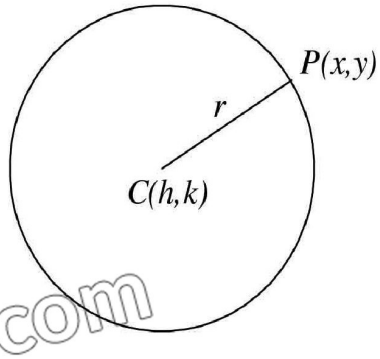
Circle

The set of all point in the plane that are equally distant from a fixed point is called a *circle*.

The fixed point is called *centre* of the circle and the distance from the centre of the circle to any point on the circle is called the *radius* of circle.

Equation of Circle

Let r be radius and $C(h,k)$ be centre of circle. Let $P(x,y)$ be any point on circle then



$$|PC| = r$$

$$\Rightarrow \sqrt{(x-h)^2+(y-k)^2} = r$$

On squaring

$$(x-h)^2+(y-k)^2 = r^2$$

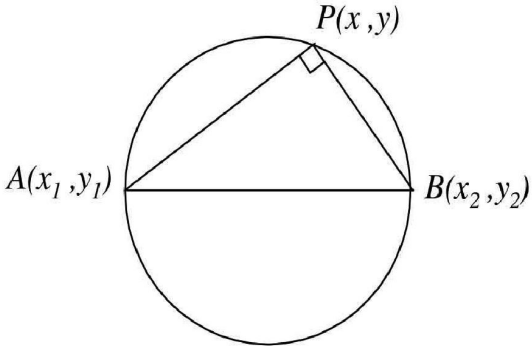
This is equation of circle in standard form.

If centre of circle is at origin i.e. $C(h,k)=C(0,0)$ then equation of circle becomes

$$x^2+y^2 = r^2$$

Equation of circle with end points of diameter

Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be end points of diameter.



Let $P(x,y)$ be any point on circle then

$m\angle APB = 90^\circ$

(***Note**: An angle in a semi circle is a right angle – see Theorem 4 at page 270*)

Thus the line AP and BP are $\perp$ar to each other and we have

$$(\text{Slope of } AP)\,(\text{Slope of } BP) = -1$$

$$\Rightarrow \left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$$

$$\Rightarrow (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

$$\Rightarrow (x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$$

is the required equation of circle with end points of diameter $A(x_1,y_1)$ & $B(x_2,y_2)$.

General form of an equation of a circle

The equation

$$x^2+y^2+2gx+2fy+c = 0$$

represents a circle.

$$\Rightarrow x^2+2gx+g^2+y^2+2fy+f^2+c = g^2+f^2$$

$$\Rightarrow (x+g)^2+(y+f)^2 = g^2+f^2-c$$

$$\Rightarrow \quad (x-(-g))^2+(y-(-f))^2 = \left(\sqrt{g^2+f^2-c}\right)^2$$

This is equation of circle in standard form with

centre at $(-g,-f)$ and radius $=\sqrt{g^2+f^2-c}$

Question # 1

In each of the following, find an equation of the circle with

(a) centre $(5,-2)$ and radius 4 (b) centre $\left(\sqrt{2},-3\sqrt{3}\right)$ and radius $2\sqrt{2}$

(c) ends of a diameter at $(-3,2)$ and $(5,-6)$.

Solution

(a) Given: centre $C(h,k) = (5,-2)$, radius $= r = 4$

Equation of circle:

$$(x-h)^2+(y-k)^2 = r^2$$
$$\Rightarrow (x-5)^2+(y+2)^2 = (4)^2$$
$$\Rightarrow x^2-10x+25+y^2+4y+4 = 16$$
$$\Rightarrow x^2+y^2-10x+4y+25+4-16 = 0$$
$$\Rightarrow x^2+y^2-10x+4y+13 = 0$$

(b) Given: centre $C(h,k) = \left(\sqrt{2},-3\sqrt{3}\right)$, radius $= r = 2\sqrt{2}$

Equation of circle:

$$(x-h)^2+(y-k)^2 = r^2$$
$$\Rightarrow \left(x-\sqrt{2}\right)^2+\left(y+3\sqrt{3}\right)^2 = \left(2\sqrt{2}\right)^2$$
$$\Rightarrow x^2-2\sqrt{2}x+2+y^2+6\sqrt{3}y+27 = 8$$
$$\Rightarrow x^2+y^2-2\sqrt{2}x+6\sqrt{3}y+2+27-8 = 0$$
$$\Rightarrow x^2+y^2-2\sqrt{2}x+6\sqrt{3}y+21 = 0$$

(c) Given end points of diameter:

$$A(x_1,y_1) = (-3,2) \quad , \quad B(x_2,y_2) = (5,-6)$$

Equation of circle with ends of diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$$
$$\Rightarrow (x-(-3))(x-5)+(y-2)(y-(-6)) = 0$$
$$\Rightarrow (x+3)(x-5)+(y-2)(y+6) = 0$$
$$\Rightarrow x^2+3x-4x-15+y^2-2y+6y-12 = 0$$
$$\Rightarrow x^2+y^2-2x+4y-27 = 0$$

Question # 2

Find the centre and radius of the circle with the given equation

(a) $x^2+y^2+12x-10y = 0$ (b) $5x^2+5y^2+14x+12y-10 = 0$

(c) $x^2+y^2-6x+4y+13 = 0$ (d) $4x^2+4y^2-8x+12y-25 = 0$

Solution

(a) $x^2+y^2+12x-10y = 0$

Here $2g=12$, $2f=-10$, $c=0$

$\Rightarrow g=6$, $f=-5$

So centre $=(-g,-f) = (-6,5)$

Radius $= \sqrt{g^2+f^2-c} = \sqrt{(6)^2+(-5)^2-0}$

$= \sqrt{36+25} = \sqrt{61}$

(b) $5x^2+5y^2+14x+12y-10 = 0$

$\Rightarrow x^2+y^2+\frac{14}{5}x+\frac{12}{5}y-2 = 0$ ÷ing by 5

Here $2g=\frac{14}{5}$, $2f=\frac{12}{5}$, $c=-2$

$\Rightarrow g=\frac{7}{5}$, $f=\frac{6}{5}$

Centre $=(-g,-f) = \left(-\frac{7}{5},-\frac{6}{5}\right)$

Radius $= \sqrt{g^2+f^2-c} = \sqrt{\left(\frac{7}{5}\right)^2+\left(\frac{6}{5}\right)^2-(-2)}$

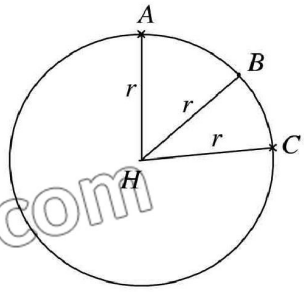
$= \sqrt{\frac{49}{25}+\frac{36}{25}+2} = \sqrt{\frac{27}{5}} = 3\sqrt{\frac{3}{5}}$

(c) *Do yourself as above.*

(d) *Do yourself as above.*

Question # 3

Written an equation of the circle that passes through the given points



(a) $A(4,5)$, $B(-4,-3)$, $C(8,-3)$

(b) $A(-7,7)$, $B(5,-1)$, $C(10,0)$

(c) $A(a,0), B(0,b), C(0,0)$ (d) $A(5,6), B(-3,2), C(3,-4)$

Solution

Given: $A(4,5)$, $B(-4,-3)$, $C(8,-3)$

Let $H(h,k)$ be centre and r be radius of circle, then

$|\overline{AH}| = |\overline{BH}| = |\overline{CH}| = r$

$\Rightarrow |\overline{AH}|^2 = |\overline{BH}|^2 = |\overline{CH}|^2 = r^2$

$\Rightarrow (h-4)^2+(k-5)^2=(h+4)^2+(k+3)^2=(h-8)^2+(k+3)^2=r^2$ …… (i)

From eq. (i)

$$(h-4)^2+(k-5)^2 = (h+4)^2+(k+3)^2$$

$$\Rightarrow h^2-8h+16+k^2-10k+25 = h^2+8h+16+k^2+6k+9$$

$$\Rightarrow h^2-8h+16+k^2-10k+25-h^2-8h-16-k^2-6k-9 = 0$$

$$\Rightarrow -16h-16k+16 = 0 \quad \Rightarrow h+k-1 = 0 \text{ ..... (ii)}$$

Again from (i)

$$(h+4)^2+(k+3)^2 = (h-8)^2+(k+3)^2$$

$$\Rightarrow (h+4)^2 = (h-8)^2 \Rightarrow h^2+8h+16 = h^2-16h+64$$

$$\Rightarrow h^2+8h+16-h^2+16h-64 = 0$$

$$\Rightarrow 24h-48 = 0 \quad \Rightarrow 24h = 48 \quad \Rightarrow \boxed{h = 2}$$

Putting value of h in (ii)

$$2+k-1 = 0 \quad \Rightarrow k+1 = 0 \quad \Rightarrow \boxed{k = -1}$$

Again from (i)

$$r^2 = (h-4)^2+(k-5)^2$$

$$= (2-4)^2+(-1-5)^2 \quad \because h=2,\ k=-1$$

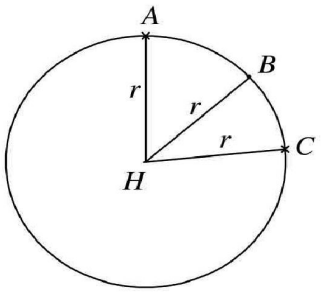
$$= (-2)^2+(-6)^2 = 4+36 = 40 \quad \Rightarrow r = \sqrt{40}$$

Now equation of circle with centre at $H(2,-1)$ & $r=\sqrt{40}$

$$(x-2)^2+(y+1)^2 = \left(\sqrt{40}\right)^2$$

$$\Rightarrow x^2-4x+4+y^2+2y+1 = 40 \Rightarrow x^2+y^2-4x+2y+4+1-40 = 0$$

$$\Rightarrow x^2+y^2-4x+2y-35 = 0 \qquad \textit{Ans.}$$



(b) Given: $A(-7,7)$, $B(5,-1)$, $C(10,0)$

Let $H(h,k)$ be centre and r be radius of circle, then

$$\left|\overline{AH}\right| = \left|\overline{BH}\right| = \left|\overline{CH}\right| = r$$

$$\Rightarrow \left|\overline{AH}\right|^2 = \left|\overline{BH}\right|^2 = \left|\overline{CH}\right|^2 = r^2$$

$$\Rightarrow (h+7)^2+(k-7)^2 = (h-5)^2+(k+1)^2 = (h-10)^2+(k-0)^2 = r^2 \text{ ..... (i)}$$

From equation (i) we have

$$(h+7)^2+(k-7)^2 = (h-10)^2+(k-0)^2$$

$$\Rightarrow h^2+14h+49+k^2-14k+49 = h^2-20h+100+k^2$$

$$\Rightarrow h^2+14h+49+k^2-14k+49-h^2+20h-100-k^2 = 0$$

$$\Rightarrow 34h-14k-2 = 0 \quad \Rightarrow 17h-7k-1 = 0 \text{ ..... (ii)}$$

Again from (i)

$$(h-5)^2+(k+1)^2 = (h-10)^2+(k-0)^2$$

$$\Rightarrow h^2-10h+25+k^2+2k+1 = h^2-20h+100+k^2$$

$$\Rightarrow\ h^2-10h+25+k^2+2k+1-h^2+20h-100-k^2 = 0$$
$$\Rightarrow\ 10h+2k-74 = 0$$
$$\Rightarrow\ 5h+k-37 = 0 \ \ldots\ldots\ldots\ldots \text{(iii)}$$

Multiplying eq. (iii) by 7 and subtracting from (ii)

$$\begin{array}{r} 17h-7k-1 = 0 \\ 35h+7k-259 = 0 \\ \hline 52h \qquad -260 = 0 \end{array}$$

$$\Rightarrow\ 52h = 260 \quad \Rightarrow\ \boxed{h = 5}$$

Putting value of h in eq. (iii)

$$5(5)+k-37 = 0 \Rightarrow 25+k-37 = 0 \Rightarrow k-12 = 0 \Rightarrow \boxed{k = 12}$$

Again from eq. (i), we have

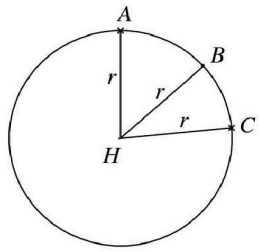
$$r^2 = (h+7)^2+(k-7)^2$$
$$= (5+7)^2+(12-7)^2 = (12)^2+(5)^2 = 144+25 = 169$$
$$\Rightarrow\ r = 13$$

Now equation of circle with centre $(5,12)$ and radius 13:

$$(x-5)^2+(y-12)^2 = (13)^2$$
$$\Rightarrow\ x^2-10x+25+y^2-24y+144 = 169$$
$$\Rightarrow\ x^2+y^2-10x-24y+25+144-169 = 0$$
$$\Rightarrow\ x^2+y^2-10x-24y = 0$$

(c) Given: $A(a,0)$, $B(0,b)$, $C(0,0)$

Let $H(h,k)$ be centre and r be radius of circle, then



$$\left|\overline{AH}\right| = \left|\overline{BH}\right| = \left|\overline{CH}\right| = r$$
$$\Rightarrow\ \left|\overline{AH}\right|^2 = \left|\overline{BH}\right|^2 = \left|\overline{CH}\right|^2 = r^2$$

$$\Rightarrow\ (h-a)^2+(k-0)^2 = (h-0)^2+(k-b)^2 = (h-0)^2+(k-0)^2 = r^2$$
$$\Rightarrow\ (h-a)^2+k^2 = h^2+(k-b)^2 = h^2+k^2 = r^2 \ \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

From equation (i)

$$(h-a)^2+k^2 = h^2+k^2 \Rightarrow h^2-2ha+a^2+k^2 = h^2+k^2$$
$$\Rightarrow\ -2ha+a^2 = 0 \Rightarrow -2ha = -a^2 \Rightarrow h = \frac{a^2}{2a} \Rightarrow \boxed{h = \frac{a}{2}}$$

Again from equation (i)

$$h^2+(k-b)^2 = h^2+k^2$$
$$\Rightarrow\ h^2+k^2-2bk+b^2 = h^2+k^2 \quad \Rightarrow -2bk+b^2 = 0$$
$$\Rightarrow\ 2bk = b^2 \quad \Rightarrow k = \frac{b^2}{2b} \quad \Rightarrow \boxed{k = \frac{b}{2}}$$

