Unit 10 Solution of Trigonometric Equations

INTRODUCTION

Trigonometric equations involve functions like sin θ, cos θ and tan θ, and are solved to find the angle(s) that satisfy a given condition. In this unit, we solve such equations using the periodic, even/odd, and translation properties of trigonometric functions. For instance, equations such as $\sin\theta = k$, $\cos\theta = k$ or $\tan\theta = k$ are solved by considering the periodic nature of these functions. Next, we also explore graphical methods to solve equations like $\sin\theta = \frac{\theta}{2}$, $\cos\theta = \theta$ or $\tan\theta = 2\theta$ within specific intervals. The general solution of these equations incorporates the periodic behavior of trigonometric functions to identify all possible values of θ. Finally, we apply the concepts of inverse trigonometric functions to real-life problems in engineering and architecture, such as determining heights, calculating angles of elevation and depression, and designing accurate scale models.

10.1 Trigonometric Equation

A trigonometric equation is an equation that involves one or more trigonometric functions such as sine, cosine, tangent etc. and their inverses. For example,

$$\sin x = \frac{1}{\sqrt{2}},\ 2\cos^2\theta - 5\cos\theta + 1 = 0,\ 2\tan^{-1}x = \frac{\pi}{2} \text{ and } \sqrt{3}\cot x - \sec x - 1 = 0$$

are trigonometric equations. An unknown value of the angle that satisfy the given trigonometric equation is called a **solution** of the equation. The solutions that lie within the interval $[0, 2\pi]$ are called the **principal solutions**. Due to the periodic nature of trigonometric functions, these equations have infinitely many solutions. Once a solution is found within one period, adding or subtracting multiples of the period generates all possible solutions called the **general solution**. For example, Consider the equation:

$$\cos x = \frac{1}{2}$$

Since cosine is positive, so x lies in the I and IV quadrant.

$$x = \frac{\pi}{3} \text{ and } x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \quad \text{where } x \in [0, 2\pi]$$

So, $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ are the solution of the equation $\cos x = \frac{1}{2}$ and are called the **principal**

Solutions. Since 2π is the period of $\cos x$.

So, all possible solutions of $\cos x=\frac{1}{2}$ are $\frac{\pi}{3}+2n\pi$ and $\frac{5\pi}{3}+2n\pi,\ n\in\mathbb{Z}$. Hence, **general Solution** is given by $x=\left\{\frac{\pi}{3}+2n\pi\right\}\cup\left\{\frac{5\pi}{3}+2n\pi\right\}$.

Example 1 Solve $\sin x=-\frac{\sqrt{3}}{2}$

Solution: Since sine is negative. So x lies in III and IV quadrants with reference angle $\frac{\pi}{3}$

Hence, $x=\pi+\frac{\pi}{3}=\frac{4\pi}{3}$ and $x=2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$ where $x\in[0,\ 2\pi]$

as 2π is the period of $\sin x$.

So, all possible solutions of $\sin x=-\frac{\sqrt{3}}{2}$ are $\frac{2\pi}{3}+2n\pi$ and $\frac{5\pi}{3}+2n\pi,\ n\in\mathbb{Z}$

General solution $=\left\{\frac{4\pi}{3}+2n\pi\right\}\cup\left\{\frac{5\pi}{3}+2n\pi\right\},\ n\in\mathbb{Z}$

Example 2 Solve $2\sec^2 x-\frac{8}{3}=0$.

Solution: $2\sec^2 x-\frac{8}{3}=0$

$$\Rightarrow\quad 2\sec^2 x=\frac{8}{3}\quad\Rightarrow\quad \sec x=\pm\frac{2}{\sqrt{3}}$$

$$\Rightarrow\quad \sec x=\frac{2}{\sqrt{3}}\quad\text{or}\quad \sec x=-\frac{2}{\sqrt{3}}$$

$$\Rightarrow\quad \cos x=\frac{\sqrt{3}}{2}\quad\text{or}\quad \cos x=-\frac{\sqrt{3}}{2}$$

When $\cos x=\frac{\sqrt{3}}{2}$

Since cosine is positive, so x lies in I and IV quadrants with reference angle $\frac{\pi}{6}$

Hence, $x=\frac{\pi}{6}$ and $x=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$ where $x\in[0,\ 2\pi]$. Period of $\cos x$ is 2π.

So, all possible solutions of $\cos x=\frac{\sqrt{3}}{2}$ are $\frac{\pi}{6}+2n\pi$ and $\frac{11\pi}{6}+2n\pi,\ n\in\mathbb{Z}$.

When $\cos x = -\dfrac{\sqrt{3}}{2}$

Since cosine is negative, so x lies in II and III quadrants with reference angle $\dfrac{\pi}{6}$.

Hence, $x = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$ and $x = \pi + \dfrac{\pi}{6} = \dfrac{7\pi}{6}$. Period of $\cos x$ is 2π.

