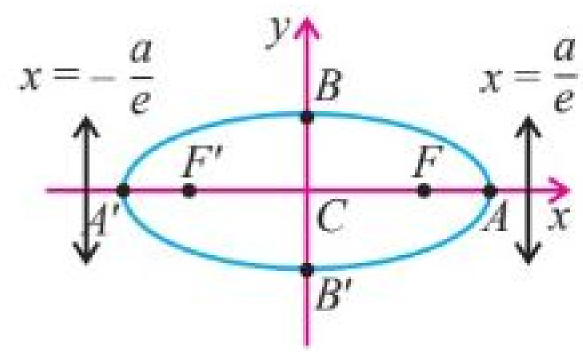
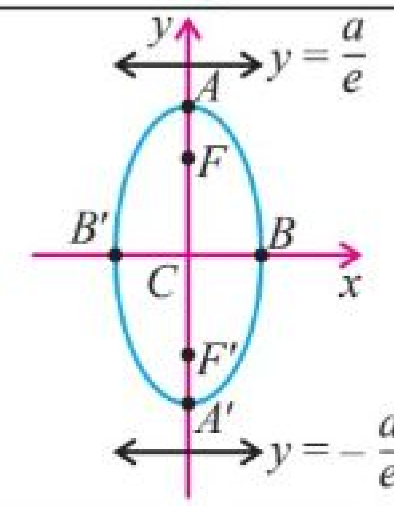
Figure 6.16

That is, if $\dfrac{|PF|}{|PM|} = e$, where $0 < e < 1$, the locus of P is an ellipse.

Note The equation of ellipse can also be derived using the above definition.

6.4.2 Summary of Important Results

Ellipse in standard form	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,\ a>b$	$\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1,\ a>b$
Centre	$(0, 0)$	$(0, 0)$
Equation of major axis	$y = 0, -a \le x \le a$	$x = 0, -a \le y \le a$
Length of major axis	$2a$	$2a$
Equation of minor axis	$x = 0, -b \le y \le b$	$y = 0, -b \le x \le b$
Length of minor axis	$2b$	$2b$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Covertices	$(0, \pm b)$	$(\pm b,\ 0)$
Foci	$(\pm c, 0), c = ae$	$(0, \pm c), c = ae$
Eccentricity	$e = \dfrac{c}{a} = \dfrac{\sqrt{a^2-b^2}}{a} < 1$	$e = \dfrac{c}{a} = \dfrac{\sqrt{a^2-b^2}}{a} < 1$
Equations of directrices	$x = \pm\dfrac{a}{e} = \pm\dfrac{c}{e^2}$	$y = \pm\dfrac{a}{e} = \pm\dfrac{c}{e^2}$
Equations of latera recta	$x = \pm ae$	$y = \pm ae$

Length of latus rectum	$\dfrac{2b^2}{a}$	$\dfrac{2b^2}{a}$
Sketch of ellipse in standard form		
Note : In both cases of ellipses, the major axis is the longer line segment and foci lie on the major axis.		

Theorem 1: (Translated forms of Ellipses)

The equation $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, with $a>b$ describes a horizontal ellipse with foci at $F'(h-c, k)$ and $F(h+c, k)$, where $c=\sqrt{a^2-b^2}$.

The centre of the ellipse is at the point $C(h,k)$ and the vertices are located at $(h\pm a,k)$ on the major axis.

The co-vertices are located at $(h,k\pm b)$.

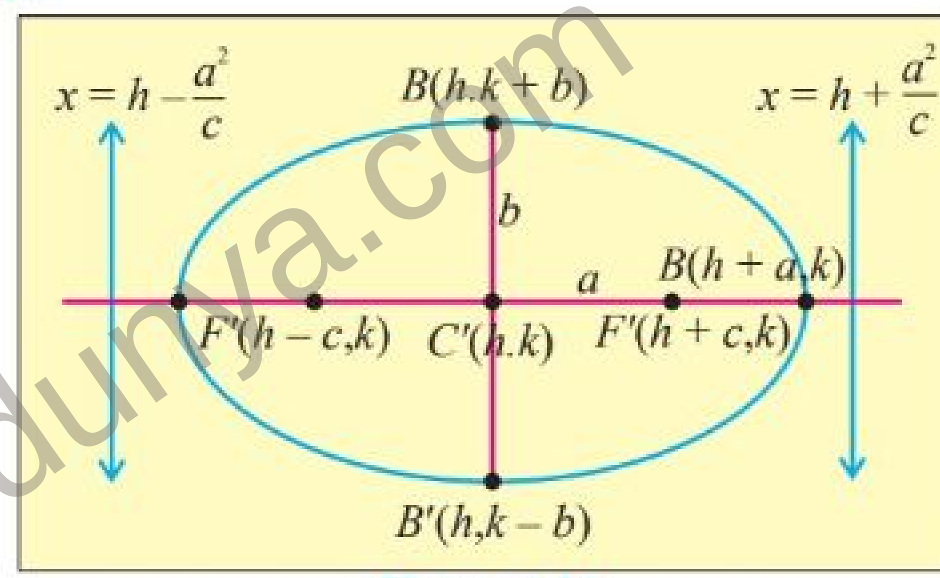


Figure 6.16

Theorem 2:

The equation $\dfrac{(x-h)^2}{b^2}+\dfrac{(y-k)^2}{a^2}=1$, with $a>b$ describes a vertical ellipse with foci at $F'(h, k-c)$ and $F(h, k+c)$, where $c=\sqrt{a^2-b^2}$. The centre of the ellipse is at the point $C(h,k)$ and the vertices are located at $(h,k\pm a)$ on the major axis. The co-vertices are located at $(h\pm b,k)$.

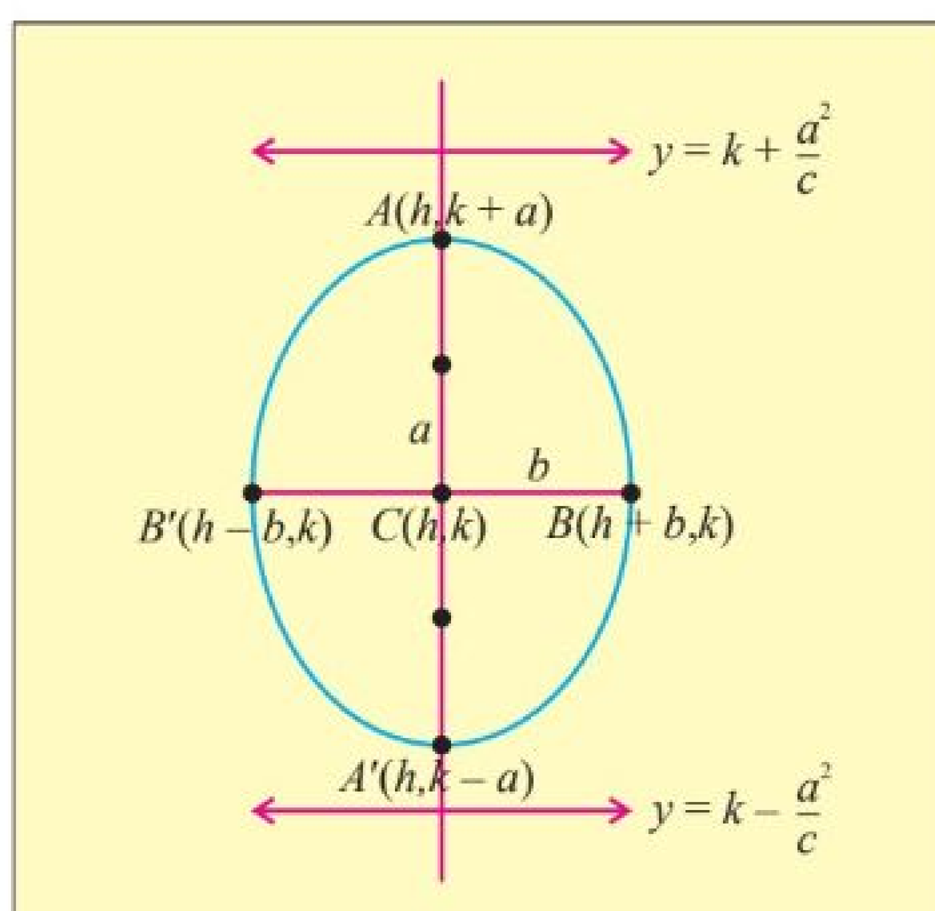


Figure 6.17

Example 18 Find an equation of the ellipse with foci (2, 1), (2, 7), major axis of length 10 and sketch its graph.

Solution: Given the foci: (2, 1) and (2, 7).
Both foci have the same x-coordinate
$\Rightarrow$ Major axis is vertical.
Major axis length = 10 $\Rightarrow 2a = 10 \Rightarrow a = 5$.
Centre is the midpoint of the foci:

$$\text{Centre} = \left(\frac{2+2}{2}, \frac{1+7}{2}\right) = (2, 4).$$

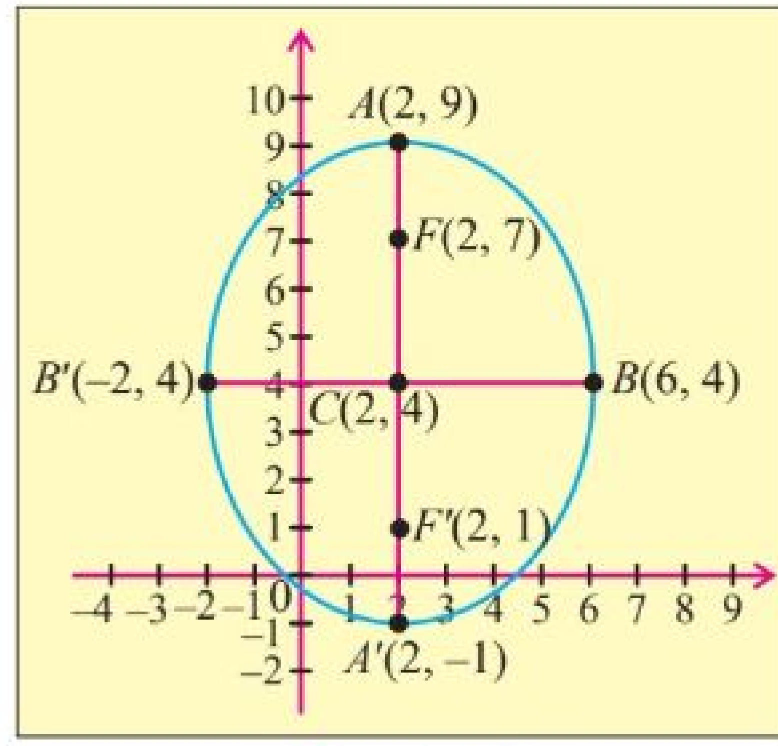


Figure 6.18

Distance from centre to either focus:

$$c = |4 - 1| = 3$$

For an ellipse: $c^2 = a^2 - b^2 \Rightarrow 3^2 = 5^2 - b^2 \Rightarrow 9 = 25 - b^2 \Rightarrow b^2 = 16 \Rightarrow b = 4$.
Thus, required equation of the ellipse with centre $(h, k) = (2, 4)$, major axis vertical, $a = 5$, and $b = 4$ is: $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, that is, $\frac{(x-2)^2}{16} + \frac{(y-4)^2}{25} = 1$.

