UNIT 9

Tangent and Angles of a Circle

Students' Learning Outcomes

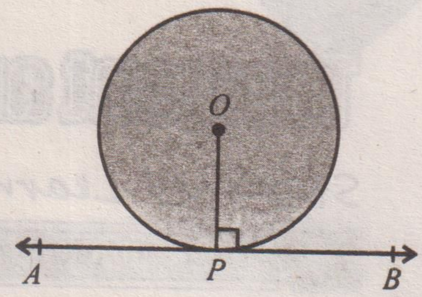
After completing this unit, the students will be able to:

- Solve problems by using the tangent and angle properties of a circle:
  - If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.
  - The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.
  - The two tangents drawn to a circle from a point outside it, are equal in length.
  - If two circles touch externally or internally, then the distance between their centres is respectively equal to the sum or difference of their radii.
  - The measure of a central angle of a minor arc of a circle is double that of the angle subtended by the corresponding major arc.
  - Any two angles in the same segment of a circle are equal.
  - The angle in a semi-circle is a right angle, in a segment greater than a semi-circle is less than a right angle, in a segment less than a semi-circle is greater than a right angle.
  - The opposite angles of any quadrilateral inscribed in a circle are supplementary.
- Find the arc length and area of sector of a circle using angle in both degrees and radians.
- Apply concepts of tangents and angles of a circle to real life world problems (such as architecture, monuments, pyramids).
- Apply trigonometry to solve real life problems in arc length and area of the sector of circle.

INTRODUCTION

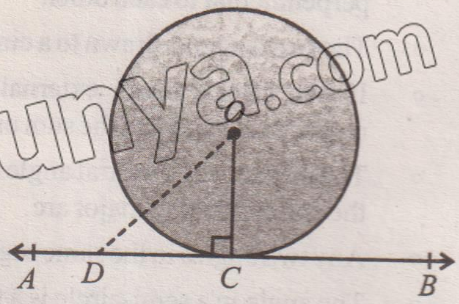
This unit focuses on solving problems using the tangent and angle properties of a circle, which are essential for understanding the geometry of circular shapes in both mathematical and real-life contexts. Students will explore key concepts such as the perpendicular relationship between a radius and a tangent, the equality of tangents drawn from an external point and the geometric properties of angles formed within and around circles. The unit also examines how the positioning of angles affects their measures and how these properties relate to cyclic quadrilaterals. Practical applications include solving problems involving architecture, engineering structures, monuments and pyramids, where circular and angular relationships are commonly found. Students will also learn how to calculate arc lengths and areas of sectors using angles measured in both degrees and radians and apply trigonometric concepts to real-world scenarios involving sectors of a circle.

9.1 Tangent to a Circle



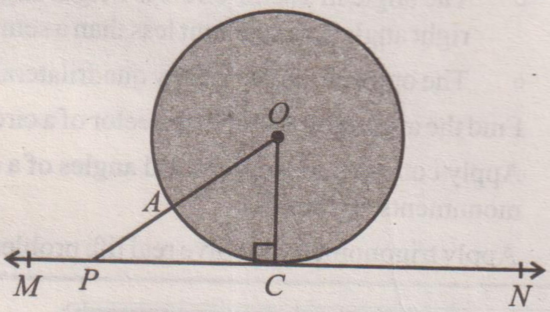
In a plane, a line is a tangent to a circle that touches only one point on the circumference of a circle. Here $\overleftrightarrow{AB}$ is a tangent at point P on the circle. Point P is called **point of tangency** or **point of contact.**

9.1.1 (a) If a Line is Drawn Perpendicular to a Radial Segment of a Circle at its Outer End Point, it is Tangent to the Circle at that Point



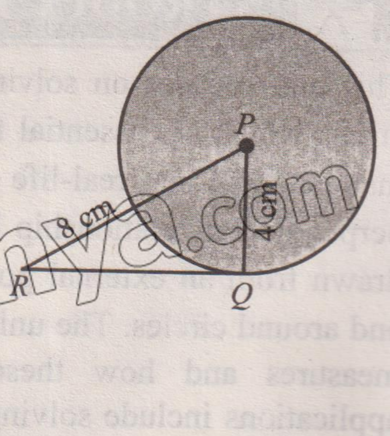
If a circle with centre O and $\overline{OC}$ is its radial segment. $\overleftrightarrow{AB} \perp \overline{OC}$ at C, then $\overleftrightarrow{AB}$ is tangent to the circle at point C.

b The Tangent to a Circle and the Radial Segment Joining the Point of Contact and the Centre, are Perpendicular to Each Other.



