Unit 03 Integration

INTRODUCTION

In previous units, we studied differentiation and learned how to determine rates of change, slopes of curves, maxima and minima, and solutions to various real-world problems. In this unit, we will study integration, which is the reverse process of differentiation and an important branch of calculus used to find accumulation, area, volume, and total change. We will learn the concepts of antiderivatives, indefinite and definite integrals, their notation, properties, and fundamental rules, including the power rule and the Fundamental Theorem of Calculus. We will also study the relationship between definite integrals and the net area under a curve, and use integration to find areas bounded by curves and lines, as well as volumes of solids of revolution. Furthermore, we will explore important integration techniques such as substitution, integration by parts, trigonometric substitution, partial fractions, and integration involving exponential and logarithmic functions. The concepts of integration have wide applications in mathematics, physics, engineering, economics, biology, and technology, including calculating the volume of containers, population growth, consumer and producer surplus, distance and velocity, drug dosage, moment of inertia, and sensor network measurements. Through this unit, we will develop both conceptual understanding and practical problem-solving skills related to integration and its real-life applications.

3.1 Antiderivatives and The Indefinite Integral

Definition: A function F is called an **antiderivative** of a given function f on an interval I if $F'(x) = f(x)$ for all x in I. In other words, the process of recovering a function F from its derivative f is called antidifferentiation.

To illustrate, let $f(x) = 2x$. If $F(x) = x^2$ then $F'(x) = 2x = f(x)$. Moreover, the function $G(x) = x^2 + 10$ also satisfies $G'(x) = 2x = f(x)$. Therefore, if $F(x)$ is any antiderivatives of $f(x)$ on a given interval, then for any value of c, the function $F(x)+c$ is also antiderivative of $f(x)$ on that interval.

Definition: Let F be an antiderivative of f on an interval I. The set (or family) of all antiderivatives of f is called the **indefinite integral** of $f(x)$ with respect to x, denoted by $\int f(x)\,dx$ and defined as $\int f(x)\,dx = F(x)+c$, where $F'(x)=f(x)$ and c is an arbitrary constant.

In the notation $\int f(x)\,dx = F(x)+c$.

(i) The symbol $\int$ is called the **integral sign.**

(ii) $\int f(x)\,dx$ is called the **indefinite integral** of $f(x)$.

(iii) The function $f(x)$ inside the integral sign is the **integrand**.

(iv) The symbol dx is called the **differential** of x. It indicates the **variable of integration**–the variable with respect to which the integration is performed.

(v) $F(x)$ is an **antiderivative** of $f(x)$.

(vi) c is the **constant of integration**.

3.1.1 Basic Rules of Integration

Rule-1: $\int k\,f(x)\,dx = k\int f(x)\,dx$ (Integration of function with constant (k)).

Rule-2: $\int [f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$ (Sum and Difference Rule).

Rule-3: $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}+c$, where $n\neq -1$ (The Power Rule).

The **General Power Rule:** $\int [f(x)]^n\, f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1}+c$, where $n\neq -1$.

Rule-4: $\int \dfrac{f'(x)}{f(x)}\,dx = \ln|f(x)|+c$

Rule-5: If f' is the derivative of f, and $\int f'(x)\,dx = f(x)+c$, then for any constants $a \neq 0$ and b, $\int f'(ax+b)\,dx = \dfrac{f(ax+b)}{a}+c$. (Linear Substitution Rule).

3.1.2 Some Useful Formulae

From (the above Definitions), we obtain integration formulas directly from differentiation formulas, as shown in the following table.

Derivative $\left(\frac{d}{dx}(f(x))\right)$	Indefinite Integral $\left(\int \frac{d}{dx}(f(x))dx = f(x)+c\right)$
$\frac{d}{dx}(c)=0$	$\int 0\,dx = c$
$\frac{d}{dx}(x)=1$	$\int 1\,dx = \int dx = x+c$
$\frac{d}{dx}(x^n)=nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1}+c, n\neq -1$ (Power Rule)
$\frac{d}{dx}(\ln\lvert x\rvert)=\frac{1}{x}, x\neq 0$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + c, x\neq 0$
$\frac{d}{dx}(e^x)=e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}(a^x)=a^x \ln a$	$\int a^x dx = \frac{a^x}{\ln a}+c; a>0, a\neq 1$
$\frac{d}{dx}(\sin x)=\cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}(\cos x)=-\sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\tan x)=\sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}(\cot x)=-\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + c$
$\frac{d}{dx}(\sec x)=\sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + c$
$\frac{d}{dx}(\csc x)=-\csc x\cot x$	$\int \csc x\cot x\,dx = -\csc x + c$
$\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + c$
$\frac{d}{dx}(\sec^{-1} x)=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}\lvert x\rvert + c$
$\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = \tan^{-1} x + c$

Example 1 Integrate the following:

(i) $\int x^{99}\,dx$ (ii) $\int -98x^{-99}\,dx$

Solution: (i) $\int x^{99}\,dx = \dfrac{x^{99+1}}{99+1}+c = \dfrac{x^{100}}{100}+c$ (By using Rule 3: The Power Rule)

We can verify this result by differentiation: $\dfrac{d}{dx}\left(\dfrac{x^{100}}{100}+c\right) = \dfrac{100x^{99}}{100}+c = x^{99}+c$

(ii) $-98\int x^{-99}\,dx = -98\dfrac{x^{-98}}{-98}+c = x^{-98}+c = \dfrac{1}{x^{98}}+c$ (By using Rule 1 and Rule 3)

Example 2 Integrate: $\int\left(8x^7-6x^5+4x^3-2x-3\right)dx$

Solution: By using Rule 2, we have:

$$\int\left(8x^7-6x^5+4x^3-2x-3\right)dx = \int 8x^7dx-\int 6x^5dx+\int 4x^3dx-\int 2x\,dx-\int 3\,dx$$

$$= 8\int x^7dx-6\int x^5dx+4\int x^3dx-2\int x\,dx-3\int 1\,dx \quad \text{(By using Rule 1)}$$

$$= 8\left(\frac{x^8}{8}\right)-6\left(\frac{x^6}{6}\right)+4\left(\frac{x^4}{4}\right)-2\left(\frac{x^2}{2}\right)-3x+c = x^8-x^6+x^4-x^2-3x+c \quad \text{(By Rule 3)}$$

Example 3 Integrate: (i) $\int\dfrac{7}{\sqrt{x}}dx$ (ii) $\int\dfrac{(1+x^2)^2}{x^4}dx$

Solution: (i) $\int\dfrac{7}{\sqrt{x}}dx = 7\int x^{-1/2}dx = 7\dfrac{x^{1/2}}{1/2}+c = 14\sqrt{x}+c$ (By using Rule 1 & 3)

(ii) $\int\dfrac{\left(1+x^2\right)^2}{x^4}dx = \int\dfrac{1+2x^2+x^4}{x^4}dx = \int\left(\dfrac{1}{x^4}+\dfrac{2x^2}{x^4}+\dfrac{x^4}{x^4}\right)dx$ (By Rule 2)

$$= \int(x^{-4}+2x^{-2}+1)\,dx = \frac{x^{-3}}{-3}+2\frac{x^{-1}}{-1}+x+c = -\frac{1}{3x^3}-\frac{2}{x}+x+c \quad \text{(By Rule 3)}$$

Example 4 (i) $\int(5x^2+7x-3)^7(10x+7)\,dx = \dfrac{(5x^2+7x-3)^8}{8}+c$

(ii) $\int\left(x+\dfrac{1}{x}\right)\left(1-\dfrac{1}{x^2}\right)dx = \dfrac{1}{2}\left(x+\dfrac{1}{x}\right)^2+c$

Example 5 Evaluate $\int \tan x\, dx$

Solution:

$$\int \tan x\, dx = \int \frac{\sec x \cdot \tan x}{\sec x} dx$$

$$= \ln|\sec x| + c \quad \text{(By Rule 4)}$$

or $$= -\ln|\cos x| + c$$

Example 6 Evaluate $\int \sec x\, dx$

Solution: $$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x} dx$$

$$= -\text{In}|\cos x| + c \quad \text{(By Rule 4)}$$

Example 7 Evaluate $\int \tan^2 x\, dx$

Solution: $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \int \sec^2 x\, dx - \int 1\, dx = \tan x - x + c$

Example 8 Evaluate $\int \cos^2 x\, dx$

Solution: $\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2} dx = \frac{1}{2}\int (1+\cos 2x)\, dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c$

Example 9 Evaluate $\int \frac{\tan x - \sec x}{1+\tan^2 x} dx$

Solution: $\int \frac{\tan x - \sec x}{1+\tan^2 x} dx = \int \frac{\tan x - \sec x}{\sec^2 x} dx = \int \left(\frac{\tan x}{\sec^2 x} - \frac{\sec x}{\sec^2 x}\right) dx$

$$= \int \left(\frac{\sin x}{\cos x}\cos^2 x - \frac{1}{\sec x}\right) dx = \int (\sin x \cos x - \cos x)\, dx = \frac{\sin^2 x}{2} - \sin x + c$$

Example 10 Evaluate $\int \sin 3x \cos 7x\, dx$

Solution: We know that, $\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$

Put $m = 3$ and $n = 7$, we get

$$\int \sin 3x \cos 7x\, dx = \frac{1}{2}\int [\sin(10x) + \sin(-4x)]\, dx$$

$$= \frac{1}{2}\int (\sin 10x - \sin 4x)\, dx = \frac{1}{2}\left[-\frac{\cos 10x}{10} - \left(-\frac{\cos 4x}{4}\right)\right] + c = -\frac{\cos 10x}{20} + \frac{\cos 4x}{8} + c$$

Example 11 Evaluate $\int \frac{\sin(x+a)}{\sin(x+b)} dx$

Solution: $\int \frac{\sin(x+a)}{\sin(x+b)} dx = \int \frac{\sin[(x+b)+(a-b)]}{\sin(x+b)} dx$

$$= \int \frac{\sin(x+b)\cos(a-b) + \cos(x+b)\sin(a-b)}{\sin(x+b)} dx$$

$$= \int\left[\cos(a-b)+\sin(a-b)\frac{\cos(x+b)}{\sin(x+b)}\right]dx = \cos(a-b)\int 1\, dx + \sin(a-b)\int\frac{\cos(x+b)}{\sin(x+b)}dx$$

$$= \cos(a-b)\cdot x + \sin(a-b)\cdot \ln\left|\sin(x+b)\right| + c$$

Example 12 Evaluate $\int \frac{a^{2x}-b^{2x}}{(ab)^x}dx$

Solution: $\int \frac{a^{2x}-b^{2x}}{(ab)^x}dx = \int \frac{a^{2x}-b^{2x}}{a^x b^x}dx = \int\left[\frac{a^{2x}}{a^x b^x}-\frac{b^{2x}}{a^x b^x}\right]dx = \int\left[\frac{a^{2x-x}}{b^x}-\frac{b^{2x-x}}{a^x}\right]dx$

$$= \int\left[\frac{a^x}{b^x}-\frac{b^x}{a^x}\right]dx = \int\left(\frac{a}{b}\right)^x dx - \int\left(\frac{b}{a}\right)^x dx = \frac{\left(\frac{a}{b}\right)^x}{\ln\left(\frac{a}{b}\right)} - \frac{\left(\frac{b}{a}\right)^x}{\ln\left(\frac{b}{a}\right)} + c$$

EXERCISE 3.1

1. Find all the antiderivatives for each derivative.

   (i) $\frac{dy}{dx} = 100x^4$ (ii) $\frac{dy}{dt} = 2 + 24t^3 - 10t^4$

2. Find each indefinite integral. (Check answer by differentiating.)

   (i) $\int(1+2t)\,dt$ (ii) $\int(3u^2+4u)\,du$

   (iii) $\int 100x^8\,dx$ (iv) $\int\left(2x^{-2}+3x^{-3}\right)dx$

3. Evaluate the following:

   (i) $\int(5x^3+10x+20)\,dx$ (ii) $\int x^2\left(x+\frac{1}{x}\right)dx$

   (iii) $\int\left(\sqrt{y}+\frac{1}{y}\right)^2 dy,\ (y>0)$ (iv) $\int\frac{\sqrt{x}(x+2)}{x}dx,\ (x>0)$

   (v) $\int\frac{x+1}{\sqrt{x+2}}dx$ (vi) $\int\frac{ax+b}{\sqrt{ax^2+2bx+c}}dx$ (vii) $\int\frac{1}{\sqrt{x+1}+\sqrt{x}}dx,\ (x>0)$

   (viii) $\int x\sqrt{1+\sqrt{x}}\,dx$ (ix) $\int\frac{x}{(a+bx)^3}dx$ (x) $\int x^5\sqrt{x^3+1}\,dx$

   (xi) $\int(3x+3^x+x^3)\,dx$ (xii) $\int 7^{\ln x}dx$ (xiii) $\int\frac{1}{1+\sqrt{2+x}}dx$

4. Evaluate the following:

   (i) $\int\csc x\,dx$ (ii) $\int\cot^2 x\,dx$ (iii) $\int\sec^4 x\,dx$ (iv) $\int\sin x\cdot\cos 7x\,dx$

(v) $\int \frac{\cos^3 x-\sin^3 x}{\cos x-\sin x}dx$ (vi) $\int \frac{\cos 2x}{\sin x+\cos x}dx$ (vii) $\int \sqrt{1-\cos 2x}\,dx$

(viii) $\int \frac{1}{1+\cos x}dx$ (ix) $\int \frac{\sin 2x}{\sqrt{1+\sin^2 x}}dx$ (x) $\int \frac{1-\cos 2x}{1+\cos 2x}dx$

3.2 Method of Substitution or Change of Variable

The method stems from reversing the Chain Rule for derivatives. Recall that for the differentiable functions $y = F(u)$ and $u = g(x)$ the Chain Rule states that

$$\frac{d}{dx}\left[F\left(g(x)\right)\right] = F'\left(g(x)\right)g'(x).$$

In terms of definite integrals $\int F'\left(g(x)\right)g'(x) = F\left(g(x)\right)+c.$

Theorem 3.1: Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f $(F' = f)$ on I, then $\int f(g(x)\cdot g'(x)dx = F(u)+c.$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and $\int f(u)\,du = F(u)+c.$

This equation $\int f(u)\,du = F(u)+c$ has the same form as the basic antiderivative formula, $\int f(x)\,dx = F(x)+c$, but the independent variable is now u.

The process of replacing x with a new variable $u = g(x)$ (and dx with the corresponding du) is called a **change of variable** or **the method of substitution**.

Recall! The Rule 4

$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$

Let $u = f(x)$. Then $du = f'(x)\,dx$.

Substituting into the given integral, we have $\int \frac{f'(x)}{f(x)}dx = \int \frac{1}{u}du$

We know from basic integration rules that $\int \frac{1}{u}du = \ln|u|+c$

Finally, substituting $u = f(x)$, we obtain $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$

Example 13 Evaluate $\int (5x^5+5)^3 x^4 dx$

Solution: When an integrand contains an expression raised to a power, such as $\left(5x^5+5\right)^3$, a useful strategy is to substitute u for that expression.

Let $u = 5x^5 + 5$. Then $du = 25x^4 dx$.

$$\int (5x^5+5)^3 x^4 dx = \frac{1}{25}\int (5x^5+5)^3 \cdot 25x^4 dx = \frac{1}{25}\int u^3 du = \frac{1}{25}\cdot\frac{u^4}{4} + c = \frac{1}{100}u^4 + c$$

Finally, substitute back $u = 5x^5 + 5$: $\int (5x^5+5)^3 x^4 dx = \frac{1}{100}\left(5x^5+5\right)^4 + c.$

Example 14 Integrate $\int \sqrt{x}\ \sqrt{1+\sqrt{x}}\,dx$

Solution: Let $1+\sqrt{x} = u$. Then, $\sqrt{x} = u-1$ and $x = (u-1)^2 \Rightarrow dx = 2(u-1)\,du$

Substitute $\sqrt{x} = u-1$ and $dx = 2(u-1)\,du$ into the integral:

$$\int \sqrt{x}\ \sqrt{1+\sqrt{x}}\,dx = \int (u-1)\sqrt{u}\ 2(u-1)\ du = 2\int (u-1)^2 u^{\frac{1}{2}} du$$

$$= 2\int (u^2-2u+1)u^{\frac{1}{2}} du = 2\int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du = 2\left[\frac{2}{7}u^{\frac{7}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right] + c$$

$$= \frac{4}{7}u^{\frac{7}{2}} - \frac{8}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + c = \frac{4}{7}\left(1+\sqrt{x}\right)^{\frac{7}{2}} - \frac{8}{5}\left(1+\sqrt{x}\right)^{\frac{5}{2}} + \frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}} + c.\ \because u = 1+\sqrt{x}$$

Example 15 Evaluate $\int \frac{x}{\sqrt{1+x^2}+1+x^2}\,dx$

Solution: Let $u = \sqrt{1+x^2}$. Then $u^2 = 1+x^2 \Rightarrow 2u\ du = 2x\ dx \Rightarrow u\,du = x\,dx$

$$\int \frac{x}{\sqrt{1+x^2}+1+x^2}\,dx = \int \frac{u}{u+u^2}\,du = \int \frac{u}{u(1+u)}\,du = \int \frac{1}{1+u}\,du = \ln|1+u| + c$$

$$= \ln\left(1+\sqrt{1+x^2}\right) + c.$$

where we can remove the absolute value signs since $1+\sqrt{1+x^2} > 0$ for all x.