So, all possible solutions of $\cos x = -\dfrac{\sqrt{3}}{2}$ are $\dfrac{5\pi}{6} + 2n\pi$ and $\dfrac{7\pi}{6} + 2n\pi$, $n \in \mathbb{Z}$

General solution $= \left\{\dfrac{\pi}{6} + 2n\pi\right\} \cup \left\{\dfrac{5\pi}{6} + 2n\pi\right\} \cup \left\{\dfrac{7\pi}{6} + 2n\pi\right\} \cup \left\{\dfrac{11\pi}{6} + 2n\pi\right\}$, $n \in \mathbb{Z}$

Example 3 Solve $\sqrt{3}\cot x - \csc x - 1 = 0$.

Solution: $\sqrt{3}\cot x - \csc x - 1 = 0$ ...(1)

$\Rightarrow \quad \sqrt{3}\dfrac{\cos x}{\sin x} - \dfrac{1}{\sin x} - 1 = 0$

$\Rightarrow \quad \sqrt{3}\cos x - 1 - \sin x = 0$

$\Rightarrow \quad \sqrt{3}\cos x - 1 = \sin x$

Taking square on both sides

$\left(\sqrt{3}\cos x - 1\right)^2 = \sin^2 x$

$\Rightarrow \quad 3\cos^2 x + 1 - 2\sqrt{3}\cos x = \sin^2 x$

$\Rightarrow \quad 3\cos^2 x + 1 - 2\sqrt{3}\cos x = 1 - \cos^2 x$

$\Rightarrow \quad 4\cos^2 x - 2\sqrt{3}\cos x = 0$

$\Rightarrow \quad 2\cos x\left(2\cos x - \sqrt{3}\right) = 0$

$\Rightarrow \quad 2\cos x = 0 \quad$ or $\quad 2\cos x - \sqrt{3} = 0$

$\Rightarrow \quad \cos x = 0 \quad$ or $\quad \cos x = \dfrac{\sqrt{3}}{2}$

Remember!

Sometimes it is necessary to square both sides of a trigonometric equation to solve it. However, this operation can introduce extraneous roots i.e., solutions that appear valid but don't actually satisfy the original equation. Therefore, it is essential to verify every potential solution by substituting it back into the original equation before accepting it as valid.

When $\cos x = 0$

$\Rightarrow \quad x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$ where $x \in [0, 2\pi]$

To check extraneous roots, put $x = \dfrac{\pi}{2}$ in equation (1)

$$\sqrt{3}\cot x - \csc x - 1 = 0$$

L.H.S= $\sqrt{3}\cot\frac{\pi}{2}-\csc\frac{\pi}{2}-1=\sqrt{3}(0)-1-1=-2\neq 0=$ R.H.S

$\Rightarrow\quad x=\frac{\pi}{2}$ does not satisfy the equation, so $\frac{\pi}{2}$ is extraneous root.

Now put $x=\frac{3\pi}{2}$ in equation (1)

L.H.S= $\sqrt{3}\cot\frac{3\pi}{2}-\csc\frac{3\pi}{2}-1=\sqrt{3}(0)-(-1)-1=-1+1=0=$ R.H.S

$\Rightarrow\quad x=\frac{3\pi}{2}$ is the solution of the equation. Period of $\cos x$ is 2π.

So, general values of x are $\frac{3\pi}{2}+2n\pi,\ n\in\mathbb{Z}$.

When $\cos x=\frac{\sqrt{3}}{2}$

$\Rightarrow$ Since cosine is positive, so x lies in I and IV quadrants with reference angle $\frac{\pi}{6}$.

So, $x=\frac{\pi}{6}$ and $x=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$ where $x\in[0,\ 2\pi]$

again, to check extraneous roots, put $x=\frac{\pi}{6}$ in equation ...(1)

$$\sqrt{3}\cot x-\csc x-1=0$$

L.H.S= $\sqrt{3}\cot\frac{\pi}{6}-\csc\frac{\pi}{6}-1=\sqrt{3}\cdot\sqrt{3}-2-1=3-3=0=$ R.H.S

$\Rightarrow\quad x=\frac{\pi}{6}$ is solution of equation (1).

Now put $x=\frac{11\pi}{6}$ in equation (1)

L.H.S= $\sqrt{3}\cot\frac{11\pi}{6}-\csc\frac{11\pi}{6}-1=\sqrt{3}(-\sqrt{3})-(-2)-1=-3+2-1=-2\neq 0=$ R.H.S

$\Rightarrow\quad x=\frac{11\pi}{6}$ does not satisfy the equation (1), so $\frac{11\pi}{6}$ is an extraneous root. Period of $\cos x$ is 2π.

So, general values of x are $\frac{\pi}{6}+2n\pi$, $n\in\mathbb{Z}$.