Again from equation (i)

$$r^2 = h^2 + k^2$$

$$= \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} \quad \Rightarrow r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

Now equation of circle with centre $\left(\frac{a}{2}, \frac{b}{2}\right)$ and radius $\frac{\sqrt{a^2+b^2}}{2}$

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right) = \left(\sqrt{\frac{a^2}{4} + \frac{b^2}{4}}\right)^2$$

$$\Rightarrow x^2 - ax + \frac{a^2}{4} + y^2 - by + \frac{b^2}{4} = \frac{a^2}{4} + \frac{b^2}{4}$$

$$\Rightarrow x^2 - ax + \frac{a^2}{4} + y^2 - by + \frac{b^2}{4} - \frac{a^2}{4} - \frac{b^2}{4} = 0 \Rightarrow x^2 + y^2 - ax - by = 0$$

Alternative Method

Given point on circle: $A(a,0)$, $B(0,b)$, $C(0,0)$

Consider an equation of circle in standard form

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ ..... (i)}$$

Since $A(a,0)$ lies on circle, therefore

$$(a)^2 + (0)^2 + 2g(a) + 2f(0) + c = 0$$

$$\Rightarrow a^2 + 2ga + c = 0 \text{ ..... (ii)}$$

Also $B(0,b)$ lies on the circle, then

$$(0)^2 + (b)^2 + 2g(0) + 2f(b) + c = 0$$

$$\Rightarrow b^2 + 2fb + c = 0 \text{ ..... (iii)}$$

Also $C(0,0)$ lies on the circle, therefore

$$(0)^2 + (0)^2 + 2g(0) + 2f(0) + c = 0 \quad \Rightarrow c = 0$$

Putting value of c in (ii)

$$a^2 + 2ga + 0 = 0 \quad \Rightarrow 2ga = -a^2 \quad \Rightarrow g = -\frac{a^2}{2a} \quad \Rightarrow g = -\frac{a}{2}$$

Putting value of c in (iii)

$$b^2 + 2fb + 0 = 0 \quad \Rightarrow 2fb = -b^2$$

$$\Rightarrow f = -\frac{b^2}{2b} \quad \Rightarrow f = -\frac{b}{2}$$

Putting value of g, f and c in (i)

$$x^2 + y^2 + 2\left(-\frac{a}{2}\right)x + 2\left(-\frac{b}{2}\right)y + 0 = 0$$

$$\Rightarrow x^2 + y^2 - ax - by = 0$$

Question # 4

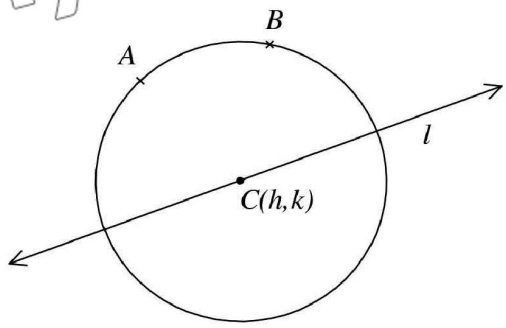
In each of the following, find an equation of the circle passes through

(a) $A(3,-1)$, $B(0,1)$ and have centre at $4x-3y-3=0$

(b) $A(-3,1)$ with radius 2 and centre at $2x-3y+3=0$

(c) $A(5,1)$ and tangent to the line $2x-y-10=0$ at $B(3,-4)$

(d) $A(1,4), B(-1,8)$ and tangent to the line $x+3y-3=0$



Solution

(a) Given: $A(3,-1)$, $B(0,1)$

$l:\ 4x-3y-3 = 0$

Let $C(h,k)$ be centre and r be radius of circle

$\because$ A & B lies on circle

$\therefore$ $\left|\overline{CA}\right| = \left|\overline{CB}\right| = r$

$\Rightarrow \sqrt{(h-3)^2+(k+1)^2} = \sqrt{(h-0)^2+(k-1)^2} = r$ …….. (i)

$\Rightarrow (h-3)^2+(k+1)^2 = h^2+(k-1)^2$ on squaring

$\Rightarrow h^2-6h+9+k^2+2k+1 = h^2+k^2-2k+1$

$\Rightarrow h^2-6h+9+k^2+2k+1-h^2-k^2+2k-1 = 0$

$\Rightarrow -6h+4k+9 = 0$ …… (ii)

Now since $C(h,k)$ lies on given equation l

$\therefore$ $4h-3k-3 = 0$ …… (iii)

$\times$ing equation (ii) by 3 & (iii) by 4 then adding

$$\begin{array}{rcl} -18h+12k+27 &=& 0 \\ 16h-12k-12 &=& 0 \\ \hline -2h \qquad +15 &=& 0 \end{array}$$

$\Rightarrow 2h = 15 \quad \Rightarrow \boxed{h = \frac{15}{2}}$

Putting in (iii)

$4\left(\frac{15}{2}\right)-3k-3 = 0 \quad \Rightarrow 30-3k-3 = 0$

$\Rightarrow -3k+27 = 0 \Rightarrow 3k = 27 \quad \Rightarrow \boxed{k = 9}$

Now from eq. (i)

$r = \sqrt{h^2+(k-1)^2}$

$= \sqrt{\left(\frac{15}{2}\right)^2+(9-1)^2} = \sqrt{\frac{225}{4}+64} = \sqrt{\frac{448}{4}}$

Now equation of circle with centre at $C(h,k)=\left(\frac{15}{2},9\right)$ and radius $\sqrt{\frac{481}{4}}$

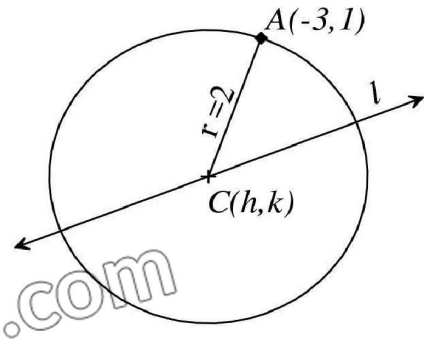
$$\left(x-\frac{15}{2}\right)^2+(y-9)^2 = \left(\sqrt{\frac{481}{4}}\right)^2$$

$$\Rightarrow\ x^2-15x+\frac{225}{4}+y^2-18y+81-\frac{481}{4} = 0$$

$$\Rightarrow\ x^2+y^2-15x-18y+17 = 0$$

(b) Given: $A(-3,1)$ lies on circle , radius $= r = 2$

$l:\ 2x-3y+3 = 0$



Let $C(h,k)$ be centre of circle.

Since $A(-3,1)$ lies on circle

$$\therefore\ r = \left|\overline{AC}\right|$$

$$\Rightarrow\ 2 = \sqrt{(h+3)^2+(k-1)^2}$$

$$\Rightarrow\ 4 = (h+3)^2+(k-1)^2$$

$$\Rightarrow\ 4 = h^2+6h+9+k^2-2k+1 \Rightarrow\ h^2+6h+9+k^2-2k+1-4 = 0$$

$$\Rightarrow\ h^2+6h+9+k^2-2k-3 = 0 \ \ldots\ldots\ \text{(i)}$$

Since centre $C(h,k)$ lies on l

$$\therefore\ 2h-3k+3 = 0$$

$$\Rightarrow\ 2h = 3k-3 \quad\Rightarrow\ h = \frac{3k-3}{2} \ \ldots\ldots\ldots\ \text{(ii)}$$

Putting value of h in (i)

$$\left(\frac{3k-3}{2}\right)^2+k^2+6\left(\frac{3k-3}{2}\right)-2k+6 = 0$$

$$\Rightarrow\ \frac{9k^2-18k+9}{4}+k^2+9k-9-2k+6 = 0$$

$$\Rightarrow\ 9k^2-18k+9+4k^2+36k-36-8k+24 = 0 \qquad \times\text{ing by } 4$$

$$\Rightarrow\ 13k^2+10k-3 = 0 \quad\Rightarrow\ 13k^2+13k-3k-3 = 0$$

$$\Rightarrow\ 13k(k+1)-3(k+1) = 0 \Rightarrow\ (k+1)(13k-3) = 0$$

$\Rightarrow\ k = -1$ or $k = \frac{3}{13}$,

$$h = \frac{3(-1)-3}{2} = \frac{-6}{2} = -3$$

$\Rightarrow\ (-3,-1)$ is centre of circle

Putting value of k in (ii)

$$h = \frac{3\left(\frac{3}{13}\right)-3}{2} = \frac{\frac{9}{13}-3}{2} = \frac{-\frac{30}{13}}{2} = \frac{-15}{13}$$

$\Rightarrow\ \left(-\frac{15}{13},\frac{3}{13}\right)$ is centre of circle.

Now equation of circle with centre at $(-3,1)$ and radius 2

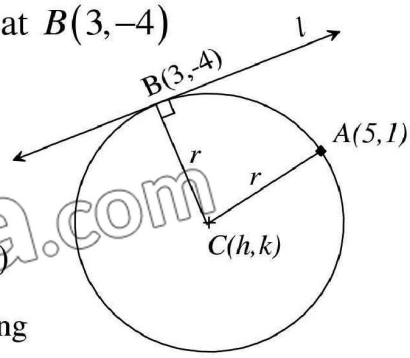
$$(x+3)^2+(y+1)^2 = (2)^2 \quad \Rightarrow \quad (x+3)^2+(y+1)^2 = 4$$

Now equation of circle with centre at $\left(-\frac{15}{13},\frac{3}{13}\right)$ and radius 2

$$\left(x+\frac{15}{13}\right)^2+\left(y-\frac{3}{13}\right)^2 = (2)^2$$

$$\Rightarrow \left(x+\frac{15}{13}\right)^2+\left(y-\frac{3}{13}\right)^2 = 4$$

(c) Given: $A(5,1)$ and $l:\ 2x-y-10=0$ is tangent at $B(3,-4)$



Let $C(h,k)$ be centre and r be radius of circle.

$\because$ $A(5,1)$ and $B(3,-4)$ lies on circle

$\therefore$ $|AC| = |BC| = r$

$\Rightarrow \sqrt{(h-5)^2+(k-1)^2} = \sqrt{(h-3)^2+(k+4)^2} = r$ ..... (i)

$\Rightarrow (h-5)^2+(k-1)^2 = (h-3)^2+(k+4)^2$ On squaring

$\Rightarrow h^2-10h+25+k^2-2k+1 = h^2-6h+9+k^2+8k+16$

$\Rightarrow h^2-10h+25+k^2-2k+1-h^2+6h-9-k^2-8k-16 = 0$

$\Rightarrow -4h-10k+1 = 0$ ..... (ii)

Now slope of tangent $l = m_1 = -\frac{a}{b} = -\frac{2}{-1} = 2$

And slope of radial segment $\overline{CB} = m_2 = \frac{k+4}{h-3}$

Since radial segment is perpendicular to tangent therefore

$$m_1m_2 = -1$$

$$\Rightarrow 2\left(\frac{k+4}{h-3}\right) = -1 \quad \Rightarrow \quad 2k+8 = -h+3$$

$$\Rightarrow h-3+2k+8 = 0$$

$$\Rightarrow h+2k+5 = 0 \quad \text{..... (iii)}$$

Multiplying eq. (iii) by 4 and adding in (ii)

$$\begin{array}{rcl} 4h+8k+20 &=& 0 \\ -4h-10k+1 &=& 0 \\ \hline -2k+21 &=& 0 \end{array}$$

$$\Rightarrow 2k = 21 \quad \Rightarrow \quad \boxed{k = \frac{21}{2}}$$

Putting value of k in (iii)

$$h+2\left(\frac{21}{2}\right)+5 = 0 \quad \Rightarrow \quad h+21+5 = 0$$

$$\Rightarrow \quad h+26 = 0 \quad \Rightarrow \quad \boxed{h = -26}$$

Now from eq. (i)

$$r = \sqrt{(h-3)^2+(k+4)^2}$$

$$= \sqrt{(-26-3)^2+\left(\frac{21}{2}+4\right)^2} \quad = \sqrt{(-29)^2+\left(\frac{29}{2}\right)^2}$$

$$= \sqrt{841+\frac{841}{4}} \quad = \sqrt{\frac{4205}{4}}$$

Now equation of circle with centre at $\left(-26, \frac{21}{2}\right)$ and radius $\sqrt{\frac{4205}{4}}$

$$(x+26)^2+\left(y-\frac{21}{2}\right)^2 = \left(\sqrt{\frac{4205}{4}}\right)^2$$

$$\Rightarrow \quad x^2+52x+676+y^2-21y+\frac{441}{4}-\frac{4205}{4} = 0$$

$$\Rightarrow \quad x^2+y^2+52x-21y-265 = 0$$

(d) Given; $A(1,4)$, $B(-1,8)$

$$l: \quad x+3y-3 = 0$$

Let $C(h,k)$ be centre and r be radius of circle then

$$\left|\overline{AC}\right| = \left|\overline{BC}\right| = r$$

$$\Rightarrow \quad \sqrt{(h-1)^2+(k-4)^2} = \sqrt{(h+1)^2+(k-8)^2} = r \quad \text{..... (i)}$$

