بِسۡمِ اللّٰهِ الرَّحۡمٰنِ الرَّحِيۡمِ

(In the Name of Allah, the Most Merciful, the Most Compassionate.)

MATHEMATICS

12

PECTAA

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CONTENTS

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Unit 01 Graphical Representation of Functions

INTRODUCTION

This unit introduces students to the fundamental ideas of graphical representation of functions, enabling them to visualize and interpret mathematical relationships effectively. Students begin by learning how to draw graphs using factor form and how to predict equations from given graphs, particularly for quadratic functions. The unit further develops understanding by classifying functions into algebraic and transcendental types, followed by a detailed study of important transcendental functions such as trigonometric, inverse trigonometric, hyperbolic, logarithmic, and exponential functions. Emphasis is placed on understanding logarithms, their laws, and their applications in real-life contexts like growth, decay, and sound intensity. Students also explore exponential functions and their applications, including compound interest, and learn to solve related equations and inequalities. The graphs of all said functions will be discussed. In particular, graphical behaviors of exponential and logarithmic functions will be studied in details. In addition, the unit covers graph of inverse functions, and graphical transformations including shifts and scaling. Overall, this unit equips students with both analytical and graphical skills necessary for higher-level mathematics and practical problem solving.

1.1 Graph of Quadratic Function Using Factors

We have already sketched the graph of the quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ in Grade-XI. We know that its graph is a parabola, which has a vertex and opens either upward or downward depending on the value of a.

Now, we will learn how to sketch its graph when it is expressed in factorized form as a product of linear factors. This approach will also help you in higher classes to sketch the graphs of higher-degree polynomials that can be factorized into linear factors.

Method to Sketch the Graph

Suppose, after factorizing $y = f(x) = ax^2 + bx + c$, we obtain $y = a(x-h)(x-k)$, where $h \leq k$. To sketch the graph, follow the given steps:

Case-I: If $h < k$

Step-1: The graph will cut the x-axis at $x = h$ and $x = k$.

Step-2: Find value of y at $x = \dfrac{h+k}{2}$.

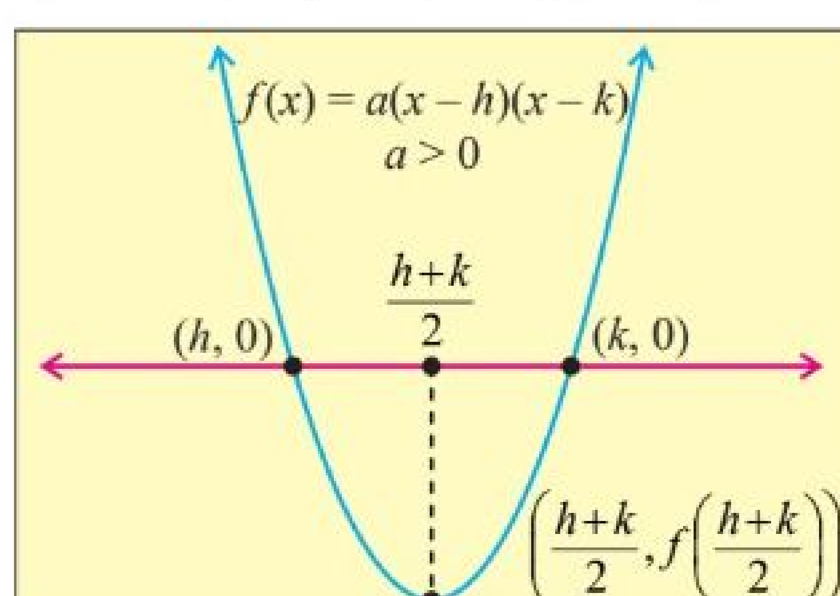


Figure 1.1

The point $\left(\frac{h+k}{2}, f\left(\frac{h+k}{2}\right)\right)$ represents the vertex of the parabola (see Figure 1.1).

Step-3: Determine the behaviour of y as $x \to \pm\infty$.

- If $a > 0$, then $y \to +\infty$.
- If $a < 0$, then $y \to -\infty$.

This statement means the graph behaves the **same way on both ends** (left and right).

Challenge!

If $y = a(x - h_1)(x - h_2)(x - h_3)$, then what will be the behaviour of y as $x \to -\infty$ and $x \to +\infty$

Step-4: Find the y-intercept:

Put $x = 0$ in the function $y = a(x - h)(x - k)$, the graph will cut the y-axis at $y = ahk$.

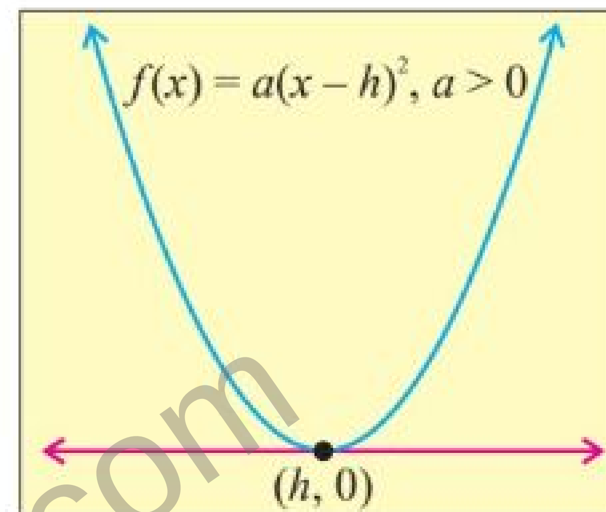


Figure 1.2

Case-II: If $h = k$, then the function becomes $y = a(x - h)^2$.

To sketch the graph, follow these steps:

Step-1: The graph **touches the x-axis** at $x = h$, therefore $(h, 0)$ is the vertex (see Figure 1.2).

Step-2: Then follow Steps 3 and 4 of Case-I to complete the sketch of the graph.

Note: Following Steps 1, 3, and 4, and by taking some points between the two x-intercepts, we can draw the graph of a polynomial of higher degree that can be factorized into linear factors.

Recall!

$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a$ $_1x + a_n$ is a polynomial of degree n, whe n is a non-negative integer and $a_0 \neq 0$.

Example 1 Sketch the graph of $y = 2(x - 1)(x - 3)$.

Solution: To find the x-intercepts, set each factor equal to zero:

$$x - 1 = 0 \text{ and } x - 3 = 0$$
$$x = 1 \text{ and } x = 3$$

Hence, the graph meets the x-axis at the points (1, 0) and (3, 0).

To find y component of the vertex, put $x = \frac{1+3}{2} = 2$ into the equation:

$$y = 2(2 - 1)(2 - 3) = -2$$

So, the vertex of the graph is (2, –2).

Since $a = 2 > 0$, so $y \to +\infty$ as $x \to -\infty$ or $x \to +\infty$.

The graph cuts the y-axis at $x = 0$ and $y = 2(-1)(-3) = 6$

So, the y-intercept is (0, 6).

Using the intercepts and the vertex, we can sketch the graph of the parabola as shown in Figure 1.3.

Figure 1.3

Example 2 Sketch the graph of $y = -4x^2 - 8x - 4$.

Solution: First, we factorize $y = -4x^2 - 8x - 4$.

$$y = -4(x^2 + 2x + 1)$$

$y = -4(x + 1)^2$ To find the x-intercepts, set $x + 1$ equal to zero:

$$x + 1 = 0 \quad \Rightarrow \quad x = -1$$

Hence, the graph meets the x-axis at the point $(-1, 0)$.

Since the function is of the form $a(x - h)^2$, that is, $y = -4(x + 1)^2$, so $(-1, 0)$ is its vertex.

Since, $a = -4 < 0$, so $y \to -\infty$ as $x \to -\infty$ or $x \to +\infty$

The graph cuts the y-axis at $x = 0$:

$$y = -4(0 + 1)^2 = -4$$

So, the y-intercept is $(0, -4)$.

The sketch of the graph is shown in Figure 1.4.

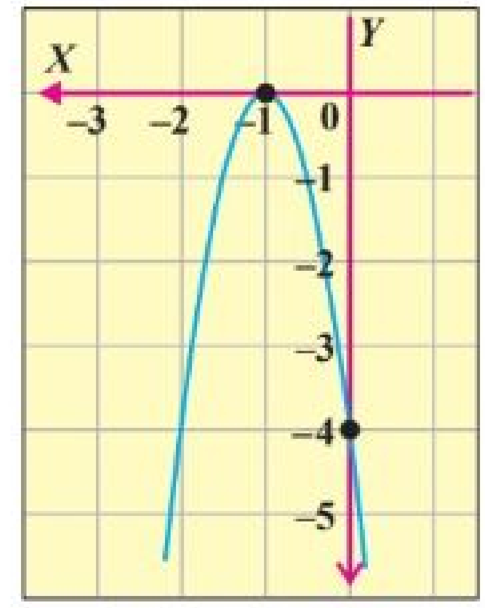


Figure 1.4

1.2 Predicting a Quadratic Function from its Graph

A quadratic function $f(x) = ax^2 + bx + c$ can be determined from its graph by using the **factor form** if the x-intercepts and one additional point on the curve are known.

If the graph crosses the x-axis at $x = h$ and $x = k$, then the function can be written as:

$$f(x) = a(x - h)(x - k)$$

Here, a is a constant that determines the shape and direction of the parabola. To find a, substitute the coordinates of a third known point into the equation.

Example 3 Find the function whose graph is shown in Figure 1.5.

Solution: From figure it is clear that x-intercepts of the graph are $h = -1$ and $k = 1$.

Therefore, the function can be written as:

$$y = a(x - (-1))(x - 1) = a(x + 1)(x - 1) \quad \ldots(1)$$

To find the value of a, substitute the point $(0, 4)$ into equation (1):

$$4 = a(0 + 1)(0 - 1)$$

$$4 = -a \quad \Rightarrow \quad a = -4$$

Substituting the value of a into the equation (1):

$$y = -4(x + 1)(x - 1)$$

$$= -4(x^2 - 1)$$

$$y = -4x^2 + 4$$

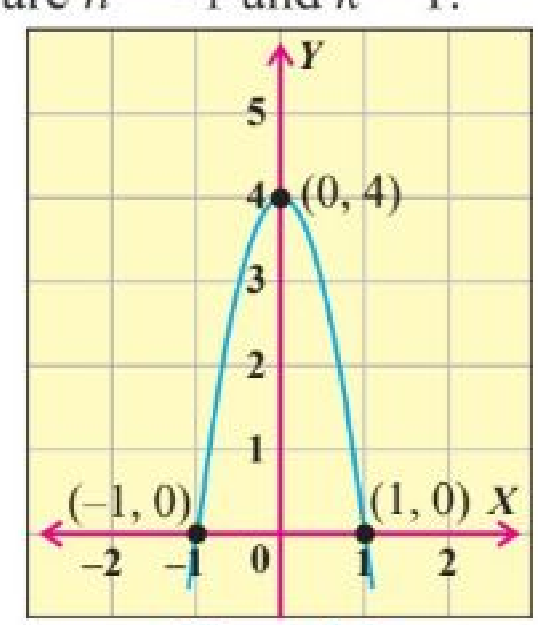


Figure 1.5

Example 4 Find an equation of the graph whose x-intercepts are $x = 1$ and $x = 2$ and passes through the point(3, 4).

