Unit 02 Further Differentiation

INTRODUCTION

Differentiation, a fundamental concept in calculus introduced by Newton and Leibniz, allows us to study how quantities change and evolve. In our previous class, we studied the definition of differentiation along with sum, difference, product, and quotient rules, and applied them to find derivatives of polynomials and rational functions. Building on that foundation, we will now explore how to find the derivatives of trigonometric, inverse trigonometric, exponential, and logarithmic functions. We will apply differentiation to identify increasing and decreasing functions and to find equations of tangents and normals to curves at specific points. Higher-order derivatives of algebraic, implicit, parametric, and transcendental functions will also be studied. We will use linear approximations and differentials to estimate function values, changes, and calculate relative and percentage errors in real-life contexts like volumes of geometric shapes. The unit also delves into finding local and global extrema using critical points

2.1 Review of Differentiation Basics

Definition: The derivative of the function $y = f(x)$ is denoted by $\frac{dy}{dx}$ or $f'(x)$ and is defined as:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ if exists or } \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} \text{ if exists.}$$

The derivative gives the instantaneous rate of change of the function at a point.

Recall!

- Power Rule: $\frac{d}{dx}\left(x^n\right) = n.x^{n-1}$ where $n \in \mathbb{R}$.
- Constant Rule: $\frac{d}{dx}(c) = 0$, for any $c \in \mathbb{R}$.

In the following rules $u = u(x), v = v(x)$ are differentiable functions and c is any arbitrary constant.

- Constant Multiple Rule: $\frac{d}{dx}(cu) = c\frac{du}{dx} = cu'$.

- Sum and Difference Rule: $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx} = u' \pm v'$
- Product Rule: This rule helps to differentiate the product of two functions.
$$\frac{d}{dx}(uv) = u'v + uv'$$
- Quotient Rule: This rule is used to differentiate the division of two functions.
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$
where $v(x) \neq 0$ for all x in the domain.

2.2 Derivatives of Trigonometric Functions

Derivative of the Sine Function

While finding derivatives of trigonometric functions, we assume that the angle is measured in radians. To calculate the derivative of $f(x) = \sin x$.

Here, $f(x+h) = \sin(x+h)$, and

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(By the definition of derivative)}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \quad \text{(By substituting the values)}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h} \quad \because \sin p - \sin q = 2\cos\frac{p+q}{2}\sin\frac{p-q}{2}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h \to 0} \cos\left(\frac{2x+h}{2}\right) \cdot \lim_{h \to 0} \frac{\sin(h/2)}{h/2} \quad \text{applying limit } h \to 0$$

$$= \cos x(1) = \cos x \quad \left(\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1,\ \theta \text{ is measured in radians}\right)$$

Thus $\dfrac{d}{dx}(\sin x) = \cos x$

The derivative of the sine function is the cosine function
$$\frac{d}{dx}(\sin x) = \cos x$$

Try yourself!

Prove by definition $\dfrac{d}{dx}(\cos x) = -\sin x.$

Derivative of the Other Trigonometric Functions

After finding derivatives of $\sin x$ and $\cos x$ by definition, the derivatives of the remaining trigonometric functions can be found using product rule and quotient rule of differentiation.

Example 1 Find $f'(x)$ if $f(x)=\tan x$ (By using definition)

Solution: $f(x)=\tan x=\dfrac{\sin x}{\cos x}$

Differentiating w.r.t. 'x', we have

$$f'(x)=\frac{(\sin x)'\cos x-\sin x(\cos x)'}{\cos^2 x} \qquad \because \left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$$

$$=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x} \qquad \because (\sin x)'=\cos x,\ (\cos x)'=-\sin x$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$

$$=\frac{1}{\cos^2 x}=\sec^2 x \qquad \because \sin^2 x+\cos^2 x=1$$

Thus, $\dfrac{d}{dx}(\tan x)=\sec^2 x$.

The Derivatives of Six Trigonometric Functions

(i) $\dfrac{d}{dx}(\sin x)=\cos x$ (ii) $\dfrac{d}{dx}(\cos x)=-\sin x$

(iii) $\dfrac{d}{dx}(\tan x)=\sec^2 x$ (iv) $\dfrac{d}{dx}(\cot x)=-\csc^2 x$

(v) $\dfrac{d}{dx}(\sec x)=\sec x\tan x$ (vi) $\dfrac{d}{dx}(\csc x)=\csc x\cot x$

Example 2 Find $\dfrac{dy}{dx}$ if $y=x^2\csc x$

Solution: $y=x^2\csc x$

Differentiating w.r.t. 'x'

$$\frac{dy}{dx}=(x^2)'\csc x+x^2(\csc x)'=2x\csc x+x^2(-\csc x.\cot x) \quad =x\csc x(2-x\cot x)$$

2.3 The Chain Rule

2.3.1 Composition of Functions

Let $g: X\to Y$, and $f: Y\to Z$, then their composition $f\circ g: X\to Y$ is defined as:

$$f\circ g(x)=f(g(x))$$

For example, consider two real valued functions $g(x)=x^2-x$ and $f(x)=\sin x$, then

$$f\circ g(x)=f\left(g(x)\right)=f\left(x^2-x\right)=\sin\left(x^2-x\right)$$

Theorem 1: (The Chain Rule)

If $f(u)$ is differentiable at the point $u=g(x)$ and $g(x)$ is differentiable at x, then the composite function $f\circ g(x)=f(g(x))$ is differentiable at x, an $f\circ g'(x)=f'(g(x))\cdot g'(x)$

In Leibniz's notation, if $y=f(u)$ and $u=g(x)$, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

Critical Thinking

Suppose $f(x)=x^2$ and $g(x)=|x|$. Then the composition $(f^{\circ}g)(x)=x^2$ is differentiable at $x=0$, although $g(x)$ is not differentiable at $x=0$. Does this contradict the Chain Rule?

Example 3 Differentiate $y=\sin(x^2-x)$ w.r.t 'x'.

Solution: Let $u=x^2-x$ ...(1)

Substituting $u=x^2-x$ into $y=\sin(x^2-x)$, we have $y=\sin u$.

Now, we have $y=\sin u$ and $u=x^2-x$

Differentiate with respect to u.

$$\frac{dy}{du}=\cos u$$

Differentiate with respect to x.

$$\frac{du}{dx}=2x-1$$

By Chain Rule $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\cos u\cdot(2x-1)$

$$\boxed{\frac{dy}{dx}=\cos(x^2-x)(2x-1)}$$ (By using equation 1)

2.3.2 Derivative of a Function Given in the Form of Parametric Equations

Let a function be represented by parametric equations

$$x=f(t) \quad \text{and} \quad y=g(t)$$

From $x=f(t)$, we have

$$t=f^{-1}(x) \qquad \text{(Provided } f \text{ is bijective)}$$

Now, by Chain Rule applying to $y=g(t)$ and $t=f^{-1}(x)$, we get:

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \left(\because \frac{dt}{dx}=\frac{1}{\frac{dx}{dt}}\right)$$

Example 4 Find $\dfrac{dy}{dx}$ if $y = a\cos t$ and $x = a\sin t$.

Solution: Differentiate $y = a \cos t$ and $x = a \sin t$ w.r.t 't'

$$\frac{dy}{dt} = -a\sin t \quad \text{and} \quad \frac{dx}{dt} = a\cos t$$

Apply the Chain Rule $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{a\sin t}{a\cos t} = -\tan t$$

2.3.3 Implicit Differentiation

When an implicit relation cannot be easily solved for y (or when multiple explicit functions are involved), we can still compute $\dfrac{dy}{dx}$ using implicit differentiation. The steps are as follows:

1. Differentiate both sides of $F(x, y) = 0$ with respect to x, treating y as an implicit function of x.
2. Apply the Chain Rule to terms containing y.
3. Solve the resulting equation for $\dfrac{dy}{dx}$.

Example 5 Find $\dfrac{dy}{dx}$ if $x + y^2 + \sin(xy^2) = 0$

Solution: Differentiate $x + y^2 + \sin(xy^2) = 0$ with respect to x to obtain:

$$\frac{d}{dx}(x) + \frac{d}{dx}(y^2) + \frac{d}{dx}\left(\sin(xy^2)\right) = 0$$

$$1 + 2y\frac{dy}{dx} + \cos(xy^2)\cdot\frac{d}{dx}(xy^2) = 0$$

$$2y\frac{dy}{dx} + \cos\left(xy^2\right)\left\{y^2 + x\left(2y\frac{dy}{dx}\right)\right\} = -1$$

$$2y\frac{dy}{dx} + y^2\cos(xy^2) + 2xy\frac{dy}{dx}\cos(xy^2) = -1$$

$$2y\frac{dy}{dx}\left\{1 + x\cos\left(xy^2\right)\right\} = -1 - y^2\cos\left(xy^2\right)$$

$$\boxed{\frac{dy}{dx} = -\frac{1 + y^2\cos(xy^2)}{2y\left\{1 + x\cos(xy^2)\right\}}}$$

2.4 Derivatives of Inverse Trigonometric Functions

Here, we will prove that:

(i) $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$

(ii) $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$

(iii) $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$, for $x \in \mathbb{R}$

(iv) $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$, for $x \in \mathbb{R}$

(v) $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{|x|\sqrt{x^2-1}}$, for $x \in (-\infty,-1) \cup (1,\infty)$

(vi) $\dfrac{d}{dx}(\csc^{-1} x) = -\dfrac{1}{|x|\sqrt{x^2-1}}$, for $x \in (-\infty,-1) \cup (1,\infty)$

Proof (1): Let $y = \sin^{-1}(x)$, where $x \in [-1,1]$ and $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

$$\sin y = x \quad \ldots (1)$$

Differentiate equation (1) w.r.t. *x*, we have

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{\cos^2 y}} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } x \in (-1,1) \quad (\because \sin y = x)}$$

Proof (2): Let $y = \cot^{-1}(x)$, where $y \in (0,\pi)$ and $x \in \mathbb{R}$.

$$\cot y = x \quad \ldots (2)$$

We differentiate equation (1) w.r.t. *x*,

$$-\csc^2 y \frac{dy}{dx} = 1 \qquad \left((\cot y)' = -\csc^2 y\right)$$

$$\frac{dy}{dx} = -\frac{1}{\csc^2 y} \qquad \left(\csc y \neq 0, \text{ for } y \in (0,\pi)\right)$$

$$= -\frac{1}{1+\cot^2 y} \qquad (\because \ 1+\cot^2 y = \csc^2 y)$$

$$\boxed{\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}} \text{ for } x \in R \qquad (\because \cot y = x)$$

Proof (3): Let $y = \sec^{-1} x$, where $x \in (-\infty,-1] \cup [1,\infty)$ and $y \in [0,\pi] - \left\{\dfrac{\pi}{2}\right\}$

$$\sec y = x \quad \ldots(1) \quad , \quad |x| \geq 1$$

We differentiate equation (1) w.r.t. *x*:

$$\sec y \tan y \frac{dy}{dx} = 1 \qquad \left(\because (\sec y)' = \sec y \tan y\right)$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \qquad \text{where, } \sec y \neq 0 \text{ and } \tan y \neq 0$$

$\tan y \neq 0 \;\Rightarrow\; y \neq 0$ and $y \neq \pi \;\Rightarrow\; x \neq -1, 1$ and from equation (1).

$$\frac{dy}{dx} = \frac{1}{x\tan y} \qquad \ldots(2) \qquad (\because \sec y = x)$$

$$\tan y = \begin{cases} \sqrt{\tan^2 y}, & y \in \left(0, \frac{\pi}{2}\right) \\ -\sqrt{\tan^2 y}, & y \in \left(\frac{\pi}{2}, \pi\right) \end{cases} = \begin{cases} \sqrt{\sec^2 y - 1}, & y \in \left(0, \frac{\pi}{2}\right) \\ -\sqrt{\sec^2 y - 1}, & y \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

$$\tan y = \begin{cases} \sqrt{x^2-1}, & x \in (1, \infty) \\ -\sqrt{x^2-1}, & x \in (-\infty, -1) \end{cases}$$

Challenge!

