Unit 05 Analytical Geometry

INTRODUCTION

Figure 5.1

Analytical geometry, also called coordinate geometry, is the branch of mathematics that combines algebra and geometry to study shapes, lines and figures using a coordinate system. Instead of just drawing or imagining figures, we describe them with equations and points on the *xy*-plane, which makes it easier to calculate distances, midpoints, slopes and intersections. This approach allows us to solve practical problems, like finding the shortest path, checking if two lines are parallel or locating the exact position of an object. Analytical geometry was first developed in the 17th century by the French mathematician René Descartes, who introduced the idea of using coordinates to represent geometric figures. His work laid the foundation for modern mathematics, as analytical geometry became the link between algebra and geometry, making it possible to study space and shapes with greater precision.

In this unit, we will study about straight lines, angles between coplanar intersecting lines, concurrency of medians, right bisectors and altitudes, area of triangular region and applications of analytical geometry to real life problems.

5.1 Equation of Straight Lines

We have learnt that in coordinate geometry, a straight line can be represented in different forms depending on the information given.

General Form

The general form of a straight line in two variables is written as $ax + by + c = 0$, where a, b and c are real numbers and a and b are not simultaneously zero.

Slope-Intercept Form

If m is the slope and c is the y-intercept of a line, then equation of this straight line in slope-intercept form is $y = mx + c$.

Point-Slope Form

If a line passes through a point (x_1, y_1) and has slope m, then equation of straight line in point slope form is $y - y_1 = m(x - x_1)$.

Two-Point Form

If a line passes through two points (x_1, y_1) and (x_2, y_2), then equation of straight line in two-point form is $\frac{y-y_2}{y_2-y_1}=\frac{x-x_2}{x_2-x_1}$ or $\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}$ or $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

Intercept Form

If a and b are the x-intercept and y-intercept of the line, then equation of straight line in intercept form is $\frac{x}{a}+\frac{y}{b}=1$.

Normal Form

$x\cos\theta + y\sin\theta = p$ is an equation of the straight line, where p is the perpendicular distance of the line from the origin and θ is the angle made by perpendicular with the positive x-axis.

Example 1 Find an equation of the straight line if its slope is 3 and y-intercept is 7.

Solution: The slope and y-intercept of the line are respectively:

$$m = 3, c = 7$$

Thus, slope-intercept form of a line is:

$$y = mx + c$$

$$y = 3x + 7$$

Example 2 Write down an equation of the straight line passing through (–4, 3) and having slope –5.

Solution: As the point (–4, 3) lies on the required line having slope –5, so by point-slope form of equation of the straight line, we have

$$y-3=-5(x-(-4)) \text{ or } y-3=-5(x+4) \text{ or } y-3=-5x-20 \text{ or } 5x+y+17=0$$

which is an equation of the required line.

Example 3 Find an equation of the straight line passing through the points (–3, –2) and (5, 4).

Solution: Using two-point form of an equation of the straight line, the required equation is

$$\frac{y-(-2)}{4-(-2)}=\frac{x-(-3)}{5-(-3)}$$

$$\frac{y+2}{6}=\frac{x+3}{8}$$

or $$y+2=\frac{6}{8}(x+3) \text{ or } 8(y+2)=6(x+3)$$

or $$8y+16=6x+18 \quad \text{or} \quad 6x-8y+2=0 \quad \text{or} \quad 3x-4y+1=0$$

Example 4 Write down an equation of the straight line which cuts the x-axis at (3, 0) and y-axis at (0, –6).

Solution: As 3 and –6 are x and y-intercepts respectively of the required line, so by two-intercepts form of an equation of the straight line, we have

$$\frac{x}{3}+\frac{y}{-6}=1$$

$$2x-y=6$$

$$2x-y-6=0$$

which is the required equation.

Example 5 The length of perpendicular from the origin to a line is 3 units and the inclination of this perpendicular is 120°. Find the equation of the straight line.

Solution: Here, $p = 3,\ \theta = 120^\circ$.

Equation of the straight line in normal form is:

$$x\cos\theta+y\sin\theta=p$$

$$x\cos 120^\circ+y\sin 120^\circ=3$$

$$-\frac{1}{2}x+\frac{\sqrt{3}}{2}y=3$$

$$x-\sqrt{3}y=-6$$

$$x-\sqrt{3}y+6=0$$

5.2 A Linear Equation in Two Variables Represents a Straight Line

Theorem 1: The linear equation $ax + by + c = 0$ in two variables x and y represents a straight line, where a, b and c are real numbers and a and b are not simultaneously zero.

Proof: Here, a and b cannot be both zero. So, the following cases arise:

Case-I: When $a = 0$

Then $ax + by + c = 0$ becomes

$$by+c=0 \quad \text{or} \quad y=-\frac{c}{b}$$

which is an equation of the straight line parallel to x-axis.

Case-II: When $b = 0$

Then $ax + by + c = 0$ becomes

$$ax+c=0 \quad \text{or} \quad x=-\frac{c}{a}$$

which is an equation of straight line parallel to y-axis.

Case-III: When $a \neq 0,\ b \neq 0$

Then $ax + by + c = 0$ takes the form

$$by=-ax-c \quad \text{or} \quad y=-\frac{a}{b}x-\frac{c}{b}=mx+k$$

which is slope-intercept form of the straight line with slope $-\frac{a}{b}$ and y-intercept $-\frac{c}{b}$.

Thus, equation $ax + by + c = 0$ always represents a straight line.

5.2.1 Transform the General Linear Equation to Standard Forms

Theorem 2: To transform the equation $ax + by + c = 0$ in the standard forms.

(i) Slope-Intercept Form

We have $ax + by + c = 0$

or $by = -ax - c$

or $y = -\frac{a}{b}x - \frac{c}{b} = mx + k$ (where, $b \neq 0$)

where $m = -\frac{a}{b}$, $k = -\frac{c}{b}$

(ii) Point-Slope Form

We note from above that slope of the line $ax + by + c = 0$ is $-\frac{a}{b}$ and y–intercept is $-\frac{c}{b}$.

So, the point $\left(-\frac{c}{a}, 0\right)$ lies on the line.

Equation of the straight line in point-slope form is $y - 0 = -\frac{a}{b}\left(x + \frac{c}{a}\right)$

(iii) Symmetric Form

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \text{ (say)}$$

$$m = \tan\theta = -\frac{a}{b},\ \sin\theta = \frac{a}{\pm\sqrt{a^2 + b^2}},\ \cos\theta = \frac{b}{\pm\sqrt{a^2 + b^2}}$$

A point on $ax + by + c = 0$ is $\left(-\frac{c}{a}, 0\right)$.

So, equation of the straight line in symmetric form is:

$$\frac{x - \left(-\frac{c}{a}\right)}{\frac{b}{\pm\sqrt{a^2 + b^2}}} = \frac{y - 0}{\frac{a}{\pm\sqrt{a^2 + b^2}}} = r$$

Sign of the radical to be properly chosen.

(iv) Two-Point Form

We choose two arbitrary points on $ax + by + c = 0$. Two such points are $\left(-\frac{c}{a}, 0\right)$ and $\left(0, -\frac{c}{b}\right)$. Equation of the straight line through these points is

$$\frac{y-0}{0+\frac{c}{b}}=\frac{x+\frac{c}{a}}{-\frac{c}{a}-0} \quad \text{or} \quad y-0=\frac{\frac{c}{b}}{-\frac{c}{a}}\left(x+\frac{c}{a}\right) \quad \text{or} \quad y=-\frac{a}{b}\left(x+\frac{c}{a}\right)$$

(v) Two-Intercept Form

$$ax+by=-c$$

$$\text{or} \quad \frac{ax}{-c}+\frac{by}{-c}=1 \quad \text{or} \quad \frac{x}{\left(-\frac{c}{a}\right)}+\frac{y}{\left(-\frac{c}{b}\right)}=1$$

where x-intercept $=-\frac{c}{a}$, y-intercept $=-\frac{c}{b}$

which is an equation of straight line in two-intercept form.

(vi) Normal Form

The equation $ax+by+c=0$ ...(i)

can be written in the normal form as:

$$\frac{ax+by}{\pm\sqrt{a^2+b^2}}=\frac{-c}{\pm\sqrt{a^2+b^2}} \quad \text{...(ii)}$$

The sign of the radical to be such that the right hand side of (ii) is positive.

