UNIT 3

Matrices and Determinants

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Display information in the form of matrix of order 2.
- Solve situations involving sum, difference and product of two matrices.
- Calculate the product of the scalar quantity and a matrix.
- Evaluate the determinant and inverse of a matrix of order 2-by-2.
- Solve the simultaneous linear equations in two variables using matrix inversion method and Cramer's rule.
- Explain, with examples, how mathematics plays a key role in the development of new scientific theories and technologies. (e.g., Mathematical models and simulations are used to design and optimize new materials and drugs, and to understand the behaviours of complex systems such as the human brain).
- Apply concepts of matrices to real world problems (such as engineering, economics, computer graphics and physics).

INTRODUCTION

The evolution of the theory of 'matrices' is the result of attempts to obtain compact and simple methods for solving system of linear equations. In 1850, it was **James Joseph Sylvester** an English Mathematician and lawyer who used the word matrix. In 1858, Arthur Cayley published **Memoir on the theory of matrices** which was remarkable for containing the first abstract definition of a matrix. The mathematicians James Joseph Sylvester (1814-1897), William Rowan Hamilton (1805-1865) and **Arthur Cayley** (1821-1895) played important role in the development of matrix theory. English Mathematician **Cullis** was the first to use modern brackets notation for matrices in 1913. The knowledge of matrices is absolutely necessary not only within the branches of Mathematics but also in other areas of science, genetics, economics, sociology, modern psychology and industrial management.

3.1 Matrices

A matrix is a rectangular array in shape, whose elements (entries) are written within square brackets in a specific order, in rows and columns. The arrays obey certain algebraic operations.

Generally, the matrices are denoted by the capital letters of the English alphabets while their elements are denoted by the small letters.

e.g $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B=[5 \quad 7]$, $C=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$D=\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$, $E=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $F=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The time table in schools, flight schedule at air-port or arrival and departure time of trains at the railway station are written in the matrices form.

3.1.1 Rows and Columns of a Matrix

Row: Entry or entries written in the horizontal line or lines form the row or rows of a matrix.

Column: Entry or entries written in vertical line or lines form the column or columns of a matrix.

For example,

Column 1, Column 2, Column 3, Column 4

$$A=\begin{bmatrix} 2 & 3 & 7 & 1 \\ 1 & 5 & 3 & 2 \\ 9 & 1 & 8 & 4 \end{bmatrix} \begin{matrix} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \\ \leftarrow \text{Row 3} \end{matrix}$$

Matrix A has 3 rows and 4 columns.

Column 1, Column 2, Column 3

$$B=\begin{bmatrix} 7 & 4 & 8 \\ 3 & 0 & 10 \\ 2 & 1 & 11 \end{bmatrix} \begin{matrix} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \\ \leftarrow \text{Row 3} \end{matrix}$$

Matrix B has 3 rows and 3 columns.

$$C=[\,5\,]$$

Matrix C has 1 row and 1 column.

3.1.2 Elements or Entries

If $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then 1, 2, 3, 4, 5, 6 are called elements or entries of matrix A.

3.1.3 Order or Size of a Matrix

If a matrix A has 'm' number of rows and 'n' number of columns, then the order or size of the matrix A is m-by-n. Order of matrix $B=\begin{bmatrix} 5 & -1 & 0 \\ 9 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ is 3-by-3. If order of a matrix is 3-by-4, it means it has 3 rows and 4 columns. It does not mean 3 by 4 = 12, however 12 gives the number of elements in the matrix.

3.1.4 Equal Matrices

Two matrices A and B are said to be equal if and only if they are of same order and their corresponding elements are same.

For example, $A=\begin{bmatrix} 2 & 3 \\ 4 & -2 \end{bmatrix}$, $B=\begin{bmatrix} 4-2 & 3 \\ 4 & 4-6 \end{bmatrix}$ are equal matrices.

Similarly, $C=\begin{bmatrix} a & b \end{bmatrix}$, $D=\begin{bmatrix} 3 & 4 \end{bmatrix}$ will be equal matrices if and only if $a=3$ and $b=4$.

EXERCISE 3.1

1. Write the number of rows and number of columns in each matrix.

$A=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$, $C=\begin{bmatrix} 8 & -10 & 11 \end{bmatrix}$,

$D=\begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $E=\begin{bmatrix} 5 & 9 & -2 \\ -3 & 4 & 5 \end{bmatrix}$

2. Write the order of each matrix.

$A=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 4 \end{bmatrix}$, $C=\begin{bmatrix} 3 & 5 \\ 5 & 6 \end{bmatrix}$, $D=\begin{bmatrix} 0 & 3 & 4 \\ 5 & 0 & 9 \end{bmatrix}$, $E=\begin{bmatrix} 0 & 1 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $F=\begin{bmatrix} 5 \end{bmatrix}$

3. Which of the following matrices are equal?

$A=\begin{bmatrix} 0 & 3 \\ 4 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 9 \\ -7 \end{bmatrix}$, $C=\begin{bmatrix} 2\times 3 & 2-1 \\ 2\times 2 & 4-2 \\ 4+4 & 3+0 \end{bmatrix}$,

$D=\begin{bmatrix} 5+4 \\ -8+1 \end{bmatrix}$, $E=\begin{bmatrix} 6 & 1 \\ 4 & 2 \\ 8 & 3 \end{bmatrix}$, $F=\begin{bmatrix} 3 & 3 & 3 \\ 3+1 & 1 \end{bmatrix}$

4. If $\begin{bmatrix} a+2 & c-3 \\ b-1 & d+4 \end{bmatrix}=\begin{bmatrix} 5 & 8 \\ 6 & 4 \end{bmatrix}$, then find the values of a, b, c and d.

5. If $\begin{bmatrix} 2x+1 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 7 & y \end{bmatrix}$, then find the values of x and y.

6. If $\begin{bmatrix} a+b & 2d-1 \\ 3b+2 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 11 & c \end{bmatrix}$, then find the values of a, b, c and d.

7. If $\begin{bmatrix} p+q & 5 \\ 11 & p-2q \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 11 & 0 \end{bmatrix}$, then find the values of p and q.

3.2 Types of Matrices

(i) Row Matrix: A matrix having only one row is called a row matrix.

For example: $A = [0\ \ 1\ \ 4]$, $B = [a\ \ b]$, $C = [a]$ are row matrices of order 1-by-3, 1-by-2 and 1-by-1 respectively.

(ii) Column Matrix: A matrix having only one column is called a column matrix.

For example: $A = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, $B = \begin{bmatrix} a \\ b \end{bmatrix}$, $C = [2]$ are column matrices of order 3-by-1, 2-by-1 and 1-by-1 respectively.

(iii) Zero matrix or Null matrix

If all the entries in a matrix are zero, it is called zero matrix or null matrix. It is represented by O.

For example: $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices of order 1-by-1, 2-by-2 and 2-by-3 respectively.

(iv) Square Matrix: A matrix in which number of rows is equal to the number of columns is called a square matrix. A matrix of order n-by-n is often referred to as a square matrix of order n.

For example: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & l \end{bmatrix}$ is a square matrix of order 3.

In a square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

a_{11}, a_{22}, a_{33} are called entries of principal diagonal or main diagonal or leading diagonal.

(v) Rectangular Matrix: If the number of rows and number of columns in a matrix are not equal, then it is called a rectangular matrix.

For example: $A=\begin{bmatrix} a & b \end{bmatrix}$, $B=\begin{bmatrix} a \\ b \end{bmatrix}$, $C=\begin{bmatrix} 4 & 0 & 7 \\ 8 & 3 & 1 \end{bmatrix}$ are rectangular matrices.