Solution: Since the x-intercepts are $x = 1$ and $x = 2$, the function can be written in factorized form as:

$$y = a(x - 1)(x - 2) \quad \ldots(1)$$

To find a, use the point (3, 4):

$$4 = a(3-1)(3-2) = 2a \quad \Rightarrow \quad a = \frac{4}{2} = 2$$

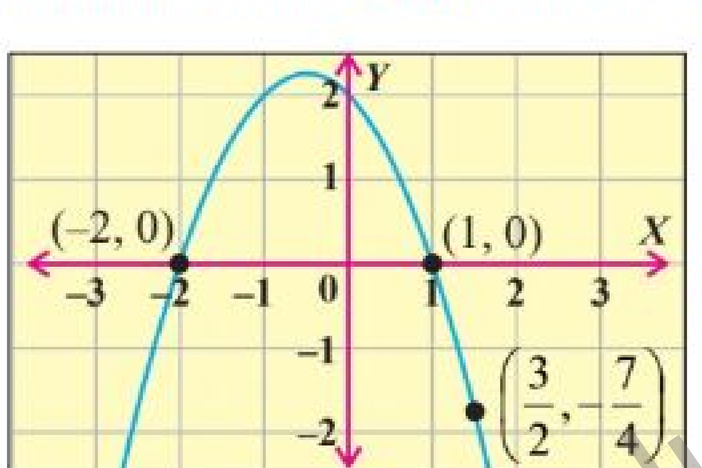
Substituting the value of a into the equation (1):

$$y = 2(x-1)(x-2)$$
$$y = 2(x^2 - 3x + 2) = 2x^2 - 6x + 4$$

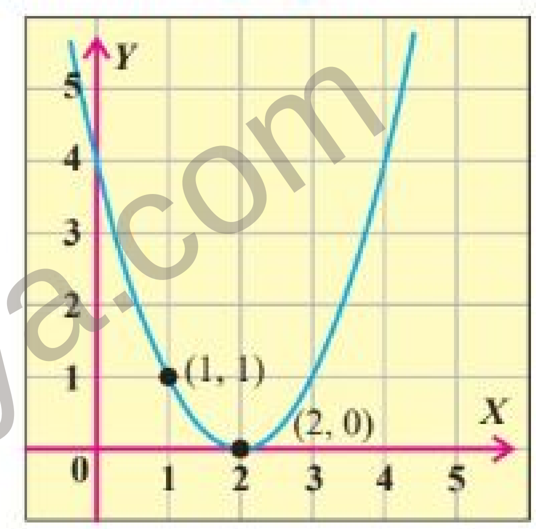
EXERCISE 1.1

1. Sketch the graph of the following functions using the method of factorization.
   (i) $y = 3(x+1)(x-1)$ (ii) $f(x) = -2(x-1)(x-2)$
   (iii) $y = (2x-1)(2x+1)$ (iv) $y = 2x^2 + x - 3$
   (v) $f(x) = x^2 + 2x + 1$ (vi) $y = 3(x-1)^2$
2. Find the functions whose graphs are shown in the following figures.

(i)



(ii)



3. Find an equation of the graph whose x-intercepts are $x = -1$ and $x = -2$ and which passes through the point $(-3, 2)$.
4. Find an equation of the graph which touches x-axis at $x = -1$ and cuts y-axis at $y = -2$.
5. Find an equation of the function whose graph cuts x-axis at $x = -1$ and $x = 1$, and whose vertex is at $(0, -2)$.

1.3 Algebraic and Transcendental Functions

Algebraic and transcendental functions are two fundamental classes of functions studied in mathematics. Together, they play a vital role in modeling and understanding a wide range of real-world phenomena. In this section, we focus on the fundamental transcendental functions, namely exponential and logarithmic functions.

(i) Algebraic Functions

A function $y = f(x)$ is **algebraic** if it satisfies a polynomial equation of the form

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \cdots + P_1(x)y + P_0(x) = 0$$

where each $P_i(x)$ is a polynomial in x for $i = 0, 1, 2, \cdots, n$ and atleast one $P_i(x)$ is not identically zero for $i = 1, 2, 3, \ldots, n$.

In simple words, a function $y = f(x)$ is algebraic if it is obtained from x using a finite number of algebraic operations such as addition, subtraction, multiplication, division, and n^{th} roots.

$f(x) = 2x^2 - 3x$, $g(x) = \dfrac{x^2+1}{x-3}$, $x \neq 3$, $h(x) = (x+1)^{\frac{3}{2}}$ and $k(x) = \sqrt{x+1}$ are examples of algebraic functions.

(ii) Transcendental Functions

A function is called a **transcendental function** if it is not algebraic.

$y = e^x$, $y = 2^x$, $y = \ln x$, $y = \sin x$, $y = \sin^{-1} x$, $y = \sin x + \ln x$, $y = e^{x^2}$ are examples of transcendental functions.

1.3.1 Fundamental Transcendental Functions

Fundamental transcendental functions are the commonly used standard transcendental functions like $y = a^x$ $(a > 0, a \neq 1)$, $y = \ln x$, $y = \sin x$, $y = \sin^{-1} x$, which form the base for more complex transcendental functions.

1.3.2 Non-Fundamental Transcendental Functions

Functions formed by combining transcendental functions with algebraic functions, or by applying transformations and composition, are non-fundamental transcendental functions.

$y = \sin x + \ln x$, $y = e^{x^2}$, $y = x \sin x$, $y = \ln 2x$ are examples of functions which are not fundamental transcendental functions. On the other hand, $y = 10^{2x}$ is a standard transcental function, because, it can be written as $y = 100^x$.

Example 5 Given the following functions:

$$f(x) = \sqrt{x+1}, g(x) = e^x, h(x) = \frac{x+1}{x^2-2x+1}, \; q(x) = \ln 2x, r(x) = \sin x$$

Classify each function as algebraic or transcendental.

Solution: The following are algebraic functions:

$$f(x) = \sqrt{x+1}, \; h(x) = \frac{x+1}{x^2-2x+1}$$

The following are transcendental functions:

$$g(x) = e^x, q(x) = \ln 2x, r(x) = \sin x$$

Example 6 Given the following functions:

$$f(x) = \cos 2x + 1, g(x) = 3^x, h(x) = \tan^{-1}x^2, q(x) = \ln x + e^x, r(x) = \tan x$$

Classify each function as fundamental or non-fundamental transcendental functions.

Solution: The following are fundamental transcendental functions:

$$g(x) = 3^x, r(x) = \tan x$$

The following are non-fundamental transcendental functions:

$f(x) = \cos 2x + 1$, $h(x) = \tan^{-1}(x^2)$, $q(x) = \ln x + e^x$

1.4 Exponential Functions

The function defined by $f(x) = a^x$ for all $x \in \mathbb{R}$, where $a > 0$ and $a \neq 1$ is called an **exponential function**. The number a is called the **base** of the exponential function.

The function $f(x) = e^x$ is called the **natural exponential function**, where e is a number defined by $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ and its approximate value is $e \approx 2.71828$.

The exponential functions $f(x) = e^x$ and $g(x) = 2^x$ represent exponential growth, while the function $f(x) = \left(\frac{1}{2}\right)^x$ represents exponential decay.

1.4.1 Graph of Exponential Function

Before sketching the graph of an exponential function, we study its essential properties that determine its overall shape and behaviour. These include the domain and range, intercepts, injectivity, monotonicity, and asymptotic behaviour. Understanding these features help us to predict the graph without plotting many points. After this analysis, we use a few representative values to draw a smooth and accurate sketch of the function.

1.4.2 Properties of the Exponential Function

Consider the exponential function:

$$y = a^x \, (a > 0, a \neq 1) \quad \ldots(1)$$

(i) Domain, Continuity, Positivity and Range of Exponential Function

Domain: The exponential function is defined for all real values of x. We can substitute any real number (positive, negative, or zero) in place of x, and the expression a^x is always meaningful. Hence, the domain of the exponential function is $]-\infty, \infty[$.

Continuity: Continuity at a point $c \in]-\infty, \infty[$ is shown using the equation:

$$\lim_{x \to c} a^x = a^c$$

This shows that the exponential function is continuous on the whole domain $]-\infty, \infty[$. Thus, graph of exponential function has no jump or hole.

Positivity: If $x = n$, a positive integer, then $a^n = a \cdot a \cdot \ldots \cdot a$ (n times).

Since, $a > 0$, product of positive numbers is positive. Thus, $a^n > 0$ also, $a^{-n} = \frac{1}{a^n}$.

Since, $a^n > 0$, its reciprocal is also positive. So, $a^{-n} > 0$.

Moreover, $a^0 = 1 > 0$.

If x is rational, then $x = \frac{p}{q}$, where p is an integer and q is a natural number. Thus:

$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

From integer case, $a^p > 0$. Since, the q^{th} root of a positive number is also positive, we have $a^{\frac{p}{q}} > 0$.

Thus, $a^x > 0$ for all rational numbers x. For irrational values of x, we accept that a^x is also positive. However, a rigorous proof of this fact is beyond the scope of this level.

Hence, $y = a^x > 0$ for all real numbers x. It follows that the graph of exponential function always lies above the x-axis.

Range: By observing the following facts:

1. $\lim_{x\to\infty} a^x = \infty$ and $\lim_{x\to-\infty} a^x = 0^+$ if $a > 1$.
2. $\lim_{x\to-\infty} a^x = \infty$ and $\lim_{x\to\infty} a^x = 0^+$ if $0 < a < 1$.
3. $y = a^x$ is continuous on $]-\infty, \infty[$.
4. $y = a^x > 0$ for all real numbers x.

We conclude that, the range of the exponential function (1) is $]0, \infty[$.

(ii) Intercepts of Exponential Function

***y*-intercept:** To find y-intercept, put $x = 0$ in equation (1):

$$y = a^0 = 1$$

So, the exponential function $y = a^x$ cuts the y-axis at the point (0, 1).

***x*-intercept:** To find x-intercept, put $y = 0$ in equation (1):

$$a^x = 0$$

But, $a^x > 0$ for all real x. It follows that there is no x-intercept.

Strictly Increasing and Strictly Decreasing Functions:

Let $y = f(x)$ be a real-valued function and let $x_1, x_2 \in D_f$ be any two points such that $x_1 < x_2$. The function f is called:

(i) **Strictly increasing** if $f(x_1) < f(x_2)$. In this case, the graph of the function rises as x moves from left to right along the x-axis.

(ii) **Strictly decreasing** if $f(x_1) > f(x_2)$. In this case, the graph of the function falls as x moves from left to right along the x-axis.

For Example: Consider the function $y = x^2$.

This function is strictly decreasing on $]-\infty, 0[$ and strictly increasing on $]0, \infty[$.

(iii) Exponential Function is Strictly Increasing or Strictly Decreasing

Let $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

Case-1: If $a > 1$

Let $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$. Then

$$a^{x_2} = a^{x_1} \cdot a^{x_2 - x_1}$$

Since $x_2 - x_1 > 0$ and $a > 1$, we have $a^{x_2 - x_1} > 1$. So:

$$a^{x_2} = a^{x_1} \cdot a^{x_2 - x_1} \Rightarrow a^{x_2} > a^{x_1}$$

Hence, $x_1 < x_2 \Rightarrow a^{x_1} < a^{x_2}$

Therefore, $y = a^x$ is **strictly increasing** when $a > 1$.

Case-2: If $0 < a < 1$

Let $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$. Then

$$a^{x_2} = a^{x_1} \cdot a^{x_2 - x_1}$$

Since $0 < a < 1$ and $x_2 - x_1 > 0$, we have $a^{x_2 - x_1} < 1$. So:

$$a^{x_2} = a^{x_1} \cdot a^{x_2 - x_1} \Rightarrow a^{x_2} < a^{x_1}$$

Hence, $x_1 < x_2 \Rightarrow a^{x_1} > a^{x_2}$

Therefore, $y = a^x$ is **strictly decreasing** when $0 < a < 1$.

(iv) Injectivity of Exponential Functions

Let $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

Further let, $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \,]-\infty, \infty[$.

So: $a^{x_1} = a^{x_2}$

Case-1: If $x_1 < x_2$, then from the function of $f(x) = a^x$ either

$a^{x_1} < a^{x_2}$ (when $a > 1$) or $a^{x_1} > a^{x_2}$ (when $0 < a < 1$).

Which is impossible.

Case-2: If $x_1 > x_2$, then from the function of $f(x) = a^x$ either $a^{x_1} > a^{x_2}$ (when $a > 1$) or $a^{x_1} < a^{x_2}$ (when $0 < a < 1$). Which is again impossible.

> **Definition of Injectivity (One-to-One Function)**
> Recall that, a function $y = f(x)$ is called **injective (one-to-one)** if different inputs always give different outputs. Formally, a function is one-to-one if:
> whenever for $x_1, x_2 \in$ Dom f such that $f(x_1) = f(x_2)$, we have $x_1 = x_2$.
> A function is injective if every horizontal line meets its graph at most once; equivalently, its graph is strictly increasing or strictly decreasing.

Hence, we must have $x_1 = x_2$ and consequently, $f(x) = a^x$ is injective. It follows that any horizontal line meets the graph of $f(x) = a^x$ at most once.

(v) Horizontal Asymptote of Exponential Function

Note that if $a > 1$, then $a^x \to 0^+$ as $x \to -\infty$ and if $0 < a < 1$, then $a^x \to 0^+$ as $x \to \infty$. It follows that the horizontal line $y = 0$ represents the horizontal asymptote for the exponential function.

> **Note: (Horizontal Asymptote)**
> A **horizontal asymptote** is a horizontal line $y = c$ which a function approaches as x becomes very large (or very small), but the graph does not necessarily touch or cross it.
> Formally the vertical line $y = c$ is called **horizontal asymptote** for the function $y = f(x)$ if $f(x) \to c$ as $x \to -\infty$ or $x \to \infty$.

Example 7 **Graph of $f(x) = e^x$**

Solution: Let $f(x) = e^x$.

- Domain of the function is $R = (-\infty, \infty)$.
- Range of $f(x) = e^x$ is $(0, \infty)$.
- The graph cuts y-axis at $(0, 1)$.
- $y = 0$ is the horizontal asymptote.
- f is continuous and strictly increasing, because $e > 1$.