Prove formulae (2), (3) and (5).

$$\tan y = \begin{cases} \sqrt{x^2-1} & x \in (1, -1) \\ -\sqrt{x^2-1}, & x \in (-\infty, 1) \end{cases} \quad \left(\begin{array}{l} \because \sec y = x, \; and\; x \in (1,\infty) \text{ when } y \in \left(0, \frac{\pi}{2}\right), \\ \text{while } x \in (-\infty, -1) \text{ when } y \in \left(\frac{\pi}{2}, \pi\right) \end{array}\right)$$

Substitute the value of $\tan y$ in equation (2)

$$\frac{dy}{dx} = \begin{cases} \dfrac{1}{x\sqrt{x^2-1}}, & x \in (1, \infty) \\ \dfrac{1}{-x\sqrt{x^2-1}}, & x \in (-\infty, -1) \end{cases}$$

Recall!

$$\sqrt{x^2} = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Hence, $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{|x|\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$ $\left(\because |x| = \begin{cases} x & if\; x \geq 0 \\ -x & if\; x < 0 \end{cases}\right)$

Example 6 Find $\dfrac{dy}{dx}$ if $y = x\cos^{-1}\left(\dfrac{x}{a}\right) - \sqrt{a^2 - x^2}$.

Solution: Differentiate $y = x\cos^{-1}\left(\dfrac{x}{a}\right) - \sqrt{a^2 - x^2}$ w.r.t. x to obtain:

$$\frac{dy}{dx} = \cos^{-1}\left(\frac{x}{a}\right) + x\left(-\frac{1}{\sqrt{1-(x/a)^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)\right) - \frac{1}{2}(a^2 - x^2)^{-1/2} \cdot \frac{d}{dx}(a^2 - x^2)$$

$$= \cos^{-1}\left(\frac{x}{a}\right) - \frac{x}{\sqrt{\dfrac{a^2 - x^2}{a^2}}} \cdot \frac{1}{a} - \frac{1}{2} \cdot \frac{1}{\sqrt{a^2 - x^2}}(-2x)$$

$$\frac{dy}{dx} = \cos^{-1}\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right)$$

EXERCISE 2.1

1. Derive the formula for the derivative of the following functions w.r.t. x:
   (a) $\sec x$ (b) $\csc x$ (c) $\cot x$

In 2 - 6, differentiate with respect to the variable involved.

2. $y = x^2 \cos x$
3. $y = x \sec x + \frac{1}{x}$
4. $y = \frac{\sin t}{1 - \cos t}$
5. $y = (1 + \csc t)\sin t$
6. $y = \sec x + 4\sqrt{x} - 10$

In 7 - 9, find $\frac{dy}{dx}$ by making suitable substitutions and applying the Chain Rule.

7. $y = \sqrt{\frac{1+x}{1-x}}$
8. $y = (\sin x + \cos x)^{3/2}$

In 9 - 12, differentiate w.r.t the variable involved.

9. $y = \sec(\tan x)$
10. $y = \sqrt{\sin x} + \cos\sqrt{x}$
11. $y = \tan^2 x + \cot(x^2)$
12. $y = \sin^3(\cos 2v)$
13. Find $\frac{dy}{dx}$ for the following parametric equations:
    (i) $x = \sin^2 t$ and $y = \cos 2t$ (ii) $x = \sqrt{1 + \sec t}$, and $y = \sqrt{\tan t}$
14. If $x = a\sec t$ and $y = b\tan t$, show that $\frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$.
15. Use implicit differentiation to find $\frac{dy}{dx}$ if:
    (i) $ax^2 + by^2 + 2gx + 2fy + c = 0$ (ii) $x + \tan(xy) = 0$
16. Differentiate w.r.t. the variables involved.
    (i) $\sin^{-1}\left(\frac{x}{a}\right)$ (ii) $\csc^{-1}\left(\frac{x^2+1}{x^2-1}\right)$
    (iii) $\cot^{-1}(\sqrt{x}) + \sqrt{\cot^{-1} x}$
    (iv) $(\tan t)^{-1} + \frac{1}{\tan^{-1} t}$

Point to Note

$\left|\frac{x^2+1}{x^2-1}\right| = \frac{x^2+1}{x^2-1} > 1$ for $x \in (-\infty, -1) \cup (1, \infty)$, and $\frac{x^2+1}{x^2-1} < -1$ has no solution.

2.5 Derivative of Exponential Functions

Recall!

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a$$

A function defined by

$$f(x) = a^x, \quad \text{where } a > 0 \text{ and } a \neq 1$$

is called an exponential function.

If $a = e \approx 2.71828$ then $f(x) = e^x$, is called natural exponential function.

Now, we shall find the derivative of $f(x) = a^x$, using the definition.

Let $f(x) = a^x$ … (1)

From equation (1) we have $f(x+h) = a^{x+h}$ …(2)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \text{(definition of derivative)}$$

After substituting the expression for $f(x)$ and $f(x+h)$ from equations (1) and (2), we get:

$$= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x(a^h-1)}{h}$$

$$= a^x \lim_{h\to 0}\frac{a^h-1}{h} = a^x \ln a$$

Challenge!

- Prove that $\frac{d}{dx}(e^x) = e^x$
- Do you know $\ln e = 1$?

Hence, we have proved that $\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$.

Example 7 Find $\frac{dy}{dx}$ if (i) $y = e^{\sin^2 x}$ (ii) a^{x^3}

Solution: **(i)** Differentiate $y = e^{\sin^2 x}$ w.r.t. x and apply the Chain Rule, to obtain:

$$\frac{dy}{dx} = e^{\sin^2 x}\frac{d}{dx}(\sin^2 x) \qquad \left(\because \frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}\right)$$

$$\frac{dy}{dx} = e^{\sin^2 x} 2\sin x \frac{d}{dx}(\sin x) = 2\sin x\, e^{\sin^2 x}\cos x$$

$$= 2\sin x \cos x\, e^{\sin^2 x}$$

(ii) Differentiate $y = a^{x^3}$ w.r.t. x and apply the Chain Rule, to obtain:

$$\frac{dy}{dx} = a^{x^3}\ln a \frac{d}{dx}(x^3) \qquad \left(\because (a^u)' = a^u \ln a \cdot \frac{du}{dx}\right)$$

$$= a^{x^3}\ln a(3x^2)$$

$$= 3x^2 a^{x^3}\ln a$$

2.6 Derivative of Logarithmic Functions

If $a^y = x$, where $a > 0$ and $a \neq 1$, then y is called the *logarithm* of x to the base a, written as:

$$y = \log_a x \quad , \quad x > 0 \quad , \quad y \in \mathbb{R}$$

If $a = e$, where $e \approx 2.71828$, then the function

$$y = \log_e x = \ln x \ \text{(whenever } e^y = x\text{)}$$

is called the *natural logarithm* of x.

Now, we will find the derivative of $f(x)=\log_a x$, using the definition.

Let $\quad f(x)=\log_a x \quad \ldots (1)$

From equation (1) we have: $f(x+h)=\log_a(x+h) \quad \ldots (2)$

We know $\quad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ (definition of derivative)

After substituting the expression for $f(x)$ and $f(x+h)$ from equations (1) and (2), we get:

$$=\lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} = \lim_{h\to 0}\frac{1}{h}\log_a\left(\frac{x+h}{x}\right)$$

$$=\lim_{h\to 0}\frac{1}{x}\cdot\frac{x}{h}\log_a\left(1+\frac{h}{x}\right) = \lim_{h\to 0}\frac{1}{x}\log_a\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

Recall!

$$\lim_{h\to 0}\left(1+\frac{h}{a}\right)^{\frac{a}{h}}=e$$

$$=\frac{1}{x}.\log_a e$$

$$=\frac{1}{x}.\frac{1}{\log_e a} \qquad \left(\because \log_a b=\frac{1}{\log_b a}\right)$$

$$=\frac{1}{x\ln a} \qquad (\because \log_e x=\ln x)$$

Hence, we have proved that $\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}$.

Example 8 Find $\frac{dy}{dx}$ if $y=\frac{a^x}{\log_a x}$.

Solution: Differentiate $y=\frac{a^x}{\log_a x}$ with respect to 'x', and apply the Quotient Rule to obtain:

$$\frac{dy}{dx}=\frac{(a^x)'\log_a x-a^x(\log_a x)'}{(\log_a x)^2}$$

$$=\frac{a^x\ln a\,\log_a x-a^x\cdot\frac{1}{x\ln a}}{(\log_a x)^2}$$

$$\boxed{=\frac{a^x\left(x(\ln a)^2\log_a x-1\right)}{x\ln a(\log_a x)^2}}$$

Example 9

Find $\frac{dy}{dx}$, if $y=\ln(\cos(e^x))$.

Solution: Differentiate $y = \ln(\cos(e^x))$ with respect to 'x', and apply the Chain Rule to obtain:

$$\frac{dy}{dx}=\frac{1}{\cos(e^x)}\frac{d}{dx}(\cos e^x)$$

$$=\frac{1}{\cos(e^x)}(-\sin e^x)\frac{d}{dx}(e^x)$$

$$=-\frac{\sin(e^x)}{\cos(e^x)}e^x$$

$$\boxed{=-e^x\tan(e^x)}$$

Challenge!

Prove that $\frac{d}{dx}(\ln x)=\frac{1}{x}$

EXERCISE 2.2

1. Find $f'(x)$ if:

(i) $f(x)=(x^3+4x)e^{1/x}$ (ii) $f(x)=\sin(2^x)$

(iii) $f(x)=e^{x\cos x}$ (iv) $f(x)=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

2. Find $\dfrac{dy}{dx}$ if:

(i) $y=\ln(e^x-e^{-x})$ (ii) $y=\log_{10}\sqrt{\dfrac{1-x^2}{1+x^2}}$ (iii) $y=\ln\left(x-\sqrt{x^2-1}\right)$

(iv) $y=\ln(1-\cos^2 x)$ (v) $y=\dfrac{\ln x}{x^2}$ (vi) $y=\dfrac{\sqrt{x^2-1}(2x+1)}{(x^3-1)^{3/2}}$

2.7 Equations of Tangent and Normal Lines to a Curve

Tangent Line: Suppose that x_0 is a point in the domain of the function $f(x)$. The tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is the line with equation

$$y-y_0=m(x-x_0)$$

where $m=f'(x_0)=\left.\dfrac{dy}{dx}\right|_{x=x_0}$, provided $f'(x_0)$ exists.

