UNIT 1

Complex Numbers

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Identify complex numbers, complex conjugate, absolute value or modulus of a complex number.
- Apply algebraic properties and perform basic operations on complex numbers.
- Demonstrate additive identity and multiplicative identity for the set of complex numbers.
- Find additive inverse and multiplicative inverse of a complex number z.
- Demonstrate the following properties of a complex number z.
  - $|z| = |-z| = |\bar{z}| = |-\bar{z}|$
  - $\bar{\bar{z}} = z,\ z\bar{z} = |z|^2$
  - $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
  - $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$; $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$
- Apply the Geometric interpretation of a complex number, modulus of a complex number and algebraic operations of a complex number.
- Find real and imaginary parts of complex numbers of the type $(x + iy)^n$ and $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^n$, $x_2 + iy_2 \neq 0$, where $n = \pm 1$ and $n = \pm 2$.
- Solve the simultaneous linear equations with complex coefficients.

INTRODUCTION

This unit introduces the fascinating world of complex numbers, beginning with their identification and the exploration of related concepts such as the complex conjugate, modulus and their representation in both algebraic and geometric forms. Students will apply core algebraic operations and understand how complex numbers obey properties. The unit also extends to evaluate expressions involving powers and quotients of complex numbers, extracting real and imaginary parts and solving simultaneous linear equations with complex coefficients.

1.1 Complex Numbers

Before we explore complex numbers, let's first consider the question: *"Is there a real number whose square is negative?"* To answer this question, we can examine a few simple examples. Let's take a look at equations 1 and 2 for better understanding.

Equation 1: $x^2 - 1 = 0$

$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm\sqrt{1} \Rightarrow x = \pm 1$$

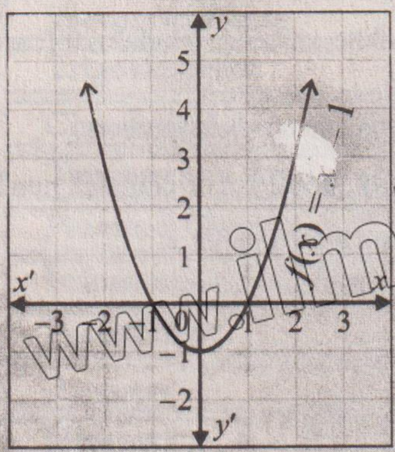


Figure 1.1

Equation 1 has two real solutions $x = -1, 1$. The graph of $f(x) = x^2 - 1$ crosses the x-axis at $(-1, 0)$ and $(1, 0)$.

Equation 2: $x^2 + 1 = 0$

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = \pm\sqrt{-1}$$

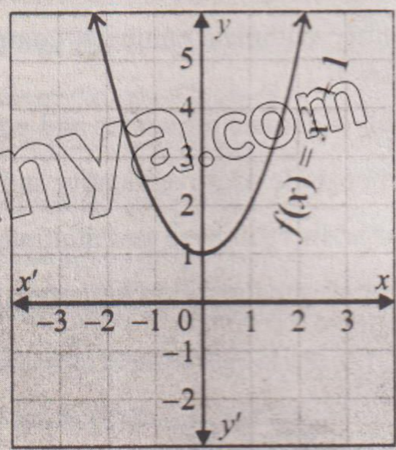


Figure 1.2

Equation 2 has no real solution, since the graph of $f(x) = x^2 + 1$ does not cross the x-axis.

This is because squaring any real number never results in a negative value. Therefore, if equation 2 has a solution, we need to introduce a new kind of number, an imaginary number, defined as the square root of −1. This imaginary unit is represented by the symbol i (iota). The imaginary number i tells us that $i^2 = -1$.

Example 1 Simplify the following:

(i) i^7 (ii) i^8 (iii) i^{17} (iv) i^{-25}

Solution

(i) $i^7 = i^6 \times i = (i^2)^3 \times i = (-1)^3 \times i = -i$

(ii) $i^8 = (i^2)^4 = (-1)^4 = 1$

Do you know?

Complex numbers are essential for many technologies like smartphone signal processing and MRI imaging.

(iii) $i^{17} = i^{16}\times i = (i^2)^8 \times i = (-1)^8 \times i = i$

(iv) $i^{-25} = \dfrac{1}{i^{25}} = \dfrac{1}{i^{24}\times i} = \dfrac{1}{(i^2)^{12}\times i} = \dfrac{1}{(-1)^{12}\times i} = \dfrac{1}{i} = \dfrac{1}{i}\times\dfrac{i}{i} = \dfrac{i}{i^2} = \dfrac{i}{-1} = -i$

We have observed that the equation $x^2 + 1 = 0$ has no solution within the real number system. To address equations like this, we expand the real number system to include new types of numbers, leading to the development of the complex number system in rectangular form.

History!

The idea of complex numbers first emerged in 16th century, when Italian Mathematician Gerolamo Cardano discovered that equations could still be solved when they involved the square root of negative number. In the 18th century, Carl Friedrich Gauss (German Mathematician and astronomer) expanded this early idea.

Gerolamo Cardano (1501 - 1576)

A complex number is a number expressed in the form $x + iy$, where x and y are real numbers and i is the imaginary unit, defined by $i^2 = -1$.

1.1.1 Rectangular Form of a Complex Number

A complex number is of the form $x + iy$ (or $x + yi$), where x and y are real numbers, x is called the real part and y is called the imaginary part of the complex number.

(i) If $x = 0$, the complex number is said to be pure imaginary.

(ii) If $y = 0$, the complex number is said to be real.

(iii) It is customary to denote the standard rectangular form of a complex number $x + iy$ as z and we write $x = Re(z)$ and $y = Im(z)$.