Some approximate functional values are given in the following tables.

x	$f(x) = e^x$
1	$e = 2.718$
2	7.389
3	20.085
10	22026.465
100	2.688×10^{43}
$\to +\infty$	$\to +\infty$

x	$f(x) = e^x$
–1	0.367
–2	0.135
–3	0.0497
–10	0.0000454
–100	3.720×10^{-44}
$\to -\infty$	$\to 0$

With all above key points, the graph of $f(x) = e^x$ is shown in Figure 1.6.

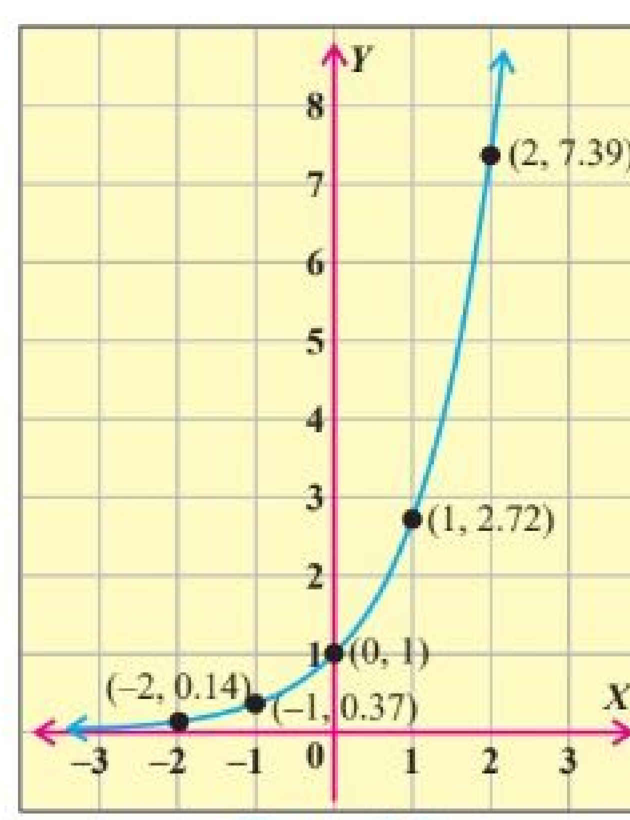


Figure 1.6

Note that the graph of all exponential functions $f(x) = a^x$, $a \in R$ and $a > 1$ is almost similar to the graph in Figure 1.6.

Example 8: Graphs of $f(x) = \left(\frac{1}{2}\right)^x = \frac{1}{2^x}$

Solution: The graph of the exponential function $f(x) = \left(\frac{1}{2}\right)^x$ has following important properties:

- Domain of the function is $R = (-\infty, \infty)$.
- Range of $f(x) = \left(\frac{1}{2}\right)^x$ is $(0, \infty)$.
- The graph cuts y-axis at $(0, 1)$.
- $y = 0$ is the horizontal asymptote.
- f is continuous and strictly decreasing because $a = \frac{1}{2} < 1$.

Some approximate functional values are given in the following tables.

x	$f(x) = \left(\frac{1}{2}\right)^x$
1	0.5
2	0.25
3	0.125
10	0.0009766
100	7.89×10^{-31}
$\to +\infty$	$\to 0$

x	$f(x) = \left(\frac{1}{2}\right)^x$
−1	2
−2	4
−3	8
−10	1024
−100	1.267×10^{30}
$\to -\infty$	$\to \infty$

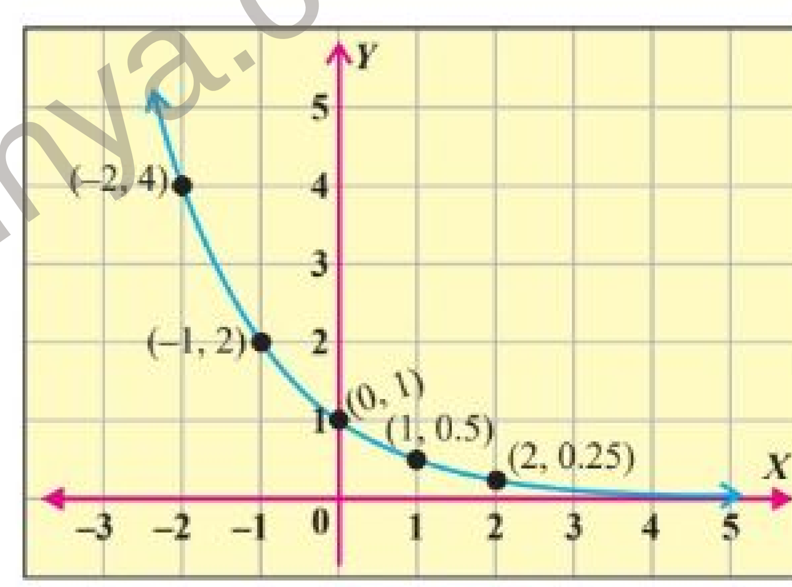


Figure 1.7

With all the above key points, the graph of $f(x) = \left(\frac{1}{2}\right)^x$ is shown in Figure 1.7.

Note that the graph of all exponential functions $f(x) = a^x$, $a \in R$ and $0 < a < 1$ is almost similar to the graph in Figure 1.7.

1.5 Introduction to Hyperbolic and Inverse Hyperbolic Functions

Hyperbolic functions are a class of transcendental functions closely related to exponential functions. They are defined using combinations of e^x and e^{-x}, and their graphs resemble trigonometric functions, although they arise from hyperbolas rather than circles. The basic hyperbolic functions include $\sinh x$, $\cosh x$ and $\tanh x$ which have important applications in mathematics, physics and engineering.

Inverse hyperbolic functions are the inverse functions of hyperbolic functions, defined on suitable restricted domains. They are useful in solving equations involving hyperbolic functions and appear in various applications such as integration, differential equations, and modeling real-world phenomena.

1.5.1 Hyperbolic Functions

The hyperboli c functions are defined as follows:

Function	Definition	Domain	Range
$\sinh x$	$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$R = (-\infty, \infty)$	$R = (-\infty, \infty)$
$\cosh x$	$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$R = (-\infty, \infty)$	$[1, \infty)$
$\tanh x = \dfrac{\sinh x}{\cosh x}$	$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	$R = (-\infty, \infty)$	$(-1, 1)$
$\coth x = \dfrac{\cosh x}{\sinh x}$	$\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$	$R\backslash\{0\}$	$(-\infty, -1) \cup (1, \infty)$
$\text{sech}\, x = \dfrac{1}{\cosh x}$	$\text{sech}\, x = \dfrac{2}{e^x + e^{-x}}$	$R = (-\infty, \infty)$	$(0, 1]$
$\text{csch}\, x = \dfrac{1}{\sinh x}$	$\text{csch}\, x = \dfrac{2}{e^x - e^{-x}}$	$R\backslash\{0\}$	$(-\infty, 0) \cup (0, \infty)$

Hyperbolic functions are so named because the parametric equations $x = \cosh t$ and $y = \sinh t$, satisfies the equation of the hyperbola $x^2 - y^2 = 1$.

Example 9 Prove that $\cosh^2 x - \sinh^2 x = 1$.

Solution: We know $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$

Therefore, $\cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2$

$$= \left(\frac{e^{2x} + e^{-2x} + 2}{4}\right) - \left(\frac{e^{2x} + e^{-2x} - 2}{4}\right)$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4} = \frac{4}{4} = 1$$

Hence, $\cosh^2 x - \sinh^2 x = 1$

1.5.2 Graphs of Hyperbolic Functions

(i) Graph of sinh x

We know that $\sinh x = \dfrac{e^x - e^{-x}}{2}$

This can be written as $\sinh x = \dfrac{e^x - \dfrac{1}{e^x}}{2} = \dfrac{e^x - \left(\dfrac{1}{e}\right)^x}{2}$

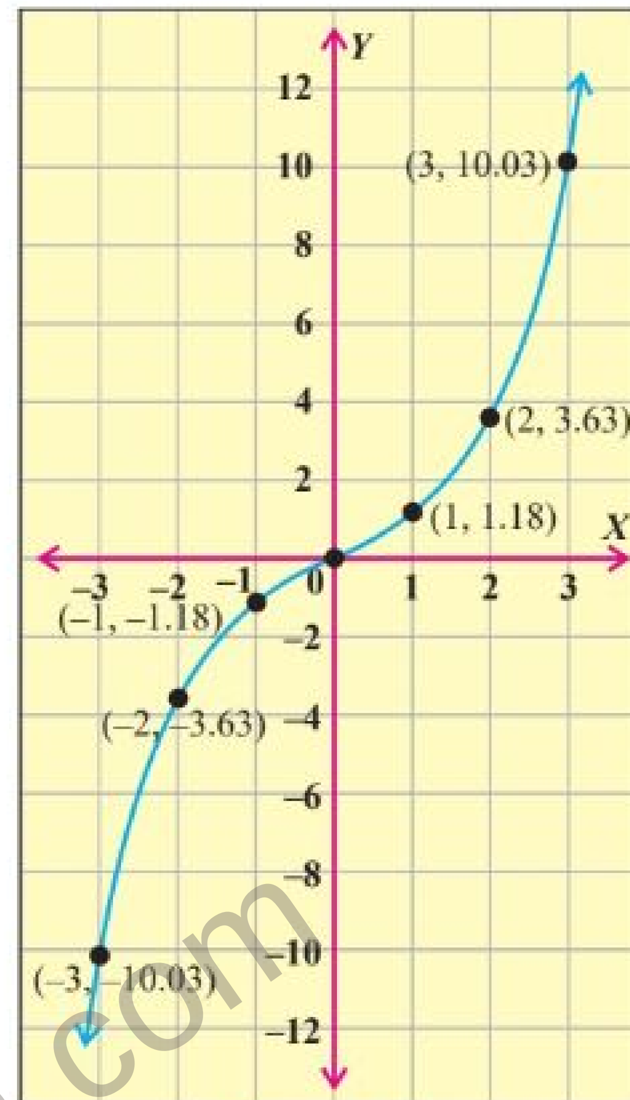


Figure 1.8

Since, domain of e^x and $\left(\dfrac{1}{e}\right)^x$ is R, therefore, the domain of $\sinh x$ is $R = (-\infty, \infty)$.

As $x \to \infty$, $e^x \to \infty$ and $\left(\dfrac{1}{e}\right)^x \to 0$.

Hence, $\sinh x = \dfrac{e^x - \left(\dfrac{1}{e}\right)^x}{2} \to \infty$

As $x \to -\infty$, $e^x \to 0$ and $\left(\dfrac{1}{e}\right)^x \to \infty$.

Hence, $\sinh x = \dfrac{e^x - \left(\dfrac{1}{e}\right)^x}{2} \to -\infty$

Therefore, the range of $\sinh x$ is $R = (-\infty, \infty)$.

At $x = 0$, $\sinh 0 = \dfrac{e^0 - e^{-0}}{2} = 0$, so the graph of $\sinh x$ passes through the point (0, 0).

x	$-\infty \leftarrow$	-3	-2	-1	0	1	2	3	$\to \infty$
$y = \sinh x$	$-\infty \leftarrow$	-10.03	-3.63	-1.18	0	1.18	3.63	10.03	$\to \infty$

The graph of $\sinh x$ is shown in Figure 1.8.

(ii) Graph of cosh *x*

We know that $\cosh x = \dfrac{e^x + e^{-x}}{2}$

This can be written as $\cosh x = \dfrac{e^x + \dfrac{1}{e^x}}{2} = \dfrac{e^x + \left(\dfrac{1}{e}\right)^x}{2}$.

Since domain of e^x and $\left(\dfrac{1}{e}\right)^x$ is R, therefore, the domain of $\cosh x$ is $R = (-\infty, \infty)$.

As $x \to \infty$, $e^x \to \infty$ and $\left(\dfrac{1}{e}\right)^x \to 0$.

Hence, $\cosh x = \dfrac{e^x + \left(\dfrac{1}{e}\right)^x}{2} \to \infty$

As $x \to -\infty$, $e^x \to 0$ and $\left(\dfrac{1}{e}\right)^x \to \infty$.

Hence, $\cosh x = \dfrac{e^x + \left(\dfrac{1}{e}\right)^x}{2} \to \infty$

At $x = 0$, $\cosh 0 = \dfrac{e^0 + e^{-0}}{2} = 1$, so the graph of $\cosh x$ passes through the point (0, 1) and $y = 1$ is minimum value of the graph.