Need to Know!

Vertical Tangent
If m is ∞ or $-\infty$ at $x = x_0$, then $x = x_0$ is tangent to the curve $y = f(x)$ is called vertical tangent

Normal Line: A line through point $(x_0, f(x_0))$ of the curve $y=f(x)$ and perpendicular to tangent line is called normal line to the curve $y=f(x)$.

Note
Slope of tangent line to a curve at a point is also known as slope of the curve at the point.

If $m=f'(x_0)$ is non zero, then equation of the normal line is $y-y_0=-\dfrac{1}{m}(x-x_0)$ and if $m=f'(x_0)=0$, then $x=x_0$ is the equation of the normal line.

Example 10 Find the equations of the tangent and normal to the curve $y=\sec x$ at the point where $x=\dfrac{\pi}{4}$.

Solution: Here, $y = \sec x$ ...(1)

Differentiating (1) with respect to x:

$$\frac{dy}{dx}=\sec x\tan x$$

$$m=\left.\frac{dy}{dx}\right|_{x=\pi/4}=\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)=\sqrt{2}$$ (Slope of tangent line)

Put $x=\frac{\pi}{4}$ in (1), $y=\sec\left(\frac{\pi}{4}\right)=\sqrt{2}$

The equation of tangent at $\left(\frac{\pi}{4},\sqrt{2}\right)=(x_0,y_0)$ with slope $m=\sqrt{2}$ is:

$$y-\sqrt{2}=\sqrt{2}\left(x-\frac{\pi}{4}\right) \quad \because y-y_0=m(x-x_0)$$

$$y-\sqrt{2}=\sqrt{2}x-\frac{\pi\sqrt{2}}{4}$$

$$4y-4\sqrt{2}=4\sqrt{2}x-\pi\sqrt{2}$$

$$4\sqrt{2}x-4y+4\sqrt{2}-\pi\sqrt{2}=0$$

The equation of normal to given curve at $\left(\frac{\pi}{4},\sqrt{2}\right)=(x_0,y_0)$ is:

$$y-\sqrt{2}=-\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right)$$

$$x+4\sqrt{2}y-8-\pi=0$$

2.8 Higher-Order Derivatives

The derivative f' of a function f is itself a function and hence may have a derivative of its own. If f' is differentiable, then its derivative is denoted by f'' and is called the *second derivative* of f. As long as the new function remains differentiable, we can continue the process of differentiating to obtain the third, fourth, fifth, and even higher derivatives of f. These successive derivatives are called *higher-order* derivatives.

The following are the commonly used notations for higher-order derivatives:

1st derivative	2nd derivative	3rd derivative	…	nth derivative
y'	y''	y'''	…	$y^{(n)}$
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	…	$\frac{d^ny}{dx^n}$
$f'(x)$	$f''(x)$	$f'''(x)$	…	$f^{(n)}(x)$
y_1	y_2	y_3	…	y_n
Dy	D^2y	D^3y	…	D^ny
$\frac{df}{dx}$	$\frac{d^2f}{dx^2}$	$\frac{d^3f}{dx^3}$	…	$\frac{d^nf}{dx^n}$

Example 11 Find the higher-order derivatives of the function

$$f(x)=\frac{1}{4}x^4-x^3+\frac{1}{2}x^2-3$$

Solution:

Differentiate $f(x)=\frac{1}{4}x^4-x^3+\frac{1}{2}x^2-3$ to find its higher-order derivatives.

$$f'(x)=\frac{1}{4}.4x^3-3x^2+\frac{1}{2}\cdot 2x-0$$

$$f''(x) = 3x^2-6x+1$$

$$f'''(x) = 6x-6$$

$$f^{(4)}(x) = 6$$

$$f^{(5)}(x) = 0$$

$$\vdots$$

$$f^{(n)}(x) = 0\ (n\geq 5)$$

Example 13 Find all higher-order derivatives of the function $y=e^{ax}$.

Solution: We differentiate $y=e^{ax}$ to find its higher-order derivatives.

$$\frac{dy}{dx}=e^{ax}.\frac{d}{dx}(ax)$$

$$=e^{ax}\cdot a=ae^{ax}$$

Differentiate with respect to 'x', to obtain:

$$\frac{d^2y}{dx^2}=a(a^{ax}a)=a^2e^{ax}$$

$$\frac{d^3y}{dx^3}=a^2(e^{ax}a)=a^3e^{ax}$$

$$\frac{d^4y}{dx^4}=a^4e^{ax}$$

$$\vdots$$

$$\frac{d^ny}{dx^n}=a^ne^{ax} \qquad (n\geq 1)$$

$$=a^ny$$

Example 12 If $x=e^{at}$ and $y=e^{2at}$, show that $x^2\frac{d^2y}{dx^2}-2y=0$.

Solution: Given: $x=e^{at}$ ... (1)

and $y=e^{2at}$ ... (2)

Differentiate equations (1) and (2) with respect to 't', to get:

$$\frac{dx}{dt}=ae^{at} \text{ and } \frac{dy}{dt}=2ae^{2at}$$

Using equations (1) and (2), we have:

$$\frac{dx}{dt}=ax \text{ and } \frac{dy}{dt}=2ay$$

Apply the Chain Rule to obtain:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2ay}{ax}$$

$$\frac{dy}{dx}=\frac{2y}{x} \qquad ...(3)$$

Differentiate with respect to 'x', to obtain:

$$\frac{d^2y}{dx^2}=2\frac{\frac{dy}{dx}x-y.1}{x^2}$$

Using equation (3), we get:

$$=2\frac{\frac{2y}{x}x-y}{x^2}$$

$$=2\frac{y}{x^2}$$

$$x^2\frac{d^2y}{dx^2}=2y$$

$$\Rightarrow\ x^2\frac{d^2y}{dx^2}-2y=0$$

Example 14: Find y_4 if $y=\ln\sqrt{\frac{2x-3}{x+1}}$.

> **Challenge!**
> Find y_{10} in example 16.

Solution: We are given: $y=\ln\sqrt{\frac{2x-3}{x+1}}$

Before differentiating we apply properties of logarithm:

$$y=\frac{1}{2}\ln\left(\frac{2x-3}{x+1}\right)$$

$$=\frac{1}{2}\left[\ln(2x-3)-\ln(x+1)\right]$$

$$y_1=\frac{1}{2}\left(\frac{2}{2x-3}-\frac{1}{x+1}\right) \qquad \text{(By differentiating w.r.t. } x\text{)}$$

$$=\frac{1}{2}\left(2(2x-3)^{-1}-(x+1)^{-1}\right)$$

$$y_2=\frac{1}{2}\left(-2(2x-3)^{-2}+(x+1)^{-2}\right) \qquad \text{(By differentiating w.r.t. } x\text{)}$$

$$y_3=\frac{1}{2}\left(4(2x-3)^{-3}-2(x+1)^{-3}\right) \qquad \text{(By differentiating w.r.t. } x\text{)}$$

$$y_4=\frac{1}{2}\left(-12(2x-3)^{-4}+6(x+1)^{-4}\right) \qquad \text{(By differentiating w.r.t. } x\text{)}$$

$$y_4=\frac{1}{2}\left[\frac{-12}{(2x-3)^4}+\frac{6}{(x+1)^4}\right]=\frac{-6}{(2x-3)^4}+\frac{3}{(x+1)^4}$$

EXERCISE 2.3

1. Find the equation of the tangent line to the curve $y=\sin^{-1}\left(\frac{x}{2}\right)+\cos x$ at the point where $x=\frac{\pi}{2}$.

2. Find a point on the curve $y=\tan^{-1}\left(\frac{x}{3}\right)-\frac{x}{3}$, where tangent is horizontal. Also find the equations of the tangent and normal at that point.

3. Find a point on the curve $x+y-e^y=0$ where the tangent line is vertical. Also, find the equations of the tangent and normal at that point.

4. Find all points on the ellipse $x^2-xy+y^2=1$ at which the tangent line is horizontal.

5. Find y_2 if:

(i) $y = x^4 - 3x^2 + 4x - 5$ (ii) $y = \ln(2x+3)$ (iii) $y = x^3 e^{2x}$

(iv) $x^2 - y^2 = 4$ (v) $x^2 - xy + y^2 = 7$

(vi) $x = a\cos t, \quad y = b\sin t$ (vii) $x = t^2, \quad y = 2t^3$

6. Find y_4 and write expressions for y_{10} and y_{11} without evaluating them:

(i) $y = \sin\left(\frac{x}{2}\right)$ (ii) $y = \ln(x^2 - a^2)$

7. If $x^2 + y^2 = 25$, show that $y^3 \frac{d^2y}{dx^2} + 25 = 0$.

8. If $y = e^{ax}\cos bx$, show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$

9. If $x = \sin t$ and $y = \cos^2 t$, show that $\frac{d^2y}{dx^2} + 2 = 0$

10. If $x = \ln t$ and $y = \frac{1}{t}$, show that $\frac{d^2y}{dx^2} - y = 0$

11. If $y = a\cos(\ln x) + b\sin(\ln x)$, show that $x^2 y_2 + xy_1 + y = 0$

2.9 Increasing and Decreasing Functions

Increasing Function	Decreasing Function
Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f is increasing if for any $x_1, x_2 \in I$. $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$	Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f is decreasing if for any $x_1, x_2 \in I$. $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

Recall!

(a,b), $[a,b]$, $(a,b]$, $[a,b)$, $(-\infty,a)$, $(-\infty,\infty)$ $(-\infty,a]$, (a,∞), $[a,\infty)$ all are intervals.

Let f be a differentiable function defined on an interval I. Then

(i) f is increasing on I, if $f'(x) \ge 0$ for each $x \in I$.

(ii) f is decreasing on I, if $f'(x) \le 0$ for each $x \in I$.

Example 15 Prove that $f(x) = e^x$ defined on $R = (-\infty, \infty)$ is increasing.

Solution: We differentiate $f(x) = e^x$ with respect to 'x' to obtain.

$$f'(x) = e^x > 0 \text{ for each } x \in (-\infty, \infty)$$

Hence, $f(x) = e^x$ is an increasing function on $R = (-\infty, \infty)$.

Example 16 Show that $f(x)=2^{-x}$ defined on $R=(-\infty,\infty)$ is decreasing.

Solution: We differentiate $f(x)=2^{-x}$ with respect to 'x' to obtain.

$$f'(x)=2^{-x}\frac{d}{dx}(-x)$$

$$f'(x)=-2^{-x}<0 \text{ for each } x\in(-\infty,\infty)$$

So, $f(x)=2^{-x}$ is decreasing function on $R=(-\infty,\infty)$.

2.9.1 Stationary Point

A point on the graph where its derivative is zero is called a stationary point.

Example 16 Find the intervals on which the function $f(x)=|x|$, defined on $\mathbb{R}=(-\infty,\infty)$, is increasing or decreasing.

Solution: We know that:

$$f(x)=|x|=\begin{cases} x & if\ x\geq 0 \\ -x & if\ x<0 \end{cases}.$$

Critical Thinking

- Every stationary point is a critical point.
- Do you think every critical point is a stationary point?