For example, $Re(5 - 7i) = 5$ and $Im(5 - 7i) = -7$

1.1.2 Equality of Complex Numbers

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$. i.e. $x_1 = x_2$ and $y_1 = y_2$.

For example, if $\alpha + i\beta = -2 + 5i$, then $\alpha = -2$, $\beta = 5$.

Example 2 If $(2x - 1) + (y + 4)i = 5 + 7i$, then find the values of x and y.

Solution Given that $(2x - 1) + (y + 4)i = 5 + 7i$

Since the two complex numbers are equal, their real and imaginary parts must be equal.

$$\therefore \quad 2x - 1 = 5 \quad , \quad y + 4 = 7$$
$$2x = 6 \quad , \quad y = 7 - 4$$
$$x = 3 \quad , \quad y = 3$$

EXERCISE 1.1

1. Simplify the following:
   (i) i^5 (ii) i^{16} (iii) $(-i)^{-19}$ (iv) $27i^{-26}$
   (v) $i^{11} + i^5$ (vi) $(i^4 + i^3 + i^2 + i)^2$ (vii) $\left(\frac{i^8}{i^5}\right)^{-5}$ (viii) $i^{13} \times i^{29}$
2. Write in terms of i.
   (i) $2 + \sqrt{-4}$ (ii) $3 - \sqrt{-7}$ (iii) $\frac{2}{5} + \frac{\sqrt{-16}}{5}$ (iv) $\sqrt{2} - \sqrt{-3}$
3. Find the values of x and y.
   (i) $(2x + 5) + (y - 3)i = 1 + 2i$ (ii) $(3x + 2) - (4 - y)i = 5 + 3i$
   (iii) $(2+i)x + (1-2i)y = 3+4i$ (iv) $(1-i)x + (2+i)y = 4-i$
   (v) $(3x - 1) + (2y - 3)i = 8 + 7i$

1.2 Algebraic Operations on Complex Numbers

1.2.1 Scalar Multiplication of Complex Numbers

If $z = x + iy$ and $k \in R$, then we define $kz = (kx) + (ky)i$ or $(kx) + i(ky)$

The following diagram shows kz for $k = 2, \frac{1}{2}, -1$

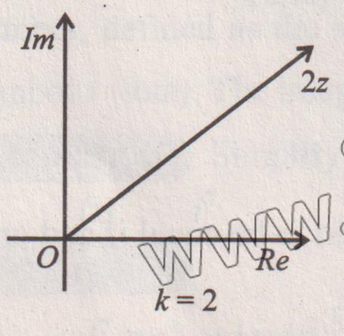


Figure 1.3

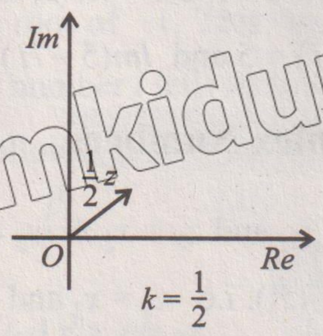


Figure 1.4

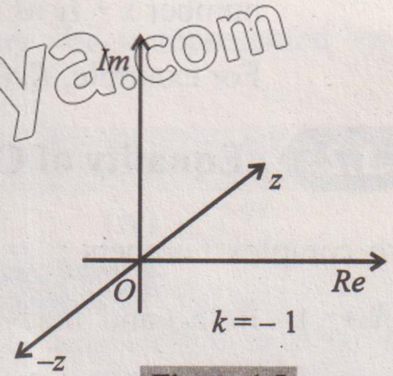


Figure 1.5

1.2.2 Addition of Two Complex Numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where x_1, x_2, y_1 and $y_2 \in R$, then

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

From figure 1.6, also by the parallelogram law of addition,

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

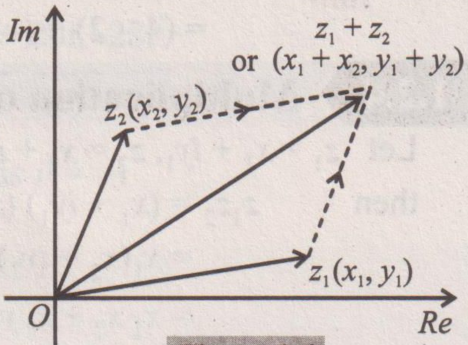


Figure 1.6

History!

The fundamental rules for addition, subtraction, multiplication and division of complex numbers were formulated by the Italian mathematician Rafael Bombelli (1526 – 1572). He is widely recognized as the first to establish a systematic algebra for complex numbers.

Example 3

Add $(3 + 4i)$ and $(5 - 2i)$.

Solution

$$(3 + 4i) + (5 - 2i)$$
$$= (3 + 5) + (4 - 2)i$$
$$= 8 + 2i$$

1.2.3 Subtraction of Two Complex Numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where $x_1, x_2, y_1, y_2 \in R$,

then $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

$$z_1 - z_2 = z_1 + (-z_2)$$
$$= (x_1 + iy_1) + (-x_2 - iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2)$$

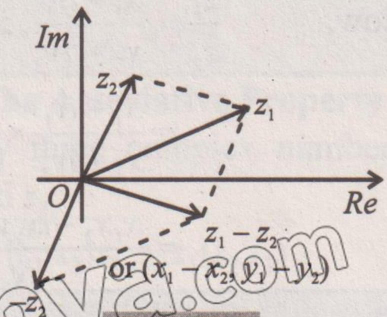


Figure 1.7

Example 4 Simplify: (i) $(8 + 2i) - (5 - 6i)$ (ii) $(4 - 3i) - (2 - 5i)$

Solution (i) $(8 + 2i) - (5 - 6i)$

$$= 8 + 2i - 5 + 6i$$
$$= (8 - 5) + (2i + 6i)$$
$$= 3 + 8i$$

Skilled Practice!