(vi) Diagonal Matrix: A square matrix is called a diagonal matrix if at least any one of the elements of its main diagonal is non-zero and non-diagonal elements are zero.

For example: $A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ are diagonal matrices of order 3-by-3 and 2-by-2 respectively.

(vii) Scalar Matrix: A diagonal matrix is called a scalar matrix if all the entries in the main diagonal are the same and non-zero.

For example: $A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, $C=\begin{bmatrix} \sqrt{3} \end{bmatrix}$ are scalar matrices of order 3-by-3, 2-by-2 and 1-by-1 respectively.

(viii) Identity Matrix or Unit Matrix: A scalar matrix is called an identity or unit matrix if all of its main diagonal entries are 1.

For example: $P=\begin{bmatrix} 1 \end{bmatrix}$, $Q=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1-by-1, 2-by-2 and 3-by-3 respectively. We represent these matrices as:

$$I_1=\begin{bmatrix} 1 \end{bmatrix}, \quad I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remember!
A scalar matrix and identity matrix are also diagonal matrices but all diagonal matrices are not scalar or identity matrices.

(ix) Transpose of a Matrix: A matrix obtained from a given matrix by interchanging the rows and columns is called the transpose of the given matrix. If A is a matrix, then its transpose is represented as A'.

For example: $A=\begin{bmatrix} 1 & 2 & 4 \\ 8 & 0 & 5 \end{bmatrix} \Rightarrow A'=\begin{bmatrix} 1 & 8 \\ 2 & 0 \\ 4 & 5 \end{bmatrix}$

(x) Symmetric Matrix: A square matrix A is called symmetric if it is equal to its transpose. i.e., $A^t = A$.

For example: If $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, then $A^t = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = A$

Thus, A is a symmetric matrix.

(xi) Skew-Symmetric Matrix: A square matrix A is called skew-symmetric if $A^t = -A$.

For example: If $B = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$, then $B^t = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = -B$

Thus, B is a skew-symmetric matrix.

(xii) Negative of a Matrix: The negative of a matrix A is obtained by multiplying each of its elements by -1. It is denoted by $-A$.

For example: If $A = \begin{bmatrix} 3 & -6 \end{bmatrix}$, then

$-A = \begin{bmatrix} -3 & 6 \end{bmatrix}$

EXERCISE 3.2

1. From the following matrices identify unit matrices, row matrices, column matrices and null matrices.

$A = \begin{bmatrix} 5 & 7 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}$, $F = \begin{bmatrix} 7 \\ 1 \\ 9 \end{bmatrix}$

2. Identify type of the given matrices as row, column, square and rectangular matrices.

$A = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 5 & 8 \\ 0 & 4 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & -5 \\ 2 & 7 \end{bmatrix}$,

$E = \begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 5 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 5 & -3 & 7 \end{bmatrix}$, $G = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 4 \\ 5 & 2 & -3 \end{bmatrix}$, $H = \begin{bmatrix} 3 & 5 \\ 4 & 4 \\ 5 & 2 \end{bmatrix}$

3. Identify diagonal, scalar and unit matrices.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$,

$D=\begin{bmatrix}7&0\\0&6\end{bmatrix}$, $E=\begin{bmatrix}2&0\\0&2\end{bmatrix}$

4. Find transpose of each of the following matrices:

$A=\begin{bmatrix}2&3\\4&5\end{bmatrix}$, $B=\begin{bmatrix}3\\7\\6\end{bmatrix}$, $C=\begin{bmatrix}5&-2&4\end{bmatrix}$, $D=\begin{bmatrix}2&5\\3&6\\4&7\end{bmatrix}$

5. Find negative of the following matrices:

$A=\begin{bmatrix}-3&0\\5&6\end{bmatrix}$, $B=\begin{bmatrix}-3&3\\-2&2\end{bmatrix}$, $C=\begin{bmatrix}-9&1\\1&-7\end{bmatrix}$

6. If $A=\begin{bmatrix}2&3\\4&5\end{bmatrix}$ and $B=\begin{bmatrix}7&6\\5&8\end{bmatrix}$, then verify that

(i) $(A^t)^t=A$ (ii) $(B^t)^t=B$

7. Show that $L=\begin{bmatrix}2&3&4\\3&-2&5\\4&5&0\end{bmatrix}$ is a symmetric matrix.

3.3 Algebraic Operations on Matrices

Basic operations on matrices are:

(i) Multiplication of a matrix by a scalar.
(ii) Addition / subtraction of two matrices.
(iii) Multiplication of two matrices.

There is no concept of dividing a matrix by another matrix.

3.3.1 Multiplication of a Matrix by a Scalar

For a given matrix $A=\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}$ and a scalar k, we define a new matrix kA

as $kA=\begin{bmatrix}ka&kb&kc\\kd&ke&kf\end{bmatrix}$, that is, each element of A has been multiplied by a scalar

k. In particular if $k=-1$, then $-A=\begin{bmatrix}-a&-b&-c\\-d&-e&-f\end{bmatrix}$.

3.3.2 Addition of Matrices

If A and B are two matrices of the **same order,** these can be added, then their sum denoted by $A + B$, is a matrix of the **same order**. They are said to be **conformable for addition.** $A + B$ is obtained by adding the **corresponding entries** of A and B.

Example 1 If $A = \begin{bmatrix} 3 & 8 & 2 \\ 5 & 3 & 1 \\ 6 & 0 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 & -2 \\ 3 & 0 & 6 \\ 2 & 3 & 4 \end{bmatrix}$, then find $A + B$ and $B + A$.

Solution Given that $A = \begin{bmatrix} 3 & 8 & 2 \\ 5 & 3 & 1 \\ 6 & 0 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 & -2 \\ 3 & 0 & 6 \\ 2 & 3 & 4 \end{bmatrix}$

Order of A = 3-by-3 and order of B = 3-by-3, A and B are conformable for addition

Now, $$A + B = \begin{bmatrix} 3+4 & 8+1 & 2-2 \\ 5+3 & 3+0 & 1+6 \\ 6+2 & 0+3 & 7+4 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 0 \\ 8 & 3 & 7 \\ 8 & 3 & 11 \end{bmatrix}$$

and $$B + A = \begin{bmatrix} 4+3 & 1+8 & -2+2 \\ 3+5 & 0+3 & 6+1 \\ 2+6 & 3+0 & 4+7 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 0 \\ 8 & 3 & 7 \\ 8 & 3 & 11 \end{bmatrix}$$

3.3.3 Subtraction of Matrices

If A and B are two matrices of the **same order,** they can be subtracted and their difference is denoted as $A - B$ or $B - A$, it is again a matrix of the **same order.** They are said to be **conformable for subtraction.** $A - B$ obtained by subtracting the corresponding entries of B from A.

Example 2 If $A = \begin{bmatrix} 5 & 9 \\ 4 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 6 \\ 2 & 1 \end{bmatrix}$, then find $A - B$ and $B - A$.

Solution $A = \begin{bmatrix} 5 & 9 \\ 4 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 6 \\ 2 & 1 \end{bmatrix}$

Now, $$A - B = \begin{bmatrix} 5-3 & 9-6 \\ 4-2 & 8-1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 7 \end{bmatrix}$$

and $$B - A = \begin{bmatrix} 3-5 & 6-9 \\ 2-4 & 1-8 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -2 & -7 \end{bmatrix}$$

We see $A - B \neq B - A$

Example 3 If $A=\begin{bmatrix}2 & 4\\1 & 5\end{bmatrix}$, $B=\begin{bmatrix}3 & 4 & 1\\5 & 2 & 7\end{bmatrix}$, then find $A+B$ and $B-A$, if possible.