Also l is tangent to circle

$\therefore$ radius of circle $=$ $\perp$ar distance of $C(h,k)$ form l

$$\Rightarrow \quad r = \frac{|h+3k-3|}{\sqrt{(1)^2+(3)^2}}$$

$$\Rightarrow \quad r = \frac{|h+3k-3|}{\sqrt{10}} \quad \text{..... (ii)}$$

Now from (i)

$$\sqrt{(h-1)^2+(k-4)^2} = \sqrt{(h+1)^2+(k-8)^2}$$

On squaring

$$(h-1)^2+(k-4)^2 = (h+1)^2+(k-8)^2$$

$$\Rightarrow \quad h^2-2h+1+k^2-8k+16 = h^2+2h+1+k^2-16k+64$$

$$\Rightarrow \quad h^2-2h+1+k^2-8k+16-h^2-2h-1-k^2+16k-64 = 0$$

$$\Rightarrow \quad -4h+8k-48 = 0$$

$$\Rightarrow \quad h-2k+12 = 0 \quad \text{..... (iii)}$$

Now from (i) & (ii)

$$\sqrt{(h-1)^2+(k-4)^2} = \frac{|h+3k-3|}{\sqrt{10}}$$

On squaring

$$(h-1)^2+(k-4)^2 = \frac{|h+3k-3|^2}{10}$$

$$\Rightarrow 10\left[(h-1)^2+(k-4)^2\right] = h^2+9k^2+9+6hk-18k-6h$$

$$\Rightarrow 10\left[h^2-2h+1+k^2-8k+16\right] = h^2+9k^2+9+6hk-18k-6h$$

$$\Rightarrow 10h^2-20h+10+10k^2-80k+160-h^2-9k^2-9-6hk+18k+6h = 0$$

$$\Rightarrow 9h^2+k^2-14h-62k-6hk+161 = 0 \text{ ..... (iv)}$$

From (iii)

$$h = 2k-12 \text{ ..... (v)}$$

Putting in (iv)

$$9(2k-12)^2+k^2-14(2k-12)-62k-6(2k-12)k+161 = 0$$

$$\Rightarrow 9(4k^2-48k+144)+k^2-28k+168-62k-12k^2+72k+161 = 0$$

$$\Rightarrow 36k^2-432k+1296+k^2-28k+168-62k-12k^2+72k+161 = 0$$

$$\Rightarrow 25k^2-450k+1625 = 0$$

$$\Rightarrow k^2-18k+65 = 0 \quad \div\text{ing by } 25$$

$$\Rightarrow k^2-13k-5k+65 = 0 \quad \Rightarrow k(k-13)-5(k-13) = 0$$

$$\Rightarrow (k-13)(k-5) = 0$$

$$\Rightarrow k=13 \text{ or } k = 5$$

Putting in eq. (v)

$$h = 2(13)-12$$
$$= 26-12 = 14$$

Now from (i)

$$r = \sqrt{(h-1)^2+(k-4)^2}$$

$$\Rightarrow r = \sqrt{(14-1)^2+(13-4)^2}$$

$$= \sqrt{(13)^2+(9)^2} = \sqrt{169+81}$$

$$= \sqrt{250}$$

Now eq. of circle with centre $(14,13)$ and radius $\sqrt{170}$

$$(x-14)^2+(y-13)^2 = \left(\sqrt{250}\right)^2$$

$$\Rightarrow (x-14)^2+(y-13)^2 = 250$$

Putting in (v)

$$h = 2(5)-12$$
$$= 10-12 = -2$$

Now from (i)

$$r = \sqrt{(h-1)^2+(k-4)^2}$$
$$= \sqrt{(2-1)^2+(5-4)^2}$$
$$= \sqrt{1+1} = \sqrt{2}$$

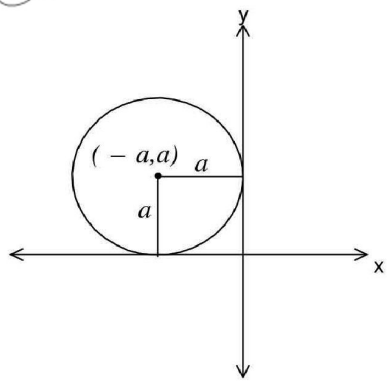
Now eq. of circle with centre $(2,5)$ and radius $\sqrt{2}$

$$(x-2)^2+(y-5)^2 = \left(\sqrt{2}\right)^2$$

$$\Rightarrow (x-2)^2+(y-5)^2 = 2$$

Question # 5

Find an equation of a circle of radius a and lying in the second quadrant such that it is tangent to both the axes.



Solution

Radius of circle $= r = a$

$\because$ circle lies in second quadrant and touching both the axis therefore centre of circle is $(-a, a)$

So equation of circle

$$(x-(-a))^2 + (y-a)^2 = (a)^2$$

$$\Rightarrow \quad x^2 - 2ax + a^2 + y^2 - 2ay + a^2 - a^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2ax - 2ay + a^2 = 0$$

Question # 6

Show that the lines $3x - 2y = 0$ and $2x + 3y - 13 = 0$ are tangents to the circle $x^2 + y^2 + 6x - 4y = 0$

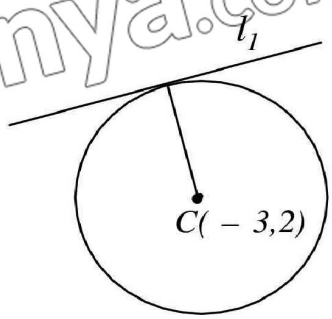
Solution

Suppose

$l_1: \; 3x - 2y = 0$

$l_2: \; 2x + 3y - 13 = 0$

$S: \; x^2 + y^2 + 6x - 4y = 0$



From S

$2g = 6 \;, \quad 2f = -4 \;, \quad c = 0$

$\Rightarrow \quad g = 3 \;, \quad f = -2 \;,$

Centre $C(-g, -f) = C(-3, 2)$

Radius $= r = \sqrt{g^2 + f^2 - c}$

$= \sqrt{(3)^2 + (-2)^2 - 0}$

$= \sqrt{9 + 4 = 0} = \sqrt{13}$

Now to check l_1 is tangent to circle, we find

$\perp$ ar distance of l_1 from centre $= \dfrac{|3(-3) - 2(2) + 0|}{\sqrt{(3)^2 + (-2)^2}}$

$= \dfrac{|-9-4|}{\sqrt{9+4}} = \dfrac{|-13|}{\sqrt{13}} = \dfrac{13}{\sqrt{13}}$

$= \sqrt{13}$ = radius of circle

$\Rightarrow$ l_1 is tangent to given circle.

Now to check l_2 is tangent to circle, let

$\perp$ ar distance of l_2 from centre $= \dfrac{|2(-3) + 3(2) - 13|}{\sqrt{(2)^2 + (3)^2}}$

$$= \frac{|-6+6-13|}{\sqrt{4+9}} = \frac{|-13|}{\sqrt{13}}$$

$$= \frac{13}{\sqrt{13}} = \sqrt{13} = \text{Radius of circle}$$

$\Rightarrow$ l_2 is also tangent to given circle.

Circles touching each other externally or internally

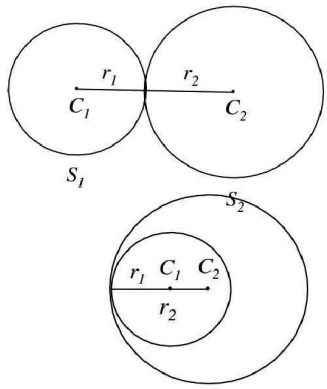
Let C_1 be centre and r_1 be radius of circle S_1 and C_2 be centre and r_2 be radius of circle S_2.

Then they touch each other externally if

$$\left|\overline{C_1C_2}\right| = r_1 + r_2$$

And they touch each other internally if

$$\left|\overline{C_1C_2}\right| = |r_2 - r_1|$$



Question # 7

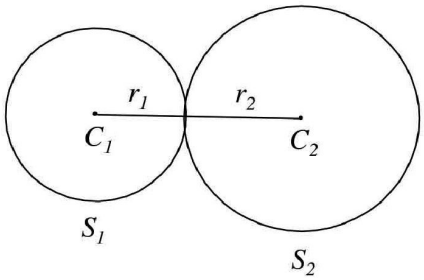
Show that the circles

$x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x + 4y + 9 = 0$ touch externally.

Solution

Let $S_1: \; x^2 + y^2 + 2x - 2y - 7 = 0$

$S_2: \; x^2 + y^2 - 6x + 4y + 9 = 0$



For S_1:

$$2g = 2 \quad , \quad 2f = -2 \quad , \quad c = -7$$

$$\Rightarrow \quad g = 1 \quad , \quad f = -1 \; ,$$

Let C_1 be centre and r_1 be radius of circle S_1, then

$$C_1(-g, -f) = C_1(-1, 1)$$

$$\text{Radius} = r_1 = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(1)^2 + (-1)^2 - (-7)} = \sqrt{1+1+7} = \sqrt{9} = 3$$

For S_2:

$$2g = -6 \quad , \quad 2f = 4 \quad , \quad c = 9$$

$$\Rightarrow \quad g = -3 \quad , \quad f = 2$$

Let C_2 be centre and r_2 be radius of circle S_2 then

$$C_2(-g, -f) = C_2(3, -2)$$

$$\text{Radius} = r_2 = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(-3)^2 + (2)^2 - 9} = \sqrt{9+4-9} = \sqrt{4} = 2$$

Now circles touch each other externally if

$$|C_1C_2| = r_1 + r_2$$

$$\Rightarrow \sqrt{(3+1)^2 + (-2-1)^2} = 3 + 2$$

$\Rightarrow \sqrt{16+9} = 5 \Rightarrow \sqrt{25} = 5 \Rightarrow 5 = 5$

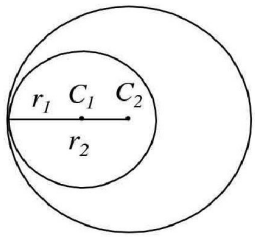
Hence both circles touch each other externally.

Question # 8

Show that the circles

$x^2 + y^2 + 2x - 8 = 0$ and $x^2 + y^2 - 6x + 6y - 46 = 0$ touches internally.

Solution



Suppose $S_1: \ x^2 + y^2 + 2x - 8 = 0$

$S_2: \ x^2 + y^2 - 6x + 6y - 46 = 0$

For S_1: $2g = 2 \ , \ 2f = 0 \ , \ c = -8$

$\Rightarrow g = 1 \ , \ f = 0$

Let C_1 be centre and r_1 be radius of circle S_1 then

$C_1(-g, -f) = C(-1, 0)$

Radius $= r_1 = \sqrt{g^2 + f^2 - c}$

$= \sqrt{(1)^2 + (0)^2 + 8} = \sqrt{9} = 3$

For S_2: $2g = -6 \ , \ 2f = 6 \ , \ c = -46$

$\Rightarrow g = -3 \ , \ f = 3$

Let C_2 be centre and r_2 be radius of circle S_2 then

$C_2(-g, -f) = C_2(3, -3)$

Radius $= r_2 = \sqrt{g^2 + f^2 - c}$

$= \sqrt{(3)^2 + (-3)^2 - (-46)} = \sqrt{9 + 9 + 46} = \sqrt{64} = 8$

Now circles touch each other internally if

$\left|\overline{C_1C_2}\right| = |r_2 - r_1| \Rightarrow \sqrt{(3+1)^2 + (-3-0)^2} = |8-3|$

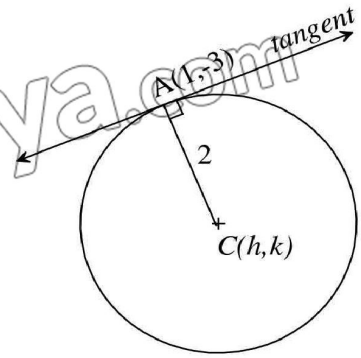
$\Rightarrow \sqrt{16+9} = |5| \Rightarrow \sqrt{25} = 5 \Rightarrow 5 = 5$

Hence circles are touching each other internally.

Question # 9

Find an equation of the circle of radius 2 and tangent to the line $x - y - 4 = 0$ at $A(1, -3)$.

Solution



Given: Radius $r = 2$,

Tangent: $x - y - 4 = 0$ at $A(1, -3)$

Suppose $C(h, k)$ be the centre then

$|AC| = 2$

$\Rightarrow \sqrt{(h-1)^2 + (k+3)^2} = 2$

On squaring

$(h-1)^2 + (k+3)^2 = 4$

$\Rightarrow h^2 - 2h + 1 + k^2 + 6k + 9 - 4 = 0 \Rightarrow h^2 + k^2 - 2h + 6k + 6 = 0$ …….. (i)

Now slope of radial line $AC = \dfrac{k+3}{h-1}$

Slope of line tangent $= -\dfrac{1}{-1} = 1$

Since radial line is $\perp$ ar to tangent, therefore

(Slope of radial line) (Slope of tangent) $= -1$

$$\Rightarrow \left(\frac{k+3}{h-1}\right)(1) = -1$$

$$\Rightarrow k+3 = -(h-1) \quad \Rightarrow k = -h+1-3 \quad \Rightarrow k = -h-2 \text{ ..... (ii)}$$

Putting in (i) $h^2 + (-h-2)^2 - 2h + 6(-h-2) + 6 = 0$

$$\Rightarrow h^2 + h^2 + 4h + 4 - 2h - 6h - 12 + 6 = 0 \Rightarrow 2h^2 - 4h - 2 = 0 \quad \Rightarrow h^2 - 2h - 1 = 0$$

$$\Rightarrow h = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

Putting $h = 1 + \sqrt{2}$ in (ii)

$k = -1 - \sqrt{2} - 2$

$\Rightarrow k = -3 - \sqrt{2}$

Now equation of circle with centre $\left(1+\sqrt{2}, -3-\sqrt{2}\right)$ and radius 2.

$$\left(x - (1+\sqrt{2})\right)^2 - \left(y - (-3-\sqrt{2})\right)^2 = (2)^2$$

$$\Rightarrow \left(x - 1 - \sqrt{2}\right)^2 - \left(y + 3 + \sqrt{2})\right)^2 = 4$$

Putting $h = 1 - \sqrt{2}$ in (ii)

$k = -1 + \sqrt{2} - 2 \Rightarrow k = -3 + \sqrt{2}$

Now equation of circle with centre $\left(1-\sqrt{2}, -3+\sqrt{2}\right)$ and radius 2.

$$\left(x - (1-\sqrt{2})\right)^2 - \left(y - (-3+\sqrt{2})\right)^2 = (2)^2$$

$$\Rightarrow \left(x - 1 + \sqrt{2}\right)^2 - \left(y + 3 - \sqrt{2})\right)^2 = 4$$

Equation of tangent and normal to the circle

Consider an equation of circle

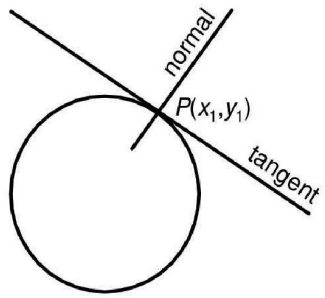
$$x^2+y^2+2gx+2fy+c = 0$$

Then equation of tangent at (x_1, y_1) is

$$x_1x+y_1y+g(x+x_1)+f(y+y_1)+c = 0$$

The equation of normal at (x_1, y_1) is

$$(y-y_1)(x_1+g) = (x-x_1)(y_1+f)$$



(*See proof at page 257*)

Question # 1

Write down equations of tangent and normal to the circle

(i) $x^2+y^2 = 25$ at $(4,3)$ and at $(5\cos\theta, 5\sin\theta)$

(ii) $3x^2+3y^2+5x-13y+2 = 0$ at $\left(1, \frac{10}{3}\right)$

Solution $x^2+y^2 = 25$

Differentiating w.r.t. x

$$2x+2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x \qquad \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