If $z_1 - z_2 = 4 + 6i$ and $z_2 = 3 - 2i$, then find z_1.

(ii) $(4 - 3i) - (2 - 5i)$
$= 4 - 3i - 2 + 5i$
$= (4 - 2) + (-3 + 5)i = 2 + 2i$

1.2.4 Multiplication of Two Complex Numbers

Let $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$

then $z_1z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2)$

$= x_1x_2 + x_1y_2i + y_1x_2i + i^2y_1y_2$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$ $\quad \because\ i^2 = -1$

Example 5

(i) Simplify: $(3 - 4i)(5 - 6i)$

(ii) If $z_1 = 2 + 3i$ and $z_2 = 4 + 7i$, then find z_1z_2.

Solution

(i) $(3 - 4i)(5 - 6i)$

$= 3(5 - 6i) - 4i(5 - 6i)$

$= 15 - 18i - 20i + 24i^2$

$= 15 - (18 + 20)i - 24$

$= 15 - 38i - 24$

$= (15 - 24) - 38i = -9 - 38i$

(ii) $z_1z_2 = (2 + 3i)(4 + 7i)$

Using

$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

$z_1z_2 = (2 \times 4 - 3 \times 7) + i(2 \times 7 + 3 \times 4)$

$= (8 - 21) + (14 + 12)i$

$= -13 + 26i$

1.2.5 Division of Two Complex Numbers

Let $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$

Now, $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2},\ z_2 \neq 0$

$= \dfrac{x_1 + iy_1}{x_2 + iy_2} \times \dfrac{x_2 - iy_2}{x_2 - iy_2}$

$= \dfrac{x_1x_2 - ix_1y_2 + iy_1x_2 - i^2y_1y_2}{(x_2)^2 - (iy_2)^2}$

$= \dfrac{x_1x_2 - i(x_1y_2 - y_1x_2) + y_1y_2}{x_2^2 + y_2^2}$ $\quad \because\ i^2 = -1$

$= \dfrac{x_1x_2 + y_1y_2 - i(x_1y_2 - y_1x_2)}{x_2^2 + y_2^2}$

$= \dfrac{x_1x_2 + y_1y_2}{x_2^{\,2} + y_2^{\,2}} + \dfrac{i(y_1x_2 - x_1y_2)}{x_2^{\,2} + y_2^{\,2}}$

Skilled Practice!

1. Simplify:
$$\left(\frac{1+2i}{3-i}\right)(2+i)$$

2. Find the complex number z:
if $\dfrac{z}{2+i} = 3 - i$

Example 6 Express $\frac{3+4i}{5-7i}$ in the form of $x+iy$ or $x+yi$.

Solution $\frac{3+4i}{5-7i}=\frac{3+4i}{5-7i}\times\frac{5+7i}{5+7i}=\frac{15+21i+20i+28i^2}{(5)^2-(7i)^2}$

$$=\frac{15-28+(21+20)i}{25+49} \quad \text{as } i^2=-1$$

$$=\frac{-13+41i}{74}$$

$$=\frac{-13}{74}+\frac{41}{74}i$$

1.2.6 Properties of Complex Numbers

The complex numbers satisfy the following properties under **addition.**	The complex numbers satisfy the following properties under **multiplication.**
(i) Closure Property For any two complex numbers z_1 and z_2, the sum z_1+z_2 is also a complex number.	**(i) Closure Property** For any two complex numbers z_1 and z_2, the product z_1z_2 is also a complex number.
(ii) The Commutative Property For any two complex numbers z_1 and z_2, $z_1+z_2=z_2+z_1$	**(ii) The Commutative Property** For any two complex numbers z_1 and z_2, $z_1z_2=z_2z_1$
(iii) The Associative Property For any three complex numbers z_1, z_2 and z_3, $(z_1+z_2)+z_3=z_1+(z_2+z_3)$	**(iii) The Associative Property** For any three complex numbers z_1, z_2 and z_3, $(z_1z_2)z_3=z_1(z_2z_3)$
(iv) The Additive Identity There exists a complex number $0=0+0i$ such that, for every complex number z, $z+0=0+z=z$ The complex number $0=0+0i$ is known as additive identity.	**(iv) The Multiplicative Identity** There exists a complex number $1=1+0i$ such that, for every complex number z, $z\times1=1\times z=z$ The complex number $1=1+0i$ is known as multiplicative identity.

(v) The Additive Inverse	(v) The Multiplicative Inverse
For every complex number z there exists a complex number $-z$ such that, $z+(-z)=(-z)+z=0$. $-z$ is called the additive inverse of z.	For any non-zero complex number z, there exists a complex number w such that, $zw = wz = 1$. w is called the multiplicative inverse of z. w and it is denoted by z^{-1}.

(vi) Distributive Property of Multiplication Over Addition

For any three complex numbers z_1, z_2, and z_3

$$z_1(z_2+z_3)=z_1z_2+z_1z_3 \text{ and } (z_1+z_2)z_3=z_1z_3+z_2z_3$$

Example 7 If $z=-7-4i$, then show that $z+0=z$.

Solution

$$z+0=(-7-4i)+(0+0i)$$
$$=-7-4i+0+0i$$
$$=-7-4i=z$$

Shows $z+0=z$ (0 is the additive identity)

Example 8 Verify the multiplicative identity for $z=3-2i$.

