Unit 11 Vector Valued Functions and Their Derivatives

INTRODUCTION

Vector-valued functions return vectors with both magnitude and direction to model multidimensional quantities such as position, velocity, or force in 2D or 3D, unlike scalar functions which give a single value. Their derivative provides the instantaneous rate of change, yielding the velocity vector essential for analyzing motion, with applications in engineering, computer graphics, and physics, from robotic arms to planetary orbits. This chapter covers definitions, component forms, operations, and differentiation techniques for analyzing curves and solving real-world problems involving velocity and acceleration.

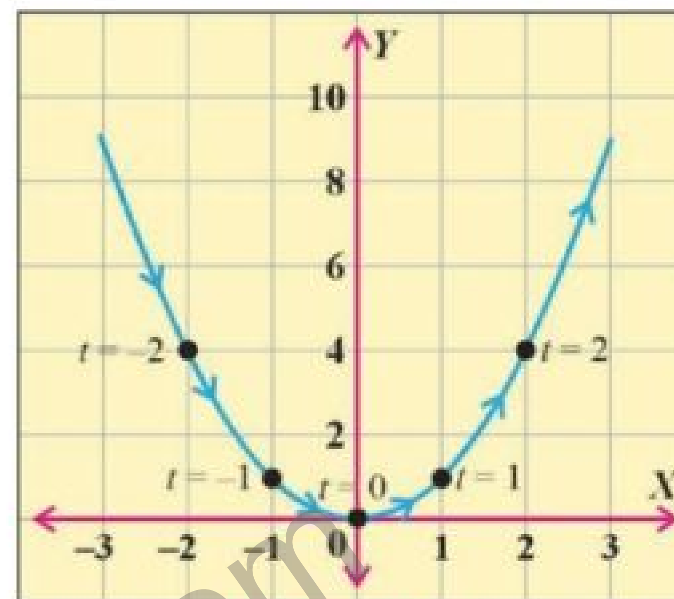


Figure 11.1

For example, a particle moving along a path in 2D might have its position at time t given by:

$\mathbf{r}(t) = t\hat{i} + t^2\hat{j}$. This means at time t, the particle is at the point (t, t^2) (Figure 11.1).

A real-world example of a vector-valued function in three-dimensional space is the position of a satellite orbiting Earth. Suppose the vector-valued function describes a satellite's position at time t (in hours): $\mathbf{r}(t) = (4000\cos t)\hat{i} + (4000\sin t)\hat{j} + 3000t\hat{k}$

Here, the components represent the satellite's coordinates in space: $x = 4000\cos t$, $y = 4000\sin t$, and $z = 3000t$ (in kilometers). The x- and y-components model the satellite's circular orbit in the xy-plane with a radius of 4000 km, while the z-component indicates a linear ascent along the z-axis at a rate of 3000 km per hour. This function maps time to a position vector, describing the satellite's trajectory through space, which is essential for tracking and communication in aerospace engineering (Figure 11.2).

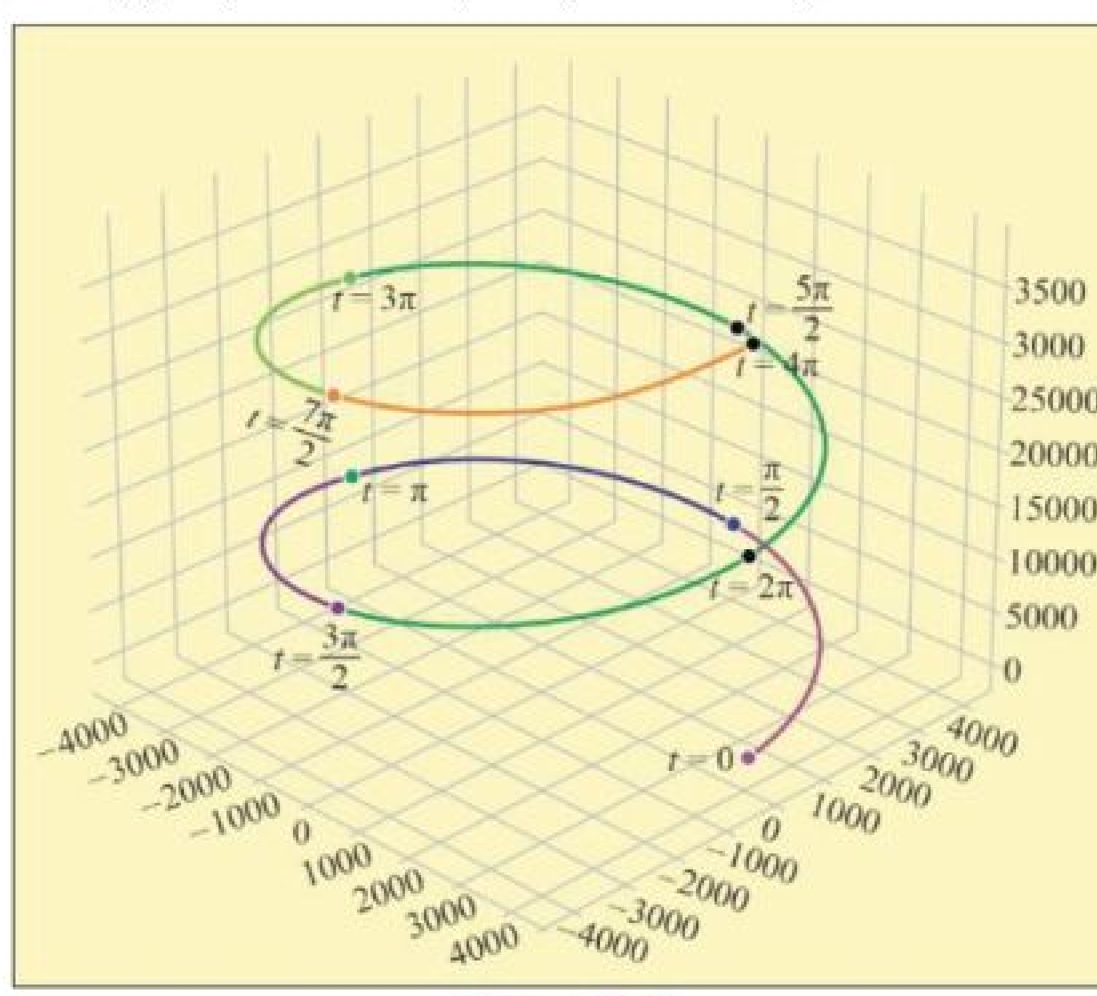


Figure 11.2

11.1 Scalar And Vector-Valued Functions Of A Single Variable

11.1.1 Scalar Function

A scalar function is a function that assigns a single real number (a scalar) to each point in its domain. Think of it as a rule that assigns one value to a position in space, like measuring temperature at a specific point.

Mathematically, if to each value of a scalar variable t in the domain of a function $f(t)$, there corresponds a unique scalar f, then f is called a scalar function of t. A real-valued function is commonly called a scalar function. For example, $f(t)=3t+2\cos t$ is a scalar function.

11.1.2 Vector-Valued Function (Vector Function)

A vector-valued function in a single variable is a function that takes one variable as input and returns a vector as output. Each component of the output vector is itself a scalar function of the input. Typically, a vector-valued function is written as:

$$\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle=f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$$

Where $\hat{i},\hat{j},\hat{k}$ are the unit vectors associated with the rectangular coordinates $f(t),g(t)$, and $h(t)$ are scalar functions of t, are the components of the vector-valued function $\mathbf{r}(t)$.

11.1.3 Constructing A Vector-Valued Function

Step-1: Determine the Components: Identify how each coordinate changes with respect to t.

Step-2: Combine into Vector Form:

Express the components as a single vector using either component notation $\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle$ or unit vectors $\mathbf{r}(t)=f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$.

Step-3: Specify the Domain: Define the range of t where the function is valid (e.g., $t\geq 0$) for time-dependent motion.

Example 1: Projectile Motion

When a ball is thrown at an angle, its position over time follows a parabolic path under the influence of gravity. If the initial velocity is $v_\circ$ at an angle θ, the position vector is:

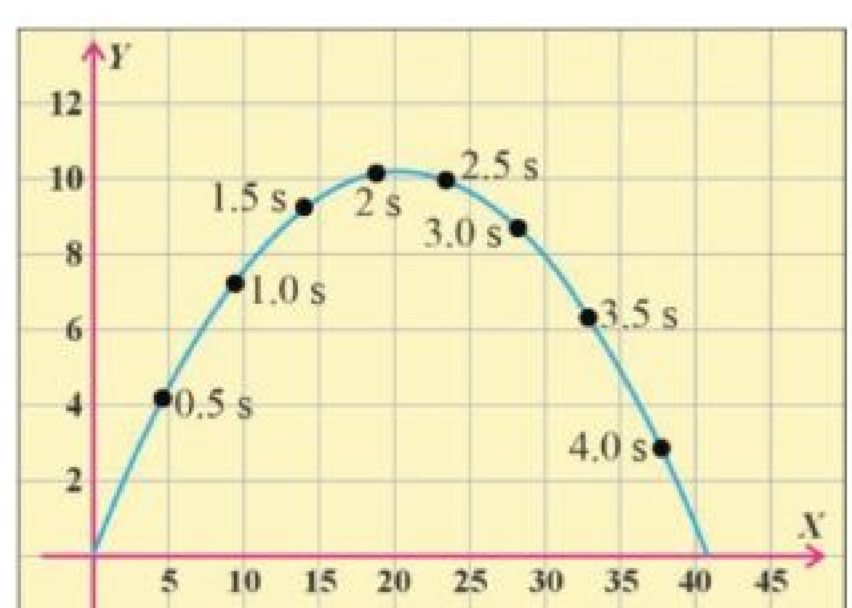


Figure 11.3

$$\mathbf{r}(t)=(v_{\circ}\cos\theta)t\hat{i}+\left((v_{\circ}\sin\theta)t-\frac{1}{2}gt^{2}\right)\hat{j}$$

Where, $(v_{\circ}\cos\theta)t$ = x-component (horizontal motion at constant speed) and

$(v_{\circ}\sin\theta)t-\frac{1}{2}gt^{2}=y$-component (vertical motion with gravity deceleration).

This model simulates the trajectories of objects like soccer balls, arrows, or rockets in 2D.

Example 2: Flight Path of a Drone

A drone taking off and moving in a helical search pattern can be modeled as:

$$\mathbf{r}(t)=5\cos t\hat{i}+5\sin t\hat{j}+0.5t\hat{k}$$

Where x- and y-components represent the circular motion with a 5-meter radius, and the z–component represents the steady ascent at 0.5 meters per second. This describes real-world applications like surveillance, search missions, or aerial photography.