If a circle with centre O, $\overline{OC}$ is a radial segment. $\overleftrightarrow{MN}$ is a tangent at point C, then $\overline{OC} \perp \overleftrightarrow{MN}$

Example 1 $\overline{QR}$ is a tangent to a circle with centre P at point Q of radius 4 cm. It meets the line segment PR such that $m\overline{PR} = 8$ cm. What is the length of $\overline{QR}$?



Solution Since $\overline{QR}$ is tangent to the circle at Q and it is perpendicular to the $\overline{PQ}$.

ΔPQR is a right-angled triangle and $\overline{PR}$ is its hypotenuse.

By using Pythagoras theorem,

$$\left(m\overline{PR}\right)^2 = \left(m\overline{PQ}\right)^2 + \left(m\overline{QR}\right)^2$$

$$8^2 = 4^2 + \left(m\overline{QR}\right)^2$$

$$64 = 16 + \left(m\overline{QR}\right)^2$$

$$\left(m\overline{QR}\right)^2 = 64 - 16 = 48$$

$$m\overline{QR} = 6.93 \text{ cm}$$

Real Life Examples of Tangents Perpendicular to Radius

1- The entry or exit roads of a roundabout are often designed tangential to the circular structure to allow smooth transitions and minimize turning effort.

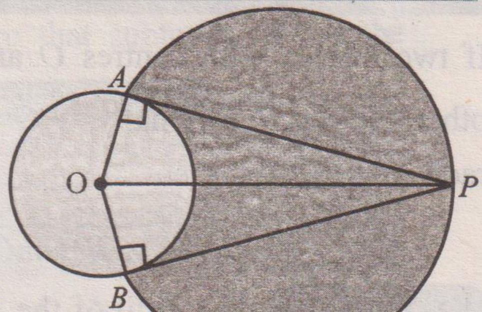
2- Each seating cabin on a Ferris wheel hangs such that its suspension arm is perpendicular to the wheel radius, ensuring the cabin stays tangent to the rotational path.

3- The tip of a clock hand touches the circular perimeter of the clock and if a tangent is drawn at the tip, it is perpendicular to the radius (hand) at that point.

9.1.2 The Two Tangents Drawn to a Circle From a Point Outside it, are Equal in Length

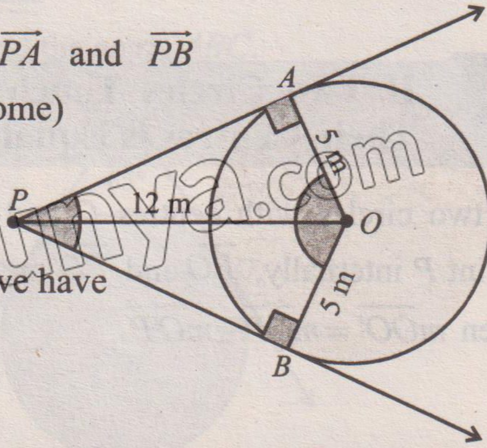
If $\overline{PA}$ and $\overline{PB}$ are two tangents drawn from an external point P to a circle having centre at O, where A and B are points of contact, then

$$m\overline{PA} = m\overline{PB}.$$



Example 2 An architect designs a circular dome on a building. A lighting rod is placed 12 m away from the centre of the dome which has a radius of 5 m. Support cables are attached from the rod to the dome, touching the dome at two points. What is the length of each cable?