Example 19 Find an equation of the ellipse with centre (0, 0), focus (5, 0) and vertex (6, 0). Sketch its graph.

Solution: Given centre is at (0, 0).
Focus: (5, 0) $\Rightarrow$ one focus is to the right; the other focus is (–5, 0) by symmetry.
Vertex: (6, 0) $\Rightarrow$ one vertex is to the right; the other vertex is (–6, 0).
All points have same y-coordinate 0, so major axis horizontal.

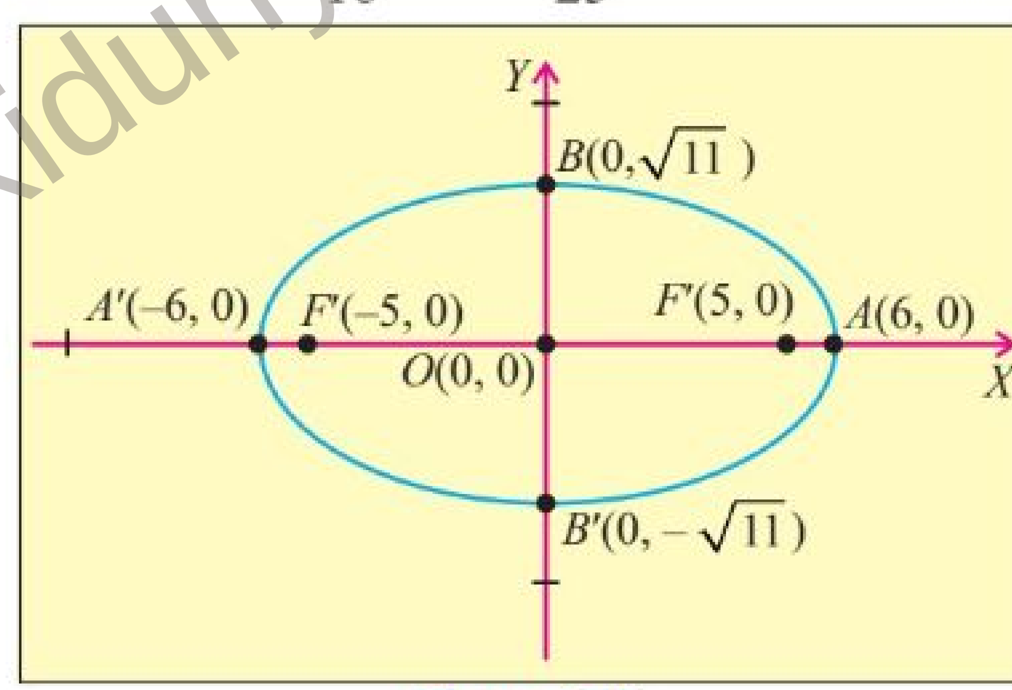


Figure 6.19

From vertices: $(\pm 6, 0) \Rightarrow a = 6$, so $a^2 = 36$.
From foci: $(\pm 5, 0) \Rightarrow c = 5, c^2 = 25$.
To find b, using the ellipse relation:
$c^2 = a^2 - b^2 \Rightarrow 25 = 36 - b^2 \Rightarrow b^2 = 11 \Rightarrow b = \sqrt{11}$
Required equation with horizontal major axis, centre at origin, $a^2 = 36$ and $b^2 = 11$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ that is, } \frac{x^2}{36} + \frac{y^2}{11} = 1.$$

Example 20 Find the centre, foci, eccentricity, vertices and directrices of the ellipse whose equation is $16x^2 + 9y^2 - 64x + 54y + 1 = 0$. Also sketch its graph.

Solution:

$$16x^2 + 9y^2 - 64x + 54y + 1 = 0$$
$$(16x^2 - 64x) + (9y^2 + 54y) + 1 = 0$$
$$16(x^2 - 4x) + 9(y^2 + 6y) + 1 = 0$$
$$16[(x^2 - 4x + 4) - 4] + 9[(y^2 + 6y + 9) - 9] + 1 = 0$$
$$16[(x-2)^2 - 4] + 9[(y+3)^2 - 9] + 1 = 0$$
$$16(x-2)^2 - 64 + 9(y+3)^2 - 81 + 1 = 0$$
$$16(x-2)^2 + 9(y+3)^2 - 144 = 0$$
$$16(x-2)^2 + 9(y+3)^2 = 144$$

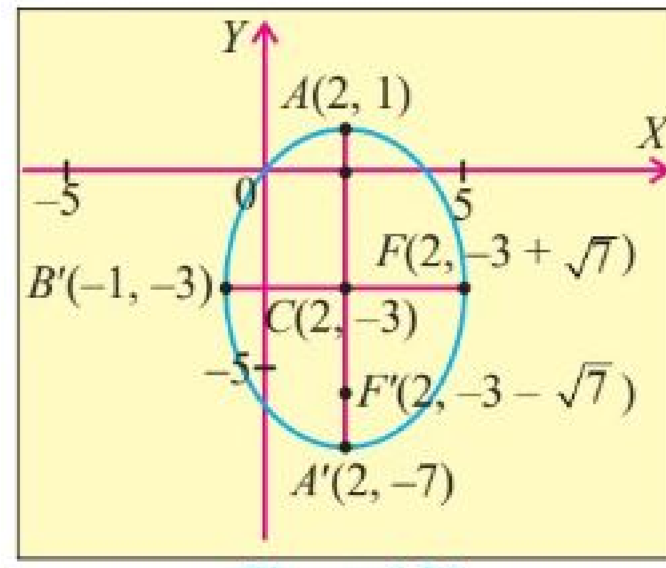


Figure 6.20

Divide both sides by 144:

$$\frac{16(x-2)^2}{144} + \frac{9(y+3)^2}{144} = 1$$
$$\frac{(x-2)^2}{9} + \frac{(y+3)^2}{16} = 1$$

Comparing with $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, we obtain

$h = 2, k = -3, b^2 = 9 \Rightarrow b = 3, a^2 = 16 \Rightarrow a = 4.$

Since $a > b$ and a^2 is under y-term, major axis is vertical.

Centre: $C(2, -3)$.

Using the ellipse relation: $c^2 = a^2 - b^2$; $c^2 = 16 - 9 = 7 \Rightarrow c = \sqrt{7}$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{7}}{4}$

Vertices: Vertices are $a = 4$ units above and below centre along vertical axis:

First vertex: $A(2, -3+4) = A(2, 1)$.

Second vertex: $A'(2, -3-4) = A'(2, -7)$.

Foci: Foci are $c = \sqrt{7}$ units above and below centre:

First focus: $F(2, -3+\sqrt{7})$, Second focus: $F'(2, -3-\sqrt{7})$.

Directrices: For vertical major axis, directrices are horizontal lines: $y = k \pm \frac{a}{e}$

$$y = k \pm \frac{a}{e} = -3 \pm \frac{4}{\sqrt{7}/4} = -3 \pm \frac{16}{\sqrt{7}}$$

Example 21 An ellipse has the following properties:

(i) Its major axis is vertical and has a length of 10 units.

(ii) The distance between its foci is 6 units.

(iii) The centre of the ellipse is at the point (3, –2).

Find the standard equation of this ellipse.

Solution:

(i) The major axis is vertical and has a length 10, so: $2a = 10 \Rightarrow a = 5 \Rightarrow a^2 = 25$

(ii) The distance between the foci is 6, so: $2c = 6 \Rightarrow c = 3 \Rightarrow c^2 = 9$

(iii) The centre is at $(h, k) = (3, -2)$

For an ellipse, the relationship between a, b and c is: $c^2 = a^2 - b^2$

Substitute $a^2 = 25$ and $c^2 = 9$

$$9 = 25 - b^2 \quad \Rightarrow \quad b^2 = 25 - 9 \quad \Rightarrow \quad b^2 = 16$$

Since the major axis is vertical, the standard form of the ellipse equation is:

$\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$ Substitute $h = 3,\ k = -2,\ a^2 = 25$, and $b^2 = 16$:

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} = 1.$$

6.4.3 Tangents and Normals to Ellipse

We are given the ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \ldots(1)$ and the line: $y = mx + c \quad \ldots(2)$

We want to find their **points of contact** and the **condition for tangency**.

Substitute $y = mx + c$ into the ellipse equation:

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

Multiplying through by a^2b^2 to clear denominators:

$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

Expand: $\quad b^2x^2 + a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$

$$b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$$

$$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0 \quad \ldots(3)$$

The equation is quadratic in x and it gives the x-coordinates of the two points where line and ellipse intersect. The corresponding values of y are obtained by setting the values of x obtained from last equation into $y = mx + c$. Thus line and ellipse intersect in two points.