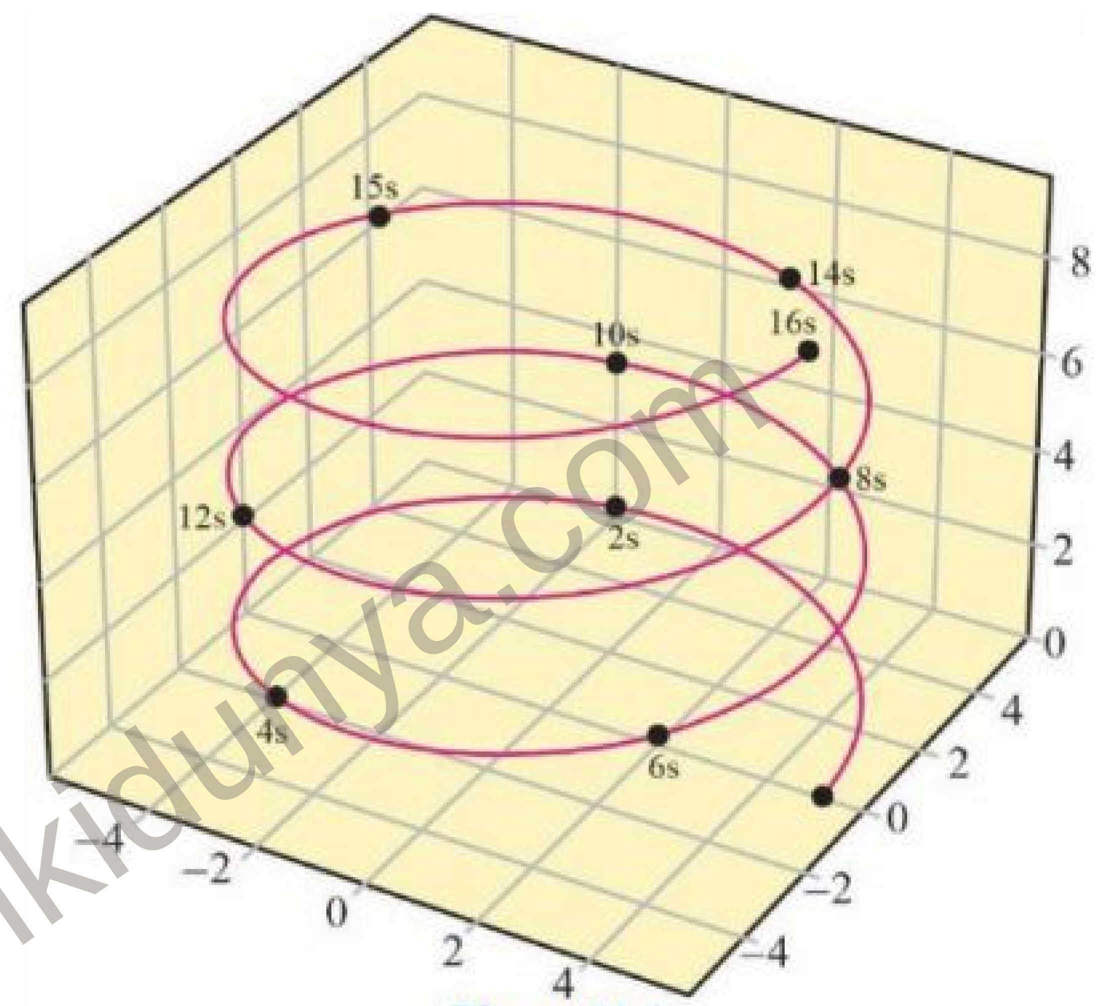


Figure 11.4

11.2 Domain and Range of a Vector-Valued Function

The domain of a vector–valued function $\mathbf{r}(t)=f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$ is the set of all real numbers t for which every component function $f(t), g(t)$, and $h(t)$ is defined.

11.2.1 Steps to Find the Domain of a Vector-Valued Function

Step-1: Determine the domain of each component separately.

Step-2: Find the intersection of the domains of each component $f(t), g(t)$, and $h(t)$. The domain of $\mathbf{r}(t)$ is the intersection of the domains of $f(t), g(t)$, and $h(t)$

The range of a vector-valued function is the set of all possible output vectors $\mathbf{r}(t)$ as t varies over the domain.

Example 1 Find the domain and range of the vector-valued function

$$\mathbf{r}(t)=\sqrt{4-t^{2}}\,\hat{i}+\frac{1}{t-1}\hat{j}\ .$$

Solution: Step 1: The domain of $\mathbf{r}(t)=\sqrt{4-t^2}\,\hat{i}+\frac{1}{t-1}\hat{j}$ is the set of all t for which both components are defined.

For the first component $\sqrt{4-t^2}$, the square root requires $4-t^2\geq 0$.

$\Rightarrow t^2\leq 4 \Rightarrow -2\leq t\leq 2 \quad \Rightarrow t\in[-2,2]$.

For the second component $\frac{1}{t-1}$, the denominator cannot be zero.

i.e. $t-1\neq 0\Rightarrow t\neq 1$

Hence, the final domain: $t\in[-2,1)\cup(1,2]$.

The range is the set of all possible output vectors $\mathbf{r}(t)=\langle x(t),y(t)\rangle$.

For $4-t^2$, range is $[0,4]$. Thus, the range of $x(t)$ is $[0,2]$, and $y(t)$ can be any real number except 0 (since $\frac{1}{t-1}$ never equals zero).

Finally, range: $\{\langle x,y\rangle | 0\leq x\leq 2, y\in\mathbb{R}, y\neq 0, x\neq\sqrt{3}\}$

11.3 Identifying Scalar and Vector Valued Functions

Understanding the distinction between scalar and vector-valued functions is crucial for advanced studies in calculus, physics, and engineering. These functions are fundamental in describing quantities that vary with respect to one or more variables, such as time or position.

11.3.1 Key Identification Criteria

Scalar Function

(i) Scalar-valued functions produce a single numerical value (e.g., a real number).

(ii) Scalar-valued functions are typically denoted by lowercase letters (e.g., $f(x)$).

(iii) Scalar-valued functions describe quantities like temperature or potential energy.

Example 2 In a heat transfer problem, the temperature $T(x,y,z)=100-x^2-y^2-z^2$ describes the temperature at a point (x,y,z) in a three-dimensional space.

Identification: The function takes a position (x,y,z) and outputs a single number T (temperature in degrees), making it scalar-valued.

Engineers use such functions to model heat distribution in materials, such as in designing cooling systems for electronic devices.

Vector-valued function:

(i) Vector-valued functions produce a vector, often written as an ordered tuple or in terms of component functions (e.g., $\mathbf{f}(t)=\langle f_1(t), f_2(t)\rangle$ (2D) or $\mathbf{g}(t)=\langle g_1(t), g_2(t), g_3(t)\rangle$ (3D)).

(ii) Vector-valued functions are often denoted by boldface letters or with an arrow (e.g., $\mathbf{f}(t)$ or $\vec{f}(t)$).

(iii) Vector-valued functions describe quantities such as position, velocity, and force.

Example 3 Consider the function $\mathbf{r}(t)=\langle \cos(t), \sin(t), t\rangle$. Here, the input is a single scalar variable t (often representing time), and the output is a 3-dimensional vector. For instance, if $t=\frac{\pi}{2}$ then $\mathbf{r}\left(\frac{\pi}{2}\right)=\left\langle \cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right), \frac{\pi}{2}\right\rangle=\left\langle 0, 1, \frac{\pi}{2}\right\rangle$. The output is a vector, indicating a point in 3D space. This function traces out a helix.

EXERCISE 11.1

1. Give two real-life examples where vector-valued functions are necessary.
2. Construct a vector function for a particle moving in a circle.
3. Write a vector function for a car moving along a parabolic path.
4. Create a 3D vector function for a drone flying upward in a spiral.
5. Given $x(t) = t^2 + 1$, $y(t) = \sqrt{t}$, construct $\mathbf{r}(t)$.
6. Given $x(t)=\frac{1}{t+4}, y(t)=\ln\sqrt{t}, z(t)=\sec^{-1}(t)$, construct $\mathbf{r}(t)$.
7. Find the domain and range of the following vector-valued functions.
   (i) $\mathbf{r}(t)=\sqrt{t}\hat{i}+\ln t\hat{j}$ (ii) $\mathbf{r}(t)=\cos(t)\hat{i}+\sin(t)\hat{j}$
   (iii) $\mathbf{r}(t)=t\hat{i}+t^2\hat{j}+e^t\hat{k}$ (iv) $\mathbf{r}(t)=\sin(t)\hat{i}+\cos(t)\hat{j}+t\hat{k}$
8. Determine which of the following functions are scalar functions or vector-valued functions. Give appropriate reasoning to support the answer.
   (i) $f(t)=(3t^2-5)^2$ (ii) $\mathbf{h}(t)=\langle t^2, e^t, \ln\sqrt{t}\rangle$
   (iii) $g(t)=\tan(t)+\csc(t)$ (iv) $\mathbf{j}(t)=\left\langle \frac{1}{t^2+1}, \frac{1}{t^2+1}\right\rangle$

11.4 Derivative Of A Vector-Valued Function

A vector-valued function of a single variable is a function that maps a scalar (usually time t) to a vector in $\mathbb{R}^n$. For a function in three-dimensional space, it is typically written as:

$$\mathbf{r}(t)=\langle f_1(t),f_2(t),f_3(t)\rangle=f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}$$

Where $\hat{i},\hat{j},\hat{k}$ are the unit vectors associated with the rectangular coordinates and $f_1(t),f_2(t)$, and $f_3(t)$ are scalar functions of t, are the components of the vector-valued function $\mathbf{r}(t)$.

The derivative of a vector-valued function $\mathbf{r}(t)$ with respect to t is defined as:

$$\frac{d\mathbf{r}}{dt}=\mathbf{r}'(t)=\lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}$$

If the limit exists, then $\mathbf{r}'(t)$ is computed by differentiating each component:

$$\mathbf{r}'(t)=\langle f_1'(t),f_2'(t),f_3'(t)\rangle$$
$$=f_1'(t)\hat{i}+f_2'(t)\hat{j}+f_3'(t)\hat{k}$$

Example 4 Find the first derivative of $\mathbf{r}(t)=t^2\hat{i}+\sin t\hat{j}+e^t\hat{k}$ with respect to t.

Solution: Differentiating each component separately, we have,

$$\frac{d}{dt}(t^2)=2t,\frac{d}{dt}(\sin t)=\cos t,\frac{d}{dt}(e^t)=e^t$$

$$\Rightarrow \mathbf{r}'(t)=2t\hat{i}+\cos t\hat{j}+e^t\hat{k}$$

Example 5 Find the first derivative of $\mathbf{r}(t)=\sec t\hat{i}+\tan t\hat{j}-\cos t\hat{k}$ at $t=\frac{\pi}{4}$.