At $(4,3)$:

Slope of tangent at $(4,3)$ $= m = \left.\frac{dy}{dx}\right|_{(4,3)} = -\frac{4}{3}$

Now equation of tangent at $(4,3)$ having slope $-\frac{4}{3}$

$$y-3 = -\frac{4}{3}(x-4) \qquad \Rightarrow 3y-9 = -4x+16$$

$$\Rightarrow 4x-16+3y-9 = 0 \qquad \Rightarrow \boxed{4x+3y-25 = 0}$$

Since normal is $\perp$ar to tangent therefore

Slope of normal at $(4,3)$ $= -\frac{1}{m} = -\frac{1}{-4/3} = \frac{3}{4}$

Now equation of normal at $(4,3)$ having slope $\frac{3}{4}$

$$y-3 = \frac{3}{4}(x-4)$$

$$\Rightarrow 4y-12 = 3x-12 \qquad \Rightarrow 3x-12-4y+12 = 0$$

$$\Rightarrow \boxed{3x-4y = 0}$$

At $(5\cos\theta, 5\sin\theta)$

Slope of tangent at $(5\cos\theta, 5\sin\theta) = m = \left.\frac{dy}{dx}\right|_{(5\cos\theta, 5\sin\theta)} = -\frac{5\cos\theta}{5\sin\theta} = -\frac{\cos\theta}{\sin\theta}$

Now equation of tangent at $(5\cos\theta, 5\sin\theta)$ having slope $-\frac{\cos\theta}{\sin\theta}$

$$y - 5\sin\theta = -\frac{\cos\theta}{\sin\theta}(x - 5\cos\theta)$$

$$\Rightarrow \quad y\sin\theta - 5\sin^2\theta = -x\cos\theta + 5\cos^2\theta$$

$$\Rightarrow \quad x\cos\theta + y\sin\theta = 5\sin^2\theta + 5\cos^2\theta$$

$$\Rightarrow \quad x\cos\theta + y\sin\theta = 5\left(\sin^2\theta + \cos^2\theta\right)$$

$$\Rightarrow \quad x\cos\theta + y\sin\theta = 5(1)$$

$$\Rightarrow \quad \boxed{x\cos\theta + y\sin\theta = 5}$$

Since normal are $\perp$ ar to tangent therefore

Slope of normal $= -\frac{1}{m} = \frac{\sin\theta}{\cos\theta}$

Now equation of normal at $(5\cos\theta, 5\sin\theta)$ having slope $\frac{\sin\theta}{\cos\theta}$

$$y - 5\sin\theta = \frac{\sin\theta}{\cos\theta}(x - 5\cos\theta)$$

$$\Rightarrow \quad y\cos\theta - 5\sin\theta\cos\theta = x\sin\theta - 5\sin\theta\cos\theta$$

$$\Rightarrow \quad x\sin\theta - 5\sin\theta\cos\theta - y\cos\theta + 5\sin\theta\cos\theta = 0$$

$$\Rightarrow \quad \boxed{x\sin\theta - y\cos\theta = 0}$$

(ii) **[Alternative Method]**

$$3x^2 + 3y^2 + 5x - 13y + 2 = 0$$

$$\Rightarrow \quad x^2 + y^2 + \frac{5}{3}x - \frac{13}{3}y + \frac{2}{3} = 0$$

Comparing it with general equation of circle

$$2g = \frac{5}{3}, \quad 2f = -\frac{13}{3}, \quad c = \frac{2}{3}$$

$$\Rightarrow \quad g = \frac{5}{6}, \quad f = -\frac{13}{6}$$

Now equation of tangent at (x_1, y_1)

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

Here $(x_1, y_1) = \left(1, \frac{10}{3}\right)$

$$\Rightarrow 1 \cdot x + \frac{10}{3} \cdot y + \frac{5}{6}(x+1) - \frac{13}{6}\left(y + \frac{10}{3}\right) + \frac{2}{3} = 0$$

$$\Rightarrow \quad x + \frac{10}{3}y + \frac{5}{6}x + \frac{5}{6} - \frac{13}{6}y - \frac{130}{18} + \frac{2}{3} = 0$$

$$\Rightarrow \frac{11}{6}x+\frac{7}{6}y+\frac{157}{18} = 0$$
$$\Rightarrow 33x+21y+157 = 0$$

is the required tangent.

Now equation of normal at (x_1, y_1)

$$(y-y_1)(x_1+g) = (x-x_1)(y_1+f)$$

Here $(x_1, y_1) = \left(1, \frac{10}{3}\right)$

$$\Rightarrow \left(y-\frac{10}{3}\right)\left(1+\frac{5}{6}\right) = (x-1)\left(\frac{10}{3}-\frac{13}{6}\right)$$
$$\Rightarrow \left(y-\frac{10}{3}\right)\left(\frac{11}{6}\right) = (x-1)\left(\frac{7}{6}\right) \qquad \Rightarrow 11y-\frac{110}{3} = 7x-7$$
$$\Rightarrow 7x-7-11y+\frac{110}{3} = 0 \qquad \Rightarrow 7x-11y+\frac{89}{3} = 0$$
$$\Rightarrow 21x-33y+89 = 0$$

is required equation of normal.

Question # 2

Write down equations of tangent and normal to the circle
$4x^2+4y^2-16x+24y-117 = 0$
at the point on the circle whose abscissa is -4.

Solution

$$4x^2+4y^2-16x+24y-117 = 0 \text{ ..... (i)}$$

Since abscissa $= -4$, so putting $x=-4$ in given eq.

$$4(-4)^2+4y^2-16(-4)+24y-117 = 0$$
$$\Rightarrow 64+4y^2+64+24y-117 = 0$$
$$\Rightarrow 4y^2+24y+11 = 0$$
$$\Rightarrow y = \frac{-24\pm\sqrt{(24)^2-4(4)(11)}}{2(4)}$$
$$= \frac{-24\pm\sqrt{576-176}}{8} \qquad = \frac{-24\pm\sqrt{400}}{8}$$
$$\Rightarrow y = \frac{-24\pm 20}{8}$$
$$\Rightarrow y = \frac{-24+20}{8} \quad \text{or} \quad y = \frac{-24-20}{8}$$
$$\Rightarrow y = -\frac{1}{2} \quad \text{or} \quad y = -\frac{11}{2}$$

So we have points $\left(-4, -\frac{1}{2}\right)$ & $\left(-4, -\frac{11}{2}\right)$

$$\text{(i)} \Rightarrow x^2+y^2-4x+6y-\frac{117}{4} = 0$$

Comparing it with general equation of circle

$$2g = -4 \quad , \quad 2f = 6 \quad , \quad c = -\frac{117}{4}$$

$$\Rightarrow \quad g = -2 \quad , \quad f = 3$$

Now equation of tangent at (x_1, y_1)

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

For $(x_1, y_1) = \left(-4, -\frac{1}{2}\right)$

Solve yourself as Q # 1(ii)

For $(x_1, y_1) = \left(-4, -\frac{11}{2}\right)$

Solve yourself as Q # 1(ii)

Position of the point with a circle

Consider the general equation of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

The point $P(x_1, y_1)$ lies on the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

The point $P(x_1, y_1)$ lies outside the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$$

And the point $P(x_1, y_1)$ lies inside the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$$

Question # 3

Check the position of the point $(5,6)$ with respect to the circle

(i) $x^2 + y^2 = 81$ (ii) $2x^2 + 2y^2 + 12x - 8y + 1 = 0$

Solution

(i) $$x^2 + y^2 = 81$$

$$\Rightarrow \quad x^2 + y^2 - 81 = 0 \quad \text{..... (i)}$$

To check the position of point $(5,6)$, Putting $x = 5$ & $y = 6$ on L.H.S of (i)

$$(5)^2 + (6)^2 - 81 = 25 + 36 - 81$$

$$= -20 < 0$$

Hence $(5,6)$ lies inside the circle.

(ii) $$2x^2 + 2y^2 + 12x - 8y + 1 = 0$$

$$\Rightarrow \quad x^2 + y^2 + 6x - 4y + \frac{1}{2} = 0 \quad \text{..... (i)}$$

To check the position of point put $x = 5$ & $y = 6$ on L.H.S of (i)

$$(5)^2 + (6)^2 + 6(5) - 4(6) + \frac{1}{2}$$

$$= 25+36+30-24+\frac{1}{2}$$

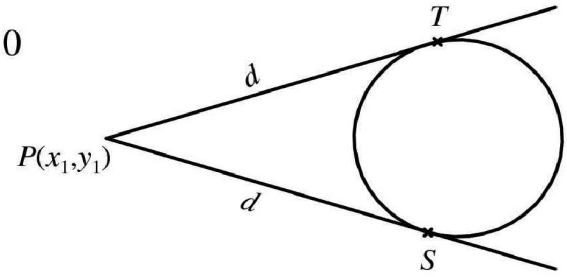
$$= \frac{135}{2} > 0$$

Hence $(5,6)$ lies outside the circle.

Length of tangent to the circle

Consider equation of circle

$$x^2+y^2+2gx+2fy+c = 0$$



If d denotes length of tangent from point $P(x_1, y_1)$ to the circle then

$$d = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

Question # 4

Find the length of the tangent drawn from the point $(-5,4)$ to the circle

$5x^2+5y^2-10x+15y-131 = 0$

Solution $5x^2+5y^2-10x+15y-131 = 0$

$$\Rightarrow x^2+y^2-2x+3y-\frac{131}{3} = 0$$

Now length of tangent from point $P(x_1, y_1)$ is

$$d = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

For $(x_1, y_1) = (-5,4)$

$$d = \sqrt{(-5)^2+(4)^2-2(-5)+3(4)-\frac{131}{5}}$$

$$= \sqrt{25+16+10+12-\frac{131}{5}}$$

$$= \sqrt{\frac{184}{5}} = 2\sqrt{\frac{46}{5}} \text{ units}$$

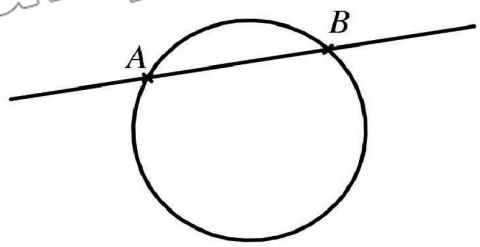
Question # 5

Find the length of the chord cut off from the line $2x+3y = 13$ by the circle

$x^2+y^2 = 26$

Solution $2x+3y = 13$ ..... (i)

$x^2+y^2 = 26$ ..... (ii)



From (i)

$$2x = 13-3y$$

$$\Rightarrow x = \frac{13-3y}{2} \text{ ..... (iii)}$$

Putting in (ii)

$$\left(\frac{13-3y}{2}\right)^2 + y^2 = 26$$

$$\Rightarrow \frac{169-78y+9y^2}{4} + y^2 = 26$$

$$\Rightarrow \frac{169-78y+9y^2+4y^2}{4} = 26$$

$$\Rightarrow 13y^2 - 78y + 169 = 104$$

$$\Rightarrow 13y^2 - 78y + 169 - 104 = 0$$

$$\Rightarrow 13y^2 - 78y + 65 = 0$$

$$\Rightarrow y^2 - 6y + 5 = 0$$

$$\Rightarrow y^2 - 5y - y + 5 = 0$$

$$\Rightarrow y(y-5) - 1(y-5) = 0$$

$$\Rightarrow (y-5)(y-1) = 0$$

$$\Rightarrow y = 5 \quad \text{or} \quad y = 1$$

Putting in (iii)

$$x = \frac{13-3(5)}{2} = -1 \qquad x = \frac{13-3(1)}{2} = 5$$

$\Rightarrow (-1,5)$ and $(5,1)$ are end points of chord intercepted.

So length of chord $= \sqrt{(5+1)^2 + (1-5)^2}$

$= \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$

Question # 6

Find the coordinates of the points of intersection of the line $x + 2y = 6$ with the circle :

$x^2 + y^2 - 2x - 2y - 39 = 0$

Solution $x^2 + y^2 - 2x - 2y - 39 = 0$ ……. (i)

$x + 2y = 6$ ……. (ii)

Just solve (i) & (ii) *to get the points*

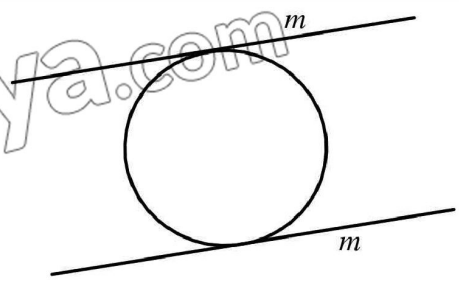
Equation of tangent to the circle having slope m

Consider an equation of circle

$$x^2 + y^2 = a^2$$

Then equations of tangents parallel to the line having slope m are

$$y = mx \pm a\sqrt{1+m^2}$$



Question # 7

Find equations of the tangents to the circle $x^2 + y^2 = 2$

(i) parallel to the $x - 2y + 1 = 0$

(ii) perpendicular to the line $3x + 2y = 6$

Solution $x^2 + y^2 = 2$

Centre of circle is at origin with radius $a = \sqrt{2}$

i) Let l: $x - 2y + 1 = 0$

Slope of $l = m = -\dfrac{1}{-2} = \dfrac{1}{2}$

Since required tangent is parallel to l

$\therefore$ Slope of tangent $= m = \dfrac{1}{2}$

Now equations of tangents are

$$y = mx \pm a\sqrt{1+m^2}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{1+\left(\frac{1}{2}\right)^2} \qquad \Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{1+\frac{1}{4}}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{\frac{5}{4}}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{\frac{10}{4}} \qquad \Rightarrow y = \frac{1}{2}x \pm \frac{\sqrt{10}}{2}$$

$$\Rightarrow 2y = x \pm \sqrt{10}$$

$$\Rightarrow x - 2y \pm \sqrt{10} = 0$$ are the req. tangents.

(ii) *Do yourself as above*

Question # 8

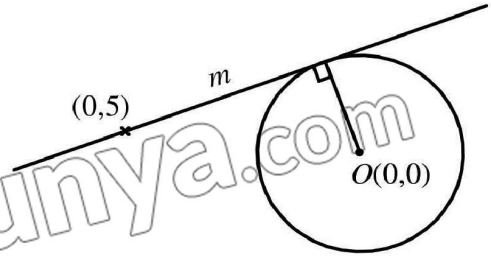
Find equations of the tangent drawn from

(i) $(0,5)$ to $x^2 + y^2 = 16$

(ii) $(-1,2)$ to $x^2 + y^2 + 4x + 2y = 0$

(iii) $(-1,2)$ to $(x+1)^2 + (y-2)^2 = 26$

Also find the point of contact.