Differentiating w. r. t. 'x':

$$f'(x)=\begin{cases} 1 & if\ x>0 \\ -1 & if\ x<0 \end{cases}$$

We now evaluate $f'(0)$ by definition:

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \quad \text{(By definition of derivative)}$$

$$=\lim_{h\to 0}\frac{f(h)-0}{h}=\lim_{h\to 0}\frac{|h|}{h}$$

Since $f(x)=|x|$ is defined different on left and right of $x=0$, we compute one sided limits:

$$\lim_{h\to 0^-}\frac{|h|}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1 \quad \text{and} \quad \lim_{h\to 0^+}\frac{|h|}{h}=\lim_{h\to 0^+}\frac{h}{h}=1$$

Since left-hand and right-hand limits are different, the limit $\lim_{h\to 0}\frac{|h|}{h}$ does not exist.

Hence $f'(0)$ is not defined. ($\therefore$ x is a critical point but not stationary)

The function has no stationary point because there is no point where the derivative is equal to zero.

Since $f'(x)=-1<0$ for each $x\in(-\infty, 0)$ $\Rightarrow$ the function is decreasing on $(-\infty, 0)$ and $f'(x)=1>0$ for each $x\in(0,\infty)$ $\Rightarrow$ the function is increasing on $(0,\infty)$

Hence $f(x)=|x|$ is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$.

In the above example, the function has no stationary point, but it changes behavior around a point where its derivative is undefined.

Conclusion: From these examples, we conclude that a function may change its:

1. Increasing/decreasing behavior around:
   - Stationary points where $f'(x)=0$, or
   - Points where $f'(x)$ is undefined.

2.9.2 Critical Point

For a function $f(x)$ a point $x=c$, $c \in \text{Dom} f$ is called a critical point if:

$$f'(c)=0 \text{ or } f'(c) \text{ does not exist.}$$

Examples:

1. For $f(x)=|x|$, the derivative does not exist at $x=0$, so $x=0$ is a critical point.
2. For $f(x)=x^2$, the derivative $f'(x)=2x$ is zero at $x=0$, so $x=0$ is a critical point.

Method to Find Intervals Where a Function is Increasing or Decreasing

To find the intervals where a function is increasing or decreasing, follow these steps:

Step 1: Differentiate the given function $f(x)$ to find $f'(x)$.

Step 2: Find critical points by solving $f'(x)=0$ and identifying points where $f'(x)$ does not exist.

Step 3: If critical points exist, divide the domain into sub-intervals using the critical points.

Step 4: Select a test point from each sub-interval and substitute it into $f'(x)$:

- If $f'(x)>0$ in a sub-interval, the function is increasing there.
- If $f'(x)<0$ in a sub-interval, the function is decreasing there.

Point to Note

In each sub-interval of the domain determined by the critical points, the sign of the derivative remains the same.

Example 12 Find the intervals in which the function $f(x)=2x^3-3x^2-12x$ is increasing or decreasing.

Solution: The function $f(x)=2x^3-3x^2-12x$ is defined for all $x \in R=(-\infty,\infty)$.

We differentiate $f(x)=2x^3-3x^2-12x$ with respect to x to obtain:

$$f'(x)=6x^2-6x-12 \quad \dots (1)$$

It is clear from equation (1) that $f'(x)$ is defined for each $x \in (-\infty,\infty)$.

To find other critical points, we solve $f'(x)=0$:

$$6x^2-6x-12=0$$
$$6(x^2-x-2)=0$$
$$6(x+1)(x-2)=0$$
$$x=-1 \quad \text{and} \quad x=2$$

Hence $x=-1$ and $x=2$ are the critical points of the given function.
Mark these points on the real line. These critical points divide the domain into intervals $(-\infty,-1)$, $(-1, 2)$ and $(2, \infty)$.

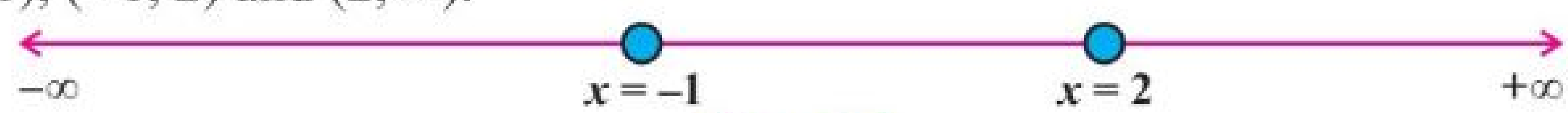

Figure 2.1

Now we check the sign of $f'(x)$ in each subinterval.

Interval	Test point	Sign of $f'(x)$	Conclusion
$(-\infty,-1)$	$x=-2$	$f'(-2)=6(-2)^2-6(-2)-12=24>0$	Increasing
$(-1,2)$	$x=0$	$f'(0)=6(0)^2-6(0)-12=-12<0$	Decreasing
$(2,\infty)$	$x=3$	$f'(3)=6(3)^2-6(3)-12=24>0$	Increasing

Hence the function $f(x)=2x^3-3x^2-12x$:

- Increases on $(-\infty,-1)$ and $(2,\infty)$.
- Decreases on $(-1,2)$.

Example 19 Find the intervals where the function $f(x)=x^{\frac{2}{3}}$ is increasing or decreasing on the domain $x\in(-\infty,\infty)$.

Solution: We differentiate $f(x)=x^{\frac{2}{3}}$ with respect to 'x' to obtain:

$$f'(x)=\frac{2}{3}x^{-\frac{1}{3}}=\frac{2}{3x^{\frac{1}{3}}} \qquad \ldots (1)$$

From equation (1), we conclude that $f'(0)$ is not defined. Thus $x=0$ is a critical point of the function $f(x)=x^{\frac{2}{3}}$.

To find other critical points we solve $f'(x)=0$.

$$\frac{2}{3x^{\frac{1}{3}}}=0$$

The above equation has no solution. Thus, the only critical point is $x=0$. This critical point divides the domain in the following intervals.

$$(-\infty, 0) \text{ and } (0, \infty)$$

Choose $x = -1 \in (-\infty, 0)$ and substitute into equation (1).

$$f'(-1) = \frac{2}{3(-1)^{\frac{1}{3}}} = -\frac{2}{3} < 0 \Rightarrow f'(x) < 0 \text{ for all } x \in (-\infty, 0). \left(\because (-1)^{\frac{1}{3}} = -1\right)$$

Thus, the function is decreasing on $(-\infty, 0)$.

Choose $x = 1 \in (-\infty, 0)$, and substitute into equation (1).

$$f'(1) = \frac{2}{3(1)^{\frac{1}{3}}} = \frac{2}{3} > 0 \Rightarrow f'(x) > 0 \text{ for all } x \in (0, \infty).$$

Thus, the function is increasing on $(0, \infty)$.

Hence the function $f(x) = x^{\frac{2}{3}}$ defined on $(-\infty, \infty)$:

- Decreases on $(-\infty, 0)$.
- Increases on $(0, \infty)$.

2.10 Relative and Absolute Extrema

2.10.1 Global (Absolute) Maximum

Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f has a global (absolute) maximum at $x = c \in I$, if $f(c) \geq f(x)$ for all $x \in I$.

If $f(x)$ has a global (absolute) maximum at $x = c$, then $f(c)$ is called global (absolute) maximum value of the function.

2.10.2 Global (Absolute) Minimum

Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f has a global (absolute) minimum at $x = c \in I$, if $f(c) \leq f(x)$ for all $x \in I$.

If $f(x)$ has a global (absolute) minimum at $x = c$, then $f(c)$ is called global (absolute) minimum value of the function.

The global maximum and minimum of a function are collectively called its global extrema.

Example: (a) Let $f(x) = x^2$ be defined on the interval $[0, 2]$.

The function f has a global maximum at $x = 2$ and a global minimum at $x = 0$. The corresponding maximum and minimum values of the function are:

$f(2) = 4$ (global maximum value)

and $f(0) = 0$ (global minimum value) (See Figure 2.2).

(b) The function $f(x) = x^2$ defined on $(-\infty, \infty)$ has an absolute minimum value at $x = 0$, where $f(0) = 0$. It has no absolute maximum value. (See Figure 2.3)

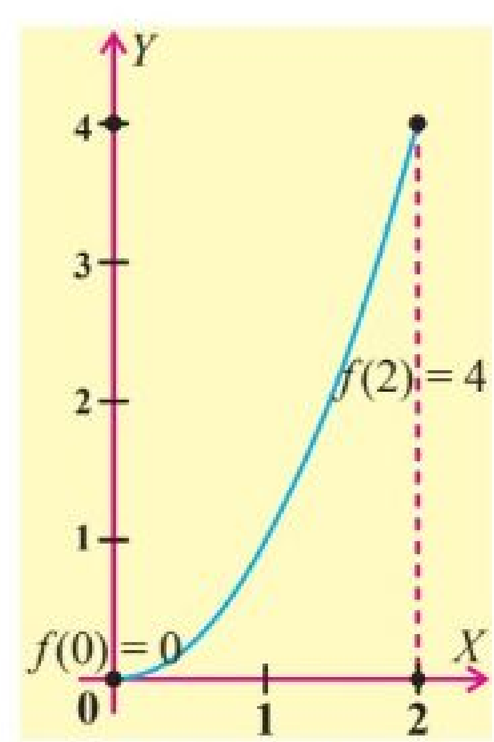

Figure 2.2

(c) The function $f(x) = -x^2 + 2$ defined on $(-\infty, \infty)$ has an absolute maximum value at $x = 0$, where $f(0) = 2$. It has no absolute minimum value. (See Figure 2.4)