Proof: We know that an equation of a straight line in normal form is:

$$x\cos\theta+y\sin\theta=p \quad \text{...(iii)}$$

If (i) and (iii) are identical, we must have

$$\frac{a}{\cos\theta}=\frac{b}{\sin\theta}=-\frac{c}{p}$$

$$\text{i.e.} \quad \frac{p}{-c}=\frac{\cos\theta}{a}=\frac{\sin\theta}{b}=\frac{\sqrt{\cos^2\theta+\sin^2\theta}}{\pm\sqrt{a^2+b^2}}=\frac{1}{\pm\sqrt{a^2+b^2}}$$

Hence, $\cos\theta=\frac{a}{\pm\sqrt{a^2+b^2}}$ and $\sin\theta=\frac{b}{\pm\sqrt{a^2+b^2}}$ and $p=\frac{-c}{\pm\sqrt{a^2+b^2}}$

Substituting values of $\cos\theta$, $\sin\theta$ and p in (iii), we have:

$$\frac{ax+by}{\pm\sqrt{a^2+b^2}}=\frac{-c}{\pm\sqrt{a^2+b^2}}$$

Thus (i) can be reduced to the form (ii) by dividing it by $\pm\sqrt{a^2+b^2}$. The sign of the radical to be chosen so that the right hand side of (ii) is positive.

Example 6 Transform the equation $3x - 10y + 13 = 0$ into

(i) Slope-intercept form (ii) Two-intercept form (iii) Normal form
(iv) Point-slope form (v) Two-point form (vi) Symmetric form

Solution: (i) $3x - 10y + 13 = 0$

$$10y = 3x + 13$$

$$y = \frac{3}{10}x + \frac{13}{10} \quad \left(\text{where,} \quad m = \frac{3}{10},\ c = \frac{13}{10}\right)$$

(ii) $3x - 10y + 13 = 0$

or $3x - 10y = -13$

or $\frac{3x}{-13} + \frac{10}{13}y = 1$

or $\frac{x}{-\frac{13}{3}} + \frac{y}{\frac{13}{10}} = 1$

(iii) $3x - 10y = -13$

Dividing both sides by $\pm\sqrt{(3)^2 + (-10)^2} = \pm\sqrt{109}$.

Since R.H.S is to be positive, we have to take negative sign.

Hence, $\frac{-3x}{\sqrt{109}} + \frac{10y}{\sqrt{109}} = \frac{13}{\sqrt{109}}$ is the normal form of the given equation.

(iv) A point on the line is $\left(-\frac{13}{3}, 0\right)$ and its slope is $\frac{3}{10}$.

Equation of line in point slope-form is:

$$y - 0 = \frac{3}{10}\left(x + \frac{13}{3}\right)$$

(v) We take any two points $\left(-\frac{13}{3}, 0\right)$ and $\left(0, \frac{13}{10}\right)$ on the line.

So, equation of the line in two-point form is:

$$\frac{y-0}{\frac{13}{10} - 0} = \frac{x + \frac{13}{3}}{0 + \frac{13}{3}}$$

(vi) We have $\tan\theta = \frac{3}{10} = m$, $\sin\theta = \frac{3}{\sqrt{109}}$, $\cos\theta = \frac{10}{\sqrt{109}}$ and a point on the line is $\left(-\frac{13}{3}, 0\right)$.

Equation of the line in symmetric form is:

$$\frac{x + \frac{13}{3}}{\frac{10}{\sqrt{109}}} = \frac{y - 0}{\frac{3}{\sqrt{109}}} = r \quad (\text{say})$$

Challenge!

Transform the equation $2x + 5y - 10 = 0$ into slope-intercept form, two-intercept form, normal form, point-slope form, two-point form and symmetric form.

5.3 Equations of Medians, Altitudes and Right Bisectors

Example 7 Triangle ABC has vertices $A(2, 3)$, $B(4, 5)$ and $C(-2, 7)$. Find the equations of medians of the triangle ABC.

Solution: Given that $A(2, 3)$, $B(4, 5)$ and $C(-2, 7)$

Let D, E and F be the mid-points of the sides BC, AC and AB respectively.

$$D\left(\frac{4+(-2)}{2}, \frac{5+7}{2}\right) = D(1,6)$$

$$E\left(\frac{2+(-2)}{2}, \frac{3+7}{2}\right) = E(0,5)$$

$$F\left(\frac{2+4}{2}, \frac{3+5}{2}\right) = F(3,4)$$

Remember!

A median joins the vertex of a triangle to the mid-point of the opposite side.

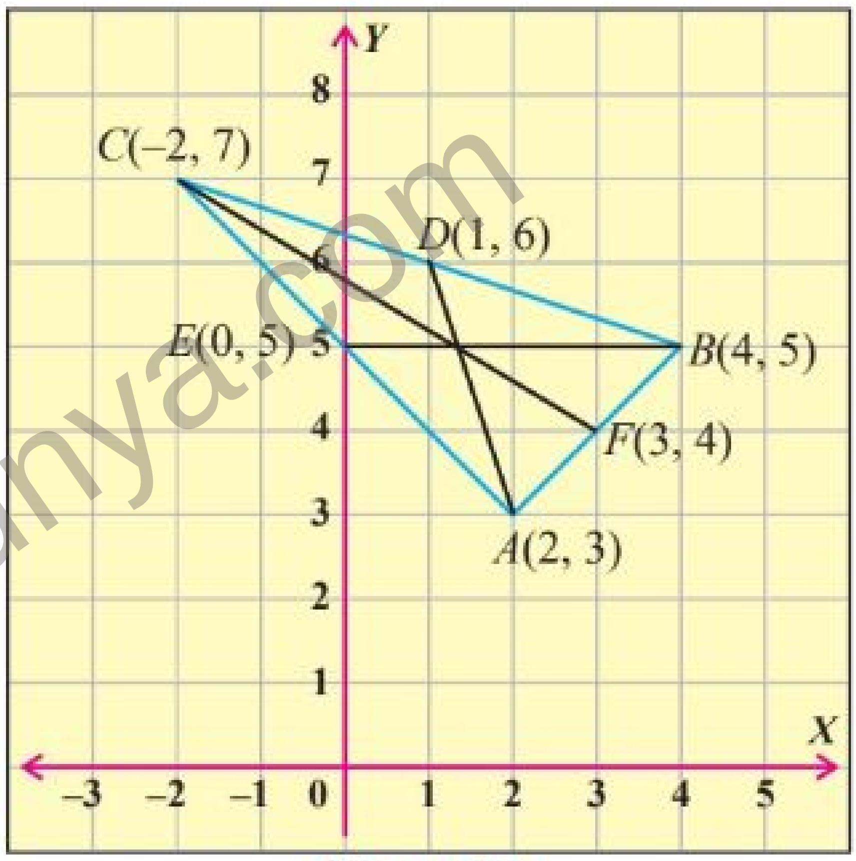

Figure 5.2

Equation of the median AD is

$$\frac{y-3}{3-6} = \frac{x-2}{2-1}$$

or $$y-3 = -3(x-2)$$

or $$3x + y - 9 = 0$$

Equation of the median BE is

$$\frac{y-5}{5-5} = \frac{x-4}{4-0}$$

or $$y-5 = 0$$

Equation of the median CF is

$$\frac{y-7}{7-4} = \frac{x+2}{-2-3}$$

or $$-5(y-7) = 3(x+2)$$

or $$3x + 5y - 29 = 0$$

Example 8 $A(1, 2)$, $B(5, -1)$ and $C(-2, -4)$ are the vertices of ΔABC. Find the equations of altitudes of triangle ABC.

Solution: Given that: $A(1, 2)$, $B(5, -1)$, $C(-2, -4)$.

Let $\overline{AE}$, $\overline{BF}$ and $\overline{CD}$ be the altitudes of the sides BC, CA and AB of the triangle respectively.