Therefore, the range of $\cosh x$ is $[1, \infty)$. To sketch the graph, we take a table of values:

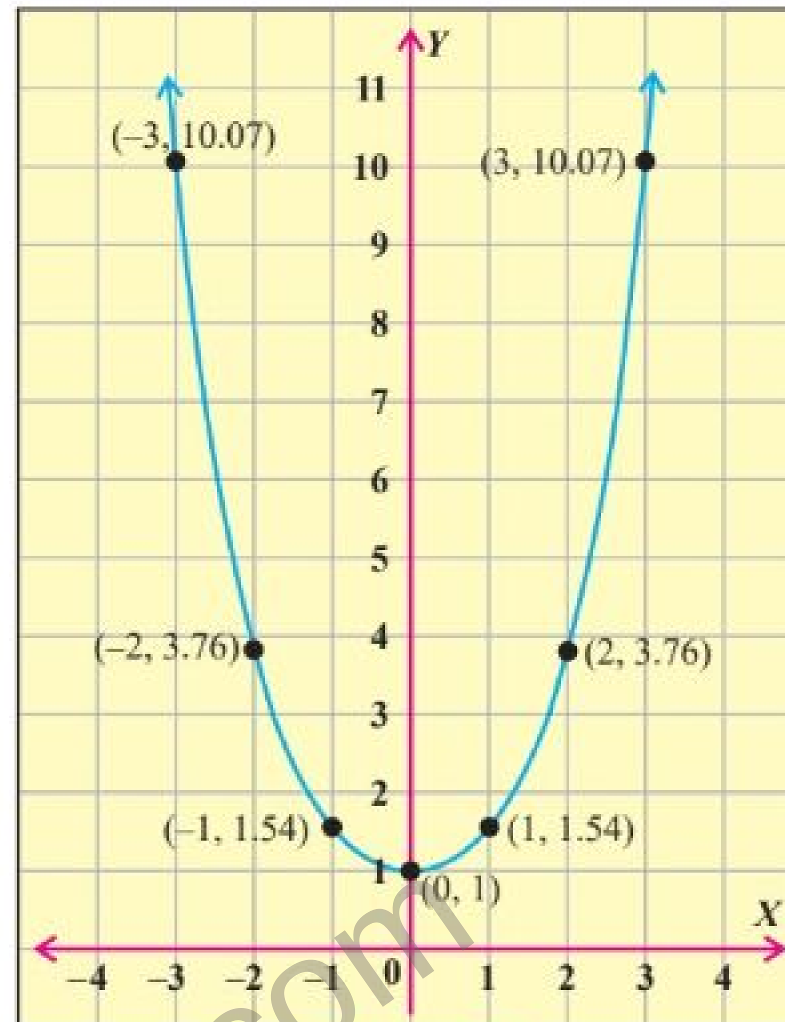


Figure 1.9

x	$-\infty \leftarrow$	−3	−2	−1	0	1	2	3	$\to \infty$
$y = \cosh x$	$+\infty \leftarrow$	10.07	3.76	1.54	1	1.54	3.76	10.07	$\to \infty$

The graph of $\cosh x$ is shown in Figure 1.9.

1.5.3 Inverse Hyperbolic Functions

The formulae, domains and ranges of all inverse hyperbolic functions are given in the following table.

Function	Formula	Domain	Range
$\sinh^{-1} x$	$\ln(x + \sqrt{x^2+1})$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\cosh^{-1} x$	$\ln(x + \sqrt{x^2-1})$	$[1, \infty)$	$[0, \infty)$
$\tanh^{-1} x$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$(-1, 1)$	$(-\infty, \infty)$
$\coth^{-1} x$	$\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\text{sech}^{-1} x$	$\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$(0, 1]$	$(0, \infty[$
$\text{csch}^{-1} x$	$\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right)$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

1.5.2 Inverse Hyperbolic Functions

(i) Inverse Hyperbolic sine Functions

From the graph of $y = \sinh x$ (see Figure 1.8), it is clear from the horizontal line test that the function is bijective. Hence, its inverse function exists, and we define it as follows:

$$x = \sinh^{-1} y \text{ if and only if } y = \sinh x$$

Example 10 Prove that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$.

Solution: Let $y = \sinh^{-1} x$, where $x, y \in]-\infty, \infty[$.

$$x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2} \quad \Rightarrow \quad 2x = e^y - \frac{1}{e^y}$$

$$\Rightarrow \quad 2x = \frac{e^{2y} - 1}{e^y} \quad \Rightarrow \quad 2xe^y = e^{2y} - 1$$

$$\Rightarrow \quad (e^y)^2 - 2xe^y - 1 = 0$$

> **Note**
> If $y = f(x)$ is a bijective function, then $x = f^{-1}(y)$ is its inverse.

Using quadratic formula:

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Note that, $x - \sqrt{x^2+1} < 0 \ \forall \ x \in]-\infty, \infty[$. Since $e^y > 0$, we take the positive sign:

$$e^y = x + \sqrt{x^2+1} \quad \Rightarrow \quad y = \ln(x + \sqrt{x^2+1})$$

Hence, $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

(ii) Inverse Hyperbolic cosine Function

From the graph of $y = \cosh x$ (see Figure 1.9), it is clear from the horizontal line test that the function is not one-to-one on the domain $(-\infty, \infty)$. However, if we restrict the domain to $[0, \infty)$, the range set is not disturbed. It follows that $y = \cosh x$ becomes one-to-one for the restricted domain $[0, \infty)$ and thus bijective. Therefore, on the domain $[0, \infty)$, the inverse hyperbolic cosine function exists and we define it as follows:

$$x = \cosh^{-1} y \text{ if and only if } y = \cosh x$$

Example 11 Prove that $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ for all $x \in [1, \infty)$.

Solution: For $y \in [0, \infty)$ = Restricted Dom of $\cosh x$ and $x \in [1, \infty)$ = Range of $\cosh x$.

Let $y = \cosh^{-1} x$

$$x = \cosh y$$

$$x = \frac{e^y + e^{-y}}{2} \quad \Rightarrow \quad 2x = e^y + \frac{1}{e^y}$$

$$\Rightarrow \quad 2x = \frac{e^{2y} + 1}{e^y} \quad \Rightarrow \quad 2xe^y = e^{2y} + 1$$

$$\Rightarrow \quad (e^y)^2 - 2xe^y + 1 = 0$$

> **Remember!**
> Dom f = Range f^{-1}
> Range f = Dom f^{-1}

Using quadratic formula:

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$e^y = x \pm \sqrt{x^2 - 1}, \text{ where } x \geq 1,\ y \geq 0 \text{ and } e^y > 0.$$

$$y = \ln(x \pm \sqrt{x^2 - 1})$$

It can easily be observed that $0 < x - \sqrt{x^2 - 1} < 1$ for all $x \geq 1$. Thus, $y = \ln\left(x - \sqrt{x^2 - 1}\right) < 0$. But, $y \geq 0$, which is possible only if the positive sign is taken.

Hence,

$$y = \ln(x + \sqrt{x^2 - 1})$$

$$\Rightarrow \quad \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

EXERCISE 1.2

1. Which of the following functions are algebraic?

   $f(x) = \sec x,\ g(x) = \dfrac{2x-3}{x+1},\ h(x) = \sqrt{x^2 + 2x},\ k(x) = \dfrac{e^x - e^{-x}}{2},\ r(x) = (x-1)^3$

2. Which of the following functions are transcendental?

   $f(x) = \dfrac{1}{x},\ g(x) = \cot x,\ h(x) = e^{2x} + \sqrt{x^2 + 2x},\ k(x) = \dfrac{e^x + e^{-x}}{2},\ r(x) = (x-1)^{-\frac{3}{2}}$

3. Which of the following functions are fundamental transcendental functions?

   $f(x) = e^x + \cos x,\ g(x) = \sin x,\ h(x) = e^{2x},\ k(x) = \dfrac{e^x - e^{-x}}{2},\ r(x) = 15^x$

4. Which of the following functions are non-fundamental transcendental functions?

   $f(x) = e^{2x},\ g(x) = \cos x,\ h(x) = 2^x,\ k(x) = \cos x^2,\ r(x) = 2^{-x}$

5. Discuss domain, range, and graph of the function $f(x) = 3^x$.

6. Discuss domain, range, and graph of the function $f(x) = \left(\dfrac{1}{3}\right)^x$.

7. Prove the following hyperbolic identities:

   (i) $1 - \tanh^2 x = \operatorname{sech}^2 x$ (ii) $\operatorname{csch}^2 x = \coth^2 x - 1$

   (iii) $\sinh 2x = 2 \sinh x \cosh x$ (iv) $\cosh 2x = \cosh^2 x + \sinh^2 x$

8. Prove the following inverse hyperbolic formulae.

   (i) $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), x \neq 1$ (ii) $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), x \neq 1$

(iii) $\text{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), x \neq 0$ (iv) $\text{csch}^{-1} x = \ln\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}+1}\right), x \neq 0$

1.6 Logarithmic Function

A logarithmic function is defined by $y = \log_a x$, $a > 0$, $a \neq 1$, $x > 0$. It assigns to each positive real number x a unique real number y, such that $a^y = x$.

Logarithmic Function as Inverse of Exponential Function

We know that the exponential function

$$y = a^x, (a > 0, a \neq 1)$$

is a one-to-one function with domain $]-\infty, \infty[$ and range $]0, \infty[$. Thus, it is invertible. Interchanging x and y, we obtain:

$$x = a^y \quad \Rightarrow \quad y = \log_a x$$

Hence, the logarithmic function is the inverse of the exponential function with domain $]0, \infty[$ and range $]-\infty, \infty[$.

A logarithm with base 10 is called a common logarithm and a logarithm with base e is called a natural logarithm and we write:

$$\log_{10} b = \log b, \log_e b = \ln b$$

Basic Logarithmic Identities

For $a > 0$, $a \neq 1$, $x > 0$, we have: $\log_a 1 = 0$, $\log_a a = 1$

$\log_a(a^x) = x$, $\log_a x = x$

Laws of Logarithms and Change of Base Formula

For $a > 0$, $a \neq 1$, $x > 0$, $y > 0$, $t \in \mathbb{R}$, we have studied the following laws of logarithms and change of base formula:

1. Product Law: $\log_a(xy) = \log_a x + \log_a y$
2. Quotient Law: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. Power Law: $\log_a(x^t) = t \log_a x$
4. Change of Base Formula: $\log_a x = \frac{\log_b x}{\log_b a}$ for every $b > 0$, $b \neq 1$. In particular:

$$\log_a x = \frac{\ln x}{\ln a} \quad \text{or} \quad \log_a x = \frac{\log x}{\log a}$$

1.6.1 Graphical Behaviour of Logarithmic Functions

In this section, we study the graphical behaviour of logarithmic functions in a structured manner. For a logarithmic function $y = \log_a x$ with domain $]0, \infty[$ and range

$]-\infty, \infty[$, we analyze its vertical asymptotic behaviour, and its overall shape for different values of the base a. We also examine important graphical features such as intercepts, continuity, injectivity and whether the function is increasing or decreasing. These properties together help us to understand how the logarithmic function behaves as a curve and provide the necessary foundation for sketching and interpreting its graph accurately.

(i) Injectivity of Logarithmic Functions

Let $f(x) = \log_a x$, where $a > 0$ and $a \neq 1$, $x \in]0, \infty[$. Let $s, t \in]0, \infty[$ be arbitrary numbers such that, $f(s) = f(t)$

$$\Rightarrow \quad \log_a s = \log_a t \quad \Rightarrow \quad a^{\log_a s} = a^{\log_a t} \quad \Rightarrow \quad s = t$$

Hence, the logarithmic function is one to one.

(ii) Vertical Asymptote of Logarithmic Functions

Let $f(x) = \log_a x$, where $a > 0$, $a \neq 1$ and $x > 0$. Then $x = a^y$. We consider the following cases:

Case-1: $a > 1$

Note that: $x \to 0^+$

$\Rightarrow a^y \to 0^+ \Rightarrow y \to -\infty$

$\Rightarrow \log_a x \to -\infty$.

Hence, $\lim_{x \to 0^+} f(x) = -\infty$.

Case-2: $0 < a < 1$

Note that: $x \to 0^+$

$\Rightarrow a^y \to 0^+ \Rightarrow y \to \infty$

$\Rightarrow \log_a x \to \infty$.

Hence $\lim_{x \to 0^+} f(x) = \infty$.

> **Vertical Asymptote of a Function**
>
> A **vertical asymptote** is a vertical line $x = c$ such that the function $f(x)$ increases or decreases without bound as $x \to c$. In limit form, $x = c$ is a vertical asymptote of the function $y = f(x)$ if:
>
> $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$

In both cases, we have: $\lim_{x \to 0^+} f(x) = \pm\infty$.

Hence, the line $x = 0$ is the vertical asymptote of $f(x) = \log_a x$. Since, $y = \log_a x$ is defined for all positive values of x, there is no other asymptote except $x = 0$.