Solution $A=\begin{bmatrix}2 & 4\\1 & 5\end{bmatrix}$, $B=\begin{bmatrix}3 & 4 & 1\\5 & 2 & 7\end{bmatrix}$

Order of A is 2-by-2 and order of B is 2-by-3. Therefore, A and B are not conformable for addition and subtraction.

3.3.4 Commutative and Associative Laws of Addition of Matrices

a Commutative Law of Addition of Matrices

For any two matrices A and B of same order $A+B=B+A$. This law is called commutative law of matrices with respect to addition.

Let $A=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$, $B=\begin{bmatrix}3 & 1 & 3\\4 & 2 & 2\\0 & -1 & 0\end{bmatrix}$

Then $A+B=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}+\begin{bmatrix}3 & 1 & 3\\4 & 2 & 2\\0 & -1 & 0\end{bmatrix}$

$$=\begin{bmatrix}1+3 & 2+1 & 3+3\\4+4 & 5+2 & 6+2\\7+0 & 8-1 & 9+0\end{bmatrix}=\begin{bmatrix}4 & 3 & 6\\8 & 7 & 8\\7 & 7 & 9\end{bmatrix} \quad ...(i)$$

and $B+A=\begin{bmatrix}3 & 1 & 3\\4 & 2 & 2\\0 & -1 & 0\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$

$$=\begin{bmatrix}3+1 & 1+2 & 3+3\\4+4 & 2+5 & 2+6\\0+7 & -1+8 & 0+9\end{bmatrix}=\begin{bmatrix}4 & 3 & 6\\8 & 7 & 8\\7 & 7 & 9\end{bmatrix} \quad ...(ii)$$

Hence, from (i) and (ii) $A+B=B+A$

Commutative law of addition of matrices holds.

b Associative Law of Addition of Matrices

For any three matrices A, B and C of same order $(A+B)+C=A+(B+C)$. This law is called associative law of matrices with respect to addition.

Let $A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$, $B=\begin{bmatrix}3&1&3\\4&2&2\\0&-1&0\end{bmatrix}$, $C=\begin{bmatrix}0&2&0\\3&0&3\\1&1&1\end{bmatrix}$

then $(A+B)+C=\left(\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}+\begin{bmatrix}3&1&3\\4&2&2\\0&-1&0\end{bmatrix}\right)+\begin{bmatrix}0&2&0\\3&0&3\\1&1&1\end{bmatrix}$

$$=\begin{bmatrix}1+3&2+1&3+3\\4+4&5+2&6+2\\7+0&8-1&9+0\end{bmatrix}+\begin{bmatrix}0&2&0\\3&0&3\\1&1&1\end{bmatrix}$$

$$=\begin{bmatrix}4&3&6\\8&7&8\\7&7&9\end{bmatrix}+\begin{bmatrix}0&2&0\\3&0&3\\1&1&1\end{bmatrix}=\begin{bmatrix}4&5&6\\11&7&11\\8&8&10\end{bmatrix} \quad ...(i)$$

and $A+(B+C)=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}+\left(\begin{bmatrix}3&1&3\\4&2&2\\0&-1&0\end{bmatrix}+\begin{bmatrix}0&2&0\\3&0&3\\1&1&1\end{bmatrix}\right)$

$$=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}+\begin{bmatrix}3+0&1+2&3+0\\4+3&2+0&2+3\\0+1&-1+1&0+1\end{bmatrix}$$

$$=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}+\begin{bmatrix}3&3&3\\7&2&5\\1&0&1\end{bmatrix}=\begin{bmatrix}4&5&6\\11&7&11\\8&8&10\end{bmatrix} \quad ...(ii)$$

Hence, from (i) and (ii) $(A+B)+C=A+(B+C)$

Associative law of addition of matrices holds.

3.3.5 Additive Identity of a Matrix

The additive identity of a matrix is a matrix that, when added to any matrix A of the same order, leaves it unchanged.

Mathematically, $A+O=A=O+A$, where O is the zero matrix (also called the **additive identity matrix**).

Let $A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$, $O=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

Then $A+O=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=A$

and $O+A=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=A$

3.3.6 Additive Inverse of a Matrix

The additive inverse of a matrix A is another matrix $-A$ such that

$$A+(-A)=O=(-A)+A.$$

Let $A=\begin{bmatrix}-2&1&3\\4&2&0\\-3&0&5\end{bmatrix}$, then $-A=\begin{bmatrix}2&-1&-3\\-4&-2&0\\3&0&-5\end{bmatrix}$

Now $A+(-A)=\begin{bmatrix}-2&1&3\\4&2&0\\-3&0&5\end{bmatrix}+\begin{bmatrix}2&-1&-3\\-4&-2&0\\3&0&-5\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O$

$(-A)+A=\begin{bmatrix}2&-1&-3\\-4&-2&0\\3&0&-5\end{bmatrix}+\begin{bmatrix}-2&1&3\\4&2&0\\-3&0&5\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O$

Thus, A and $-A$ are additive inverse of each other.

EXERCISE 3.3

1. Which of the following matrices are conformable for addition and subtraction?

$A=\begin{bmatrix}3&4\\-2&1\end{bmatrix}$, $B=\begin{bmatrix}3\\4\end{bmatrix}$, $C=\begin{bmatrix}5&2\end{bmatrix}$, $D=\begin{bmatrix}1&7\\2&5\end{bmatrix}$, $E=\begin{bmatrix}2\end{bmatrix}$,

$F=\begin{bmatrix}7&11\end{bmatrix}$, $G=\begin{bmatrix}a\\b\end{bmatrix}$, $H=\begin{bmatrix}3\end{bmatrix}$, $M=\begin{bmatrix}l\\m\end{bmatrix}$

2. If $X=\begin{bmatrix}1&-1\\-2&2\end{bmatrix}$, $Y=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ and $Z=\begin{bmatrix}3&0\\0&-2\end{bmatrix}$, then find the following:

(i) $X+Y$ (ii) $Y+7Z$ (iii) $4X-Z$

(iv) $X + 2Y + 3Z$ (v) $X - 4Y + Z$ (vi) $Z - Z$

3. Find the additive inverse of the following matrices:

(i) $P = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$ (ii) $Q = \begin{bmatrix} 9 & -3 \end{bmatrix}$ (iii) $R = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$ (iv) $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$, then verify the following:

(i) $A + B = B + A$ (ii) $(A + B) + C = A + (B + C)$

(iii) $(2A + B) + C = 2A + (B + C)$ (iv) $3(A + B) = 3A + 3B$

5. If $A = \begin{bmatrix} 5 & 6 \\ 7 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -6 \\ -7 & 2 \end{bmatrix}$, then show that B is additive inverse of A and A is additive inverse of B.

6. If $A = \begin{bmatrix} 6 & -2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -1 \\ 3 & 0 \end{bmatrix}$, then verify that

(i) $(A + B)^t = A^t + B^t$ (ii) $(A - B)^t = A^t - B^t$

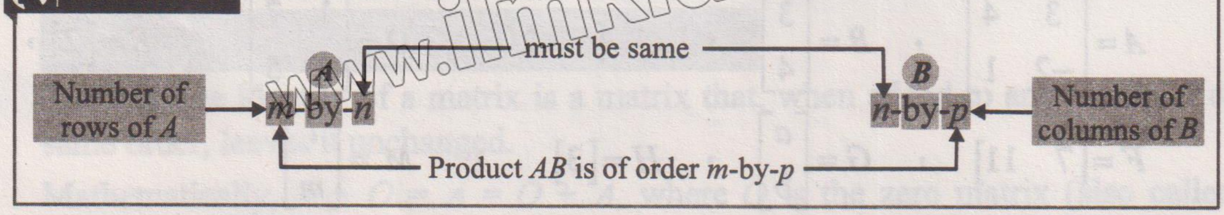
3.4 Multiplication of Matrices

A matrix A is said to be conformable for multiplication with a matrix B if the number of columns of A is equal to the number of rows of B.