Example 16 Evaluate $\int \frac{\sin x}{e^{\cos x}}\,dx$

Solution: Let $u = \cos x$. Then $du = -\sin x\,dx$.

$$\int \frac{\sin x}{e^{\cos x}}\,dx = \int \frac{-du}{e^u} = -\int e^{-u}\ du = -\frac{e^{-u}}{-1} + c = e^{-u} + c = e^{-\cos x} + c$$

Example 17 Evaluate $\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

Solution: Let $u = \tan^{-1}x$, then $du = \frac{1}{1+x^2}\ dx$

$$\int \frac{e^{\tan^{-1}x}}{1+x^2}\ dx = \int e^u\ du = e^u + c = e^{\tan^{-1}x} + c$$

Example 18 Evaluate $\int \frac{e^{2x}+2e^x}{(e^x+1)^2}dx$

Solution: Let $e^x+1=u \Rightarrow e^x=u-1 \Rightarrow e^x dx=du$

$$\int \frac{e^{2x}+2e^x}{(e^x+1)^2}dx=\int \frac{e^x+2}{\left(e^x+1\right)^2}\cdot e^x dx=\int \frac{u-1+2}{u^2}du=\int \frac{u+1}{u^2}du=\int\left(\frac{u}{u^2}+\frac{1}{u^2}\right)du$$

$$=\int\left(\frac{1}{u}+u^{-2}\right)du=\ln|u|+\frac{u^{-1}}{-1}+c=\ln|u|-\frac{1}{u}+c=\ln\left|e^x+1\right|-\frac{1}{e^x+1}+c$$

Example 19 Evaluate $\int \frac{e^x \ln(1+e^x)}{1+e^x}dx$

Solution: Let $u=\ln(1+e^x)$, then $du=\frac{e^x}{1+e^x}dx$

$$\int \frac{e^x \ln(1+e^x)}{1+e^x}dx=\int u\, du=\frac{u^2}{2}+c=\frac{1}{2}\left[\ln(1+e^x)\right]^2+c$$

Example 20 Evaluate $\int \frac{\ln(1+x)-\ln x}{x(1+x)}dx$

Solution: Let $u=\ln(1+x)-\ln x$

Then, $\frac{du}{dx}=\frac{1}{1+x}-\frac{1}{x}=\frac{x-(1+x)}{x(1+x)}=\frac{x-1-x}{x(1+x)}=\frac{-1}{x(1+x)}$

$$\frac{du}{dx}=\frac{-1}{x(1+x)} \Rightarrow -du=\frac{1}{x(1+x)}dx$$

$$\int \frac{\ln(1+x)-\ln x}{x(1+x)}dx=\int u(-du)=-\int u\, du=-\frac{u^2}{2}+c=-\frac{1}{2}[\ln(1+x)-\ln x]^2+c$$

Example 21 Evaluate $\int \frac{x^2\tan^{-1}(x^3)}{1+x^6}dx$

Solution: Let $u=\tan^{-1}(x^3)$,

then $\frac{du}{dx}=\frac{1}{1+(x^3)^2}\frac{d}{dx}(x^3) \Rightarrow du=\frac{3x^2}{1+x^6}dx \Rightarrow \frac{1}{3}du=\frac{x^2}{1+x^6}dx$

Thus, $\int \frac{x^2\tan^{-1}(x^3)}{1+x^6}dx=\int u\frac{1}{3}du=\frac{1}{3}\int u\, du=\frac{u^2}{6}+c=\frac{1}{6}\left[\tan^{-1}(x^3)\right]^2+c.$

3.3 Trigonometric Substitution

Integrals involving the radicals $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, and $\sqrt{x^2-a^2}$ (where $a>0$) can often be evaluated by making a substitution involving a trigonometric function. The objective of trigonometric substitution is to eliminate the radical in the integrand. We do this by using the Pythagorean identities:

$$\sin^2\theta+\cos^2\theta=1 \text{ and } \sec^2\theta-\tan^2\theta=1$$

Case 1: Integrals involving $\sqrt{a^2-x^2}$ (where, $a>0$) can frequently be reduced to a simpler form by using the substitution $x=a\sin\theta$.

Case 2: Integrals involving $\sqrt{a^2+x^2}$ (where, $a>0$) can frequently be reduced to a simpler form by using the substitution $x=a\tan\theta$.

Case 3: Finally, integrals involving $\sqrt{x^2-a^2}$ (where, $a>0$) can frequently be reduced to a simpler form by using the substitution $x=a\sec\theta$.

Some Special Integrals as Formulae

(i) $\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c$ **(ii)** $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+c$, where $a>0$

Proof: **(i)** We want to evaluate $\int\frac{1}{a^2-x^2}dx$

The denominator $a^2-x^2=(a-x)(a+x)$, so

$$\int\frac{1}{a^2-x^2}dx=\int\frac{1}{(a-x)(a+x)}dx$$

$$=\frac{1}{2a}\int\frac{(a-x)+(a+x)}{(a-x)(a+x)}dx$$

$$=\frac{1}{2a}\int\left[\frac{(a-x)}{(a-x)(a+x)}+\frac{(a+x)}{(a-x)(a+x)}\right]dx$$

$$=\frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx$$

$$=\frac{1}{2a}\int\left(\frac{1}{a+x}-\frac{-1}{a-x}\right)dx$$

$$=\frac{1}{2a}(\ln|a+x|-\ln|a-x|)+c$$

$$=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c$$

Challenge!

Prove the given formulae.

(i) $\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$

(ii) $\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

(iii) $\int\frac{dx}{\sqrt{a^2+x^2}}=\ln\left|x+\sqrt{a^2+x^2}\right|+c$, where $a>0$

(iv) $\int\frac{dx}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+c$, where $a>0$

(v) $\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c$, where $a>0$

(ii) We want to evaluate $\int \frac{1}{\sqrt{a^2-x^2}}dx$, where a is a positive constant and the domain is $-a < x < a$ so that the square root is real.

Let $x = a\sin\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $dx = a\cos\theta\, d\theta$.

Also

$$\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin\theta)^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|$$

On the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\cos\theta > 0$, so $|\cos\theta| = \cos\theta$. Thus, $\sqrt{a^2-x^2} = a\cos\theta$.

Substitute into the integral: $\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{a\cos\theta}\cdot a\cos\theta\, d\theta = \int 1d = \theta + c$

We have $x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a}$, and since θ is in the principal range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

Thus, $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Example 22 Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}dx$

Solution: Let $x = 3\sin\theta$, where $\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, then $dx = 3\cos\theta\, d\theta$.

$$\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{3\cos\theta}{(3\sin\theta)^2}\cdot 3\cos\theta\, d\theta = \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta = -\cot\theta - \theta + c \qquad \ldots(1)$$

We have $x = 3\sin\theta \Rightarrow \sin\theta = \frac{x}{3}\cdot$ So, $\theta = \sin^{-1}\left(\frac{x}{3}\right)\cdot$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} = \frac{\sqrt{1-\left(\frac{x}{3}\right)^2}}{\frac{x}{3}} = \frac{\sqrt{9-x^2}}{x}$$

Thus, equation becomes $\int \frac{\sqrt{9-x^2}}{x^2}dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + c$

Example 23 Evaluate $\int \frac{1}{x\sqrt{x^4+1}}dx$

Solution: $\int \frac{1}{x\sqrt{x^4+1}}dx = \frac{1}{2}\int \frac{2x}{x^2\sqrt{(x^2)^2+1}}dx$

Let $x^2 = \tan\theta$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $2x\,dx = \sec^2\theta\,d\theta$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\tan\theta\sqrt{\tan^2\theta+1}}d\theta = \frac{1}{2}\int \frac{\sec^2\theta}{\tan\theta\sec\theta}d\theta$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}d\theta = \frac{1}{2}\int \csc\theta\,d\theta = \frac{1}{2}\ln|\csc\theta - \cot\theta| + c$$

$$= \frac{1}{2}\ln\left|\sqrt{1+\cot^2\theta} - \cot\theta\right| + c = \frac{1}{2}\ln\left|\sqrt{1+\left(\frac{1}{x^2}\right)^2} - \frac{1}{x^2}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{x^4+1}}{x^2} - \frac{1}{x^2}\right| + c = \frac{1}{2}\ln\left|\frac{\sqrt{x^4+1}-1}{x^2}\right| + c$$

Example 23 Evaluate $\int \frac{\sqrt{x^2-9}}{x}dx$, $(x>3)$

Solution: Let $x = 3\sec\theta$, then $dx = 3\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{x^2-9}}{x}dx = \int \frac{\sqrt{(3\sec\theta)^2-9}}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \sqrt{9(\sec^2\theta-1)}\cdot\tan\theta\,d\theta = \int 3\tan\theta\tan\theta\,d\theta$$

$$= 3\int \tan^2\theta\,d\theta = 3\int(\sec^2\theta-1)\,d\theta = 3(\tan\theta-\theta)+c = 3\left[\sqrt{\sec^2\theta-1}-\theta\right]+c$$

$$= 3\left[\sqrt{\left(\frac{x}{3}\right)^2-1} - \sec^{-1}\left(\frac{x}{3}\right)\right]+c = \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right)+c$$

Example 24 Evaluate $\int \frac{(x+1)e^x}{1+x^2e^{2x}}dx$

Solution: Let $u = xe^x$, then $\frac{du}{dx} = xe^x + e^x(1) \Rightarrow du = (x+1)e^x\,dx$

$$\int \frac{(x+1)e^x}{1+x^2e^{2x}}dx = \int \frac{(x+1)e^x}{1+(xe^x)^2}dx = \int \frac{du}{1+u^2} = \tan^{-1}u + c = \tan^{-1}(xe^x)+c$$

Example 26 Evaluate $\int \frac{1}{x\sqrt{1-x^3}}dx,\ x \in (-\infty, 0)\cup(0, 1)$.

Solution: $\int \frac{1}{x\sqrt{1-x^3}}dx$

$$= -\frac{1}{3}\int \frac{-3x^2}{x^3\ \sqrt{1-x^3}}\ dx \qquad \text{Put } u=\sqrt{1-x^3} \Rightarrow u^2 = 1-x^3 \Rightarrow 2u\ du = -3x^2\ dx$$

$$= -\frac{1}{3}\int \frac{2u}{(1-u^2)u}du = \frac{2}{3}\int \frac{1}{u^2-1^2}\ du = \frac{2}{3}\frac{1}{2(1)}\ \ln\left|\frac{u-1}{u+1}\right| + c = \frac{1}{3}\ln\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + c$$

Example 27 Evaluate $\int \frac{2x}{\sqrt{x^4+9}}\ dx$

Solution: Let $u=x^2$, then $du = 2xdx$

The integral becomes:

$$\int \frac{1}{\sqrt{u^2+9}}\ du = \int \frac{1}{\sqrt{u^2+3^2}}\ du = \ln\left|u+\sqrt{u^2+3^2}\right| + c = \ln\left|u+\sqrt{u^2+9}\right| + c$$

Substitute u with x^2:

$$\int \frac{1}{\sqrt{u^2+9}}\ du = \ln\left|x^2+\sqrt{x^4+9}\right| + c$$

Example 28 $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Solution: We have x^2-4x+3

$$x^2-4x+3 = (x^2-4x+4)+3-4 = (x-2)^2-1$$

$$\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}$$

Let $u = x-2$, then $du = dx$, the integral becomes

$$\int \frac{du}{u\sqrt{u^2-1}} = \int \frac{du}{u\sqrt{u^2-1^2}} = \frac{1}{1}\sec^{-1}\left|\frac{u}{1}\right| + c = \sec^{-1}|u| + c$$

Substitute u with $x-2$

$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|x-2| + c$$

EXERCISE 3.2

1. Evaluate the following integrals: (By using suitable substitution)

(i) $\int x(7x^2+8)^9\,dx$ (ii) $\int \frac{x^2}{(x^3+2)^5}\,dx$ (iii) $\int\left(1+\frac{1}{x}\right)^{-2}\left(\frac{1}{x^2}\right)dx$

(iv) $\int \frac{(1+\sqrt{x})^5}{\sqrt{x}}\,dx$ (v) $\int (7-x^5)^4\,x^4\,dx$ (vi) $\int \frac{t^2-2t}{(t^3-3t^2+5)^4}\,dt$

(vii) $\int \frac{\sec^2 x}{\tan x\cdot \ln\tan x}\,dx$ (viii) $\int \frac{\sin x-\cos x}{\sqrt{1+\sin 2x}}\,dx$ (ix) $\int \frac{\tan^3 x}{\sec^4 x+\tan^4 x}\,dx$

(x) $\int \frac{x}{1+x\tan x}\,dx$ (xi) $\int \frac{\ln\sec x}{\cot x}\,dx$ (xii) $\int \frac{\ln(\sin x)}{\tan x}\,dx$

(xiii) $\int \frac{e^{2x}}{(2+e^{2x})^2}\,dx$

2. Evaluate the following integrals: (By change of variable)

(i) $\int \frac{e^x}{9-e^{2x}}\,dx$ (ii) $\int \frac{\cos x}{\sin^2 x-4}\,dx$ (iii) $\int \frac{e^x}{1+e^{2x}}\,dx$

(iv) $\int \frac{1}{x\sqrt{4-(\ln x)^2}}\,dx$ (v) $\int \frac{1}{9x^2-25}\,dx$ (vi) $\int \frac{dx}{x\sqrt{3x^2-12}}$

(vii) $\int \frac{x^2}{x^2+a^2}\,dx$ (viii) $\int \frac{1}{x\left[1+(\ln x)^2\right]}\,dx$ (ix) $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

(x) $\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$ (xi) $\int \frac{dx}{x\sqrt{x^2-9}}$ (xii) $\int \sqrt{\frac{1+x}{1-x}}\,dx$

(xiii) $\int \frac{\sqrt{a^2-x^2}}{x}\,dx \quad (a>0)$

3.4 Integration by Parts

The rule that corresponds to the Product Rule for differentiation is called **integration by parts**. Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx$$

Product Rule in Integral Form

Suppose that $f(x)$ and $h(x)$ are two differentiable functions. According to the product rule,

$$\frac{d}{dx}\left[f(x)h(x)\right]=f(x)h'(x)+h(x)f'(x)$$

Integrating both sides with respect to x gives:

$$f(x)\ h(x)=\int\left[f(x)h'(x)+h(x)f'(x)\right]dx$$

Splitting the integral, we have:

$$f(x)\ h(x)=\int f(x)h'(x)\ dx+\int h(x)f'(x)\ dx$$

Rearranging terms, we obtain:

$$\int f(x)h'(x)\ dx=f(x)h(x)-\int h(x)f'(x)\ dx \qquad \ldots(1)$$

Now, let $h'(x)=g(x)$, then $h(x)=\int g(x)\,dx$.

Substituting into (1), we get:

$$\int f(x)g(x)dx=f(x)\left(\int g(x)dx\right)-\int\left[\left(\int g(x)dx(f'(x)\right)\right]dx \qquad \ldots(2)$$

Example 29 Evaluate $\int x\sin nx\,dx,\ n\neq 0.$

Solution: $\int x\sin nx\,dx$; Integrating by parts taking x as first function, we have

$$=x\left(\frac{-\cos nx}{n}\right)-\int\left(\frac{-\cos nx}{n}\right)(1)dx=\frac{-x\cos nx}{n}+\frac{1}{n}\int\cos nx\,dx$$

$$=\frac{-x\cos nx}{n}+\frac{1}{n}\cdot\frac{\sin nx}{n}+c=\frac{-x\cos nx}{n}+\frac{\sin nx}{n^2}+c$$

Example 30 Evaluate $\int x\,e^{ax}\,dx$

Solution: Applying the formula for integration by parts, we have

$$\int x\,e^{ax}\ dx=x\frac{e^{ax}}{a}-\int\frac{e^{ax}}{a}\cdot 1\,dx=x\frac{e^{ax}}{a}-\frac{1}{a}\int e^{ax}\ dx=x\frac{e^{ax}}{a}-\frac{1}{a}\cdot\frac{e^{ax}}{a}+c$$

$$=e^{ax}\left(\frac{x}{a}-\frac{1}{a^2}\right)+c=e^{ax}\left(\frac{ax-1}{a^2}\right)+c$$

Example 30 Evaluate $\int\frac{x}{\cos^2 x}dx$

Solution: $\int\frac{x}{\cos^2 x}dx=\int x\sec^2\ x\ dx$

Applying the formula for integration by parts, we have

$$\int x\sec^2 x\,dx=x\cdot\tan x-\int\tan x\cdot 1\,dx=x\cdot\tan x-\int\frac{\sin x}{\cos x}dx$$

$$=x\cdot\tan x+\int\frac{-\sin x}{\cos x}\ dx=x\cdot\tan x+\ln\left|\cos\ x\right|+c$$

Example 32 Evaluate $\int \ln(\sin x)\cdot \sin 2x\,dx$

Solution: $\int \ln(\sin x)\cdot \sin 2x\,dx = \int \ln(\sin x)\cdot 2\sin x\cdot\cos x\,dx$

$= 2\int \ln(\sin x)\cdot \sin x\cos x\,dx$ Put $u = \sin x$. $\frac{du}{dx} = \cos x \Rightarrow du = \cos x\,dx$

$$= 2\int \ln u\cdot u\,du = 2\left[\ln u\cdot\frac{u^2}{2} - \int\frac{u^2}{2}\cdot\frac{1}{u}du\right] = \ln u\cdot u^2 - \int u\,du = u^2\ln u - \frac{u^2}{2} + c$$

$$= \sin^2 x\,\ln(\sin x) - \frac{\sin^2 x}{2} + c$$

$$= \sin^2 x\left[\ln(\sin x) - \frac{1}{2}\right] + c$$

> **Note!**
> $\int e^{g(x)}[f(x)g'(x) + f'(x)]dx = e^{g(x)}f(x) + c$

Example 33 Evaluate $\int e^{x^2}(\cos x + 2x\sin x)\,dx$

Solution: $\int e^{x^2}(\cos x + 2x\sin x)\,dx = \int e^{x^2}\cos x\,dx + \int e^{x^2}\,2x\sin x\,dx$

$$= e^{x^2}(\sin x) - \int(\sin x)e^{x^2}\,2x\,dx + \int e^{x^2}\,2x\sin x\,dx = e^{x^2}\sin x + c$$

Example 34 Evaluate $\int \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)dx$

Solution: $\int \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)dx$; Putting $x = \tan\theta$; $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $dx = \sec^2\theta\,d\theta$

$$= \int \sin^{-1}\left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\right)\cdot\sec^2\theta\,d\theta = \int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\cdot\sec^2\theta\,d\theta$$

$$= \int \sin^{-1}\left(\frac{\sin\theta}{\cos\theta}\cos\theta\right)\cdot\sec^2\theta\,d\theta = \int \sin^{-1}(\sin\theta)\cdot\sec^2\theta\,d\theta$$

$$= \int \theta\cdot\sec^2\theta\,d\theta = \theta\tan\theta - \int(\tan\theta)(1)\,d\theta = \theta\tan\theta - \ln|\sec\theta| + c$$

$$= \theta\tan\theta - \ln|\sqrt{\sec^2\theta}| + c = \theta\tan\theta - \frac{1}{2}\ln|1+\tan^2\theta| + c = \left(\tan^{-1}x\right)\cdot x - \frac{1}{2}\ln|1+x^2| + c$$

$$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$$

Example 35 Show that $\int e^{ax}\sin bx\,dx = \frac{1}{\sqrt{a^2+b^2}}\,e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