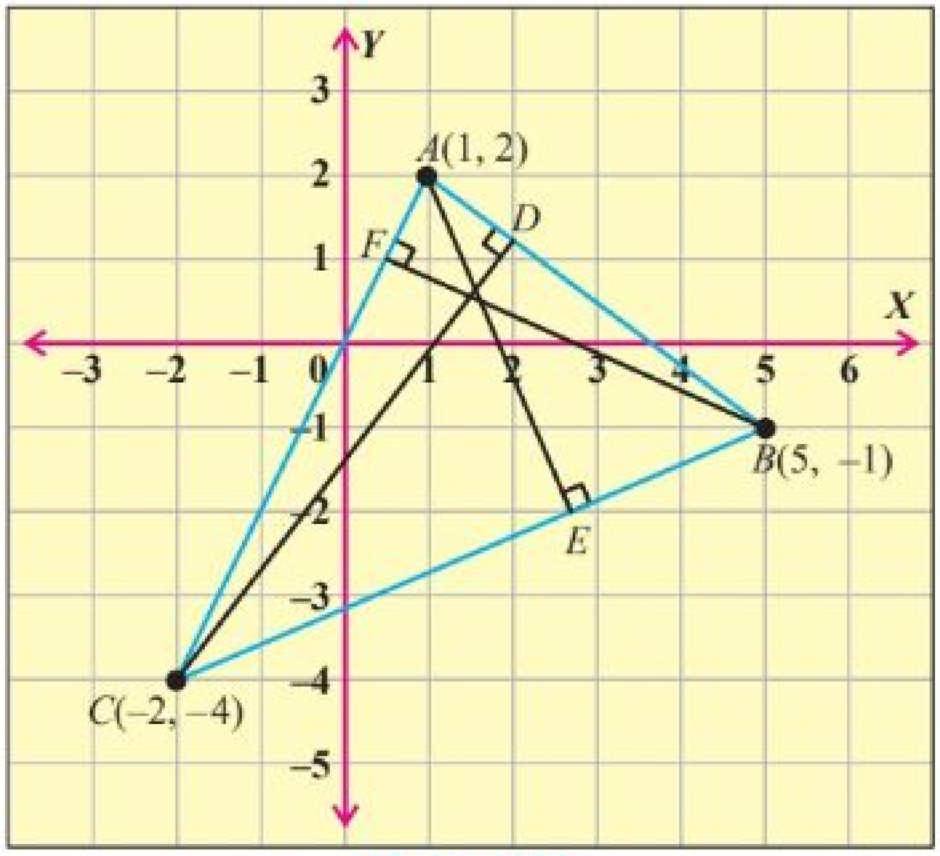

Figure 5.3

$$\text{Slope of } \overline{BC} = \frac{-4-(-1)}{-2-5} = \frac{3}{7}$$

$$\text{Slope of } \overline{AE} = \frac{-1}{\text{Slope of } \overline{BC}} = \frac{-7}{3}$$

Equation of the altitude AE is

$$y-2 = \frac{-7}{3}(x-1)$$

or $\quad 3y-6 = -7x+7$

$$7x+3y-13=0$$

$$\text{Slope of } \overline{AB} = \frac{-1-2}{5-1} = -\frac{3}{4}$$

$$\text{Slope of } \overline{CD} = \frac{-1}{\text{Slope of } \overline{AB}} = \frac{4}{3}$$

Equation of the altitude CD is

$$y+4 = \frac{4}{3}(x+2)$$

or $\quad 4x-3y-4=0$

$$\text{Slope of } \overline{CA} = \frac{2-(-4)}{1-(-2)} = \frac{6}{3} = 2$$

$$\text{Slope of } \overline{BF} = \frac{-1}{\text{Slope of } \overline{CA}} = \frac{-1}{2}$$

Equation of the altitude BF is

$$y+1 = \frac{-1}{2}(x-5)$$

or $\quad x+2y-3=0$

Remember!

An altitude is the line drawn from a vertex perpendicular to the opposite side.

Remember!

If m_1 and m_2 be the slopes of perpendicular lines then $m_1 m_2 = -1$.

Example 9: Triangle ABC has vertices $A(-3, 2)$, $B(5, 4)$ and $C(3, -8)$. Find the equations of right bisectors of triangle ABC.

Solution: Given that

$A(-3, 2)$, $B(5, 4)$, $C(3, -8)$

Let D, E and F be the mid-points of the sides BC, CA and AB respectively.

$$D\left(\frac{5+3}{2}, \frac{4+(-8)}{2}\right) = D(4,-2)$$

$$E\left(\frac{-3+3}{2}, \frac{2+(-8)}{2}\right) = E(0,-3)$$

$$F\left(\frac{-3+5}{2}, \frac{2+4}{2}\right) = F(1,3)$$

$$\text{Slope of } \overline{AB} = \frac{4-2}{5-(-3)} = \frac{2}{8} = \frac{1}{4}$$

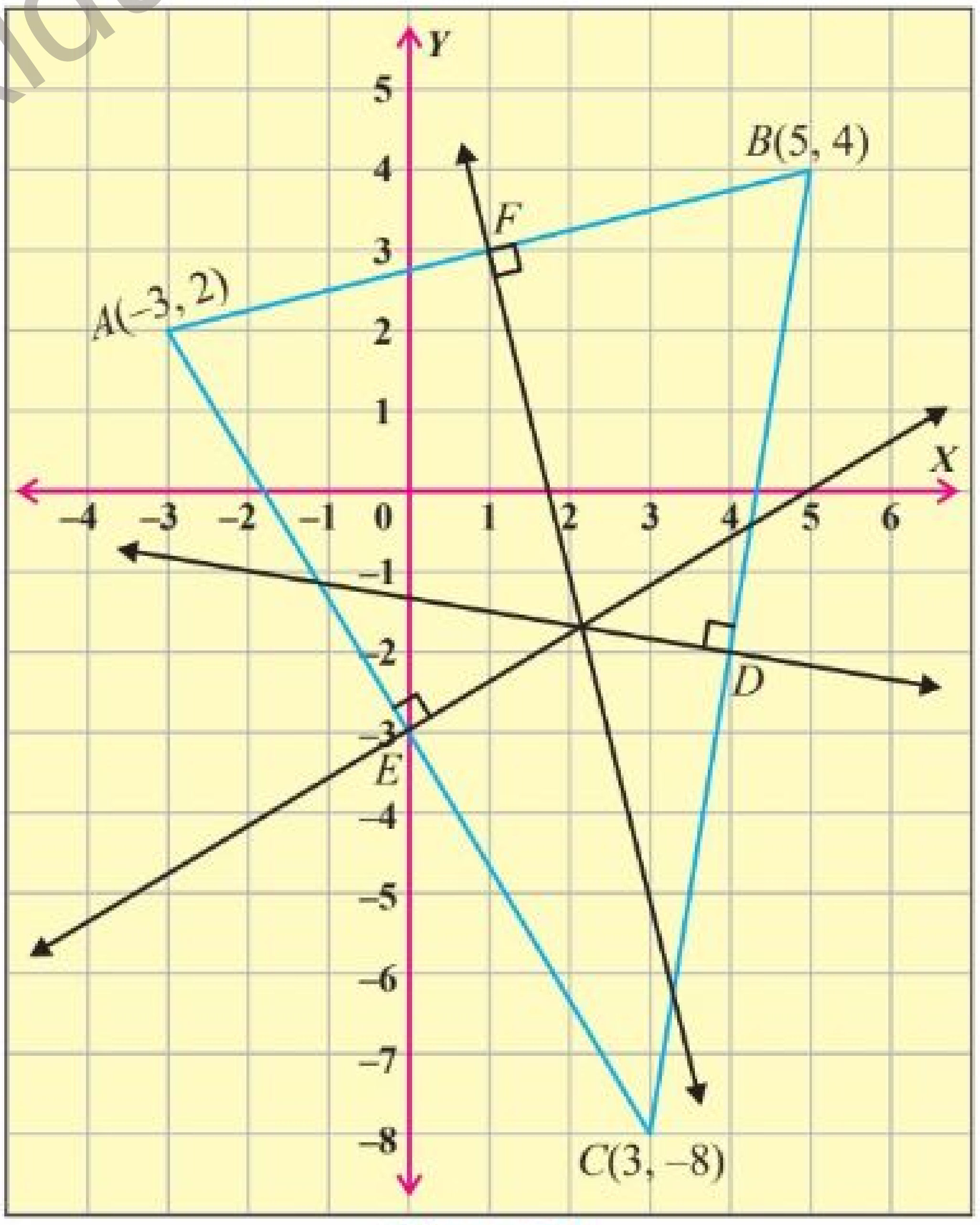

Figure 5.4

Slope of right bisector to $\overline{AB} = \dfrac{-1}{\text{Slope of } \overline{AB}} = -4$

Equation of the right bisector to $\overline{AB}$ is:

$$y - 3 = -4(x - 1) \quad \text{or} \quad 4x + y - 7 = 0$$

$$\text{Slope of } \overline{BC} = \frac{-8-4}{3-5} = \frac{-12}{-2} = 6$$

Slope of right bisector to $\overline{BC} = \dfrac{-1}{\text{Slope of } \overline{BC}} = -\dfrac{1}{6}$

Equation of the right bisector to $\overline{BC}$ is:

$$y + 2 = -\frac{1}{6}(x - 4)$$

or $\quad x + 6y + 8 = 0$

$$\text{Slope of } \overline{CA} = \frac{2-(-8)}{-3-3} = \frac{10}{-6} = -\frac{5}{3}$$

Slope of right bisector to $\overline{CA} = \dfrac{-1}{\text{Slope of } \overline{CA}} = \dfrac{3}{5}$

Equation of the right bisector to $\overline{CA}$ is

$$y + 3 = \frac{3}{5}(x - 0) \quad \text{or} \quad 3x - 5y - 15 = 0$$

Remember!

A perpendicular or right bisector of a triangle is a line that is perpendicular to one side of the triangle and passes through the mid point of that side.