Solution Suppose two tangents (cables) $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are drawn to the circumference of the circle (dome) from the point P (on the rod). Here,



$$m\overline{OP} = 12\,m,\ m\overline{OA} = m\overline{OB} = 5\text{ m}$$

In the right triangle OAP, by Pythagoras theorem, we have

$$\left(m\overline{OP}\right)^2 = \left(m\overline{OA}\right)^2 + \left(m\overline{AP}\right)^2$$

$$12^2 = 5^2 + \left(m\overline{AP}\right)^2$$

$$144 = 25 + \left(m\overline{AP}\right)^2$$

$$\left(m\overline{AP}\right)^2 = 144 - 25 = 119$$

$$m\overline{AP} = 10.91 \text{ m}$$

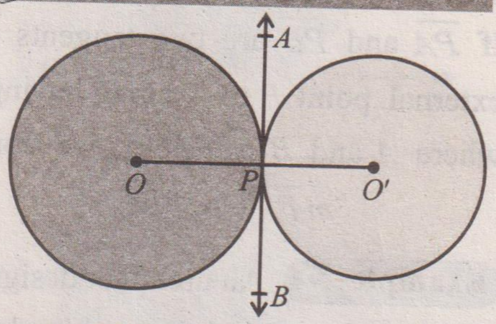
So, the length of each cable is 10.91 m

Real-Life Examples of Equal Tangents to a Circle

1. A lamp post is erected at a point outside a circular garden. Two paths are constructed from the lamp post to the edge of the circular garden such that they just touch the garden. These paths represent tangents to the circle.
2. A security camera is placed such that it monitors a circular restricted zone. The lines of sight just touch the boundary of the zone.
3. Two ropes from a peg outside a circular tent are tied to two points on the base circle of the tent.

9.1.3 (a) If Two Circles Touch Externally, then the Distance Between Their Centres is Equal to the Sum of their Radii.

If two circles with centres O and O' touch each other at point P externally, $\overline{PO}$ and $\overline{PO'}$ are their radii respectively, then



$$m\overline{OO'} = m\overline{OP} + m\overline{O'P}$$

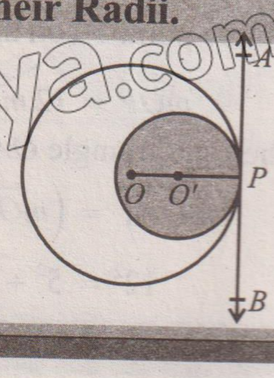
Example 3 The radii of the two circles are 4cm and 6cm. What is the distance between their centres if they touch externally?

Solution Since the circles touch externally,
So, the distance between their centres = sum of their radii

$$= 4 + 6 \quad = 10 \text{ cm}$$

b If Two Circles Touch Internally, Then the Distance Between Their Centres is Equal to the Difference of Their Radii.

If two circles with centres O and O' touch each other at point P internally, $\overline{PO}$ and $\overline{PO'}$ are their radii respectively, then $m\overline{OO'} = m\overline{OP} - m\overline{O'P}$.



Example 4 An inner decorative circle touches the outer clock face internally. If the outer circle has radius 12 cm and the inner has radius 7.5 cm, find the distance between their centres.

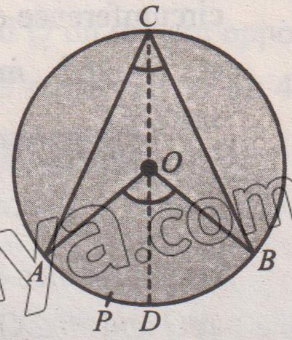
Solution Since, Radius of outer circle = 12 cm

Radius of inner circle = 7.5 cm

Thus, the distance between their centres = 12 – 7.5 = 4.5 cm

9.1.4 The Measure of a Central Angle of a Minor Arc of a Circle is Double that of the Angle Subtended by the Corresponding Major Arc.

In the circle with centre O, $\widehat{APB}$ is a minor arc. It subtends an angle AOB at the centre and an angle ACB is subtended by the major arc, then $m\angle AOB = 2m\angle ACB$



Example 5 A curved running track forms an arc that subtends 60° at the centre of the circular ground. What angle does it subtend at a point on the boundary?

Solution Since curved running track forms an arc that subtends 60° at the centre of the circular ground.