Thus, if A and B are given two matrices of order m-by-n and n-by-p respectively, then the product of matrices A and B is denoted by AB and its order will be m-by-p.

The order of AB is m-by-p = number of rows of A by number of columns of B.

Remember!



Let us explain the concept of multiplication of matrices with the help of an example from our daily life.

Talmeez purchased 7 books and 6 pencils at the rate of Rs. 20 and Rs. 3 per article respectively from shop No. 1 and he purchased same number of books and pencils at the rate of Rs. 22 and Rs. 4 per article respectively from Shop No. 2. Majid purchased 5 books and 8 pencils from Shop No. 1 at the same rate Talmeez purchased. Majid purchased same number of books and pecils from Shop No. 2 at the same rate Talmeez purchased.

We show their purchase in the form of tables.

Table 1

	Books	Pencils
Talmeez	7	6
Majid	5	8

Table 2

	Shop No. 1	Shop No. 2
Rate per book	20	22
Rate per pencil	3	4

Amount paid by Talmeez at Shop No. 1.

$7\times 20+6\times 3=158$ rupees ...(i)

Amount paid by Talmeez at Shop No. 2

$7\times 22+6\times 4=178$ rupees ...(ii)

Amount paid by Majid at shop No. 1

$5\times 20+8\times 3=124$ rupees ...(iii)

Amount paid by Majid at shop No. 2

$5\times 22+8\times 4=142$ rupees ...(iv)

We show entries of the two tables in the form of matrices.

Table 1 $\begin{bmatrix}7 & 6\\ 5 & 8\end{bmatrix}$ **Table 2** $\begin{bmatrix}20 & 22\\ 3 & 4\end{bmatrix}$

Calculations given above is shown in the form of matrix.

$$\begin{bmatrix}7\times 20+6\times 3 & 7\times 22+6\times 4\\ 5\times 20+8\times 3 & 5\times 22+8\times 4\end{bmatrix} \text{ see (i), (ii), (iii) and (iv)}$$

$$=\begin{bmatrix}158 & 178\\ 124 & 142\end{bmatrix}$$

For example, $A=\begin{bmatrix}9 & 7\\ 4 & 3\end{bmatrix}, B=\begin{bmatrix}2 & 6\\ 5 & 8\end{bmatrix}$

Now, $AB=\begin{bmatrix}9 & 7\\ 4 & 3\end{bmatrix}\begin{bmatrix}2 & 6\\ 5 & 8\end{bmatrix}$

$$= \begin{bmatrix} 9\times 2+7\times 5 & 9\times 6+7\times 8 \\ 4\times 2+3\times 5 & 4\times 6+3\times 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 18+35 & 54+56 \\ 8+15 & 24+24 \end{bmatrix} = \begin{bmatrix} 53 & 110 \\ 23 & 48 \end{bmatrix}$$

Remember!

For Multiplication

If A and B are two matrices and we multiply A with B, when:

(i) Number of columns in A = Number of rows in B.

(ii) Multiplication of two matrices A and B is written as AB.

(iii) To find AB, we start with the first row of A and multiply its each element with the corresponding elements of the first column of B and add the products. This gives the first element of the first row of the product matrix AB.

Next, we multiply each element of the first row of A with the corresponding elements of second column of B and at the product. This give the second element of first row of the product matrix AB.

Similarly, we multiply all the rows of A with each column of B and add the products to get remaining elements of the matrix AB.

Example 4 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} l & m \\ n & p \end{bmatrix}$, then find:

(i) AB, if possible. (ii) BA, if possible.

Solution (i) Number of columns in matrix $A = 2$

Number of rows in matrix $B = 2$

Thus, AB is possible

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} l & m \\ n & p \end{bmatrix}$$

$$= \begin{bmatrix} al+bn & am+bp \\ cl+dn & cm+dp \end{bmatrix}$$

(ii) Number of columns in matrix $B = 2$

Number of rows in matrix $A = 2$

Thus, BA is also possible

$$BA = \begin{bmatrix} l & m \\ n & p \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} la+mc & lb+md \\ na+pc & nb+pd \end{bmatrix}$$

We note that $AB \neq BA$

Example 5 If $A = \begin{bmatrix} 2 & 0 \\ 5 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, then find:

(i) AB, if possible. (ii) BA, if possible.

Solution (i) Number of columns in matrix $A = 2$

Number of rows in matrix $B = 2$

Thus, AB is possible

Now $AB = \begin{bmatrix} 2 & 0 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 \times (-2) + 0 \times 1 \\ 5 \times (-2) + (-3) \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 + 0 \\ -10 - 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -13 \end{bmatrix}$$

(ii) $BA = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 5 & -3 \end{bmatrix}$

Number of columns in matrix $B = 1$

Number of rows in matrix $A = 2$

Thus, BA is not possible

> **Skilled Practice!**
>
> If $A = [5 \quad 3]$ and $B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, then find AB and its order.

3.4.1 Commutative Law of Multiplication of Matrices

Consider $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix}$

$$AB = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \times 1 + 2 \times 0 & 5 \times 5 + 2 \times (-2) \\ 3 \times 1 + 4 \times 0 & 3 \times 5 + 4 \times (-2) \end{bmatrix}$$

$$= \begin{bmatrix} 5 + 0 & 25 - 4 \\ 3 + 0 & 15 - 8 \end{bmatrix} = \begin{bmatrix} 5 & 21 \\ 3 & 7 \end{bmatrix} \quad \text{...(i)}$$

Now $BA = \begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 1 \times 5 + 5 \times 3 & 1 \times 2 + 5 \times 4 \\ 0 \times 5 + (-2) \times 3 & 0 \times 2 + (-2) \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5+15 & 2+20 \\ 0-6 & 0-8 \end{bmatrix} = \begin{bmatrix} 20 & 22 \\ -6 & -8 \end{bmatrix} \quad ...(ii)$$

From (i) and (ii), it is verified that $AB \neq BA$

Thus, commutative law under multiplication does not hold in general in the multiplication of matrices.

3.4.2 Associative Law of Multiplication of Matrices

If A, B and C are any three matrices, then $(AB)C = A(BC)$ is called associative law of multiplication of matrices.

Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1(2)+0(0) & 1(1)+0(3) \\ 2(2)+1(0) & 2(1)+1(3) \end{bmatrix} = \begin{bmatrix} 2+0 & 1+0 \\ 4+0 & 2+3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$$

Now, $(AB)C = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2(1)+1(4) & 2(2)+1(0) \\ 4(1)+5(4) & 4(2)+5(0) \end{bmatrix} = \begin{bmatrix} 2+4 & 4+0 \\ 4+20 & 8+0 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 24 & 8 \end{bmatrix} \quad ...(i)$$

$$BC = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+1(4) & 2(2)+1(0) \\ 0(1)+3(4) & 0(2)+3(0) \end{bmatrix} = \begin{bmatrix} 2+4 & 4+0 \\ 0+12 & 0+0 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 12 & 0 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 6 & 4 \\ 12 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(6)+0(12) & 1(4)+0(0) \\ 2(6)+1(12) & 2(4)+1(0) \end{bmatrix} = \begin{bmatrix} 6+0 & 4+0 \\ 12+12 & 8+0 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 24 & 8 \end{bmatrix} \quad ...(ii)$$

Hence, from (i) and (ii)

$(AB)C = A(BC)$

Associative law of multiplication of matrices holds.