5.4 Condition of Concurrency of Three Straight Lines

Theorem 3: Three non-parallel lines

$$\ell_1 : a_1x + b_1y + c_1 = 0 \qquad \text{...(i)}$$

$$\ell_2 : a_2x + b_2y + c_2 = 0 \qquad \text{...(ii)}$$

$$\ell_3 : a_3x + b_3y + c_3 = 0 \qquad \text{...(iii)}$$

are concurrent iff $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Proof: If the lines are concurrent, then they have a common point of intersection $P(x_1, y_1)$ say. As $\ell_1 \nparallel \ell_2$, so their point of intersection (x, y) is:

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \; y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} \qquad (\because a_1b_2 - a_2b_1 \neq 0)$$

This point also lies on (iii), so

$$a_3\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}\right)+c_3=0$$

or $\quad a_3(b_1c_2-b_2c_1)+b_3(a_2c_1-a_1c_2)+c_3(a_1b_2-a_2b_1)=0$

An easier way to write the above equation is in the following determinant form:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

This is necessary and sufficient condition iff of concurrency of given three lines.

Example 10: Check whether the following lines are concurrent or not. If concurrent, then find the point of concurrency.

$$3x-2y-2=0$$
$$x+y-4=0$$
$$2x-y-2=0$$

Solution: The determinant of the coefficients of the given equations is:

$$\begin{vmatrix} 3 & -2 & -2 \\ 1 & 1 & -4 \\ 2 & -1 & -2 \end{vmatrix}=3\begin{vmatrix} 1 & -4 \\ -1 & -2 \end{vmatrix}-(-2)\begin{vmatrix} 1 & -4 \\ 2 & -2 \end{vmatrix}-2\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$$

$$=3(-2-4)+2(-2+8)-2(-1-2)$$

$$=3(-6)+2(6)-2(-3)=-18+12+6=0$$

Thus, the given lines are concurrent.

As we know that the point of intersection of any two lines is the point of concurrency.

From $x+y-4=0$ and $2x-y-2=0$, we have

$3x-6=0 \;\Rightarrow\; x=2$

Put $x=2$ in $x+y-4=0$, we have

$2+y-4=0 \quad\Rightarrow\quad y=2$

So, (2, 2) is the point of concurrency.

Challenge!

If the lines $2x \quad 3y+5=0$, $3x+4y-1=0$ and $kx-7y+6=0$ are concurrent, then find the value of k.

5.5 Equation of Lines Through the Point of Intersection of Two Lines

We can find a family of lines through the point of intersection of two non-parallel lines ℓ_1 and ℓ_2.

Let $\quad \ell_1 : a_1x+b_1y+c_1=0 \quad$ ...(i)

and $\quad \ell_2 : a_2x+b_2y+c_2=0 \quad$ ...(ii)

For a non-zero real k, consider the equation

$$a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0 \qquad ...(iii)$$

This, being a linear equation, represents a straight line. For different values of k, (iii) represents different lines. Thus (iii) is a family of lines.

If (x_1, y_1) is any point lying on both (i) and (ii), then it is their point of intersection. Since, (x_1, y_1) lies on both (i) and (ii), we have

$$a_1x_1+b_1y_1+c_1=0 \quad \text{and} \quad a_2x_1+b_2y_1+c_2=0$$

From the above two equations, we note that (x_1, y_1) also lies on (iii).

Thus, (iii) is the required family of lines through the point of intersection of (i) and (ii). Since, k can assume an infinite number of values, (iii) represents an infinite number of lines.

A particular line of the family (iii) can be determined if one more condition is given.

Example 11 Find the family of lines through the point of intersection of the lines

$$2x-3y-14=0 \quad \text{and} \quad 2x+y-10=0$$

Find the member of the family which is:

(a) parallel to the line with slope $-\frac{1}{2}$

(b) perpendicular to the line $2x-3y+2=0$

Solution: (a) Given that $2x-3y-14=0$ ...(i)

$2x+y-10=0$ ...(ii)

A family of lines through the point of intersection of equations (i) and (ii) is:

$$2x-3y-14+k(2x+y-10)=0$$

or $(2+2k)x+(k-3)y+(-14-10k)=0$ ...(iii)

Slope of (iii) is given by: $m=-\frac{2+2k}{k-3}$

This is the slope of any member of the family (iii). If (iii) is parallel to the line with slope $-\frac{1}{2}$, then

$$-\frac{2+2k}{k-3}=-\frac{1}{2} \quad \text{or} \quad 4+4k=k-3 \quad \text{or} \quad 3k=-7 \quad \text{or} \quad k=-\frac{7}{3}$$

Put $k=-\frac{7}{3}$ in (iii), we have

$$\left(2+2\left(-\frac{7}{3}\right)\right)x+\left(-\frac{7}{3}-3\right)y+\left(-14-10\left(-\frac{7}{3}\right)\right)=0$$

$$\left(\frac{6-14}{3}\right)x+\left(\frac{-7-9}{3}\right)y+\left(\frac{-42+70}{3}\right)=0$$

$$\left(-\frac{8}{3}\right)x+\left(-\frac{16}{3}\right)y+\left(\frac{28}{3}\right)=0$$

$$-8x-16y+28=0$$

$$2x+4y-7=0$$

which is the required equation of the member of the family.

(b) Slope of $2x - 3y + 2 = 0$ ...(iv)

is $\frac{2}{3}$. Since (iii) is perpendicular to (iv) given,

we have $-\frac{2+2k}{k-3}\times\frac{2}{3}=-1$ or $4 + 4k = 3k - 9$

$$k=-13$$

Substituting $k = -13$ in (iii), we get

$-24x - 16y + 116 = 0$ or $6x + 4y - 29 = 0$

which is the required equation of line.

Theorem 4: Altitudes of a triangle are concurrent.

Proof: Let the coordinates of the vertices of ΔABC be as shown in figure.

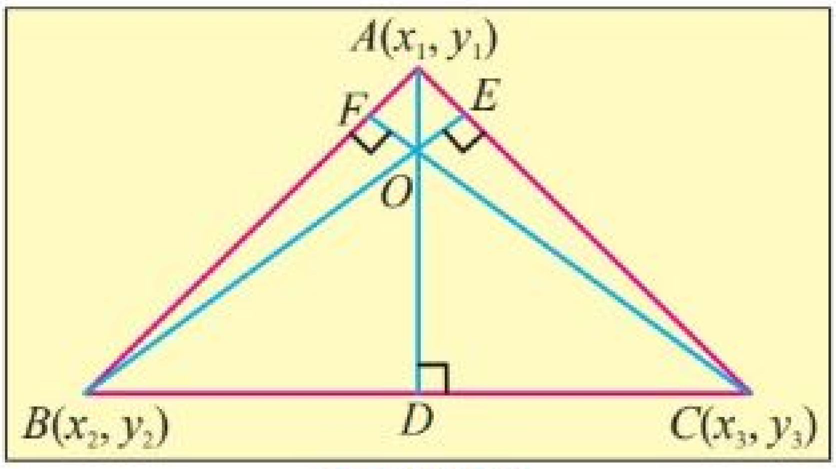

Figure 5.5

Then, slope of $\overline{BC}=\frac{y_2-y_3}{x_2-x_3}$

Therefore, slope of the altitude $AD=-\frac{x_2-x_3}{y_2-y_3}$

Equation of the altitude AD is

$y-y_1=-\frac{x_2-x_3}{y_2-y_3}(x-x_1)$ (point-slope form)

or $x(x_2-x_3)+y(y_2-y_3)-x_1(x_2-x_3)-y_1(y_2-y_3)=0$ ...(i)

Equations of the altitudes BE and CF are respectively (by symmetry)

$x(x_3-x_1)+y(y_3-y_1)-x_2(x_3-x_1)-y_2(y_3-y_1)=0$ ...(ii)

and $x(x_1-x_2)+y(y_1-y_2)-x_3(x_1-x_2)-y_3(y_1-y_2)=0$ ...(iii)

The three lines (i), (ii) and (iii) are concurrent if and only if

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -x_1(x_2-x_3)-y_1(y_2-y_3) \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-x_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix}$$

Adding 2nd and 3rd rows to the 1st row of the determinant, we have

$$\begin{vmatrix} 0 & 0 & 0 \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-x_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix}=0$$

Thus, the altitudes of the triangle are concurrent.

Theorem 5: Right bisectors of a triangle are concurrent.

Proof: Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC.

The mid-point D of BC has coordinates $\left(\dfrac{x_2+x_3}{2}, \dfrac{y_2+y_3}{2}\right)$.

Since the slope of $\overline{BC}$ is $\dfrac{y_2-y_3}{x_2-x_3}$, the slope of the right bisector DO of BC is $-\dfrac{x_2-x_3}{y_2-y_3}$.