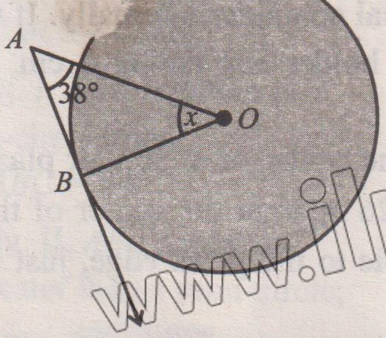
As we know that the measure of an angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

Therefore, angle at circumference $= \dfrac{60°}{2} = 30°$

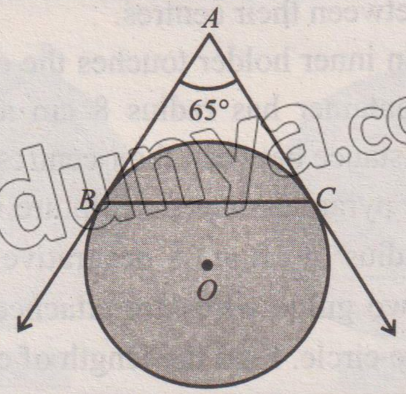
> **Skilled Practice!**
> A radar antenna scans an arc subtending 90° at the centre. What is the angle subtended by the arc at the edge of its circular path?

EXERCISE 9.1

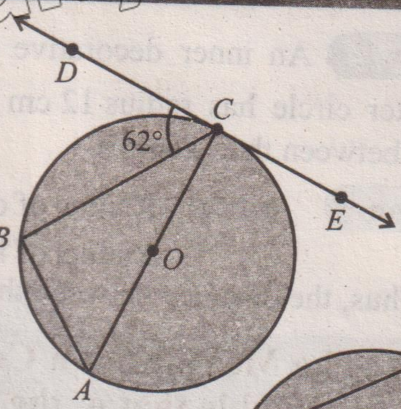
1. Find the value of x.



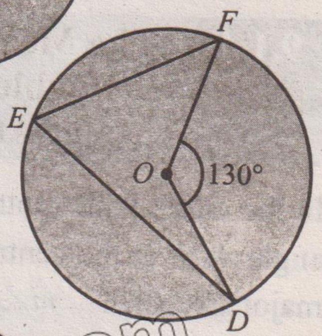
2. Find the angle ABC.



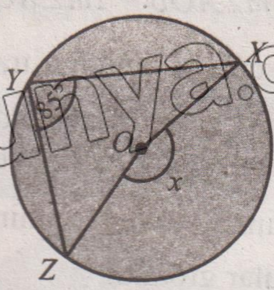
3. A, B and C are points on the circumference of a circle with centre at O. $\overline{AC}$ is the diameter of the circle and $\overleftrightarrow{DE}$ is the tangent to the circle at the point C and $m\angle BCD = 62°$. Find
(i) $m\angle BCA$ (ii) $m\angle BAC$



4. D, E and F are points on the circumference of a circle with centre at O and $m\angle DOF = 130°$. Find $m\angle DEF$.



5. X, Y and Z are points on the circumference of a circle with centre at O and $m\angle XYZ = 85°$. Find x.



6. In a historical monument, a circular fountain with a radius of 3 m is built. A flagpole is erected 7 m away from the centre of the fountain. Two ropes from the pole are tied to the edge of the fountain, just touching it. Find the length of each rope.

7. Two circular gears touch each other externally for proper rotation in a machine. The radii of the two circular gears are 5cm and 7cm. What is the distance between their centres if they touch externally?

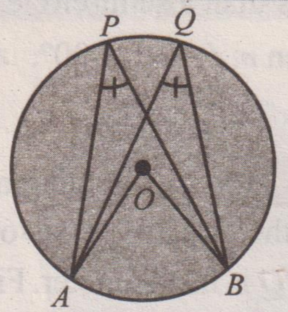
8. A small sensor lies inside a satellite dish and touches its wall internally. If the dish has radius 15 cm and the sensor has radius 2.5 cm, find the distance between their centres.

9. An inner holder touches the outer cylindrical container internally. If the outer container has radius 8 cm and the inner holder has radius 6 cm, find the distance between their centres.

10. A pyramid-shaped sculpture is placed in the center of a circular plaza with a radius of 10 m. A decorative pole stands 26 m from the center of the circle. Two guide wires are attached from the pole to the plaza edge, just touching the circle. Find the length of each wire.

9.1.5 Any Two Angles in the Same Segment of a Circle are Equal.

If $\angle APB$ and $\angle AQB$ be angles in the same segment $APQB$, of a circle whose centre is O, then $m\angle APB = m\angle AQB$

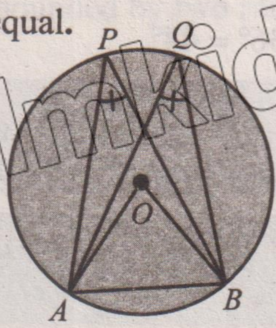


Example 6 In a large circular stained-glass window of a library, two decorative beams are drawn from the same chord AB to two points P and Q on the arc opposite to $\widehat{AB}$ such that $\angle APB$ and $\angle AQB$ are formed. If $\angle APB$ is measured to be 40°, what is $m\angle AQB$?