Solution: Let $I = \int e^{ax}\sin bx\,dx$ Integrating by parts, we have

$$= e^{ax}\left(-\frac{\cos bx}{b}\right) - \int\left(-\frac{\cos bx}{b}\right)e^{ax}\cdot a\,dx$$

$$=-\frac{e^{ax}\cos bx}{b}+\frac{a}{b}\int e^{ax}\cos bx\,dx$$ Again, integrating by parts, we have

$$=-\frac{e^{ax}\cos bx}{b}+\frac{a}{b}\left[e^{ax}\left(\frac{\sin bx}{b}\right)-\int\left(\frac{\sin bx}{b}\right)e^{ax}\cdot a\,dx\right]$$

$$=-\frac{e^{ax}\cos bx}{b}+\frac{ae^{ax}\sin bx}{b^2}-\frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

$$I=-\frac{e^{ax}\cos bx}{b}+\frac{ae^{ax}\sin bx}{b^2}-\frac{a^2}{b^2}I$$

$$I+\frac{a^2}{b^2}I=-\frac{e^{ax}\cos bx}{b}+\frac{ae^{ax}\sin bx}{b^2}$$

$$\left(\frac{b^2+a^2}{b^2}\right)I=\frac{e^{ax}}{b^2}(a\sin bx-b\cos bx)\ \Rightarrow\ I=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)$$

Substituting $a=r\cos\theta$ and $b=r\sin\theta$, we have

$$I=\frac{e^{ax}}{a^2+b^2}(r\cos\theta\sin bx-r\sin\theta\cos bx)$$

$$=\frac{1}{a^2+b^2}e^{ax}r(\sin bx\cos\theta-\cos bx\sin\theta)=\frac{1}{a^2+b^2}e^{ax}r\sin(bx-\theta)$$

Since $a=r\cos\theta$ and $b=r\sin\theta$

Squaring and adding, we have $r^2=a^2+b^2\ \Rightarrow\ r=\sqrt{a^2+b^2}$ and $\frac{r\sin\theta}{r\cos\theta}=\frac{b}{a}$

$$\Rightarrow\ \tan\theta=\frac{b}{a}\ \Rightarrow\ \theta=\tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore\ I=\frac{1}{a^2+b^2}e^{ax}\sqrt{a^2+b^2}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)=\frac{1}{\sqrt{a^2+b^2}}e^{ax}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)+c$$

Example 36 Evaluate $\int x\sin^2 x\,dx$

Solution: $\int x\sin^2 x\,dx=\int x.\frac{1-\cos 2x}{2}dx=\frac{1}{2}\int(x-x\cos 2x)\,dx$

$$=\frac{1}{2}\int x\,dx-\frac{1}{2}\int x\cos 2x\,dx$$ Integrating the second integral by parts

$$=\frac{1}{2}\left(\frac{x^2}{2}\right)-\frac{1}{2}\left[x\left(\frac{\sin 2x}{2}\right)-\int\left(\frac{\sin 2x}{2}\right)\cdot 1\,dx\right]=\frac{x^2}{4}-\frac{x\sin 2x}{4}+\frac{1}{4}\int\sin 2x\,dx$$

$$=\frac{x^2}{4}-\frac{x\sin 2x}{4}+\frac{1}{4}\left(-\frac{\cos 2x}{2}\right)+c=\frac{x^2}{4}-\frac{x\sin 2x}{4}-\frac{1}{8}\cos 2x+c$$

Special Integral

$$\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c$$

Proof: Let $I=\int\sqrt{x^2+a^2}\,dx$

$=\int\sqrt{x^2+a^2}\,(1)\,dx$ Integrating by parts, we have

$$=\sqrt{x^2+a^2}\,(x)-\int x\frac{d}{dx}\left[\left(x^2+a^2\right)^{\frac{1}{2}}\right]dx=x\sqrt{x^2+a^2}-\int x\cdot\frac{1}{2}\left(x^2+a^2\right)^{-\frac{1}{2}}\cdot(2x)\,dx$$

$$=x\sqrt{x^2+a^2}-\int\frac{x^2}{\sqrt{x^2+a^2}}dx=x\sqrt{x^2+a^2}-\int\frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$$

$$=x\sqrt{x^2+a^2}-\int\frac{x^2+a^2}{\sqrt{x^2+a^2}}+\int\frac{a^2}{\sqrt{x^2+a^2}}dx$$

$$=x\sqrt{x^2+a^2}-\int\sqrt{x^2+a^2}\,dx+a^2\int\frac{1}{\sqrt{x^2+a^2}}\,dx$$

$$I=x\sqrt{x^2+a^2}-I+a^2\int\frac{1}{\sqrt{x^2+a^2}}dx$$

$$2I=x\sqrt{x^2+a^2}+a^2\ln\left|x+\sqrt{x^2+a^2}\right|+c_1$$

$$I=\frac{1}{2}x\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c$$

Thus, $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c$

Challenge!

Prove that: (i) $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

(ii) $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+c$

Example 37 Evaluate $\int\sqrt{x^2-4x+13}\,dx$

Solution: $\int\sqrt{x^2-4x+13}\,dx=\int\sqrt{(x^2-4x+4)+13-4}\,dx=\int\sqrt{(x-2)^2+9}\,dx$

$=\int\sqrt{(x-2)^2+(3)^2}\,dx$

Let $x-2=u$, so $du=dx$

$$\int\sqrt{(x-2)^2+(3)^2}\,dx=\int\sqrt{u^2+3^2}\,du=\frac{u}{2}\sqrt{u^2+3^2}+\frac{3^2}{2}\ln\left|u+\sqrt{u^2+3^2}\right|+c$$

Replace u with $x-2$

$$=\frac{x-2}{2}\sqrt{(x-2)^2+9}+\frac{9}{2}\ln\left|(x-2)+\sqrt{(x-2)^2+9}\right|+c$$

$$=\frac{x-2}{2}\sqrt{x^2-4x+13}+\frac{9}{2}\ln\left|(x-2)+\sqrt{x^2-4x+13}\right|+c$$

Example 38 Evaluate $\int\sqrt{8x-x^2-7}\,dx$

Solution: $\int\sqrt{8x-x^2-7}\,dx=\int\sqrt{-(x^2-8x)-7}\,dx$

$$=\int\sqrt{-(x^2-8x+16-16)-7}\,dx=\int\sqrt{-[(x-4)^2-16]-7}\,dx$$

$$=\int\sqrt{3^2-(x-4)^2}\,dx \qquad \text{Let } x-4=u, \text{ so } dx=du$$

$$=\int\sqrt{3^2-u^2}\,du=\frac{u}{2}\sqrt{3^2-u^2}+\frac{3^2}{2}\sin^{-1}\left(\frac{u}{3}\right)+c$$

Substitute back $u=x-4$

$$=\frac{x-4}{2}\sqrt{9-(x-4)^2}+\frac{9}{2}\sin^{-1}\left(\frac{x-4}{3}\right)+c=\frac{x-4}{2}\sqrt{8x-x^2-7}+\frac{9}{2}\sin^{-1}\left(\frac{x-4}{3}\right)+c$$

Example 39 Evaluate $\int\sqrt{x^2-8x+7}\,dx$

Solution: $\int\sqrt{x^2-8x+7}\,dx=\int\sqrt{(x^2-8x+16)+7-16}\,dx$

$$=\int\sqrt{(x-4)^2-9}\,dx=\int\sqrt{(x-4)^2-3^2}\,dx \quad \text{Let } u=x-4, \text{ so } du=dx$$

$$=\int\sqrt{u^2-3^2}\,du=\frac{u}{2}\sqrt{u^2-3^2}-\frac{3^2}{2}\ln\left|u+\sqrt{u^2-3^2}\right|+c$$

Substitute back $u=x-4$

$$=\frac{x-4}{2}\sqrt{(x-4)^2-9}-\frac{9}{2}\ln\left|x-4+\sqrt{(x-4)^2-9}\right|+c$$

$$=\frac{x-4}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\ln\left|x-4+\sqrt{x^2-8x+7}\right|+c$$

EXERCISE 3.3

Evaluate the following integrals:

1. $\int\ln x\,dx$
2. $\int x^7\cdot\ln x\,dx$
3. $\int x\cos 2x\,dx$
4. $\int\frac{x}{\sin^2 x}dx$
5. $\int xe^{2x}\,dx$
6. $\int x^2e^{ax}\,dx$

7. $\int x\tan^2 x\,dx$ 8. $\int \ln(\sin x)\cdot\cos x\,dx$ 9. $\int e^x \ln(1+e^x)\,dx$

10. $\int (2x^3+2x)e^{x^2}\,dx$ 11. $\int e^x \tan^{-1}(e^x)\,dx$ 12. $\int \frac{x}{1+\cos x}\,dx$

13. $\int \frac{x}{1-\sin x}\,dx$ 14. $\int \frac{x\cos x-\sin x}{x^2}\,dx$ 15. $\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\,dx$

16. $\int \sqrt{x}\sin\sqrt{x}\,dx$ 17. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ 18. $\int \frac{e^{\tan^{-1}x}}{(1+x^2)^2}\,dx$

19. $\int \sin^{-1}x\,dx$ 20. $\int \frac{\ln x\tan^{-1}(\ln x)}{x}\,dx$ 21. $\int \csc^3 x\,dx$

22. $\int e^{5x}(x^5+x^4+1)\,dx$ 23. $\int \frac{e^{\sqrt{x}}(1+\sqrt{x})}{\sqrt{x}}\,dx$ 24. $\int e^{3x}(\sin x-3\cos 3x)\,dx$

3.5 Integration by Using Partial Fractions

If $P(x), Q(x)$ are polynomial functions and the denominator $Q(x)$ $(x\neq 0)$, in the rational function $\frac{P(x)}{Q(x)}$, can be factorized into linear and quadratic (irreducible) factors, then the rational function $\frac{P(x)}{Q(x)}$ is written as a sum of simpler rational functions, each of which can be integrated by methods already known to us.

Here, we will give examples of the following four cases when the denominator $Q(x)$ contains

Case-I: Non-repeated linear factors.
Case-II: Repeated and non-repeated linear factors.
Case-III: Linear and non repeated irreducible quadratic factors.
Case-IV: Linear and repeated irreducible quadratic factors.

Case-I: Integration of $\int \frac{P(x)}{Q(x)}\,dx$ where $Q(x)$ has non-repeated linear factors

Example 40 Integrate $\int \frac{3x-2}{x(x-1)(x-2)}\,dx$

Solution: $\int \frac{3x-2}{x(x-1)(x-2)}\,dx$

Let $\frac{3x-2}{x(x-1)(x-2)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-2}$

Multiplying both sides, by $x(x-1)(x-2)$ we have

$$3x-2=A(x-1)(x-2)+B(x)(x-2)+C(x)(x-1) \quad ...(1)$$

Putting $x=0$ in (1): $3(0)-2=A(0-1)(0-2)+B(0)(0-2)+C(0)(0-1)$

$$-2=A(-1)(-2)+B(0)+C(0) \Rightarrow -2=2A \Rightarrow \boxed{A=-1}$$

Putting $x=1$ in (1): $3(1)-2=A(1-1)(1-2)+B(1)(1-2)+C(1)(1-1)$

$$3-2=A(0)+B(-1)+C(0) \Rightarrow 1=-B \Rightarrow \boxed{B=-1}$$

Putting $x=2$ in (1): $3(2)-2=A(2-1)(2-2)+B(2)(2-2)+C(2)(2-1)$

$$6-2=A(0)+B(0)+C(2)(1) \Rightarrow 4=2C \Rightarrow \boxed{C=2}$$

$$\therefore \quad \frac{3x-2}{x(x-1)(x-2)}=\frac{-1}{x}+\frac{-1}{x-1}+\frac{2}{x-2}$$

Thus, $\int \frac{3x-2}{x(x-1)(x-2)}dx=\int\left[\frac{-1}{x}+\frac{-1}{x-1}+\frac{2}{x-2}\right]dx$

$$=-\int\frac{1}{x}dx-\int\frac{1}{x-1}dx+2\int\frac{1}{x-2}\,dx=-\ln|x|-\ln|x-1|+2\ln|x-2|+C$$

Case-II Integration of $\int\frac{P(x)}{Q(x)}dx$ where $Q(x)$ has repeated linear factors:

Example 41 Evaluate $\int\frac{6x-10}{(x+1)(x-3)^2}dx$

Solution: Let $\frac{6x-10}{(x+1)(x-3)^2}=\frac{A}{x+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$

Multiplying both sides by $(x+1)(x-3)^2$:

$$6x-10=A(x-3)^2+B(x+1)(x-3)+C(x+1) \quad \ldots(1)$$

From equation (1),

$$0.x^2+6x-10=A(x^2-6x+9)+B(x^2-2x-3)+C(x+1) \quad \ldots(2)$$

Let $x+1=0 \Rightarrow x=-1$ Putting $x=-1$ in (1)

$$6(-1)-10=A(-1-3)^2+B(0)+C(0)$$

$$-6-10=A(-4)^2 \Rightarrow -16=16A \Rightarrow \boxed{A=-1}$$

Let $x-3=0 \Rightarrow x=3$ Putting $x=3$ in (1)

$$6(3)-10=A(0)^2+B(0)+C(3+1)$$

$$18-10=4C \Rightarrow 8=4C \Rightarrow \boxed{C=2}$$

Comparing coefficient of x^2 on both sides of equation (2)

$$A+B=0 \Rightarrow B=-A \Rightarrow B=-(-1) \Rightarrow \boxed{B=1}$$

So (1) becomes

$$\frac{6x-10}{(x+1)(x-3)^2}=\frac{-1}{x+1}+\frac{1}{x-3}+\frac{2}{(x-3)^2}$$

Thus, $\int\frac{6x-10}{(x+1)(x-3)^2}dx=\int\left[\frac{-1}{x+1}+\frac{1}{x-3}+\frac{2}{(x-3)^2}\right]dx$

$$=-\int\frac{1}{x+1}\,dx+\int\frac{1}{x-3}\,dx+2\int(x-3)^{-2}\,dx=-\ln|x+1|+\ln|x-3|+2\frac{(x-3)^{-2+1}}{-2+1}+c$$

$$=\ln|x-3|-\ln|x+1|+2\frac{(x-3)^{-1}}{-1}+c=\ln\left|\frac{x-3}{x+1}\right|-\frac{2}{x-3}+c$$

Case-III Integration of $\int\frac{P(x)}{Q(x)}dx$ where $Q(x)$ has at least one non-repeated irreducible quadratic factor.

Example 42 Find $\int\frac{5x^2+3x+2}{(x+1)(x^2+1)}dx$

Solution: $\frac{5x+3x+2}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$

Multiplying both sides by $(x+1)(x^2+1)$ to clear fractions:

$5x^2+3x+2=A(x^2+1)+(Bx+C)(x+1)$ …(1)

Put $x+1=0 \Rightarrow x=-1$ in equation (1)

$5(-1)^2+3(-1)+2=A[(-1)^2+1]+[B(-1)+C](-1+1)$

$5-3+2=A(2)+0 \Rightarrow 4=2A \Rightarrow \boxed{A=2}$

From (1)

$5x^2+3x+2=Ax^2+A+Bx^2+Bx+Cx+C$

$5x^2+3x+2=(A+B)x^2+(B+C)x+C+A$

Equating coefficients of x^2, we get

$5=A+B \Rightarrow 5=2+B \Rightarrow \boxed{B=3}$

Equating constants, we get

$2=A+C \Rightarrow 2=C+2 \Rightarrow \boxed{C=0}$

Now, the integrand becomes,

$$\frac{5x+3x+2}{(x+1)(x^2+1)}=\frac{2}{x+1}+\frac{3x+0}{x^2+1}$$

$$\int\frac{5x+3x+2}{(x+1)(x^2+1)}=2\int\frac{1}{x+1}\,dx+\int\frac{3x}{x^2+1}\,dx=2\int\frac{1}{x+1}\,dx+\frac{3}{2}\int\frac{2x}{x^2+1}\,dx$$

$$=2\ln|x+1|+\frac{3}{2}\ln|x^2+1|+c=2\ln|x+1|+\frac{3}{2}\ln(x^2+1)+c$$

Case-IV Integration of $\int \frac{P(x)}{Q(x)}dx$ where $Q(x)$ has repeated irreducible quadratic factors:

Example 43 Evaluate $\int \frac{x^3+2x+1}{(x^2+1)^2}\,dx$

Solution: $\frac{x^3+2x+1}{(x^2+1)^2} = \frac{x^3+x+x+1}{(x^2+1)^2} = \frac{x(x^2+1)+x+1}{(x^2+1)^2} = \frac{x(x^2+1)}{(x^2+1)^2} + \frac{x+1}{(x^2+1)^2}$

$$= \frac{x}{x^2+1} + \frac{x}{(x^2+1)^2} + \frac{1}{(x^2+1)^2}$$

Thus, $\int \frac{x^3+2x+1}{(x^2+1)^2}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx + \int \frac{1}{(x^2+1)^2}\,dx$

$$= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int (x^2+1)^{-2}(2x)\,dx + \int \frac{1}{(x^2+1)^2}\,dx$$

$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\frac{(x^2+1)^{-1}}{-1} + \int \frac{1}{(x^2+1)^2}\,dx \quad \ldots(1)$$

To evaluate $\int \frac{1}{(x^2+1)^2}\,dx$, put $x=\tan\theta$, $dx = \sec^2\theta\, d\theta$

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{1}{(\tan^2\theta+1)^2}\cdot\sec^2\theta\, d\theta = \int \frac{1}{(\sec^2\theta)^2}\cdot \sec^2\theta\, d\theta = \int \frac{1}{\sec^2\theta}\, d\theta$$

$$= \int \cos^2\theta\, d\theta = \int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] = \frac{1}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right]$$

$$= \frac{1}{2}[\theta + \sin\theta\cos\theta] = \frac{1}{2}\left[\theta + \frac{\sin\theta\cos^2\theta}{\cos}\right] = \frac{1}{2}\left[\theta + \frac{\tan\theta}{\sec^2\theta}\right] = \frac{1}{2}\left[\theta + \frac{\tan\theta}{1+\tan^2\theta}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}x + \frac{x}{1+x^2}\right]$$ (1) becomes

Finally: $\int \frac{x^3+2x+1}{(x^2+1)^2}\,dx = \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\frac{1}{x^2+1} + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{1+x^2} + c$

$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right) + c$$

$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{x-1}{2(x^2+1)} + c$$

EXERCISE 3.4

Evaluate the following integrals:

1. $\int \frac{5x+7}{(x+1)(x+2)}dx$
2. $\int \frac{3x+5}{x^2+4x+3}dx$
3. $\int \frac{x^3+2x^2-x-1}{x^2-1}dx$
4. $\int \frac{x^2-12x-21}{x^2-6x-7}dx$
5. $\int \frac{3-16x}{4+5x-6x^2}dx$
6. $\int \frac{6x^2-22x+18}{x(x-1)(x-2)(x-3)}dx$
7. $\int \frac{x^2-3}{(x+2)^3}dx$
8. $\int \frac{e^x-1}{e^x\left(e^x+1\right)}dx$
9. $\int \frac{25(x+2)}{(x-2)(x+3)^2}dx$
10. $\int \frac{x^3+4x^2+5x+2}{(x+1)^3(x+2)}dx$
11. $\int \frac{x^2+2x-1}{(x-1)(x^2+1)}dx$
12. $\int \frac{x^3+2x+1}{(x^2+1)(x^2+4)}dx$
13. $\int \frac{x^4+3x^2+x+1}{(x^2+1)^2(x-1)}dx$
14. $\int \frac{\cos x}{\sin^3 x-1}dx$
15. $\int \frac{x^2+1}{(x^2-2x+2)^2}dx$

3.6 The Definite Integral

Definition: Let f be a continuous function on the closed interval $[a, b]$. The **definite integral** of f from a to b is defined as $\int_a^b f(x)\,dx = F(b)-F(a)$, where $F'(x) = f(x)$.