Solution: Differentiating each component separately and finding the value at $t=\frac{\pi}{4}$, we have,

$$\frac{d}{dt}(\sec t)=\sec t.\tan t\Rightarrow \left.\frac{d(\sec t)}{dt}\right|_{t=\frac{\pi}{4}}=\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)=\sqrt{2}.1=\sqrt{2}$$

$$\frac{d}{dt}(\tan t)=\sec^2 t\Rightarrow \left.\frac{d(\tan t)}{dt}\right|_{t=\frac{\pi}{4}}=\sec^2\left(\frac{\pi}{4}\right)=\left(\sqrt{2}\right)^2=2$$

$$\frac{d}{dt}(-\cos t)=\sin t \Rightarrow \left.\frac{d(-\cos t)}{dt}\right|_{t=\frac{\pi}{4}}=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \mathbf{r}'\left(\frac{\pi}{4}\right)=\sqrt{2}\hat{i}+2\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$$

EXERCISE 11.2

1. Find the first-order derivatives of the following vector-valued functions with respect to t.
   (i) $\mathbf{r}(t)=(t^2+1)\hat{i}+(t^3+t^2+3t)\hat{j}$
   (ii) $\mathbf{r}(t)=(t\sin t-\cos t)\hat{i}+(\cot(t)+\ln t)\hat{j}+(\ln(1+t^2))\hat{k}$
   (iii) $\mathbf{r}(t)=\frac{2}{\sqrt{1-t^2}}\hat{i}+\frac{\sqrt{3}}{1+t^2}\hat{k}$
2. Find the first-order derivatives of the following vector-valued functions with respect to t at the specified values of t.
   (i) $\mathbf{r}(t)=\left(\frac{t^2+1}{\sin t}\right)\hat{i}+\left(\sqrt{t}e^t\right)\hat{j}$ at $t=\frac{\pi}{2}$
   (ii) $\mathbf{r}(t)=\left(t\sqrt{t+2}\right)\hat{i}+\ln\left(t+\sqrt{t^2+1}\right)\hat{j}$ at $t=0$
   (iii) $\mathbf{r}(t)=\arctan(t)\hat{i}+\frac{\sin(t^2)}{t}\hat{j}+\sqrt[3]{1+t}\hat{k}$ at $t=0$
   (iv) $\mathbf{r}(t)=te^{-t}\hat{i}+\cos(3t)\hat{j}+\ln(1+t^2)\hat{k}$ at $t=0$

11.5 Velocity and Acceleration

If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then $\frac{d}{dt}\mathbf{r}(t)$ represent the velocity vector of the particle, which is tangent to the curve, i.e. $\frac{d}{dt}\mathbf{r}(t)=\mathbf{v}(t)$, and the derivative of the velocity vector with respect to time $\boldsymbol{t}$ is the acceleration of the particle, denoted by $\mathbf{a}$ i.e. $\frac{d}{dt}\mathbf{v}(t)=\mathbf{a}(t)=\frac{d^2}{dt^2}\mathbf{r}(t)$.

Example 6 Vibrating conveyor belt in a factory sorting system

A small metal part moves on a conveyor belt in a factory. The belt transports the part eastward (x-direction) while simultaneously vibrating sideways (north-south, y-direction) to help sort items into bins. The position of the part at time t seconds is given by $\mathbf{r}(t)=3t^2\hat{i}+2\sin t\hat{j}$ in meters. At a specific moment, $t=\frac{\pi}{2}$ an engineer needs to analyze the part's motion to ensure it enters the correct bin. Hence, find the following quantities at $t=\frac{\pi}{2}$.

(a) velocity (b) acceleration (c) speed (d) direction of motion.

Solution:

(a) Velocity: The velocity vector is the first derivative of the position function.

$\mathbf{v}(t)=\mathbf{r}'(t)=6t\hat{i}+2\cos t\hat{j}$, at $t=\frac{\pi}{2}$ is, $\mathbf{v}\left(\frac{\pi}{2}\right)=6\left(\frac{\pi}{2}\right)\hat{i}+2\cos\left(\frac{\pi}{2}\right)\hat{j}=3\pi\hat{i}+0\hat{j}$

At this instant, the part is moving purely eastward with no north–south velocity.

(b) Acceleration: The acceleration vector is the derivative of the velocity vector.

$$\mathbf{a}(t)=\mathbf{v}'(t)=6\hat{i}-2\sin t\hat{j}\text{, at } t=\frac{\pi}{2} \text{ is, } \mathbf{a}\left(\frac{\pi}{2}\right)=6\hat{i}-2\sin\left(\frac{\pi}{2}\right)\hat{j}=6\hat{i}-2\hat{j}$$

The part is accelerating eastward at 6 m/sec^2 and southward at 2 m/sec^2. Even though its north-south velocity is momentarily zero, it is starting to move southward.

(c) Speed: The speed is the magnitude of the velocity vector.

$|\mathbf{v}(t)|=\sqrt{36t^2+4\cos^2 t}$, at $t=\frac{\pi}{2}$,

$$\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right|=\sqrt{\left(6.\frac{\pi}{2}\right)^2+\left(2\cos\left(\frac{\pi}{2}\right)\right)^2}=\sqrt{(3\pi)^2+0}=3\pi$$

The part is moving along its path at about 9.42 meters per second.

(d) Direction of Motion: The direction is given by the unit tangent vector,

$$\hat{\mathbf{v}}(t)=\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}\text{, at } t=\frac{\pi}{2} \text{ is, } \hat{\mathbf{v}}\left(\frac{\pi}{2}\right)=\frac{3\pi\hat{i}+0\hat{j}}{3\pi}=\hat{i}+0\hat{j}$$

The direction of motion is due east (positive x-direction). The part has momentarily stopped its north-south oscillation and is moving purely horizontally along the belt.

Example 7 Drone inspecting a cylindrical water tower

A drone is programmed to fly in a helical (spiral) path around a large cylindrical water tower to inspect its outer surface for cracks or rust. The drone maintains a constant horizontal distance of 2 meters from the tower's center while climbing upward at a steady rate. The drone's position in 3D space at time t seconds is given by: $\mathbf{r}(t)=2\cos t\hat{i}+2\sin t\hat{j}+3t\hat{k}$

Here:

- $x(t)=2\cos t$: east-west position (amplitude 2 m)
- $y(t)=2\cos t$: north-south position (amplitude 2 m)
- $z(t)=3t$: Vertical height (climbs 3 m/s)

At a specific moment $t=\pi$ seconds, the drone operator needs to analyze its motion to ensure smooth inspection. Find the following quantities at $t=\pi$ (a) velocity, (b) acceleration, (c) speed, and (d) direction of motion at $t=\pi$.

Solution:

(a) Velocity: Compute the first derivatives of each component.

$$\mathbf{v}(t)=\mathbf{r}'(t)=-2\sin t\hat{i}+2\cos t\hat{j}+3\hat{k}$$

at $t=\pi$, $\mathbf{v}(\pi)=-2\sin(\pi)\hat{i}+2\cos(\pi)\hat{j}+3\hat{k}=0\hat{i}-2\hat{j}+3\hat{k}$

At this instant, the drone has no east–west motion, is moving southward at 2 m/s, and is climbing at 3 m/s.

(b) Acceleration: Compute the second derivative.

$$\mathbf{a}(t)=\mathbf{v}'(t)=-2\cos t\hat{i}-2\sin t\hat{j}+0\hat{k}$$

at $t=\pi$ $\mathbf{a}(\pi)=-2\cos(\pi)\hat{i}-2\sin(\pi)\hat{j}+0\hat{k}=2\hat{i}+0\hat{j}+0\hat{k}$

The drone is accelerating eastward at 2 m/s². This centripetal acceleration points toward the center of the circular path (the tower's axis), keeping the drone in its circular horizontal trajectory.

(c) Speed: Calculate the magnitude of the velocity.

$$|\mathbf{v}(t)|=\sqrt{4\cos^2 t+4\sin^2 t+9}=\sqrt{13}$$

at $t=\pi$, $|\mathbf{v}(\pi)|=\sqrt{13}\approx 3.6056$ m/sec

The drone's speed is constant at about 3.61 m/s throughout its helical climb, ideal for capturing stable inspection footage.

(d) Direction of Motion: The unit tangent vector is,

$$\hat{\mathbf{v}}(t)=\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}=\frac{-2\sin t\hat{i}+2\cos t\hat{j}+3\hat{k}}{\sqrt{13}}$$

at $t=\pi$ $\hat{\mathbf{v}}(\pi)=\dfrac{0\hat{i}-2\hat{j}+3\hat{k}}{\sqrt{13}}=0\hat{i}-\dfrac{2}{\sqrt{13}}\hat{j}+\dfrac{3}{\sqrt{13}}\hat{k}$

The drone is moving in the south-up vertical plane. The direction has no east-west component.

EXERCISE 11.3

In the following exercises (Questions 1-5), each problem involves a vector-valued function representing the particle's position in the *xy*-plane or in 3D space. Find the velocity, acceleration, speed, and direction of motion at a specified value of ***t***. Interpret the results by describing the particle's motion at that instant, including how fast and in what direction it is moving, whether it is speeding up or slowing down, and how its path is curving.

1. $\mathbf{r}(t) = 2\sin t\hat{i} + 3t\hat{j}$ at $t = \frac{\pi}{2}$
2. $\mathbf{r}(t) = e^t\hat{i} + e^{-t}\hat{j}$ at $t = 0$
3. $\mathbf{r}(t) = \cos t\hat{i} + \sin t\hat{j} + 2t\hat{k}$ at $t = \pi$
4. $\mathbf{r}(t) = t^2\hat{i} + t\hat{j} + t^3\hat{k}$ at $t = 2$
5. $\mathbf{r}(t) = e^t \sin t\hat{i} + (t^3 - 2t)\hat{j} + \arctan(2t)\hat{k}$ at $t = 0$
6. A cannonball is fired from the ground at an angle 30° with an initial velocity of 100 m/s. Assuming gravity acts downward ($g = 9.8$ m/s²) and neglecting air resistance.
   (a) Formulate the position vector $\mathbf{r}(t)$ of the cannonball as a function of time.
   (b) Find the velocity vector at $t = 5$ sec.
   (c) Determine the time of the flight (when the cannonball hits the ground).
7. A comet orbits the Sun in an elliptical path such that its position (in AU) at time **t** (in years) is described by, $x = 2\cos t, y = \sin t$.
   (a) Write the position vector $\mathbf{r}(t)$
   (b) Find the velocity and speed at $t = \frac{\pi}{2}$
   (c) Show that the acceleration vector always points toward the origin.
8. A robotic arm moves such that: Its *x*-position increases linearly at 2 m/s, Its *y*-position follows $y(t) = t^2$, Its *z*-position is given by $z(t) = 1 - e^{-t}$.
   (a) Construct the position vector $\mathbf{r}(t)$
   (b) Find the velocity vector at t=1 sec
   (c) Compute the acceleration at t=*1* sec
9. A particle moves along a circular path, described by the position vector $\mathbf{r}(t) = R\cos\omega t\hat{i} + R\sin\omega t\hat{j}$. Prove that its velocity is perpendicular to its position vector.