Solution

(i) $x^2 + y^2 = 16$

$\Rightarrow$ radius of circle is 4 with centre $O(0,0)$

Let slope of required tangent be m, then eq. of tangent passing through $(0,5)$

$$y - 5 = m(x - 0)$$

$$\Rightarrow y - 5 = mx$$

$$\Rightarrow mx - y + 5 = 0 \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Now since (i) is tangent to circle, therefore

Radius of circle = $\perp$ar distance of (i) from centre $O(0,0)$

$$\Rightarrow\ 4 = \frac{|m(0)-0+5|}{\sqrt{m^2+(-1)^2}}$$

$$\Rightarrow\ 4 = \frac{|5|}{\sqrt{m^2+1}} \qquad \Rightarrow\ 4\sqrt{m^2+1} = |5|$$

On squaring

$$\left(4\sqrt{1+m^2}\right)^2 = |5|^2$$

$$\Rightarrow\ 16\left(m^2+1\right) = 25 \qquad \Rightarrow\ 16m^2+16 = 25$$

$$\Rightarrow\ 16m^2 = 25-16 \qquad \Rightarrow\ 16m^2 = 9$$

$$\Rightarrow\ m^2 = \frac{9}{16} \qquad \Rightarrow\ m = \pm\frac{3}{4}$$

When $m=\frac{3}{4}$, putting in (i)

$$\frac{3}{4}x-y+5 = 0$$

$$\Rightarrow\ 3x-4y+20 = 0$$

When $m=-\frac{3}{4}$, putting in (i)

$$-\frac{3}{4}x-y+5 = 0$$

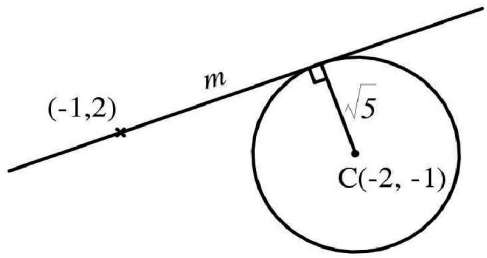
$$\Rightarrow\ 3x+4y-2 = 0$$

(ii) $$x^2+y^2+4x+2y = 0$$

Comparing it with general equation of circle

We have $2g = 4$, $2f = 2$, $c = 0$

$$\Rightarrow\ g = 2 \quad , \quad f = 1$$



Centre $C(-g,-f) = C(-2,-1)$

Radius $= \sqrt{g^2+f^2-c} = \sqrt{(2)^2+(1)^2-0}$

$= \sqrt{4+1} = \sqrt{5}$

Let m be a slope of required tangent, so equation of tangent passing thorough $(-1,2)$

$$y-2 = m(x+1)$$

$$\Rightarrow\ y-2 = mx+m$$

$$\Rightarrow\ mx-y+(m+2) = 0 \ \ldots\ldots\ldots\ldots \text{(i)}$$

$\because$ (i) is tangent ot circle,

$\therefore$ $\perp$ar distance of tangent from centre $(-2,-1)$ = Radius of circle

$$\Rightarrow\ \frac{|m(-2)-(-1)+(m+2)|}{\sqrt{(m)^2+(-1)^2}} = \sqrt{5}$$

$$\Rightarrow\ \frac{|-2m+1+m+2|}{\sqrt{m^2+1}} = \sqrt{5}$$

$$\Rightarrow \left|-m+3\right| = \sqrt{5}\cdot\sqrt{m^2+1}$$

On squaring

$$m^2-6m+9 = 5\left(m^2+1\right)$$

$$\Rightarrow 5m^2+5-m^2+6m-9 = 0$$

$$\Rightarrow 4m^2+6m-4 = 0$$

$$\Rightarrow 2m^2+3m-2 = 0$$

$$\Rightarrow 2m^2+4m-m-2 = 0$$

$$\Rightarrow 2m(m+2)-1(m+2) = 0$$

$$\Rightarrow (m+2)(2m-1) = 0$$

$$\Rightarrow m+2 = 0 \quad \text{or} \quad 2m-1 = 0$$

$$\Rightarrow m=-2 \quad \text{or} \quad m=\frac{1}{2}$$

Putting value of m in (i)

$-2x-y+(-2+2) = 0$	$\frac{1}{2}x-y+\left(\frac{1}{2}+2\right) = 0$
$\Rightarrow -2x-y+0 = 0$	$\Rightarrow \frac{1}{2}x-y+\frac{5}{2} = 0$
$\Rightarrow 2x+y = 0$	$\Rightarrow x-2y+5 = 0$

(iii) $$(x+1)^2+(y-2)^2 = 26$$

$$\Rightarrow (x-(-1))^2+(y-2)^2 = \left(\sqrt{26}\right)^2$$

Centre of circle is $(-1,2)$ and radius $\sqrt{26}$

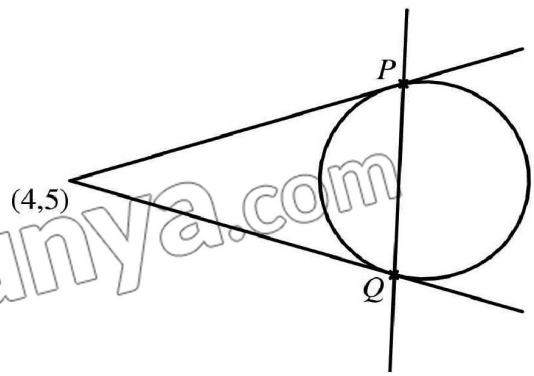
Now do yourself as above.

Note: *To find point of contact, solve equation of tangent and circle.*

Question # 9

Find an equation of the chord of contact of the tangents drawn from $(4,5)$ to the circle

$2x^2+2y^2-8x+12y+21 = 0$



Solution

Given: $2x^2+2y^2-8x+12y+21 = 0$

$$\Rightarrow x^2+y^2-4x+6y+\frac{21}{2} = 0$$

Comparing it with general equation of circle

$$2g = -4 \;,\quad 2f = 6 \;,\quad c = \frac{21}{2}$$

$$\Rightarrow g = -2 \;,\quad f = 3$$

Let the point of contact of two tangent be $P(x_1,y_1)$ and $Q(x_2,y_2)$

Eq. of tangent at $P(x_1, y_1)$

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

$$\Rightarrow x_1x + y_1y - 2(x + x_1) + 3(y + y_1) + \frac{21}{2} = 0$$

$$\Rightarrow x_1x + y_1y - 2x - 2x_1 + 3y + 3y_1 + \frac{21}{2} = 0$$

Since tangent is drawn from $(4,5)$, therefore

$$\Rightarrow x_1(4) + y_1(5) - 2(4) - 2x_1 + 3(5) + 3y_1 + \frac{21}{2} = 0$$

$$\Rightarrow 4x_1 + 5y_1 - 8 - 2x_1 + 15 + 3y_1 + \frac{21}{2} = 0$$

$$\Rightarrow 2x_1 + 8y_1 + \frac{35}{2} = 0$$

$$\Rightarrow 4x_1 + 16y_1 + 35 = 0 \text{ ..... (i)}$$

Similarly equation of tangent passing through $Q(x_2, y_2)$ and $(4,5)$ gives

$$4x_2 + 16y_2 + 35 = 0 \text{ ..... (ii)}$$

Eqs. (i) and (ii) show that both points P & Q lies on the line

$$4x + 16y + 35 = 0$$

So it is the required equation of chord of contact.

Question # 1

Prove that normal lines of a circle pass through the centre of the circle.

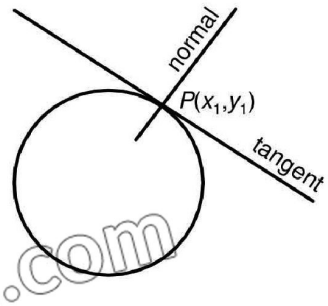
Solution Consider a circle with centre at origin and radius r.

$$x^2 + y^2 = r^2.$$

Differentiating w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad 2y\frac{dy}{dx} = -2x \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Slope of tangent at $(x_1, y_1) = m = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{x_1}{y_1}.$



Since normal is $\perp$ar to tangent therefore

Slope of normal at $(x_1, y_1) = -\frac{1}{m} = -\frac{1}{-x_1/y_1} = \frac{y_1}{x_1}.$

Now equation of normal at (x_1, y_1) having slope $\frac{y_1}{x_1}$

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

$$\Rightarrow \quad x_1 y - x_1 y_1 = y_1 x - y_1 x_1$$

$$\Rightarrow \quad x_1 y = y_1 x \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

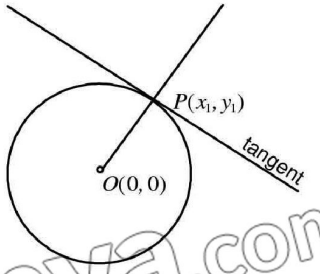
Clearly centre of circle $(0,0)$ satisfies (i), hence normal lines of the circles passing through the centre of the circle.

Question # 2

Prove that the straight line drawn from the centre of a circle perpendicular to a tangent passes through the point of tangency.

Solution Consider a circle with centre at origin and radius r.

$$x^2 + y^2 = r^2.$$



Differentiating w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -2x \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Slope of tangent at $(x_1, y_1) = m = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{x_1}{y_1}.$

Slope of line $\perp$ar to tangent $= -\frac{1}{m} = -\frac{1}{-x_1/y_1} = \frac{y_1}{x_1}.$

Now equation of line perpendicular to tangent passing through centre $(0,0)$

$$y - 0 = \frac{y_1}{x_1}(x - 0)$$

$$\Rightarrow \quad x_1 y = y_1 x \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Clearly the point of tangency (x_1, y_1) satisfy (i), hence the straight line drawn from the centre of circle perpendicular to a tangent passes through the point of tangency.

Question # 3

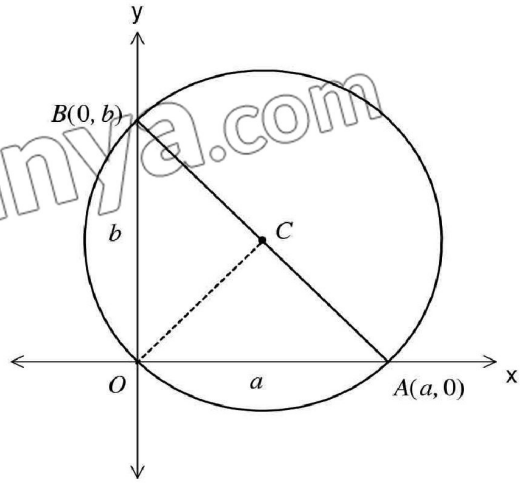
Prove that the mid-point of the hypotenuse of a right triangle is the circumcentre of the triangle.

Solution Let OAB be a right triangle with $|OA| = a$, $|OB| = b$.

Then the coordinates of A and B are $(a,0)$ and $(0,b)$ respectively.

Let C be the mid-point of hypotenuse AB. Then

$$\text{coordinate of } C = \left(\frac{a+0}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right).$$



Now
$$|CA| = \sqrt{\left(\frac{a}{2} - a\right)^2 + \left(\frac{b}{2} - 0\right)^2}$$
$$= \sqrt{\left(-\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$
$$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$|CB| = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - b\right)^2}$$
$$= \sqrt{\left(\frac{a}{2}\right)^2 + \left(-\frac{b}{2}\right)^2}$$
$$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$|CO| = \sqrt{\left(0 - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2}$$
$$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}.$$

Since $|CA| = |CB| = |CO|$, therefore C is the centre of the circumcircle.

Hence the mid-point of the hypotenuse of a right triangle is the circumcentre of the triangle.

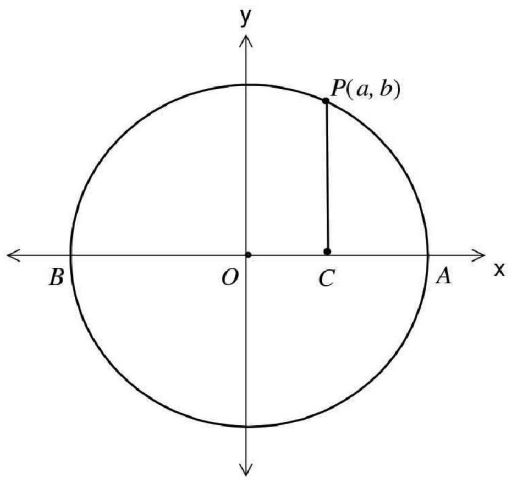
Mean proportional

Let a, b and c be three numbers. The number b is said to be *mean proportional* between a and b if a, b, c are in geometric means or

$$b^2 = ac \quad \text{or} \quad \frac{b}{a} = \frac{a}{c}.$$

Question # 4

Prove that the perpendicular dropped from a point of a circle on a diameter is a mean proportional between the segments into which it divides the diameter.



Solution Consider a circle of radius r and centre $(0,0)$, then equation of circle

$$x^2 + y^2 = r^2$$

Let A and B are end-points of diameter of circle along x-axis, then coordinate of A and B are $(-r,0)$ and $(0,r)$ respectively.

Also let $P(a,b)$ be any point on circle and $\perp$ar from P cuts diameter at C. Then coordinate of C are $(a,0)$.

Since $P(a,b)$ lies on a circle, therefore

$$a^2 + b^2 = r^2 \quad \text{..... (i)}$$

Now

$$|AC| = \sqrt{(r+a)^2 - (0-0)^2} = r+a.$$

$$|CB| = \sqrt{(r-a)^2 - (0-0)^2} = r-a.$$

$$|PC| = \sqrt{(a-a)^2 + (b-0)^2} = \sqrt{0+b^2} = b.$$

Now

$$\begin{aligned} |AC|\cdot|CB| &= (r+a)(r-a) \\ &= r^2 - a^2 \\ &= a^2 + b^2 - a^2 \qquad \text{from (i)} \\ &= b^2 = |PC|^2 \end{aligned}$$

$$\Rightarrow |AC|\cdot|CB| = |PC|\cdot|PC| \qquad \Rightarrow \frac{|AC|}{|PC|} = \frac{|PC|}{|CB|}$$

$\Rightarrow |PC|$ is a mean proportional to $|AC|$ and $|CB|$.