(iii) Logarithmic Functions are Strictly Increasing or Strictly Decreasing

Consider the logarithmic function:

$$y(x) = f(x) = \log_a x \quad (a > 0, a \neq 1, x > 0)$$

Let $x_1, x_2 \in]0, \infty[$ such that $x_1 < x_2$. Then, $y_1 = f(x_1) = \log_a x_1$ and $y_2 = f(x_2) = \log_a x_2$.

So that: $x_1 = a^{y_1}$ and $x_2 = a^{y_2}$

Since, $x_1 < x_2$, we have $a^{y_1} < a^{y_2}$. We consider the following cases:

Case-1:

If $a > 1$, then $a^{y_1} < a^{y_2} \Rightarrow y_1 < y_2$.

It follows that:

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Hence, $y = \log_a x$ is strictly increasing when $a > 1$.

Case-2:

If $0 < a < 1$, then $a^{y_1} < a^{y_2} \Rightarrow y_1 > y_2$.

It follows that:

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

Hence, $y = \log_a x$ is strictly decreasing when $0 < a < 1$.

(iv) Intercepts and Continuity of Logarithmic Functions

For the logarithmic function: $y = \log_a x \ (a > 0, a \neq 1, x > 0)$

Intercepts:

(i) The x-intercept is obtained by substituting $y = 0$:

$$\log_a x = 0 \quad \Rightarrow \quad x = a^0 = 1$$

Hence, the graph of $y = \log_a x$ intersects x-axis at the point $(1, 0)$.

(ii) Since, the function $y = \log_a x$ is undefined at $x = 0$, it has no y-intercept.

Continuity:

The function $y = \log_a x$ is continuous for all $x > 0$. This can be seen from the fact that:

$\lim\limits_{x \to c} \log_a x = \log_a c$ for all $c > 0$.

Hence, the function has no breaks or jumps in its domain.

1.6.2 Graph of Logarithmic Function

The graph of the logarithmic function $y = \log_a x$ is constructed using the properties discussed in the above subsections. In particular, the domain $x > 0$, the vertical asymptote $x = 0$, the intercept $(1, 0)$, the continuity, the injectivity and the monotonic behaviour depending on the value of a together determine the shape and position of the curve. Using these features, the graph can be drawn accurately for different values of the base a.

Figure 1.10 shows the graphs of logarithmic functions with bases greater than 1, whereas Figure 1.11 represents the graphs of logarithmic functions with bases between 0 and 1.

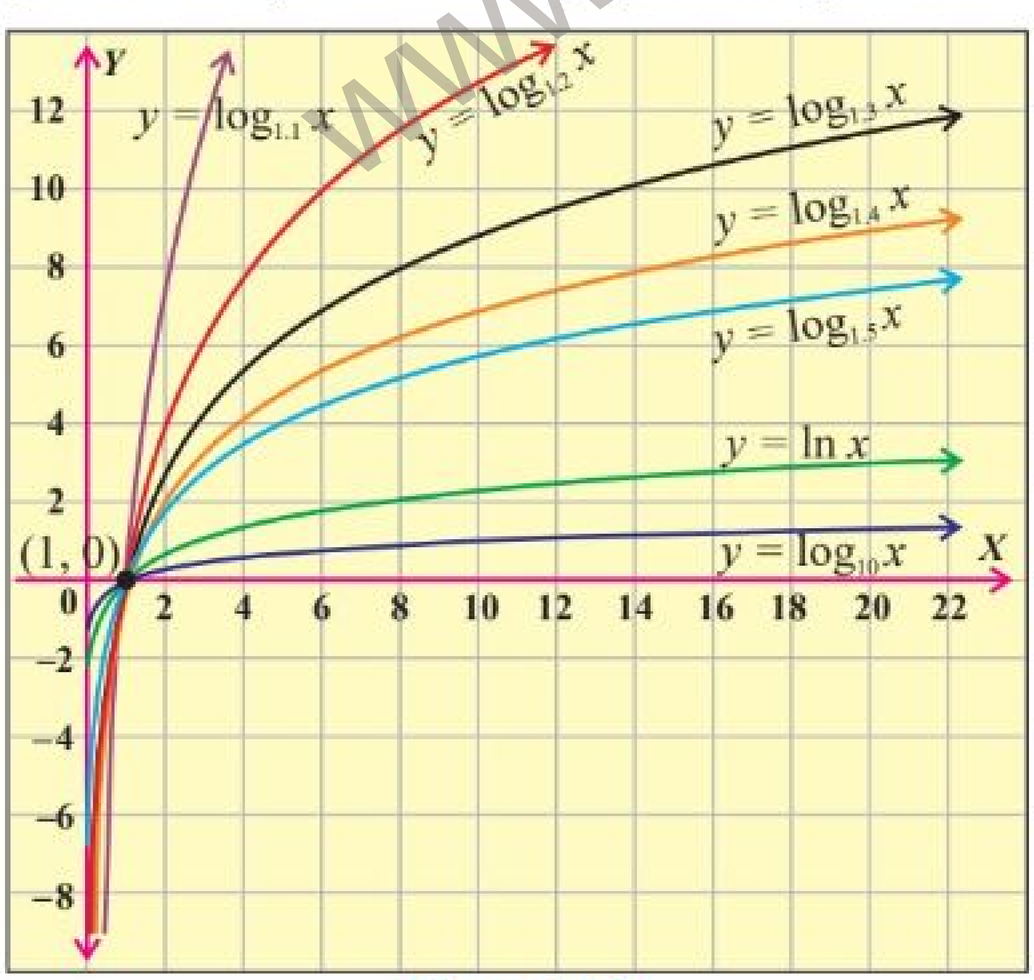


Figure 1.10

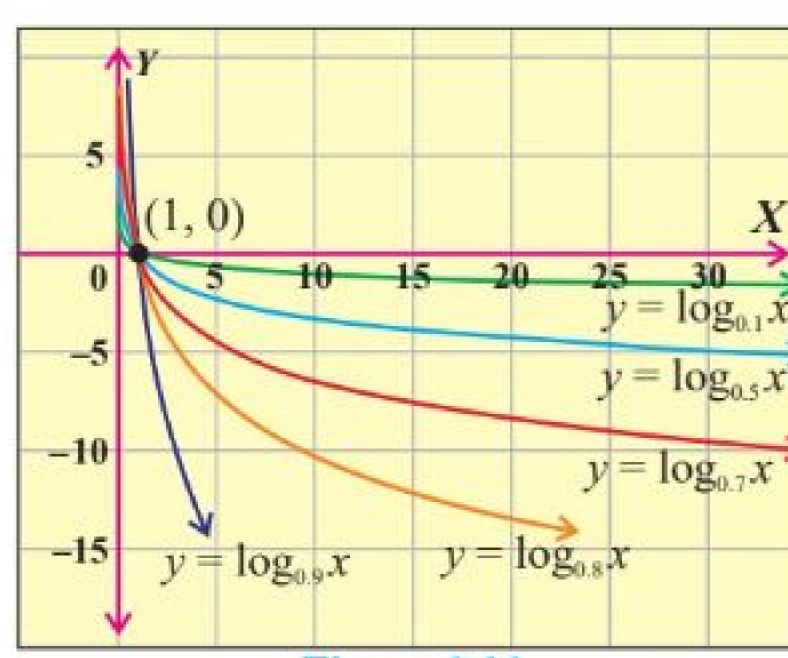


Figure 1.11

1.7 Modulus Function and Its Graph

The **modulus function** or **absolute value function** is denoted and defined by:

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}$$

For example, if $x = 2 > 0$ then $|x| = |2| = 2$ and if $x = -2 < 0$ then $|x| = |-2| = -(-2) = 2$. Similarly, $x = 0$ then $|x| = |0| = 0$. Thus, the modulus of a number is always non-negative.

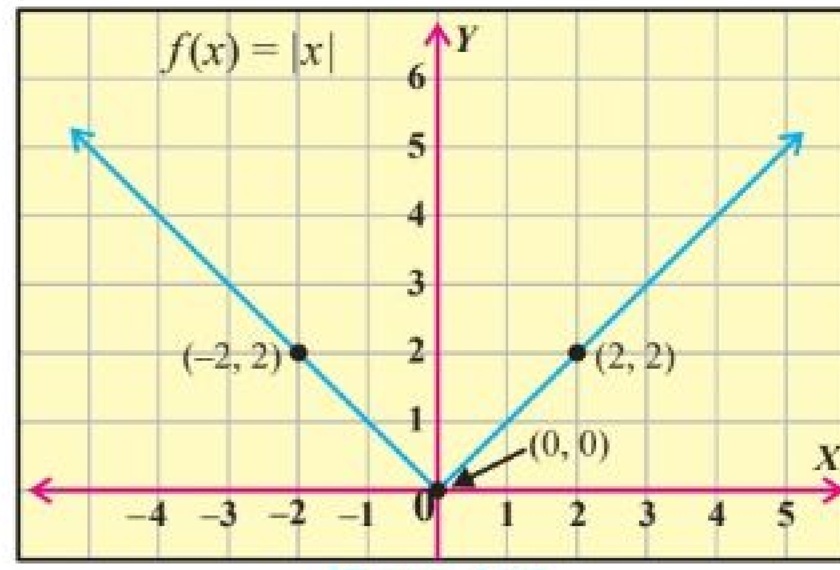


Figure 1.12

The graph of the modulus function is V-shaped. It consists of two straight lines meeting at the origin. For $x \geq 0$, the graph is the line $y = x$, while for $x < 0$, the graph is the line $y = -x$. Thus, the graph of modulus function can be sketched using the points (0, 0), (2, 2) and (–2, 2) as shown in Figure 1.12.

1.8 Interpreting the Relation Between a One-to-One Function and Its Inverse through Graph

A function has an inverse only if it is bijective, that is, both one-to-one and onto. However, if the co-domain of a function is taken equal to its range, then every value in the co-domain is already attained by the function. Therefore, the function automatically becomes onto, and only the one-to-one condition is required for the existence of its inverse.

The graph of a function and its inverse are related through a geometric symmetry about the line $y = x$. This line is used because any point (x, y) on a function represents an input-output pair, and for its inverse, the roles of input and output are interchanged. The line $y = x$ is exactly the set of all points where input and output values are equal, making it the natural axis of reflection for this interchange.

1.8.1 Method of Constructing the Graph of an Inverse Function

The graph of an inverse function is obtained geometrically as follows:

(i) Draw the graph of the given one-to-one function.

(ii) Consider the line $y = x$ on the same coordinate plane.

(iii) Reflect the graph of the function about the line $y = x$.

(iv) The resulting reflected curve represents the graph of the inverse function.

The reflection ensures that each point on the original graph is mapped to a corresponding point on the inverse graph by maintaining equal perpendicular distance from the line $y = x$, preserving the geometric symmetry between the two graphs.

Example 12 **(Logarithmic Graph from Exponential Graph)**

Show graphically that the function $y = \ln x$ is the inverse of the function $y = e^x$.

Solution:

Consider the exponential function $y = e^x$

Note that the function $y = e^x$ is one to one on $]-\infty, \infty[$ and its range is $]0, \infty[$. Hence, it is invertible and its inverse is defined on $]0, \infty[$. We draw the graph of $y = e^x$ along with the line $y = x$ (see Figure 1.13). To obtain the graph of the inverse function, points on the curve $y = e^x$ are reflected about the line $y = x$. The curve passing through these reflected points is then drawn (see Figure 1.13). The resulting curve represents the function:

$$y = \ln x$$

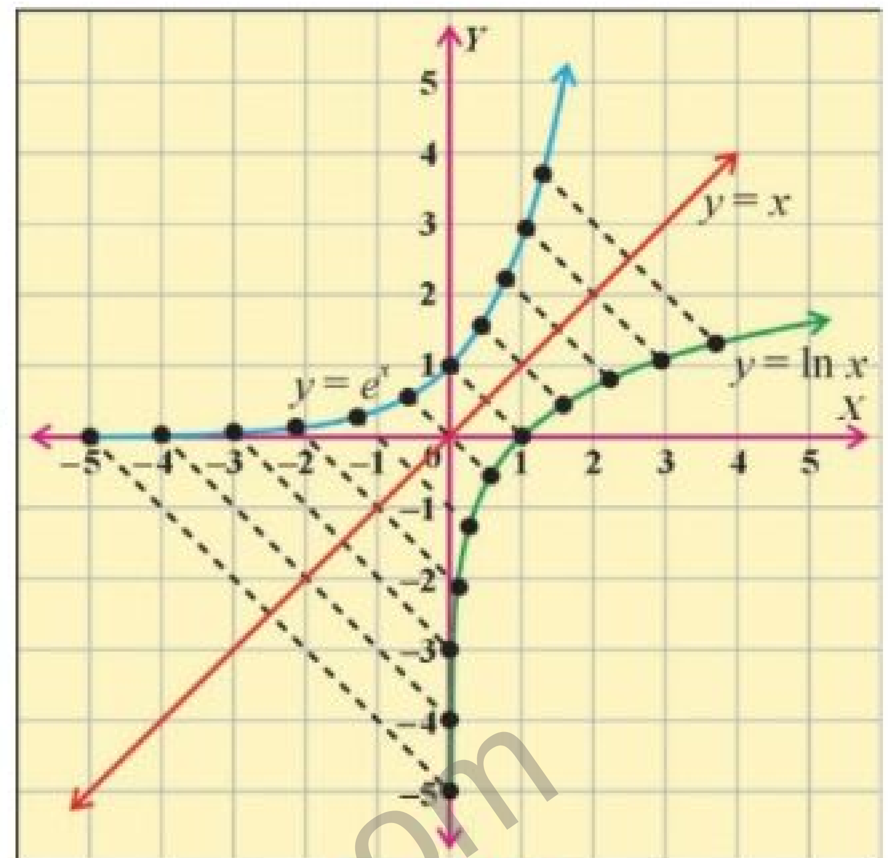


Figure 1.13

which is clearly defined on the interval $]0, \infty[$.