10. A Ferris wheel of radius 20 m rotates with an angular speed of 0.2 rad/sec.
    (a) Construct its position vector $\mathbf{r}(t)$
    (b) Find the passenger's velocity at t=10sec
    (c) Show that the acceleration is centripetal (toward the center)
11. A quarterback throws a football with a position vector

$$\mathbf{r}(t) = 20t\hat{i} + \left(20t - 4.9t^2\right)\hat{j}$$

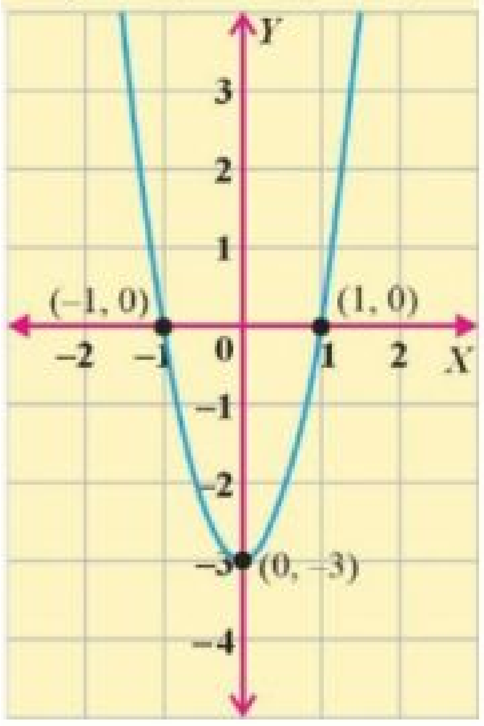
    (a) Find the time of flight before it hits the ground.
    (b) What is the maximum height reached?
    (c) At which time did football reach its maximum height?

Answers

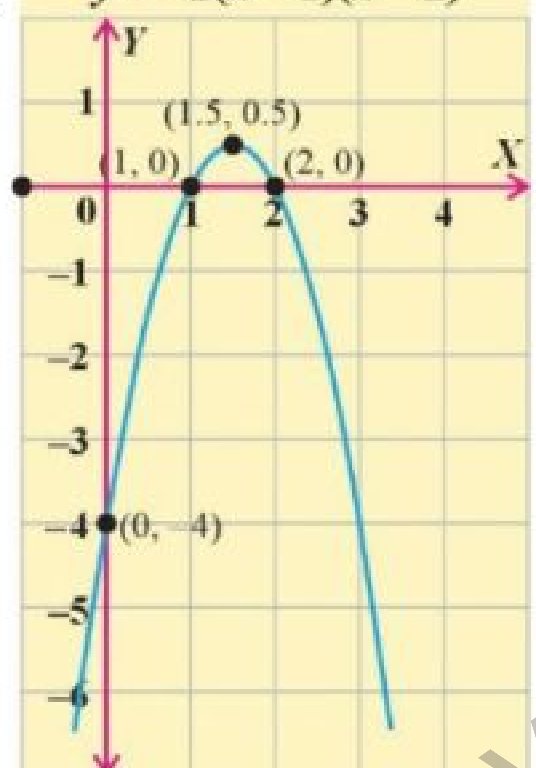
Answers to Exercise 1.1

1.

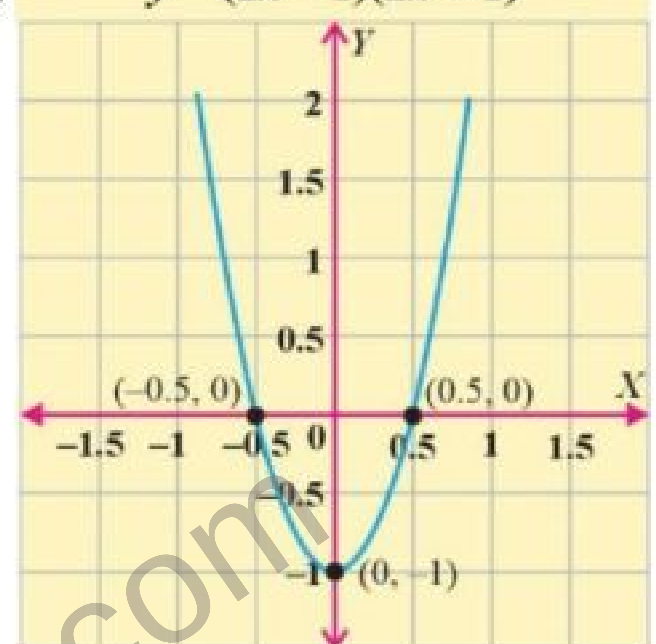
(i) $y = 3(x+1)(x-1)$



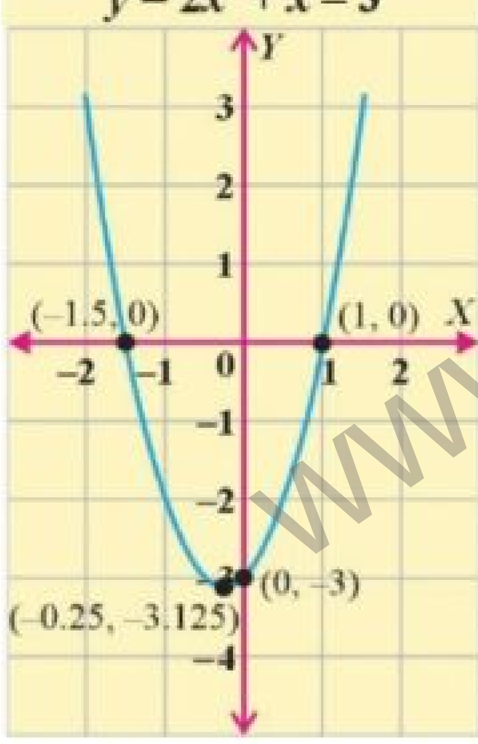
(ii) $y = -2(x-1)(x-2)$



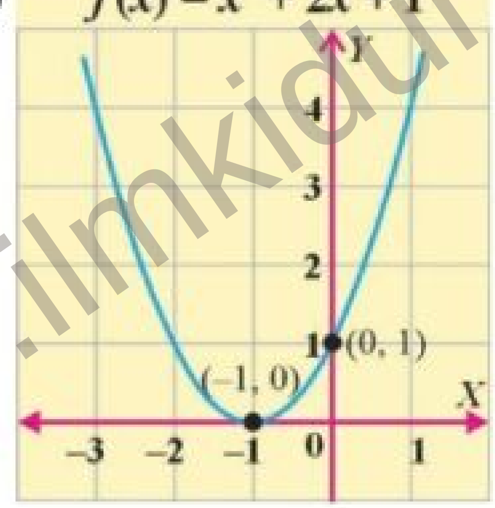
(iii) $y = (2x-1)(2x+1)$



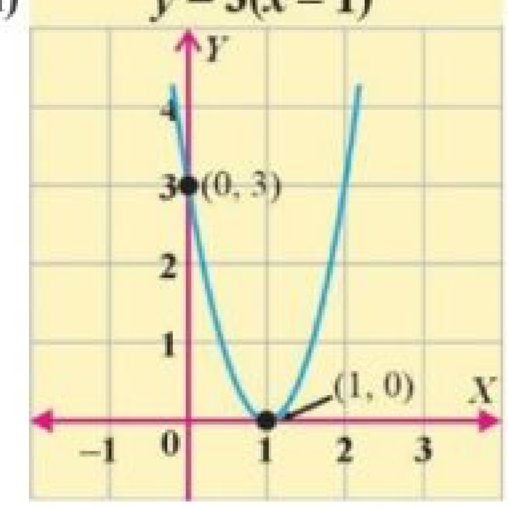
(iv) $y = 2x^2 + x - 3$



(v) $f(x) = x^2 + 2x + 1$



(vi) $y = 3(x-1)^2$



2. **(i)** $y = -(x-1)(x+2)$ **(ii)** $y = (x-2)^2$

3. $y = (x+1)(x+2)$ **4.** $y = -2(x+1)^2$

5. $y = 2x^2 - 2$

ANSWERS TO EXERCISE 1.2

1. Algebraic Functions: $g(x) = \dfrac{2x-3}{x+1}$, $h(x) = \sqrt{x^2+2x}$, $r(x) = (x-1)^3$

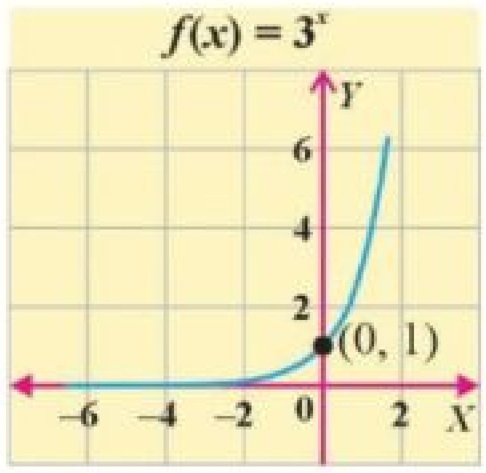
2. Transcendental Functions: $g(x) = \cot x$, $h(x) = e^{2x} + \sqrt{x^2+2x}$, $k(x) = \dfrac{e^x + e^{-x}}{2}$

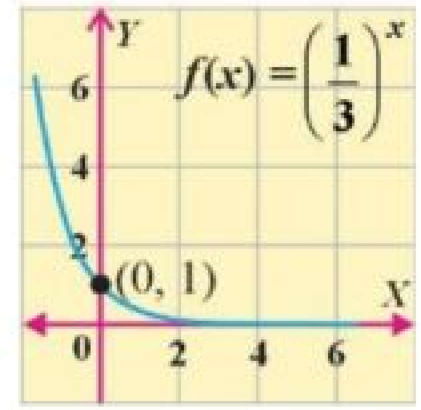
3. Fundamental Transcendental Functions: $g(x) = \sin x$, $h(x) = e^{2x}$, $r(x) = 15^x$