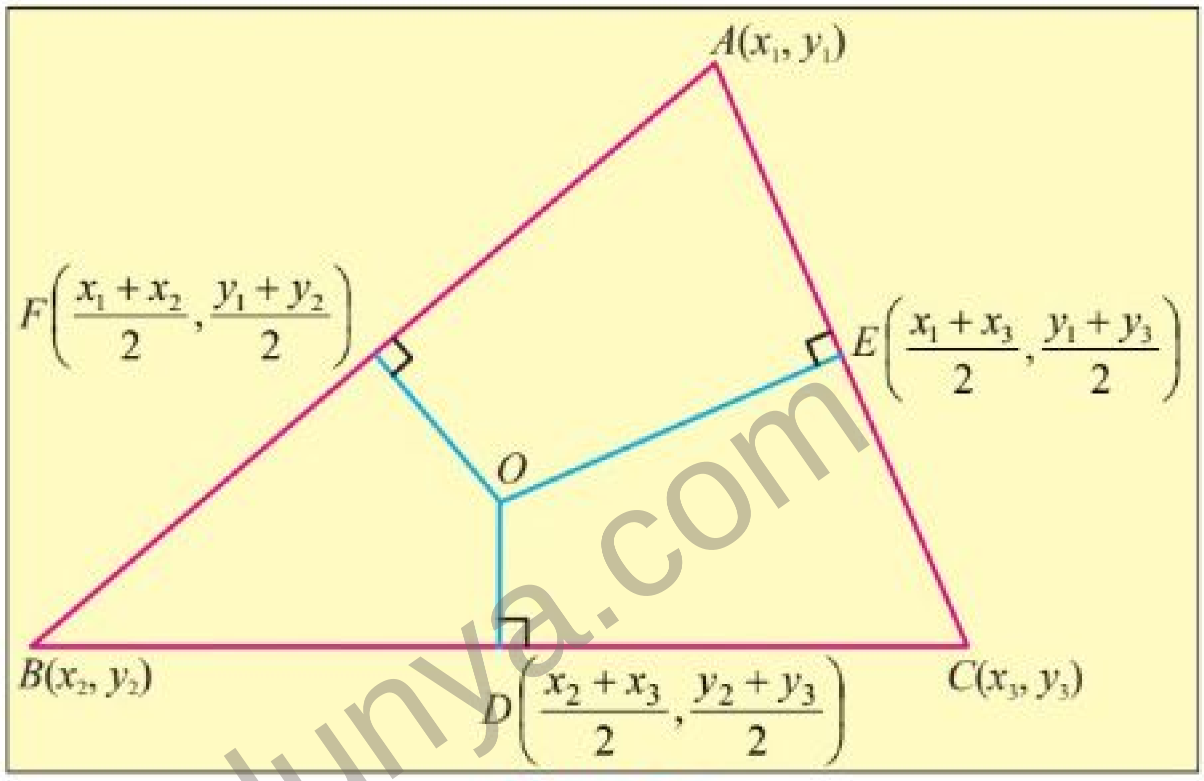

Figure 5.6

Equation of the right bisector DO of BC is:

$$y-\frac{y_2+y_3}{2}=-\frac{x_2-x_3}{y_2-y_3}\left(x-\frac{x_2+x_3}{2}\right), \text{(By using, point-slope form)}$$

or $$x(x_2-x_3)+y(y_2-y_3)-\frac{1}{2}\left(y_2^2-y_3^2\right)-\frac{1}{2}\left(x_2^2-x_3^2\right)=0$$

By symmetry, equations of the other two right bisectors EO and FO are respectively:

$$x(x_3-x_1)+y(y_3-y_1)-\frac{1}{2}\left(y_3^2-y_1^2\right)-\frac{1}{2}\left(x_3^2-x_1^2\right)=0 \quad \text{...(ii)}$$

and $$x(x_1-x_2)+y(y_1-y_2)-\frac{1}{2}\left(y_1^2-y_2^2\right)-\frac{1}{2}\left(x_1^2-x_2^2\right)=0 \quad \text{...(iii)}$$

The lines (i), (ii) and (iii) will be concurrent if and only if

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -\frac{1}{2}\left(y_2^2-y_3^2\right)-\frac{1}{2}\left(x_2^2-x_3^2\right) \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}\left(y_3^2-y_1^2\right)-\frac{1}{2}\left(x_3^2-x_1^2\right) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}\left(y_1^2-y_2^2\right)-\frac{1}{2}\left(x_1^2-x_2^2\right) \end{vmatrix}=0$$

Adding 2nd and 3rd row to 1st row of the determinant, we have

$$\begin{vmatrix} 0 & 0 & 0 \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}\left(y_3^2-y_1^2\right)-\frac{1}{2}\left(x_3^2-x_1^2\right) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}\left(y_1^2-y_2^2\right)-\frac{1}{2}\left(x_1^2-x_2^2\right) \end{vmatrix}=0$$

Thus, the right bisectors of a triangle are concurrent.

Theorem 6: Medians of the triangle are concurrent.

Proof: Let $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,\ y_3)$ be the vertices of ΔABC.

The mid-point D of BC has coordinates $\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right)$.

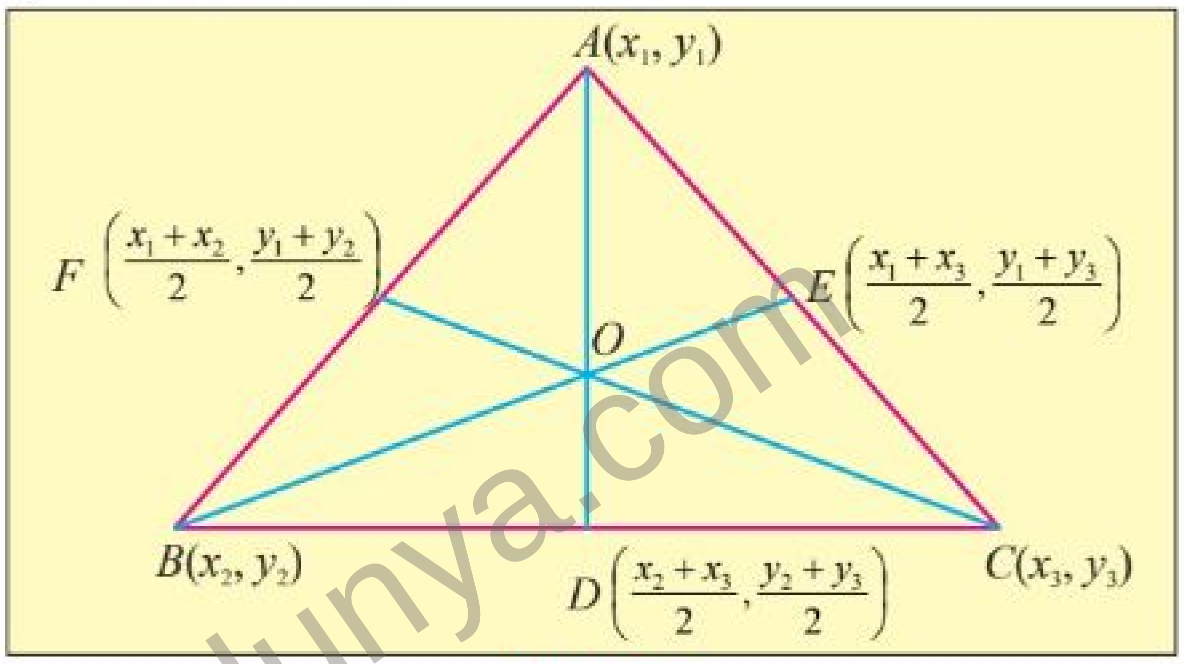

Figure 5.7

Equation of the median AD is

$$\frac{y-y_1}{\frac{y_2+y_3}{2}-y_1}=\frac{x-x_1}{\frac{x_2+x_3}{2}-x_1} \text{ (Two-point form)}$$

or $\quad x(y_2+y_3-2y_1)+y(2x_1-x_2-x_3)-x_1(y_2+y_3-2y_1)+y_1(x_2+x_3-2x_1)=0$ ...(i)

By symmetry, equations of the other two medians BE and CF are respectively:

$$x(y_3+y_1-2y_2)+y(2x_2-x_3-x_1)-x_2(y_3+y_1-2y_2)+y_2(x_3+x_1-2x_2)=0 \text{...(ii)}$$

$$x(y_1+y_2-2y_3)+y(2x_3-x_1-x_2)-x_3(y_1+y_2-2y_3)+y_3(x_1+x_2-2x_3)=0 \text{...(iii)}$$

The lines (i), (ii) and (iii) will be concurrent if and only if

$$\begin{vmatrix} y_2+y_3-2y_1 & 2x_1-x_2-x_3 & -x_1(y_2+y_3-2y_1)+y_1(x_2+x_3-2x_1) \\ y_3+y_1-2y_2 & 2x_2-x_3-x_1 & -x_2(y_3+y_1-2y_2)+y_2(x_3+x_1-2x_2) \\ y_1+y_2-2y_3 & 2x_3-x_1-x_2 & -x_3(y_1+y_2-2y_3)+y_3(x_1+x_2-2x_3) \end{vmatrix}=0$$

Adding 2nd and 3rd row to 1st row, we have

$$\begin{vmatrix} 0 & 0 & 0 \\ y_3+y_1-2y_2 & 2x_2-x_3-x_1 & -x_2(y_3+y_1-2y_2)+y_2(x_3+x_1-2x_2) \\ y_1+y_2-2y_3 & 2x_3-x_1-x_2 & -x_3(y_1+y_2-2y_3)+y_3(x_1+x_2-2x_3) \end{vmatrix}=0$$

Thus, the medians of a triangle are concurrent.