Tangency Condition

Tangency occurs when the quadratic in x has equal roots.
This means the discriminant $D = 0$:

$$(2a^2mc)^2 - 4(b^2 + a^2m^2) \cdot a^2(c^2 - b^2) = 0$$

$$4a^4m^2c^2 - 4a^2(b^2 + a^2m^2)(c^2 - b^2) = 0$$

Factor out $4a^2$:

$$4a^2\left[a^2m^2c^2 - (b^2 + a^2m^2)(c^2 - b^2)\right] = 0$$

Since $a \neq 0$:

$$a^2m^2c^2 - (b^2 + a^2m^2)(c^2 - b^2) = 0$$

Expand the second term:

$$a^2m^2c^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2m^2b^2 = 0$$

$$a^2m^2b^2 - b^2c^2 + b^4 = 0$$

Divide through b^2 $(b \neq 0)$:

$$a^2m^2 - c^2 + b^2 = 0$$

So, the tangency condition is:

$$\boxed{c^2 = a^2m^2 + b^2} \Rightarrow c = \pm\sqrt{a^2m^2 + b^2} \quad \ldots(4)$$

Putting value of c in $y = mx + c$, we obtain $y = mx \pm \sqrt{a^2m^2 + b^2}$ two tangents to ellipse for all values of m.

Do You Know?

If a glass of water is tilted, The surface of the water forms an ellipse.

Point of Tangency

For the tangent $y = mx + \sqrt{a^2m^2 -5}$ the Q.E (3) has equal roots given by

$$x = \frac{-2a^2mc}{2(b^2 + a^2 + m^2)} = \frac{-a^2mc}{c^2}$$

We have equation of $= \dfrac{-a^2m}{c}$

$$= \frac{-a^2m}{\sqrt{a^2m^2 + b^2}}$$

Substituting $x = \dfrac{-a^2m}{\sqrt{a^2m^2 + b^2}}$

Into 0 (5) we have

So, point of contact of (5) and (1) $y = \dfrac{b^2}{\sqrt{a^2m^2 + b^2}}$ is $\left(\dfrac{-a^2m}{\sqrt{a^2m^2 + b^2}}, \dfrac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$.

Similarly, point of contact of $y = mx - \sqrt{a^2m^2 + b^2}$ and (1) is

$$\left(\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$$

Example 22 Find equation of the tangent and normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1).

Solution: Equation of the ellipse is: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Differentiating implicitly with respect to x: $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$

Thus, the slope at (x_1, y_1) is: $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}=-\frac{b^2x_1}{a^2y_1}$

Equation of the tangent:

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

Multiplying through $\frac{y_1}{b^2}$: $\quad \frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=-\frac{xx_1}{a^2}+\frac{x_1^2}{a^2}$

Rearranging: $\quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$

Since (x_1, y_1) lie on ellipse, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

Hence, the equation of the tangent is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Equation of the normal:

The slope of the normal is $\frac{a^2y_1}{b^2x_1}$

Thus, $\quad y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$

Multiplying $\quad b^2x_1y-b^2x_1y_1=a^2y_1x-a^2x_1y_1$

Rearranging: $\quad a^2y_1x-b^2x_1y=x_1y_1(a^2-b^2)$

Dividing both sides by by x_1y_1 (assuming $x_1\neq 0,\ y_1\neq 0$): $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.

EXERCISE 6.4

1. Find an equation of the ellipse with given data.
   (i) Foci $(0, \pm 5)$ and minor axis of length 8.
   (ii) Vertices $(-2, 5)$, $(-2, -3)$; foci $(-2, 4)$ and $(-2, -2)$.

(iii) Vertices $(\pm 6, 0)$, eccentricity $= \frac{2}{3}$.

(iv) Centre $(-3, 1)$, major axis parallel to the x-axis and of length 10 units, minor axis parallel to the y-axis and of length 6 units.

2. Find the centre, foci, eccentricity, vertices and directrices of the ellipse whose equation is given and sketch its graph.
   (i) $4x^2 + 9y^2 = 36$ (ii) $4x^2 + 24x + 9y^2 - 36y + 36 = 0$
3. Find the equation of the ellipse as the locus of points $P(x, y)$ such that the sum of the distances from P to the points $(-3, 0)$ and $(3, 0)$ is 10.
4. Find the equation of an ellipse centered at $(5, 3)$ that is tangent to both the x-axis and y-axis.
5. What are the dimensions of the smallest rectangle aligned to the axes that completely contains the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$?
6. Find the dimensions of the rectangle with greatest perimeter that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.
7. Find the minimum distance of the point $P(3\cos\theta, 2\sin\theta)$ on the ellipse from the centre of the ellipse.
8. Find the minimum distance from the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ to the line $x + y = 6$.
9. Find the value of k for which the line $y = 2x + k$ is tangent to the ellipse $\frac{x^2}{9} + y^2 = 1$.
10. Determine whether the line $y = 2x - 3$ is a tangent, secant, or external to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.
11. Find the equation of an ellipse with centre at $(0, 0)$, a horizontal major axis of length 8, and that touches the line $y = x + 5$.
12. Show that the family of lines given by: $x\cos\theta + y\sin\theta = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$ where θ is a parameter and $a > b$, are all tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
13. If a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre C cuts the major and minor axes at P and Q respectively. Prove that $\frac{a^2}{|CP|^2} + \frac{b^2}{|CQ|^2} = 1$.

6.5 Hyperbola

The definition of a hyperbola is similar to that of an ellipse. For an ellipse, the sum of the distances between the foci and a point on the ellipse is fixed, whereas for a hyperbola, the absolute value of the difference between these distances is fixed. One distinguishing feature of a hyperbola is that its graph has two separate branches (see Figure 21).

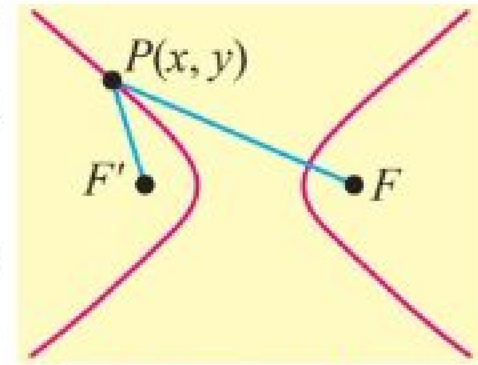


Figure 6.21

Derivation of the Standard Equation of Hyperbola

Thus, we can define hyper bola as a **hyperbola** is the set of all points (x, y) for which the absolute value of the difference between the distances from two distinct fixed points called **foci** is constant. The hyperbolas with simplest equations are those with foci on one of the coordinate axes. To obtain the simplest equation for a hyperbola, we place the foci on the x-axis at the points $F'(-c, 0)$ and $F(c, 0)$, so that the origin is the midpoint of the segment joining them (see Figure 22).

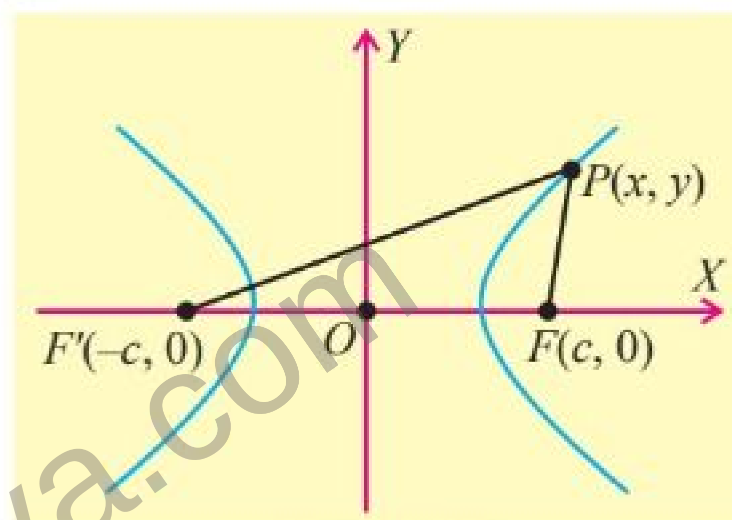


Figure 6.22

Let the absolute value of constant difference of distances from any point $P(x, y)$ on the hyperbola to the foci be $2a$, where $a > 0$. Thus,

$$\left||PF'|-|PF|\right| = 2a \quad \text{or} \quad |PF'|-|PF| = \pm 2a$$

Using the distance formula,

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a$$

To simplify, isolate one radical and square both sides:

$$\sqrt{(x+c)^2+y^2} = \pm 2a + \sqrt{(x-c)^2+y^2}$$

Squaring, $(x+c)^2+y^2 = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$

Expanding the squares,

$$x^2+2cx+c^2+y^2 = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2} + x^2-2cx+c^2+y^2$$

Cancelling the common terms x^2, c^2, and y^2 from both sides,

$$2cx = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2} - 2cx$$

Bring the terms involving cx together: $4cx = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2}$

Divide by 4 and isolate the remaining radical:

$$\mp a\sqrt{(x-c)^2+y^2} = a^2 - cx$$

Now we square both sides again:

$$a^2[(x-c)^2+y^2]=a^4-2a^2cx+c^2x^2$$

Expanding the left side: $a^2(x^2-2cx+c^2+y^2)=a^4-2a^2cx+c^2x^2$

$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$$

Adding $2a^2cx$ to both sides:

$$a^2x^2+a^2c^2+a^2y^2=a^4+c^2x^2$$

Rearranging: $a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

From triangle $PF'F$ in Figure 2, we note that $|PF'|-|PF|<|F'F|$, that is, $2a<2c$, so $a<c$ and thus $a^2-c^2<0$. For convenience, we define a new constant b such that $-b^2=a^2-c^2$.

Substituting, the equation of the hyperbola becomes $(-b^2)x^2+a^2y^2=a^2(-b^2)$

Finally, dividing both sides by $-a^2b^2$ gives the standard form of the equation of a hyperbola: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ where $c^2=a^2+b^2$.