Solution

$$z\times 1=(3-2i)(1+0i)$$
$$=3+0i-2i-0i^2$$
$$=3-2i=z$$
$$1\times z=(1+0i)(3-2i)$$
$$=3-2i+0i-0i^2$$
$$=3-2i=z$$

Hence, verified that $z\times 1=1\times z=z$

Example 9 Find the additive inverse of $z=7-10i$.

Solution $z=7-10i$

Additive inverse of $z=-7+10i$

because $z+(-z)=7-10i+(-7+10i)$
$$=7-10i-7+10i=0$$

Example 10 Find multiplicative inverse of $4-3i$.

Solution Let $z=4-3i$

Multiplicative inverse of $z=\dfrac{1}{4-3i}$

Then $z^{-1}=\dfrac{1}{4-3i}$

$$=\frac{4+3i}{(4-3i)(4+3i)} \quad =\frac{4+3i}{(4)^2-(3i)^2}$$

$$z^{-1}=\frac{4+3i}{16+9}=\frac{4+3i}{25}=\frac{4}{25}+\frac{3i}{25}$$

EXERCISE 1.2

1. Simplify and write in the form $a+bi$:
   (i) $(2+5i)+(3-zi)$ (ii) $(16-3i)+(9+2i)$
   (iii) $(9-2i)-(7-3i)$ (iv) $(11+9i)-(9-7i)$
   (v) $(3+4i)(2-3i)$ (vi) $(5-2i)(3-4i)$
   (vii) $(3-5i)\div(2-4i)$ (viii) $(5+2i)\div(6-3i)$
2. Write additive inverse for each complex number:
   (i) $3+2i$ (ii) $4-3i$ (iii) $5-7i$ (iv) $-\dfrac{2}{3}+\dfrac{5}{4}i$

3. Find multiplicative inverse for each complex number:

(i) $4+5i$ (ii) $6+2i$ (iii) $7-3i$ (iv) $\sqrt{5}-4i$

4. If $z_1 = 2+5i$, $z_2 = 1-3i$ and $z_3 = 2+i$, then verify that

(i) $z_1+z_2=z_2+z_1$ (ii) $z_1z_2=z_2z_1$

(iii) $(z_1+z_2)+z_3=z_1+(z_2+z_3)$ (iv) $(z_1z_2)z_3=z_1(z_2z_3)$

(v) $z_1+(-z_1)=(-z_1)+z_1=0$

5. If $\dfrac{(1+i)^2}{2-i}=x+iy$, then find the values of x and y.

6. If $(2x+iy)(1-i)=4+2i$, then find the values of x and y.

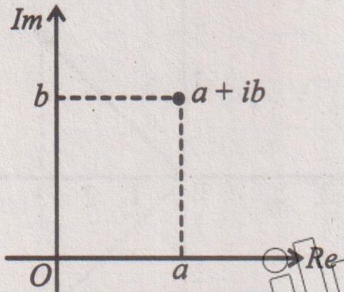
7. Find the values of a and b, if $(a+bi)(1+3i)=-8+11i$.

1.3 Complex or Argand Plane

A complex number $z = x+iy$ is exclusively determined by an ordered pair of real number (x, y). The numbers $2-5i$, 8 and $-7i$ are equivalent to $(2,-5)$, $(8, 0)$ and $(0,-7)$ respectively. In this manner, a complex number $z = x+iy$ can be represented by the point (x, y) in the coordinate plane.

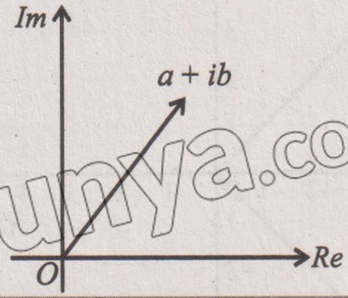
If we consider x-axis as real axis and y-axis as imaginary axis to represent a complex number, then the xy-plane is called complex plane or Argand plane. It is named in honour of the Swiss mathematician Jean Argand (1768–1822).

A complex number is represented not only by a point, but also by a position vector pointing from the origin to the point. The complex number, the corresponding point and the vector are all typically denoted by the same symbol, z. Geometrically, a complex number can be interpreted either as a point in the complex plane C or as a vector in the Argand plane.



Complex number as a point.

Figure 1.8



Complex number by a position vector pointing from origin to the point.

Figure 1.9

Here some complex numbers are plotted on the complex plane.

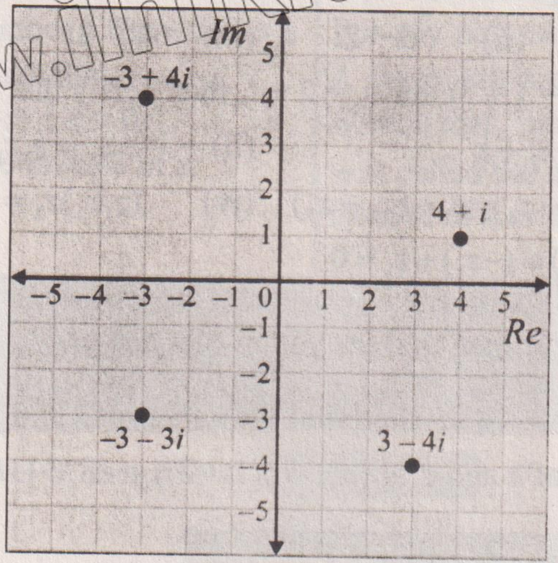


Figure 1.10

1.3.1 Conjugate of a Complex Number

The conjugate of the complex number $x + iy$ is defined as the complex number $x - iy$. The complex conjugate of z is denoted by $\bar{z}$. To get the conjugate of the complex number z, simply change i by $-i$ in z. For instance $3 - 8i$ is the conjugate of $3 + 8i$. The product of a complex number with its conjugate is a real number.