Hence, general solution $=\left\{\frac{3\pi}{2}+2n\pi\right\}\cup\left\{\frac{\pi}{6}+2n\pi\right\},\ n\in\mathbb{Z}$

EXERCISE 10.1

Solve the following trigonometric equation:

1. $\sin x=\frac{1}{\sqrt{2}}$
2. $2\cos x=-1$
3. $3\tan^2 x=1$
4. $9-3\csc^2 x=5$
5. $2\cos^2 x-3\cos x+1=0$
6. $\sin^2 x-3\cos^2 x=0$
7. $\sqrt{2}\cos x\sin x-\sin x=0$
8. $3\cot^2 x+3\csc^2 x=5$
9. $\sin 2x-\sqrt{3}\cos x=0$
10. $\cos 2x=\cos x$
11. $\tan^2 x-\left(\sqrt{3}+1\right)\tan x=-\sqrt{3}$
12. $\tan x=\sin 2x$
13. $\sin 3x=\cos 2x$
14. $\sqrt{2\sin^2 x+1}=\sin x+\cos x$
15. $\sin^3 x\cos x+\cos^3 x\sin x=\frac{1}{2}$
16. $\sin^3 2x\cos 2x-\cos^3 2x\sin 2x=\frac{\sqrt{2}}{8}$
17. $\sin 3x\cos x+\cos 3x\sin x=\frac{1}{\sqrt{2}}$
18. $\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}=4$
19. $\frac{\tan\left(7x+\frac{\pi}{6}\right)+\tan 3x}{1-\tan\left(7x+\frac{\pi}{6}\right)\tan 3x}=1$
20. $\frac{\cot\left(2x+\frac{\pi}{12}\right)-\cot\left(5x+\frac{\pi}{12}\right)}{1+\cot\left(2x+\frac{\pi}{12}\right)\cot\left(5x+\frac{\pi}{12}\right)}=\frac{1}{\sqrt{3}}$
21. $\frac{\sin^3 x+\cos^3 x}{\sin x+\cos x}-\frac{\sin x}{\sqrt{1+\tan^2 x}}=\frac{1}{2}$
22. $\sin 3x-\sin 5x+\sin 7x=0$
23. $\cos 2x-\cos 6x=\sin 4x$
24. $\sin x+\sin 3x+\sin 5x+\sin 7x=0$

10.2 Graphical Solution of Trigonometric Equation

The graphical solution of trigonometric equations involves plotting the graphs of both sides of the equation and identifying their points of intersection. Each point where the curves intersect represents a solution to the equation.

When dealing with trigonometric equations of the form $f(x) = g(x)$, it can be challenging to find exact solutions. While some equations can be solved algebraically using trigonometric identities and properties, many are too complex for such methods. In these cases, graphical methods serve as a valuable alternative. By plotting both functions $y = f(x)$ and $y = g(x)$ on the coordinate plane, we can visually observe their points of intersection. The x-coordinates of these intersection points are the solutions that satisfy $f(x) = g(x)$. This graphical approach not only helps in determining the number of solutions but also provides deeper insight into the behavior of the functions and the nature of their interactions.

Example 4 Solve the equation $\sin x = \frac{x}{2}$ graphically over the interval $[-\pi, \pi]$

Solution: To find the values of x that satisfy $\sin x = \frac{x}{2}$, we identify the points where the graphs of $y = \sin x$ and $y = \frac{x}{2}$ intersect.

Now, graph of both the functions in the interval $[-\pi, \pi]$ represented as follows:

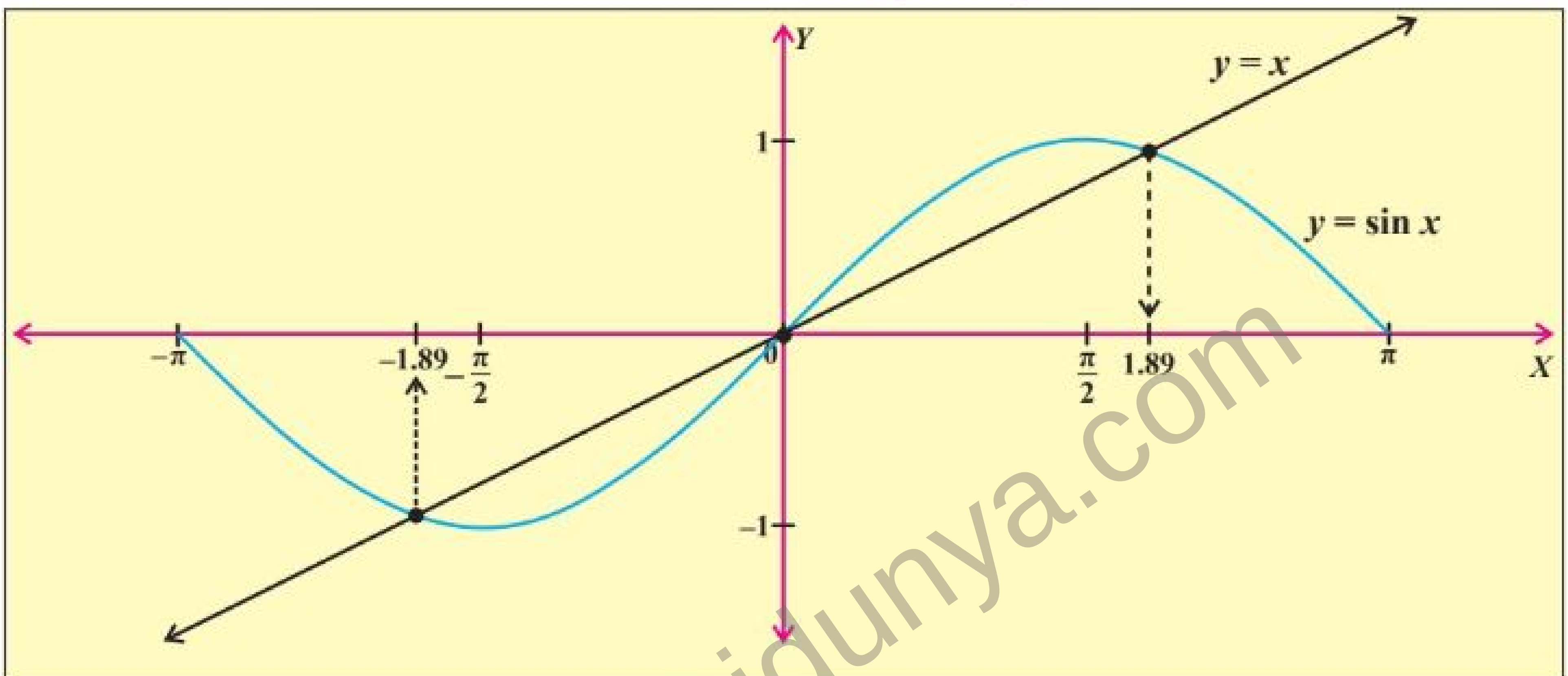

Figure 10.1

From the graph (see Figure 10.1), we observe that the curves intersect at three points within the interval $[-\pi, \pi]$. Thus, the solutions to $\sin x = \frac{x}{2}$ in $[-\pi, \pi]$ are:

$x = -1.89, 0$ or 1.89 radians

Note

The exact values can be refined using numerical methods, but the graphical approach provides a clear visual approximation

Example 5 Solve the equation $\cos x = \frac{x}{8}$ graphically.

Solution: To find the values of x that satisfy $\cos x = \frac{x}{8}$, we identify the points where the graphs of $y = \cos x$ and $y = \frac{x}{8}$ intersect.

First, note that the range of $\cos x$:

$$-1 \le \cos x \le 1 \Rightarrow -1 \le \frac{x}{8} \le 1 \Rightarrow -8 \le x \le 8$$

Next, we plot both functions over the interval $[-8, 8]$

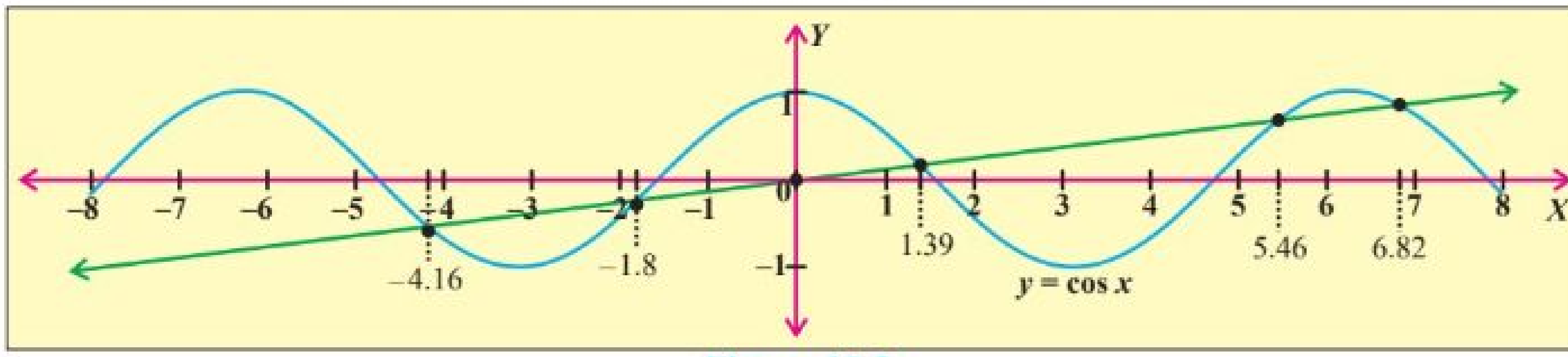

Figure 10.2

From the graph (see Figure 10.2), we observe that the curves intersect at six points within the interval $[-8, 8]$.

Thus, the solutions to $\cos x = \frac{x}{8}$ in $[-\pi, \pi]$ are approximately:

$x = -4.16,\ -1.8,\ 0,\ 1.39,\ 5.46$ and 6.82 radians

Example 6: Solve the equation $\tan x = \frac{x}{2}$ graphically in $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$.

Solution: To find the values of x that satisfy $\tan x = \frac{x}{2}$, we identify the points where the graphs of $y = \tan x$ and $y = \frac{x}{2}$ intersect.