6.5.1 Key Features of the Hyperbola

- **Vertices**: Setting $y=0$ in the equation of hyperbola, gives $x=\pm a$. The points $A'(-a, 0)$ and $A(a, 0)$ are the **vertices** of the hyperbola.
- **Transverse Axis**: The line segment $A'A$ connecting the vertices is the **transverse axis** (or **focal axis**) of the hyperbola. Length of the transverse axis is $2a$.
- **Conjugate Axis:** A line passes through the centre and perpendicular to the transverse axis is called conjugate axis. Setting $x=0$ in the equation of hyperbola, gives $y^2=-b^2$, which is impossible, so hyperbola does not meet y-axis in real points. If we take two points $B'(0, -b)$ and $B(0, b)$ on the y-axis, then the line segment $B'B$ is called the **conjugate axis** of the hyperbola. Length of the **conjugate axis** is $2b$.
- **Foci**: The foci are at $(\pm c, 0)$, they always lie on the transverse axis. In case of hyperbola $c^2=a^2+b^2$, so that, unlike the ellipse, we may have $a>b$ or $a<b$ or $a=b$.
- **Centre**: The midpoint $C(0, 0)$ of the segment joining the foci is called the **centre** of the hyperbola. Note that it is same as the midpoint of the transverse axis.
- **Symmetry**: The equation is unchanged if x is replaced by $-x$ and y by $-y$; hence the hyperbola is symmetric with respect to both the axis, and the origin.
- **Eccentricity:** The **eccentricity** e of a hyperbola is defined as $e=\dfrac{c}{a}\cdot$

  Since $c>a$, we have $e>1$.

- **Directrices:** For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, there are two directrices given by the vertical lines $x=\pm\frac{a}{e}$.
- **Latus Rectum:** Each of the focal chords perpendicular to the transverse axis is called a **latus rectum** (plural: latera recta). The length of each latus rectum is $\frac{2b^2}{a}$.
- **Parametric Equations:** For all real θ such that $\theta\neq(2n+1)\frac{\pi}{2}$ (where n is an integer), the point $(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Thus, the **parametric equations** of the hyperbola are: $x=a\sec\theta, y=b\tan\theta$.
- **Foci on the *y*-axis:** If the foci are on the y-axis at $(0,\pm c)$, the roles of x and y are interchanged, and the standard equation becomes $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, vertices $(0,\pm a)$.
- **Relation between *a*, *b* and *e*:**

  Since $b^2=c^2-a^2$, setting $c=ae \Rightarrow b^2=a^2e^2-a^2 \Rightarrow b^2=a^2(e^2-1)$.
- **Branches:** To analyze the hyperbola further, we write $\frac{x^2}{a^2}=1+\frac{y^2}{b^2}\geq 1$. This shows that $x^2\geq a^2$ so $|x|\geq a$. Therefore, we have $x\geq a$ or $x\leq -a$. This means that the hyperbola consists of two parts, called its branches. Note that no portion of the curve lies between $-a<x<a$, because in this interval y is imaginary.
- **Asymptotes:** An important feature of hyperbola that is not shared by ellipses is the presence of asymptotes. An asymptote of a curve $y=f(x)$ is a line $y=g(x)$ such that

  $$\lim_{x\to\pm\infty}[f(x)-g(x)]=0$$

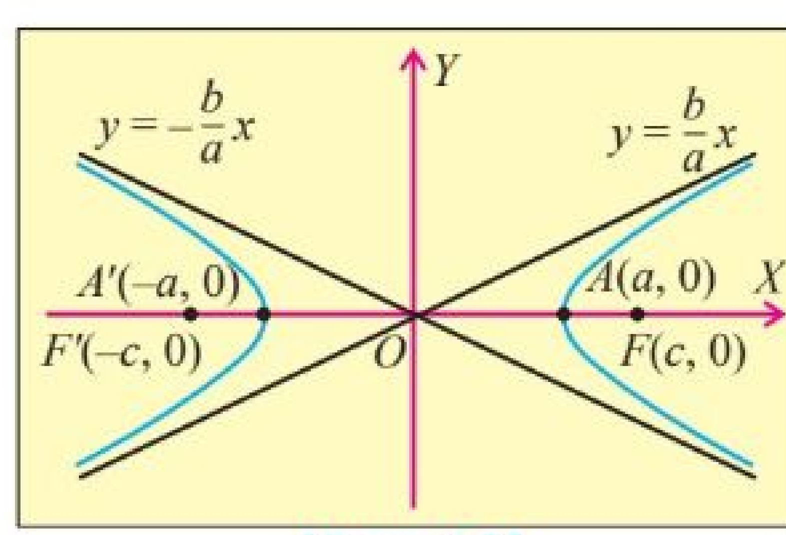


Figure 6.23

For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we have $\frac{y^2}{b^2}=\frac{x^2}{a^2}-1$ or $y^2=\frac{b^2}{a^2}x^2-b^2$. Notice that

$$\lim_{x\to\pm\infty}\frac{y^2}{x^2}=\lim_{x\to\pm\infty}\left(\frac{b^2}{a^2}-\frac{b^2}{x^2}\right)=\frac{b^2}{a^2}.$$

That is, as $x \to \pm\infty, \frac{y^2}{x^2} \to \frac{b^2}{a^2}$, so that $\frac{y}{x} \to \pm\frac{b}{a}$ and so, $y = \pm\frac{b}{a}x$ are the (slant) asymptotes as shown in Figure 6.23. The asymptotes of a curve do not meet the curve but distance of any point on the curve from any of the two lines approaches zero.

Note. The joint equation of the two asymptotes is

$$\left(y+\frac{b}{a}x\right)\left(y-\frac{b}{a}x\right)=0, \text{ that is, } y^2-\frac{b^2}{a^2}x^2=0 \text{ or } \frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

Rectangular Hyperbola as a special case: If in $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ the lengths a and b become equal then the resulting hyperbola $x^2 - y^2 = a^2$ is called rectangular hyperbola. In this case asymptotes are perpendicular to each other.

This is obtained by replacing the constant term 1 with 0 in the standard equation.

- **Second definition of Hyperbola**

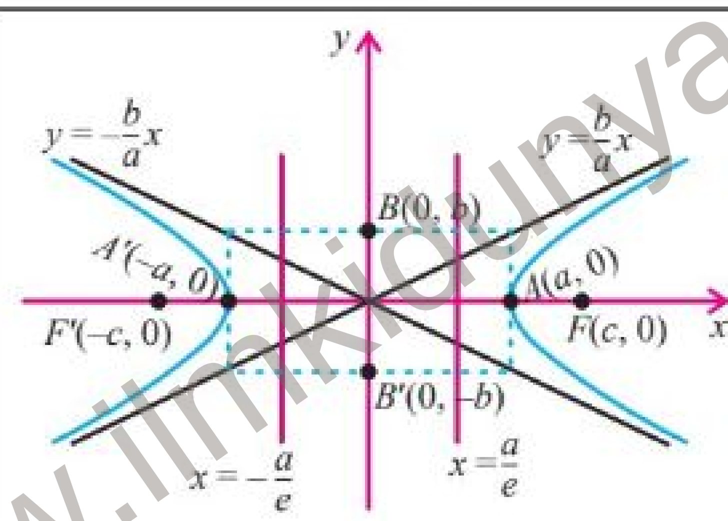
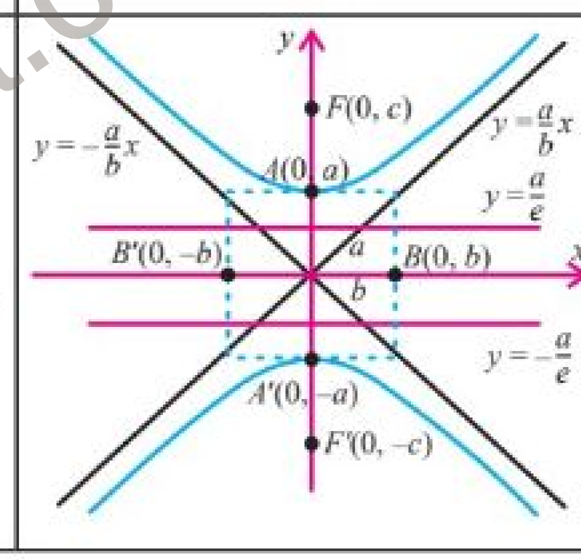
The hyperbola can also be defined as, if a point P moves so that the ratio of its distance from a fixed point F (the focus) to its distance from a fixed straight line (the directrix) is equal to a constant, e (greater than 1), the locus of P is a hyperbola of eccentricity e.

That is, if $\frac{|PF|}{|PM|}=e$, where $e > 1$, the locus of P is a hyperbola.

Note. The equation of hyperbola can also be derived using the above definition.

6.5.2 Summary of Important Results

Hyperbola in standard form	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Centre	(0, 0)	(0, 0)
Equation of transverse axis	$y=0, -a \le x \le a$	$x=0, -a \le y \le a$
Length of transverse axis	$2a$	$2a$
Equation of conjugate axis	$x=0, -b \le y \le b$	$y=0, -b \le x \le b$
Length of conjugate axis	$2b$	$2b$
Vertices	$(\pm a, 0)$	$(0, \pm a)$

Conjugate points	$(0, \pm b)$	$(\pm b,\ 0)$
Foci	$(\pm c,\ 0), c = ae$	$(0,\ \pm c), c = ae$
Eccentricity	$e = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a} > 1$	$e = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a} > 1$
Equations of directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{a}{e}$
Equations of latera recta	$x = \pm ae$	$y = \pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Equations of asymptotes	$y = \pm\frac{b}{a}x$	$y = \pm\frac{a}{b}x$
Sketch of hyperbola in standard form		

Theorem 1: Translated Forms of Hyperbola's

The equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

describes a hyperbola with foci at the points $F'(h-c,\ k)$ and $F(h+c, k)$, where $c = \sqrt{a^2+b^2}$.

The centre of the hyperbola is at the point $C(h,\ k)$ and the vertices are located at $(h \pm a,\ k)$.

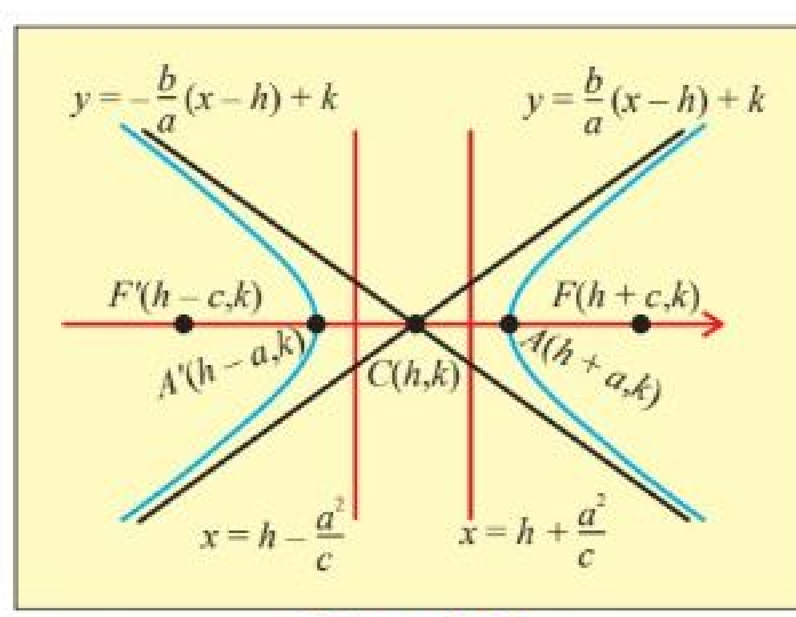


Figure 6.25

The asymptotes are $y = \pm\frac{b}{a}(x-h)+k$.