For example, $(1 + 2i)(1 - 2i) = (1)^2 - (2i)^2 = 1 + 4 = 5$

1.3.2 Geometrical Representation of Conjugate of a Complex Number

Geometrically, the conjugate of z is obtained by reflecting z on the real axis.

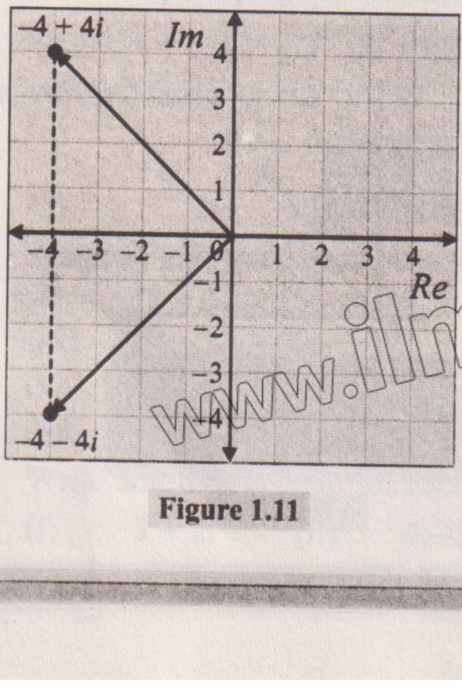


Figure 1.11

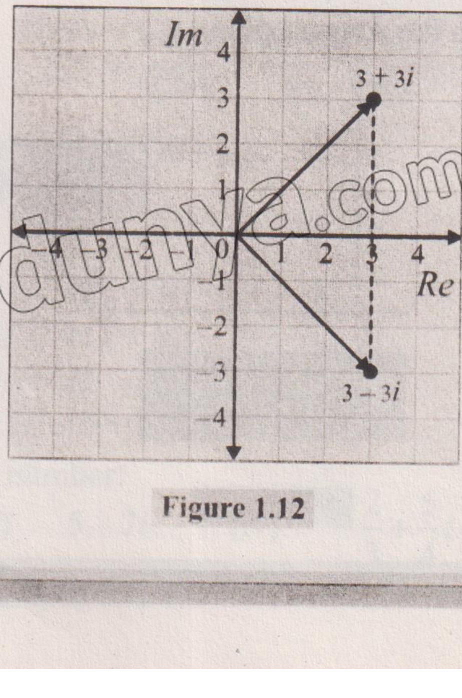


Figure 1.12

Note

Two complex numbers, $x + iy$ and $x - iy$, are known as conjugates of each other. Conjugates are particularly useful when dividing complex numbers. By multiplying both the numerator and the denominator by the conjugate of the denominator, the complex number in the denominator can be transformed into a real number.

1.3.3 Properties of Complex Conjugate

For any two complex numbers z_1 and z_2, we have

Property 1 $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Proof: Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, where $x_1, y_1, x_2, y_2 \in R$

Now, $\overline{z_1 + z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)}$

$= \overline{(x_1 + x_2) + i(y_1 + y_2)}$

$= (x_1 + x_2) - i(y_1 + y_2) = x_1 + x_2 - iy_1 - iy_2$

$= (x_1 - iy_1) + (x_2 - iy_2) = \overline{z_1} + \overline{z_2}$

Example 11 If $z_1 = 4 + 3i$, $z_2 = 5 + 2i$, then prove that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

Solution $z_1 + z_2 = 4 + 3i + 5 + 2i = 9 + 5i$

$\overline{z_1 + z_2} = 9 - 5i$ ... (i)

$\overline{z_1} = 4 - 3i$

$\overline{z_2} = 5 - 2i$

$\overline{z_1} + \overline{z_2} = 4 - 3i + 5 - 2i$

$\overline{z_1} + \overline{z_2} = 9 - 5i$ ... (ii)

From (i) and (ii), we get

$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Property 2 $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

Proof: Let $z_1 = x_1 + iy_1$

and $z_2 = x_2 + iy_2$ where $x_1, y_1, x_2, y_2 \in R$

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

Therefore: $\overline{z_1 z_2} = \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)}$

$= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)$ ... (i)

and $\overline{z_1}\,\overline{z_2} = (\overline{x_1 + iy_1})(\overline{x_2 + iy_2})$

$= (x_1 - iy_1)(x_2 - iy_2)$

$= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)$ ... (ii)

From (i) and (ii), we get

$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

Example 12 If $z_1 = 3 + 4i$, $z_2 = 2 + 3i$, then prove that $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$.

Solution Now, $z_1 z_2 = (3 + 4i)(2 + 3i)$

$$= 6 + 9i + 8i + 12i^2$$
$$= 6 + 17i - 12$$
$$= -6 + 17i$$
$$\overline{z_1 z_2} = -6 - 17i \quad \ldots \text{(i)}$$

and $\overline{z_1}\ \overline{z_2} = (3 - 4i)(2 - 3i)$

$$= 6 - 9i - 8i + 12i^2$$
$$= 6 - 9i - 8i - 12$$
$$\overline{z_1}\ \overline{z_2} = -6 - 17i \quad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$$

Property 3 $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, $(z_2 \neq 0)$

Proof: Let $z_1 = x_1 + iy_1$

and $z_2 = x_2 + iy_2$

Now, $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} \times \dfrac{x_2 - iy_2}{x_2 - iy_2} = \dfrac{(x_1x_2 + y_1y_2) - i(x_1y_2 - y_1x_2)}{x_2^2 + y_2^2}$

and $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{(x_1x_2 + y_1y_2) + i(x_1y_2 - y_1x_2)}{x_2^2 + y_2^2} \quad \ldots \text{(i)}$