3.4.3 Distributive Laws of Multiplication Over Addition and Subtraction

(a) Let A, B and C be any three matrices, then distributive laws of multiplication over addition are as follows:

(i) $A(B+C) = AB + AC$ (Left distributive law over addition)

(ii) $(A+B)C = AC + BC$ (Right distributive law over addition)

Let $A=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$, $B=\begin{bmatrix}3 & 1\\2 & 4\end{bmatrix}$, $C=\begin{bmatrix}1 & 0\\5 & 2\end{bmatrix}$

$$A(B+C)=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\left(\begin{bmatrix}3 & 1\\2 & 4\end{bmatrix}+\begin{bmatrix}1 & 0\\5 & 2\end{bmatrix}\right)$$

$$=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}4 & 1\\7 & 6\end{bmatrix}$$

$$=\begin{bmatrix}1(4)+2(7) & 1(1)+2(6)\\0(4)+1(7) & 0(1)+1(6)\end{bmatrix}$$

$$=\begin{bmatrix}4+14 & 1+12\\0+7 & 0+6\end{bmatrix}=\begin{bmatrix}18 & 13\\7 & 6\end{bmatrix} \quad ...(i)$$

$$AB=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\2 & 4\end{bmatrix}=\begin{bmatrix}3+4 & 1+8\\0+2 & 0+4\end{bmatrix}=\begin{bmatrix}7 & 9\\2 & 4\end{bmatrix}$$

$$AC=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\5 & 2\end{bmatrix}=\begin{bmatrix}1+10 & 0+4\\0+5 & 0+2\end{bmatrix}=\begin{bmatrix}11 & 4\\5 & 2\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}7 & 9\\2 & 4\end{bmatrix}+\begin{bmatrix}11 & 4\\5 & 2\end{bmatrix}=\begin{bmatrix}7+11 & 9+4\\2+5 & 4+2\end{bmatrix}=\begin{bmatrix}18 & 13\\7 & 6\end{bmatrix} \quad ...(ii)$$

Hence, from (i) and (ii)

$$A(B+C)=AB+AC$$

Similarly, we can verify right distributive law over addition.

(b) If A, B and C be any three matrices, then distributive laws of multiplication over subtraction are as follows:

(i) $A(B-C)=AB-AC$ (Left distributive law over subtraction)

(ii) $(A-B)C=AC-BC$ (Right distributive law over subtraction)

Let $A=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$, $B=\begin{bmatrix}4 & 3\\1 & 2\end{bmatrix}$, $C=\begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}$

$$A(B-C)=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\left(\begin{bmatrix}4 & 3\\1 & 2\end{bmatrix}-\begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}\right)$$

$$=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}4-2 & 3-1\\1-0 & 2-1\end{bmatrix}=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 2\\1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1(2)+2(1) & 1(2)+2(1)\\0(2)+1(1) & 0(2)+1(1)\end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & 2+2 \\ 0+1 & 0+1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \quad ...(i)$$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1(4)+2(1) & 1(3)+2(2) \\ 0(4)+1(1) & 0(3)+1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 4+2 & 3+4 \\ 0+1 & 0+2 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 1 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(2)+2(0) & 1(1)+2(1) \\ 0(2)+1(0) & 0(1)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 1+2 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$AB - AC = \begin{bmatrix} 6 & 7 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6-2 & 7-3 \\ 1-0 & 2-1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \quad ...(ii)$$

Hence, from (i) and (ii)

$$A(B-C) = AB - AC$$

Similarly, we can verify right distributive law over subtraction.

3.4.4 Multiplicative Identity of a Matrix

The multiplicative identity of a matrix is a matrix that, when multiplied with any matrix A, does not change it. i.e., $AI = IA = A$

Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

$$AI = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2(1)+3(0) & 2(0)+3(1) \\ 4(1)+1(0) & 4(0)+1(1) \end{bmatrix} = \begin{bmatrix} 2+0 & 0+3 \\ 4+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1(2)+0(4) & 1(3)+0(1) \\ 0(2)+1(4) & 0(3)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 3+0 \\ 0+4 & 0+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = A$$

Hence, I is multiplicative identity of a matrix A.

EXERCISE 3.4

1. Find AB and BA, if possible.

(i) $A=\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}$, $B=\begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix}$ (ii) $A=[1 \quad -2]$, $B=\begin{bmatrix}3\\ -4\end{bmatrix}$

(iii) $A=\begin{bmatrix}4\\ 4\end{bmatrix}$, $B=[2 \quad 5]$ (iv) $A=\begin{bmatrix}1 & 2\\ -1 & 1\\ 3 & 0\end{bmatrix}$, $B=\begin{bmatrix}-1 & 4 & 1\\ 2 & 3 & 1\end{bmatrix}$

(v) $A=\begin{bmatrix}1 & 2 & 2\\ 3 & 1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 5\\ 2 & 4\\ 1 & 6\end{bmatrix}$

2. Verify each statement, using $A=\begin{bmatrix}5 & 1\\ -1 & 4\end{bmatrix}$, $B=\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}$ and $C=\begin{bmatrix}5 & 1\\ 1 & 4\end{bmatrix}$.

(i) $AB \neq BA$ (ii) $A(B-C)=AB-AC$

(iii) $A(BC)=(AB)C$ (iv) $(BC)^t=C^tB^t$

(v) $(B+C)A=BA+CA$

3. If $\begin{bmatrix}4 & a\\ b & 3\end{bmatrix}\begin{bmatrix}6\\ 6\end{bmatrix}=\begin{bmatrix}6\\ 3\end{bmatrix}$, then find the values of a and b.

4. If $\begin{bmatrix}x & 1\\ y & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 3 & -1\end{bmatrix}=\begin{bmatrix}7 & -1\\ 4 & -2\end{bmatrix}$, then find the values of x and y.

3.5 Multiplicative Inverse of a Matrix

3.5.1 Determinant of a Matrix of Order 2

Let $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ be a matrix of order 2. Then the determinant of A is defined as:

$$|A|=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$$

For example, Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ be a square matrix. Then $|A|=\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$ is called the determinant of matrix A and $ad-bc$ is called value of determinant. We can note that the elements of A and the elements of $|A|$ are the same. The brackets [ ] of the matrix is replaced by two vertical lines "| |" in the determinant.

Example 6 Find value of the determinant of $B = \begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$.

Solution

$$|B| = \begin{vmatrix} 8 & 2 \\ 4 & 3 \end{vmatrix}$$
$$= 8 \times 3 - 4 \times 2$$
$$= 24 - 8 = 16$$

Remember!
The value of the determinant is not altered by changing the rows into columns and columns into rows.

3.5.2 Singular and Non-Singular Matrices

A square matrix A is said to be **singular** if $|A| = 0$. A square matrix A is said to be **non-singular** if $|A| \neq 0$.

For example, Let $A = \begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 4 & 8 \\ 2 & 4 \end{vmatrix} = 4 \times 4 - 8 \times 2 = 16 - 16 = 0$

Thus, matrix A is a singular matrix.