4. Non-Fundamental Transcendental Function: $k(x) = \cos x^2$

5. Dom $f =]-\infty, \infty[$, Range $f =]0, \infty[$. Graph is given in the following figure.

6. Dom $f =]-\infty, \infty]$, Range $f =]0, \infty[$. Graph is given in the following figure.





Exercise 1.3

1. (ii) $f^{-1}(x) = 4^x + 2$

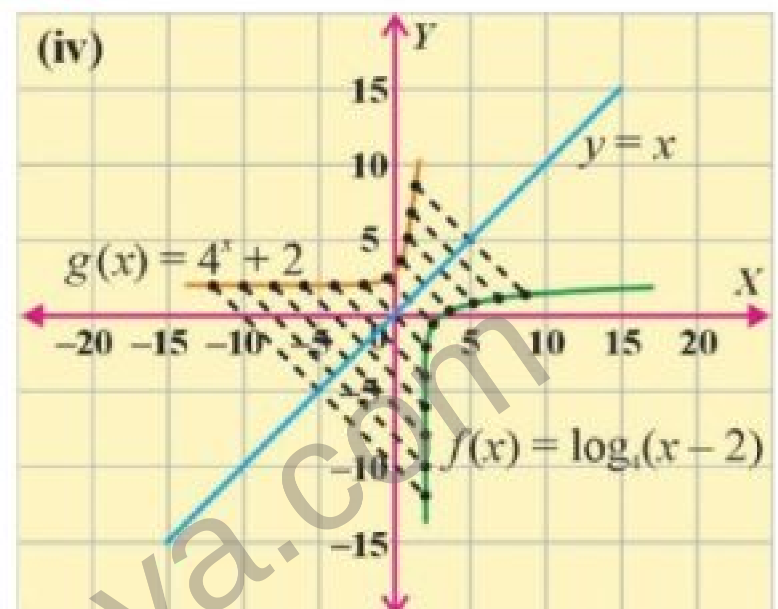
(iii) $D_f = R_{f^{-1}} =]2, \infty[, R_f = D_{f^{-1}} =]-\infty, \infty[$

(iii) Graphs of $f(x) = \log_4(x-2)$ and $f^{-1}(x) = 4^x + 2$ are drawn in the figure.

Note that:

(a) any horizontal line crosses the curve of f only once. Which indicates that f is one to one

(b) both functions are symmetric about the line $y = x$.

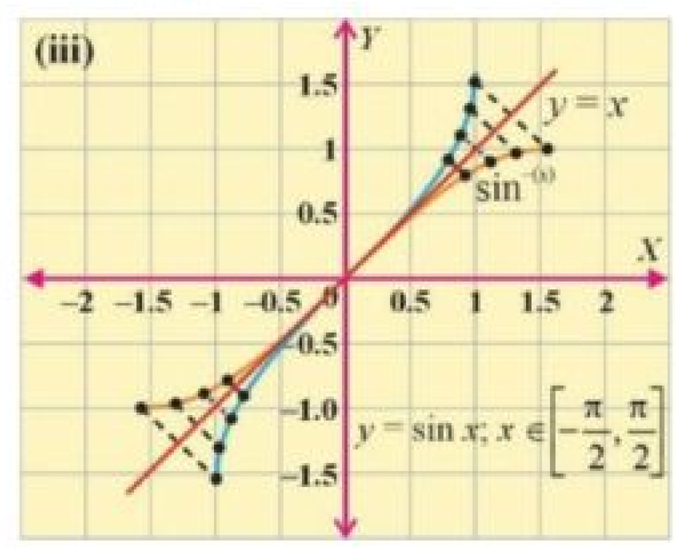


2. (i) $D_f = R_{f^{-1}} = [-1, 1], R_f = D_{f^{-1}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) Graphs of $f(x) = \sin^{-1}(x)$ and $f^{-1}(x) = \sin(x)$ are drawn in the figure, which clearly indicates that:

(a) any horizontal line crosses the curve of f only once. Which indicates that f is one to one.

(b) both functions are symmetric about the line $y = x$.



3. The graph of f^{-1} is given in the figure.

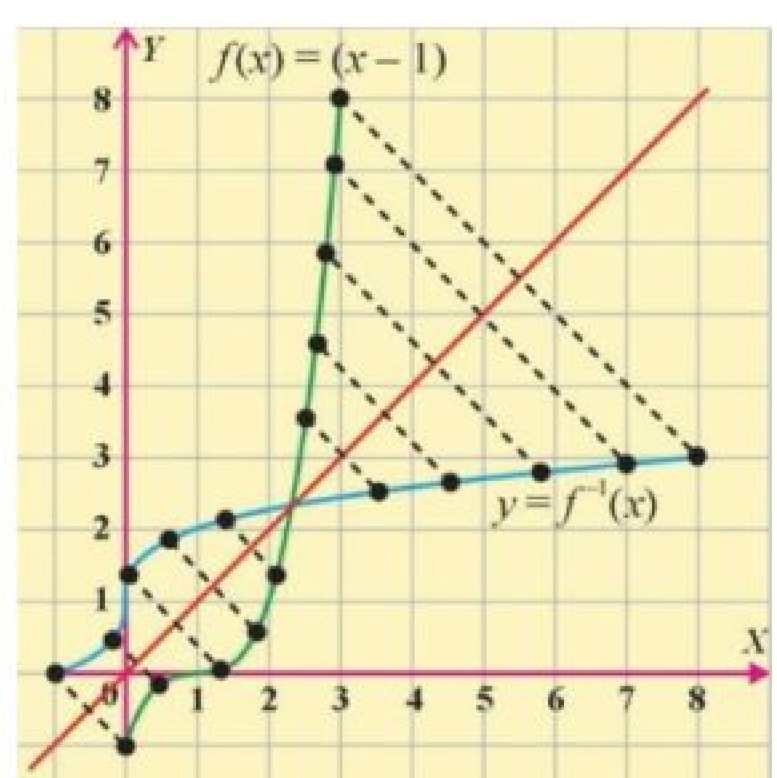
4. (i) $y = 2\sqrt{x}$ (ii) $y = \sqrt{x-3}$

(iii) $y = 2\sqrt{x-3}$

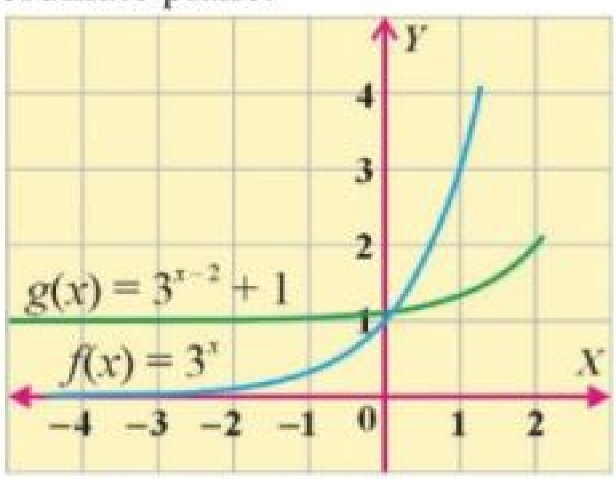
5. $y = \dfrac{2x+5}{2x+2}$

6. $y = \ln\sqrt{x-1} + 3$

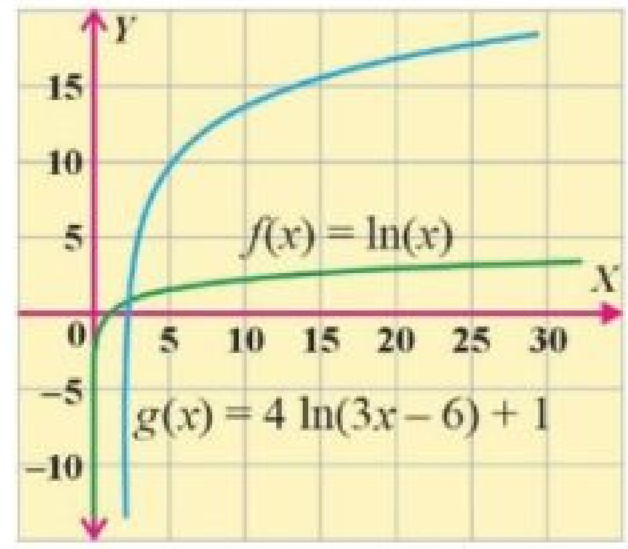
7. $y = \dfrac{1}{2}\sqrt{\dfrac{6x+3}{3x+5}} - 1$



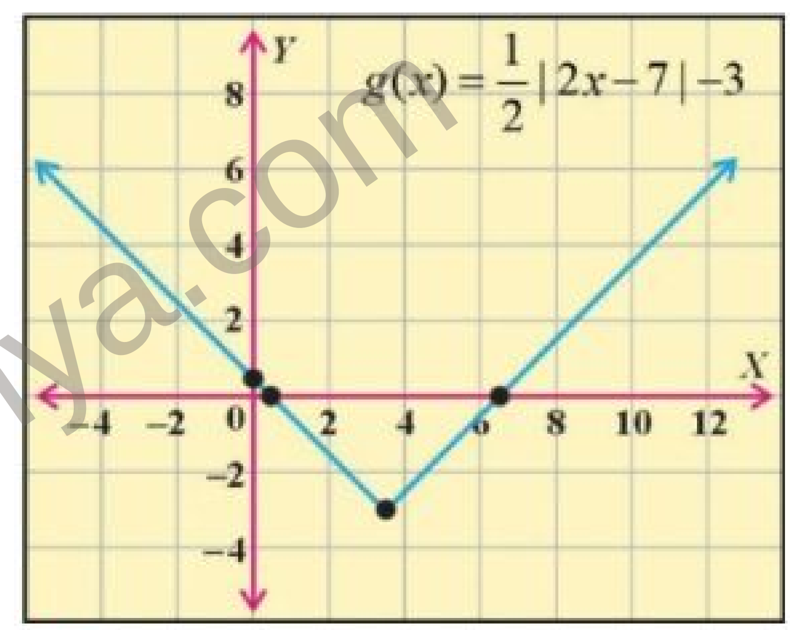
8. **Comparison:** From the figure, it is clear that the graph of $g(x)$ is obtained from $f(x)$ by shifting it 2 units to the right and 1 unit upward. Therefore, both graphs are identical in shape but differ in their position on the coordinate plane.



9. **Transformed function:** $g(x) = 4\ln(3x - 6) + 1$
Graphs of $f(x)$ and $g(x)$ are given in the figure.