Theorem 2: The equation

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

describes a hyperbola with foci at the points $F'(h, k-c)$ and $F(h, k+c)$, where $c=\sqrt{a^2+b^2}$.

The centre of the hyperbola is at the point (h, k) and the vertices are located at $(h, k\pm a)$.

The asymptotes are $x=\pm\frac{b}{a}(y-h)+k$.

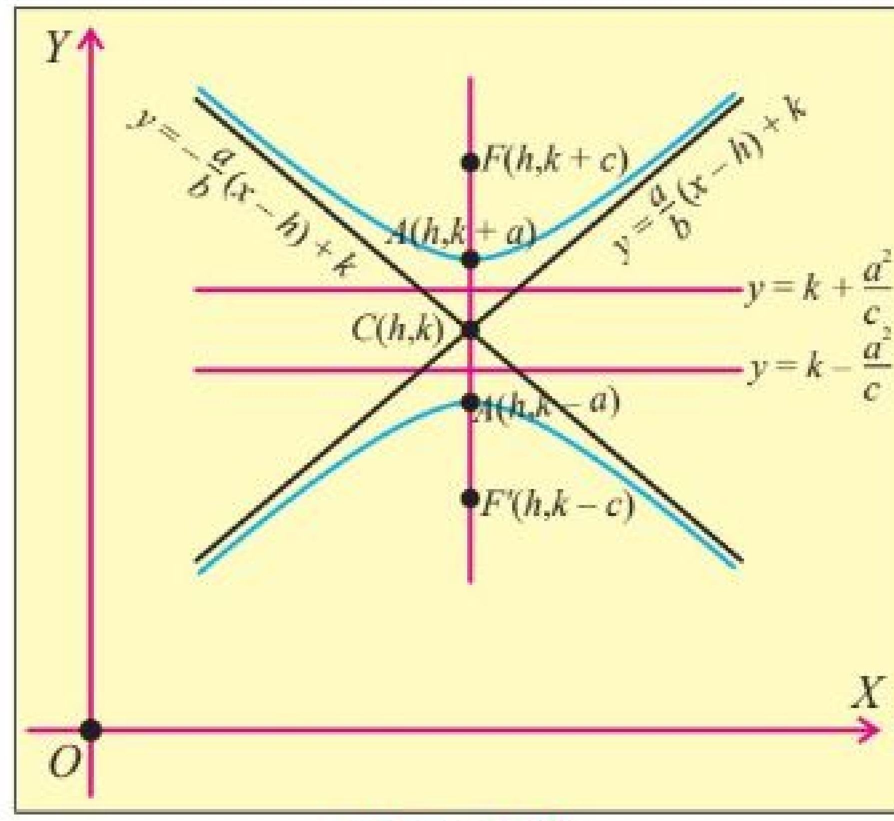


Figure 6.26

Example 23 A hyperbola has a horizontal transverse axis of length 12, centre at (–2, 5), and eccentricity $\frac{5}{3}$. Find the equation of the hyperbola.

Solution: Since the transverse axis is horizontal, the hyperbola opens left and right.

The standard form is: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, where (h, k) is the centre.

Given centre: $(h, k)=(-2, 5)$.

Length of transverse axis $=2a=12 \Rightarrow a=6$

Eccentricity: $e=\frac{c}{a}=\frac{5}{3} \Rightarrow \frac{c}{6}=\frac{5}{3} \Rightarrow c=10$

Using the relationship $c^2=a^2+b^2: 100=36+b^2 \Rightarrow b^2=64$.

Substitute $h=-2$, $k=5$, $a^2=36$, $b^2=64$ into the standard form:

$$\frac{(x+2)^2}{36}-\frac{(y-5)^2}{64}=1$$

Example 24 Find an equation of the hyperbola with given data and sketch the graph.

(i) Foci: $(-3, 5\pm5\sqrt{2})$ (ii) Length of transverse axis $=6$.

Solution: Midpoint of foci $(-3, 5+5\sqrt{2})$, $(-3, 5-5\sqrt{2})$ is the centre: $C(-3, 5)$.

Transverse axis is vertical, so hyperbola has the form: $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

$2a=6 \Rightarrow a=3$

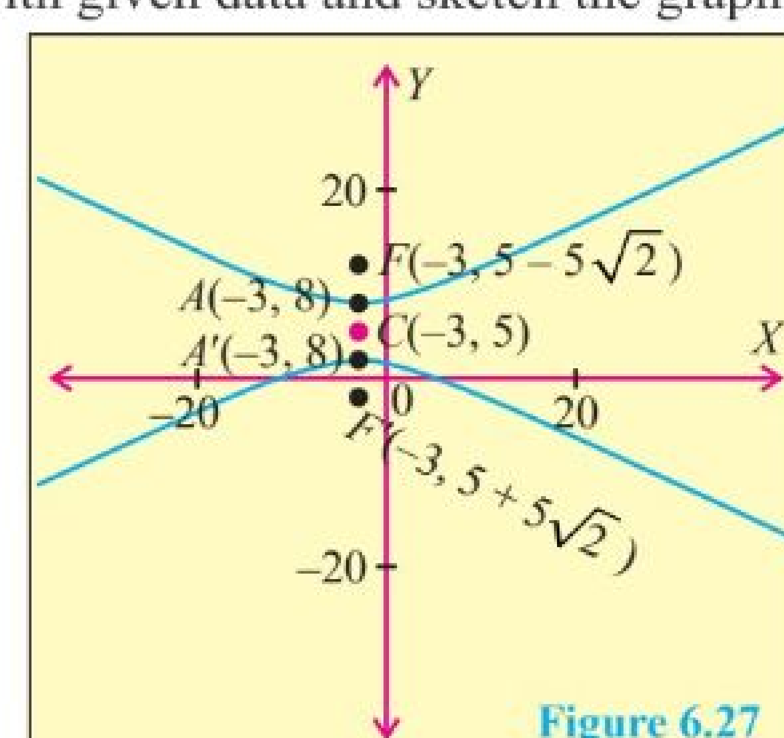


Figure 6.27

$2c = \text{distance between foci} = 10\sqrt{2}$

$c = 5\sqrt{2} \Rightarrow c^2 = 50$

$c^2 = a^2 + b^2 \Rightarrow 50 = 9 + b^2 \Rightarrow b^2 = 41$

Required equation: $\frac{(y-5)^2}{9} - \frac{(x+3)^2}{41} = 1.$

Example 25 Find an equation of the hyperbola with given data and sketch the graph. (i) Foci $(4,-3)$, $(4,\ 7)$ and (ii) one vertex $(4,\ 5)$.

Solution:

Centre (h,k) is the midpoint of foci $(4, -3)$ and $(4, 7)$

$$h = \frac{4+4}{2} = 4,\ k = \frac{-3+7}{2} = \frac{4}{2} = 2$$

So centre: $(4,\ 2)$.

Both foci have same x-coordinate, this implies transverse axis is vertical.

$c =$ Distance from centre (4, 2) to a focus (4, 7)

$= 7 - 2 = 5.$

Given vertex: (4, 5).

$a =$ Distance from centre (4, 2) to vertex (4, 5)

$= 5 - 2 = 3.$

$c^2 = a^2 + b^2$

$5^2 = 3^2 + b^2$

$b^2 = 25 - 9$

$= 16$

$$\frac{(y-2)^2}{9} - \frac{(x-4)^2}{16} = 1$$

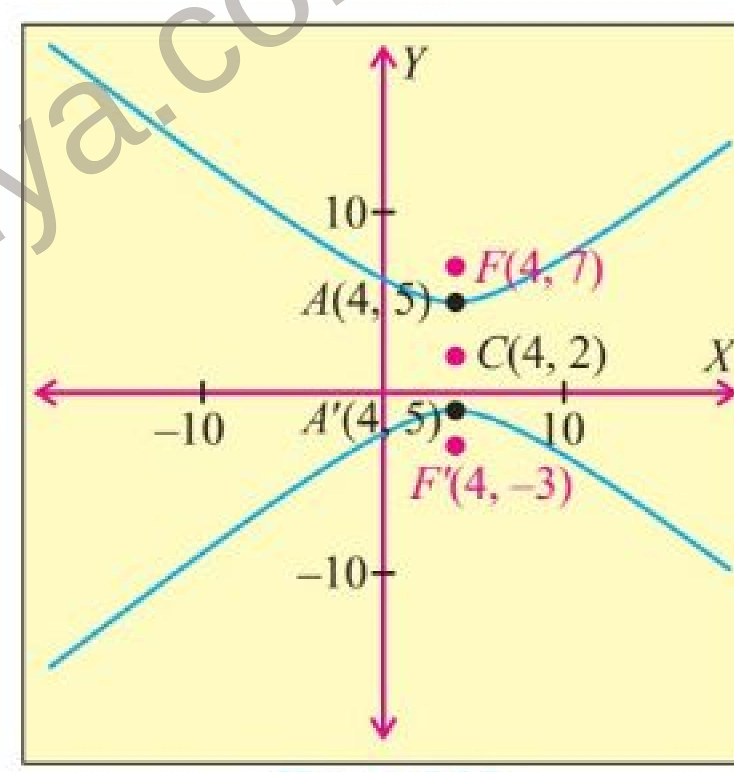


Figure 6.28

Example 29 Find the centre, foci, eccentricity, vertices and equations of the directrices of the hyperbola:

$$9x^2 - 36x - 4y^2 + 8y - 4 = 0.\ \text{Also sketch the graph.}$$

Solution:

$9x^2 - 36x - 4y^2 + 8y - 4 = 0$

$9x^2 - 36x - 4y^2 + 8y = 4$

$9(x^2 - 4x) - 4(y^2 - 2y) = 4$

$9(x^2-4x+4-4)-4(y^2-2y+1-1)=4$

$9[(x-2)^2-4]-4[(y-1)^2-1]=4$

$9(x-2)^2-36-4(y-1)^2+4=4$

$9(x-2)^2-4(y-1)^2=36$

Divide by 36

$\dfrac{9(x-2)^2}{36}-\dfrac{4(y-1)^2}{36}=1$

$\dfrac{(x-2)^2}{4}-\dfrac{(y-1)^2}{9}=1$

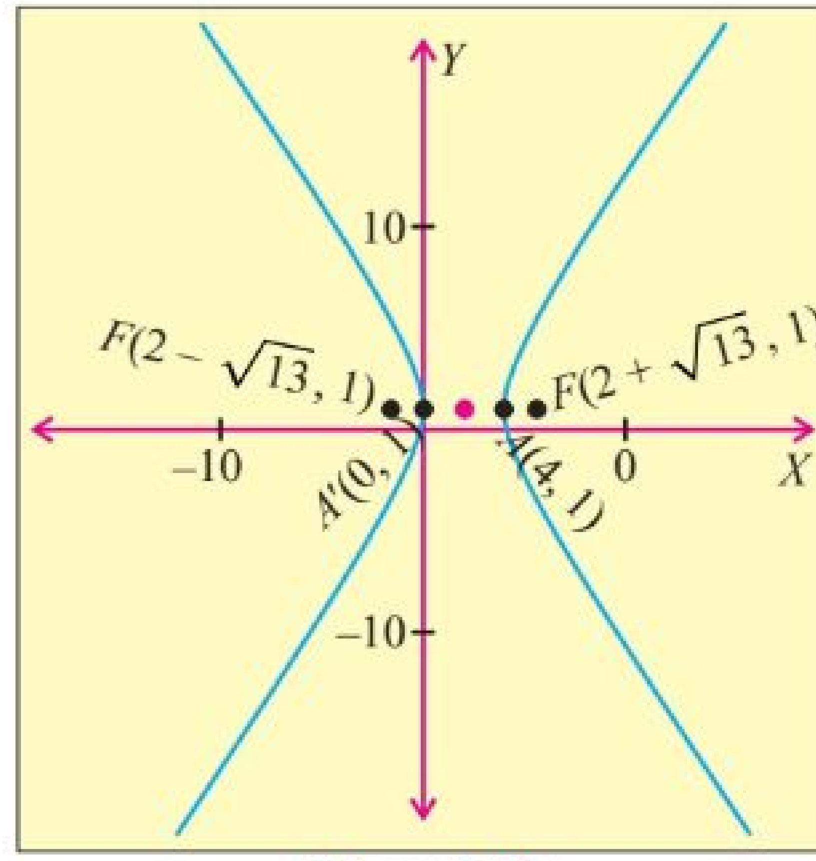


Figure 6.29

This is a horizontal hyperbola:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Here: $h=2,\ k=1,\ a^2=4 \Rightarrow a=2,\ b^2=9 \Rightarrow b=3.$

Using the relation: $c^2=a^2+b^2$: $c^2=4+9=13 \Rightarrow c=\sqrt{13}$

Centre: $(h,\ k)=(2,\ 1)$.

For horizontal hyperbola vertices are: $(h\pm a,k)$

That is vertices are $A(2+2,1)=A(4,1),\ A'(2-2,1)=A'(0,\ 1)$

Foci: $(h\pm c,k)$

That is foci are $F(2+\sqrt{13},1),\ F'(2-\sqrt{13},1)$

Eccentricity $e=\dfrac{c}{a}=\dfrac{\sqrt{13}}{2}$

Directrices: For a horizontal hyperbola: $x=h\pm\dfrac{a}{e}$

$\dfrac{a}{e}=\dfrac{2}{\frac{\sqrt{13}}{2}}=\dfrac{4}{\sqrt{13}}=\dfrac{4\sqrt{13}}{13}$. So directrices: $x=2\pm\dfrac{4\sqrt{13}}{13}$

6.5.3 Tangents and Normals to Hyperbola

We are given the hyperbola: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \quad \ldots(1)$

and the line: $y=mx+c \quad \ldots(2)$

We want to find their **points of intersection** and the **condition for tangency**.

Substitute $y=mx+c$ into the hyperbola equation:

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

Multiplying through by a^2b^2 to clear denominators: $b^2x^2 - a^2(mx+c)^2 = a^2b^2$

Expand:

$$b^2x^2 - a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$$

$$b^2x^2 - a^2m^2x^2 - 2a^2mcx - a^2c^2 - a^2b^2 = 0$$

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0 \qquad \ldots(3)$$

The equation is quadratic in x and it gives the x-coordinates of the two points where line and hyperbola intersect. The corresponding values of y are obtained by setting the values of x obtained from last equation into $y = mx + c$. Thus line and hyperbola intersect in two points.

Tangency Condition

Tangency occurs when the quadratic in x has equal roots.

This means the discriminant $D = 0$:

$$(-2a^2mc)^2 - 4(b^2 - a^2m^2)\left[-a^2(c^2 + b^2)\right] = 0$$

$$4a^4m^2c^2 + 4a^2(b^2 - a^2m^2)(c^2 + b^2) = 0$$

Factor out $4a^2$: $\quad 4a^2\left[a^2m^2c^2 + (b^2 - a^2m^2)(c^2 + b^2)\right] = 0$

Since $a \neq 0$: $\quad a^2m^2c^2 + (b^2 - a^2m^2)(c^2 + b^2) = 0$

Expand the second term:

$$a^2m^2c^2 + b^2c^2 + b^4 - a^2m^2c^2 - a^2m^2b^2 = 0$$

$$-a^2m^2b^2 + b^2c^2 + b^4 = 0$$

Divide through b^2 $(b \neq 0)$:

$$-a^2m^2 + c^2 + b^2 = 0$$

So the tangency condition is:

$$\boxed{c^2 = a^2m^2 - b^2} \Rightarrow c = \pm\sqrt{a^2m^2 - b^2} \quad \text{provided } a^2m^2 > b^2$$

Putting value of c in $y = mx + c$, we obtain $y = mx \pm \sqrt{a^2m^2 - b^2}$ two tangents to hyperbola for all values of m.

Points of Contact

The line $y = mx + c$ intersects the hyperbola in two distinct points given by solving:

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0$$

$$x = \frac{-(-2a^2mc) \pm \sqrt{(2a^2mc)^2 - 4(b^2 - a^2m^2)\cdot(-a^2(c^2 + b^2))}}{2(b^2 - a^2m^2)}$$

$$= \frac{2a^2mc \pm \sqrt{4a^4m^2c^2 + 4a^2(b^2 - a^2m^2)(c^2 + b^2)}}{2(b^2 - a^2m^2)}$$

$$= \frac{a^2mc \pm \sqrt{a^4m^2c^2 + a^2(b^2 - a^2m^2)(c^2 + b^2)}}{b^2 - a^2m^2}$$

The y-coordinates follow from $y = mx + c$.

Example 27 Find equation of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1).

Solution: Equation of the hyperbola is: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Differentiating implicitly with respect to x:

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}$$

Thus the slope at (x_1, y_1) is: $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{b^2x_1}{a^2y_1}$

Equation of the tangent: $y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$

Multiplying through $\frac{y_1}{b^2}$: $\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$

Rearranging: $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$

Since (x_1, y_1) lie on hyperbola, so $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

Hence, the equation of the tangent is: $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Hence, equation of normal at $(x - y_1)$ is:

The slope of the normal is $-\frac{a^2y_1}{b^2x_1}$

Thus, $y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1)$

Multiplying: $b^2x_1y - b^2x_1y_1 = -a^2y_1x + a^2x_1y_1$

Rearranging: $a^2y_1x + b^2x_1y = x_1y_1(a^2 + b^2)$

Dividing both sides by $x_1\ y_1$ (assuming $x_1 \neq 0,\ y_1 \neq 0$): $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

Example 28 Find the equations of the tangents to the hyperbola $2x^2 - 3y^2 = 24$ that pass through the point (2, –4).

Solution: $2x^2 - 3y^2 = 24$

Divide through by 24: $\frac{x^2}{12} - \frac{y^2}{8} = 1$. So $a^2 = 12,\ b^2 = 8$.

For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, tangent with slope m is: $y = mx \pm \sqrt{a^2m^2 - b^2}$

Substitute $a^2 = 12,\ b^2 = 8:\ y = mx \pm \sqrt{12m^2 - 8}$

Substitute $x = 2, y = -4:\ -4 = 2m \pm \sqrt{12m^2 - 8} \Rightarrow -4 - 2m = \pm\sqrt{12m^2 - 8}$

Square: $16 + 16m + 4m^2 = 12m^2 - 8 \Rightarrow 8m^2 - 16m - 24 = 0 \Rightarrow m^2 - 2m - 3 = 0$

$\Rightarrow (m-3)(m+1) = 0 \quad \Rightarrow \quad m = 3$ or $m = -1$.

For m = 3:

$$y = 3x \pm \sqrt{12(9) - 8} \Rightarrow y = 3x \pm \sqrt{108 - 8} \Rightarrow y = 3x \pm \sqrt{100} \Rightarrow y = 3x \pm 10.$$

Check which sign satisfies the point (2, – 4):

Check: $y = 3x + 10$:

At $x = 2$, $y = 3(2) + 10 = 16$. So (2, – 4) do not lie on $y = 3x + 10$.

Check: $y = 3x - 10$:

At $x = 2$, $y = 3(2) - 10 = -4$. So (2, – 4) lie on $y = 3x - 10$.

So tangent is $3x - y - 10 = 0$.