Now, $\dfrac{\overline{z_1}}{\overline{z_2}} = \dfrac{x_1 - iy_1}{x_2 - iy_2}$

$$= \frac{(x_1 - iy_1)(x_2 + iy_2)}{(x_2 - iy_2)(x_2 + iy_2)}$$

$$= \frac{x_1x_2 + ix_1y_2 - iy_1x_2 - i^2y_1y_2}{x_2^2 + y_2^2}$$

$$\frac{\overline{z_1}}{\overline{z_2}} = \frac{(x_1x_2 + y_1y_2) + i(x_1y_2 - y_1x_2)}{x_2^2 + y_2^2} \quad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

Example 13 If $z_1 = 5 + 4i$, $z_2 = 3 + 2i$, then prove that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

Solution

$$\frac{z_1}{z_2} = \frac{5+4i}{3+2i}$$

$$= \frac{(5+4i)(3-2i)}{(3+2i)(3-2i)}$$

$$= \frac{15-10i+12i-8i^2}{(3)^2-(2i)^2}$$

$$= \frac{23+2i}{9+4}$$

$$\frac{z_1}{z_2} = \frac{23+2i}{13} = \frac{23}{13} + \frac{2}{13}i$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{23}{13} - \frac{2}{13}i \quad \text{... (i)}$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{5-4i}{3-2i} = \frac{(5-4i)(3+2i)}{(3-2i)(3+2i)}$$

$$= \frac{15+10i-12i-8i^2}{9+4} = \frac{15-2i+8}{9+4}$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{23-2i}{13} = \frac{23}{13} - \frac{2}{13}i \quad \text{... (ii)}$$

From (i) and (ii), we get

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

Skilled Practice!

Take any two complex numbers and prove that:

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(ii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $(z_2 \neq 0)$

Property 4 $\bar{\bar{z}} = z$

Proof: Let $z = x + iy$

Then $\bar{z} = x - iy$

$\bar{\bar{z}} = x + iy$

$= z$

Example 14 If $z = 7 + 3i$, then prove that $\bar{\bar{z}} = z$.

Solution $z = 7 + 3i$

Then $\bar{z} = 7 - 3i$

$\bar{\bar{z}} = 7 + 3i = z$

Hence, proved $\bar{\bar{z}} = z$

1.4 Modulus of a Complex Number

Just as the absolute value of a real number measures its distance from the origin on the real number line, the modulus of a complex number measures the distance from the origin in the complex plane. Notice that the distance from the origin to the point $P(x, y)$ lies along a radial line and forms the hypotenuse of a right triangle, where the horizontal and vertical sides have length x and y respectively.

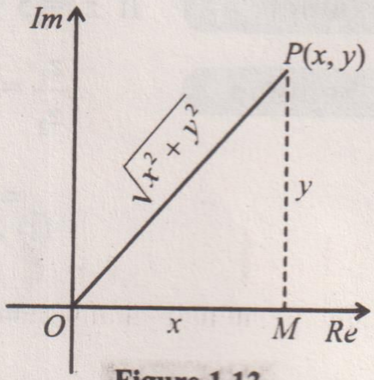


Figure 1.13

Definition

If $z = x + iy$, then the modulus of z is denoted by $|z|$ and defined as $|z| = \sqrt{x^2 + y^2}$.

For example, (i) $|i| = |0 + 1i| = \sqrt{0^2 + 1^2} = 1$

(ii) $|3 - 5i| = \sqrt{(3)^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$

1.4.1 Properties of Modulus of a Complex Number

Property 1 $|z| = |\bar{z}|$

Proof: Let $z = x + iy$

Then $|z| = \sqrt{x^2 + y^2}$ ... (i)

Now $\bar{z} = x - iy$

$|\bar{z}| = \sqrt{(x)^2 + (-y)^2} = \sqrt{x^2 + y^2}$ ... (ii)

From (i) and (ii), we get

$|\bar{z}| = |z|$

Example 15 If $z = 5 + 4i$, then show that $|z| = |\bar{z}|$.

Solution

$z = 5 + 4i$

$|z| = \sqrt{(5)^2 + (4)^2}$

$= \sqrt{25 + 16} = \sqrt{41}$ ... (i)

Now $\bar{z} = 5 - 4i$

$|\bar{z}| = \sqrt{(5)^2 + (-4)^2}$

$= \sqrt{25 + 16} = \sqrt{41}$ ... (ii)

From (i) and (ii), we get

$|z| = |\bar{z}|$

Property 2 $|z| = |-z| = |\bar{\bar{z}}| = |-\bar{z}|$

Proof: Let $z = x + iy$

Then $|z| = \sqrt{x^2 + y^2}$ ... (i)

$|-z| = |-x - iy| = \sqrt{(-x)^2 + (-y)^2}$

$= \sqrt{x^2 + y^2}$ ... (ii)

$\bar{z} = x - yi \Rightarrow \bar{\bar{z}} = x + yi$

$|\bar{\bar{z}}| = \sqrt{x^2 + y^2}$ ... (iii)

Here $\bar{z} = x - iy$ as $z = x + iy$

$-\bar{z} = -x + iy$

$|-\bar{z}| = \sqrt{(-x)^2 + (y)^2}$

$= \sqrt{x^2 + y^2}$ ... (iv)

From (i), (ii), (iii) and (iv), we have

$|z| = |-z| = |\bar{\bar{z}}| = |-\bar{z}|$

Property 3 $z\bar{z} = |z|^2$

Proof: Let $z = x + iy$

$\therefore$ $\bar{z} = x - iy$

Now $z\bar{z} = (x + iy)(x - iy)$

$= (x)^2 - (iy)^2$

$= x^2 - (i^2 y^2)$

$= x^2 - (-y^2)$ as $i^2 = -1$

$= x^2 + y^2 = |z|^2$ as $|z| = \sqrt{x^2 + y^2}$

Hence, $z\bar{z} = |z|^2$

Example 16 If $z = 5 + 3i$, then show that $z\bar{z} = |z|^2$.