Here, a is the **lower limit** and b the **upper limit** of the integration.

Remark: The value of a definite integral is a real number, not a family of functions, as is the case for indefinite integrals. The numbers a and b are the **limits of integration**.

Theorem 3.2: First Fundamental Theorem of Calculus (FTC)

If f is continuous on $[a, b]$ and $F(x)=\int_a^x f(t)\,dt$ for every x in $[a, b]$,

then $F'(x)=f(x)$, on $[a, b]$.

The derivative with respect to upper limit of the integral of f is just $f(x)$.

Theorem 3.3: Second Fundamental Theorem of Calculus (FTC)

If f is continuous on $[a, b]$ and $F'(x)=f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

Example 44 Evaluate: $\int_{-1}^{1}(x^{\frac{1}{5}}+1)\,dx$

Solution: $\int_{-1}^{1}(x^{\frac{1}{5}}+1)\,dx=\left(\dfrac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1}+x\right)_{-1}^{1}=\left(\dfrac{5}{6}x^{\frac{6}{5}}+x\right)_{-1}^{1}$

$$=\left(\frac{5}{6}(1)^{\frac{6}{5}}+1\right)-\left(\frac{5}{6}(-1)^{\frac{6}{5}}+(-1)\right)=\frac{5}{6}+1-\frac{5}{6}+1=2$$

Example 45 Evaluate $\int_{0}^{\pi/4}\dfrac{\sec\theta}{\sin\theta+\cos\theta}d\theta$

Solution: $\int_{0}^{\pi/4}\dfrac{\sec\theta}{\sin\theta+\cos\theta}d\theta=\int_{0}^{\pi/4}\dfrac{\sec\theta}{\cos\theta(\tan\theta+1)}d\theta=\int_{0}^{\pi/4}\dfrac{\sec^2\theta}{1+\tan\theta}d\theta$

$$=\left[\ln|1+\tan\theta|\right]_0^{\pi/4}=\ln\left|1+\tan\frac{\pi}{4}\right|-\ln|1+0|=\ln|1+1|-0=\ln 2$$

Example 46 Evaluate $\int_{0}^{\sqrt{7}}\dfrac{x}{\sqrt{x^2+9}}dx$

Solution: Let $f(x)=x^2+9$, then $f'(x)=2x$, so

$$\int_{0}^{\sqrt{7}}\frac{x}{\sqrt{x^2+9}}dx=\frac{1}{2}\int_{0}^{\sqrt{7}}(x^2+9)^{-\frac{1}{2}}(2x)\,dx=\frac{1}{2}\left|\frac{(x^2+9)^{\frac{1}{2}}}{\frac{1}{2}}\right|_0^{\sqrt{7}}$$

$$=\left[\sqrt{x^2+9}\right]_0^{\sqrt{7}}=\left[\sqrt{7+9}-\sqrt{0+9}\right]=\sqrt{16}-\sqrt{9}=4-3=1$$

Substitution in a Definite Integral (Change of variables)

Suppose that g is differentiable function on $[a, b]$. Also suppose that f is continuous on the range of g. Then $\int_{a}^{b}f(g(x))g'(x)\,dx=\int_{g(a)}^{g(b)}f(u)\,du.$

Example 47 Evaluate $\int_{0}^{2}2xe^{x^2}dx$

Solution: Let $g(x)=x^2$. Then $g'(x)=2x$

The integrand is $e^{x^2}\cdot 2x=f(g(x))\cdot g'(x)$ with $f(u)=e^u$

When $x=0$, $u=g(0)=0^2=0$

When $x = 2$, $u = g(2) = 2^2 = 4$

Therefore, $\int_0^2 2xe^{x^2}dx = \int_{u=0}^{u=4} e^u\,du = \int_0^4 e^u\,du = \left[e^u\right]_0^4 = e^4 - e^0 = e^4 - 1$

Integration by Parts Formula for Definite Integrals

Let $f(x)$ and $g(x)$ be functions with continuous derivatives on the interval $[a, b]$. Then the integration by parts formula for definite integrals is:

$$\int_a^b f(x)g(x)\,dx = \left[f(x)\left(\int g(x)dx\right)\right]_a^b - \int_a^b\left[\left(\int g(x)dx\right)\left(f'(x)\right)\right]dx$$

Example 48 If $f(0) = f(\pi) = 0$, show that $\int_0^\pi f(x)\sin x\,dx = -\int_0^\pi f''(x)\sin x\,dx$

Solution: $\int_0^\pi f(x)\sin x\,dx$ Applying integration by parts:

$$= \left|f(x)(-\cos x)\right|_0^\pi - \int_0^\pi f'(x)(-\cos x)\,dx = -f(\pi)\cos\pi + f(0)\cos 0 + \int_0^\pi f'(x)\cos x\,dx$$

$$= -(0)(-1) + (0)(1) + \left|f'(x)\sin x\right|_0^\pi - \int_0^\pi \sin x\cdot f''(x)dx$$

$$= 0 + 0 + f'(\pi)\sin(\pi) - f'(0)\sin(0) - \int_0^\pi f''(x)\sin x dx$$

$$= f'(\pi)(0) - f'(0)\ (0) - \int_0^\pi f''(x)\sin x\,dx = -\int_0^\pi f''(x)\sin x\,dx$$

3.6.1 Properties of the Definite Integrals

Throughout this section, assume f is a continuous function on the given interval.

Property-I: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

Property-II: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

Remember!

$\int_a^a f(x)dx = 0$

Property-III: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$.

This result can be generalized as follows:

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \cdots + \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^b f(x)\,dx$$

where $a < c_1 < c_2 < c_3 < \cdots < c_{n-1} < c_n < b$

Property-IV: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$

(i) As a special case, if we take $a=0$, we obtain: $\int_0^b f(x)\,dx = \int_0^b f(b-x)\,dx$

(ii) As another special case, if we take $a=-b$, we obtain: $\int_{-b}^b f(x)\,dx = \int_{-b}^b f(-x)\,dx.$

(iii) For any integrand of the form $\dfrac{f(x)}{f(x)+g(x)}$, if there exists a substitution $x \to a+b-x$ such that $f(a+b-x)=g(x)$ and $g(a+b-x)=f(x)$, then $\int_a^b \dfrac{f(x)}{f(x)+g(x)}\,dx = \dfrac{b-a}{2}.$

Property-V: $\int_{-a}^a f(x)\,dx = \int_0^a \big(f(x)+f(-x)\big)\,dx$

In particular, if $f(-x)=f(x)$ (that is, f is an even function), then

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

and if $f(-x)=-f(x)$ (that is, f is an odd function), then $\int_{-a}^a f(x)\,dx = 0$

Property-VI: $\int_0^a f(x)\,dx = \int_0^{a/2} \big(f(x)+f(a-x)\big)\,dx$

In particular, $\int_0^a f(x)\,dx = \begin{cases} 0 & \text{if} \quad f(a-x)=-f(x) \\ 2\int_0^{a/2} f(x)\,dx & \text{if} \quad f(a-x)=f(x) \end{cases}$

Property-VII: If $f(x) \geq 0$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$ and if $f(x) \leq 0$ on the interval $[c, d]$, then $\int_c^d f(x)\,dx \leq 0.$

Property-VIII: If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \leq \int_c^b g(x)\,dx.$

Property-IX: $\int_a^b \big(\alpha f_1(x) + \beta f_2(x)\big)\,dx = \alpha\int_a^b f_1(x)\,dx + \beta\int_a^b f_2(x)\,dx.$

Example 49 Let $f(x)=\begin{cases} x^2 & \text{if } 0\le x<2 \\ 4-x & \text{if } 2\le x\le 4 \end{cases}$ Compute: $\int_0^4 f(x)dx.$

Solution: $\int_0^4 f(x)\,dx=\int_0^2 f(x)\,dx+\int_2^4 f(x)\,dx=\int_0^2 x^2\,dx+\int_2^4 (4-x)\,dx$

$$=\left[\frac{x^3}{3}\right]_0^2+\left[4x-\frac{x^2}{2}\right]_2^4=\left[\frac{(2)^3}{3}-\frac{(0)^3}{3}\right]+\left[\left(4(4)-\frac{(4)^2}{2}\right)-\left(4(2)-\frac{(2)^2}{2}\right)\right]$$

$$=\left[\frac{8}{3}-0\right]+(16-8)-(8-2)=\frac{8}{3}+8-6=\frac{8}{3}+2=\frac{8+6}{3}=\frac{14}{3}.$$

Example 50 Evaluate: $\int_{-6}^{4} f(x)\,dx$ where $f(x)=|x-1|+|x+2|$.

Solution: To find where the expressions inside the absolute values changes:

$$x-1=0 \Rightarrow x=1 \quad \text{and} \quad x+2=0 \Rightarrow x=-2$$

So, break the interval [–6, 4] at $x = -6, x = -2, x = 1, x = 4$.

These points divide the interval [–6, 4] into three intervals [–6, 2], [–2, 1], [1, 4].

Now, express $f(x)$ without absolute values and compute the integral over each subinterval:

(a) For $x\in[-6,-2]$: $|x-1|=-(x-1)=-x+1$ and $|x+2|=-(x+2)=-x-2$

$$f(x)=|x-1|+|x+2|=-x+1-x-2=-2x-1$$

$$\int_{-6}^{-2} f(x)\,dx=\int_{-6}^{-2}(-2x-1)\,dx=\left[-\frac{2x^2}{2}-x\right]_{-6}^{-2}=\left[-x^2-x\right]_{-6}^{-2}$$

$$=(-4+2)-(-36+6)=-2-(-30)=-2+30=28$$

(b) For $x\in[-2,1]$: $|x-1|=-(x-1)=-x+1$ and $|x+2|=x+2$

$$f(x)=|x-1|+|x+2|=-x+1+x+2$$

$$\int_{-2}^{1} f(x)\,dx=\int_{-2}^{1} 3\,dx=|3x|_{-2}^{1}=3(1)-3(-2)=3+6=9$$

(c) For $x\in[1,4]$: $|x-1|$ and $|x+2|=x+2$

$$f(x)=|x-1|+|x+2|=x-1+x+2=2x+1$$

$$\int_{1}^{4} f(x)\,dx=\int_{1}^{4}(2x+1)\,dx=[x^2+x]_1^4=[(4)^2+4]-[(1)^2+1]=16+4-2=18$$

From (a), (b) and (c), we have:

$$\int_{-6}^{4} f(x)\,dx=\int_{-6}^{-2} f(x)\,dx+\int_{-2}^{1} f(x)\,dx+\int_{1}^{4} f(x)\,dx=28+9+18=57.$$

Example 51 Evaluate $\int_0^1 x^2(1-x)^{\frac{3}{2}}\,dx$.

Solution: Let $I=\int_0^1 x^2(1-x)^{\frac{3}{2}}\,dx$

$$I=\int_0^1 (1-x)^2\left(1-(1-x)\right)^{\frac{3}{2}}dx \qquad \left(\text{By using } \int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right)$$

$$=\int_0^1 (1-2x+x^2)(1-1+x)^{\frac{3}{2}}dx=\int_0^1 (1-2x+x^2)x^{\frac{3}{2}}dx=\int_0^1\left(x^{\frac{3}{2}}-2x\,x^{\frac{3}{2}}+x^2x^{\frac{3}{2}}\right)dx$$

$$=\int_0^1\left(x^{\frac{3}{2}}-2x^{\frac{5}{2}}+x^{\frac{7}{2}}\right)dx=\left[\frac{2}{5}x^{\frac{5}{2}}-\frac{4}{7}x^{\frac{7}{2}}+\frac{2}{9}x^{\frac{9}{2}}\right]_0^1=\left(\frac{2}{5}(1)^{\frac{5}{2}}-\frac{4}{7}(1)^{\frac{7}{2}}+\frac{2}{9}(1)^{\frac{9}{2}}\right)-(0)$$

$$=\frac{2}{5}-\frac{4}{7}+\frac{2}{9}=\frac{126-180+70}{315}=\frac{16}{315}$$

Example 52 $\int_0^{2\pi}\cos^5 x\,dx$

Solution: Let $I=\int_0^{2\pi}\cos^5 x\,dx$

Here $f(x)=\cos^5 x \Rightarrow f(2\pi-x)=[\cos(2\pi-x)]^5=(\cos x)^5=\cos^5 x=f(x)$

$\therefore\ I=2\int_0^{\pi}\cos^5 x\,dx$

Also $f(\pi-x)=(\cos(\pi-x))^5=(-\cos x)^5=-\cos^5 x=-f(x)$

Thus, $I=\int_0^{2\pi}\cos^5 x\,dx=0.$

Example 53 Show that $\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2+\sin^2 x\ \cos^2 x}}dx=0.$

Solution: Let $I=\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2+\sin^2 x\ \cos^2 x}}dx$

Using the property: $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$ of definite integrals, we substitute x

with $\frac{\pi}{5}+\frac{3\pi}{10}-x=\frac{\pi}{2}-x$:

$$I=\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}-\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{2}+\sin^2\left(\frac{\pi}{2}-x\right)\cos^2\left(\frac{\pi}{2}-x\right)}dx=\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\cos x}-\sqrt{\sin x}}{\sqrt{2}+\sin^2 x\ \cos^2 x}dx$$

$$=-\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2}+\sin^2 x\ \cos^2 x}dx.\ \text{Clearly } I=-I\Rightarrow 2I=0\Rightarrow I=0.$$

That is, $\int_{\pi/5}^{3\pi/10}\frac{\sqrt{\sin x}-\sqrt{\cos x}}{\sqrt{2}+\sin^2 x\ \cos^2 x}dx=0.$

Example 54 Given $\int_{-3}^{3}\frac{f(x)}{1+e^x}dx=4$ and $\int_{-3}^{3}f(x)\,dx=10$. Evaluate $\int_{-3}^{3}\frac{f(x)}{1+e^{-x}}dx$.

Solution: Key identity: $\frac{1}{1+e^x}+\frac{1}{1+e^{-x}}=1.$

Multiplying the identity by $f(x)$, we get: $\frac{f(x)}{1+e^x}+\frac{f(x)}{1+e^{-x}}=f(x)$

Integrate over [–3, 3], we get: $\int_{-3}^{3}\frac{f(x)}{1+e^x}dx+\int_{-3}^{3}\frac{f(x)}{1+e^{-x}}dx=\int_{-3}^{3}f(x)\,dx$

Substitute the given values: $4+\int_{-3}^{3}\frac{f(x)}{1+e^{-x}}dx=10\Rightarrow\int_{-3}^{3}\frac{f(x)}{1+e^{-x}}dx=10-4=6$

Thus, $\int_{-3}^{3}\frac{f(x)}{1+e^{-x}}dx=6.$

Example 55 Evaluate $\int_{-a}^{a}\frac{x^3\sin^2 x}{x^4+2x^2+1}dx.$

Solution: Let $f(x)=\frac{x^3\sin^2 x}{x^4+2x^2+1}$

$$f(-x)=\frac{(-x)^3\sin^2(-x)}{(-x)^4+2(-x)^2+1}=\frac{-x^3\sin^2 x}{x^4+2x^2+1}=-f(x)$$

This shows that $f(x)$ is odd over the symmetric interval $[-a, a]$.

Hence, $\int_{-a}^{a}\frac{x^3\sin^2 x}{x^4+2x^2+1}dx=0$

Example 56 Evaluate $\int_1^3 f(x)\,dx$.

Solution: Since $f(2-x)=f(x) \Rightarrow \int_0^2 f(x)dx = 4 \Rightarrow 2\int_0^1 f(x)\,dx = 4 \Rightarrow \int_0^1 f(x)\,dx = 2$

Given $\int_0^2 f(x)dx = 4$ and $f(2-x)=f(x)=f(-x)$.

To evaluate $\int_1^3 f(x)\,dx$. Put $x = 2-t,\ dx = -dt$. When $x=1,\ t=1$; When $x=3,\ t=-1$

Therefore, $\int_1^3 f(x)\,dx = \int_1^{-1} f(2-t)(-dt) = -\int_{-1}^{1} f(2-t)(-dt) = \int_{-1}^{1} f(2-t)\,dt$

$= \int_{-1}^{1} f(2-x)\,dx = \int_{-1}^{1} f(x)\,dx = 2\int_0^1 f(x)\,dx = 2(2) = 4.$ Since $f(2-x)=f(x)=f(-x)$

EXERCISE 3.5

1. Evaluate the following definite integrals:

(i) $\int_1^2 (x^2+x)\,dx$ (ii) $\int_{-2}^{3} (4x^3+3x^2+2x)\,dx.$ (iii) $\int_0^{\pi/2} \sin^2 x\,dx$

(iv) $\int_0^{\pi/4} \frac{1}{1+\sin x}\,dx$ (v) $\int_0^{\pi/2} \cos^3\theta\,d\theta$ (vi) $\int_0^1 \frac{\sqrt{x}}{1+x}\,dx$

(vii) $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$ (viii) $\int_0^1 \tan^{-1} x\,dx$ (ix) $\int_0^{2\pi} \frac{\sin^2 x}{1+e^{\cos x}}\,dx$

2. Evaluate the following definite integrals:

(i) $\int_{-1}^{1} f(x)dx$, where $f(x)=\begin{cases} 5-3x & \text{if} \quad x\le 0 \\ 5+3x & \text{if} \quad x>0 \end{cases}$

(ii) $\int_{-3}^{5} f(x)dx$, where $f(x)=|x-4|+|x|$ (iii) $\int_{-3}^{4} |4-x^2|\,dx.$

3. Let $f(x)=\begin{cases} x^2 & \text{if} \quad 0\le x\le 2 \\ 2x+1 & \text{if} \quad 2<x\le 4 \end{cases}$, Here $g(x)$ is defined as: $g(x)=f(x)+f(4-x)$.