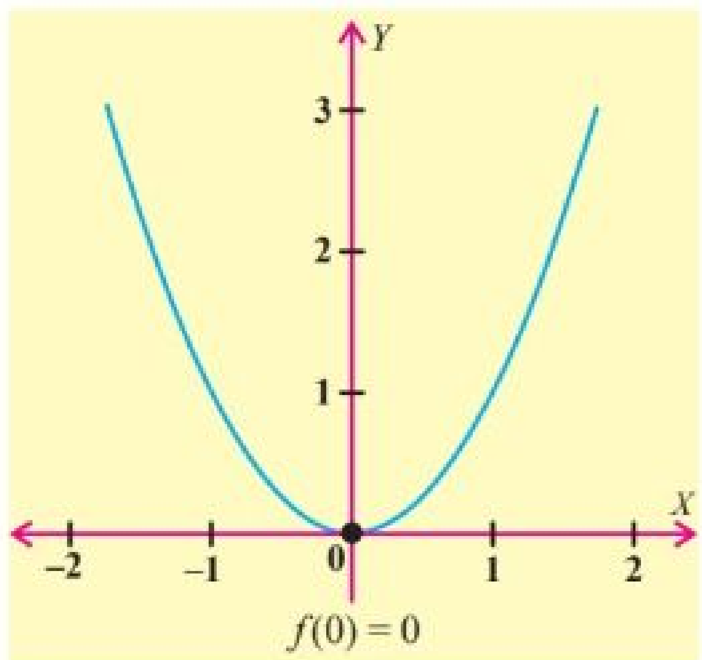

Figure 2.3

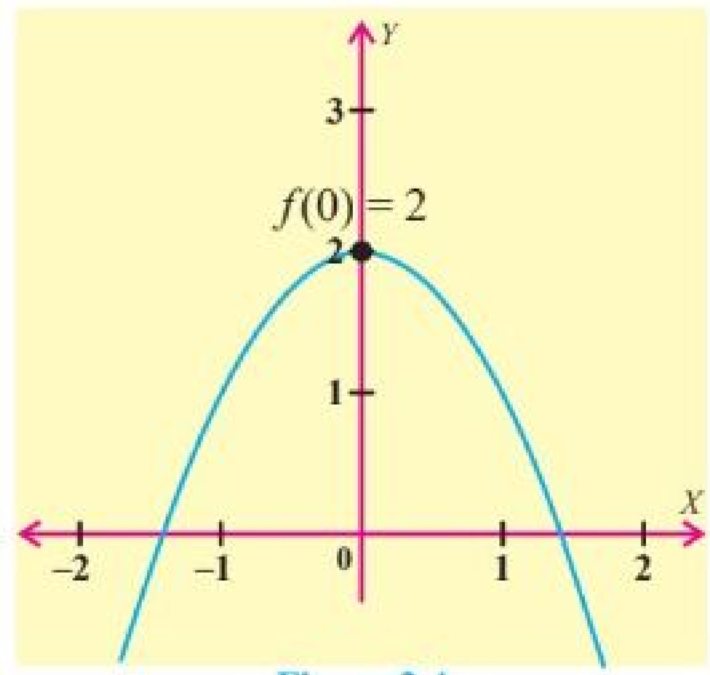

Figure 2.4

Local (Relative) Maxima	Local (Relative) Minima
Let I be an interval and let $f : I \to \mathbb{R}$ be a function. We say that f has a local (relative) maxima at $x = c \in I$, if there exists an open interval I_1 such that $c \in I_1$ and: $f(c) \geq f(x)$ for all $x \in I \cap I_1$. If $f(x)$ has a local (relative) maximum at $x = c$, then $f(c)$ is called local (relative) maximum value of the function. (See Figure 2.5)	Let I be an interval and let $f : I \to \mathbb{R}$ be a function. We say that f has a local (relative) minima at $x = e \in I$, if there exists an open interval I_1 such that $e \in I_1$ and: $f(e) \leq f(x)$ for all $x \in I \cap I_1$. If $f(x)$ has a local (relative) minimum at $x = e$, then $f(e)$ is called local (relative) minimum value of the function. (See Figure 2.5)

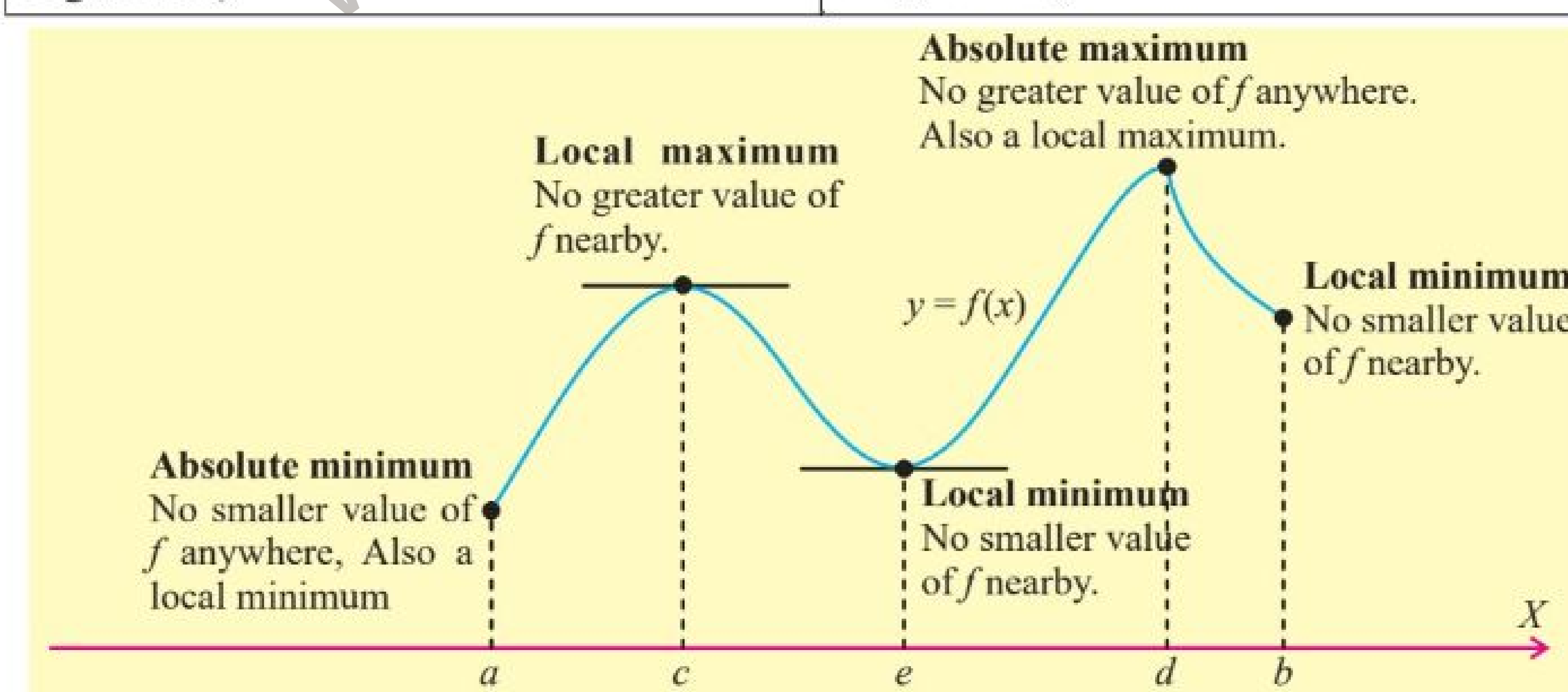

Figure 2.5

2.10.3 Method to Find Global Extrema of a Continuous Functions Defined on a Closed Interval [a, b]

1. Find all critical points of the function within the interval [a, b].
2. Find values of the function at critical points and at end points of the interval.
3. The largest of these values is the global maximum value, and the smallest is the global minimum value.

Example 20 Find the global extreme values of $f(x) = x^4 - 4x$ on [–2, 3].

Solution: We are given: $f(x) = x^4 - 4x$ … (1)

We differentiate equation (1) with respect to 'x' to get:

$$f'(x) = x^4 - 4x \quad \ldots (2)$$

From equation (2) it is clear that $f'(x)$ exists on [–2,3].

For stationary points, we solve $f'(x) = 0$:

$$4x^3 - 4 = 0$$

$$4(x^3 - 1) = 0$$

$$(x-1)(x^2 + x + 1) = 0$$

Either: $x - 1 = 0$ or $x^2 + x + 1 = 0$

The equation $x^2 + x + 1 = 0$ has complex roots, so the only real stationary point is $x = 1$.

Critical Point Value: $f(1) = 1^4 - 4(1) = -3$

End Point Values: $f(-2) = (-2)^4 - 4(-2) = 16 + 8 = 24$

$$f(3) = 3^4 - 4(3) = 81 - 12 = 69$$

From the values –3, 24, 69:

- The global maximum is $f(3) = 69$
- The global minimum is $f(1) = -3$

2.10.4 Second Derivative Test

Let f be a twice differentiable function at $x = c$, where c lies in the domain of f. Suppose that $f'(c) = 0$. Then:

Point to Note

If first derivative is not defined at a critical point, can we apply second derivative test at that critical point.

1. f has relative maximum at c if $f''(c) < 0$.
2. f has relative minimum at c, if $f''(c) > 0$.
3. If $f''(c) = 0$, then the test is inclusive, and the nature of the critical point cannot be determined using the second derivative test alone.

Example 21 Examine the function $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ for relative extreme values.

Solution: We are given: $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ …(1)

Differentiate w.r.t. 'x': $f'(x) = 2x^3 - 4x^2 - 2x + 4$ … (2)

Set $f'(x) = 0$:

$$2x^3 - 4x^2 - 2x + 4 = 0$$
$$x^3 - 2x^2 - x + 2 = 0$$
$$x^3 - x - 2x^2 + 2 = 0$$
$$x(x^2 - 1) - 2(x^2 - 1) = 0$$
$$(x - 2)(x^2 - 1) = 0$$
$$(x - 2)(x - 1)(x + 1) = 0$$

Thus, the critical points are: $x = -1, x = 1, x = 2$

We differentiate the function $f'(x) = 2x^3 - 4x^2 - 2x + 4$ w.r.t. 'x':

$$f''(x) = 6x^2 - 8x - 2$$
$$= 2(3x^2 - 4x - 1) \quad \ldots(3)$$

Substitute $x = -1$ into equation (3):

$$f''(-1) = 2(3(-1)^2 - 4(-1) - 1) = 12 > 0$$

Thus, the function has a relative minimum at $x = -1$.

x = 1, substitute into equation (3): $f''(1) = 2(3(1)^2 - 4(1) - 1) = -4 < 0$

Thus, the function has a relative maximum at $x = 1$.

Substitute $x = 2$ into equation (3): $f''(2) = 2(3(2)^2 - 4(2) - 1) = 6 > 0$

Thus, the function has a relative minimum at $x = 2$.

Substitute $x = -1, x = 1$ and $x = 2$ into equation (1):

$$f(-1) = \frac{1}{2}(-1)^4 - \frac{4}{3}(-1)^3 - (-1)^2 + 4(-1) + 1 = \frac{1}{2} + \frac{4}{3} - 4 = -\frac{13}{6}$$

$$f(1) = \frac{1}{2}(1)^4 - \frac{4}{3}(1)^3 - (1)^2 + 4(1) + 1 = -\frac{5}{6} + 4 = \frac{19}{6}$$

$$f(2) = \frac{1}{2}(2)^4 - \frac{4}{3}(2)^3 - (2)^2 + 4(2) + 1 = \frac{7}{3}$$

Finally:
- The relative minimum value at $x = -1$ is $f(-1) = -\frac{13}{6}$.
- The relative maximum value at $x = 1$ is $f(1) = \frac{19}{6}$.
- The relative minimum value at $x = 2$ is $f(2) = \frac{7}{3}$.

EXERCISE 2.4

1. Determine the intervals on which f is increasing or decreasing for the domain mentioned in each case.

   (i) $f(x) = \sin x \;;\; x \in (0, 2\pi)$ (ii) $f(x) = 4 - x^2 \;;\; x \in \mathbb{R} = (-\infty, \infty)$

   (iii) $f(x) = x^2 + 3x + 2 \;;\; x \in [-4, 1]$ (iv) $f(x) = \sin x + \frac{1}{4}\cos 2x \;;\; x \in [0, 2\pi]$

2. Using the Second Derivative Test, find local extreme values of the following functions:

(i) $f(x)=8-2x^2$ (ii) $f(x)=x^2-x-2$ (iii) $f(x)=x^4-8x^2+2$

(iv) $f(x)=(x+2)^2(x-3)$ (v) $f(x)=(x^2-2x)e^x$

3. Discuss the global extrema of the following functions over specified domains.

(i) $f(x)=\sin x+\cos x$ on $[0,2\pi]$ (ii) $f(x)=x^{4/3}$ on $[-1,8]$

4. Find the global and local extrema of the following functions over the specified domains. (Hint: Use the increasing and decreasing behavior of the functions including the end points).