10. Graph of $g(x) = \frac{1}{2}|2x - 7| - 3$ is given in the figure.



Answers to Exercise 1.4

1. (i) $S = \{4 - 2\sqrt{2}, 4 + 2\sqrt{2}\}$ (ii) $S = \left\{\frac{3+\sqrt{5}}{2}\right\}$ (iii) $S = \{1, 3\}$

(iv) $S = \{1\}$ (v) $S = \left\{\frac{1}{2}\log_2\left(\frac{1+\sqrt{13}}{2}\right)\right\}$ (vi) $S = \left\{\log_2\left(\frac{5+\sqrt{21}}{2}\right), \log_2\left(\frac{5-\sqrt{21}}{2}\right)\right\}$

2. (i) $S = [0, 1]$ (ii) $S = [2, \infty]$ (iii) $S = [3, 5]$

(iv) $S = \left]\frac{1}{2}, 1\right[\cup \left]3, 3+\sqrt{5}\right[$ (v) $S = \left[\frac{5+\sqrt{33}}{2}, \infty\right[$ (vi) $S = \varnothing$

Answers to Exercise 1.5

1. 34.7 hours 2. $k \approx 0.0511$ 3. 27.73 hours 4. $k \approx -0.0507$

5. 60 dB 6. $I = 10^{-4}$ W/m^2 7. (i) $k \approx -0.13863$ (ii) $N = 125$

8. $\frac{I_1}{I_2} = 100$ 9. $A =$ Rs. 4764.06 10. $t \approx 3$ years 11. $A \approx$ Rs. 3664.2 12. $t \approx 8.42$ years

EXERCISE 3.1

1. (i) $y = 20x^5 + c$ (ii) $y = 2t + 6t^4 - 2t^5 + c$ 2. (i) $t + t^2 + c$ (ii) $u^3 + 2u^2 + c$

(iii) $\frac{100}{9}x^9 + c$ (iv) $-\frac{2}{x} - \frac{3}{2x^2} + c$ 3. (i) $\frac{5}{4}x^4 + 5x^2 + 20x + c$

(ii) $\frac{x^4}{4}+\frac{x^2}{2}+c$ (iii) $\frac{y^2}{2}+4\sqrt{y}-\frac{1}{y}+c$ (iv) $\frac{2}{3}x^{3/2}+4\sqrt{x}+c$

4. (i) $\frac{1}{140}\left(7x^2+8\right)^{10}+c$ (ii) $-\frac{1}{12}\left(x^3+2\right)^{-4}+c$ (iii) $-\frac{1}{3}\left(1+\frac{1}{x}\right)^3+c$

(iv) $\frac{1}{3}\left(1+\sqrt{x}\right)^6+c$ (v) $\frac{2}{21}\left(7x+8\right)^{3/2}+c$ (vi) $-\frac{1}{25}\left(7-x^5\right)^5+c$

(vii) $-\frac{1}{9}\left(t^3-3t^2+5\right)^{-3}+c$ **5.** (i) $\frac{2}{3}\sqrt{x+2}\left(x-1\right)+c$ (ii) $\sqrt{ax^2+2bx+c}+C$

(iii) $\frac{2}{3}\left(x+1\right)^{3/2}-\frac{2}{3}x^{3/2}+c$ (iv) $\frac{4}{9}\left(1+\sqrt{x}\right)^{9/2}-\frac{12}{7}\left(1+\sqrt{x}\right)^{7/2}+\frac{12}{5}\left(1+\sqrt{x}\right)^{5/2}-\frac{4}{3}\left(1+\sqrt{x}\right)^{3/2}+c$

(v) $-\frac{a+2bx}{2b^2\left(a+bx\right)^2}+c$ (vi) $\frac{2}{15}\left(x^3+1\right)^{5/2}-\frac{2}{9}\left(x^3+1\right)^{3/2}+c$

Exercise 3.2

1. (i) $\frac{23}{6}$ (ii) 89 **2.** (i) 13 (ii) 42 (ii) $\frac{71}{3}$ **3.** $\frac{100}{3}$ **4.** (i) 0 (ii) $\int_0^5 f\left(x\right)dx$

5. $\frac{1}{132}$ **6.** (i) $\frac{128}{15}$ (ii) $\frac{16}{15}$ (iii) 0 **7.** $\frac{8}{3}$ **8.** 25 **9.** 38 **13.** $\frac{a^3}{3}$

14. 18 **16.** (i) 1 (ii) $\frac{1}{2}\ln 2$ **17.** (i) $\frac{\sin t}{1+\sqrt{t}}$ (ii) $\frac{b}{1+b^2x^2}-\frac{a}{1+a^2x^2}$

Exercise 3.3

1. (i) $\ln\left|\csc x-\cot x\right|+c$ (ii) $-\cot x-x+c$ (iii) $2\sqrt{1+\sin^2 x}+c$

(iv) $x+\frac{\sin^2 x}{2}+c$ (v) $\sin x+\cos x+c$ (vi) $-\sqrt{2}\cos x+c$

(vii) $\tan\frac{x}{2}+c$ (viii) $\frac{\cos 6x}{12}-\frac{\cos 8x}{16}+c$ (ix) $\tan x-x+c$

(x) $2-\sqrt{2}$ (xi) $\frac{2}{3}$ (xii) $\tan x+\frac{\tan^3 x}{3}+c$ (xiii) $\frac{3x^2}{2}+\frac{3^x}{\ln 3}+\frac{x^4}{4}+c$

(xiv) $4\sqrt[4]{x}-4\ln\left(1+\sqrt[4]{x}\right)+c$ (xv) $2\sqrt{2+x}-2\ln\left(1+\sqrt{2+x}\right)+c$

(xvi) $\frac{x^{\ln 7+1}}{\ln 7+1}+c$ **2.** (i) $\ln\left|\ln\tan x\right|+c$ (ii) $-\ln\left|\sin x+\cos x\right|+c$

(iii) $\frac{1}{4}\ln\left(1+\sin^4 x\right)+c$ (iv) $\ln\left|\cos x+x\sin x\right|+c$ (v) $\frac{\left(\ln\cos x\right)^2}{2}+c$

(vi) $\frac{\left(\ln\sin x\right)^2}{2}+c$ (vii) $-\frac{1}{2\left(2+e^{2x}\right)}+c$ (viii) $\frac{\pi}{4}$ (ix) $\frac{\pi}{2}$

Exercise 3.4

1. (i) $\frac{1}{6}\ln\left|\frac{3+e^x}{3-e^x}\right|+c$ (ii) $\frac{1}{4}\ln\left|\frac{\sin x-2}{\sin x+2}\right|+c$ (iii) $\arctan\left(e^x\right)+c$

(iv) $\arcsin\left(\frac{\ln x}{2}\right)+c$ **(v)** $\frac{1}{30}\ln\left|\frac{3x-5}{3x+5}\right|+c$ **(vi)** $\frac{1}{2\sqrt{3}}\arctan\left(\frac{\sqrt{x^2-4}}{2}\right)+c$

(vii) $x-a\arctan\left(\frac{x}{a}\right)+c$ **(viii)** $\arctan(\ln x)+c$ **(ix)** $2-\frac{\pi}{2}$

(x) $\frac{x}{\sqrt{1+x^2}}+c$ **(xi)** $\frac{1}{3}\arctan\left(\frac{\sqrt{x^2-9}}{3}\right)+c$ **(xii)** $\arcsin x-\sqrt{1-x^2}+c$

(xiii) $\frac{\pi}{8}\ln 2$ **(xiv)** $-\frac{\sqrt{x^2+4}}{4x}+c$ **(xv)** $a\ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right|+\sqrt{a^2-x^2}+c$

Exercise 3.5

1. **(i)** $x\ln x-x+c$ **(ii)** $\frac{x^8\ln x}{8}-\frac{x^8}{64}+c$ **(iii)** $\frac{x\sin 2x}{2}+\frac{\cos 2x}{4}+c$ **(iv)** $-x\cot x+\ln|\sin x|+c$

(v) $\frac{xe^{2x}}{2}-\frac{e^{2x}}{4}+c$ **(vi)** $\frac{x^2e^{ax}}{a}-\frac{2xe^{ax}}{a^2}+\frac{2e^{ax}}{a^3}+c$ **(vii)** $x\tan x-\ln|\sec x|-\frac{x^2}{2}+c$

(viii) $\sin x\cdot\ln(\sin x)-\sin x+c$ **(ix)** $(e^x+1)\ln(1+e^x)-e^x+c$

(x) $\frac{(\ln x)^2\tan^{-1}(\ln x)}{2}-\frac{\ln x}{2}+\frac{\tan^{-1}(\ln x)}{2}+c$ **(xi)** $\frac{e^{5x}}{5}(x^5+1)+c$ **(xii)** $2\sqrt{x}\,e^{\sqrt{x}}+c$

(xiii) $\frac{e^{3x}}{10}(3\sin x-\cos x)-\frac{e^{3x}}{2}(\cos 3x+\sin 3x)+c$ **(xiv)** $x^2e^{x^2}+c$

(xv) $e^x\tan^{-1}(e^x)-\frac{1}{2}\ln(1+e^{2x})+c$ **(xvi)** $x\tan\frac{x}{2}-2\ln\left|\sec\frac{x}{2}\right|+c$

(xvii) $x\tan x+x\sec x-\ln|\sec x|-\ln|\sec x+\tan x|+c$

(xviii) $\frac{\sin x}{x}+c$ (xix) $2\sqrt{x}\tan^{-1}\sqrt{x}-\ln(1+x)+c$

(xx) $-2x\cos\sqrt{x}+4\sqrt{x}\sin\sqrt{x}+4\cos\sqrt{x}+c$ **(xxi)** $\frac{e^x}{1+x^2}+c$

(xxii) $\frac{\pi}{4}-\frac{1}{2}\ln 2$ **(xxiii)** $\frac{e^{\tan^{-1}x}(2x^2+2x+3)}{5(1+x^2)}+c$ **(xxiv)** $x\sin^{-1}x+\sqrt{1-x^2}+c$

(xxv) $-\frac{1}{2}\csc x\cot x+\frac{1}{2}\ln|\csc x-\cot x|+c$

Exercise 3.6

(i) $2\ln|x+1|+3\ln|x+2|+c$ **(ii)** $\ln|x+1|+2\ln|x+3|+c$

(iii) $\frac{x^2}{2}+2x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+c$ **(iv)** $x-7\ln|x-7|+\ln|x+1|+c$

(v) $\frac{5}{3}\ln|4-3x|+\ln|2x+1|+c$ **(vi)** $-3\ln|x|+\ln|x-1|+\ln|x-2|+\ln|x-3|+c$

(vii) $\ln|x+2|+\frac{4}{x+2}-\frac{1}{2(x+2)^2}+c$ **(viii)** $e^{-x}+2\ln\left(\frac{e^x}{e^x+1}\right)+c$

(ix) $4\ln\left|\frac{x-2}{x+3}\right|-\frac{5}{x+3}+c$ **(x)** $\ln|x+1|+c$ **(xi)** $\ln|x-1|+2\arctan x+c$

(xii) $\frac{1}{6}\ln\left(x^2+1\right)+\frac{1}{3}\ln\left(x^2+4\right)+\frac{1}{3}\arctan x-\frac{1}{6}\arctan\left(\frac{x}{2}\right)+c$

(xiii) $\frac{3}{2}\ln|x-1|-\frac{1}{4}\ln\left(x^2+1\right)+\frac{x}{2\left(x^2+1\right)}+c$

(xiv) $\frac{1}{3}\ln|\sin x-1|-\frac{1}{6}\ln\left(\sin^2 x+\sin x+1\right)-\frac{\sqrt{3}}{3}\arctan\left(\frac{2\sin x+1}{\sqrt{3}}\right)+c$

Exercise 3.7

1. $\frac{1}{6}$ square units **2.** **(i)** Signed area = $\frac{2}{3}$, Total area = 4

(ii) Signed area = -6, Total area = $\frac{46}{3}$ **3.** **(i)** $\frac{4}{3}$ **(ii)** $\frac{1}{3}$ **(iii)** $\frac{8}{5}$ **(iv)** $\frac{1}{3}$

4. 1 square unit **5.** $\frac{4}{3}$ square units **6.** 36 square units **7.** 1:1 **8.** **(a)** 9

(b) 9 **(c)** The function is nonnegative on $[0,3]$, so signed area equals total area.