Hence, the function $y = \ln x$ is the inverse of $y = e^x$.

1.9 Transformation of Graphs of Functions

Graphs of functions can be transformed in various ways to obtain new graphs from basic functions. These transformations include reflection, rotation, stretching, compression and shifting, and are applied to represent changes in functions in algebraic, graphical and real-life contexts. One of the most important applications of these transformations is in computer graphics, where they are used to model, manipulate and animate images and shapes. However, in this section, we shall focus only on three fundamental transformations: horizontal shift, vertical shift and scaling, which are most commonly used in the graphical analysis of functions.

(i) Shifting of Graphs

Shifting of graphs refers to the movement of a graph from its original position without changing its shape or size. In shifting, the graph is translated either horizontally or vertically.

In mathematical modeling, shifting is used to represent changes in a function due to variations in input or output values. It allows the same basic function to describe different situations by adjusting its position on the coordinate plane.

Horizontal Shift

A graph of the function $y = f(x)$ is **shifted horizontally** when the input variable x is changed to $x - h$. The resulting graph is represented by the function:

$$y = f(x - h)$$

- If $h > 0$: shift h units to the right
- If $h < 0$: shift $|h|$ units to the left

Vertical Shift

A graph of the function $y = f(x)$ is **shifted vertically** when the output variable y is changed to $y - k$. The resulting graph is represented by the function:

$$y = f(x) + k$$

- If $k > 0$: shift k units upward
- If $k < 0$: shift $|k|$ units downward

Challenge!

Draw the graph of the function

$$y = (x - 2)^2 - 2$$

and observe how it can be obtained from the graph of $y = x^2$

For better understanding of horizontal and vertical shifts, we consider the function:

$$y = x^2$$

This equation represents a parabola with vertex at (0, 0), as shown in Figure 1.14.

Next, we consider the function $y = (x - 2)^2$

This also represents a parabola with vertex at (2, 0). Its graph can be seen in Figure 1.15.

Note that the graph in Figure 1.15 can be obtained directly from the graph in Figure 1.14 by shifting it 2 units horizontally to the right.

Now, consider another function $y = x^2 - 2$

This equation also represents a parabola with vertex at (0, –2). Its graph is shown in Figure 1.16.

Note that this graph can be obtained directly from the graph in Figure 1.14 by shifting it 2 units downward.

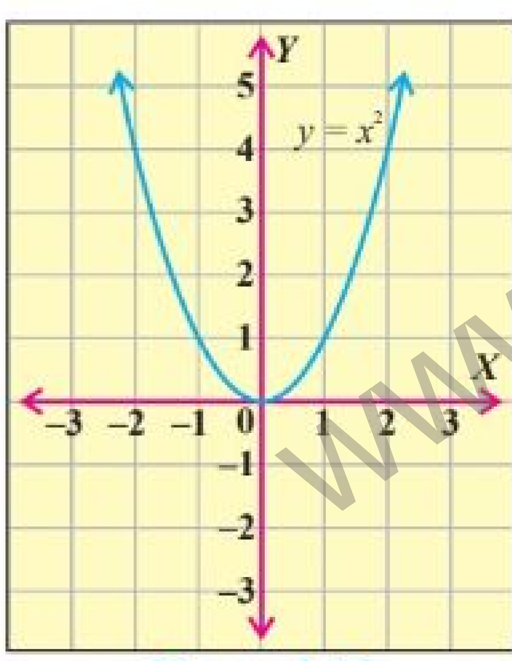


Figure 1.14

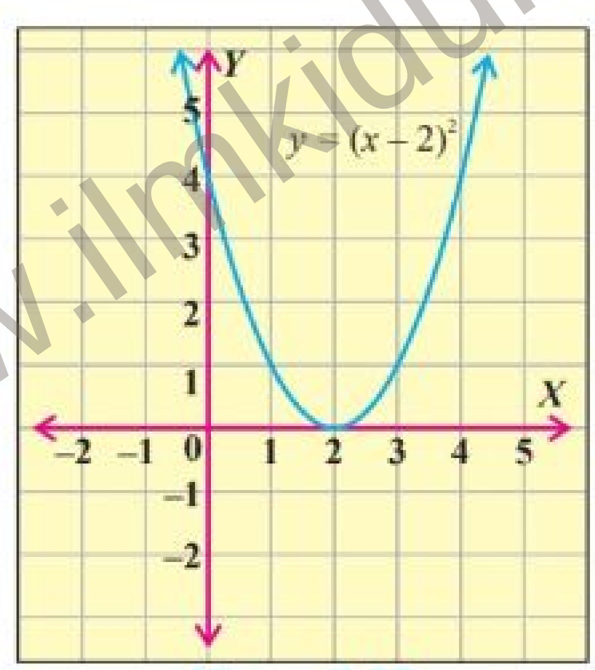


Figure 1.15

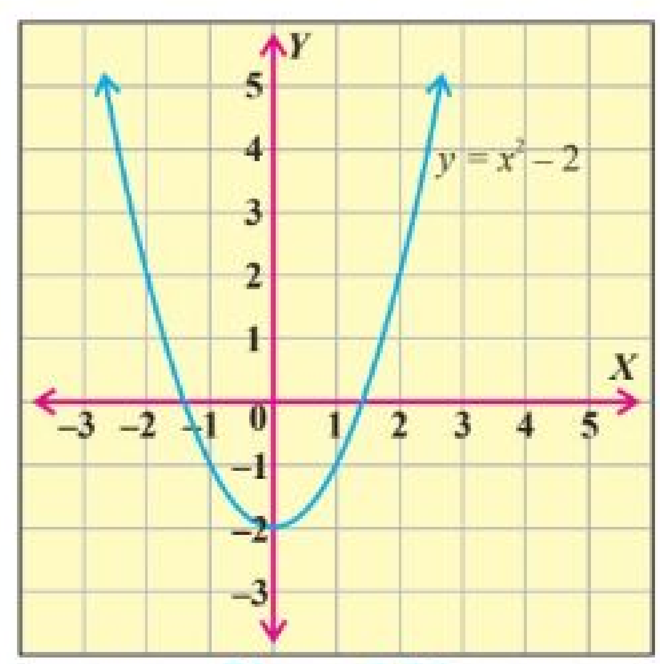


Figure 1.16

General Formula for Vertical and Horizontal Shift

If both horizontal and vertical shifts are applied together to the function $y = f(x)$, then the general form of the transformed function is given by:

$$y = f(x - h) + k$$

(ii) Scaling of Graphs

Scaling of graphs refers to the stretching or compression of a graph without changing its general shape. In scaling, the graph is enlarged or reduced either in the vertical direction or in the horizontal direction.

Scaling helps in understanding how a function changes when its input or output values are multiplied by a positive constant factor.

Vertical Scaling

If a function $y = f(x)$ is multiplied by a positive constant a, then the new function:

$$y = af(x), \quad a > 0$$

represents a **vertical scaling** of the graph of $y = f(x)$.

- If $a > 1$, then the graph is **stretched vertically** by factor a.
- If $0 < a < 1$, then the graph is **compressed vertically** by factor $\frac{1}{a} > 1$.
- Only the y-values of the graph change, while x-values remain unchanged.

Horizontal Scaling

If a function $y = f(x)$ is transformed to:

$$y = f(ax), \quad a > 0$$

then, it represents a **horizontal scaling** of the graph of $y = f(x)$.

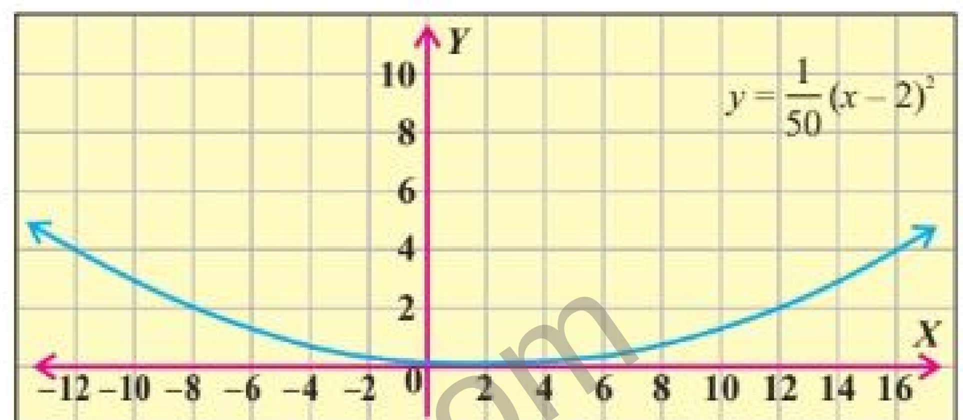


Figure 1.17

- If $a > 1$, then the graph is **compressed horizontally** by a factor a
- If $0 < a < 1$, then the graph is **stretched horizontally** by a factor $\frac{1}{a} > 1$.
- Only the x-values of the graph change, while y-values remain unchanged

Important Note:

If $a < 0$, scaling is not purely stretching or compression. It results in:

- reflection along the x-axis in $y = af(x)$
- reflection along the y-axis in $y = f(ax)$

Therefore, in this section we consider only $a > 0$ to focus on scaling alone.

For better understanding of scaling, we consider the following functions:

$$y = \frac{1}{50}(x-2)^2 \quad \text{and} \quad y = (3x-2)^2$$

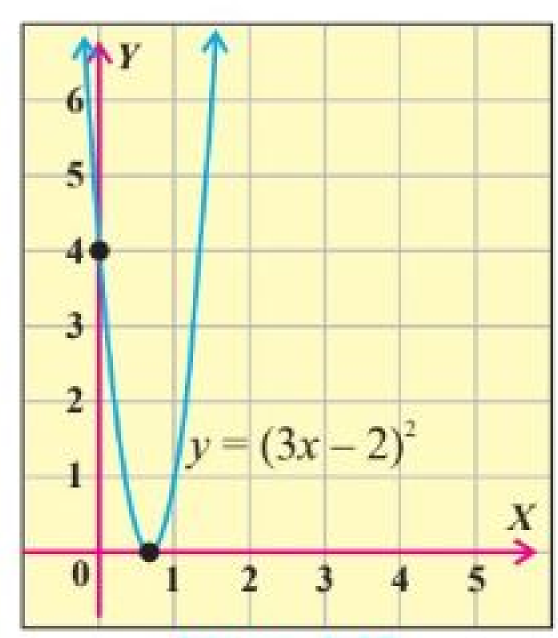


Figure 1.18

The graph of $y = \frac{1}{50}(x-2)^2$ is shown in Figure 1.17 which can be obtained from the graph of $y = (x-2)^2$ (Figure 1.15) using a vertical compression by factor of 50. Similarly, the graph of $y = (3x-2)^2$ can be obtained from the graph of $y = (x-2)^2$ by a horizontal compress, that is, by a factor of 3 (see Figure 1.18).

General Formula for Scaling of Graphs

If a function $y = f(x)$ is transformed to

$$y = a\,f(bx), \quad a > 0, b > 0$$

then it represents both vertical and horizontal scaling of the graph of $y = f(x)$.

If $a > 1$, the graph is stretched vertically by factor a. If $0 < a < 1$, the graph is compressed vertically by factor $\frac{1}{a}$. On the other hand, if $b > 1$, the graph is compressed horizontally by factor b and if $0 < b < 1$, the graph is stretched horizontally by factor $\frac{1}{b}$.

Example 13 Let $f(x) = \ln(x^2 + 1)$. Find the function obtained after:

(i) a vertical stretch by factor 2 (ii) then a shift 1 unit to the left.