(i) $f(x)=4-x^2$ on $[-1,2]$ (ii) $f(x)=\cos x-\sin x$ on $[0,2\pi]$

5. Prove that $f(x)=\dfrac{\ln(2x)}{x^2}$ has local maximum at $x=\dfrac{\sqrt{e}}{2}$.

6. Show that $f(x)=x^{x^2}$ has a relative minimum at $x=\dfrac{1}{\sqrt{e}}$.

2.11 Local Linear Approximation (Linearization)

Sometimes, complicated functions are hard to work with, so we use simpler functions to approximate them. These approximations are often accurate enough for practical purposes.

In this section, we discuss linearization, which uses tangent lines to approximate a function near a specific point.

Linearization simplifies complex or nonlinear functions with a straight-line approximation, making calculations easier. It is widely used in machine learning, physics, engineering, and economics for quick, local estimates, especially for small changes around known values.

2.11.1 Linearization

If a function $f(x)$ is differentiable at $x=a$, then the approximating function

$$L(x)=f(a)+f'(a)(x-a)$$

is the linearization of $f(x)$ at $x=a$. The approximation

$$f(x)\approx L(x)$$

of $f(x)$ by $L(x)$ is called the local linear approximation of $f(x)$ at $x=a$. The point $x=a$ is referred to as the center of the approximation.

Recall!

A function of the form $y=mx+c$, where m is slope and c is the y- intercept, is called a linear function. The graph of a linear function is a straight line.

Let $(a,f(a))$ be a point on the graph of $y=f(x)$. If $f(x)$ is differentiable, then the equation of tangent line at $(a,f(a))$ is given by:

$$y=f(a)+f'(a()x-a).$$

We can see in Fig. 2.6, the tangent line $L(x) = 2x - 1$ to the graph of the curve $f(x) = x^2$ at $(1, f(1))$ lies close to the curve near the point of tangency. For a small interval on either side of $x = 1$, y-values of the tangent line provide good approximations to the y-values of the curve $f(x) = x^2$. We observe this phenomenon by observing the table of values for the difference between the values of $L(x)$ and $f(x)$. The phenomenon is true not just for this curve; every differentiable curve behaves locally like its tangent line.

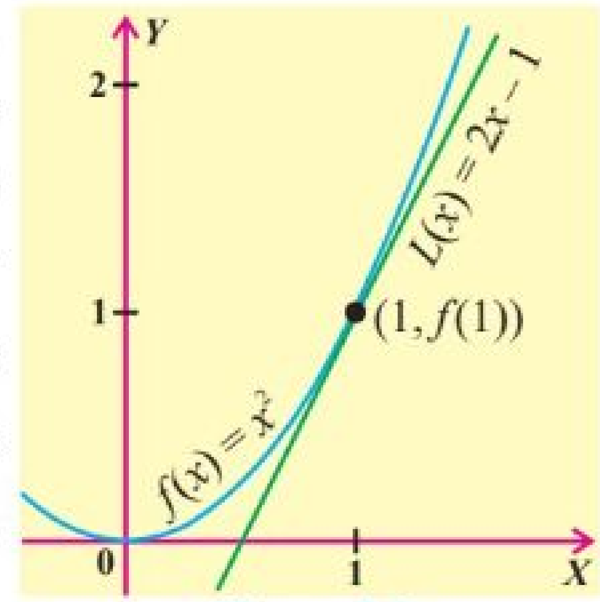

Figure 2.6

x	0.7	0.8	0.9	1	1.1	1.2	1.3
$f(x) = x^2$	0.49	0.64	0.81	1	1.21	1.44	1.69
$L(x) = 2x - 1$	0.4	0.6	0.8	1	1.2	1.4	1.6
$f(x) - L(x)$	0.09	0.04	0.01	0	0.01	0.04	0.09

Example 22 **(a)** Find the local linear approximation of $f(x) = \sqrt{2x-1}$ at $x = 5$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{9.2}$, and compare your approximation to the result produced directly by calculator.

Solution: **(a)** We are given: $f(x) = \sqrt{2x-1}$ … (1)

Differentiate $f(x)$ w.r.t. x: $f'(x) = \frac{1}{2}(2x-1)^{-\frac{1}{2}}(2)$

$$= \frac{1}{\sqrt{2x-1}} \quad \ldots (2)$$

Substitute $x = 5$ into equations (1) and (2):

$$f(5) = \sqrt{2(5)-1} = 3 \quad \text{and} \quad f'(5) = \frac{1}{\sqrt{2(5)-1}} = \frac{1}{3}$$

Now linearization is: $L(x) = f(5) + f'(5)(x-5)$

$$= 3 + \frac{1}{3}(x-5) = \frac{1}{3}(9 + x - 5) = \frac{1}{3}(x+4)$$

Hence: $L(x) = \frac{1}{3}(x+4)$

Thus, the local linear approximation at $x = 5$ is:

$$\sqrt{2x-1} \approx \frac{1}{3}(x+4) \quad \ldots (3)$$

(b) Substitute $x = 5.1$ into equation (3)

$$\sqrt{2(5.1)-1} \approx \frac{1}{3}(5.1+4)$$

$$\sqrt{9.2} \approx \frac{1}{3}(9.1)$$

$$\sqrt{9.2} \approx 3.0333 \qquad \ldots (4)$$

Calculator value $\sqrt{9.2} = 3.0331$ ... (5)

From expressions (4) and (5), we see that the calculator value is very close to the value obtained by linear approximation.

Example 23: (a) Find the local linear approximation of $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

(b) Use the local linear approximation obtained in part (a) to approximate $\cos 91°$, and compare your approximation to the result produced directly by a calculator.

Solution: (a) We are given:

$$f(x) = \cos x \qquad \ldots (1)$$

Differentiate $f(x)$ with respect to x:

$$f'(x) = -\sin x \qquad \ldots (2)$$

Substitute $x = \frac{\pi}{2}$ into the equations (1) and (2)

$$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0 \quad \text{and} \quad f'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$$

The linearization $x = \frac{\pi}{2}$ is: $L(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)$

$$= 0 + (-1)\left(x - \frac{\pi}{2}\right)$$

$$\Rightarrow \quad L(x) = -x + \frac{\pi}{2} \qquad \ldots (3)$$

Thus, the local linear approximation at $x = \frac{\pi}{2}$ is:

$$f(x) \approx L(x)$$

From equations (1) and (3): $\cos x \approx -x + \frac{\pi}{2}$ ... (4)

(b) To approximate $\cos 91°$, first convert degree into radians:

$$91° = 91 \times \frac{\pi}{180} = \frac{91\pi}{180}$$

Substitute $x = 91° = 1.588825$ into equation (4):

$$\cos 91° \approx -\frac{91\pi}{180} + \frac{\pi}{2}$$

$$\Rightarrow \cos 91° \approx -0.01745329 \qquad \ldots(5)$$

Calculator value: $\cos 91° = -0.01745240 \qquad \ldots(6)$

From expressions (4) and (5), we see that the calculator value is very close to the value obtained by linear approximation.

EXERCISE 2.5

1. Find the linear approximation of $f(x) = \sqrt{x+1}$ at $x = 3$, and use it to approximate $\sqrt{3.1}$ and $\sqrt{2.9}$.

2. (a) Find the local linear approximation of $f(x) = \sin x$ at $x = 0$.

   (b) Use the local linear approximation obtained in part (a) to approximate sin 2°, and compare your approximation to the result produced directly by a calculator.

2. Find the linear approximation of the following functions at the given points:

   (i) $f(x) = \sqrt{x^2 + 16}$ at $x = 3$ (ii) $f(x) = \tan x$ at $x = \pi$

   (ii) $f(x) = \cos x$ at $x = 0$

3. Use a local linear approximation to estimate the following:

   (i) $(82)^{\frac{1}{3}}$ (ii) $\sin 29°$

2.12 Differentials

Assume that f is differentiable at a point x, if dx is an independent variable that can take values in the neighborhood of 0 and not 0, then we define dependent variable dy, by the formula:

$$dy = f'(x)dx \qquad \ldots(1)$$

Since $dx \neq 0$, we can divide both sides of (1) by dx to obtain:

$$\frac{dy}{dx} = f'(x)$$

This achieves our goal of defining dy and dx so that their ratio is equal to the derivative $f'(x)$.

Example 24 **(a)** Find dy if $y = x^4 + 7x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.1$.

Solution: (a) Let $f(x) = x^4 + 7x$.

Differentiate $f(x)$ with respect to x:

$$f'(x) = 4x^3 + 7 \qquad \ldots(1)$$

We know: $dy = f'(x)dx$

Substitute the value of $f'(x)$ from equation (1):

$$dy = (4x^3 + 7)dx \qquad \dots (2)$$

(b) Substitute $x = 1$ and $dx = 0.1$ into equation (2):

$$dy = (4(1)^3 + 7)(0.1) = 11(0.1) = 1.1$$

Thus, the value of dy is 1.1.

2.12.1 Using dy to Approximate Δy

When a function $y = f(x)$ is differentiable at a point $P(x, y)$, we can use the differential dy to estimate the actual change Δy in y, for small change Δx in x.

The actual change Δy in y is given by:

$$\Delta y = f(x + \Delta x) - f(x) \qquad \dots (1)$$

But calculating Δy exactly often requires complicated formulas. Instead, we use the differential dy, defined as:

$$dy = f'(x)dx$$

If we take $dx = \Delta x$, then: $dy = f'(x)\Delta x \qquad \dots(2)$

Let $Q(x + \Delta x, f(x + \Delta x))$ be a neighboring point of $P(x, y)$ on $y = f(x)$.