For example, Let $M = \begin{bmatrix} 7 & 3 \\ 4 & 9 \end{bmatrix}$

Now $|M| = \begin{vmatrix} 7 & 3 \\ 4 & 9 \end{vmatrix} = 7 \times 9 - 4 \times 3$

$$= 63 - 12 = 51 \neq 0$$

Thus, matrix M is a non-singular matrix.

3.5.3 Adjoint of a Matrix

The adjoint of a square matrix is an important concept in matrix algebra, especially when finding the inverse of a matrix or solving system of linear equations.

The adjoint of a square matrix A of order 2 is obtained by interchanging the diagonal elements and changing the signs of other elements. It is denoted as Adj A.

If $A = \begin{bmatrix} 4 & 2 \\ -6 & 1 \end{bmatrix}$, then Adj $A = \begin{bmatrix} 1 & -2 \\ 6 & 4 \end{bmatrix}$

3.5.4 Multiplicative Inverse of a Non-Singular Matrix

We know $\frac{3}{4}$ and $\frac{4}{3}$ are multiplicative inverse (or reciprocal) as the product $\left(\frac{3}{4} \times \frac{4}{3}\right)$ is 1.

Similarly, if we have two square matrices of same order and their product is a unit matrix, then these matrices are called multiplicative inverse of each other.

Remember!
The inverse of a singular matrix does not exist.

Let A be any square matrix and there exists a square matrix B having the same order as A such that $AB = BA = I$ (a unit matrix). Then A and B are multiplicative inverse of each other. Hence $AA^{-1} = A^{-1}A = I$.

We denote multiplicative inverse of A by A^{-1}.

3.5.5 Adjoint Method to Find Inverse of a Square Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a non-singular matrix, then multiplicative inverse of A is:

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

i.e., $$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 7 If $A = \begin{bmatrix} 8 & 5 \\ 4 & 3 \end{bmatrix}$, then find multiplicative inverse of A and verify that

$$AA^{-1} = A^{-1}A = I$$

Challenge!
If $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}$, then verify $(AB)^{-1} = B^{-1}A^{-1}$
Can we say it is also a Reversal Law of Multiplicative Inverse?

Solution

$$A = \begin{bmatrix} 8 & 5 \\ 4 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 8 & 5 \\ 4 & 3 \end{vmatrix}$$

$$= 8 \times 3 - 5 \times 4$$

$$= 24 - 20 = 4 \neq 0$$

A is a non-singular matrix, so A^{-1} is possible.

$$A^{-1} = \frac{1}{|A|}\text{Adj } A$$

$$= \frac{1}{4}\begin{bmatrix} 3 & -5 \\ -4 & 8 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{-5}{4} \\ -1 & 2 \end{bmatrix}$$

Let us verify that A and A^{-1} are multiplicative inverse of each other.

and $\quad AA^{-1} = \begin{bmatrix} 8 & 5 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{-5}{4} \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 8 \times \frac{3}{4} + 5 \times (-1) & 8 \times \left(\frac{-5}{4}\right) + 5 \times 2 \\ 4 \times \frac{3}{4} + 3 \times (-1) & 4 \times \left(\frac{-5}{4}\right) + 3 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Now, $\quad A^{-1}A = \frac{1}{4}\begin{bmatrix} 3 & -5 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} 8 & 5 \\ 4 & 3 \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} 3 \times 8 + (-5) \times 4 & 3 \times 5 + (-5) \times 3 \\ (-4) \times 8 + 8 \times 4 & (-4) \times 5 + 8 \times 3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 24-20 & 15-15 \\ -32+32 & -20+24 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, $AA^{-1} = A^{-1}A = I$

Hence, $\begin{bmatrix} 8 & 5 \\ 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} \frac{3}{4} & \frac{-5}{4} \\ -1 & 2 \end{bmatrix}$ are multiplicative inverse of each other.

Example 8 Prove that $A = \begin{bmatrix} 4 & -2 \\ -6 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$ are multiplicative inverse of each other.

Solution Given that $A = \begin{bmatrix} 4 & -2 \\ -6 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$

Now,

$$AB = \begin{bmatrix} 4 & -2 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix} = \begin{bmatrix} 4\times 1 + (-2)\times \frac{3}{2} & 4\times \frac{1}{2} + (-2)\times 1 \\ (-6)\times 1 + 4\times \frac{3}{2} & (-6)\times \frac{1}{2} + 4\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 2-2 \\ -6+6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Remember!

If A is a singular matrix i.e; $|A| = 0$, so in such a case, there is no inverse of matrix A.

and $$BA = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 4 + \frac{1}{2}\times(-6) & 1\times(-2) + \frac{1}{2}\times 4 \\ \frac{3}{2}\times 4 + 1\times(-6) & \frac{3}{2}\times(-2) + 1\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -2+2 \\ 6-6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, $AB = BA = I$

Therefore, $A = \begin{bmatrix} 4 & -2 \\ -6 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$ are multiplicative inverse of each other.

Example 9 If $A = \begin{bmatrix} x & 8 \\ 5 & 10 \end{bmatrix}$ is a singular matrix, then find the value of x.

Solution If $A = \begin{bmatrix} x & 8 \\ 5 & 10 \end{bmatrix}$ is a singular matrix, then $|A| = 0$

Now, $$|A| = \begin{vmatrix} x & 8 \\ 5 & 10 \end{vmatrix}$$

$$= 10x - 5\times 8 = 10x - 40$$

Since, A is singular, therefore,

$$|A| = 10x - 40 = 0$$

i.e. $10x = 40$

$\Rightarrow$ $x = 4$

Challenge!

Take any 2 by 2 matrix and check whether it is singular or non-singular. Also find its adjoint.

EXERCISE 3.5

1. Find the values of each of the determinant.

(i) $\begin{vmatrix} 10 & 5 \\ 4 & 6 \end{vmatrix}$ (ii) $\begin{vmatrix} -5 & 8 \\ -3 & -7 \end{vmatrix}$ (iii) $\begin{vmatrix} 3 & 8 \\ 0 & 2 \end{vmatrix}$

2. Find whether the following matrices are singular or non-singular.

$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 21 \\ 2 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 13 & 5 \\ 7 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

3. Find the value of x when $A = \begin{bmatrix} x & 6 \\ 5 & 15 \end{bmatrix}$ is a singular matrix.

4. Find the adjoint of the following matrices:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}$, $C = \begin{bmatrix} -3 & 5 \\ -3 & 2 \end{bmatrix}$

5. Find multiplicative inverse of the following matrices:

(i) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} -4 & 8 \\ 7 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 40 & 8 \\ 5 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$ (v) $\begin{bmatrix} 10 & 8 \\ 3 & 3 \end{bmatrix}$ (vi) $\begin{bmatrix} -2 & -3 \\ 4 & 5 \end{bmatrix}$

6. If $A = \begin{bmatrix} 5 & -3 \\ 2 & -1 \end{bmatrix}$, then find A^{-1} and prove that $AA^{-1} = A^{-1}A = I$.

7. Show that the following matrices are multiplicative inverse of each other.

(i) $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 15 \\ 2 & 8 \end{bmatrix}$, $\begin{bmatrix} 4 & -\frac{15}{2} \\ -1 & 2 \end{bmatrix}$

8. Prove that $(AB)^{-1} = B^{-1}A^{-1}$, if

(i) $A = \begin{bmatrix} -3 & -2 \\ 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ 8 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$

3.6 Solution of Simultaneous Linear Equations

A **linear equation** is an algebraic equation in which the highest power of the variable is 1. A general form of a linear equation in two variables is:

$ax + by = c$, where a, b, c are real numbers, x and y are variables and $a, b \neq 0$.