Solution: We have: $f(x) = \ln(x^2 + 1)$.

(i) Formula for the vertical scaling is $y = af(x)$

Since, there is a vertical stretch by factor 2, we have $a = 2$. Thus, the resulting function becomes:

$$y = 2\ln(x^2 + 1)$$

(ii) Now we have, $g(x) = 2\ln(x^2 + 1)$.

Formula for the horizontal translation is $y = g(x - h)$.

Since, there is a shift of 1 unit to the left, we have $h = -1$. Thus, the resulting function becomes:

$$y = g(x - (-1)) = g(x + 1)$$

$$= 2\ln[(x + 1)^2 + 1] = 2\ln(x^2 + 2x + 1 + 1)$$

Hence, the transformed function is $y = 2\ln(x^2 + 2x + 2)$

Example 14 Let $f(x) = \frac{x}{x+1}$. Find the function obtained after applying all the following transformations with the given order:

(i) vertical stretch by 2 (ii) horizontal stretch by 3

(iii) shift 2 units to the right (iv) shift 1 unit downward

Solution: Given function: $f(x) = \frac{x}{x+1}$.

(i) Vertical stretch by 2.

The resulting function is $y = af(x)$ where $a = 2$. Thus, we get:

$$y = 2\frac{x}{x+1}$$

$$= \frac{2x}{x+1}$$

We set, $g(x) = \frac{2x}{x+1}$.

(ii) Horizontal stretch by 3.

The resulting function is $y = g(bx)$, where $b = \frac{1}{3}$. So, we have:

$$y = g\left(\frac{x}{3}\right) = \frac{2\frac{x}{3}}{\frac{x}{3}+1} = \frac{2x}{x+3}$$

Let $s(x) = \frac{2x}{x+3}$

(iii) Shift 2 units to the right

The resulting function is $y = (x - h)$, where $h = 2$. So, we get:

$$y = s(x-2) = \frac{2(x-2)}{(x-2)+3} = \frac{2x-4}{x+1}$$

Let $t(x) = \dfrac{2x-4}{x+1}$

(iv) Shift 1 unit downward

The resulting function is $y = t(x) + k$, where $k = -1$. Thus, we have:

$$y = t(x) - 1 = \frac{2x-4}{x+1} - 1 = \frac{x-5}{x+1}$$

Hence, transformed function is $y = \dfrac{x-5}{x+1}$.

EXERCISE 1.3

1. Let $f(x) = \log_4(x-2)$
   (i) Show that $f(x)$ is a one-to-one function using the horizontal line test.
   (ii) Find the inverse function $f^{-1}(x)$ algebraically.
   (iii) State the domain and range of $f(x)$ and $f^{-1}(x)$.
   (iv) Verify graphically that the graphs of $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = x$.
2. Let $f(x) = \sin^{-1} x$
   (i) State the domain and range of $f(x)$ and $f^{-1}(x)$.
   (ii) Sketch the graph of $f(x)$ and show that it is one-to-one using the horizontal line test.
   (iii) Draw the graph of $f^{-1}(x)$ and verify that it is the reflection of $f(x)$ in the line $y = x$.
3. A one-to-one function is defined by $f(x) = (x-1)^3$, $x \in [0, 3]$
   Using the concept of reflection in the line $y = x$, sketch the graph of the inverse function $f^{-1}(x)$.
4. Given the function $f(x) = \sqrt{x}$
   Find the equation of the function after:
   (i) vertical stretch by factor 2
   (ii) horizontal shift 3 units to the right
   (iii) applying both transformations (i) and (ii).
5. Given the function $f(x) = \dfrac{1}{x}, x \neq 0$. Find the equation of the transformed function after applying the following transformations in order.
   Horizontal shift 2 units to the left, horizontal compression by factor 2, vertical stretching by factor 3, and vertical shift 1 unit upward.
6. Given the function $f(x) = \ln x$, find the equation of the transformed function obtained after shifting the graph 1 unit to the right, vertically compressing it by a factor of 2, and then shifting it 3 units upward.

7. Given the function $f(x)=\sqrt{\dfrac{2x-1}{x+3}}$, $x\neq -3$. Find the equation of the transformed function after applying in order, the horizontal shift 2 units to the left, horizontal compression by factor 3, vertical compression by factor 2, and vertical shift 1 unit downward.
8. Given the function $f(x)=3^x$ and its transformed function $g(x) = 3^{x-2} + 1$, draw the graphs of both functions on the same coordinate plane and compare their shapes.
9. Given the function $f(x)=\ln x$. Find the equation of the transformed function after applying in order, the vertical stretch by factor 4, shift 6 units to right, horizontal compress by factor 3, and shift 1 unit upward. Draw the graphs of both the original and transformed functions on the same coordinate plane.
10. Draw the graph of $f(x) = |x|$ after applying in order, shift of 7 units to right, horizontal compress by factor 2, vertical shift 6 units downward, and vertical compress by factor 2.

1.10 Solving Problems Involving Exponential and Logarithmic Equations and Inequalities

In this section, we solve exponential and logarithmic equations and inequalities in which the variable appears in the exponent or inside a logarithmic expression. Before solving, the domain must be considered wherever the expression is not defined for all real values of the variable. The methods of solving equations and inequalities are illustrated in the following examples.

Examples of Exponential and Logarithmic Equations

Example 15 Solve the equation: $4^{x-1} = 8$.

Solution: The given equation is: $4^{x-1} = 8$.

Domain: Since the expression 4^{x-1} is defined for every real number

$$\text{Domain} =]-\infty, \infty[.$$

Since $4 = 2^2$ and $8 = 2^3$, the above equation can be written as:

$$(2^2)^{x-1} = 2^3 \text{ or } 2^{2x-2} = 2^3$$

Since the exponential function is one to one, we must have:

$$2x-2=3 \quad \Rightarrow \quad x=\frac{5}{2}$$

Solution Set: $\left\{\dfrac{5}{2}\right\}$

Example 16

Find the solution of $2^{2x} - 3^{5x+3} = 0$.

Solution: The given equation can be written as:

$$2^{2x} = 3^{5x+3} \quad \ldots(1)$$

Domain: Since the expressions 2^{2x} and 3^{5x+3} are defined for every real number.

So, Domain = $]-\infty, \infty[$.

Applying natural logarithms on both sides of equation (1):

$$\ln 2^{2x} = \ln 3^{5x+3}$$

Applying power law of logarithms:

$$2x \ln 2 = (5x + 3) \ln 3$$

$$\Rightarrow \quad 2x \ln 2 = 5x \ln 3 + 3 \ln 3$$

$$\Rightarrow \quad (2 \ln 2 - 5 \ln 3)x = 3 \ln 3$$

Applying power law of logarithms:

$$(\ln 2^2 - \ln 3^5)x = \ln 3^3$$

$$\Rightarrow \quad (\ln 4 - \ln 243)x = \ln 27$$

Applying quotient law of logarithms:

$$x \ln\left(\frac{4}{243}\right) = \ln 27$$

$$\Rightarrow \quad x = \ln 27/\ln\left(\frac{4}{243}\right) = \log_{\frac{4}{243}}(27)$$

Solution Set: $\left\{\log_{\frac{4}{243}}(27)\right\}$

Example 16 Solve the equation:

$$\log x + \log(x - 3) = 1$$

Solution:

The given equation is:

$$\log x + \log(x - 3) = 1 \quad \ldots(1)$$

Domain: Note that the expressions $\log x$ and $\log(x - 3)$ are defined only when:

$$x > 0 \text{ and } x - 3 > 0 \quad \Rightarrow \quad x > 3$$

Hence, Domain = $]3, \infty[$.

Applying product law of logarithms on the L.H.S of the given equation (1):

$$\log[x(x - 3)] = 1$$

Writing in exponential form:

$$x(x - 3) = 10^1 = 10$$

Factorizing: $x^2 - 3x - 10 = 0$

$$(x - 5)(x + 2) = 0$$

$$\Rightarrow \quad x = 5 \text{ or } x = -2$$

Since $5 \in]3, \infty[$ whereas, $-2 \notin]3, \infty[$, the solution set is $\{5\}$.

Example 17 Solve the equation: $\log_x(x^2 - 3x + 2) = 2$.

Solution: Domain:
- Base condition: $x > 0, x \neq 1$
- Argument condition: $x^2 - 3x + 2 > 0$

The given equation is: $\log_x(x^2 - 3x + 2) = 2$.

Convert to exponential form: $x^2 = x^2 - 3x + 2$

$$\Rightarrow \quad 3x - 2 = 0 \quad \Rightarrow \quad 3x - 2 = 0 \quad \Rightarrow \quad x = \frac{2}{3}$$

Check domain:
- Base condition: $x = \frac{2}{3} > 0$ and $x \neq 1$.
- Argument condition:

$$x^2 - 3x + 2 = \left(\frac{2}{3}\right)^2 - 3\left(\frac{2}{3}\right) + 2 = \frac{4}{9} - 2 + 2 = \frac{4}{9} > 0$$

Hence, $x = \frac{2}{3}$ is a solution of the given equation.

Solution set: $\left\{\frac{2}{3}\right\}$

Examples of Exponential and Logarithmic Inequations

Example 19 Solve the inequality: $2^{x+1} > 5^x$.

Solution:

Domain: Note that, the expression 2^{x+1} and 5^x are defined for every real number.

The given inequality can be written as: $2 \cdot 2^x > 5^x$

Since, $2^x > 0$ for every real number, we have:

$$2 > \frac{5^x}{2^x} \quad \Leftrightarrow \quad 2 > \left(\frac{5}{2}\right)^x$$

Since, the logarithmic function with base 10 is strictly increasing, we have:

$$2 > \frac{5^x}{2^x} \quad \Leftrightarrow \quad \log 2 > \log\left(\frac{5}{2}\right)^x$$

Applying power law of logarithms: $\log 2 > x \log\left(\frac{5}{2}\right)$

Since, $\frac{5}{2} > 0 \quad \Rightarrow \quad \log\left(\frac{5}{2}\right) > 0$, we have:

$$x < \frac{\log 2}{\log\left(\frac{5}{2}\right)} = \log_{\frac{5}{2}}(2) \approx 0.7565$$

Hence, $2^{x+1} > 5^x \quad \Rightarrow \quad x < \log_{\frac{5}{2}}(2)$

Solution Set: $]-\infty, \log_{\frac{5}{2}}(2)[$.

Example 20 Solve the inequality: $\log(2x - 1) \geq \log(5 - 3x)$

Solution:

Domain: Note that $\log(2x - 1)$ and $\log(5 - 3x)$ are defined when:

$$2x - 1 > 0 \quad \Rightarrow \quad x > \frac{1}{2} \quad \text{and} \quad 5 - 3x > 0 \quad \Rightarrow \quad x < \frac{5}{3}$$

So, we have: $\frac{1}{2} < x < \frac{5}{3}$

The given inequality is: $\log(2x - 1) \geq \log(5 - 3x)$

Since, the logarithmic function with base 10 is strictly increasing, we must have:

$$\log(2x-1) \geq \log(5-3x) \quad \Leftrightarrow \quad 2x-1 \geq 5-3x$$

Solve inequality: $2x + 3x \geq 5 + 1 \quad \Leftrightarrow \quad 5x \geq 6 \quad \Leftrightarrow \quad x \geq \frac{6}{5}$

Compatibility with domain: $\frac{6}{5} \leq x < \frac{5}{3}$

It follows that: $\log(2x-1) \geq \log(5-3x)$ iff $\frac{6}{5} \leq x < \frac{5}{3}$

Solution Set: $\left[\frac{6}{5}, \frac{5}{3}\right[$

Example 21 Solve the inequality: $\log_{\frac{7}{9}}(x^2 - 3x + 2) \geq \log_{\frac{7}{9}}(2x-1)$

Solution:

Domain: For the expression $\log_{\frac{7}{9}}(x^2 - 3x + 2)$ to be defined, we have:

$x^2 - 3x + 2 > 0 \quad \Rightarrow \quad (x-1)(x-2) > 0 \quad \Rightarrow$ either $x < 1$ and $x < 2$ (which gives $x < 1$) or $x > 1$ and $x > 2$ (which gives $x > 2$). Hence $x < 1$ or $x > 2$.