Now, we plot both functions over the interval $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

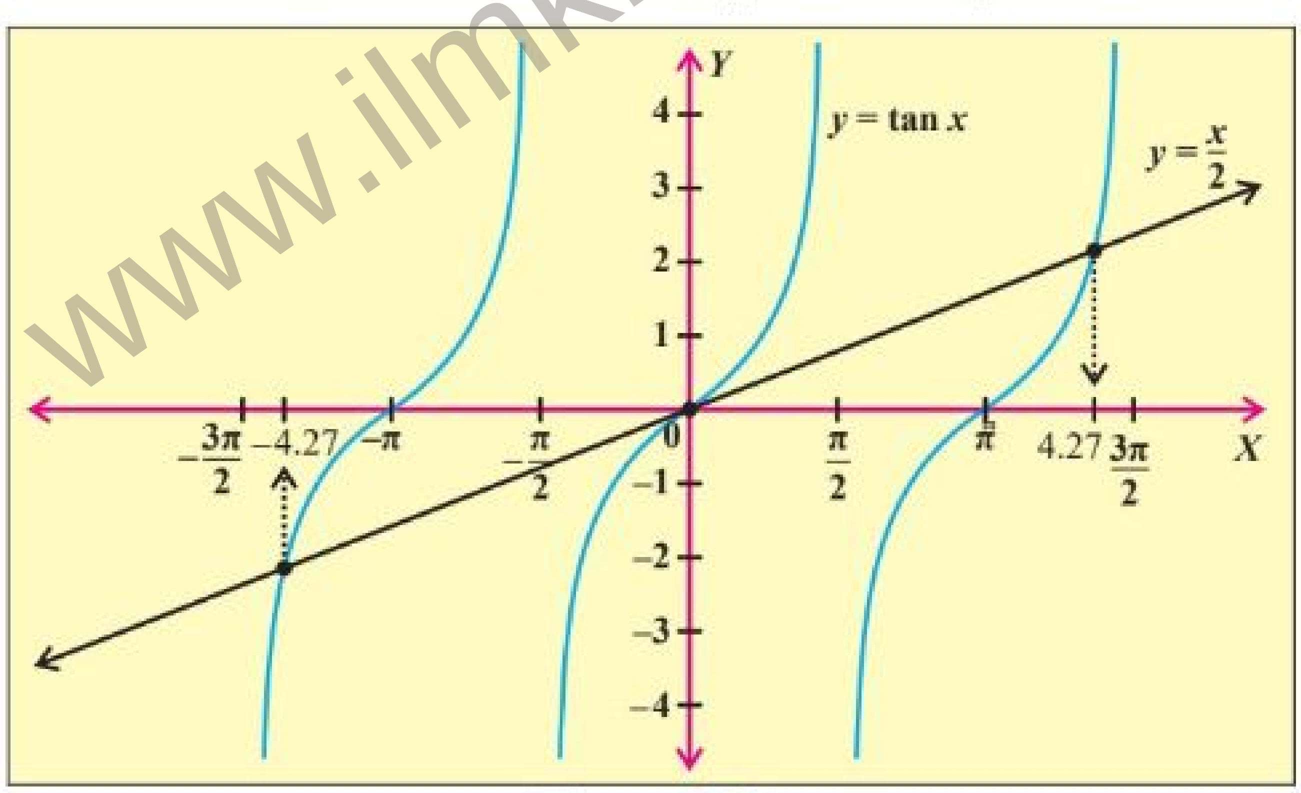

Figure 10.3

From the graph (see figure 3), we observe that the curves intersect at three points within the interval $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$.

Thus, the solutions to $\tan x = \frac{x}{2}$ in $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$ are approximately:

$x = -4.27$, 0 and 4.27 radians

Example 7 Solve the equation $3\cos x = 2\sin x - 1$ graphically in $[-2\pi, 2\pi]$.

Solution: To find the values of x that satisfy $3\cos x = 2\sin x - 1$, we identify the points where the graphs of $y = 3\cos x$ and $y = 2\sin x - 1$ intersect.

Now, we plot both functions over the interval $[-2\pi, 2\pi]$.