Simultaneous linear equations are a set of two or more linear equations with two or more variables that are solved together (simultaneously) to find a common solution. The general form of system of two linear equations in two variables is given below:

$$ax + by = m$$
$$cx + dy = n$$

where *a, b, c, d, m* and *n* are real numbers.

Here, we will find the solution of two simultaneous equations in two variables by the following methods:

(i) Matrix Inversion Method

(ii) Cramer's Rule

3.6.1 Matrix Inversion Method

Let $ax + by = m$
and $cx + dy = n$

These equations can be written in the matrix form as: $\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$

or $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$ (i)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} m \\ n \end{bmatrix}$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Now if $|A| \neq 0$, then (i) can be written as:

$AX = B$

and $A^{-1}AX = A^{-1}B$ [pre-multiply by A^{-1}]

$IX = A^{-1}B$ [as $A^{-1}A = I$]

Thus $X = A^{-1}B$ [as $IX = X$]

$$X = \frac{\text{Adj } A}{|A|} \cdot B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}}{ad-bc} = \frac{\begin{bmatrix} dm-bn \\ an-cm \end{bmatrix}}{ad-bc} = \begin{bmatrix} \dfrac{dm-bn}{ad-bc} \\ \dfrac{an-cm}{ad-bc} \end{bmatrix}$$

$$x = \frac{dm-bn}{ad-bc}, y = \frac{an-cm}{ad-bc}$$

Example 10 Solve the following by matrix inversion method:

$$2x + 3y = 13$$
$$4x - 5y = -7$$

Solution Given that $2x + 3y = 13$

$$4x - 5y = -7$$

Writing the equations in matrix form:

$$\begin{bmatrix} 2x+3y \\ 4x-5y \end{bmatrix} = \begin{bmatrix} 13 \\ -7 \end{bmatrix}$$

or $\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ -7 \end{bmatrix}$ ...(i)

Let $A = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 13 \\ -7 \end{bmatrix}$

Then (i) can be written as

$$AX = B$$

or $X = A^{-1}B$

Now $A = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix}$$
$$= (2)(-5) - (3)(4)$$
$$= -10 - 12$$
$$= -22 \neq 0$$

$$\text{Adj } A = \begin{bmatrix} -5 & -3 \\ -4 & 2 \end{bmatrix}$$

Therefore $A^{-1} = \frac{1}{|A|} \text{Adj} A$

$$= \frac{1}{-22}\begin{bmatrix} -5 & -3 \\ -4 & 2 \end{bmatrix}$$

Now $X = A^{-1}B$ can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-22}\begin{bmatrix} -5 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ -7 \end{bmatrix}$$

$$= \frac{1}{-22}\begin{bmatrix} (-5)\times 13 + (-3)\times(-7) \\ (-4)\times 13 + \quad 2\times(-7) \end{bmatrix}$$

$$= \frac{1}{-22}\begin{bmatrix} -65+21 \\ -52-14 \end{bmatrix}$$

$$= \frac{1}{-22}\begin{bmatrix} -44 \\ -66 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Thus $x = 2, y = 3$

Solution set is $\{(2, 3)\}$

Challenge!

Take any two linear equations in two variables and solve them by matrix inversion method.

History!

Gabriel Cramer was a Genevan mathematician. Cramer showed promised in Mathematics from an early age. At 18 he received his doctorate in Mathematics.

In 1750 he published Cramer's rule, giving a general formula for the solution for any unknown in linear equation system having a unique solution, in terms of determinants applied by the system. This rule is still standard.

Gabriel Cramer
(July 31, 1704 - January 4, 1752)

3.6.2 Cramer's Rule

Consider the following system of linear equations.

$$ax + by = m$$
$$cx + dy = n$$

These equations can be written in the matrix form as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$$

$$AX = B \qquad \text{...(i)}$$

Where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} m \\ n \end{bmatrix}$

If $|A| \neq 0$, then from equation (i)

$$A^{-1}AX = A^{-1}B \qquad \text{[pre multiplication by } A^{-1}\text{]}$$

$$IX = A^{-1}B$$

or $$X = A^{-1}B \qquad [\text{as } IX = X]$$

$$X = \frac{\text{Adj}\,A}{|A|} \times B \qquad \left[A^{-1} = \frac{\text{Adj}\,A}{|A|}\right]$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}}{|A|}$$

$$= \frac{\begin{bmatrix} dm - bn \\ -cm + an \end{bmatrix}}{|A|}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{dm - bn}{|A|} \\ \dfrac{-cm + an}{|A|} \end{bmatrix}$$

or $$x = \frac{dm - bn}{|A|}, \quad y = \frac{an - cm}{|A|}$$

or $$x = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{|A|} = \frac{|A_x|}{|A|}, \quad y = \frac{\begin{vmatrix} a & m \\ c & n \end{vmatrix}}{|A|} = \frac{|A_y|}{|A|}$$

Example 11 Use Cramer's rule to solve the system of equations:

$$2x + 3y = 13$$
$$4x - 5y = -7$$

Solution $2x + 3y = 13$

$4x - 5y = -7$

Challenge!

Take any two linear equations in two variables and solve them by using Cramer's rule.

Matrix form of the equations is:

$$\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ -7 \end{bmatrix}$$

Here, we have

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}, A_x = \begin{bmatrix} 13 & 3 \\ -7 & -5 \end{bmatrix}, A_y = \begin{bmatrix} 2 & 13 \\ 4 & -7 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix}$$

$$= (2)(-5) - (3)(4)$$

$$= -10 - 12$$

$$= -22 \neq 0$$

$$|A_x| = \begin{vmatrix} 13 & 3 \\ -7 & -5 \end{vmatrix}$$

$$= (13)(-5) - (3)(-7)$$

$$= -65 + 21 = -44$$

$$x = \frac{|A_x|}{|A|} = \frac{-44}{-22} = 2$$

$$|A_y| = \begin{vmatrix} 2 & 13 \\ 4 & -7 \end{vmatrix}$$

$$= (2)(-7) - (13)(4)$$

$$= -14 - 52 = -66$$

$$y = \frac{|A_y|}{|A|} = \frac{-66}{-22} = 3$$

Solution set $= \{(2, 3)\}$

3.7 Applications of Matrices in Real World Problems

Example 12 Three forces act on a particle which must be in equilibrium i.e. $F_1 + F_2 + F_3 = 0$, where $F_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, F_2 = \begin{bmatrix} -3 \\ x \end{bmatrix}, F_3 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$. Find the value of x.

Solution Since $F_1 + F_2 + F_3 = 0$

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ x \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5-3-2 \\ 2+x-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ x-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x-2=0$$

$$x=2$$

Example 13 Eight years ago Huria's was $\frac{3}{4}$ of Jannat's age. After four years Huria's age will be $\frac{6}{7}$ of Jannat's age. Find their present ages by using matrices.