Compute: $\int_0^4 g(x)dx.$

4. Simplify: (i) $\int_0^1 f(x)\,dx - \int_0^2 f(x)\,dx - \int_3^1 f(x)\,dx + \int_3^2 f(x)\,dx.$

(ii) $\int_0^3 5f(x)\,dx - \int_1^3 5f(4-x)\,dx - \int_1^5 5f(x)\,dx - \int_0^5 4f(5-x)\,dx.$

5. Prove that $\int_0^1 x^m(1-x)^n\,dx=\int_0^1 x^n(1-x)^m\,dx$ and hence evaluate $\int_0^1 x(1-x)^{10}\,dx$.

6. Evaluate the following definite integrals:

(i) $\int_0^4 x\sqrt{4-x}\,dx$ (ii) $\int_0^1 \frac{x^2}{\sqrt{1-x}}\,dx$ (iii) $\int_{-1}^1 \frac{e^x-1}{e^x+1}\,dx$

7. Let $f(x)$ be a function satisfying $f(x)+f(-x)=x^2+x^3$ for all x. Evaluate the definite integral $\int_{-2}^2 f(x)dx$.

8. Let $\int_0^3 f(x)\,dx=20$, $\int_0^{10} f(x)\,dx=45$. Evaluate $\int_3^{10} f(x)\,dx$.

9. Given that: $\int_0^9 f(x)\,dx=16$ and $\int_0^3 f(x)\,dx=12$. Evaluate the integral $\int_3^9 (2f(x)+5)\,dx$.

10. If $f(4-x)=f(x)$, show that $\int_0^2 f(2x)\,dx=\int_0^2 f(x)\,dx$.

11. Show that $\int_{-1}^1 (1-x)f\left(1-x^2\right)dx=\int_{-1}^1 f\left(1-x^2\right)dx$.

12. If $f(a+x)+f(a-x)=0$, show that $\int_0^{2a} f(x)\,dx=0$.

13. Show that $\int_{-a}^a \frac{f(x)}{1+e^x}\,dx=\int_0^a f(x)dx$, where $f(x)$ is an even function. Hence or otherwise evaluate $\int_{-a}^a \frac{x^2}{1+e^x}\,dx$.

14. Given the function $f(x)$ satisfies the following conditions:

(i) $\int_0^4 f(x)dx=12$ (ii) $f(4-x)=f(x)$ (iii) $f(x+2)=f(x)$

Evaluate $\int_{-1}^5 f(x)\,dx$.

15. Show that:

(i) $\int_{\pi/6}^{\pi/3} \ln(\tan x)\,dx=0$ (ii) $\int_0^{\pi} \frac{\sin x\cos x}{1+e^{\sin x}}\,dx=0$

(iii) $\int_{0}^{\pi} x(\pi - x)\cos(\pi - x)\,dx = 0.$ (iv) $\int_{1}^{3} \frac{(x-2)^3 + 4(x-2)}{1+e^{(x-2)^2}}\,dx = 0.$

16. Evaluate the following definite integrals:

(i) $\int_{1}^{3} \frac{x^2}{x^2+(4-x)^2}\,dx$ (ii) $\int_{0}^{\ln 2} \frac{e^x}{e^x + 2e^{-x}}\,dx.$

17. Find the indicated derivatives:

(i) $\frac{d}{dt}\int_{\pi}^{t} \frac{\sin y}{1+\sqrt{y}}\,dy$ (ii) $\frac{d}{dx}\int_{ax}^{bx} \frac{1}{1+t^2}\,dt$

3.7 Application of Definite Integrals

3.7.1 Area Under the Curve

A remarkable result in calculus is that the area under the graph of $y = f(x)$, where $f(x) \geq 0$, can be obtained simply by evaluating the definite integral of $f(x)$ over the interval $[a, b]$.

If f is continuous and $f(x) \geq 0$ over the interval $[a, b]$, then the area A of the region under the curve of f from a to b is $A = \int_{a}^{b} f(x)\,dx$

Notes:

1. If $f(x) \leq 0$ on the interval $[a, b]$, then the area bounded by the curve $y = f(x)$ and x-axis is given by $\left|\int_{a}^{b} f(x)\,dx\right|$ or equivalently $\int_{a}^{b}|f(x)|\,dx$ or equivalently $-\int_{a}^{b} f(x)\,dx$.

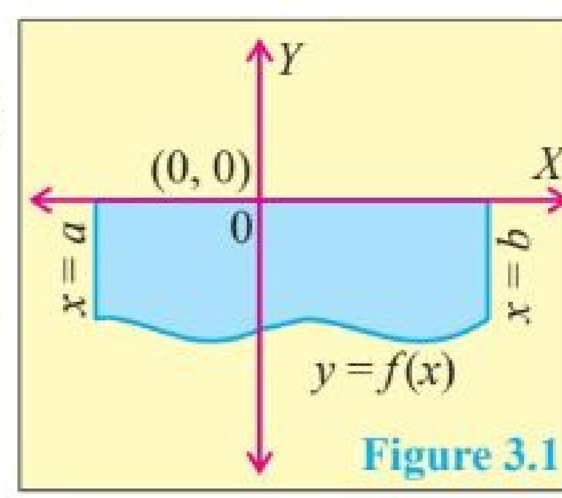


Figure 3.1

2. If $f(x) \geq 0$ on the interval $[a, c]$ and $f(x) \leq 0$ on the interval $[c, b]$, then the total area bounded by the curve $y = f(x)$ and x-axis on the interval $[a, b]$ is

$$\int_{a}^{c} f(x)dx + \left|\int_{c}^{b} f(x)dx\right|$$

or equivalently $\int_{a}^{c} f(x)\,dx + \int_{c}^{b}|f(x)|\,dx$

or equivalently $\int_{a}^{c} f(x)dx - \int_{c}^{b} f(x)dx$

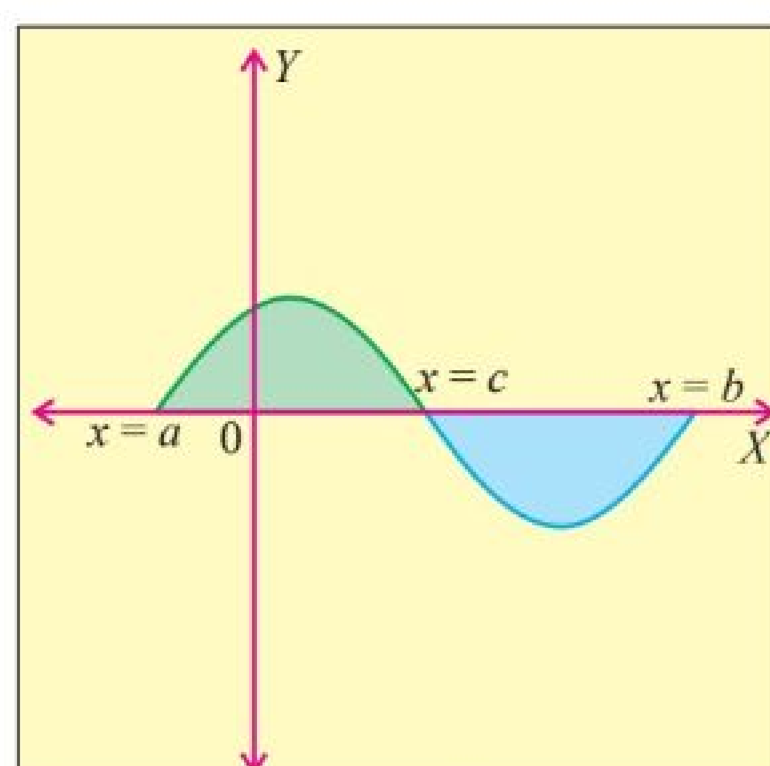


Figure 3.2

3. If $f(x)$ changes sign at $c_1, c_2, c_3, \dots, c_n$ in (a, b) where $c_1 < c_2 < \cdots < c_n$ are the roots of $f(x)=0$.

$$\text{Total Area} = \int_a^b |f(x)|\,dx = \left|\int_a^{c_1} f(x)\,dx\right| + \left|\int_{c_1}^{c_2} f(x)\,dx\right| + \cdots + \left|\int_{c_n}^{b} f(x)\,dx\right|$$

4. The area bounded by the curve $x = f(y)$, the y-axis, the horizontal lines $y = c$ and $y = d$ $(d > c)$ is given by $\int_c^d f(y)\,dy$.

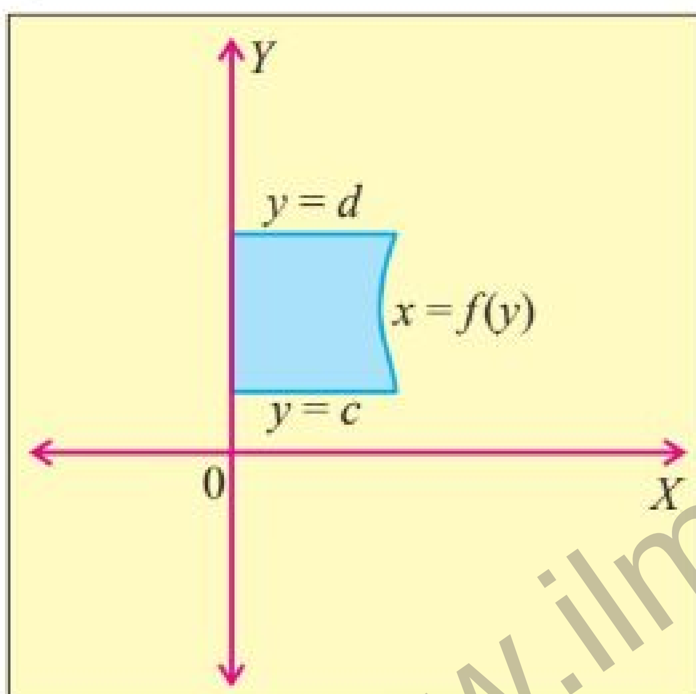


Figure 3.3

5. **Area between two curves**

If f and g are continuous and $f(x) \geq g(x)$ over the interval $[a, b]$, then the area A of the region bounded by the graphs of f, g, $x = a$, and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

The above result does not require $f(x)$ or $g(x)$ to remain positive over the interval $[a, b]$.

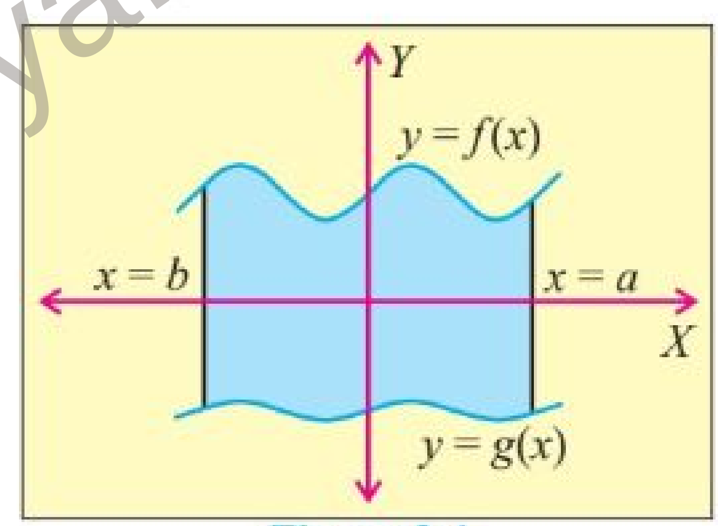


Figure 3.4

Signed Area and Net Signed Area

The definite integral $\int_a^b f(x)\,dx$ gives the area between the curve and the x-axis, but the interpretation depends on whether the curve crosses the axis.

- If the **entire curve** lies **above** the x-axis, then the definite integral directly gives the **signed area**—and it is **positive**.
- If the **entire curve** lies **below** the x-axis, then the definite integral directly gives the **signed area**—and it is **negative**.
- If **some part** of the curve is above the axis and some part is below the axis, and we add these areas **with their signs** (positive for above, negative for below), then we get the **net signed area**.

Thus:

- **Signed area** refers to the case when the curve is entirely above or entirely below the axis.
- **Net signed area** refers to the result after canceling positive and negative contributions when the curve crosses the axis.
  In both cases, the formula is the same:

$$\text{Signed Area (or Net Signed Area)} = \int_a^b f(x)\,dx$$

The result may be:

- **Positive** (more area above or entirely above),
- **Negative** (more area below or entirely below), or **Zero** (equal area above and below).

Example 57 Find the area bounded by the parabola $y = x^2 - 5x + 4$ and the x-axis.

Solution: Given the parabola: $y = x^2 - 5x + 4$

Set $y = 0$ to find x-intercepts:

$$x^2 - 5x + 4 = 0 \;\Rightarrow\; (x-1)(x-4) = 0$$

So, the x-intercepts are at $x = 1$ and $x = 4$.

The parabola opens upwards (positive leading coefficient).

Between the roots $x = 1$ and $x = 4$, y(function) is negative. The vertical lines are $x = 1$ and $x = 4$, which are exactly the intercepts.

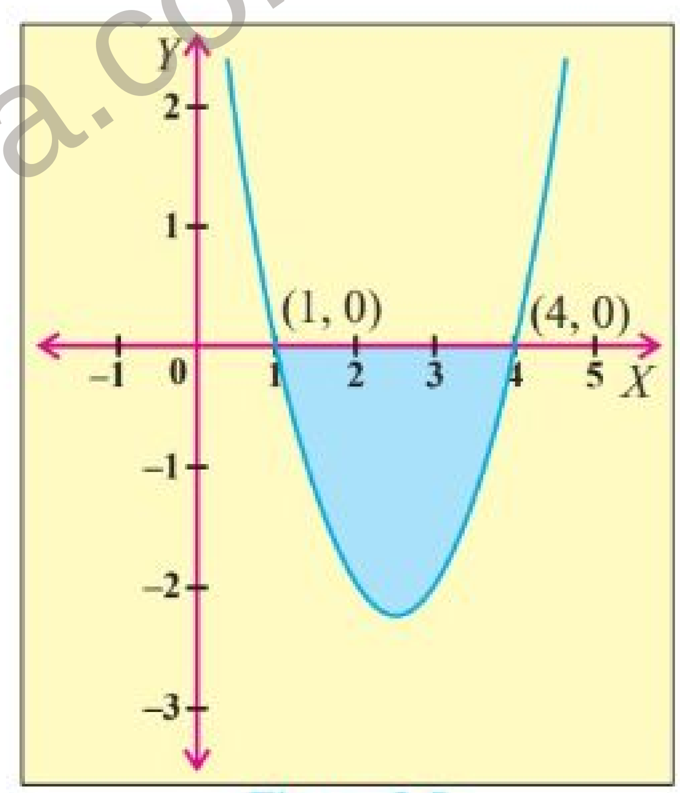


Figure 3.5

Therefore, the region bounded by the curve, the x-axis, and these lines is from $x = 1$ to $x = 4$, where the curve is below the x-axis.

Since the curve is below the x-axis from $x = 1$ to $x = 4$, the area is:

$$\text{Area} = \int_1^4 |y|\,dx = \int_1^4 -(x^2 - 5x + 4)\,dx = \int_1^4 (-x^2 + 5x - 4)\,dx$$

$$= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x\right]_1^4 = \left(-\frac{(4)^3}{3} + \frac{5(4)^2}{2} - 4(4)\right) - \left(-\frac{(1)^3}{3} + \frac{5(1)^2}{2} - 4(1)\right)$$

$$= \frac{8}{3} - \left(-\frac{11}{6}\right) = \frac{16+11}{6} = \frac{27}{6} = \frac{9}{2}$$

The area is $\frac{9}{2}$ square units.

Example 58 Find the area bounded by the curves $y=13-x^2$ and $y=2x+5$.

Solution: First, we find points of intersection of the curves $y=13-x^2$ and $y=2x+5$ as:

$$13-x^2=2x+5 \Rightarrow 13-x^2-2x-5=0 \Rightarrow 8-x^2-2x=0$$

$$\Rightarrow x^2+2x-8=0 \Rightarrow x^2+4x-2x-8=0$$

$$\Rightarrow x(x+4)-2(x+4)=0 \Rightarrow (x-2)(x+4)=0$$

So, $x=-4$ or $x=2$

Determine which curve is on the top:

Test $x=0$:

$$y(0)=13-0^2=13$$

$$y(0)=2(0)+5=5$$

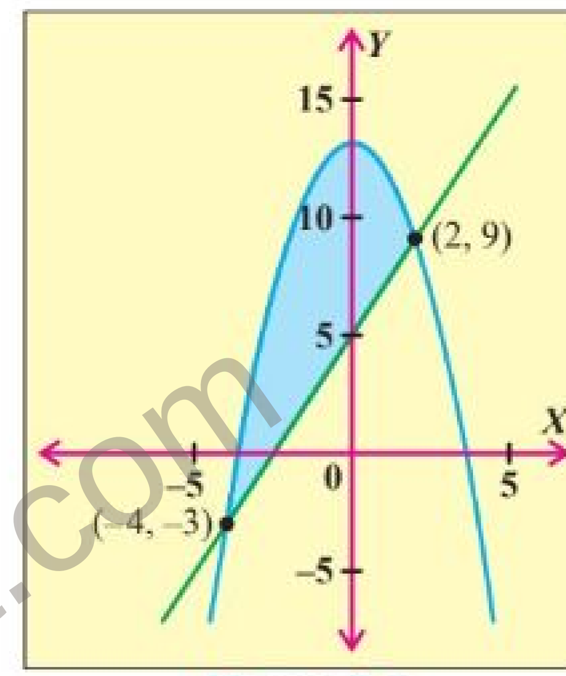


Figure 3.6

So, $y=13-x^2$ is above $y=2x+5$ in the interval [–4, 2].

$$\text{Area} = \int_{-4}^{2}[\text{upper}-\text{lower}]\,dx = \int_{-4}^{2}[(13-x^2)-(2x+5)]\,dx$$

$$=\int_{-4}^{2}(8-x^2-2x)\,dx = \left[8x-\frac{x^3}{3}-x^2\right]_{-4}^{2}$$

$$=\left(8(2)-\frac{2^3}{3}-2^2\right)-\left(8(-4)-\frac{(-4)^3}{3}-(-4)^2\right)$$

$$=60-\left(\frac{8}{3}+\frac{64}{3}\right)=60-\frac{8+64}{3}=60-\frac{72}{3}=60-24=36 \text{ square units}$$

The area between the two curves is 36 square units.