9. $\frac{37}{12}$ square units **10.** 18 square units **11.** Consumer's Surplus = 27

Producer's Surplus = 36 **12.** $D(x)=24-2x$; $S(x)=12+x$ **13.** **(a)** $p_e=2\ln 2$

(b) $p=6-(3-\ln 2)x$ **14.** CS = Rs. 324 **15.** **(i)** $\frac{32\pi}{5}$ cubic units

(ii) $\frac{8\pi}{3}$ cubic units **(iii)** 4π cubic units **(iv)** $\frac{16\pi}{15}$ cubic units

Exercise 3.8

1. $I=2048$ units units, $I=\frac{1}{6}Mh^2$ **2.** **(a)** $v(t)=6t^2-36t+30$ m/s **(b)** -50 m

(c) 78 m **(d)** $v(t)=3t+\frac{2}{(t+2)^2}+\frac{9}{2}$ cm/s **(e)** $s(t)=\frac{3}{2}t^2+\frac{9}{2}t-\frac{2}{t+2}+3$ cm

3. **(a)** $N(t)=200\cdot 3^t$ **(b)** 16200 bacteria **4.** **(a)** $P(t)=25000+800t+30t^2$ **(b)** 36000 people

(c) 5650 people **(d)** $b=3$ $\int_2^4 f(x)dx=15$ **(e)** $\int_3^4 f(x)dx=1$

(f) $\int_3^4 f(x)dx=4$ **4.** **(a)** $T=\frac{\ln 4}{0.3}\approx 4.62$ hours **(b)** AUC = 30 mg·h/L

(c) One dose insufficient; 6 doses needed in 24 hours

4. **(a)** Total at $t=10$: $12e^{-2}+12e^{-0.4}$ **(b)** Student forgot the contribution of the second dose

(c) **(i)** $12e^{-3.2}+12e^{-1.6}$ **(ii)** $12e^{-3.2}+12e^{-1.6}+12$

(d) $C_{total}(t)=12e^{-0.2t}+12e^{-0.2(t-8)}++12e^{-0.2(t-16)}$

EXERCISE 4.1

1. **(i)** order 3, degree 1 **(ii)** order 2, degree 3 **(iii)** order 2, degree undefined

(iv) order 3, degree 2 **(vi)** order 2, degree undefined **2.** $\frac{dA}{dt}-rA=0$

3. $\frac{dT}{dt}+kT=0,\ \ k>0$. **4.** $\frac{dv}{dt}=-g-\frac{k}{m}v,\ \ k\geq 0$, g is gravitational acceleration; $\frac{dv}{dt}+g=0$

5. $\frac{dN}{dt}=-kT,\ \ k>0$ **6.** $\frac{dx}{dt}=-\frac{k}{\pi r^2}\sqrt{x},\ \ k>0$ **7.** $\frac{dq}{dt}=-\frac{1}{RC}q$ **8.** $\frac{dI}{dt}=-\frac{R}{L}I$

EXERCISE 4.2

2. Separable: (*i*), (*ii*), (*v*), (*vi*) Homogeneous: (*iii*), (*iv*), (*vii*)

3. $\sin^{-1}x+\sin^{-1}x+C=0$ **4.** $e^y=e^x+C$ **5.** $2\tan\sqrt{y}=x+C$

6. $e^{-y}+2e^{\sqrt{x}}+C=0$ **7.** $y=\sin\left(x^2+C\right)$ **8.** $y=\ln\left|\frac{C\sqrt{1+x^2}}{1-x}\right|$

9. $y=x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$ **10.** $y=C(x+2)(2y-1)$ **11.** $\tan x\tan y=C$

12. $2y=\ln\left|C(e^{2x}+e^{-2x})\right|$ **13.** $y\ln y=y+x\cos x-\sin x+C$ **14.** $\tan\left(\frac{y}{x}\right)=\ln|Cx|$

15. $\cos\left(\frac{y}{x}\right)=\ln|Cx|$ **16.** $x^3y=C(y+3x)$ **17.** $x^2+y^2=Cx^3$

18. $\tan^{-1}\left(\frac{y}{x}\right)=\ln|Cx|$ **19.** 19. $\frac{x}{x+y}=-\ln|Cx|$ **20.** $e^{\frac{y}{x}}=\ln|Cx|$

21. $2y^3-3xy^2+6x^2y=6x^3\ln\left|C\left(y+x\right)\right|$ **22.** $y=3+Cx^{\frac{1}{4}}$ **23.** $xy=C$

24. $y(y+1)=\frac{1}{2}x^2+C$ **25.** $\tan^{-1}y=\frac{x^2}{2}+\frac{\pi}{4}$ **26.** $s=4e^{1-t^2}$

27. $e^y(\sin x+\cos x)=1$ **28.** $y+\sqrt{x^2+y^2}=x^2$ **29.** $y=x+x\ln|x|$

EXERCISE 4.3

1. $v=t^2-7t+10;\ \ s=\frac{1}{3}t^3-\frac{7}{2}t^2+10t$ **2.** 863 (approximately) **3.** 619 (approximately)

4. **(a)** $k\approx 0.05776$ **(b)** 20 grams **(c)** 48 hours **5.** Rs. 1491.8 **6.** Rs. 15021

7. 7.32% **8.** 16.80°C **9.** 7.12 hours **10.** $I(t)=\frac{1}{5}\left(1-e^{-30t}\right)$

11. $I(t)=2e^{-\frac{1}{18}t}$ amps **12.** $I(t)=-\frac{1}{8}e^{-\frac{1}{800}t}$ amps **13.** **(i)** $x(t)=\left(3-\frac{t}{8\pi}\right)^2$

(ii) $V(t)=4\pi\left(3-\frac{t}{8\pi}\right)^2$ **(iii)** 75.4 minutes **14. (i)** velocity: $v(t)=1470-980t$

(ii) height: $h(t)=1470t-490t^2$ **(iii)** Maximum height: 1093 cms

15. (i) velocity: $v(t)=4.9\left(1-e^{-2t}\right)$ **(ii)** height: $h(t)=502.45-4.9t-2.45e^{-2t}$

Answers 5.1

2. $4x-y-1=0$ **3.** $5x+3y-18=0$ **4.** $6x+5y-16=0, 6x+y-24=0, y+2=0$

6. $x-2y-4=0, x-4=0, x+2y-4=0$

8. Medians: $14x-y-23=0, 11x+14y-23=0, 25x+13y-46=0$

Altitudes: $3x-4y+14=0, 4x-13y+64=0, 5x+y-22=0$

Perpendicular Bisectors: $6x-8y-25=0, 8x-26y-71=0, 5x+y-2=0$

10. $(1+3\lambda)x+(2-\lambda)y+(4\lambda-3)=0$ **(i)** $21x-14y+41=0$ **(ii)** $28x-21y+59=0$

11. $(2+\lambda)x+(-3+4\lambda)y+(5-7\lambda)=0$ **(i)** $22x-11y+17=0$ **(ii)** $11x-11y+20=0$

12. Not concurrent **14.** $m_1c_2-m_2c_1+m_2c_3-m_3c_2+m_3c_1-m_1c_3=0$

15. $k=4$ **16.** $a=-\frac{1}{3}$

Exercise 5.2

1. (i) 168.7°, acute angle: 11.3° **(ii)** 7.1° **2. (i)** 90° **(ii)** 45° **(iii)** 120°

3. (i) 18.4° **(ii)** 101.3° **4.** $-2+\sqrt{3}, -2-\sqrt{3}$ **5. (i)** 66.4°, 66.4°, 53.1°

(ii) 36.9°, 19.4°, 123.7° **(iv)** 53.1°, 63.4°, 63.4° **6.** 126.9°, 74°, 106°, 53.1°

7. 112.6°, 67.4°, 112.6°, 67.4° **8.** 16.5 **9.** Collinear **10.** $\lambda=7$

Exercise 5.3

1. $(3x-4y)(2x+y)=0$, 79.7° **2.** $(2x-y)(x-3y)=0$, 45° **3.** $(5x+y)(x+y)=0$, 33.69°

4. $y=x(\tan\alpha\pm\sec\alpha)$, 90° **5.** $(7x-y)(x-y)=0$, 36.87° **6.** $k=2$

8. $2x^2+5xy-3y^2=0$ **9.** $x-y+100=0$, slope $=1$ **10.** 1, 45°

11. $2x-y=0, x+3y=0$, 81.87° **12.** 0° **13.** $3x+y=0, x-3y=0$, 90°

EXERCISE 6.1

1. (a) $x^2+y^2-10x-14y+70=0$ **(b)** $x^2+y^2-2ax\cos\theta-2ay\sin\theta=0$

(c) $x^2+y^2-5x+3y-22=0$ **2. (i)** $(x-1)^2+(y-3)^2=\frac{9}{2}$; Center: $(1,3)$, Radius: $\frac{3\sqrt{2}}{2}$

(ii) $(x-a)^2+(y+b)^2=(a+b)^2$; Center: $(a,-b)$, Radius: $|a+b|$

3. $x^2+y^2-4x-6y-12=0$; Center: $(2,3)$, Radius: 5

4. $x^2+y^2-2x-3y=0$ **5.** $5x^2+5y^2-19x-11y+2=0$; Yes, (4, 1) lies on the circle.