- The slope of the tangent line at P is $f'(x)$
- The slope of the secant line connecting P and Q is $\dfrac{\Delta y}{\Delta x}$

As $\Delta x \to 0$, the point Q approaches P, and secant line PQ approaches the tangent line at P. (See Fig. 2.7)

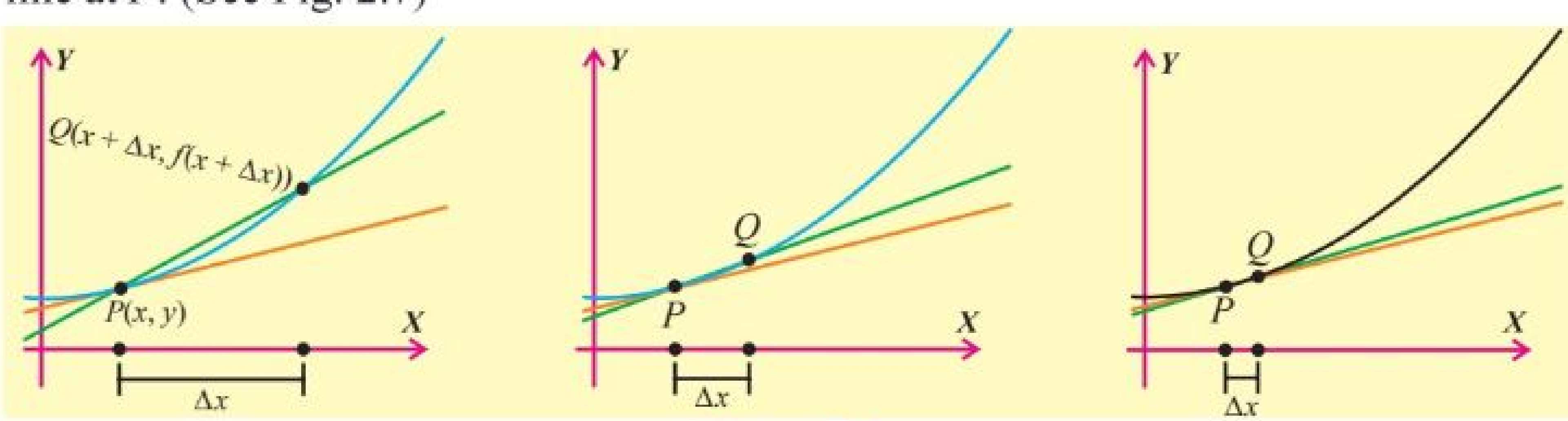

Figure 2.7

Therefore, for small Δx:

slope of tangent line at P $\approx$ Slope of secant line PQ

$$\Rightarrow \quad f'(x) \approx \frac{\Delta y}{\Delta x}$$

Substitute $f'(x) \approx \dfrac{\Delta y}{\Delta x}$ into equation (2):

$$dy \approx \frac{\Delta y}{\Delta x} \times \Delta x$$

$$dy \approx \Delta y \text{ (for } \Delta x \text{ is small)}$$

Thus, for small change Δx, the actual change Δy is approximately equal to the differential dy, that is:

$$\Delta y \approx dy = f'(x)dx$$

Using Differentials to Approximate:

Since $\Delta y = f(x+\Delta x) - f(x)$ and for small change Δx

$dx = \Delta x$ and $\Delta y \approx dy$.

$\Rightarrow \quad dy \approx f(x+\Delta x) - f(x)$

and hence: $f(x+\Delta x) \approx f(x) + dy$

2.13 Error in Differential Approximations

Suppose f is a differentiable function at a point x, and there is a small change Δx in x. If:

- Δy is actual corresponding change in y, given by

$$\Delta y = f(x+\Delta x) - f(x)$$

- dy is the approximation change in y obtained using differentials, given by

$$dy = f'(x)dx \quad \text{where } dx = \Delta x$$

Then:

Error in Δy is: Error $= \Delta y - dy$

Relative error in Δy is: Relative Error $= \dfrac{\Delta y - dy}{\Delta y} = \dfrac{\text{Error}}{\text{Original Value}}$, when $\Delta y \neq 0$

Percentage error Δy is:

$$\text{Percentage Error} = \left|\frac{\Delta y - dy}{\Delta y}\right| \times 100 = \left|\frac{\text{Error}}{\text{Original value}}\right| \times 100 \quad \text{when } \Delta y \neq 0$$

Example 25 Find dy and Δy for the function $f(x) = x^3$ when $x = 1$ and $dx = 0.1$.

Solution: We are given: $f(x) = x^3$ ... (1)

Differentiate with respect to x: $f'(x) = 3x^2$

The formula for dy is: $dy = f'(x)dx$

Substituting expression for $f'(x)$: $dy = 3x^2dx$

Substituting $x = 1$ and $dx = 0.1$: $dy = 3(1)^2(0.1) = 0.3$

We know: $\Delta y = f(x + \Delta x) - f(x)$

Substitute $x = 1$ and $\Delta x = dx = 0.1$: $\Delta y = f(1 + 0.1) - f(1) = f(1.1) - f(1)$

From equation (1), $f(x) = x^3$, so: $\Delta y = (1.1)^3 - (1)^3 = 1.331 - 1 = 0.331$

Example 26 Use differentials to approximate $\sqrt{24}$.

Solution: Let $f(x)=\sqrt{x}$ … (1)

Differentiate with respect to x: $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$ … (2)

Choose a real number that closest to 24 is a perfect square. Take $x=25$

We are approximating $\sqrt{24}$, so: $x+\Delta x=24$ and $x=25$

Thus: $24+\Delta x=25 \Rightarrow \Delta x=-1$

Since in differential approximation, $dx=\Delta x$, we take: $dx=-1$

We know: $dy=f'(x)dx$

From equation (2): $dy=\frac{1}{2\sqrt{x}}dx$

Substitute $x=25$ and $dx=-1$: $dy=\frac{1}{2\sqrt{25}}(-1)=-\frac{1}{10}=-0.1$

We know: $f(x+\Delta x)\approx f(x)+dy$

Substitute $dy=-0.1$, $x=25$ and $\Delta x=-1$:

$$f(25-1)\approx f(25)+(-0.1)$$

$$f(24)\approx f(25)-0.1$$

From equation (1): $\sqrt{24}=\sqrt{25}-0.1=5-0.1=4.9$

Example 27 Using differentials approximate the value of cos 61°3.

Solution: Let $f(x)=\cos x$ … (1)

Differentiate with respect to x: $f'(x)=-\sin x$ … (2)

Choose $x=60°$ because calculating cos 60° is easy.

We are approximating cos 61°, so: $x+\Delta x=61°$ and $x=60°$

Thus: $60°+\Delta x=61° \Rightarrow \Delta x=1°$

Since in differential approximation, $dx=\Delta x$, we take: $dx=1°=\frac{\pi}{180}$ radian

We know: $dy=f'(x)dx$

From equation (2): $dy=-\sin x\, dx$

Substitute $x=60°$ and $dx=\frac{\pi}{180}$: $dy=-\sin 60°\left(\frac{\pi}{180}\right)=-\frac{\sqrt{3}}{2}\times\frac{\pi}{180}=-0.0151$

We know: $f(x+\Delta x)\approx f(x)+dy$

Substitute $dy=-0.0151$, $x=60°$ and $x+\Delta x=61°$: $f(61°)\approx f(60°)+(-0.0151)$

From equation (1): $\cos 61°=\cos 60°-0.0151=0.5-0.0151=0.4849$

Example 28 The side of a cube is measured to be 25 cm, with a possible error of ± 1 cm.

(a) Use differentials to estimate the error in the calculated volume.

(b) Estimate the percentage errors in the side and volume.

Solution: Let x be side of the cube: $x = 25$ cm, $dx = \Delta x = \pm 1$ cm

Let V be the volume of cube: $V = x^3$

(a) The differential of V is: $dV = \frac{d}{dx}\left(x^3\right)dx$

$$= 3x^2 dx \qquad \dots (1)$$

Substitute $x = 25$ and $dx = \pm 1$ into equation (1):

$$dV = 3(25)^2(\pm 1) = \pm 1875\,\text{cm}^3$$

Thus, the estimated error in measuring volume is $\pm 1875\ cm^3$

(b) Percentage error in measuring the side is:

$$\left|\frac{dx}{x}\right| \times 100 = \frac{1}{25} \times 100 = 4\ \% \qquad \dots (2)$$

Since: $V = x^3$ and $dV = 3x^2 dx$

The percentage error in measuring the volume is:

$$\left|\frac{dV}{V}\right| \times 100 = \left|\frac{3x^2 dx}{x^3}\right| \times 100 = 3 \times \left|\frac{dx}{x}\right| \times 100$$

Using equation (2): $\left|\frac{dV}{V}\right| \times 100 = 3 \times 4\% = 12\%$

Example 29 The radius r of a circle is measured with an error of at most 2%. What is the maximum corresponding percentage error in computing the circle's

(a) circumference? **(b)** area?

Solution: Since r is the radius. The percentage error in measuring the radius is given as:

$$\left|\frac{dr}{r}\right| \times 100 = 2\ \% \qquad \dots (1)$$

(a) Let C be the circumference of the circle:

$$C = 2\,\pi r$$

The differential of C is: $dC = \frac{d}{dr}(2\pi r)dr = 2\pi dr \qquad \dots (2)$

Percentage error in measuring the circumference:

$$\left|\frac{dC}{C}\right| \times 100 = \left|\frac{2\pi dr}{2\pi r}\right| \times 100 = \left|\frac{dr}{r}\right| \times 100$$

Using equation (1): $\left|\frac{dC}{C}\right| \times 100 = 2\ \%$

(b) Let A be the area of the circle: $A = \pi r^2$

The differential of A is: $dA = \frac{d}{dr}\left(\pi r^2\right)dr = 2\pi r dr$

Percentage error in measuring the area:

$$\left|\frac{dA}{A}\right| \times 100 = \left|\frac{2\pi r dr}{\pi r^2}\right| = 2 \times \left|\frac{dr}{r}\right| \times 100$$

Using equation (1): $\left|\frac{dA}{A}\right| \times 100 = 2 \times 2\ \% = 4\ \%$

EXERCISE 2.6

1. Find the differential dy in the following cases:
   (i) $y = x^3 - 3\sqrt{x}$ (ii) $y = \cos x^2$
2. Use dy to approximate Δy for the following functions:
   (i) $y = x^3 - x$ when x changes from 3 to 3.02.
   (ii) $y = \sin x + \cos x$ when x changes from 30° to 29°.
   (iii) $y = \frac{1}{x}$ when $x = 0.5$ and $dx = -0.1$
3. Use differentials to approximate the value of:
   (i) $\sqrt{26}$ (ii) $(33)^{\frac{1}{5}}$ (iii) $\tan 46°$
4. The side of a cube is measured to be 30 cm with a maximum error of 0.14 cm in its measurement. Find the maximum error in the calculated volume of the cube.
5. The radius of a sphere is measured as 100 ± 0.5 cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

2.14 Applying Extrema to Real-World Problems

To solve extrema problems in real-life scenarios, follow these steps:

1. **Understand the Problem –** Identify what needs to be optimized.
2. **Define Variables –** Assign variables to unknown quantities.
3. **Formulate the Function –** Write an equation representing the quantity to optimize.
4. **Find Constraints –** If applicable, express limitations.
5. **Find Critical Points –** Take the derivative and find critical points.
6. **Verify Extrema –** Use the second derivative test to confirm maxima/minima.
7. **Interpret the Solution –** Relate the mathematical answer back to the real-world context.

Example 30 A rectangular garden is to be fenced on three sides with 100 meters of fencing. The fourth side is along a wall and requires no fence. Find the dimensions that maximize the area and the maximum area.

Solution: Let the side opposite to the wall be x meters.

Let each side perpendicular to the wall be y meters.

Total length of the garden to be fenced:

$$x+2y=100$$

$$y=\frac{100-x}{2} \qquad \dots (1)$$

Area of the garden $= A = xy$

Substituting the expression of y from (1):

$$A=x\left(\frac{100-x}{2}\right)$$

$$A=\frac{1}{2}(100x-x^2) \qquad \dots (2)$$

Differentiating w. r. t. 'x':

$$\frac{dA}{dx}=\frac{1}{2}(100-2x)=50-x \qquad \dots (3)$$

Set $\frac{dA}{dx}=0$: $\qquad 50-x=0$

$$x=50$$

Thus, $x = 50$ is the critical point.