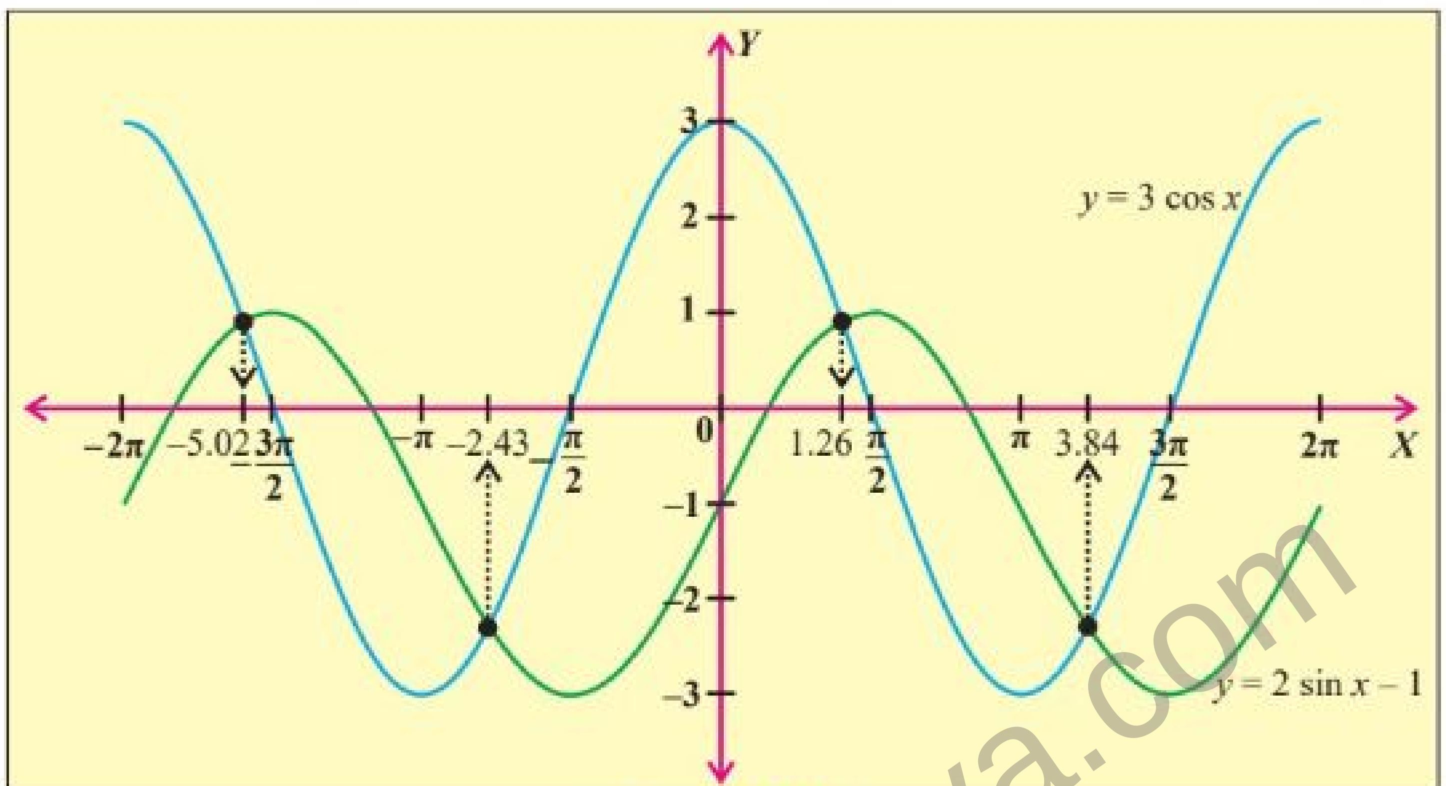

Figure 10.4

From the graph (see Figure 10.4), we observe that the curves intersect at four points within the interval $[-2\pi, 2\pi]$.

Thus, the solutions to $3\cos x = 2\sin x - 1$ in $[-2\pi, 2\pi]$ are approximately:

$x = -5.02, \ -2.43, \ 1.26$ and 3.84 radians.

Example 8 Solve the equation $2\sin^2 x = 1 - 3\cos x$ graphically in $[0, 2\pi]$.

Solution: To find the values of x that satisfy $2\sin^2 x = 1 - 3\cos x$, we identify the points where the graphs of $y = 2\sin^2 x$ and $y = 1 - 3\cos x$ intersect.

Now, we plot both functions over the interval $[0, 2\pi]$.

From the graph (see figure 5), we observe that the curves intersect at four points within the interval $[0, 2\pi]$.

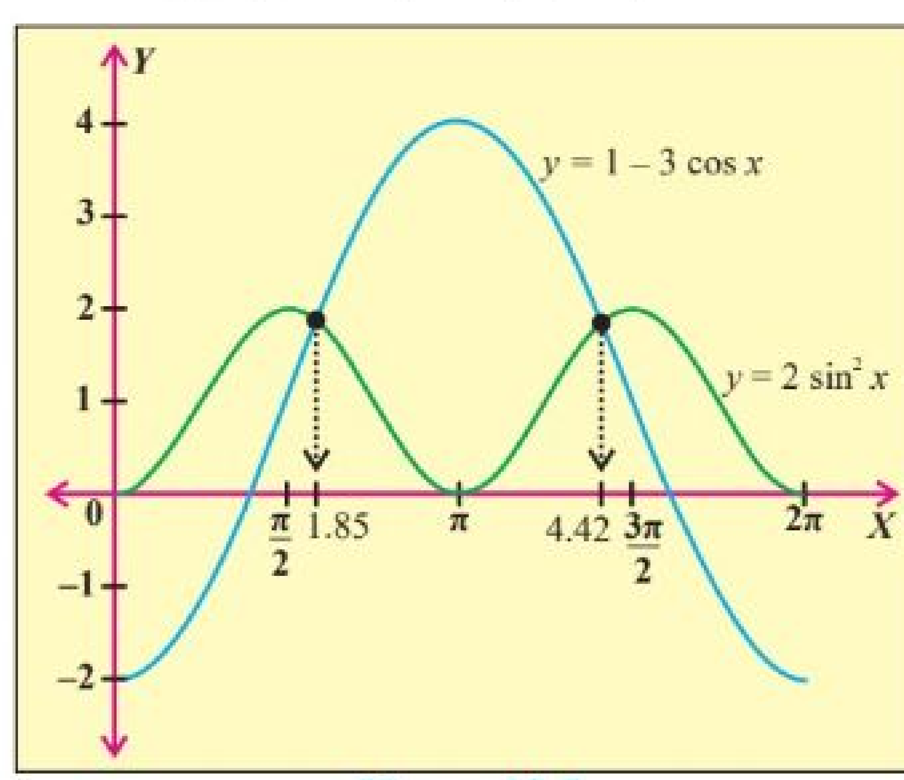

Figure 10.5

Thus, the solutions to $2\sin^2 x = 1 - 3\cos x$ in $[0, 2\pi]$ are approximately:

$$x = 1.855 \text{ and } 4.427 \text{ radians}$$

10.3 Real Life Problems

Example 9 A lighthouse with a 15 meter tall lantern stands atop a cliff. From a ship 120 meters offshore, the lantern subtends the same angle as a 1.5 meter buoy placed at the base of the cliff.

(a) Find the exact height of the cliff.

(b) Calculate the subtended angle (θ).

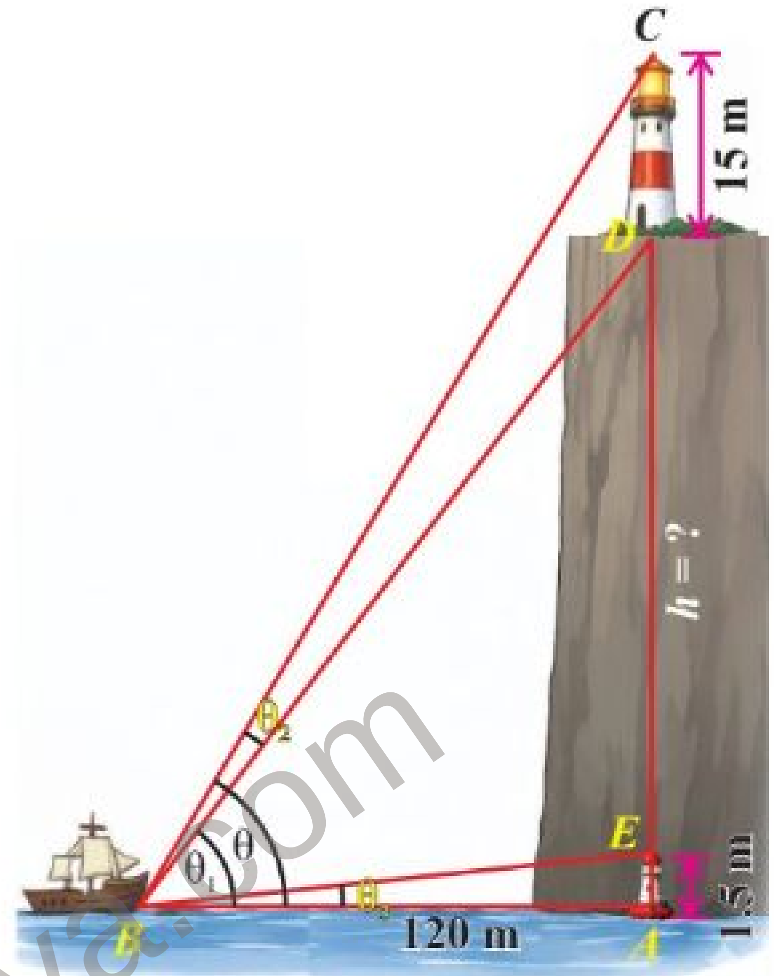

Figure 10.6

Solution: Let h be the height of the cliff then from triangle ABD

$$\tan\theta_1 = \frac{h}{120} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{h}{120}\right)$$

Also, from triangle ABC

$$\tan\theta = \frac{h+15}{120} \Rightarrow \theta = \tan^{-1}\left(\frac{h+15}{120}\right)$$

The subtended angle for the lantern

$$\theta_2 = \theta - \theta_1 = \tan^{-1}\left(\frac{h+15}{120}\right) - \tan^{-1}\left(\frac{h}{120}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{h+15}{120}\right) - \tan^{-1}\left(\frac{h}{120}\right)$$

height of the buoy = 1.5 m, then from triangle ABE

$$\tan\theta_3 = \frac{1.5}{120} \Rightarrow \theta_3 = \tan^{-1}\left(\frac{1.5}{120}\right)$$

given that the lantern subtends the same angle as a 1.5-meter buoy placed at the base of the cliff, that is, $\theta_2 = \theta_3$

$$\tan^{-1}\left(\frac{h+15}{120}\right) - \tan^{-1}\left(\frac{h}{120}\right) = \tan^{-1}\left(\frac{1.5}{120}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{h+15}{120} - \frac{h}{120}}{1 + \frac{h+15}{120}\cdot\frac{h}{120}}\right) = \tan^{-1}\left(\frac{1.5}{120}\right)$$

$$\left(\because \tan^{-1}\alpha - \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha-\beta}{1+\alpha\cdot\beta}\right)\right)$$

$$\Rightarrow \quad \frac{\frac{15}{120}}{\frac{14400+h^2+15h}{14400}}=0.0125$$

$$\Rightarrow \quad \frac{15}{120}\cdot\frac{14400}{14400+h^2+15h}=0.0125$$

$$\Rightarrow \quad h^2+15h-129600=0$$

Using quadratic formula

$$\Rightarrow \quad h=\frac{-15\pm\sqrt{15^2-4(1)(-129600)}}{2(1)}=\frac{-15\pm\sqrt{225+518400}}{2}$$

$$\Rightarrow \quad h=\frac{-15\pm\sqrt{518625}}{2}=\frac{-15\pm 720.2}{2} \quad \Rightarrow \quad h=\frac{705.2}{2}=352.6$$

Thus, height of the cliff is 352.6 m.