Solution According to the article 9.1.5, angles in the same segment are equal.

Since $m\angle APB = 40°$

Therefore, $m\angle AQB = 40°$



Skilled Practice!

A projector casts an image over an arc on a circular wall. Two observers on the arc segment report one observing a 50° angle. What is the other angle in the same segment?

9.1.6

(a) The Angle in a Semi-Circle is a Right Angle.

(b) The Angle in a Segment Greater than a Semi-Circle is Less than a Right-Angle.

(c) The Angle in a Segment Less than a Semi-Circle is Greater than a Right-Angle.

A circle with centre at O and $\overline{AC}$ is a chord of the circle.

(a) In Fig. I, ABC segment is a semi-circle, then $m\angle ABC = 90°$.

(b) In Fig. II, ABC segment is greater than a semi-circle, then $m\angle ABC < 90°$.

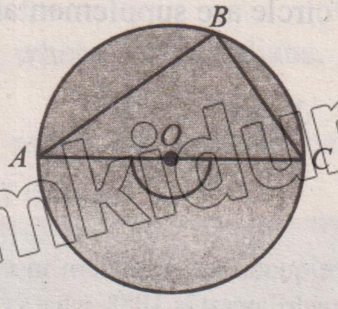


Figure I

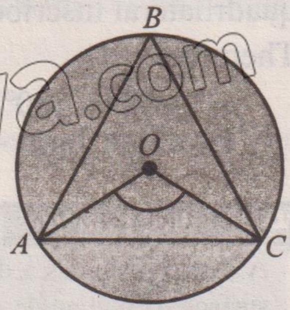


Figure II

(c) In Fig. III, ABC segment is less than a semi-circle, then $m\angle ABC > 90°$.

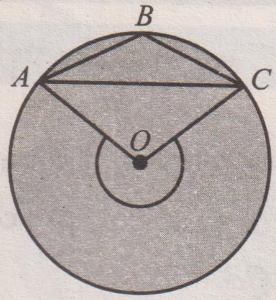
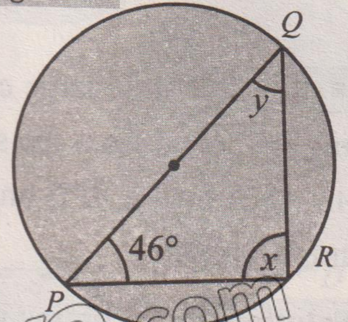


Figure III

Example 7 In the diagram P, Q and R are points on the circumference of a circle and the line segment PQ is the diameter. Find the angles x and y.



Solution Since angle in semi-circle is 90°.

Therefore, $x = 90°$

As we know that the sum of the interior angles of any triangle is 180°.

$$46° + 90° + y = 180°$$

i.e., $$136° + y = 180°$$

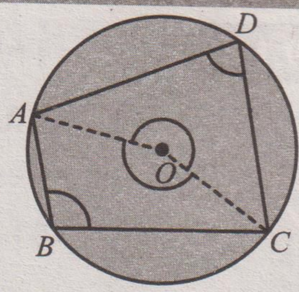
$$y = 180° - 136° = 44°$$

Skilled Practice!

A cable in a semicircular bridge forms a triangle with the diameter of the arch. If the cable meets the endpoints of the diameter and the top of the arch, what is the angle at the top?

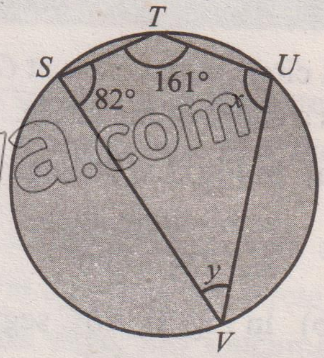
9.1.7 The Opposite Angles of Any Quadrilateral Inscribed in a Circle are Supplementary.