Question # 1

Find the focus, vertex and directix of the parabola sketch its graph.

(i) $y^2 = 8x$ (ii) $x^2 = -16y$

(iii) $x^2 = 5y$ (iv) $y^2 = -12x$

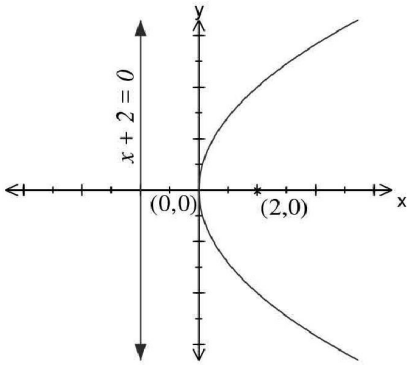
(v) $x^2 = 4(y-1)$

(vi) $y^2 = -8(x-3)$

(vii) $(x-1)^2 = 8(y+2)$ (viii) $y = 6x^2 - 1$

(ix) $x + 8 - y^2 + 2y = 0$

(x) $x^2 - 4x - 8y + 4 = 0$



Solution

(i) $y^2 = 8x$

Here $4a = 8 \Rightarrow a = 2$

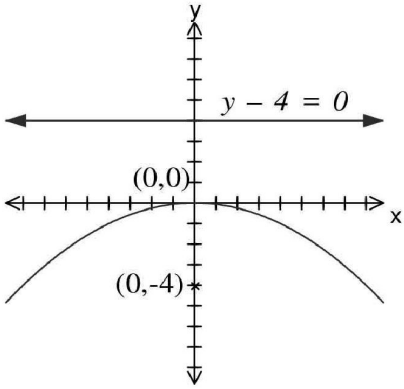
Vertex: $O(0,0)$

The axis of parabola is along x-axis and opening of parabola is to the right side.

Focus: $(a,0) = (2,0)$

Directrix: $x + a = 0$

$\Rightarrow x + 2 = 0$



(ii) $x^2 = -16y$

Here $4a = 16 \Rightarrow a = 4$

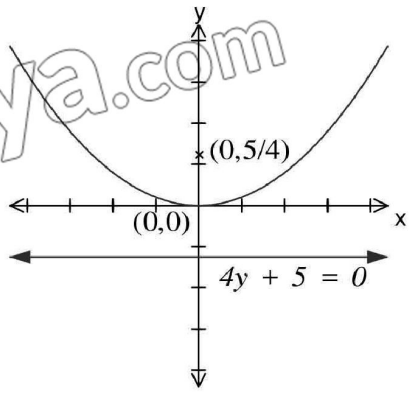
Vertex: $O(0,0)$

The axis of parabola is along $y-$axis and opening of the parabola is downward.

So Focus: $F(0,-a) = F(0,-4)$

Directrix: $y - a = 0$

$\Rightarrow y - 4 = 0$



(ii) $x^2 = 5y$

Here $4a = 5 \Rightarrow a = \frac{5}{4}$

And vertex: $O(0,0)$

The axis of the parabola is along y-axis and opening of the parabola is upward.

Focus: $F(0,a) = F\left(0,\frac{5}{4}\right)$

Directrix: $y+a=0 \Rightarrow y+\frac{5}{4}=0$

$\Rightarrow 4y+5=0$

(iv) $y^2=-12x$

Here $4a=12 \Rightarrow a=3$

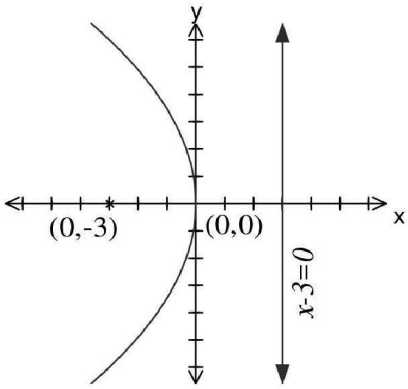
And vertex: $O(0,0)$

The axis of the parabola is along x-axis and opening of the parabola is to the left side.

Focus: $F(-a,0)=(-3,0)$

Directrix: $x-a=0$

$\Rightarrow x-3=0$



(v) $x^2=4(y-1)$ ..... (i)

Put $X=x$, $Y=y-1$

$\Rightarrow X^2=4Y$ ..... (ii)

Here $4a=4 \Rightarrow a=1$

And vertex of parabola (ii) is $O(0,0)$ with axis of parabola is along Y-axis open upward.

$\because$ Vertex: $O(0,0)$

$\Rightarrow X=0$, $Y=0$

$\Rightarrow x=0$, $y-1=0 \Rightarrow y=1$

$\Rightarrow (0,1)$ is vertex of parabola (i)

Now focus: $F(0,a)=F(0,1)$

$\Rightarrow X=0$, $Y=1$

$\Rightarrow x=0$, $y-1=1$

$y=1+1 \Rightarrow y=2$

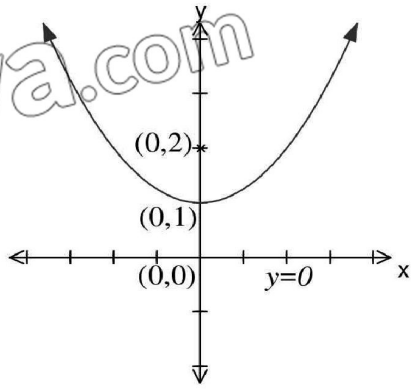
$\Rightarrow (0,2)$ is focus of parabola (i)

Directrix of parabola (ii) is

$Y+a=0 \Rightarrow Y+1=0$

$\Rightarrow y-1+1=0$

$\Rightarrow y=0$ is directrix of parabola (i)

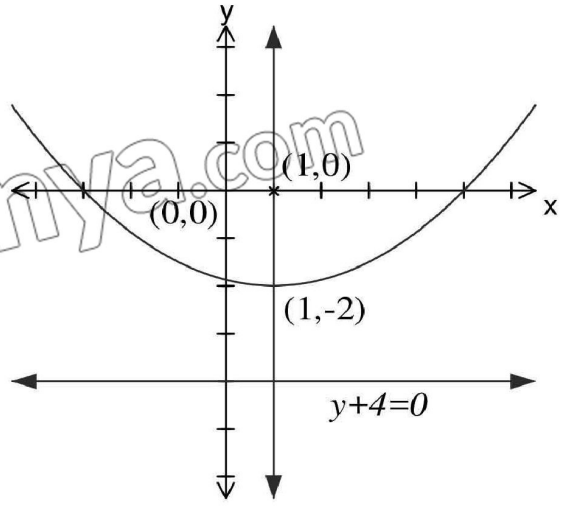


(vi) *Do yourself*

(vii) $(x-1)^2=8(y+2)$ ..... (i)

Put $X=x-1$, $Y=y+2$ in (i)

$X^2=8Y$ ..... (ii)



Here $4a = 8 \Rightarrow a = 2$

Axis of parabola is along Y-axis open upward with vertex of $(0,0)$

$\Rightarrow X = 0 \quad , \quad Y = 0$

$\Rightarrow x-1 = 0 \quad , \quad y+2 = 0$

$\Rightarrow x = 2 \quad , \quad y = -2$

$\Rightarrow (1,-2)$ is vertex of parabola (i)

Focus of (ii) is $(0,a) = (0,2)$

$\Rightarrow X = 0 \quad , \quad Y = 2$

$\Rightarrow x-1 = 0 \quad , \quad y+2 = 2$

$\Rightarrow x = 1 \quad , \quad y = 0$

$\Rightarrow (1,0)$ is the focus of given parabola (i)

Directrix of (ii)

$Y+a = 0 \Rightarrow Y+2=0$

$\Rightarrow y+2+2 = 0 \Rightarrow y+4 = 0$ is directrix of given parabola.

(viii) $y = 6x^2 - 1$

$\Rightarrow 6x^2 = y+1 \Rightarrow x^2 = \frac{1}{6}(y+1)$

Now try yourself

(ix) $x+8-y^2+2y = 0$

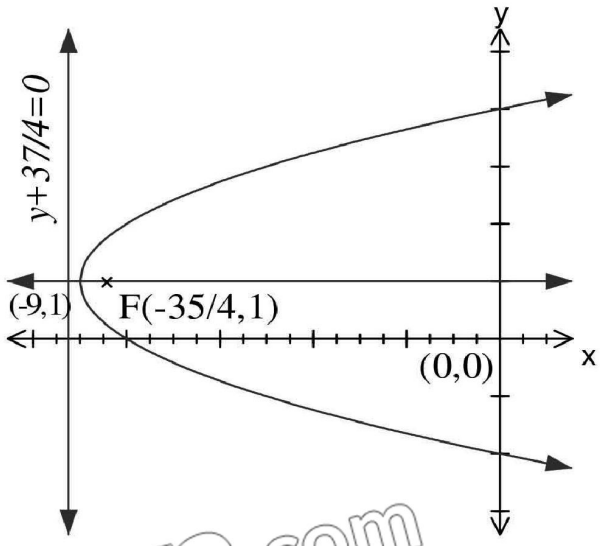
$\Rightarrow y^2-2y = x+8$

$\Rightarrow y^2-2y+1 = x+8+1$

$\Rightarrow (y-1)^2 = x+9$

Put $X = x+9 \quad , \quad Y = y-1$

$Y^2 = X$



Here $4a = 1 \Rightarrow a = \frac{1}{4}$

The axis of parabola is along x-axis and it is opening to the right side.

Vertex of parabola (ii) is $(0,0)$

$\Rightarrow X = 0 \quad , \quad Y = 0$

$\Rightarrow x+9 = 0 \quad , \quad y-1 = 0$

$\Rightarrow x = -9 \quad , \quad y = 1$

$\Rightarrow (-9,1)$ is vertex of the parabola (i)

Focus: $(a,0) = \left(\frac{1}{4},0\right)$

$X = \frac{1}{4}$, $Y = 0$

$\Rightarrow \quad x+9 = \frac{1}{4}$, $y-1 = 0$

$\Rightarrow \quad x = \frac{1}{4}-9$, $y-1 = 0$

$\Rightarrow \quad x = -\frac{35}{4}$, $y = 1$

$\Rightarrow \quad \left(-\frac{35}{4},1\right)$ is focus of parabola (i)

Directrix of parabola (ii)

$X + a = 0$

$\Rightarrow \quad X + \frac{1}{4} = 0$

$\Rightarrow \quad x+9+\frac{1}{4} = 0$

$\Rightarrow \quad x+\frac{37}{4} = 0$ is directrix of parabola (i)

(x) *Do yourself as above*

Question # 2

Write an equation of the parabola with given elements.

(i) Focus $(-3,1)$; directrix $x=3$

(ii) Focus $(2,5)$; directrix $y=1$

(iii) Focus $(-3,1)$; directrix $x-2y-3=0$

(iv) Focus $(1,2)$; vertex $(3,2)$

(v) Focus $(-1,0)$; vertex $(-1,2)$

(vi)Directrix $x=-2$,Focus $(2,2)$

(vii)Directrix $y=3$; vertex $(2,2)$

(viii) Directrix $y=1$, length of latusrectum is 8. Opens downward.

(ix)Axis $y=0$, through $(2,1)$ and $(11,-2)$

(x)Axis parallel to y-axis, the points $(0,3),(3,4)$ *and* $(4,11)$ lie on the graph.

Solution

(i) Focus: $F(-3,1)$

Directrix: $x = 3$ i.e. $x-3 = 0$

Let $P(x,y)$ be any point on parabola then by definition

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$\Rightarrow \quad \sqrt{(x+3)^2+(y-1)^2} = \frac{|x-3|}{\sqrt{(1)^2+(0)^2}}$

$\Rightarrow \quad \sqrt{x^2+6x+9+y^2-2y+1} = |x-3|$

On squaring

$\Rightarrow \quad x^2+6x+9+y^2-2y+1 = x^2-6x+9$

$\Rightarrow \quad x^2+6x+9+y^2-2y+1-x^2+6x-9=0$

$\Rightarrow \quad y^2+12x-2y+1=0$

is required equation of parabola.

(ii) *Do yourself as above.*

(iii) Focus: $F(-3,1)$

Directrix: $x-2y-3=0$

Let $P(x,y)$ be any point on parabola, then by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix.

$$\Rightarrow \sqrt{(x+3)^2+(y-1)^2} = \frac{|x-2y-3|}{\sqrt{(1)^2+(-2)^2}}$$

$$\Rightarrow \sqrt{x^2+6x+9+y^2-2y+1} = \frac{|x-2y-3|}{\sqrt{5}}$$

$$\Rightarrow \sqrt{5}\sqrt{x^2+6x+9+y^2-2y+1} = |x-2y-3|$$

On squaring both sides

$$5(x^2+6x+9+y^2-2y+1) = x^2+4y^2+9-4xy+12y-6x$$

$$\Rightarrow 5x^2+30x+45+5y^2-10y+5-x^2-4y^2-9+4xy-12y+6x = 0$$

$$\Rightarrow 4x^2+y^2+36x-22y+4xy+41 = 0$$

is required equation

(iv) Given: Focus $(1,2)$, Vertex $(3,2)$

Focus and vertex implies that axis of parabola is parallel to x-axis and opening to left side. Therefore eq. of parabola with vertex $(3,2)$

$$(y-2)^2 = -4a(x-3) \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Now $a =$ Distance between focus and vertex

$$= \sqrt{(3-1)^2+(2-2)^2} = \sqrt{4+0} = 2$$

Putting in (i)

$$(y-2)^2 = -4(2)(x-3) \quad \Rightarrow \quad y^2-4y+4 = -8x+24$$

$$\Rightarrow \quad y^2-4y+4+8x-20 = 0 \quad \Rightarrow \quad y^2-4y+8x-20 = 0 \quad \text{is req. eq.}$$

(v) *Do yourself*

(vii) Directrix: $y = 3$ i.e. $y-3=0$

Vertex $(2,2)$

Since axis of parabola is parallel to y-axis (because directrix is parallel to x-axis). And opening is downward.

So equation of parabola with vertex $(h,k) = (2,2)$

$$(x-h)^2 = -4a(y-k)$$

$$\Rightarrow (x-2)^2 = -4a(y-2)$$

Now a = Distance of vertex $(2,2)$ from directrix

$$= \frac{|2-3|}{\sqrt{(0)^2+(1)^2}} = \frac{|-1|}{1} = 1$$

Putting in (i)

$$(x-2)^2 = -4(1)(y-2)$$

$$\Rightarrow x^2-4x+4 = -4y+8 \qquad \Rightarrow x^2-4x+4+4y-8 = 0$$

$$\Rightarrow x^2-4x+4y-4 = 0$$ is require equation.