EXERCISE 5.1

1. Find equation of the altitude of triangle XYZ from vertex $X(1, 3)$, $Y(5, 7)$ and $Z(-3, 9)$.
2. $A(-3, -7)$, $B(4, 5)$ and $C(-1, 2)$ are the vertices of ΔABC. Find the equation of right bisector of side BC.
3. The vertices of ΔPQR are $P(1, 2)$, $Q(5, -6)$ and $R(7, -2)$. Find the equations of all three medians.
4. Show that the medians of the triangle with vertices $A(3, 2)$, $B(-1, -5)$ and $C(7, 2)$ are concurrent.
5. The vertices of a triangle are $A(2, -1)$, $B(4, 3)$ and $C(6, -1)$. Find the equations of altitudes and show that they are concurrent.
6. Find the equations of medians, altitudes and right bisectors of the triangle whose vertices are $A(2, 5)$, $B(-3, 4)$ and $C(6, -8)$.
7. Convert each of the following equation into

   (a) $2x + 3y - 6 = 0$ (b) $x - 2y + 4 = 0$

   (c) $4x + 5y - 20 = 0$ (d) $3x + 4y - 12 = 0$

   (i) Slope-intercept form (ii) Two-point form

   (iii) Intercept form (iv) Point-slope form

   (v) Normal form (vi) Symmetric form
8. Find the equation of family of lines through the point of intersection of the lines $x + 2y - 3 = 0$, $3x - y + 4 = 0$. Find the member of the family which is:

   (i) parallel to a line with slope $\frac{3}{2}$ (ii) perpendicular to the line $3x + 4y = 0$
9. Find the family of lines through the point of intersection of the lines $2x - 3y + 5 = 0$, $x + 4y - 7 = 0$. Find the member of the family which is

   (i) parallel to $y = 2x + 1$ (ii) perpendicular to $y = x$
10. Check whether the following lines are concurrent or not. If concurrent, then find the point of concurrency $x + 2y - 7 = 0$; $3x - y - 4 = 0$; $2x + 35y - 145 = 0$
11. Find the condition that the lines $y = m_1 x + c_1$; $y = m_2 x + c_2$ and $y = m_3 x + c_3$ are concurrent.
12. Find the value of k if the lines $x + y - 4 = 0$, $2x - y + 1 = 0$ and $kx + 2y - 10 = 0$ are concurrent.
13. If the lines $2x + 3y - 7 = 0$, $4x - y - 5 = 0$ and $6x + ay - 9 = 0$ meet at a point, then find the value of a.

5.6 Angle Between Two Lines

Let ℓ_1 and ℓ_2 be two intersecting lines $(\ell_1 \not\parallel \ell_2)$, which meet at a point P. At the point P two supplementary angles are formed by the lines ℓ_1 and ℓ_2.

Unless $\ell_1 \perp \ell_2$, one of the two angles is acute. The angle from ℓ_1 to ℓ_2 is the angle θ through which ℓ_1 is rotated counterclockwise about the point P so that it coincides with ℓ_2. In the figure below, θ is the angle of intersection. It is clear that the inclination of a line is the angle measured in the counterclockwise direction from the positive x-axis to the line.

Theorem 7: Let ℓ_1 and ℓ_2 be two non-vertical lines such that they are not perpendicular to each other. If m_1 and m_2 are the slopes of ℓ_1 and ℓ_2 respectively, then the angle θ from ℓ_1 to ℓ_2 is given by $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

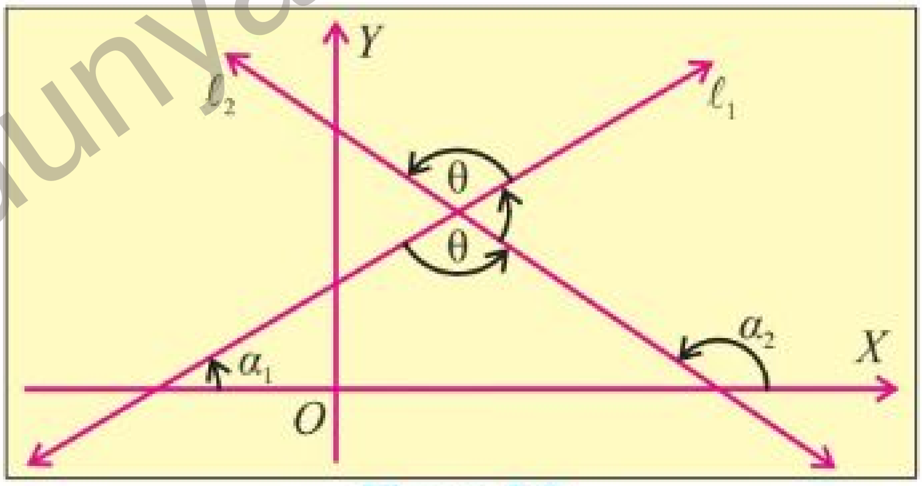

Figure 5.8

Proof: From the figure, we have

$$\alpha_2 = \alpha_1 + \theta$$

or

$$\theta = \alpha_2 - \alpha_1$$

$$\tan\theta = \tan(\alpha_2 - \alpha_1)$$

$$\tan\theta = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Corollary 1: $\ell_1 \parallel \ell_2$ if and only if $m_1 = m_2$. Snice, $\ell_1 \parallel \ell_2$ so, $\theta = 0$

$$\Leftrightarrow \quad \tan\theta = 0 = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\Leftrightarrow \quad m_2 = m_1$$

Corollary 2: $\ell_1 \perp \ell_2$ iff $1 + m_1 m_2 = 0$. Since, $\ell_1 \perp \ell_2$ so, $\theta = \dfrac{\pi}{2}$

$$\Leftrightarrow \quad \tan\frac{\pi}{2} = \frac{m_2 - m_1}{1 + m_1 m_2} = \infty$$

$$\Leftrightarrow \quad \infty = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad \boxed{\because \tan\frac{\pi}{2} = \infty}$$

$$\Leftrightarrow \quad 1 + m_1 m_2 = 0$$

Example 12 Find the angle from the line with slope $-\frac{7}{3}$ to the line with slope $\frac{5}{2}$.

Solution: Here, $m_1 = -\frac{7}{3}$ and $m_2 = \frac{5}{2}$

Let θ be the required angle, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{5}{2} - \left(-\frac{7}{3}\right)}{1 + \left(\frac{5}{2}\right)\left(-\frac{7}{3}\right)} = \frac{29}{-29} = -1 \qquad \Rightarrow \tan\theta = -1$$

Thus, $\theta = 135^o$

Example 13: Find the angles of the triangle whose vertices are $A(1, 2)$, $B(7, 3)$ and $C(4, 9)$.

Solution: Let the slopes of the sides AB, BC and CA be denoted by m_c, m_a and m_b respectively. Then

$$m_a = \frac{9-3}{4-7} = -2,\ m_b = \frac{2-9}{1-4} = \frac{7}{3},\ m_c = \frac{3-2}{7-1} = \frac{1}{6}$$