If a circle with centre at O and $ABCD$ is an inscribed quadrilateral, then



$$m\angle CBA + m\angle ADC = 180°, \quad m\angle BAD + m\angle BCD = 180°$$

Example 8 S, T, U and V are points on the circumference of a circle. Find the angles x and y.



Solution As we know that opposite angles of any quadrilateral inscribed in a circle are supplementary.

Therefore,

$x + 82° = 180°$	$y + 161° = 180°$
$x = 98°$	$y = 19°$

Skilled Practice!

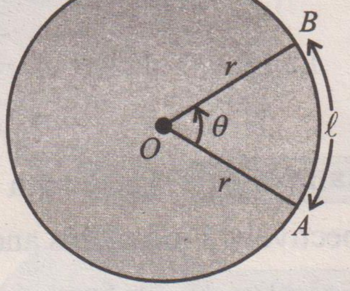
A round table has a decorative quadrilateral pattern inscribed on its surface. If one angle of the quadrilateral is 105°, what is the measure of the opposite angle?

9.2 Area of Sector and Arc Length of a Circle

9.2.1 Arc Length and Area of Sector

Let ℓ be the length of a circular arc AB of a circle of radius r and θ be its central angle measure in radians. Then

Ratio of ℓ to the circumference $2\pi r$ = Ratio of θ to 2π



or $\quad \ell : 2\pi r = \theta : 2\pi$

or $$\frac{\ell}{2\pi r} = \frac{\theta}{2\pi}$$

$$\ell = \frac{\theta}{2\pi} \times 2\pi r$$

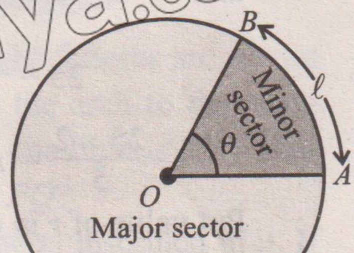
$\ell = r\theta \quad$, where θ is in radian.

9.2.2 Area of a Sector

A sector of a circle is a region bounded by two radii and corresponding arc.

A slice of Pizza and region between the hands of a watch are examples of a sector of circle.

A pair of radii divides the circle into two regions namely major sector and minor sector.



The area of a sector of a circular region of radius r is $\frac{1}{2}r^2\theta$ where θ is the central angle and it is measured in radians.

Proof: As shown in adjoining figure,

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\text{Central angle of sector}}{\text{Complete angle in circle}}$$

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$A = \frac{\pi r^2\theta}{2\pi}$$

$\Rightarrow \quad$ Area of sector $= \frac{1}{2}r^2\theta$, where θ is in radians.

Note

$$A = \frac{1}{2}r^2\theta$$

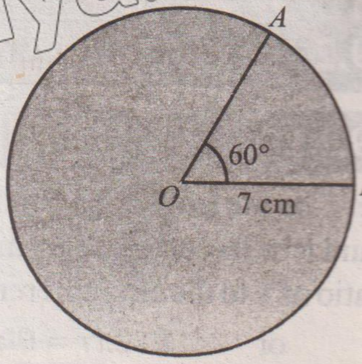
$\because \quad \ell = r\theta$

So, $\quad A = \frac{1}{2}r(r\theta)$

$$A = \frac{1}{2}r\ell$$

Example 9 Find the area of sector of a circle having central angle 60° and radius 7 cm. Also find the length of the arc.

Solution Here $\theta = 60° = \frac{60 \times \pi}{180} = 1.047\,\text{rad}, \; r = 7\text{ cm}$



$$\text{Area of sector} = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}(7)^2(1.047)$$

$$= \frac{1}{2}(49)(1.047) = 25.65\text{ cm}^2$$

$$\text{Length of arc } AB = r\theta$$

$$= (7)(1.047) = 7.33\text{ cm}$$

Example 10 The length of an arc and area of sector of a circle are 5cm and 25cm^2 respectively. Find radius and the central angle of sector.