Example 59 Find the area of the region bounded by the parabolas $y=4-x^2$ and $y=x^2-2x$.

Solution: To find the point of intersection equating both equations.

$$4-x^2=x^2-2x$$

$$4-x^2-x^2+2x=0$$

$$4-2x^2+2x=0$$

$$-2x^2+2x+4=0$$

Divide by –2:

$$x^2-x-2=0$$

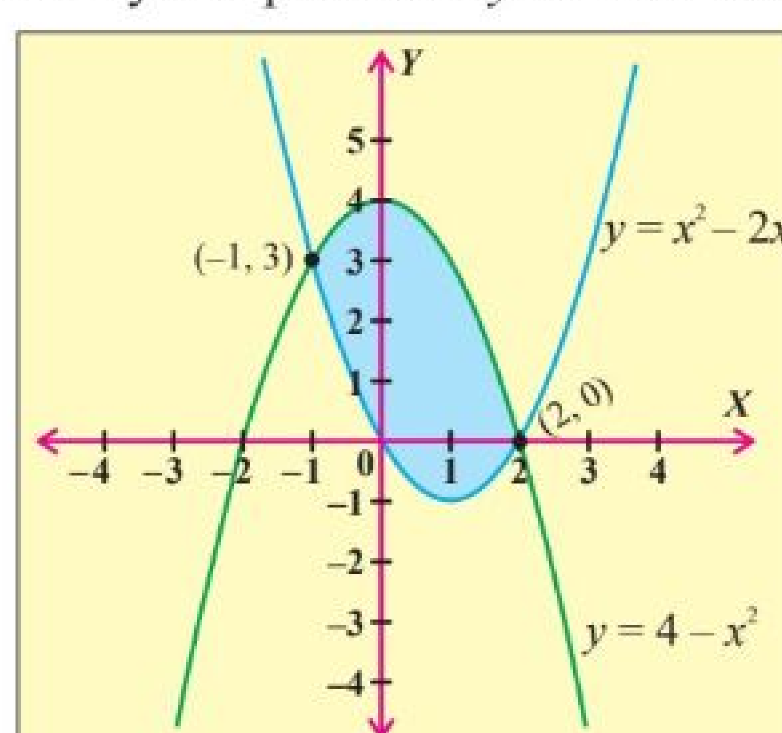


Figure 3.7

$$(x+1)(x-2)=0$$

So, $x=-1$ and $x=2$

Intersection points: $(-1,\ 3)$ and $(2,\ 0)$

Determine which curve is on top at $[-1, 2]$

Take $x=0 \in [-1, 2]$. Put it on both curves, we have

Test $x=0$: $y(0)=4-0^2=4$ and $y(0)=0^2-2(0)=0$

So, $4-x^2 > x^2-2x$ for $x=0$

Thus, $f(x)=4-x^2$ is the upper curve, $g(x)=x^2-2x$ is the lower curve.

$$\text{Area} = \int_{-1}^{2}[f(x)-g(x)]\,dx = \int_{-1}^{2}\left[(4-x^2)-(x^2-2x)\right]dx$$

$$= \int_{-1}^{2}(4-2x^2+2x)\,dx = \left[4x-\frac{2}{3}x^3+x^2\right]_{-1}^{2}$$

$$= \left(4(2)-\frac{2(8)}{3}+2^2\right)-\left(4(-1)-\frac{2(-1)}{3}+(-1)^2\right) = \frac{20}{3}-\left(-\frac{7}{3}\right)=9 \text{ square units}$$

3.7.2 Consumer and Producer Surplus

One place in economics where area between two curves is important is in the calculation of a quantity called **consumer surplus** and a quantity called **producer surplus**.

Demand Function

The demand curve for a commodity shows the quantity consumers are willing to buy at any given price. Let p be the price that a consumer is willing to pay for a quantity x of a certain commodity. Then p and x are related to each other through the demand function $p = D(x)$. The graph of the demand function generally slopes downwards, as the demand for a commodity decreases when its price increases; that is, with an increase in price, the consumer is likely to buy less. [Figure 3.8]

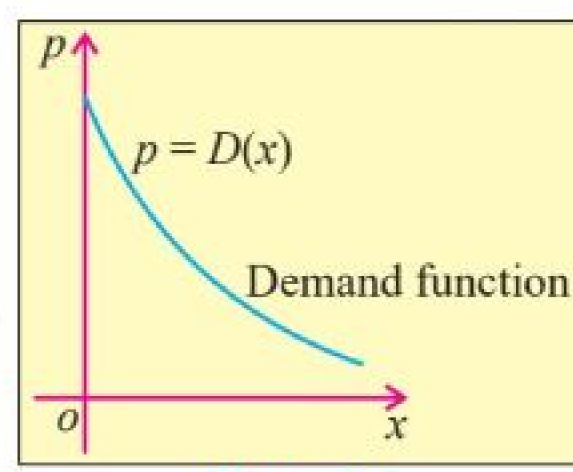


Figure 3.8

Supply Function

Let p be the price at which a producer is willing to sell a quantity x of a certain commodity. Then p and x are related to each other through the supply function $p = S(x)$. The graph of the supply function generally slopes upwards, as the supply of a commodity increases when its price increases; that is, with an increase in price, the producer is willing to supply more. [Figure 3.9]

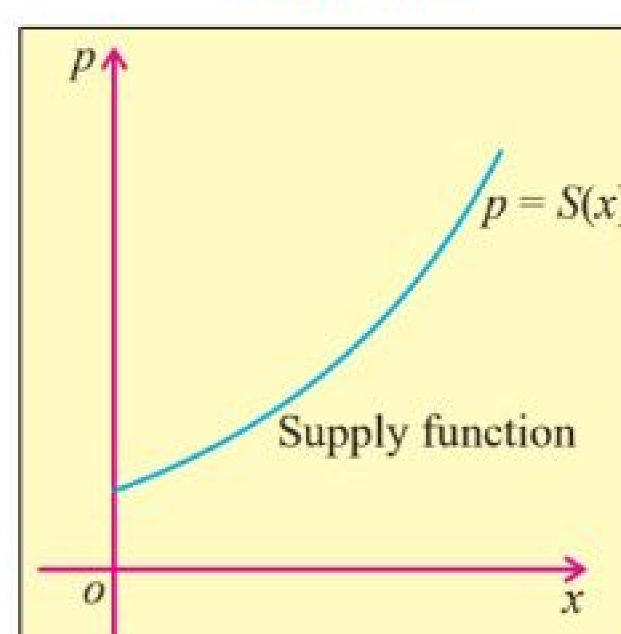


Figure 3.9

Market Equilibrium

The point where the demand and supply curves intersect is called the market equilibrium. At this point the quantity that consumers want to buy equals the quantity that producers want to sell.

If (x_e, p_e) is the equilibrium point, the quantity x_e is called the equilibrium quantity and the price p_e is called the equilibrium price.

Algebraically to find the equilibrium point, we solve the system of equations formed by the demand equation and supply equation.

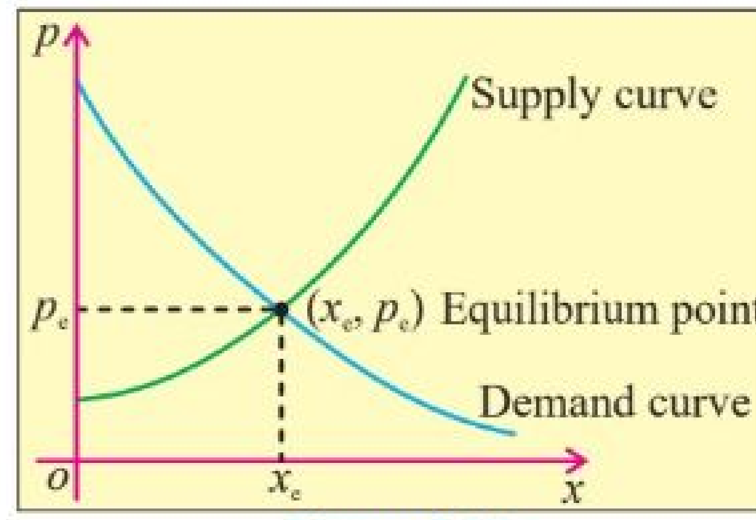


Figure 3.10

Consumer Surplus (CS)

Let the demand function be $p = D(x)$, and let x_e be the quantity demanded when the market price is p_e. However, some consumers who are unaware of equilibrium price are willing to pay more than p_e for the quantity x_e of the commodity. The difference between what a consumer is willing to pay and what they actually pay is a benefit to the consumer.

Thus, consumer surplus can be defined as:

3.4 Consumer Surplus (CS)

Consumer surplus is the total benefit gained by consumers when they purchase an item at the equilibrium price, rather than at the higher price they would have been willing to pay.

Geometrically, it is the area under the demand curve and above the horizontal line drawn through the equilibrium point.

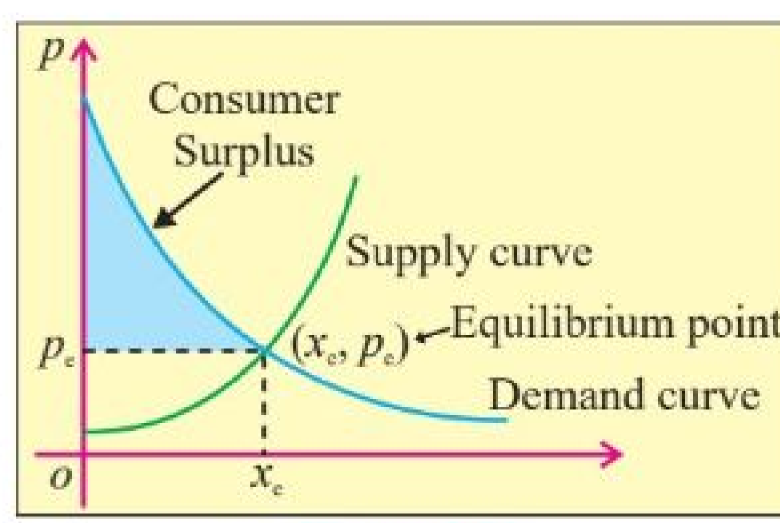


Figure 3.11

$$\text{Thus, CS} = \begin{pmatrix}\text{Area under demand function}\\ \text{from } x = 0 \text{ to } x = x_e\end{pmatrix} - \begin{pmatrix}\text{Area under equilibrium price}\\ \text{from } x = 0 \text{ to } x = x_e\end{pmatrix}$$

$$= \int_0^{x_e} (\text{Demand function})\,dx - \int_0^{x_e} (\text{Equilibrium price})\,dx$$

$$= \int_0^{x_e} D(x)\,dx - \int_0^{x_e} p_e\,dx = \int_0^{x_e} \left(D(x) - p_e\right)dx$$

Producer Surplus (PS)

Let the supply function be $p = S(x)$, and let x_e be the quantity supplied when the market price is p_e. However, some producers are willing to accept less than p_e for the quantity x_e of the commodity. The difference between what producers actually receive and what they would have been willing to accept is a benefit to the producer.
The great economist Alfred Marshall called this benefit producer surplus.

3.5 Producer Surplus (PS)

Producer surplus is the total benefit gained by producers when they sell an item at the equilibrium price, rather than at the lower price they would have been willing to accept.
Geometrically, it is the area above the supply curve and below the horizontal line drawn through the equilibrium point.

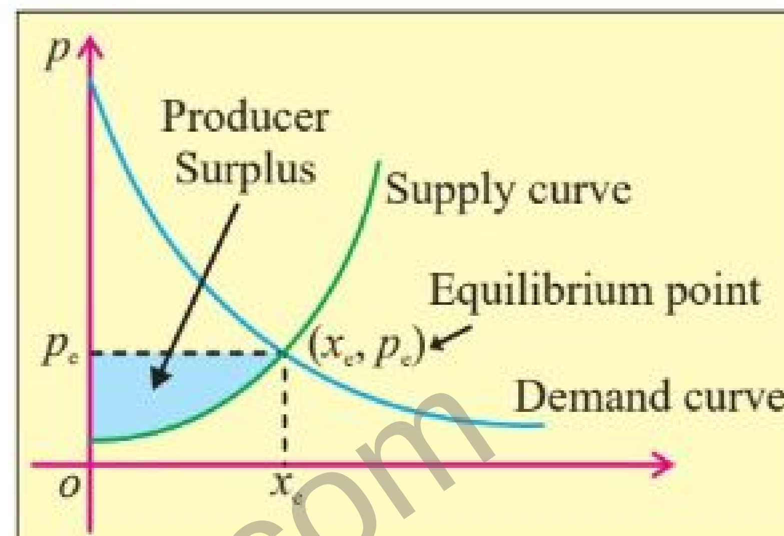


Figure 3.12

$$\text{Thus, PS} = \begin{pmatrix}\text{Area under equilibrium price}\\ \text{from } x=0 \text{ to } x=x_e\end{pmatrix} - \begin{pmatrix}\text{Area above supply function}\\ \text{from } x=0 \text{ to } x=x_e\end{pmatrix}$$

$$= \int_0^{x_e} (\text{Equilibrium price})\,dx - \int_0^{x_e} (\text{Supply function})\,dx$$

$$= \int_0^{x_e} p_e\,dx - \int_0^{x_e} S(x)\,dx = \int_0^{x_e} \left(p_e - S(x)\right)dx$$

Consumer Surplus (CS) and Producer Surplus (PS) Formulae

1. With Respect to Quantity (x)

Let:
- $p = D(x)$ = demand price function
- $p = S(x)$ = supply price function
- x_c = equilibrium quantity
- x_e = equilibrium price $= D(x_e) = S(x_e)$

Consumer Surplus (CS) = area between demand curve and equilibrium price line, from $x = 0$ to $x = x_e$: $\text{CS} = \int_0^{x_e} \left(D(x) - p_e\right)dx$

Producer Surplus (PS) = area between demand curve and equilibrium price line, from $x = 0$ to $x = x_e$: $\text{PS} = \int_0^{x_e} \left(p_e - S(x)\right)dx$

Total surplus (TS): $\text{TS} = \text{CS} + \text{PS} = \int_0^{x_e} \left(D(x) - S(x)\right)dx$

2. With Respect to Price (p)

Let:
- $x_d(p) = D^{-1}(p)$ = quantity demanded at price p
- $X_s(p) = S^{-1}(p)$ = quantity supplied at price p
- $P_{\max}$ = choke price (demand = 0 for higher prices)
- $P_{\min}$ = minimum price (supply = 0 for lower prices)
- P_e = equilibrium price

Consumer Surplus (CS) = area between p_e and $p_{\max}$, under demand curve $x_d(p)$:

$$\text{CS} = \int_{p=p_e}^{p_{\max}} x_d(p)\,dp$$

Producer Surplus (PS) = area between $p_{\min}$ and p_e, under supply curve $x_s(p)$:

$$\text{PS} = \int_{p=p_{\min}}^{p_e} x_s(p)\,dp$$

3. Summary Table

Surplus	Integration w.r.t. Quantity	Integration w.r.t. price
CS	$\int_0^{x_e}(D(x)-p_e)\,dx$	$\int_{p=p_e}^{p_{\max}} x_d(p)\,dp$
PS	$\int_0^{x_e}(p_e - S(x))\,dx$	$\int_{p=p_{\min}}^{p_e} x_s(p)\,dp$

4. Key Notes

- Choke price $p_{\max} = D(0)$ (price when $x = 0$ in demand).
- Minimum supply price $p_{\min} = S(0)$ (price when $x = 0$ in supply)
- When integrating w.r.t. x, limits are quantities, integrand is price difference.

Example 60 The supply curve for a good is $p_s = \sqrt{16+x}$. The demand curve is linear with vertical intercept 73 and passes through the equilibrium point. If the quantity sold is $x_e = 33$, find the equation of the demand curve (p_d).

Solution: Supply curve: $p_s = \sqrt{16+x}$

At $x_e = 33$, $p_e = \sqrt{16+33} = \sqrt{49} = 7$. So intersecting point is $(x_e, p_e) = (33, 7)$.

Demand is linear $p_d = mx + c$.

Vertical intercept $c = 73 \Rightarrow p_d = mx + 73$

It passes through (33, 7): $7 = m \cdot 33 + 73 \Rightarrow 33m = -66 \Rightarrow m = -2$

So, $p_d = -2x + 73$.

Example 61 If the supply function for a particular commodity is $25p=(x+10)^2$ and the market price is $Rs.64$, find the producer's surplus by two methods:

(i) Integrate with respect to x (ii) Integrate with respect to p

Solution: Given: $25p=(x+10)^2 \Rightarrow p=\dfrac{(x+10)^2}{25}$

At $p_e=64: 25\times 64=(x_e+10)^2 \Rightarrow (x_e+10)^2=5^2\times 8^2$

Take positive root: $x_e+10=5\times 8 \Rightarrow x_e+10=40 \Rightarrow x_e=30$

Method (i) Integrate w.r.t. x

$$\text{PS}=\int_0^{x_e}(p-p(x))\,dx=\int_0^{30}\left[64-\frac{(x+10)^2}{25}\right]dx=\left[64x-\frac{(x+10)^3}{25\times 3}\right]_0^{30}$$

$$=\left(64\times 30-\frac{(30+10)^3}{75}\right)-\left(0-\frac{10^3}{75}\right)=1920-\frac{(40)^3}{75}+\frac{(10)^3}{75}=1920-840=\text{Rs. }1080$$

Method (ii) Integrate w.r.t. p

From: $25p=(x+10)^2 \Rightarrow x+10=5\sqrt{p} \Rightarrow x=5\sqrt{p}-10$

Find $p_{\min}$ when $x=0$:

$$0=5\sqrt{p}-10 \Rightarrow 5\sqrt{p}=10 \Rightarrow \sqrt{p}=\frac{10}{5} \Rightarrow \sqrt{p}=2 \Rightarrow p_{\min}=4$$

$$\text{PS}=\int_{p_{\min}}^{p_e} x_s(p)\,dp=\int_4^{64}(5\sqrt{p}-10)\,dp=\left[5\frac{p^{3/2}}{3/2}-10p\right]=\left[\frac{10}{3}p^{3/2}-10p\right]$$

$$=\left[\frac{10}{3}(64)^{3/2}-10(64)\right]-\left[\frac{10}{3}(4)^{3/2}-10(4)\right]=\left[\frac{10}{3}(8^2)^{3/2}-640\right]-\left[\frac{10}{3}(2^2)^{3/2}-10(4)\right]$$

$$=\frac{5120}{3}-\frac{80}{3}-640+40=\frac{5120-80}{3}-600=\frac{5040}{3}-600=1680-600=1080$$

Both methods give PS = Rs.1080.