For $m = -1$:

$$y = -x \pm \sqrt{12(1) - 8} \Rightarrow y = -x \pm \sqrt{12 - 8} \Rightarrow y = -x \pm \sqrt{4} \Rightarrow y = -x \pm 2.$$

Check which sign satisfies the point (2, –4):

Check: $y = -x + 2$:

At $x = 2$, $y = -2 + 2 = 0$. So (2, – 4) do not lie on $y = -x + 2$

Check: $y = -x - 2$::

At $x = 2$, $y = -2 - 2 = -4$. So (2, – 4) lie on $y = -x - 2$

So tangent is $x + y + 2 = 0$.

EXERCISE 6.5

1. Find an equation of the hyperbola with the given data. Also sketch the graph.
   (i) Centre (0, 0), Focus (0, 10), Vertex (0, 8)
   (ii) Foci (0, ±8), Eccentricity = 4 (iii) Foci (0, ±10), Directrices $y = \pm 5$
2. Find the centre, foci, eccentricity, vertices and equations of directrices of each of the following. Also sketch the graph.
   (i) $\frac{x^2}{36} - \frac{y^2}{64} = 1$ (ii) $\frac{(x+3)^2}{5} - \frac{(y-2)^2}{4} = 1$ (iii) $\frac{y^2}{25} - \frac{x^2}{16} = 1$

3. A hyperbola has its centre at the origin, transverse axis of length 14 along the y-axis, and conjugate axis of length 10 along the x-axis. Find the equation of the hyperbola.
4. A hyperbola has a horizontal transverse axis of length 8. One of its foci is at the point $(1+2\sqrt{5}, 3)$ and its eccentricity is $\frac{\sqrt{5}}{2}$. Find the equation of the hyperbola.
5. A hyperbola has a vertical transverse axis of length 8, centre at (2, −1), and eccentricity $\frac{\sqrt{13}}{2}$. Find the equation of the hyperbola.
6. A hyperbola has its centre at (−1, 4), a focus at (5, 4), and the corresponding directrix given by the line $x = -3$. Find the equation of the hyperbola.
7. A hyperbola has a vertical transverse axis, eccentricity $\frac{\sqrt{19}}{4}$, and latus rectum of length $\frac{3}{2}$. The hyperbola passes through the points (2, 8) and (2, 0). Find the equation of the hyperbola.
8. Find the equation of the hyperbola in standard form (centred at the origin) that passes through the points $(\sqrt{80}, 4)$ and $(\sqrt{32}, 2)$.
9. Find equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$.
10. Find the length of the latus rectum of the hyperbola $9x^2-16y^2=144$.
11. Find the eccentricity of the hyperbola whose asymptotes are perpendicular.
12. If the transverse axis of a hyperbola is 8 and the latus rectum is 9, find its eccentricity.
13. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. If the eccentricity of the hyperbola is 2, find its equation.
14. Find the eccentricity of the hyperbola whose latus rectum is half its transverse axis.
15. Find the equations of the normals to the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ which are parallel to the line $2x-y=5$.
16. If the distance between the foci of a hyperbola is 16 and the distance between its vertices is 12, find its eccentricity.
17. Find the length of the transverse axis of the hyperbola $4x^2-9y^2=36$.

18. A hyperbola for which $a = b$ is called rectangular. Show that a hyperbola is rectangular if and only if its asymptotes are perpendicular to one another.
19. Find the equations of the tangents to the hyperbola $7x^2 - 3y^2 = 105$ that pass through the point (3, 1).
20. Find the condition that the line $lx + my = 1$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

6.6 Applying Concepts of Conics To Real Life World Problems

Example 29 A comet has a parabolic orbit with the earth at the focus. When the comet is h km from the earth, the line joining the comet and the earth makes an angle of 30° with the axis of the parabola. How close will the comet come to the earth?

Solution: Let equation of the parabolic orbit be $y^2 = 4ax$. Let the centre of Earth $E(a, 0)$ be the focus. As point a parabola that is closest to the focus is it vertex. So, we need to find $|OE| = a$

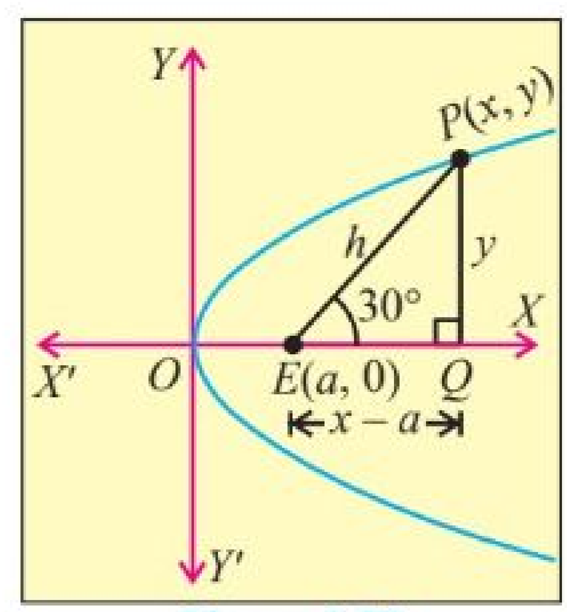


Figure 6.30

From right ΔEQP, $\cos 30° = \frac{x-a}{h} \Rightarrow \frac{\sqrt{3}}{2} = \frac{x-a}{h}$

$$\Rightarrow x - a = \frac{\sqrt{3}}{2}h \Rightarrow x = a + \frac{\sqrt{3}}{2}h$$

also $\sin 30° = \frac{y}{h} \Rightarrow \frac{1}{2} = \frac{y}{h} \Rightarrow y = \frac{h}{2}$

Since $\left(a + \frac{\sqrt{3}}{2}h, \frac{h}{2}\right)$ lies on the parabola $y^2 = 4ax$, it must satisfy it.

$$\left(\frac{h}{2}\right)^2 = 4a\left(a + \frac{\sqrt{3}}{2}h\right) \Rightarrow \frac{h^2}{4} = 4a^2 + 2\sqrt{3}ah$$

$$4a^2 + 2\sqrt{3}ah - \frac{h^2}{4} = 0$$

$$\Rightarrow \quad a = \frac{-2\sqrt{3}h \pm \sqrt{(2\sqrt{3}h)^2 - 4(4)\left(-\frac{h^2}{4}\right)}}{2(4)} = \frac{-2\sqrt{3}h \pm \sqrt{12h^2 + 4h^2}}{8}$$

$$= \frac{-2\sqrt{3}h \pm \sqrt{16h^2}}{8} = \frac{-2\sqrt{3}h \pm 4h}{8}$$

$$= \frac{2\left(-\sqrt{3} \pm 2\right)h}{8} = \frac{\left(-\sqrt{3} \pm 2\right)h}{4} \text{ Since, } \frac{-\sqrt{3}+2}{4} < 0 \text{ so, neglect it.}$$

Since minimum value of a is required, we have $a = \frac{\left(-\sqrt{3}+2\right)h}{4}$ km

Example 30 Elliptical Electron Orbit

In the Bohr–Sommerfeld model of the atom, electrons move in elliptical orbits around the nucleus, which is located at one focus. Consider an electron in an elliptical orbit with a semi-major axis $a = 5.29 \times 10^{-11}$ m (the Bohr radius) and an eccentricity $e = 0.20$.

(a) Find the distance from the nucleus to the centre of the ellipse.

(b) Determine the minimum distance (perihelion) and maximum distance (apoapsis) of the electron from the nucleus.

Solution: (a) For an ellipse, the distance from the centre to each focus is $c = ae$.

$$c = (5.29 \times 10^{-11}) \times 0.20 = 1.058 \times 10^{-11} \text{m}$$

Thus, the nucleus is 1.058×10^{-11} m from the centre of the ellipse.

(b) The minimum distance (periapsis) occurs when the electron is closest to the nucleus at the focus:

$$r_{\min} = a(1-e) = (5.29 \times 10^{-11}) \times (1 - 0.20) = (5.29 \times 10^{-11}) \times 0.80 = 4.232 \times 10^{-11} \text{ m}$$

The maximum distance (apoapsis) occurs when the electron is farthest from the nucleus:

$$r_{max} = a\left(1+e\right) = (5.29 \times 10^{-11}) \times (1 + 0.20) = (5.29 \times 10^{-11}) \times 1.20 = 6.348 \times 10^{-11} \text{ m}$$

Example 30 The cable of a suspension bridge is parabolic. The roadway is 10 m, below the lowest point of the cable. The span of the bridge is 200 m, and the tops of the piers are 50 m above the roadway. Find the equation of the parabola. If the load is supported by vertical chains at intervals of 10 m, find the length of the chain that is 20 m away from either pier.

Solution: Let BAC be the cable, $A(0, 0)$ the origin at the lowest point of the cable (vertex of the parabola), and DE be the road way.

Taking Ax as the x-axis the parabola opens upwards with vertex at (0, 0) and its equation is of the form:

$$x^2 = 4ay \qquad \ldots(1)$$

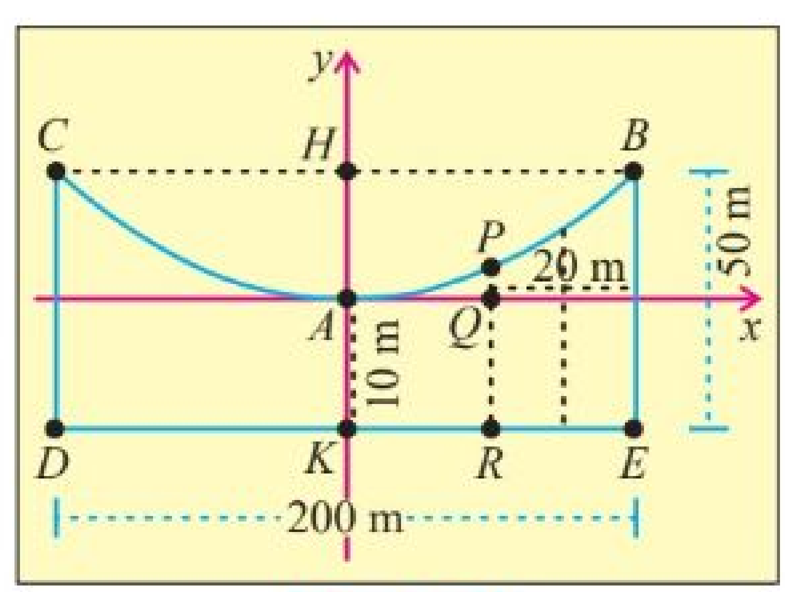


Figure 6.31

Since $HB = 100$ m and $AH = 50 - 10 = 40$ m, the coordinates of B are $(100, 40)$ which lie on (1).