Solution $\ell = 5\text{cm}, A = 25\text{cm}^2, r = ?, \theta = ?$

As $\ell = r\theta \Rightarrow 5 = r\theta$

$$\theta = \frac{5}{r} \quad \text{.......... (i)}$$

and $A = \frac{1}{2}r^2\theta$

$$25 = \frac{1}{2}r^2\left(\frac{5}{r}\right) \quad \text{using eq. (i)}$$

$$\frac{25 \times 2}{5} = r \Rightarrow r = 10\text{cm}$$

Put value of r in (i), we have

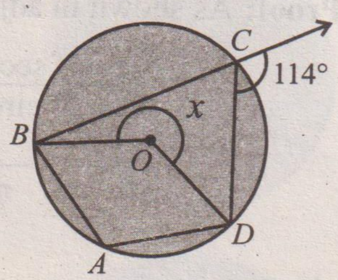
$$\theta = \frac{5}{10} = \frac{1}{2}\text{ radian}$$

Skilled Practice!

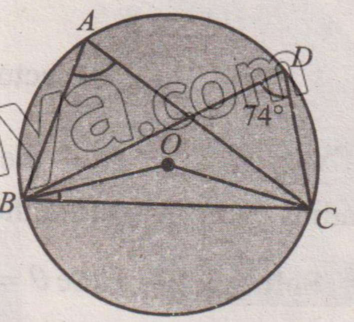
If ℓ is the arc length and r is the radius of the circle where $\ell = 14\text{m}, r = 4\text{m}$, then find the central angle. Also find the area of the sector of the circle.

EXERCISE 9.2

1. A, B, C and D are points on circumference of a circle with centre at O as shown in figure. Find the angle x.

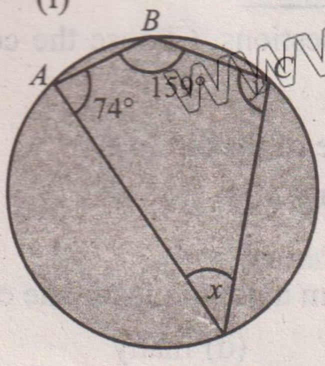


2. In the adjoining figure $m\angle BDC = 74°$, find $m\angle BAC$, $m\angle BOC$ and $m\angle OBC$.

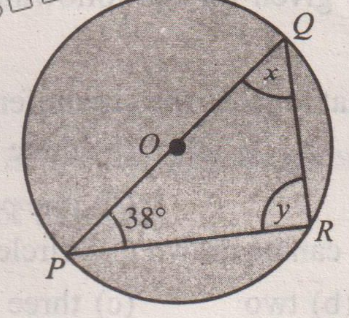


3. Find the angles x and y in the following figures.

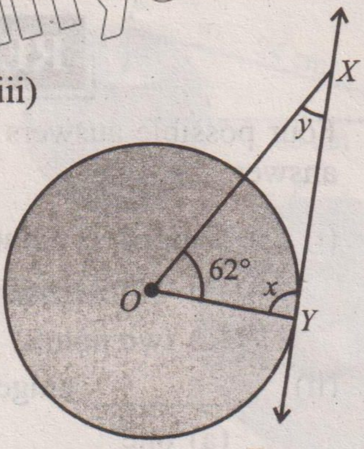
(i)



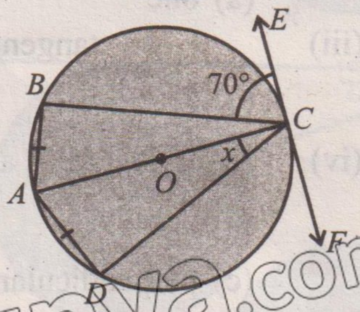
(ii)



(iii)



4. Find the angle x in the given figure.



5. In a circular arch over a monument entrance, two spotlight fixtures are placed such that they each shine from two different points on the arch to the same chord PQ on the base of the arch. If the angle formed at one light is 55°, what is the angle at the second light on the same side of chord PQ?

6. A circular garden has a walking path forming a quadrilateral inscribed in it. If one angle is 87°, what is the opposite angle?

7. A ferris wheel has a radius of 12 m. If a passenger travels a distance of 18 m along the circumference of the ferris wheel, what is the angle (in radians) swept by the passenger's position from the starting point?

8. Find θ, when
   (i) $\ell = 3$ cm, $r = 2.2$ cm (ii) $\ell = 5.6$ cm, $r = 2$ cm

9. Find r, when
   (i) $\ell = 5.5$ cm, $\theta = 40°20'$ (ii) $\ell = 13$ cm, $\theta = 70°$

10. Find ℓ and area of sector, when
    (i) $r = 1.7$ cm, $\theta = 0.25$ radian (ii) $r = 3$ cm, $\theta = 45°$

11. Uzma cut a pizza of radius 14 cm into 8 equal slices. What is the area of one slice (sector).

12. The perimeter and area of a sector are 14 cm and 10 cm^2 respectively. Find the radius of the circle and the central angle of the sector.