3.7.3 Solids of Revolution

The volume of an object plays an important role in many problems in the physical sciences. In this section, we consider methods for computing volumes of solid of revolution.

If a region in a plane is revolved about a line in the plane, the resulting solid is a **solid of revolution**, and we say that the solid is **generated** by the region. The line about which we revolve is an axis of **revolution**. In particular, if the region shown in Figure 3.13(a) is revolved about the x-axis, we obtain the solid illustrated in

Figure 3.13(b). If a plane perpendicular to the x-axis intersects the solid shown in Figure 3.13(b), a circular section is obtained and volume in this case can be found by disk method which is explained below in theorem 3.10 and 3.11.

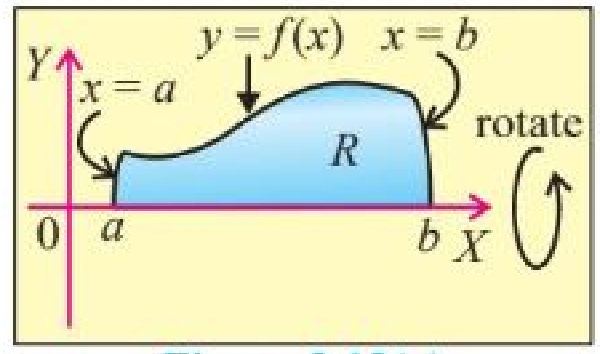


Figure 3.13(a)

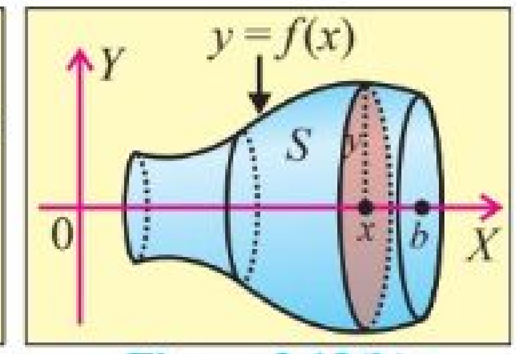


Figure 3.13(b)

Let f be continuous and non-negative on $[a, b]$, and let R be the region bounded by the graph of $y = f(x)$, the x-axis and the vertical lines $x = a$, and $x = b$. The **volume** V of the solid of revolution generated by revolving R about the x-axis is $V = \int_a^b \pi [f(x)]^2 \, dx.$

Similarly, let g be continuous and non-negative on $[c, d]$, and let R be the region bounded by the graph of $x = g(x)$, the y-axis and the horizontal lines $y = c$, and $y = d$. The **volume** V of the solid of revolution generated by revolving R about the y-axis is

$$V = \int_c^d \pi [g(y)]^2 \, dy$$

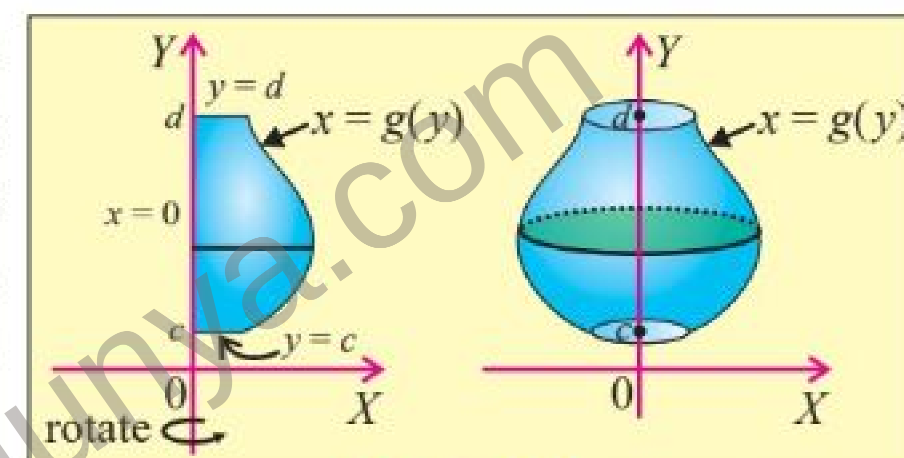


Figure 3.14

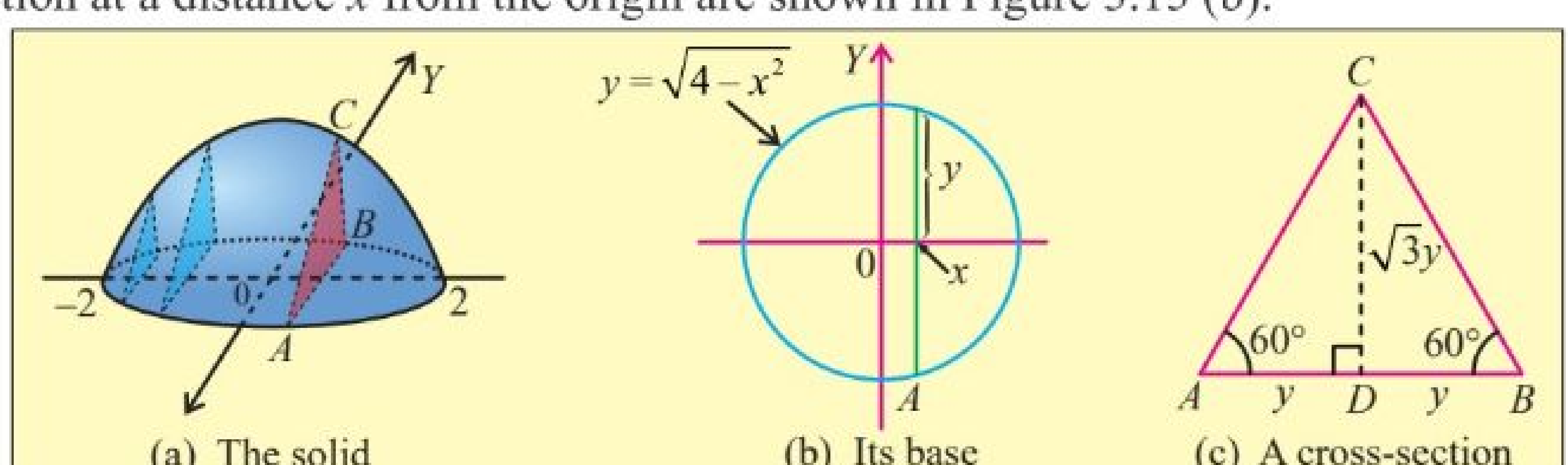
Volume of Solid of Revolution by Cross-sectional Method

Let S be a solid that lies between the planes P_a and P_b. If the cross-sectional area of S in the plane P_x is $A(x)$, where A is a continuous function, then **volume** of S is

$$V = \int_a^b A(x) \, dx$$

Example 62 A solid has a circular base of radius 2. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

Solution: Let us take the circle $x^2 + y^2 = 4$. The solid, its base, and a typical cross-section at a distance x from the origin are shown in Figure 3.15 (b).



(a) The solid (b) Its base (c) A cross-section

Figure 3.15

Since B lies on the circle, we have $y=\sqrt{4-x^2}$ and so the base of the triangle ABC is $|AB|=2y$. Since the triangle is equilateral, we see from the Figure 3.15 that its height is $\sqrt{3}y$. Therefore, the cross-sectional area is

$$A(x)=\frac{1}{2}(\text{Base})(\text{Height})=\frac{1}{2}\cdot 2y\cdot\sqrt{3}y$$
$$=\sqrt{3}y^2=\sqrt{3}\left(4-x^2\right)$$

and the volume of the solid is

$$V=\int_{-2}^{2}A(x)\,dx=\int_{-2}^{2}\sqrt{3}\left(4-x^2\right)dx=\sqrt{3}\left[4x-\frac{x^3}{3}\right]_{-2}^{2}$$
$$=\sqrt{3}\left[\left(4(2)-\frac{2^3}{3}\right)-\left(4(-2)-\frac{(-2)^3}{3}\right)\right]=\sqrt{3}\left[8-\frac{8}{3}+8-\frac{8}{3}\right]=\frac{32\sqrt{3}}{3}\text{ cubic units}$$

Example 63 Find the volume of the solid obtained by rotating the region bounded by $y=x^3$, $y=64$, and $x=0$ around the y-axis.

Solution: Since the axis of rotation is the y-axis, we write the curve $y=x^3$ in the form of $x=y^{1/3}$. The region and the solid with a typical disk are sketched in the Figure 16. Using the Formula

$$V=\int_{c}^{d}\pi\left[g(y)\right]^2dy \text{ with } c=0,\ d=64 \text{ and}$$

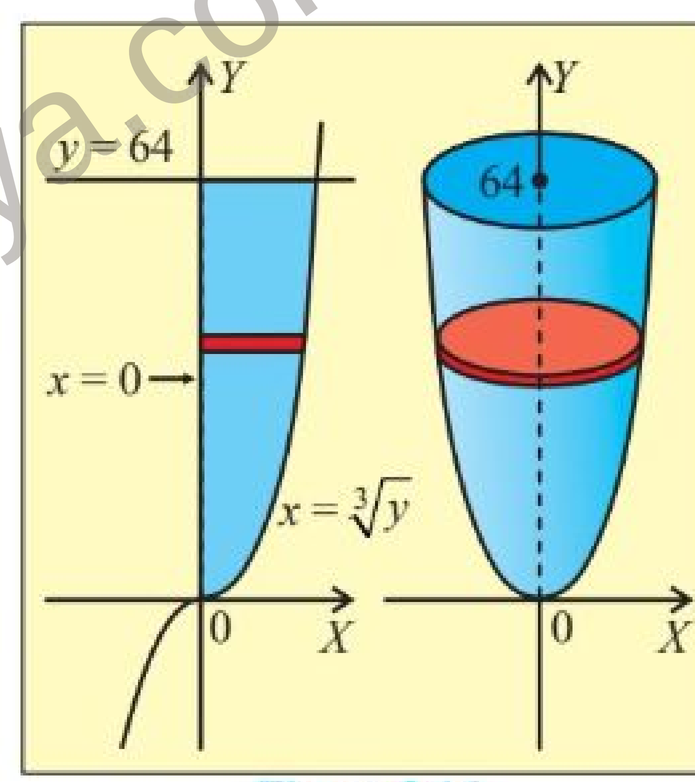


Figure 3.16

$g(y)=y^{1/3}$, we have

$$V=\int_{0}^{64}\pi\left(y^{1/3}\right)^2dy=\pi\int_{0}^{64}y^{2/3}dy=\pi\left[\frac{3}{5}\,y^{5/3}\right]_{0}^{64}$$
$$=\frac{3\pi}{5}\left[(64)^{5/3}-0\right]=\frac{3\pi}{5}\left[(4^3)^{5/3}\right]=\frac{3\pi}{5}(4^5)=\frac{3\pi}{5}(1024)=\frac{3072\pi}{5}\quad\text{cubic units}$$

Example 64 A cone is formed by rotating the straight line $y=2x$ around the x-axis from $x=0$ to $x=5$ cm. Find the volume of the cone.

Solution: The volume V of a solid formed by rotating $y=f(x)$ around the x-axis from $x=a$ to $x=b$ is: $V=\int_{a}^{b}\pi\left[f(x)\right]^2dx$

Here $f(x)=2x, a=0$ and $b=5$.

So, $V=\pi\int_0^5 (2x)^2\,dx=\pi\int_0^5 4x^2 dx=4\pi\int_0^5 x^2\,dx=4\pi\left[\frac{x^3}{3}\right]_0^5$

$$=4\pi\left(\frac{5^3}{3}-0\right)=4\pi\left(\frac{125}{3}\right)=\frac{500\pi}{3}\approx 523.6\text{ cm}^3$$

Example 65: A mug is designed by rotating the line $y=3x+1$ around the x-axis from $x=0$ to $x=4$ cm. Find the volume of the mug.

Solution: The Volume V is given by the formula: $V=\int_a^b \pi\left[f(x)\right]^2\,dx$

Here $f(x)=3x+1$, $a=0$, and $b=4$

$$V=\pi\int_0^4 (3x+1)^2dx=\pi\int_0^4 (9x^2+6x+1)\,dx=\pi\left[\frac{9x^3}{3}+\frac{6x^2}{2}+x\right]_0^4$$

$$=\pi\left[3x^3+3x^2+x\right]_0^4=\pi\left[\left(3(4)^3+3(4)^2+4\right)-(0)\right]=\pi[192+48+4]=244\pi\text{ cm}^3$$

EXERCISE 3.6

1. Find the area of the region that is above the x-axis, but below the curve $y=(x-2)(3-x)$.
1. Find the total area and the signed area between the graph of $y=x^2-4$ and the x-axis over the interval (i) $[-3,-1]$ (ii) $[-3,3]$.
2. Sketch the region bounded by the given curves and find the area of the region.
   (i) $y=2x,\ y=x^2$ (ii) $y=\sqrt{x},\ y=x^2$
   (iii) $y=x^4-x^2,\ y=1-x^2$ (iv) $y=x^2$ and $y=|x|$.
3. Find the area of the region bounded by $y=|x-3|, x=2, x=4$ and the x-axis.
4. Find the area bounded by the parabola $y=x^2-6x+8, x=2, x=4$ and the x-axis.
5. Find the area bounded by the parabola $x=9-y^2$, and the y-axis.
6. Find the ratio of the areas of the regions bounded by the parabola $y=x^2-4x+3$ and the coordinate axes.
7. For the function $f(x)=2x$ over the interval $[0,3]$:
   (a) Find the signed area (b) Find the total (physical) area
   (c) Explain why they are equal in this case.
8. Find the area of the region bounded by the curve $y=(x-1)(x+1)(x-2)$ and the x-axis.

9. Find the area of the region bounded by the line $y = x + 1$ and the parabola $y^2 = 2x + 10$.
10. The demand function is $p = 30 - 2x - x^2$ and supply function is $p = -6 + 4x + x^2$. Find consumer's surplus and producer's surplus.
11. At equilibrium, the quantity sold is $x_e = 4$ and the price is $p_e = 16$. Consumer's surplus is 16 and producer's surplus is 8. Both the demand and supply curves are linear. Find the demand function $D(x)$ and the supply function $S(x)$.
12. The supply function is $p = \ln(x + 2)$ and the quantity sold is $x_e = 2$. Consumer's surplus is $6 - 2\ln 2$. Find:
    (a) Equilibrium price (b) Linear demand function $p = a - bx$.
13. If the demand function for a particular commodity is $p = 100 - 2x$ and the market price is p_e = Rs.64, find the consumer's surplus using two methods:
    (a) Integrate with respect to x. (b) Integrate with respect to p.
15. Find the volume of the solid obtained by rotating the region bounded by the given curves about the indicated axis.
    (i) $y = x^2, x = 2, y = 0$; x-axis (ii) $x + y = 2, x = 0, y = 0$; x-axis
    (iii) $y = \sqrt{x-5}, x = 6, x = 8, y = 0$; x-axis (iv) $y = x^2 - 2x, y = 0$; x-axis

3.8 Applying Concepts of Integration to Real-Life World Problems

3.8.1 Moment of Inertia

Definition: The moment of inertia of a single particle of mass m about an axis AB is defined to be mr^2, where r is its distance from the particle to the axis AB. For a system of particles $m_1, m_2, \ldots, m_n$ placed at distances $r_1, r_2, \ldots, r_n$ respectively from a given axis, the moment of inertia of the system is defined as

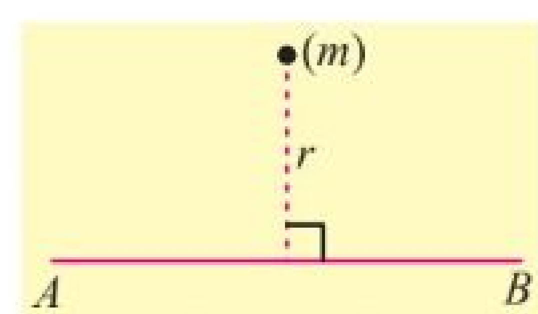


Figure 17

$$m_1r_1^2 + m_2r_2^2 + \cdots + m_nr_n^2 = \sum_{i=1}^{n} m_ir_i^2$$

The moment of inertia of a rigid body having continuous matter is:

$$\lim_{\delta m \to 0} \sum r^2 \cdot \delta m = \int r^2 \, dm$$

the integral being taken all over the body.

1. If the moment of inertia of a body of mass M about any axis be MR^2, then R is called the radius of gyration of the body about that axis.
2. The above definitions and the following two theorems enable us to determine the moment of inertia of many other bodies.

Example 66 **A uniform Rod of Length 2*a*:**

Find the moment of inertia of a uniform rod of length $2a$ and mass M about a line through one end and perpendicular to the rod.

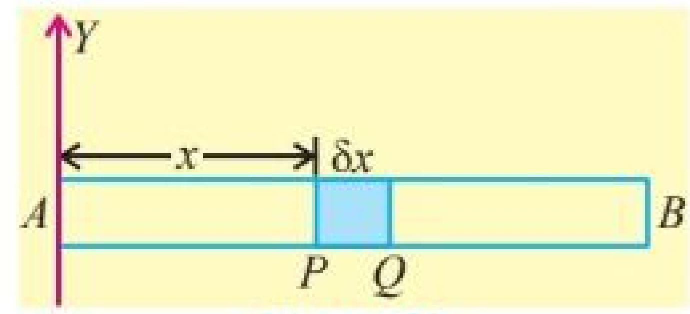


Figure 20

Solution: Let M be the mass of a uniform rod AB of length $2a$, then the mass per unit length of the rod is $\frac{M}{2a}$.