(b) The subtended angle for the lantern

$$\theta_2=\tan^{-1}\left(\frac{367.6}{120}\right)-\tan^{-1}\left(\frac{352.6}{120}\right)=71.93^\circ-71.21^\circ=0.72^\circ$$

$$\Rightarrow \quad \theta_2=0.72^\circ$$

Thus, the subtended angle for the lantern is 0.72°.

Example 10 A projectile is launched from the ground with an initial velocity $v_\circ=30\,m/s$. The range R of the projectile is given by the formula $R=\frac{v_\circ^2}{g}\sin 2\theta$, where θ is the launch angle to the horizontal and $g=9.8$ m/s^2 is the acceleration due to gravity.

(a) Find the angle θ (in degrees) required for the projectile to hit a target exactly 80 meters away.

(b) If the maximum range achievable is $R_{max}=\frac{v_\circ^2}{g}$, determine the angle θ that gives the maximum range. Justify your answer mathematically.

Solution: (a) Given that: $v_\circ=30\,m/s$, $R=80\ m$, $g=9.8\,m/s^2$.

Substituting these values in $R=\frac{v_\circ^{2}}{g}\sin 2\theta$, we have

$$80=\frac{(30)^2\sin 2\theta}{9.8} \qquad \Rightarrow \qquad 80\times 9.8=900\sin 2\theta$$

$$\Rightarrow \quad 784 = 900\sin 2\theta \quad \Rightarrow \quad \frac{784}{900} = \sin 2\theta$$

$$\Rightarrow \quad \sin 2\theta = 0.8711 \quad \Rightarrow \quad 2\theta = \sin^{-1}(0.8711)$$

$$\Rightarrow \quad 2\theta = 60.5^\circ \quad \Rightarrow \quad \theta = \frac{60.5^\circ}{2} = 30.25^\circ$$

Since $\sin\theta = \sin(180^\circ - \theta,$, second possible solution in $[0, 2\pi]$ is:

$$\Rightarrow \quad 2\theta = 180^\circ - 60.5^\circ = 119.5^\circ$$

$$\text{or} \quad \theta = \frac{119.5^\circ}{2} = 59.75^\circ$$

Thus, the projectile will hit the ground 80 meters away at angles 30.25° or 59.75°.

(b) The range is maximized when $\sin 2\theta$ is maximized. Since the maximum value of $\sin 2\theta$ is 1, so we have:

$$\sin 2\theta = 1 \quad \Rightarrow \quad 2\theta = \sin^{-1}(1)$$

$$\Rightarrow \quad 2\theta = 90^\circ \quad \Rightarrow \quad \theta = \frac{90^\circ}{2}$$

$$\Rightarrow \quad \theta = 45^\circ$$

Now, the maximum range can be calculated as:

$$R_{max} = \frac{v_\circ^2}{g} = \frac{900}{9.8} = 91.84\,m$$

Thus, the maximum range is achieved at 45°, as sin 2θ reaches its maximum value of 1 at this angle, and no other angle yields a greater range.

EXERCISE 10.2

Solve graphically:

1. $3\sin x = \frac{x}{2} + 1, \; x \in [-\pi, \pi]$
2. $\cos(2x-1) = \frac{x}{5}, \; x \in [0, \pi]$
3. $3\tan x = 2x + 1, \; x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
4. $\sec(x+3) = -2x + \frac{1}{3}, \; x \in [-\pi, \pi]$
5. $2\cos x = x^2 + 2x - 1, \; x \in [-\pi, \pi]$
6. $\sin 3x = \frac{x}{2}$
7. $\sin 2x = 2\cos 3x, \; x \in [0, \pi]$
8. $\sin x + \cos x = -2\cos x, \; x \in [0, \pi]$
9. A hiker pulls a sled using a rope that exerts a force of 80 newtons over a distance of 25 meters. The total work done is 1200 joules. At what angle to the direction of motion is the force being applied?

10. A radio tower stands on flat ground with a small antenna mounted at its top. From a point 50 meters away from the base of the tower. The antenna subtends the same angle as a 1 meter vertical rod placed at the base of the tower. The total height of the tower and antenna is 10 meters.
    (a) Find the height of the tower (without the antenna).
    (b) Calculate the subtended angle.
11. A person standing 20 meters away from the base of a tree measures the angle of elevation to the top of the tree as 30°.
    (a) Find the height of the tree.
    (b) If the person walks 10 meters closer to the tree, what will be the new angle of elevation to the top of the tree?