Differentiating equation (1) w. r. t. 'x':

Since $\frac{d^2A}{dx^2}=-1<0$, the area A is maximum at $x = 50$ meters

Substitute $x = 50$ into equations (1) and (2):

$$y=\frac{100-50}{2}=25 \text{ meters and } A=\frac{1}{2}\left(100\times 50-50^2\right)=1250 \text{ m}^2$$

The maximum fenced area of the garden is 1250 m^2, which occurs when:

- The side opposite the wall is 50 meters
- Each of the sides perpendicular to the wall is 25 meters

Example 31 Find the dimensions of a rectangle of maximum area that can be inscribed under the curve $y=4-x^2$ and above the x-axis.

Solution: Let one vertex of the rectangle on the parabola in first quadrant be $A(x, y)$.

The other vertex in second quadrant is $B(-x, y)$.

Height of the rectangle = y

Width of the rectangle = $2x$

Area of the rectangle: $A = 2xy$ … (1)

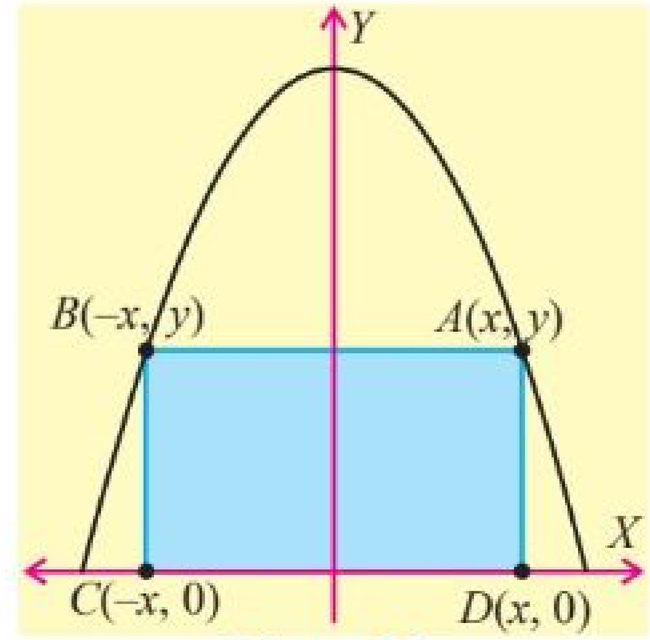

Figure 2.8

The vertex $A(x, y)$ of the rectangle satisfies $y = 4 - x^2$.

Substitute this expression of y into equation (1):

$$A = 2x(4 - x^2) = 2(4x - x^3)$$

Differentiate with respect to x: $\dfrac{dA}{dx} = 2(4 - 3x^2)$ … (2)

Set $\dfrac{dA}{dx} = 0$:

$$2(4 - 3x^2) = 0 \Rightarrow 3x^2 = 4 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \frac{2}{\sqrt{3}}$$

We consider only positive root since x in first quadrant is positive.

Differentiating $\dfrac{dA}{dx} = 2(4 - 3x^2)$ w.r.t. x:

$$\frac{d^2A}{dx^2} = 2(-6x) = -12x$$

At $x = \dfrac{2}{\sqrt{3}}$:

$$\frac{d^2A}{dx^2} = -12\left(\frac{2}{\sqrt{3}}\right) = -\frac{24}{\sqrt{3}} < 0$$

Negative second derivative implies that area is maximum at $x = \dfrac{2}{\sqrt{3}}$.

Substitute $x = \dfrac{2}{\sqrt{3}}$ into $y = 4 - x^2$:

$$y = 4 - \left(\frac{2}{\sqrt{3}}\right)^2 = 4 - \frac{4}{3} = \frac{8}{3}$$

The dimensions of the rectangle of maximum area are:

- Width $= 2x = \dfrac{4}{\sqrt{3}}$ units
- Height $= \dfrac{8}{3}$ units

Example 32 A cylindrical can must hold 500 cm³ of liquid. Find the radius and height that minimizes the material cost.

Solution: The cost of the material will be minimized if the surface area of the cylinder is minimized. Our objective is to minimize the surface area of the cylinder.

Let:
- Radius of the base = r cm
- Height of the cylinder = h cm

Volume of the cylinder: $V = \pi r^2 h = 500\,\text{cm}^3$

$$h = \frac{500}{\pi r^2} \qquad \dots (1)$$

Surface area of cylinder:

- Surface area of two circular ends $= 2\pi r^2$
- Curved surface area $= 2\pi rh$

Total surface area of the cylinder:

$$S = 2\pi r^2 + 2\pi rh \qquad \dots (2)$$

Substitute the expression of h from equation (1):

$$S = 2\pi r^2 + 2\pi r \times \frac{500}{\pi r^2} = 2\pi r^2 + \frac{1000}{r}$$

Differentiate 'S' w.r.t 'r':

$$\frac{dS}{dr} = 4\pi r - \frac{1000}{r^2} \qquad \dots (3)$$

Set $\frac{dS}{dr} = 0$:

$$4\pi r - \frac{1000}{r^2} = 0 \quad \Rightarrow \quad 4\pi r = \frac{1000}{r^2} \Rightarrow \quad r^3 = \frac{1000}{4\pi}$$

$$r = \left(\frac{250}{\pi}\right)^{1/3} \qquad \dots (4)$$

Differentiate equation (3) w.r.t. r:

$$\frac{d^2S}{dr^2} = 4\pi + \frac{2000}{r^3}$$

Substitute $r = \left(\frac{250}{\pi}\right)^{1/3}$ into $\frac{d^2S}{dr^2}$:

$$\frac{d^2S}{dr^2} = 4\pi + \frac{2000}{\left(\left(\frac{250}{\pi}\right)^{1/3}\right)^3} > 0$$

Since $\frac{d^2S}{dr^2} > 0$, S is minimum at $r = \left(\frac{250}{\pi}\right)^{\frac{1}{3}}$.

Substitute $r=\left(\frac{250}{\pi}\right)^{\frac{1}{3}}$ into $h=\frac{500}{\pi r^2}$:

$$h=\frac{500}{\pi\left(\left(\frac{250}{\pi}\right)^{\frac{1}{3}}\right)^2}=\frac{500}{(\pi)^{\frac{1}{3}}(250)^{\frac{2}{3}}}$$

The material cost is minimized when:

- Radius $=r=\left(\frac{250}{\pi}\right)^{\frac{1}{3}}$ cm
- Height $=h=\frac{500}{(\pi)^{\frac{1}{3}}(250)^{\frac{2}{3}}}$ cm

Example 33 A car moves along the parabolic road $y=4-x^2$. A mobile tower is at the origin. Find the position of the car where the signal strength is strongest.

Solution: Signal strength is strongest when the distance between the car and the tower is minimum.

Let the position of the car be $P(x, y)$ on the parabola $y=4-x^2$

The distance D of the car from the tower at origin (0, 0) is:

$$D=\sqrt{x^2+y^2} \quad \ldots (1)$$

Since the car lies on the parabola $y=4-x^2$

$$\Rightarrow \quad x^2=4-y \quad \ldots (2)$$

Substitute $x^2=4-y$ into equation (1):

$$D=\sqrt{4-y+y^2}$$

Since minimizing D is equivalent to minimizing D^2:

$$D^2=4-y+y^2$$

Let $D^2=f(y)$:

$$f(y)=y^2-y+4$$

Differentiate with respect to y:

$$\frac{df}{dy}=2y-1 \quad \ldots (3)$$

Set $\frac{df}{dy}=0$:

$$2y-1=0$$

$$y=\frac{1}{2}$$

Differentiate equation (3) with respect to x:

$$\frac{d^2 f}{dy^2} = 2 > 0$$

Since second derivative is positive, f is minimum at $y = \frac{1}{2}$.

Substitute $y = \frac{1}{2}$ into $x^2 = 4 - y$:

$$x^2 = 4 - \frac{1}{2} = \frac{7}{2}$$

$$x = \pm\sqrt{\frac{7}{2}}$$

The positions of the car where signal strength are $\left(\sqrt{\frac{7}{2}}, \frac{1}{2}\right)$ and $\left(-\sqrt{\frac{7}{2}}, \frac{1}{2}\right)$.

Example 34 A software company is designing a data processing algorithm for analyzing batches of customer transactions. The runtime $T(n)$ in milliseconds depends on the size of the batch n (the number of transactions processed together) and is modeled by the equation:

$$T(n) = n^3 - 12n^2 + 36n$$

- The term n^3 reflects the time complexity of a sorting step in the algorithm.
- The term $-12n^2$ accounts for optimizations that reduce processing time for moderate batch sizes.
- The term $36\,n$ represents overhead that scales linearly with batch size.

The company wants to find the optimal batch size n that minimizes the algorithm's total runtime.

Solution: The runtime of the algorithm is:

$$T(n) = n^3 - 12n^2 + 36n \qquad \dots (1)$$

We need to find the input size n (the number of transactions processed together) that minimizes the runtime $T(n)$.

Differentiate equation (1) with respect to n:

$$\frac{dT}{dn} = 3n^2 - 24n + 36 \qquad \dots (2)$$

To find critical points set $\frac{dT}{dn} = 0$:

$$3n^2 - 24n + 36 = 0$$

$$n^2 - 8n + 12 = 0$$

$$(n-2)(n-6) = 0$$

Thus, the critical points are:

$$n = 2 \text{ and } n = 6$$

Differentiate equation (2) with respect to n:

$$\frac{d^2T}{dn^2} = 6n - 24 \qquad \dots (3)$$

Substitute $n = 2$ and $n = 6$ into equation (3):

$$\frac{d^2T}{dn^2} = 6 \times 2 - 24 = -12 < 0 \quad \text{and} \quad \frac{d^2T}{dn^2} = 6 \times 6 - 24 = 12 > 0$$

Since $\frac{d^2T}{dn^2} < 0$ at $n = 2$, the function $T(n)$ has a local maximum at $n = 2$.

Since $\frac{d^2T}{dn^2} > 0$ at $n = 6$, the function $T(n)$ has a local minimum at $n = 6$.

The runtime is minimized when the input size is $n = 6$.

EXERCISE 2.7

1. A window consists of a rectangle surmounted by a semicircular opening. If the total perimeter is 21 meter, find the dimensions that maximize the area.
2. A rectangular sheet of metal with dimensions 40 cm by 25 cm is to be made into an open box by cutting equal squares from the corners and folding up the sides. Find the size of the squares that maximize the volume.
3. A closed cylindrical tank is to be constructed to hold 54π liters of water. Find the dimensions that minimize the construction cost of the tank.
4. A wire of 20 meters is cut into two pieces. One piece is bent into a square and the other into a circle. How should the wire be cut to maximize the total area enclosed?
5. A team of environmental scientists is monitoring the oxygen level in a lake over time to ensure the water quality remains suitable for aquatic life. The oxygen level in the lake, measured in milligrams per liter (mg/L), is modeled by the equation:

   $$O(t) = t^3 - 18t^2 + 81t + 100$$

   Where $O(t)$ is the oxygen level at time t (in days after monitoring began).
   Scientists want to determine when during the monitoring period the oxygen level is at its lowest, as low oxygen levels can be harmful to fish and other aquatic organisms.
   Find the time t when the oxygen level is at its minimum.
6. The perimeter of a triangle is 16 cm. If one side is of length 6 cm, what are the lengths of the other sides for maximum area of the triangle.
7. A poster must have 200 cm^2 printed area with margins of 2 cm on top/bottom and 3 cm on sides. Find poster dimensions to minimize paper used.
8. Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.