Let us consider an element PQ of length δx at a distance x from the line AY. Let AY be a line through A and perpendicular to AB.

$$\therefore\ \delta m = \text{Mass of an element } PQ = (\text{Mass per unit length}) \times (\delta x) = \frac{M}{2a} \times \delta x$$

The moment of inertia of this element PQ about the line AY is

$$x^2 \delta m = x^2 \cdot \frac{M}{2a} \cdot \delta x$$

Thus, the moment of inertia of the whole rod about AY is

$$\int_0^{2a} x^2 \frac{M}{2a} dx = \frac{M}{2a}\int_0^{2a} x^2 dx = \frac{M}{2a}\left[\frac{x^3}{3}\right]_0^{2a} = \frac{M}{2a}\left(\frac{(2a)^3}{3} - \frac{(0)^3}{3}\right) = \frac{M}{2a}\left(\frac{8a^3}{3}\right) = \frac{4}{3}Ma^2.$$

Example 67 **A Uniform Rectangular Lamina:**

Find the moment of inertia of a uniform rectangular lamina about a line through the centre and parallel to a side.

Solution: Let O be the centre of the rectangular lamina $ABCD$ with $AB = 2a, BC = 2b$ and M be the mass of this lamina.

Then the mass per unit area (density) is $\frac{\text{Mass}}{\text{Area}} = \frac{M}{(2a)(2b)} = \frac{M}{4ab}$.

Let OX be a line parallel to AB about which the moment of inertia is to be required. Take an elementary strip PQ of breadth δx perpendicular to OX at a distance x from O.

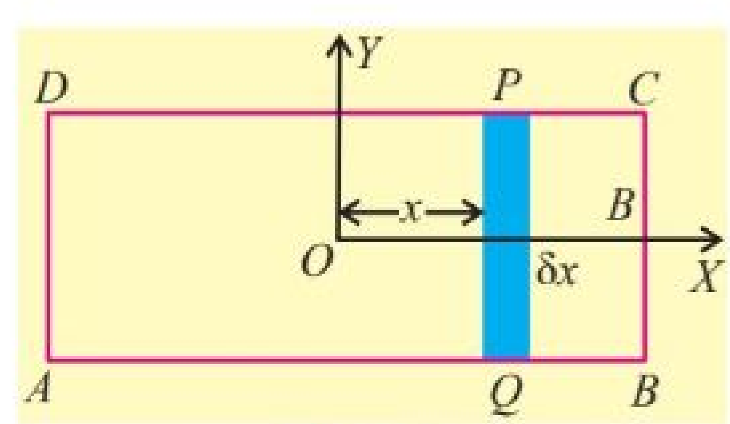


Figure 22

$\therefore\ \delta m =$ The mass of this elementary strip

$$= \text{density} \times \text{area} = \frac{M}{4ab}(2b \cdot \delta x) = \frac{M}{2a}\delta x.$$

The moment of inertia of this strip about OX is

$$\delta m \cdot \frac{b^2}{3} = \frac{M}{2a}\delta x \cdot \frac{b^2}{3} \text{ by previous example.}$$

Thus, the moment of inertia of the rectangular lamina about OX is

$$\int_{-a}^{a} \frac{M}{2a}\cdot\frac{b^2}{3}\,dx = \frac{Mb^2}{6a}\int_{-a}^{a} 1\,dx = \frac{Mb^2}{6a}[x]_{-a}^{a} = \frac{Mb^2}{6a}\left(a-(-a)\right) = \frac{Mb^2}{6a}(2a) = \frac{1}{3}Mb^2$$

Hence the moment of inertia of the rectangular lamina about a line through the centre and parallel to the side with length $2a$ is $\frac{1}{3}Mb^2$.

3.8.2 Distance, Velocity, Acceleration

Suppose a particle is moving along a straight line with position function $s(t)$ velocity function $v(t)$, and acceleration function $a(t)$, Since $s'(t) = v(t)$, the Fundamental theorem of Calculus gives:

$$\int_{t_1}^{t_2} v(t)\,dt = \int_{t_1}^{t_2} s'(t)\,dt = [s(t)]_{t_1}^{t_2} = s(t_2) - s(t_1) \qquad \ldots(1)$$

Similarly, since $v'(t) = a(t)$, the Fundamental Theorem of Calculus gives:

$$\int_{t1}^{t2} a(t)\,dt = v(t_2) - v(t_1) \qquad \ldots(2)$$

If we want to calculate the distance traveled during the time interval $[t_1, t_2]$, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore, total distance traveled $= \int_{t_2}^{t_1} |v(t)|\,dt$.

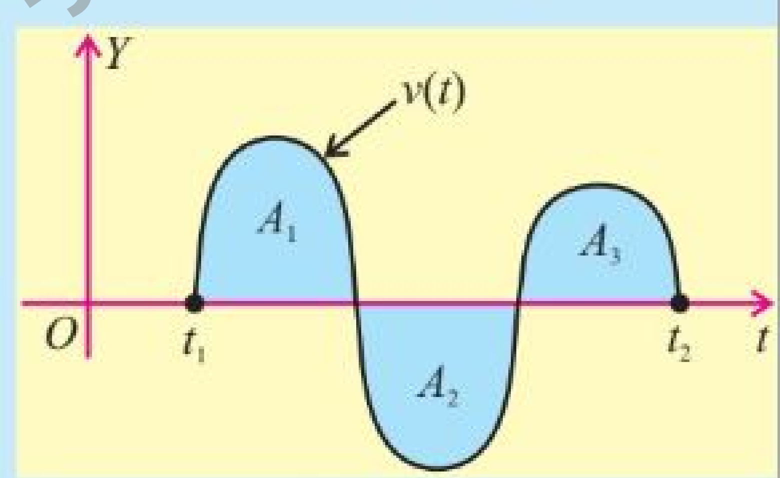


Figure 23

Given figure shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.

Displacement $= \int_{t_1}^{t_2} v(t)dt = A_1 - A_2 + A_3$

Distance $= \int_{t_1}^{t_2} v(t)\,dt = A_1 + A_2 + A_3$

Example 68 A particle moves along a straight line with acceleration $a(t) = 6t - 18(\text{m/s}^2)$ and initial velocity $v(0) = 15$ m/s.

(a) Find the velocity function $v(t)$.

(b) Find the displacement of the particle during $0 \leq t \leq 3$.

(c) Find the total distance traveled during $0 \leq t \leq 3$.

Solution: (a) Find $v(t)$

We have $a(t) = 6t - 18$

Velocity is the integral of acceleration:

$$v(t) = \int_0^t a(t)\,dt = \int_0^t (6t-18)\,dt = 6\frac{t^2}{2} - 18t + c = 3t^2 - 18t + c$$

where c is constant of integration.

As $v(0) = 15$ m/s

So, $v(0) = 3(0)^2 - 18(0) + c$

$\boxed{15 = c}$

Hence $v(t) = 3t^2 - 18t + 15$

(b) Displacement on $0 \le t \le 3$

$$S = \int_0^3 (3t^2 - 18t + 15)\,dt = [t^3 - 9t^2 + 15t]_0^3$$

$= 27 - 81 + 45 = -9$ This means that the particle moved 9 m to the left.

(c) Total distance traveled

Note that $v(t) = 3(t^2 - 6t + 5) = 3(t-1)(t-5)$

and so $v(t) \ge 0$ on the interval [0, 1] and $v(t) \le 0$ on [1, 3]. Thus the total distance traveled is

$$\int_0^3 |v(t)|\,dt = \int_0^1 |v(t)|\,dt + \int_1^3 |v(t)|\,dt = \int_0^1 v(t)\,dt + \int_1^3 (-v(t))\,dt$$

$$= \int_0^1 (3t^2 - 18t + 15)\,dt - \int_1^3 (3t^2 - 18t + 15)\,dt$$

$$= [t^3 - 9t^2 + 15t]_0^1 - [t^3 - 9t^2 + 15t]_1^3$$

$$= [(1 - 9 + 15) - (0)] - [(27 - 81 + 45) - (1 - 9 + 15)]$$

$$= 7 - (-16) = 7 + 16 = 23 \text{ m}$$

Example 69 **Growth rate of population:** A town's population growth rate (in people per year) is given by

$$P'(t) = 5000 + 200t$$

where t is the number of years after 2020. If the population in 2020 (t = 0) was 100,000:

(a) Find the population function $P(t)$ (b) What will the population be in 2030?

(c) Find the total population increase from 2022 to 2028.

Solution Given

A town's population growth rate is

$P'(t) = 5000 + 200t$ people/year, $t =$ years after 2020,

$P(0) = 100,000$

(a) Population function $P(t)$

$P(t) = \int P'(t)\,dt = \int (5000 + 200t)\,dt$

$P(t) = 5000t + 100t^2 + C$

Given $P(0) = 100000$

$100000 = 0 + 0 + C \Rightarrow C = 100000$

$P(t) = 100000 + 5000t + 100t^2$

(b) Population in 2030

$2030 \rightarrow t = 10$ (i.e. 10 years)

$P(10) = 100000 + 5000(10) + 100(100)$

$= 100000 + 50000 + 10000$

$= 160000$ people

(c) Population increase from 2022 to 2028

$2022 \rightarrow t_1 = 2$ and $2028 \rightarrow t_2 = 8$

Increase $= P(8) - P(2)$

$P(8) = 100000 + 5000(8) + 100(64) = 146400$

$P(2) = 100000 + 5000(2) + 100(4) = 110400$

Increase $= 146400 - 110400 = 36000$ people

3.8.3 Average Value of a Function

finition: If f is continuous on $[a, b]$, then its average value on $[a, b]$ (also called its mean value) is $f_{av} = \dfrac{1}{b-a}\int_a^b f(x)\,dx$.

Example 70 Find the average value of the function $f(x) = x^2 + 2$ over the interval $[-1, 2]$.

Solution: With $a = -1$ and $b = 2$, we have

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx = f_{av} = \frac{1}{2-(-1)}\int_{-1}^{2}\left(x^2+2\right)dx = \frac{1}{3}\left[\frac{x^3}{3}+2x\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[\left(\frac{2^3}{3}+2(2)\right)-\left(\frac{(-1)^3}{3}+2(-1)\right)\right] = \frac{1}{3}\left(\frac{8}{3}+\frac{1}{3}+4+2\right) = \frac{1}{3}\left(\frac{9}{3}+6\right) = 3$$

Thus, the average value of $f(x) = x^2 + 2$ over $[-1, 2]$ is 3.

Example 71 A temperature sensor moves slowly along a 10-meter track, continuously measuring temperature. The temperature at position x is given by

$$T(x) = 20 + x, \text{ where } T(x) \text{ is in } {}^\circ C$$

Find the average temperature measured by the sensor as it moves from $x = 0$ to $x = 10$ meters.

Solution: Function: $T(x)=20+x$, Limits are from $x=0$ to $x=10$.

Average Temperature $=\dfrac{1}{10-0}\int_{0}^{10}(20+x)\,dx$

$$=\frac{1}{10}\left[20x+\frac{x^2}{2}\right]_0^{10}=\frac{1}{10}\left[\left(20(10)+\frac{(10)^2}{2}\right)-\left(20(0)+\frac{(0)^2}{2}\right)\right]=25\ ^\circ\mathrm{C}$$

3.8.4 Drug Dosage Required by Integrating Concentration

Core Concept: Drug Concentration and Integration

Think of fighting an infection as a **battle** between the drug and the bacteria.

Drug Concentration $C(t)$: This is like the "number of soldiers" in your bloodstream at time t. Right after a dose, you have many soldiers. Over time, the body removes them (metabolism/excretion), so the number decreases.

Minimum Inhibitory Concentration (MIC): This is the **minimum number of soldiers** needed to hold the battlefield and prevent the bacteria from growing. If the number of soldiers drops below this, the infection can start winning again.

Area Under the Curve (AUC): If you imagine a graph of "number of soldiers" (concentration) vs. time, the area under that curve represents the total fighting effort over time.

A short time with many soldiers might give the same total "fighting effort" as a long time with fewer soldiers.

This "total fighting effort" (AUC) is what matters for killing the bacteria effectively.

Why Integrate?

Integration is just a way to **add up** the total "soldier-hours" (concentration × time).

If $C(t)$ is concentration at time t, then:

$$\text{AUC}=\int_{t_1}^{t_2}C(t)\,dt$$

This AUC (Area Under the Curve) measures **total drug exposure**.

Doctors know from experiments:

- If AUC is too small $\rightarrow$ drug won't work well.
- If AUC is too large $\rightarrow$ risk of side effects.

So they calculate the AUC to decide the right dose and frequency.

Example 72 **Basic AUC Calculation**

A patient is given an intravenous dose of an antibiotic. The concentration in the bloodstream over time is modeled by:

$C(t)=8e^{-0.15t}(\text{mg}/\text{L})$ where t is in hours.

(a) Find the total drug exposure (AUC) from $t=0$ to $t=6$ hours.

(b) What is the AUC from $t=0$ to infinity?

(c) If the minimum required total exposure over the first 6 hours is $40\,\text{mg hr/L}$, is the drug exposure from this dose sufficient for a 6-hour treatment?

Solution: (a) AUC from $t=0$ to $t=6$:

$$\text{AUC}_{0\to 6}=\int_0^6 8e^{-0.15t}\,dt=8\left[\frac{e^{-0.15t}}{-0.15}\right]_0^6=\frac{8}{-0.15}(e^{-0.15\times 6}-e^0)$$

$$=\frac{8}{0.15}(1-e^{-0.9})=53.333\times 0.5934\approx 31.65\text{ mg}\cdot\text{hr/L}$$

(b) AUC from $t=0$ to infinity:

$$AUC_{0\to\infty}=\int_0^\infty 8e^{-0.15t}\,dt 8\left[\frac{e^{-0.15t}}{-0.15}\right]_0^\infty=\frac{8}{0.15}(1-0)=53.33\text{ mg hr / L}$$

(c) Required total exposure over first 6 hours $=40\text{ mg}\cdot\text{hr/L}$

Actual AUC from $t=0$ to $t=6=31.65\text{ mg}\cdot\text{hr/L}$

$$31.65<40$$

Conclusion: No

The drug exposure from this dose over 6 hours is insufficient to meet the required minimum of $40\text{ mg}\cdot\text{hr/L}$

EXERCISE 3.7

1. Find the moment of inertia I of a uniform rectangular plate $ABCD$ about the edge AD when $AD = 3$, $AB = 8$, $BC = 3$, $CD = 8$ and the mass per unit area is 4.
2. Find the moment of inertia of a uniform triangular lamina about its base.
3. A particle moves along a straight line with acceleration $a(t) = 12t - 36$ (m/s^2) and initial velocity $v(0) = 30$ m/s.
   (a) Find the velocity function $v(t)$.
   (b) Find the displacement of the particle during the time period $1\le t\le 6$.
   (c) Find the distance traveled during this time period.
4. A particle moving along a coordinate line at time $t=0$ is at a position $2cm$ from the origin and traveling at a velocity of $5cm/\sec$. If the acceleration of the particle is given by: $a(t)=3-4(t+2)^{-3}$. Find the velocity and the position of the particle as functions of t.
5. A projectile is launched vertically upward from an initial height of s_o feet with an initial velocity of v_o ft/s. Neglecting air resistance, show that its height $s(t)$ above the ground at time t is given by $s(t)=-\frac{1}{2}gt^2+v_ot+s_o$ where, $g>0$ is the magnitude of the acceleration due to gravity.

6. In a bacterial culture, the growth rate is proportional to the number of bacteria present. Initially there are 200 bacteria. After 2 hours, there are 1800 bacteria.
   (a) Find the number of bacteria at time t hours.
   (b) How many bacteria will there be after 4 hours?
7. The rate of population growth of a small city (in people per year) is modeled by $P'(t) = 800 + 60t$, where t is the number of years after 2025. If the population in 2025 ($t = 0$) was 25,000 people:
   (a) Find the population function $P(t)$.
   (b) Estimate the population in 2035.
   (c) Calculate the total population growth from 2028 to 2033.
8. Find the integer b such that the average value of $f(x) = 2 + 4x - 3x^2$ on the interval [2, b] equals –7.
9. Let f be continuous, $\int_0^2 f(x)\,dx = 5$. If the average value of f on $[0, 4]$ is twice the average value on $[0, 2]$, find $\int_2^4 f(x)\,dx$.
10. Let f be continuous, and suppose: $\int_1^3 f(x)\,dx = 8$. If the average value of f on $[1, 4]$ is three times the average value on $[3, 4]$, find $\int_3^4 f(x)\,dx$.
11. Let f be continuous and suppose: $\int_0^2 f(x)\,dx = 10$ and $\int_2^4 f(x)\,dx = 6$. If the average value of f on $[0, 3]$ equals the average value on $[3, 4]$, find $\int_3^4 f(x)\,dx$.
12. A patient receives an IV bolus dose of a drug. The concentration in the blood is modeled by:
   $C(t) = 12e^{-0.3t}$ (mg/L) where t is the time in hours since that dose was taken.
   The minimum effective concentration (MEC) is 3 mg/L. The drug is considered effective if total AUC in a 24-hour period is at least 150 mg·h/L.
   (a) Find the time T (in hours) when the concentration reaches the MEC.
   (b) Calculate the AUC from $t = 0$ to $t = T$ for a single dose.
   (c) Determine if one dose provides enough total AUC over 24 hours. If not, how many doses (given at intervals of T hours) are needed in 24 hours to meet the requirement?

13. A patient takes a drug every 8 hours. The single-dose concentration (in mg/L) is $C(t) = 12e^{-0.2t}$ where t is time in hours since that dose was taken.

(a) Understanding Contributions

If the patient takes doses at $t = 0$, $t = 8$ and $t = 16$. Write the contribution from each dose separately at time $t = 10$ hours. Then write the total concentration at $t = 10$.

(b) Common Mistake

A student calculates concentration at $t = 10$ as $12e^{-0.2\times10} \approx 1.624$ mg/L.

What is wrong with this calculation?

(c) Just Before V/S Just After a Dose

Calculate

(i) Concentration just before the third dose at $t = 16$ hours.

(ii) Concentration just after the third dose at $t = 16$ hours.

(d) General Formula

Write the general formula for $C_{\text{total}}(t)$ for $16 \le t \le 24$, i.e., after 3 doses have been given.