Therefore, $(100)^2 = 4a(40) \Rightarrow 10000 = 4a(40)$

$$4a = \frac{10000}{40} = 250.$$

Hence equation of the parabola is $x^2 = 250y$ ...(2)

If PR be the chain at a distance 20 m from the pier BE, then

$$AQ = 100 - 20 = 80\text{m}$$

Since $P(80, PQ)$ lies on (2), $(80)^2 = 250 \cdot PQ \Rightarrow 6400 = 250 \cdot PQ$

$$\Rightarrow PQ = \frac{6400}{250} = 25.6\,\text{m}$$

Thus, the required length of the chain ($= PR$) is the vertical distance from the cable to the road way:

Length $= PR = PQ + QR = 25.6 + 10 = 35.6\,\text{m}$.

Example 32 The moon revolves around Earth in an elliptical orbit, with the centre of Earth located at one focus. The major and minor axis of the orbit have lengths of 768,800 km and 767,600 km, respectively. Find the greatest and least distances (the apogee and perigee) from Earth's centre to the moon's centre.

Solution: Begin by solving for a and b:

$2a = 768{,}800 \Rightarrow a = 384{,}400$ and $2b = 767{,}600$

$\Rightarrow b = 383{,}800$

$$c = \sqrt{a^2 - b^2} = \sqrt{(384400)^2 - (383800)^2}$$
$$= \sqrt{147763360000 - 147302440000}$$
$$= \sqrt{460920000}$$
$$= 21469.047 \approx 21469 \text{ km}$$

Figure 6.32

The greatest distance between the centre of Earth and the centre of the moon is

$$a + c \approx 384{,}400 + 21{,}469 = 405{,}869 \text{ km}$$

and the least distance is

$$a - c \approx 384{,}400 - 21{,}469 = 362{,}931 \text{km}.$$

Example 33 **Satellite Orbit**

The **apogee** (the point in orbit farthest from Earth) and the **perigee** (the point in orbit closest to Earth) of an elliptical orbit of an Earth satellite are given by A and P, respectively. Show that the eccentricity of the orbit is $e = \dfrac{A-P}{A+P}$.

Solution: Consider an elliptical orbit with Earth at one focus. Let:

a = semi-major axis of the ellipse, c = distance from the centre to each focus,

$e = \dfrac{c}{a}$ = eccentricity

The **perigee** P is the closest distance from the satellite to Earth. Since Earth is at one focus, this distance occurs when the satellite is at the vertex nearest that focus: $P = a - c$

The **apogee** A is the farthest distance from the satellite to Earth. This occurs when the satellite is at the opposite vertex: $A = a + c$

We have the system: $\begin{cases} A = a + c \\ P = a - c \end{cases}$

Add the two equations: $A + P = (a+c) + (a-c) = 2a \Rightarrow a = \dfrac{A+P}{2}$

Subtract the second equation from the first:

$$A - P = (a+c) - (a-c) = 2c \Rightarrow c = \frac{A-P}{2}$$

By definition, eccentricity is: $e = \dfrac{c}{a}$

Substitute the expressions for c and a: $e = \dfrac{\frac{A-P}{2}}{\frac{A+P}{2}} = \dfrac{A-P}{A+P}$

Thus, the eccentricity e of the elliptical orbit is given by: $e = \dfrac{A-P}{A+P}$.

Remark

- This formula holds for any elliptical orbit where the central body is at one focus

Example 34 A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 20 cm.

(a) Find an equation of the parabola.

(b) Find the diameter of the opening $|CD|$, 20 cm from the vertex.

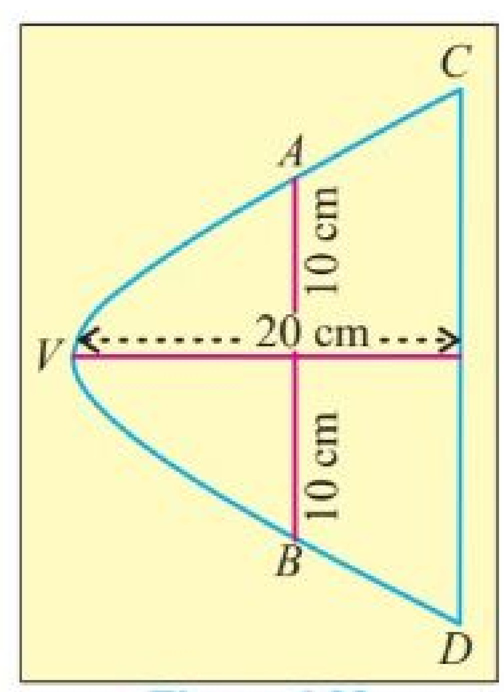


Figure 6.33

Solution: From Figure description:

- Cross-section of a parabolic reflector is a parabola
- The bulb is at the focus F
- The opening at the focus is 20 cm
  This means the width perpendicular to the axis at the focus is 20 cm
- 10 cm appears on each side of the axis in the diagram at the focus level
- Take vertex at the origin, axis horizontal (opening to the right)

Thus, the standard form is: $y^2 = 4ax$

where $4a = |AB| =$ length of latus rectum

Equation of parabola is $y^2 = 20x$

20 cm from vertex means $x = 20$ (since vertex at origin, axis along x-axis)

From $y^2 = 20x$: At $x = 20$: $y^2 = 20 \times 20 = 400 \Rightarrow y = \pm 20$

Half-width $= 20$ cm

Diameter $|CD| = 2(20) = 40$ cm.

Example 35 A parabolic flashlight reflector has the shape given by the equation:

$$x = \frac{1}{20}y^2$$

where should the lightbulb be placed so that the reflected rays emerge parallel to the axis?

Solution: The equation is given as: $x = \frac{1}{20}y^2 \Rightarrow y^2 = 20x$

This is a parabola that opens to the right with its vertex at the origin (0, 0).

The standard form of a parabola opening to the right is: $y^2 = 4ax$

Where:

- a is the distance from the vertex to the focus
- The focus is at $(a, 0)$
- The directrix is the vertical line $x = -a$

Compare $y^2 = 20x$ with $y^2 = 4ax$: $4a = 20 \Rightarrow a = 5$

For a parabola $y^2 = 4ax$, the focus is at $(a, 0)$

Substituting $a = 5$: Focus = (5, 0)

The lightbulb should be placed at the focus of the parabola. By the reflective property of parabolas, any light ray emitted from the focus will reflect off the parabolic surface and travel outward parallel to the axis (the x-axis).

Thus, the bulb should be located at: $(5, 0)$.

Remark:

The reflective property ensures that light from the focus reflects parallel to the axis, a principle used in flashlights, headlights, and satellite dishes.

EXERCISE 6.6

1. A comet has a parabolic orbit with the sun at the focus. When the comet is 100 million km from the sun, the line joining the sun and the comet makes an angle of 60° with the axis of the parabola. How close will the comet get to the sun?
2. Find an equation of the parabola formed by the cables of a suspension bridge whose span is ℓ m and the vertical height of the supporting towers is h m.
3. A parabolic arch has 120 m base and height 25 m. Find the height of the arch at a point 42 m from the centre of the base.

4. An athletic field features an elliptical running track. The track is 120 meters long along its major axis and 80 meters wide along its minor axis.

   (a) Find the distance from the centre to each focus of the ellipse.

   (b) If two water fountains are placed at the foci, how far apart are they?

5. A landscaper wants to create an elliptical flower bed with a major axis of 10 meters and a minor axis of 6 meters. Using the gardener's method, two stakes are placed at the foci, and a loop of string is stretched around them to trace the ellipse.

   (a) How far apart should the two stakes be placed?

   (b) What should be the total length of the string loop?

6. An architect designs a bridge with an elliptical arch. The arch has a span (width) of 24 meters and a height of 8 meters. The ellipse is centred at the origin with the major axis horizontal.

   (a) Write the equation of the ellipse representing the arch.

   (b) A boat needs to pass under the bridge. The boat is 6 meters wide. What is the maximum height of the boat that can pass through the arch if it must stay centred?

7. In the Bohr–Sompmerfeld model, an electron in an elliptical orbit around a nucleus obeys a modified form of Kepler's third law. For a hydrogen-like atom, the square of the orbital period T is proportional to the cube of the semi-major axis a.

   Consider an electron in an elliptical orbit with a semi-major axis $a = 2.12 \times 10^{-10}$ m (four times the Bohr radius) and an eccentricity $e = 0.30$. The constant of proportionality for hydrogen is $k = 2.96 \times 10^{-16} \text{s}^2/\text{m}^3$.

   (a) Determine the distance between the nucleus (at one focus) and the centre of the ellipse. (b) Calculate the orbital period T of the electron.

   (c) Find the maximum and minimum distances of the electron from the nucleus during its orbit.

8. LORAN (long distance radio navigation) for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci.

Assume that two stations, 500 miles apart, are positioned on a rectangular coordinate system at $(-250, 0)$ and $(250, 0)$ and that a ship is traveling on a path with coordinates $(x, 120)$ (a horizontal line 120 miles above the x-axis). Find the x-coordinate of the position of the ship when the time difference between the pulses from the transmitting stations is 1200 microseconds (0.0008 second).

9. A satellite orbits Earth in an elliptical path. The apogee (farthest distance from Earth) is $A = 42000$ km, and the perigee (closest distance to Earth) is $P = 30000$ km.
   (a) Find the semi-major axis a of the orbit.
   (b) Find the distance from the centre to each focus c.
   (c) Calculate the eccentricity e of the orbit.
10. The end towers of a suspension bridge are 100 meters high and 540 meters apart. The suspension cable hangs from the top of the towers in the form of a parabola and the lowest point of the cable is 25 meters above the horizontal road. Find the length of the supporting vertical cable at a horizontal distance of 81 meters from the centre of the bridge.
11. An architectural arch is built in the shape of the upper half of an ellipse with semi-major axis 8 meters and semi-minor axis 5 meters. A horizontal beam is to be placed across the arch at a height where the width of the arch is exactly 10 meters. Calculate the exact height above ground level where this beam should be installed.