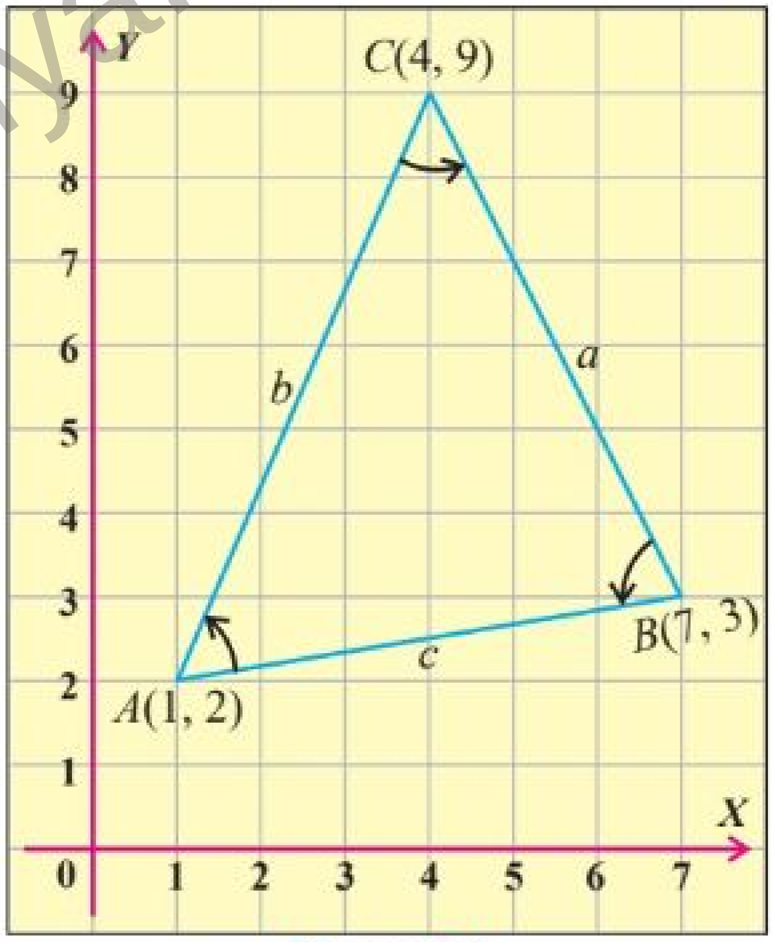

Figure 5.9

The angle A is measured from $\overline{AB}$ to $\overline{AC}$.

$$\tan A = \frac{m_c - m_b}{1 + m_c m_b} = \frac{\frac{1}{6} - \frac{7}{3}}{1 + \left(\frac{1}{6}\right)\left(\frac{7}{3}\right)} = -\frac{39}{25}$$

$$\Rightarrow \quad A = 57.35^o$$

The angle B is measured from $\overline{BC}$ to $\overline{AB}$.

$$\tan B = \frac{m_c - m_a}{1 + m_c m_a} = \frac{\frac{1}{6} - (-2)}{1 + \left(\frac{1}{6}\right)(-2)} = \frac{13}{4}$$

$$\Rightarrow \quad B = 72.87^o$$

The angle C is measured from $\overline{AC}$ to $\overline{BC}$.

$$\tan C = \frac{m_a - m_b}{1 + m_a m_b} = \frac{-2 - \frac{7}{3}}{1 + (-2)\left(\frac{7}{3}\right)} = \frac{13}{11}$$

$$\Rightarrow \quad C = 49.78^o$$

5.7 Area of a Triangular Region Whose Vertices are Given

To find the area of triangular region whose vertices are: $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$.

Draw perpendicular $\overline{PL}, \overline{QN}$ and $\overline{RM}$ on x-axis.

Area of triangular region PQR = Area of trapezoidal region $PLMR$ + Area of trapezoidal region $RMNQ$ − Area of trapezoidal region $PLNQ$

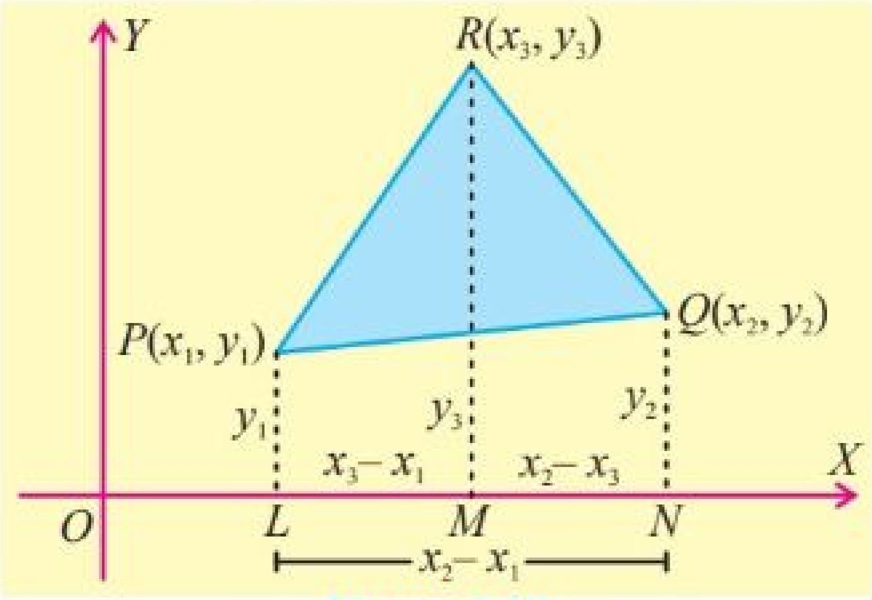

Figure 5.10

Note

A trapezium is a quadrilateral having two parallel and two non-parallel sides.

Area of trapezoidal region $= \frac{1}{2}$(sum of ∥ sides) (distance between ∥ sides).

$$= \frac{1}{2}\left(|\overline{PL}|+|\overline{RM}|\right)\left(|\overline{LM}|\right)+\frac{1}{2}\left(|\overline{RM}|+|\overline{QN}|\right)\left(|\overline{MN}|\right)-\frac{1}{2}\left(|\overline{PL}|+|\overline{QN}|\right)\left(|\overline{LN}|\right)$$

$$= \frac{1}{2}\left[(y_1+y_3)(x_3-x_1)+(y_3+y_2)(x_2-x_3)-(y_1+y_2)(x_2-x_1)\right]$$

$$= \frac{1}{2}\left(x_3y_1+x_3y_3-x_1y_1-x_1y_3+x_2y_3+x_2y_2-x_3y_3-x_3y_2-x_2y_1-x_2y_2+x_1y_1+x_1y_2\right)$$

$$= \frac{1}{2}\left(x_3y_1-x_1y_3+x_2y_3-x_3y_2-x_2y_1+x_1y_2\right)$$

Thus, the required area is given by: $\Delta = \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Corollary: If the points P, Q and R are collinear, then $\Delta = 0$

Example 14: Find the area of the region bounded by the triangle with vertices $A(a, b)$, $B(-a, b)$ and $C(0, -b)$.

Solution: $\Delta = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ -a & b & 1 \\ 0 & -b & 1 \end{vmatrix}$

$$= \frac{1}{2}\left[a(b+b)-b(-a-0)+1(ab-0)\right] = \frac{1}{2}\left[2ab+ab+ab\right] = 2ab$$

Thus, the area of ΔABC is $2ab$

Example 15: Find the area of the region bounded by the triangle with vertices $A(1, 2)$, $B(3, 6)$ and $C(5, 10)$.

Solution: Area (Δ) of the region bounded by the triangle ABC is

$$\Delta = \frac{1}{2}\begin{vmatrix}1 & 2 & 1\\ 3 & 6 & 1\\ 5 & 10 & 1\end{vmatrix} = \frac{1}{2}\begin{vmatrix}1 & 2 & 1\\ 2 & 4 & 0\\ 4 & 8 & 0\end{vmatrix} \quad \text{by} \begin{pmatrix}R_2 - R_1\\ R_3 - R_1\end{pmatrix}$$

$$= \frac{1}{2}[16 - 16] \text{ (Expanding by 3}^{\text{rd}}\text{ column)} = 0$$

Thus, the given points are collinear.

EXERCISE 5.2

1. Find the angle measured from the line ℓ_1 to the line ℓ_2, where
   - (i) ℓ_1: Joining $(2, 7)$ and $(-4, 5)$
ℓ_2: Joining $(-7, 2)$ and $(1, 3)$
   - (ii) ℓ_1: Joining $(1, 2)$ and $(3, 5)$
ℓ_2: Joining $(2, -1)$ and $(4, 3)$

   Also find the acute angle in each case.
2. Find the angle from the line with slope m_1 to the line with slope m_2, where
   - (i) $m_1 = 2, m_2 = -\frac{1}{2}$
   - (ii) $m_1 = 3, m_2 = -2$
3. Find the angle from line ℓ_1 to the line ℓ_2, where
   - (i) $\ell_1: 2x + y - 3 = 0; \ell_2: x + y - 5 = 0$
   - (ii) $\ell_1: 2x - 3y + 2 = 0; \ell_2: x + y - 4 = 0$
4. The angle from the line $ax - y + 2 = 0$ to the line $x + y - 4 = 0$ is 30°. Find the value of a.
5. Find the interior angles of the triangle whose vertices are:
   - (i) $A(2, 3), B(6, 3), C(4, 7)$
   - (ii) $L(1, 2), M(5, 5), N(7, 2)$
6. Find the interior angles of the quadrilateral whose vertices are $A(2, 1)$, $B(3, 1)$, $C(4, 5)$ and $D(-1, 5)$.
7. Find the interior angles of the kite whose vertices are $A(0, 0)$, $B(2, 3)$, $C(4, 0)$ and $D(2, -3)$.
8. Find the area of the triangle formed by the points $A(-3, 2)$, $B(4, -1)$, $C(1, 5)$.
9. By computing the area of triangle ABC check whether the points are collinear where $A(0, 4)$, $B(3, 2)$ and $C(6, 0)$.
10. The coordinates of the points are $A(0, \lambda)$, $B(-2, 1)$ and $C(-3, -2)$. By computing the area bounded by ABC, find the value of λ if points are collinear.