REVIEW EXERCISE 9

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) Tangent is a line that touches the circumference of the circle at :
(a) many points (b) three points
(c) two points (d) one point

(ii) ________ tangents can be drawn to a circle from a point outside the circle.
(a) one (b) two (c) three (d) many

(iii) ________ tangents can be drawn to a circle from centre of the circle.
(a) zero (b) one (c) two (d) three

(iv) In circle, radius and tangent are:
(a) parallel (b) equal
(c) perpendicular (d) zero

(v) An angle in a segment greater than a semicircle is ______ angle:
(a) an acute (b) right (c) an obtuse (d) straight

(vi) The angle subtended by the arc at the centre of the circle is called:
(a) acute angle (b) central angle
(c) right angle (d) complete angle

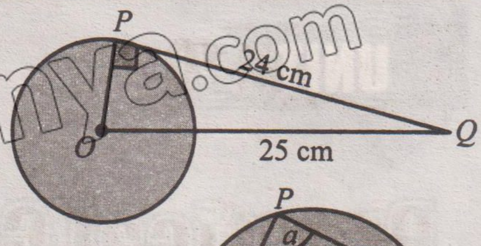
(vii) What is the measure of an angle inscribed in a semi-circle?
(a) 45° (b) 60°
(c) 90° (d) 120°

(viii) Any two angles in the same segment of the circle are:
(a) supplementary (b) complementary
(c) equal (d) zero

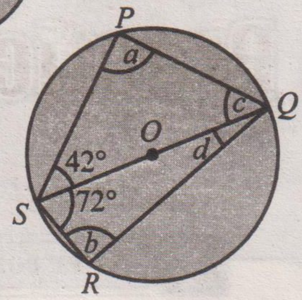
(ix) Area of the sector of the circle = ________
(a) $\frac{1}{2}r\theta$ (b) $\frac{3}{2}r^2\theta$ (c) $\frac{1}{2}r^2\theta$ (d) $\frac{3}{2}r\theta$

(x) If $r = 6$cm and $\theta = 2$ radians, then arc length is:
(a) 12 cm (b) 8 cm (c) 6 cm (d) 3 cm

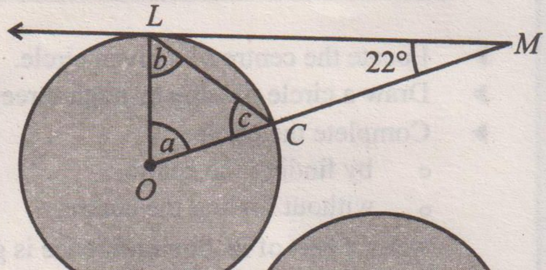
2. In the adjoining figure, find the radius of the circle.



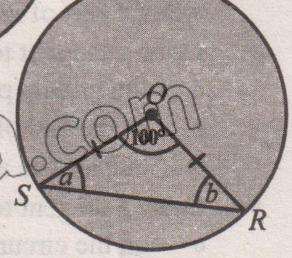
3. Find the angles a, b, c and d in the given figure, if O is the centre of the circle.



4. Find the angles a, b and c, if $\overrightarrow{ML}$ is a tangent line and O is the centre of the circle.



5. Find the angles marked in the given figure, where O is the centre of the circle.



6. Two round dining tables are arranged to touch at their edges. The radii of the two round dining tables are 10 cm and 12 cm. What is the distance between their centres if they touch externally?

7. A circular mosaic pattern is laid on the floor beneath a modern pyramid structure. Two tiles are placed such that they both connect to the same chord EF and reach two points G and H on the arc above EF. If $m\angle EGF = 65°$, find $m\angle EHF$.

8. A triangular frame within a semicircular gate joins the ends of the diameter to the peak. What is the angle formed at the peak?

9. A circular birthday cake has a radius of 15cm. A slice has an arc length of 10cm. What is the area of this slice?

10. A sector cut from a circle of radius 5cm has a perimeter of 16cm. Find area of this sector.