Solution $z = 5 + 3i$

$\therefore$ $\bar{z} = 5 - 3i$

$z\bar{z} = (5 + 3i)(5 - 3i)$

$= (5)^2 - (3i)^2$

$= (25) - (9i^2)$

$= 25-(-9)$

$= 25+9 = 34$ ... (i)

$|z| = \sqrt{(5)^2+(3)^2}$

$|z|^2 = \left[\sqrt{(5)^2+(3)^2}\right]^2$

$= 25+9 = 34$ ... (ii)

From (i) and (ii), we get

$$z\bar{z} = |z|^2$$

EXERCISE 1.3

1. Find the modulus of the following complex numbers:

   (i) $4+3i$ (ii) $-5-4i$ (iii) $\frac{3}{5}-\frac{4}{5}i$ (iv) $-\sqrt{2}-\sqrt{3}i$

2. If $z_1 = 2+7i$ and $z_2 = 4-3i$, then verify that

   (i) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$ (ii) $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$ (iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

3. If $z = 5-2i$, then verify that

   (i) $\bar{\bar{z}} = z$ (ii) $|z| = |\bar{z}|$ (iii) $|z| = |-z|$

   (iv) $z\bar{z} = |z|^2$ (v) $|z| = |-\bar{z}|$

4. If $z = 4-3i$, then verify that $|z| = |-z| = |\bar{\bar{z}}| = |-\bar{z}|$.

5. If $z_1 = 2+3i$, $z_2 = -1+i$, then evaluate:

   (i) $Re(z_1 z_2)$ (ii) $Im(z_1 z_2)$

1.5 Finding Real and Imaginary Parts of Complex Number of $(x+iy)^n$ and $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$, where $x_2+iy_2 \neq 0$ and $n = \pm 1$ and $n = \pm 2$

$(x+iy)^n$ when $n = \pm 1$

Let $z = (x+iy)^n$ be a complex number.

(i) For $n = 1$

We have $z = (x+iy)^1$

$= x+iy$

$Re(z) = x,\ Im(z) = y$

(ii) For $n = -1$

We have $z = (x+iy)^{-1}$

$$= \frac{1}{x+iy}$$

$$= \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$$

$$\therefore \quad Re(z) = \frac{x}{x^2+y^2}, \quad Im(z) = \frac{-y}{x^2+y^2}$$

Skilled Practice!

Find real and imaginary parts of $(x+iy)^n$ for $n = \pm 2$

Example 17 Find real and imaginary parts of $(4+3i)^{-1}$.

Solution Let $z = (4+3i)^{-1}$

$$= \frac{1}{(4+3i)}$$

$$= \frac{1(4-3i)}{(4+3i)(4-3i)}$$

$$= \frac{4-3i}{16+9} = \frac{4-3i}{25} = \frac{4}{25} - \frac{3}{25}i$$

$$\therefore \quad Re(z) = \frac{4}{25}, \quad Im(z) = -\frac{3}{25}$$

Example 18 Find real and imaginary parts of $z = (4+3i)^{-2}$.

Solution $z = (4+3i)^{-2}$

$$= \frac{1}{(4+3i)^2}$$

$$= \frac{1}{(4)^2 + (3i)^2 + 2(4)(3i)} = \frac{1}{16-9+24i}$$

$$= \frac{1}{7+24i}$$

$$= \frac{(7-24i)}{(7+24i)(7-24i)}$$

Do you know?

$$Re(z) = \frac{z+\bar{z}}{2},$$

$$Im(z) = \frac{z-\bar{z}}{2i}$$

$$= \frac{7-24i}{49+576}$$

$$= \frac{7-24i}{625}$$

$$= \frac{7}{625} - \frac{24}{625}i$$

$$\therefore \quad Re(z) = \frac{7}{625}, \; Im(z) = -\frac{24}{625}$$

Example 19 Find real and imaginary parts of $z = \left(\frac{4+3i}{3+2i}\right)^{-1}$.

Solution $z = \left(\frac{4+3i}{3+2i}\right)^{-1}$

$$= \frac{3+2i}{4+3i}$$

$$= \frac{(3+2i)(4-3i)}{(4+3i)(4-3i)}$$

$$= \frac{12-9i+8i-6i^2}{(4)^2+(3)^2} = \frac{12-i+6}{16+9}$$

$$= \frac{18-i}{25} = \frac{18}{25} - \frac{1}{25}i$$

$$\therefore \quad Re(z) = \frac{18}{25}, \quad Im(z) = -\frac{1}{25}$$

Example 20 Find real and imaginary parts of $z = \left(\frac{1+i}{1-i}\right)^{-2}$.

Solution $z = \left(\frac{1+i}{1-i}\right)^{-2}$

$$= \left(\frac{1-i}{1+i}\right)^2 = \left(\frac{1-i}{1+i} \times \frac{1-i}{1-i}\right)^2 = \left(\frac{1-i-i+i^2}{1^2-i^2}\right)^2$$

$$= \left(\frac{1-2i-1}{1+1}\right)^2 = \left(\frac{-2i}{2}\right)^2 = (-i)^2 = -1$$

$$\therefore \quad Re(z) = -1, \; Im(z) = 0$$

1.6 Solution of Simultaneous Linear Equations with Complex Coefficients

More than one equation which are to be satisfied by the same values of the variables involved are called simultaneous equations or a system of equations.