(viii) Directrix: $y = 1$

Latusractum $= 4a = 8 \Rightarrow a = 2$

$\because$ Parabola is open downward

$\therefore$ Consider vertex $= (h,-1)$

And equation of parabola

$$(x-h)^2 = -4a(y-k)$$

$$\Rightarrow (x-h)^2 = -4(2)(y+1) \qquad \Rightarrow x^2-2hx+h^2 = -8y-8$$

$$\Rightarrow x^2-2hx+8y+h^2+8 = 0$$ is req. eq.

(ix) Axis of parabola: $y = 0$

Let vertex is (h,k)

$\because$ it lies on x-axis $\therefore$ $k=0$

Now equation of parabola with vertex $(h,0)$

$$(y-0)^2 = 4a(x-h)$$

$$\Rightarrow y^2 = 4a(x-h) \text{ ..... (i)}$$

$\because$ $(2,1)$ lies on parabola (i)

$\therefore$ $(1)^2 = 4a(2-h)$

$\Rightarrow$ $1 = 4a(2-h)$ ..... (ii)

Also $(11,-2)$ lies on parabola (i)

$$(-2)^2 = 4a(11-h)$$

$$\Rightarrow 4 = 4a(11-h)$$

$$\Rightarrow 1 = a(11-h) \text{ ..... (iii)}$$

Dividing (i) & (ii)

$$\frac{1}{1} = \frac{4a(2-h)}{a(11-h)}$$

$$\Rightarrow 1 = \frac{4(2-h)}{(11-h)} \qquad \Rightarrow 11-h = 8-4h$$

$$\Rightarrow 4h-h = 8-11 \qquad \Rightarrow 3h = -3 \qquad \Rightarrow h = -1$$

Putting in (ii)

$$1 = 4a(2-(-1))$$

$$\Rightarrow 1 = 4a(3) \qquad \Rightarrow 1 = 12a \qquad \Rightarrow a = \frac{1}{12}$$

Using in (i)

$$y^2 = 4\left(\frac{1}{12}\right)(x-(-1)) \qquad \Rightarrow 3y^2 = x+1$$

$$\Rightarrow 3y^2 - x - 1 = 0$$

is the required equation.

(x) Axis parallel to y-axis, then points (0,3), (3,4) and (4,11) lie on the graph.

Equation of parabola axis parallel to y-axis with vertex (h, k)

$$(x-h)^2 = 4a(y-k) \text{ ..... (i)}$$

$\because$ $(0,3)$ lies on parabola (i)

$$\therefore (0-h)^2 = 4a(3-k)$$

$$\Rightarrow h^2 = 12a - 4ak \text{ ..... (ii)}$$

Also $(3,4)$ lies on parabola (i)

$$(3-h)^2 = 4a(4-k)$$

$$\Rightarrow 9-6h+h^2 = 16a-4ak$$

$$\Rightarrow h^2-6h+9 = 16a-4ak \text{ ..... (iii)}$$

Also $(4,11)$ lies on parabola (i)

$$(4-h)^2 = 4a(11-k)$$

$$\Rightarrow 16-8h+h^2 = 44a-4ak$$

$$\Rightarrow h^2-8h+16 = 44a-4ak \text{ ..... (iv)}$$

Subtracting (ii) & (iii)

$$\begin{array}{rcl} h^2 & = & 12a-4ak \\ h^2-6h+9 & = & +16a-4ak \\ {\scriptstyle +} \quad {\scriptstyle -} & & {\scriptstyle -} \quad {\scriptstyle +} \\ \hline 6h-9 & = & -4a \text{ .....(v)} \end{array}$$

Now subtracting (iii) & (iv)

$$h^2-6h+9 = 16a-4ak$$
$$h^2-8h+16 = +44a-4ak$$
$$\overset{+}{-}\quad\overset{-}{-}\quad\overset{+}{-}$$
$$2h-7=-28a \text{ .....}(vi)$$

Multiplying (vi) with 3 and subtracting from (v)

$$6h-21 = -84a$$
$$+6h-9 = -4a$$
$$\overset{-}{-}\quad\overset{+}{+}\quad\overset{+}{+}$$
$$-12=-80a$$

$$\Rightarrow \boxed{a = \frac{12}{80} = \frac{3}{20}}$$

Putting in (v)

$$6h-9 = -40\left(\frac{3}{20}\right) \Rightarrow 6h = 9-\frac{3}{5} = \frac{42}{5} \Rightarrow \boxed{h = \frac{7}{5}}$$

Putting value of a and h in (ii), we get

$$\left(\frac{7}{5}\right)^2 = 12\left(\frac{3}{20}\right)-4\left(\frac{3}{20}\right)k \Rightarrow \frac{49}{25} = \frac{9}{5}-\frac{3}{5}k$$

$$\Rightarrow \frac{3}{5}k = \frac{9}{5}-\frac{49}{25} \Rightarrow \frac{3}{5}k = -\frac{4}{25} \Rightarrow \boxed{k = -\frac{4}{15}}$$

Now putting the value of a, h and k in (i), we get

$$\left(x-\frac{7}{5}\right)^2 = 4\left(\frac{3}{20}\right)\left(y+\frac{4}{15}\right) \Rightarrow x^2-\frac{14}{5}x+\frac{49}{25} = \frac{3}{5}\left(y+\frac{4}{15}\right)$$

$$\Rightarrow x^2-\frac{14}{5}x+\frac{49}{25} = \frac{3}{5}y+\frac{4}{25} \Rightarrow x^2-\frac{14}{5}x-\frac{3}{5}y+\frac{49}{25}-\frac{4}{25} = 0$$

$$\Rightarrow x^2-\frac{14}{5}x-\frac{3}{5}y+\frac{9}{5} = 0$$

$$\Rightarrow 5x^2-14x-3y+9 = 0$$

is the required equation.

Question # 3

Find an equation of the parabola having its focus at the origin and dirctrix parallel to the
(i) $x-axis$ (ii) $y-axis$

Solution

(i) When directrix is parallel to x-axis

Suppose $F(0,0)$ be focus and equation of directrix be

$y = h$ (parallel to x-axis)

i.e. $y-h = 0$

Now let $P(x,y)$ be any point on parabola the by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$$\Rightarrow \sqrt{(x-0)^2+(y-0)^2} = \frac{|y-h|}{\sqrt{(0)^2+(1)^2}}$$

$$\Rightarrow \sqrt{x^2+y^2} = \frac{|y-h|}{1}$$

On squaring

$$\Rightarrow x^2+y^2 = y^2-2hy+h^2 \quad \Rightarrow x^2+y^2-y^2+2hy-h^2 = 0$$

$$\Rightarrow x^2+2hy-h^2 = 0 \quad \text{is req. equation.}$$

ii) When directrix is parallel to y-axis.

When directrix is parallel to x-axis

Suppose $F(0,0)$ be focus and equation of directrix be

$x = h$ (parallel to y-axis)

i.e. $x-h = 0$

Now let $P(x,y)$ be any point on parabola the by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$$\Rightarrow \sqrt{(x-0)^2+(y-0)^2} = \frac{|x-h|}{\sqrt{(1)^2+(0)^2}}$$

$$\Rightarrow \sqrt{x^2+y^2} = \frac{|x-h|}{1}$$

On squaring

$$\Rightarrow x^2+y^2 = x^2-2hx+h^2 \quad \Rightarrow x^2+y^2-x^2+2hx-h^2 = 0$$

$$\Rightarrow y^2+2hx-h^2 = 0 \text{ is req. equation}$$

Question # 4

Show that an equation of parabola with focus at $(a\cos\alpha, a\sin\alpha)$ and directrix $x\cos\alpha + y\sin\alpha + a = 0$ is $(x\sin\alpha - y\cos\alpha)^2 = 4a(x\cos\alpha + y\sin\alpha)$

Solution Focus: $F(a\cos\alpha, a\sin\alpha)$

Directrix: $x\cos\alpha + y\sin\alpha + a = 0$

Let $P(x,y)$ be any point on parabola then by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$$\Rightarrow \sqrt{(x-a\cos\alpha)^2+(y-a\sin\alpha)^2} = \frac{|x\cos\alpha + y\sin\alpha + a|}{\sqrt{\cos^2\alpha+\sin^2\alpha}}$$

On squaring

$$(x-a\cos\alpha)^2+(y-\sin\alpha)^2 = \frac{|x\cos\alpha + y\sin\alpha + a|^2}{1}$$

$$\Rightarrow x^2-2ax\cos\alpha+a^2\cos^2\alpha+y^2-2ay\sin\alpha+a^2\sin^2\alpha$$
$$= x^2\cos^2\alpha+y^2\sin^2\alpha+a^2+2ax\cos\alpha+2ay\sin\alpha+2xy\sin\alpha\cos\alpha$$

$$\Rightarrow \quad x^2 - x^2\cos\alpha + y^2 - y^2\sin\alpha + a^2\left(\cos^2\alpha + \sin^2\alpha\right)$$
$$= a^2 + 2ax\cos\alpha + 2ay\sin\alpha + 2xy\sin\alpha\cos\alpha + 2ax\cos\alpha + 2ay\sin\alpha$$
$$\Rightarrow \quad x^2\left(1-\cos^2\alpha\right) + y^2\left(1-\sin^2\alpha\right) + a^2(1) - a^2 - 2xy\sin\alpha\cos\alpha$$
$$= 4ax\cos\alpha + 4ay\sin\alpha$$
$$\Rightarrow \quad x^2\cos^2\alpha + y^2\sin^2\alpha - 2xy\sin\alpha\cos\alpha = 4a\left(x\cos\alpha + y\sin\alpha\right)$$
$$\Rightarrow \quad \left(x\sin\alpha - y\cos\alpha\right)^2 = 4a\left(x\cos\alpha + y\sin\alpha\right)$$

is equation of parabola which is given.

Question # 5

Show that the ordinate at any point P of the parabola is a mean propositional between the length of the latusrectum and the abscissa of P .

Solution

Consider equation of parabola

$$y^2 = 4ax$$
$$\Rightarrow \quad y \cdot y = 4a \cdot x$$
$$\Rightarrow \quad \frac{4a}{y} = \frac{y}{x}$$
$$\Rightarrow \quad \frac{latus\ ractum}{ordinate} = \frac{ordinate}{abscissa}$$

$\Rightarrow$ ordinate is mean proportional between latus rectum and abscissa.

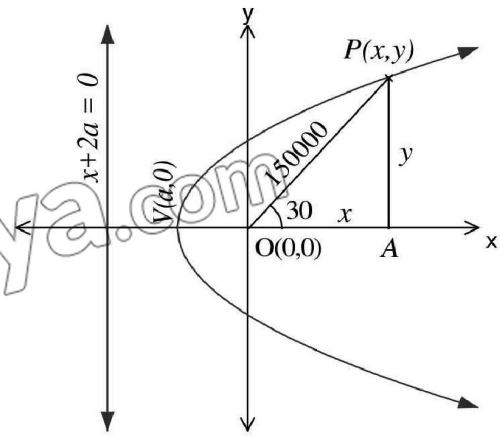
Question # 6

A comet has a parabolic orbit with the earth at the focus. When the comet is $150,000km$ from the earth, the line joining the comet and the earth makes an angle of 30 with the asix of the parabola. How close will the comet come to the earth

Solution Suppose earth be at focus which is origin and $V(-a,0)$ be vertex of parabola.

Then directrix of parabola;

$$x = -2a$$
$$\Rightarrow \quad x + 2a = 0$$



Let comet be at a point $P(x,y)$ then by definition of parabola

$$|PF| = \perp\text{ar distance of } P(x,y) \text{ from directrix}$$
$$\Rightarrow \quad \sqrt{(x-0)^2 + (y-0)^2} = \frac{|x+2a|}{\sqrt{(1)^2 + (0)^2}}$$
$$\Rightarrow \quad \sqrt{x^2 + y^2} = |x+2a|$$

On squaring

$$x^2 + y^2 = (x+2a)^2 \quad \text{..... (i)}$$

Also by Pythagoras theorem in $\triangle ABC$

$$|OA|^2 + |AP|^2 = |OP|^2$$

$$\Rightarrow \quad x^2 + y^2 = (150000)^2 \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Comparing (i) and (ii)

$$(x+2a)^2 = (150000)^2$$

$$\Rightarrow \quad x + 2a = \pm 150000 \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Now from right triangle OAP

$$\cos 30^\circ = \frac{|OA|}{|OP|} \qquad \Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{x}{150000}$$

$$\Rightarrow \quad x = \frac{\sqrt{3}}{2}(150000)$$

Using in (iii)

$$\frac{\sqrt{3}}{2}(150000) + 2a = \pm 150000$$

$$\Rightarrow \quad 2a = \pm 150000 - \frac{\sqrt{3}}{2}(150000) \qquad \Rightarrow \quad 2a = \pm 150000 - \sqrt{3}(75000)$$

$$\Rightarrow \quad 2a = 75000\left(\pm 2 - \sqrt{3}\right) \qquad \Rightarrow \quad a = 37500\left(\pm 2 - \sqrt{3}\right)$$

Since a is shortest distance and can't be negative

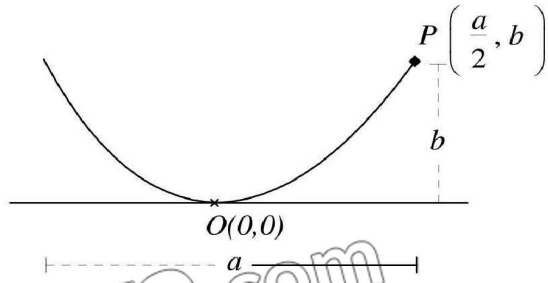
Therefore $a = 37500\left(2 - \sqrt{3}\right) Km$

Question # 7

Find an equation of the parabola formed by the cables of a suspension bridge whose span is a m and the vertical height of the supporting towards is b m.