For the expression $\log_{\frac{7}{9}}(2x-1)$ to be defined, we have:

$$2x - 1 > 0 \Rightarrow x > \frac{1}{2}$$

Combined domain: $\left]\frac{1}{2}, 1\right[\cup \left]2, \infty\right[$

The given inequality is: $\log_{\frac{7}{9}}(x^2 - 3x + 2) \geq \log_{\frac{7}{9}}(2x-1)$

Since, $0 < \frac{7}{9} < 1$, the logarithmic function with base $\frac{7}{9}$ is strictly decreasing. Thus, we have:

$$\log_{\frac{7}{9}}(x^2 - 3x + 2) \geq \log_{\frac{7}{9}}(2x-1) \quad \Leftrightarrow \quad x^2 - 3x + 2 \leq 2x - 1$$

$$\Leftrightarrow \quad x^2 - 3x + 2 - 2x + 1 \leq 0 \quad \Leftrightarrow \quad x^2 - 5x + 3 \leq 0$$

Solving the quadratic equation $x^2 - 5x + 3 = 0$:

$$x = \frac{5 \pm \sqrt{25-12}}{2} = \frac{5 \pm \sqrt{13}}{2}$$

Since, $x^2 - 5x + 3 \leq 0$, either $x \leq \dfrac{5-\sqrt{13}}{2}$ and $x \geq \dfrac{5+\sqrt{13}}{2}$ (which is impossible) or

$x \geq \dfrac{5-\sqrt{13}}{2}$ and $x \leq \dfrac{5+\sqrt{13}}{2}$ $\left(\text{which gives} \dfrac{5-\sqrt{13}}{2} \leq x \leq \dfrac{5+\sqrt{13}}{2}\right)$.

Compatibility with the domain: $\left]\dfrac{1}{2}, 1\right[\cup \left]2, \infty\right[$.

If $x \in \left]\dfrac{1}{2}, 1\right[\cup \left]2, \infty\right[$ and $\dfrac{5-\sqrt{13}}{2} \leq x \leq \dfrac{5+\sqrt{13}}{2}$ then, we have:

$$x \in \left[\frac{5-\sqrt{13}}{2}, 1\right[\cup \left]2, \frac{5+\sqrt{13}}{2}\right]$$

Hence, $\log_{\frac{7}{9}}(x^2 - 3x + 2) \geq \log_{\frac{7}{9}}(2x - 1)$ iff $x \in \left[\dfrac{5-\sqrt{13}}{2}, 1\right[\cup \left]2, \dfrac{5+\sqrt{13}}{2}\right]$.

Solution Set: $\left[\dfrac{5-\sqrt{13}}{2}, 1\right[\cup \left]2, \dfrac{5+\sqrt{13}}{2}\right]$

EXERCISE 1.4

1. Solve the following equations:
   (i) $\log(3x - 2) = \log(x^2 - 5x + 6)$ (ii) $\ln(x^2 - 1) = \ln(3x - 2)$
   (iii) $\log(x) + \log(x^2 - 5x + 7) = \log(3)$ (iv) $3^{(x+1)} - 2 \cdot 9^x + 9 = 0$
   (v) $x \log_{\frac{1}{e}} 4 = \log_{\frac{1}{e}}(16^3 - 3)$ (vi) $2^x + 2^{-x} = 5$

2. Solve the following inequalities (if its solution exists).
   (i) $2^x + 2^{1-x} \leq 3$ (ii) $2^x \geq \log_2(16)$
   (iii) $\log_2(x - 1) \geq \log_2(5 - x)$ (iv) $\log_{\frac{1}{3}}(x^2 - 4x + 3) > \log_{\frac{1}{3}}(2x - 1)$
   (v) $\log_2(x - 2) + \log_2(x - 3) \geq 3$ (vi) $\log(x) + \log(x - 3) \geq \log(2x) + \log(x - 1)$

1.11 Applications of Exponential and Logarithmic Functions in Real-Life Situations

Although there are many real-life situations involving exponential and logarithmic functions, in this section we focus on selected applications such as growth and decay processes, the measurement of sound intensity in decibels, and compound interest.

Exponential models are used to describe quantities that change over time, while logarithmic functions help in determining unknown quantities in such models. The formula for growth and decay will be used as a given model and will be derived later in the chapter on differential equations, while the formulas for sound intensity and compound interest will be taken as standard results and will not be derived here.

(i) Growth and Decay

Many natural and physical processes such as population growth and radioactive decay change in such a way that the rate of change is proportional to the current amount. These processes are known as exponential growth or decay.

The general mathematical model is:

$$N = N_0 \; e^{kt}$$

Where N_0 is the initial amount, N is the amount at time t, and k is a constant which is positive for growth and negative for decay.

In growth, the quantity increases with time, while in decay it decreases with time. Logarithms are used to determine unknown quantities such as time or rate in these models.

Example 22 **(Growth)**

A population is given by: $N = 200e^{0.04t}$

Find the time t when the population becomes 400, where t is measured in hours.

Solution: We are given that:

$$N = 200e^{0.04t}$$

Substituting $N = 400$:

$$400 = 200e^{0.04t}$$

$$2 = e^{0.04t}$$

Take natural logarithm on both sides:

$$\ln 2 = 0.04t$$

$$t = \frac{\ln 2}{0.04} \approx 17.33 \text{ hours}$$

Hence the populations become 400 in 17.33 hours approximately.

Example 23 **(Decay)**

A radioactive substance decays according to: $N = 500e^{kt}$

After 20 hours, the amount becomes 300. Find the value of k, where t is in hours.

Solution: We have:

$$N = 500e^{kt}$$

Substituting $t = 20$ and $N = 300$:

$$300 = 500e^{20k}$$

$$\frac{3}{5} = e^{20k}$$

$$\ln\left(\frac{3}{5}\right) = 20k$$

$$k = \frac{1}{20}\ln\left(\frac{3}{5}\right) \approx -0.0256$$

Hence, $k \approx -0.0256$

(ii) Sound Intensity (Decibels)

The intensity of sound is the amount of sound energy passing through a unit area per unit time. Since sound intensity varies over a very wide range, it is measured on a logarithmic scale called the decibel (dB) scale.

The sound level L is given by:

$$L = 10\log\left(\frac{I}{I_0}\right)$$

where I is the intensity of sound and I_0 is the **threshold of hearing**, which is the minimum sound intensity that a normal human ear can detect. Its standard value is given by:

$$I_0 = 10^{-12}\ \text{W/m}^2$$

where, W/m^2 (watts per square meter) is the unit of sound intensity. It represents the amount of sound power (energy per second) passing through a unit area.

Because of the logarithmic scale, even a small increase in decibels represents a large change in actual sound intensity, which makes logarithmic laws very useful in comparing sound levels in real-life situations.

Example 24 **(Finding sound level)** A sound has intensity $I = 10^{-4}$ and reference intensity $I_0 = 10^{-12}$. Find the sound level.

Solution: We have: $L = 10\log\left(\frac{I}{I_0}\right)$

Substituting, $I = 10^{-4}$ and $I = 10^{-12}$:

$$L = 10\log\left(\frac{10^{-4}}{10^{-12}}\right)$$
$$= 10\log(10^8)$$
$$= 10 \times 8$$
$$= 80\ \text{dB}$$

Example 25 A sound has level $L = 60\text{dB}$ and $I_0 = 10^{-12}$. Find the intensity I.

Solution: We have: $L = 10\log\left(\frac{I}{I_0}\right)$

Substituting $L = 60\text{dB}$ and $I_0 = 10^{-12}$:

$$60 = 10\log\left(\frac{I}{10^{-12}}\right)$$

$$6 = \log\left(\frac{I}{10^{-12}}\right)$$

Convert to exponential form:

$$10^6 = \frac{I}{10^{-12}}$$

$$I = 10^6 \times 10^{-12} = 10^{-6}\ \text{W/m}^2$$

Example 26 The sound levels at two points are $L_1 = 90$ dB and $L_2 = 60$ dB. Find the ratio of their intensities $\frac{I_1}{I_2}$, where $I_0 = 10^{-12}\ \text{W/m}^2$.

Solution: Given: $L_1 = 90$ dB, $L_2 = 60$ dB

Formula: $$L = 10\log\left(\frac{I}{I_0}\right)$$

For the given values of L_1 and L_2, we get:

$$90 = 10\log\left(\frac{I_1}{I_0}\right) \quad \text{or} \quad 9 = \log\left(\frac{I_1}{I_0}\right)$$

$$60 = 10\log\left(\frac{I_2}{I_0}\right) \quad \text{or} \quad 6 = \log\left(\frac{I_2}{I_0}\right)$$

Converting in exponential form:

$$\frac{I_1}{I_0} = 10^9 \quad \text{and} \quad \frac{I_2}{I_0} = 10^6$$

Dividing the above equations:

$$\frac{I_1}{I_2} = \frac{10^9}{10^6} \quad \text{and} \quad \frac{I_1}{I_2} = 10^3 = 1000$$

Which is the required ratio of intensities of the sound levels.

(iii) Compound Interest

Compound interest is a method of calculating interest in which the interest earned in each period is added to the **principal** (initial amount), so that interest is earned on both the principal and the accumulated interest. This results in exponential growth of money over time and is widely used in banking and financial systems to model investment growth. Compound interest is generally classified into two types: discrete compounding and continuous compounding.

Discrete Compound Interest

Discrete compound interest is a method of calculating interest in which the interest is added to the principal at fixed time intervals such as yearly, half-yearly, quarterly, or monthly. In this method, interest is not added continuously, but only at specific intervals, and the future amount is calculated using a fixed compounding frequency. This type of compounding leads to exponential growth of money over time and is commonly used in banking systems.

The discrete compound interest is given by the formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where, A = Final amount

P = Principal (initial amount)

r = Interest rate per unit time (in decimal form)

n = Number of times interest is compounded per unit time

t = Time (measured in the same unit as the compounding period)

Continuous Compound Interest

Continuous compound interest is a method of calculating interest in which the interest is assumed to be added to the principal continuously at every instant of time. In this case, the compounding occurs without any fixed intervals, and the amount increases smoothly over time. This type of compounding represents the limiting case of discrete compounding when the number of compounding periods becomes infinitely large.

The formula for continuous compounding is:

$$A = Pe^{rt}$$

where,

A = Final amount after time t (principal + interest)

P = Principal (initial amount invested or borrowed)

r = Rate of interest per unit time (in decimal form)

t = Time (measured in the same unit as the rate is based on, such as years or months)

Example 27: A sum of Rs. 1000 is invested at a rate of 5% per year compounded continuously for 3 years. Find the final amount.

Solution: Given: $P = 1000$,

$$r = 5\% = 0.05$$

$$t = 3$$

Formula for the continuous compound interest:

$$A = Pe^{rt}$$

Substituting the values:

$$A = 1000e^{0.05 \times 3}$$

$$A = 1000e^{0.15}$$

$$A \approx 1000 \times 1.1618$$

$$= 1161.8$$

Hence, the final amount after 3 years is

$$A \approx \text{Rs. } 1161.8.$$

Example 28: A sum of Rs. 5000 is invested at a rate of 8% per annum, compounded quarterly. Find the amount after 3 years.

Solution: We have: $P = 5000$

$$r = 0.08 \text{ (8\% per annum)}$$

Quarterly compounding $n = 4$

$$t = 3 \text{ years}$$

Formula for the discrete compound interest:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substituting the values:

$$A = 5000\left(1 + \frac{0.08}{4}\right)^{4\times3}$$

$$A = (5000)(1.02)^{12}$$

$$\approx 6341.2$$

Hence, the amount after 3 years is

$$A \approx \text{Rs. } 6341.2.$$

EXERCISE 1.5

1. A population is given by $N = 300e^{0.02t}$, where t is measured in hours. Find the time when the population becomes 600.
2. A culture grows according to $N = 150e^{kt}$, where time t is measured in hours. After 10 hours, the population becomes 250. Find the growth constant k.
3. A radioactive substance decays as $N = 800e^{-0.05t}$, where t is in hours. Find the time when the amount reduces to 200.
4. A substance decays according to $N = 600e^{kt}$, where time t is measured in hours. After 8 hours, the amount becomes 400. Find the decay constant k.
5. A sound has intensity $I = 10^{-6}$ W/m^2 and reference intensity $I_0 = 10^{-12}$ W/m^2. Find the sound level in decibels.
6. A sound has level $L = 80$ dB and $I_0 = 10^{-12}$ W/m^2. Find the intensity I.
7. A quantity changes according to $N = 500e^{kt}$. If $N = 250$ when $t = 5$, find:
   (i) the value of constant k.
   (ii) the quantity N when $t = 10$.
8. Two sounds have intensities I_1 and I_2 such that $L_1 = 85$ dB and $L_2 = 65$ dB.
   Find the ratio $\frac{I_1}{I_2}$, where $I_0 = 10^{-12}$ W/m^2.
9. A sum of Rs. 4000 is invested at 6% per annum, compounded annually for 3 years. Find the final amount.
10. A sum of Rs. 5000 becomes 6324.48 when invested at 8% per annum, compounded annually. Find the time t.
11. A sum of Rs. 3000 is invested at 5% per annum, compounded continuously for 4 years. Find the final amount.
12. An amount of Rs. 2000 becomes 3320.12 under continuous compounding at 6% per annum. Find the time t.