Solution Suppose Huria's present age is x years and Jannat's present age is y years. Eight years ago their ages were $x-8$ and $y-8$. According to first condition:

$$x-8 = \frac{3}{4}(y-8)$$

or $\quad 4x-32 = 3y-24$

or $\quad 4x-3y = -24+32$

$\quad 4x-3y = 8 \quad$ ...(i)

After 4 years their ages will be $x+4$ and $y+4$.
Applying the second condition:

$$x+4 = \frac{6}{7}(y+4)$$

or $\quad 7x+28 = 6y+24$

$\quad 7x-6y = -4 \quad$ ...(ii)

Matrix form of (i) and (ii) is

$$\begin{bmatrix} 4 & -3 \\ 7 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

Here, $A = \begin{bmatrix} 4 & -3 \\ 7 & -6 \end{bmatrix}$, $A_x = \begin{bmatrix} 8 & -3 \\ -4 & -6 \end{bmatrix}$, $A_y = \begin{bmatrix} 4 & 8 \\ 7 & -4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & -3 \\ 7 & -6 \end{vmatrix}$$

$$= (4)(-6)-(7)(-3)$$

$$= -24+21$$

$$= -3 \neq 0$$

Applying Cramer's rule

$$x = \frac{|A_x|}{|A|} = \frac{\begin{vmatrix} 8 & -3 \\ -4 & -6 \end{vmatrix}}{-3} = \frac{-48-12}{-3} = \frac{-60}{-3} = 20$$

$$y = \frac{|A_y|}{|A|} = \frac{\begin{vmatrix} 4 & 8 \\ 7 & -4 \end{vmatrix}}{-3} = \frac{-16-56}{-3} = \frac{-72}{-3} = 24$$

∴ Present ages of Huria and Jannat are 20 years and 24 years respectively.

3.8 The Role of Mathematics in Scientific Theories and Technological Advancement

Mathematics is often described as the language of science. It plays a fundamental role in developing scientific theories and advancing modern technologies. From understanding the laws of nature to designing life-saving drugs, Mathematics provides the tools needed to describe, predict and optimize real-world phenomena.

One major area where Mathematics contributes is in the use of mathematical models and simulations. Scientists and engineers use these models to represent complex systems such as weather patterns, population growth or the functioning of the human brain. These models allow researchers to test hypotheses, predict future outcomes, and visualize scenarios that would be too costly, dangerous or impossible to replicate in the real world.

In medicine and materials science, for example, mathematical simulations are used to design and optimize new materials or drug molecules. This saves time and resources by reducing the need for trial-and-error experiments. Similarly, in fields like Physics and astronomy, equations and models help us to understand the formation of galaxies, black holes and subatomic particles.

Drug Discovery

Mathematical algorithms help simulate how a drug will interact with cells or proteins in the body, speeding up the discovery of effective treatments.

Neuroscience

Mathematical models are used to simulate brain activity, helping researchers understand learning, memory and diseases like epilepsy.

Climate Science

Complex equations describe the interaction of air, water and energy, helping scientists to predict weather and long-term climate change.

EXERCISE 3.6

1. Solve by matrix inversion method, if possible.

(i) $2x + 5y = 19$, $4x - 3y = -1$ (ii) $3x + 2y = 7$, $5x - y = 16$

(iii) $x - 2y = 9$, $2x + 7y = 4$ (iv) $3x + 2y = 2$, $x - 2y = 2$

2. Use Cramer's rule to solve the following pair of linear equations, if possible.

(i) $x + 4y = 4$, $2x - y = 5$ (ii) $x + 2y = 7$, $3x - 2y = -3$

(iii) $2x - 5y = -6$, $4x - 3y = -12$ (iv) $3x + 2y = -1$, $5x + 6y = 5$

3. An electrical engineer wants to determine the current in two branches A and B of a simple electrical circuit. The system of the equations is:

$$x + y = 7$$
$$2x - y = 2$$

where x is the current in branch A and y is the current in branch B. Find x and y by using matrices.

4. Three forces act on a particle and must be in equilibrium i.e. $F_1 + F_2 + F_3 = 0$, where $F_1 = \begin{bmatrix} 8 \\ x \end{bmatrix}, F_2 = \begin{bmatrix} -2 \\ -7 \end{bmatrix}, F_3 = \begin{bmatrix} y \\ -1 \end{bmatrix}$. Find the value of x and y.

5. Two support beams, A and B are holding up a combined load of 100 kN. Twice the load on beam A and three times the load on beam B equals 240 kN. Find the load of beam A and beam B by using matrices.

6. In a 2D game world, two characters are moving along straight paths. One character moves along a line where the total of twice their horizontal position and vertical position is 5, while the other moves along a line where their horizontal position is one more than their vertical position. Find their point of intersection by using matrices.

7. Two years ago a man was 5 times as old as his son was. After 6 years he will be 3 times as old as his son. Find their present ages by using matrices.

8. Two cyclists are 44 km apart and start out at the same time. If they go towards one another they meet in 2 hours, but if they go in the same direction the faster overtakes the slower in $7\frac{1}{2}$ hours. Find their speeds by using matrices.

9. The numerator of a fraction is 7 less than the denominator. If the numerator is increased by 3, the new fraction can be cancelled down to $\frac{3}{4}$. Find the original fraction by using matrices.

REVIEW EXERCISE 3

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) If $\begin{bmatrix} a+2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$, then $a =$

(a) 3 (b) 5 (c) 6 (d) 7

(ii) $A = \begin{bmatrix} 3 & 5 & 0 \end{bmatrix}$ is a _______ matrix.

(a) row (b) square (c) column (d) null

(iii) $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a _______ matrix.

(a) identity (b) square (c) row (d) column

(iv) $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a/an _______ matrix.

(a) rectangular (b) identity (c) column (d) row

(v) If $A^t = -A$, then A is _______ matrix.

(a) symmetric (b) row
(c) rectangular (d) skew-symmetric

(vi) If $A = \begin{bmatrix} 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 8 \end{bmatrix}$, then $A + B =$

(a) [21 32] (b) [24 28]
(c) [10 12] (d) [11 11]

(vii) If $A = [3 \quad 4]$, $B = [7 \quad 8]$, then $B - A =$

(a) [10 12] (b) [11 11]
(c) [4 4] (d) [–4 –4]

(viii) What is the additive inverse of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$?

(a) $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ -4 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(ix) If $A = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix}$, then order of A^t is:

(a) 3-by-2 (b) 2-by-3 (c) 3-by-3 (d) 2-by-2

(x) $\begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} =$

(a) 11 (b) 12 (c) 13 (d) –11

2. If $A = \begin{bmatrix} 4 & 2 \\ 7 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 5 \\ 8 & 8 \end{bmatrix}$, then find

(i) $(A - B)^t$ (ii) $B^t - A^t$ (iii) $2A + 3B$

3. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}$, then verify that

(i) $2(A + B) = 2A + 2B$ (ii) $(A + B) + C = A + (B + C)$

(iii) $(A + B)C = AC + BC$ (iv) $C(A - B) = CA - CB$

(v) $(AB)^{-1} = B^{-1}A^{-1}$ (vi) $AA^{-1} = A^{-1}A = I$

(vii) $(AB)^t = B^tA^t$ (viii) $(AB)C = A(BC)$

4. If $A = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$, then find

(i) $|B|$ (ii) Adj$B$ (iii) A^{-1}

(iv) $A^{-1}A$ (v) $(AB)^t$ (vi) $(B^t)^t$

5. Use matrix inversion method and Cramer's rule to solve the following pair of linear equations, if possible:

(i) $3x + 4y = 7$, $5x - y = 2$ (ii) $x - 6y = -15$, $2x + 6y = -3$ (iii) $2x + y = 5$, $x + 3y = 3$

6. Find two numbers by using matrices such that twice the first added to the second makes 21 and twice the second added to the first makes 27.

7. 4 knives and 6 forks cost Rs. 136, whereas 6 knives and 5 forks cost Rs. 164. Find the cost of a knife and a fork by using matrices.

8. A shop employs, 5 men and 3 women, pays total daily wages Rs. 3500. If the number of men is reduced to 2 and 3 extra women are taken on, the daily wages amount to Rs. 5000. Find daily wages of a man and a woman by using matrices.