6. $3x^2+3y^2+72x-38y-11=0$ **7.** $x^2+y^2+2ax-2ay+a^2=0$

8. $x^2+y^2-10x-4y+25=0$ **9.** $x^2+y^2-10x+17=0$

10. $x^2+y^2-40x+20y+100=0$; $x^2+y^2-10x-20y+100=0$

11. $x^2+y^2-4x-1=0$, $x^2+y^2-14x+29=0$

12. $x^2+y^2-10x-10y+25=0$, $x^2+y^2+6x+2y-15=0$

EXERCISE 6.2

1. **(a)** At (2, 2): Tangent: $x+y=4$ Normal: $x-y=0$ At $\left(2\sqrt{2}\cos\theta, 2\sqrt{2}\sin\theta\right)$: Tangent: $x\cos\theta+y\sin\theta=2\sqrt{2}$ Normal: $x\sin\theta-y\cos\theta=0$
   (b) Tangent: $x+4y-24=0$, Normal: $4x-y-11=0$
2. Condition of tangency: $c^2=a^2\left(1+m^2\right)$ Point of contact: $\left(-\frac{am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$ or $\left(\frac{am}{\sqrt{1+m^2}}, -\frac{a}{\sqrt{1+m^2}}\right)$ depending on the sign of c.
3. $(4,3)$ **4.** **(a)** $x-y\pm2\sqrt{2}=0$ **(b)** $3x+4y\pm10=0$ (c) $4x+3y\pm10=0$
5. $3x-4y+13=0$ and $3x-4y-17=0$
6. **(a)** Tangent: $x+y-4=0$, (Point of contact:$(2,2)$)
   (b) Tangent: $41x+y+116=0$, $\left(\text{Point of contact:}\left(-\frac{82}{29}, -\frac{2}{29}\right)\right)$
   (c) Tangent: $y+4=0$, (Point of contact:$(2,-4)$)
   $4x+3y-36=0$, (Point of contact:$(6,4)$)
7. **(a)** $(x-3)^2+(y+1)^2=25$ **(b)** Centre: $(3,-1)$, Radius: 5 **(c)** P:$(3,4)$
   (d) $y=4$ **(e)** $x=3$ **(f)** $x=-2$

EXERCISE 6.3

1. **(i)** Vertex: $(0,0)$ Focus: $(3,0)$ Directrix: $x=-3$ **(ii)** Vertex: $(0,1)$ Focus: $(0,3)$ Directrix: $y=-1$ **(iii)** Vertex: $(3,0)$ Focus: $(1,0)$ Directrix: $x=5$
   (iv) Vertex: $(2,0)$ Focus: $(2,2)$ Directrix: $y=-2$ **2.** **(i)** $(y-2)^2=-16(x-1)$
   (ii) $(x-3)^2=12(y-2)$ **(iii)** $(y-3)^2=8(x-6)$ **(iv)** $4y^2=5x+6$
3. **(i)** $x^2=-2dy+d^2$ **(ii)** $y^2=-2dx+d^2$ where d is the directed distance from origin to directrix
6. $y=-\frac{5}{4}$ **7.** $a=2, b=-12, c=16$ **8.** Tangent: $ty=x+at^2$, Normal: $y=-tx+2at+at^3$
9. Tangent: $2x-y-7=0$, Normal: $x+2y-11=0$
10. Tangent: $3x-y+1=0$, Point of tangency: $\left(\frac{1}{3}, 2\right)$

EXERCISE 6.4

1. **(i)** $\frac{x^2}{16}+\frac{y^2}{41}=1$ **(ii)** $\frac{(x+2)^2}{9}+\frac{(y-1)^2}{16}=1$ **(iii)** $\frac{x^2}{36}+\frac{y^2}{20}=1$
   (iv) $\frac{(x+3)^2}{25}+\frac{(y-1)^2}{9}=1$ **2.** **(i)** Centre: $(0,0)$, Foci: $(\pm\sqrt{5},0)$, Eccentricity: $e=\frac{\sqrt{5}}{3}$,
   Vertices: $(\pm3,0)$, Directrices: $x=\pm\frac{9\sqrt{5}}{5}$

(ii) Centre: $(-3,2)$, Foci: $\left(-3\pm\sqrt{5},2\right)$, Eccentricity: $e=\dfrac{\sqrt{5}}{3}$, Vertices: $(0,2)$ and $(-6,2)$

Directrices: $x=2\pm\dfrac{9}{\sqrt{5}}$ 3. $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$ 4. $\dfrac{x^2}{36}+\dfrac{y^2}{20}=1$

5. $\dfrac{(x-5)^2}{25}+\dfrac{(y-3)^2}{9}=1$ **6.** $e=\dfrac{3}{5}$ **7.** $\dfrac{(x+2)^2}{20}+\dfrac{(y-3)^2}{16}=1$

8. 4 units **9.** 8 by 6 **10.** $\dfrac{32}{5}$ and $\dfrac{18}{5}$ 11. Minimum distance = 2

12. $11+2\sqrt{10}$ **13.** $\dfrac{6\sqrt{2}-\sqrt{26}}{2}$ **14.** $k=\pm\sqrt{37}$ **15.** The line is a secant.

16. $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$

EXERCISE 6.5

1. (i) $\dfrac{y^2}{64}-\dfrac{x^2}{36}=1$ (ii) $\dfrac{y^2}{4}-\dfrac{x^2}{60}=1$ (iii) $\dfrac{y^2}{50}-\dfrac{x^2}{50}=1$

2.(i) Centre: (0,0); Vertices: (±6,0); Foci: (±10,0); Eccentricity: $e=\dfrac{5}{3}$; Directrices: $x=\pm\dfrac{18}{5}$ (ii)

Centre: $(-3,2)$, Vertices: $\left(-3\pm\sqrt{5},2\right)$; Foci: $(0,2)$ and $(-6,2)$; Eccentricity: $e=\dfrac{3}{\sqrt{5}}$ Directrices:

$x=-3\pm\dfrac{5}{3}$

(iii) Centre: (0,0); Vertices: (0,±5); Foci: (0,± $\sqrt{41}$); Eccentricity: $e=\dfrac{\sqrt{41}}{5}$;

Directrices: $y=\pm\dfrac{25}{\sqrt{41}}$ 3. $\dfrac{y^2}{49}-\dfrac{x^2}{25}=1$ 4. $\dfrac{(x-1)^2}{16}-\dfrac{(y-3)^2}{4}=1$

5. $\dfrac{(y+1)^2}{16}-\dfrac{(x-2)^2}{36}=1$ 6. $\dfrac{(x+1)^2}{12}-\dfrac{(y-4)^2}{24}=1$

7. $\dfrac{(y-4)^2}{16}-\dfrac{(x-2)^2}{3}=1$ 8. $\dfrac{x^2}{16}-\dfrac{y^2}{4}=1$

9. Tangent: $\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1$, Normal: $ax\tan\theta+by\sec\theta=(a^2+\ b^2)\sec\theta\tan\theta$

10. $\dfrac{9}{2}$ 11. $e=\sqrt{2}$ 12. $e=\dfrac{\sqrt{34}}{4}$ 13. $\dfrac{x^2}{4}-\dfrac{y^2}{12}=1$ 14. $e=\sqrt{\dfrac{3}{2}}$

15. $5y=10x-14$ and $5y=10x+14$ 16. $e=\dfrac{4}{3}$ 17. 6

19. $y=2x-5$ and $y=-3x+10$ 20. $a^2l^2-b^2m^2=1$

EXERCISE 6.6

1. 75 million km **2.** $y=\dfrac{4h}{\ell^2}x^2$ **3.** 12.75m **4.** **(a)** $20\sqrt{5}$ m **(b)** $40\sqrt{5}$ m

5. **(a)** 8 m **(b)** 18 m **6.** (a) $\frac{x^2}{144}+\frac{y^2}{64}=1$ **(b)** $2\sqrt{15}$ m **7.** **(a)** 6.36×10^{-10} m

(b) 5.31×10^{-23} s **(c)** $r_{min}=1.384\times10^{-10}$ m and $r_{max}=2.756\times10^{-10}$ m

8. 126.65 miles **9.** **(a)** 36000 km **(b)** 6000 km **(c)** $\frac{1}{6}$ **10.** 31.75 meters

11. $\frac{5\sqrt{39}}{8}$ meters

Answers 8.1

1. **(i)** 0.36 **(ii)** 1.86 **(i)** 1.32 **(ii)** 0.85 **(iii)** 7.39

Answers 8.2

2. **(i)** 1.068 **(ii)** 0.605 **(iii)** 1.929 **(iv)** 2.613 **3.** **(i)** 2.61 **(ii)** −1.93 **(iii)** 0.85 **(iv)** 0.36 **4.** **(i)** 1.73 **(ii)** 2.153 **(iii)** 2.0946 **(iv)** 0.361 **(v)** 4.4934

Answers 8.3

1. 0.8903 **2.** 1.7212 **3.** 7.5522 **4.** 4.0625 **5.** 5.1463
6. 1.4704 **7.** 2.9411 **8.** 0.6179 **9.** Exact Area = 14.137 **10.** 2.3203×10^{6}

Answers 8.4

1. 0.12 moles per liter. **2.** 4.88 mL/s **3.** 0.539 coulombs **4.** 65.311 unit/seconds
5. 0.524 radian **6.** 20.488 volts

Answers 9.1

Inverse Trigonometric function Exercise 1 Answers

1. (i) $\frac{\pi}{3}$ (ii) $\frac{3\pi}{4}$ (iii) $\frac{-\pi}{3}$ (iv) $\frac{\pi}{6}$ (v) $\frac{-\pi}{4}$

(vi) $\frac{-\pi}{4}$ (vii) $\frac{\pi}{3}$ (viii) $\frac{3\pi}{4}$ (ix) $\frac{\pi}{2}$ (x) 0

(xi) $\frac{\pi}{3}$ **2.** (i) $\left[\frac{2}{5},\frac{4}{5}\right]$ (ii) [3, 4] (iii) $-\sqrt{2}\le x\le\sqrt{2}$

(iv) $\left[\frac{1}{e},e\right]$ (v) [−2, 2] (vi) $(-\infty,-3]\cup[-1,\infty)$ (vii) $\left(-\infty,-\frac{1}{3}\right]\cup[1,\infty)$

(viii) $\left[\frac{1}{e},e\right]$ **4.** (i) $\frac{3}{\sqrt{13}}$ (ii) $\frac{\sqrt{50}}{7}$ (iii) $\frac{1}{7}$ (vi) $\frac{1}{\sqrt{3}}$

(v) $\sqrt{\frac{11+4\sqrt{6}}{5}}$ (vi) $\sqrt{10}$

Answers 10

Solution of Trigonometric Equation Exercise 2 Answers

1. $x=-2.97$, 0.41 and 2.33 radians **2.** $x=1.16$ radians **3.** $x=0.65$ radians
4. $x=-1.66$ radians **5.** $x=-2.04$, 0.82 radians **6.** $x=-0.89$, 0, 0.89 radians

7. $x=0.40$, 1.57, 2.74 radians **8.** $x=1.89$ radians **9.** $\theta\approx53.13^\circ$
10. **(a)** $h=8.96$m **(b)** $\theta\approx1.15^\circ$ **11.** $h=11.55$ m, $\theta\approx49.1^\circ$