Example 21 Solve the given simultaneous linear equations with complex coefficients for z and w:

$$5z-(3+i)\,w=7-i$$
$$(2-i)z+2i\,w=4$$

Solution

$$5z-(3+i)\,w=7-i \quad \text{... (i)}$$
$$(2-i)z+2i\,w=4 \quad \text{... (ii)}$$

Step I From equation (i), we solve for w in terms of z.

$$5z-(3+i)\,w=7-i$$
$$(3+i)w=5z-7+i$$
$$w=\frac{5z-7+i}{3+i} \quad \text{... (iii)}$$

Step II Put the expression of w in equation (ii)

$$(2-i)z+2i\left(\frac{5z-7+i}{3+i}\right)=4$$
$$(2-i)(3+i)z+2i(5z-7+i)=4(3+i)$$
$$(6+2i-3i-i^2)z+10iz-14i+2i^2=12+4i$$
$$(6-i+1)z+10iz-14i-2=12+4i$$
$$(7-i)z+10iz-14i-2=12+4i$$
$$7z-iz+10iz-14i-2=12+4i$$
$$7z+9iz=12+4i+14i+2$$
$$7z+9iz=14+18i$$
$$z(7+9i)=2(7+9i)$$
$$z=\frac{2(7+9i)}{7+9i}$$
$$z=2$$

Step III

Put $z=2$ in (iii), we get

$$w=\frac{5(2)-7+i}{3+i}$$
$$=\frac{3+i}{3+i}=1$$

Hence, $z=2,\ w=1$

EXERCISE 1.4

1. Find the real and imaginary parts of the following complex numbers:

(i) $(8-3i)^2$ (ii) $(5+3i)^{-1}$ (iii) $(4-5i)^{-1}$

(iv) $(4-3i)^{-2}$ (v) $\left(\dfrac{3+2i}{4+3i}\right)^{-1}$ (vi) $\left(\dfrac{2-i}{2+i}\right)^{-2}$

(vii) $\left(\dfrac{1-2i}{1+i}\right)^{2}$

2. Solve the following simultaneous linear equations with complex coefficients for w and z:

(i) $3z+(2+i)w=11-i$
$(2-i)z-w=-1+i$

(ii) $2z+(3+i)w=9-i$
$-iz-iw=-1+i$

(iii) $z-4w=3i$
$2z+3w=11-5i$

(iv) $z+w=3i$
$2z+3w=2$

(v) $2z+(3+i)w=1$
$-z-(1-i)w=2$

REVIEW EXERCISE 1

1. Four possible answers are given for the following questions. Choose the correct answer:

(i) $i^2+i^4=$
(a) -1 (b) 0 (c) 1 (d) 2

(ii) Real part of $(2-3i)(2+3i)$ is:
(a) -3 (b) 1 (c) 4 (d) 13

(iii) Imaginary part of $(2-i)(2+i)$ is:
(a) 0 (b) 1 (c) 7 (d) 9

(iv) $x+iy$ will be pure imaginary number, when:
(a) $y=0$ (b) $x=0$ (c) $i=0$ (d) $x=0, y=0$

(v) What is additive inverse of $5-2i$?
(a) $5+2i$ (b) $-5-2i$ (c) $5-2i$ (d) $-5+2i$

(vi) What is multiplicative inverse of $z=1+i$?
(a) $1-i$ (b) i (c) $\dfrac{1}{2}-\dfrac{1}{2}i$ (d) $-\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}i$

(vii) If $z = 4 - 3i$, then $z\bar{z} =$

(a) 3 (b) 9 (c) 16 (d) 25

(viii) Conjugate of $9 - 4i$ is:

(a) $-9 - 4i$ (b) $9 + 4i$ (c) $9 + 9i$ (d) $4 - 9i$

(ix) If $z = 4 + 4i$, then $z + \bar{z} =$

(a) 8 (b) $8 + 8i$ (c) $8i$ (d) 0

(x) If $z = 5 + 4i$, then $|z| =$

(a) 9 (b) 25 (c) 41 (d) $\sqrt{41}$

2. (i) Is "0" a complex number? Explain.
   (ii) What is the result of multiplying a complex number by its conjugate?
   (iii) State the condition for two complex numbers to be equal.

3. Simplify:

(i) i^{37} (ii) $i^{13} \times i^{11}$ (iii) $(-i)^{-9}$

(iv) $(3 - 4i)(5 - 6i)$ (v) $(3 + 4i) \div (5 - 7i)$

4. Find additive and multiplicative inverse of $z = 8 + 9i$.

5. If $z_1 = 3 + 4i$ and $z_2 = 2 + 3i$, then verify that

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (ii) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$ (iii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

(iv) $|\bar{z}_1| = |-\bar{z}_1|$ (v) $\bar{\bar{z}}_2 = z_2$ (vi) $z_1\bar{z}_1 = |z_1|^2$

6. If $z_1 = 5 + 4i$, $z_2 = 3 + 2i$, then find

(i) $z_1 z_2$ (ii) $\dfrac{z_1}{z_2}$ (iii) $\bar{z}_1\bar{z}_2$ (iv) $|z_1 z_2|$

7. Find real and imaginary parts of $z = (2 + 7i)^{-1}$.

8. Solve the given simultaneous linear equations with complex coefficients for z and w:

$$iz + (2 - i)\,w = 4 + i$$
$$iz + (3 + i)\,w = 3 + 3i$$

9. Solve $(3 - 4i)(a + bi) = 1 + 0i$ and find the values of a and b.

10. Solve the equation for x and y:

$(3 - 2i)(x + yi) = 2(x - 2yi) + 2i - 1$