5.8 Homogeneous Equation of the Second Degree in Two Variables

We know that if a graph is a straight line in a plane, then its equation is a linear equation in two variables x and y. Conversely, the graph of any linear equation in x and y is a straight line.

Suppose we have two straight lines represented by

$$a_1x + b_1y + c_1 = 0 \qquad \dots(\text{i})$$

and

$$a_2x + b_2y + c_2 = 0 \qquad \dots(\text{ii})$$

Multiplying equations (i) and (ii), we have

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0 \qquad \dots(\text{iii})$$

It is a second degree equation in x and y.

Equation (iii) is called **joint equation of the pair of lines** (i) and (ii). On the other hand, given an equation of second degree in x and y, say

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \dots(\text{iv})$$

where $a \neq 0$ represents equations of a pair of lines if (iv) can be resolved into two linear factors. In this section, we shall study special joint equations of pairs of lines which pass through the origin.

Let $y = m_1x$ and $y = m_2x$ be two lines passing through the origin. Their joint equation is:

$$(y - m_1x)(y - m_2x) = 0$$

or

$$y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 = 0 \qquad \dots(\text{v})$$

Equation (v) is a special type of a second degree homogeneous equation.

5.8.1 Homogeneous Equation

Let $f(x, y) = 0$ be any equation in two variables x and y is called a homogeneous equation of degree n (a positive integer) if the function $f(x, y)$ is homogeneous of degree n, then, $f(kx, ky) = k^n f(x, y)$ for some real number k.

For example, in equation (v) above if we replace x and y by kx and ky respectively, we have:

$$f(kx, ky) = k^2 y^2 - k^2 (m_1 + m_2) xy + k^2 m_1 m_2 x^2$$

$$= k^2[y^2 - (m_1 + m_2)xy + m_1m_2x^3] = 0 \Rightarrow f(kx, ky) = k^2 f(x, y)$$

Thus (v) is a homogeneous equation of degree 2. A general second degree homogeneous equation can be written as: $ax^2 + 2hxy + by^2 = 0$

provided a, h and b are not simultaneously zero.

Theorem 8: Every homogeneous second degree equation

$$ax^2 + 2hxy + by^2 = 0 \qquad \ldots(1)$$

represents a pair of lines through the origin. The lines are

(i) real and distinct, if $h^2 > ab$ (ii) real and coincident, if $h^2 = ab$

(iii) imaginary, if $h^2 < ab$

Proof: Multiplying (1) by b and re-arranging the terms, we have:

$$b^2y^2 + 2bhxy + abx^2 = 0$$

$$b^2y^2 + 2bhxy + h^2x^2 - h^2x^2 + abx^2 = 0$$

$$(by + hx)^2 - x^2(h^2 - ab) = 0$$

$$\left(by + hx + x\sqrt{h^2 - ab}\right)\left(by + hx - x\sqrt{h^2 - ab}\right) = 0$$

Thus equation (1) represents a pair of lines whose equations are:

$$by + x\left(h + \sqrt{h^2 - ab}\right) = 0 \qquad \ldots(2)$$

and

$$by + x\left(h - \sqrt{h^2 - ab}\right) = 0 \qquad \ldots(3)$$

Clearly the lines (2) and (3) are (i) real and distinct if $h^2 > ab$ (ii) real and coincident if $h^2 = ab$ (iii) imaginary, if $h^2 < ab$. It is interesting to note that even in case the lines are imaginary, they intersect in a real point viz (0,0) since this point lies on their joint equation (1).

Example 16 Find an equation of each of the lines represented by $3x^2 - 10xy + 3y^2 = 0$

Solution:

$$3x^2 - 10xy + 3y^2 = 0$$

$$3x^2 - 9xy - xy + 3y^2 = 0$$

$$3x(x - 3y) - y(x - 3y) = 0$$

$$(x - 3y)(3x - y) = 0$$

$$x - 3y = 0 \text{ and } 3x - y = 0$$

5.9 Angle Between the Lines

$$ax^2 + 2hxy + by^2 = 0 \qquad \ldots(\text{i})$$

We have already seen that the lines represented by (i) are:

$$by + x\left(h + \sqrt{h^2 - ab}\right) = 0 \qquad \ldots(\text{ii})$$

and

$$by + x\left(h - \sqrt{h^2 - ab}\right) = 0 \qquad \ldots(\text{iii})$$

Now, slopes of (ii) and (iii) are respectively given by:

$$m_1 = \frac{-\left(h + \sqrt{h^2 - ab}\right)}{b} \text{ and } m_2 = \frac{-\left(h - \sqrt{h^2 - ab}\right)}{b}$$

Therefore, $m_1 + m_2 = -\dfrac{2h}{b}$ and $m_1 m_2 = \dfrac{a}{b}$

If θ is the measure of the angle between the lines (ii) and (iii), then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{\dfrac{4h^2}{b^2} - \dfrac{4a}{b}}}{1 + \dfrac{a}{b}}$$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b} \qquad \ldots\text{(iv)}$$

Thus, equation (iv) gives angle between the lines represented by equation (i).

Special Cases:

- If two lines are parallel, then $\theta = 0$, so that $\tan\theta = 0$ this implies $h^2 - ab = 0$, which is the condition for the lines to be coincident.
- If the lines are perpendicular, then $\theta = 90^o$, so that $\tan\theta$ is not defined. This implies $a + b = 0$.

Hence, the condition for (i) to represent a pair of perpendicular lines is that sum of the coefficients of x^2 and y^2 is 0.

Example 17 Find measure of the angle between the lines represented by:

$$5x^2 + 6xy + y^2 = 0.$$

Solution: Here $a = 5$, $h = 3$, $b = 1$

If θ is the measure of the angle between the given lines, then it is given by:

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b} = \frac{2\sqrt{9 - 5}}{5 + 1} = \frac{2}{3}$$

$$\theta = 33.69^o$$

5.10 Application of Analytical Geometry to Real Life

Analytical geometry connects mathematics to the real life. It allows us to analyze positions, distances, slopes and angles making it essential in aviation, astronomy, engineering and many fields of science and technology.

Example 17 Planet A is at (1, 0) AU and planet B is at (–0.5, 0.866) AU. Find the distance between them. Write answer in km.

Solution: Distance $= \sqrt{[1 - (-0.5)]^2 + (0 - 0.866)^2}$

$$= \sqrt{2.25 + 0.75} = \sqrt{3} = 1.732\ AU$$

We know that $|AU|$ = 149.6 million km, then

Distance = 1.732 × 149.6 = 259.1 million km

Remember!

An astronomical unit (AU) is a unit of length and exactly equal to 149597870700 m.

Example 19 Aircraft flies from $A(100, 200)$ km to $B(500, 800)$ km. Find slope of the track and straight line distance.

Solution: Slope = $\dfrac{800-200}{500-100} = \dfrac{600}{400} = 1.5$

$$\text{Distance} = \sqrt{(500-100)^2 + (800-200)^2}$$

$$= \sqrt{(400)^2 + (600)^2} = \sqrt{520000} = 721.11 \text{ km}$$

EXERCISE 5.3

Find the equation of lines represented by each of the following and also find measure of the angle between them (**Problem 1–5**):

1. $6x^2 - 5xy - 4y^2 = 0$
2. $2x^2 - 7xy + 3y^2 = 0$
3. $5x^2 + 6xy + y^2 = 0$
4. $x^2 + 2xy \sec \alpha + y^2 = 0$
5. $7x^2 - 8xy + y^2 = 0$
6. Find the value of k such that the lines represented by the equation $(k-8)x^2 - 9xy + 3ky^2 = 0$ are perpendicular to each other.
7. Prove that the lines represented by the equation $9y^2 + 6xy + x^2 = 0$ are coincident.
8. Find a joint equation of the lines through the origin and perpendicular to the lines $3x^2 + 5xy - 2y^2 = 0$.
9. $A(-2,-1)$ and $B(3, 4)$ are two points on the solar panel roof. Find slope and tilt angle.
10. The paths of two rays from a star are represented by the equation
    $$2x^2 + 5xy - 3y^2 = 0$$
    Find the equations of the two lines and determine the angle between the rays.
11. The trajectories of two projectiles in a physics experiment are modeled by the homogeneous equation $9x^2 - 12xy + 4y^2 = 0$. Prove that this represents two coincident lines and find angle between them.
12. The stresses in a metal plate are represented by the homogeneous equation $3x^2 - 8xy - 3y^2 = 0$. Show that this equation represents a pair of straight lines through the origin. Find:
    i. the equations of lines represented by this equation.
    ii. angle between them.