Solution Consider equation of parabola with vertex $O(0,0)$

$$x^2 = 4a'y \quad \ldots\ldots\ldots\ldots \text{(i)}$$



Since $P\left(\frac{a}{2}, b\right)$ lies on parabola

$$\left(\frac{a}{2}\right)^2 = 4a'(b) \quad \Rightarrow \quad a' = \frac{a^2}{16b}$$

Putting in (i)

$$x^2 = 4\left(\frac{a^2}{16b}\right)y$$

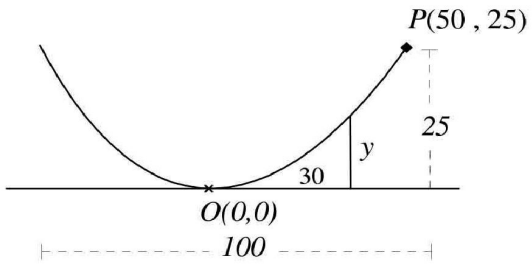
$$\Rightarrow \quad x^2 = \frac{a^2}{4b}y \text{ is required equation.}$$

Question # 8

A parabolic arch has a $100m$ base and height $25m$. Find the height of the arch at the point $30m$ from the centre of the base.

Solution Suppose equation of parabola with vertex $(0,0)$

$$x^2 = 4ay \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$



From figure, we see that $P(50,25)$ lies on parabola.

$$(50)^2 = 4a(25)$$

$$\Rightarrow \quad 2500 = 100a \qquad \Rightarrow \quad a = 25$$

Putting in (i)

$$x^2 = 4(25)y \quad \Rightarrow \quad x^2 = 100y$$

When $x = 30$

$$(30)^2 = 100y$$

$$\Rightarrow \quad y = \frac{900}{100} \quad \Rightarrow \quad y = 9$$

Hence the required height = $9m$

Question # 9

Show that tangent at any point P of a parabola makes equal angles with the line PF and the line through P parallel to the axis of the parabola, F being focus. (These angles are called respectively angle of incidence and angle of reflection).

Solution

Suppose the parabola

$$y^2 = 4ax \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Let $P(x_1, y_1)$ be any point on parabola, then

$$y_1^2 = 4ax_1 \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

Now differentiating (i) w.r.t x

$$\frac{d}{dx}y^2 = \frac{d}{dx}4ax \quad \Rightarrow \quad 2y\frac{dy}{dx} = 4a \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

Slope of tangent at $(x_1, y_1) = m_1 = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2a}{y_1}$

Now slope of $PS = m_2 = \frac{y_1 - 0}{x_1 - a}$

$$\Rightarrow \quad m_2 = \frac{y_1}{x_1 - a}$$

Now slope of line parallel to axis of parabola $= m_3 = 0$

(because axis of parabola is along x-axis)

Let θ_1 be angle between tangent and line parallel to axis of parabola, then

$$\tan\theta_1 = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{\frac{2a}{y_1} - 0}{1 + \left(\frac{2a}{y_1}\right)(0)} = \frac{2a/y_1}{1}$$

$$\Rightarrow \quad \tan\theta_1 = \frac{2a}{y_1} \quad \text{..... (iii)}$$

Let θ_2 be angle between tangent and PS, then

$$\tan\theta_2 = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{\frac{y_1}{x_1 - a} - \frac{2a}{y_1}}{1 + \left(\frac{y_1}{x_1 - a}\right)\left(\frac{2a}{y_1}\right)} = \frac{\frac{y_1^2 - 2a(x_1 - a)}{y_1(x_1 - a)}}{\frac{x_1 - a + 2a}{x_1 - a}} = \frac{y_1^2 - 2ax_1 + 2a^2}{y_1(x_1 + a)}$$

$$= \frac{4ax_1 - 2ax_1 + 2a^2}{y_1(x_1 + a)} \qquad \text{from (ii)}$$

$$= \frac{2ax_1 + 2a^2}{y_1(x_1 + a)} = \frac{2a(x_1 + a)}{y_1(x_1 + a)}$$

$$\Rightarrow \quad \tan\theta_2 = \frac{2a}{y_1} \quad \text{..... (iv)}$$

From (iii) and (iv)

$$\tan\theta_1 = \tan\theta_2 \qquad \Rightarrow \quad \theta_1 = \theta_2$$

as required

(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ii) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

Special case of an Ellipse

Circle is a special case of an Ellipse. In circle ''e'' = 0

Parametric Equations of an Ellipse

$x = a\cos\theta$, $y = b\sin\theta$ are Parametric Equations of Ellipse.

Important points about an Ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$
(1) Eccentricity	(1) Eccentricity
$e^2 = \frac{a^2 - b^2}{a^2}$	$e^2 = \frac{a^2 - b^2}{a^2}$
$(ae)^2 = a^2 - b^2$	$(ae)^2 = a^2 - b^2$
$c^2 = a^2 - b^2$	$c^2 = a^2 - b^2$ where $c = ae$
(2) Foci $(\pm ae, 0)$ or $(\pm c, 0)$	(2) Foci $(0, \pm ae)$ or $(0, \pm c)$
(3) Length of major axis = 2a	(3) Length of major axis = 2a
(4) Length of minor axis = 2b	(4) Length of minor axis = 2b
(5) Equations of directrix x : $x = \pm\frac{a}{e}$	(5) Equations of directrix : $y = \pm\frac{a}{e}$
(6) Length of latus rectum $= \frac{2b^2}{a}$	(6) Length of latus rectum $= \frac{2b^2}{a}$
(7) Center (0, 0)	(7) Center (0, 0)
(8) Vertices $(\pm a, 0)$	(8) Vertices $(0, \pm a)$
(9) Covertices $(0, \pm b)$	(9) Covertices $(\pm b, 0)$

Note:

If center is other than (0, 0) then equations of ellipse be comes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \& \quad \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

EXERCISE 6.5

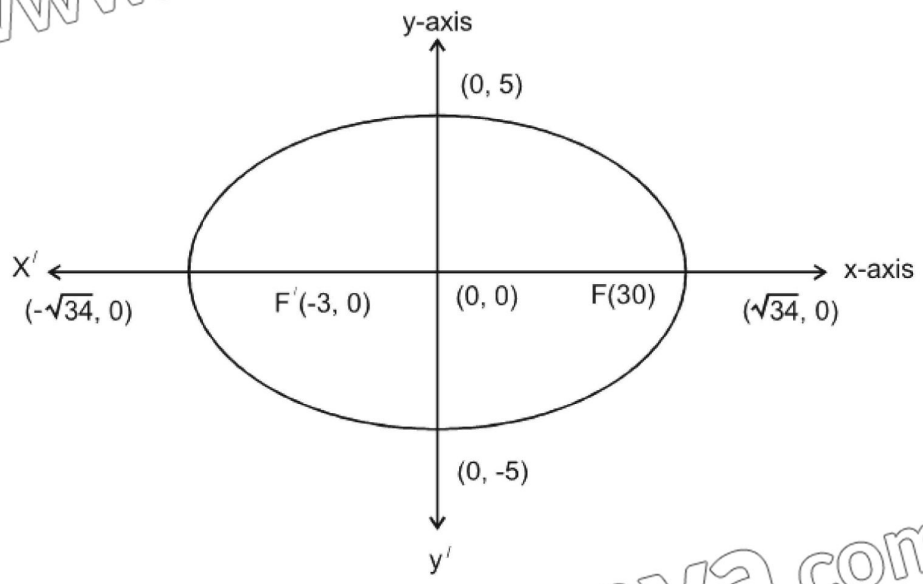
Q.1: Find an equation of the Ellipse with given data and sketch its graph.

(i) Foci (±3, 0) and minor axis of length 10.

(Lahore Board 2009)

Solution:

Given $(\pm ae, 0) = (\pm 3, 0)$



$$ae = 3$$

$$\Rightarrow \quad c = 3 \qquad \text{Also } 2b = 10 \quad \Rightarrow \quad \boxed{b = 5}$$

We know that

$$c^2 = a^2 - b^2$$

$$(3)^2 = a^2 - (5)^2$$

$$9 = a^2 - 25$$

$$9 + 25 = a^2$$

$$a^2 = 34 \quad \Rightarrow \quad \boxed{a = \pm\sqrt{34}}$$

Required equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{x^2}{34} + \frac{y^2}{25} = 1$$

Here vertices are

$$= (\pm a, 0)$$

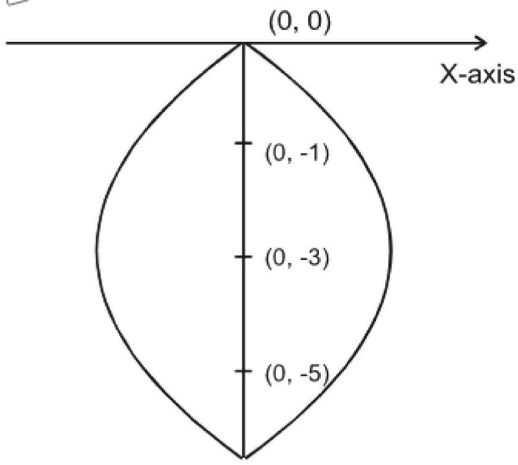
$$= (\pm\sqrt{34}, 0)$$

Covertices are $(0, \pm b)$

$$= (0, \pm 5)$$

(ii) Foci (0, – 1) & (0, – 5) and major axis of length 6.

Solution:



We know that center = mid point of foci

$$= \left(\frac{0+0}{2}, \frac{-1-5}{2}\right)$$

$$= (0, -3)$$

Also we know that

C = distance between centre and focus

$$C = \sqrt{(0-0)^2 + (-3+1)^2}$$

$$C = \sqrt{4} = 2$$

Given $2a = 6$

$a = 3$

We know for Ellipse

$$c^2 = a^2 - b^2 \quad \Rightarrow \quad (2)^2 = (3)^2 - b^2$$

$$4 - 9 = -b^2 \quad \Rightarrow \quad b^2 = 5$$

With center (0, – 3), required equation of the ellipse is

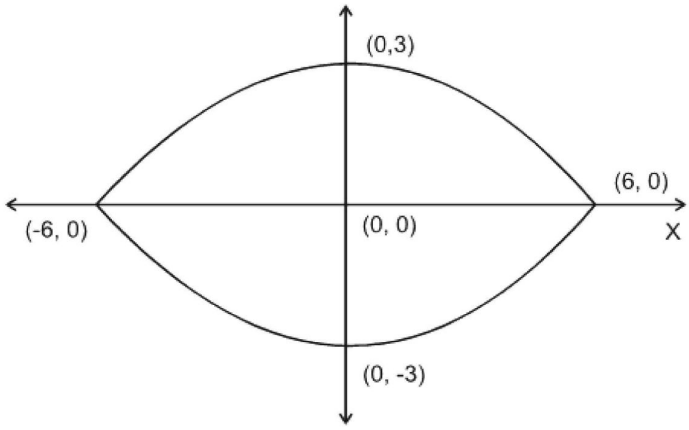
$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-0)^2}{5} + \frac{(y+3)^2}{(3)^2} = 1$$

$$\frac{x^2}{5} + \frac{(y+3)^2}{9} = 1$$

(iii) Foci $(-3\sqrt{3}, 0)$ & Vertices $(\pm 6, 0)$
(Lahore Board 2009)

Solution:



$(\pm ae, 0) = (\pm 3\sqrt{3}, 0)$ & $(\pm a, 0) = (\pm 6, 0)$

$ae = 3\sqrt{3}$ $\quad a = 6$

$c = 3\sqrt{3}$ $\quad a = 6$

We know that $c^2 = a^2 - b^2$

$(3\sqrt{3})^2 = (6)^2 - b^2$

$27 = 36 - b^2 \quad \Rightarrow \quad b^2 = 9$

Required equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$

Covertices are $= (0, \pm b)$

$= (0, \pm 3)$

(iv) Vertices (–1, 1), (5, 1) foci (4, 1) & (0, 1)

Solution:

$2a = |VV'|$ $\qquad 2c = |FF'|$

$2a = \sqrt{(5+1)^2 + (1-1)^2}$ $\qquad 2c = \sqrt{(0+4)^2 + (1-1)^2}$

$2a = \sqrt{36}$ $\qquad 2c = \sqrt{16}$

$2a = 6$ $\qquad 2c = 4$

$\boxed{a = 3}$ $\qquad \boxed{c = 2}$

We know that $c^2 = a^2 - b^2$

$(2)^2 = (3)^2 - b^2$

$$4 = 9 - b^2$$

$$\boxed{b^2 = 5}$$

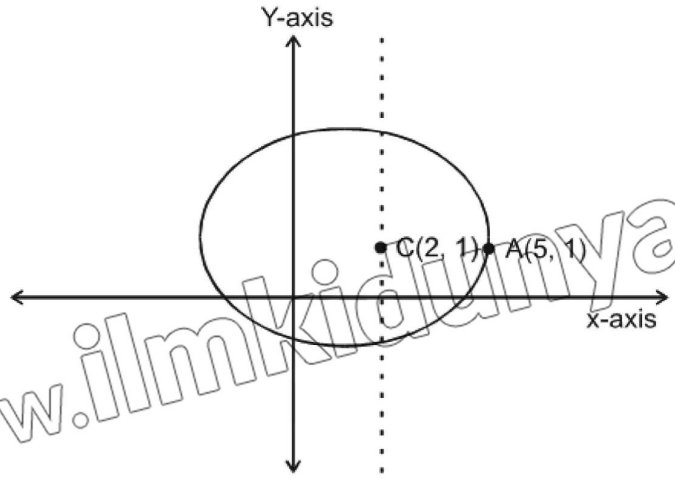
Now center of ellipse $= (h, k) =$ mid point of foci

$$= \left(\frac{4+0}{2}, \frac{1+1}{2}\right)$$

with this center required equation of ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

i.e; $$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{5} = 1$$



(v) Foci ($\pm\sqrt{5}$, 0) & passing through $\left(\frac{3}{2}, \sqrt{3}\right)$

Solution:

Given $(\pm ae, 0) = (\pm\sqrt{5}, 0)$

$c = \sqrt{5}$ we know that $c^2 = a^2 - b^2$

$(\sqrt{5})^2 = a^2 - b^2$

$5 = a^2 - b^2$

$\Rightarrow a^2 = 5 + b^2$ ..... (1)

Since Ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

and it is passing through point $\left(\frac{3}{2}, \sqrt{3}\right)$ therefore it becomes $\frac{9}{4a^2} + \frac{3}{b^2} = 1$ .....(2)

from (1) we have $a^2 = 5 + b^2$ Put in (2)

$$\frac{9}{4(5+b^2)} + \frac{3}{b^2} = 1$$

$$\frac{9b^2 + 12(5+b^2)}{4b^2(5+b^2)} = 1$$

$$9b^2 + 60 + 12b^2 = 20b^2 + 4b^4$$

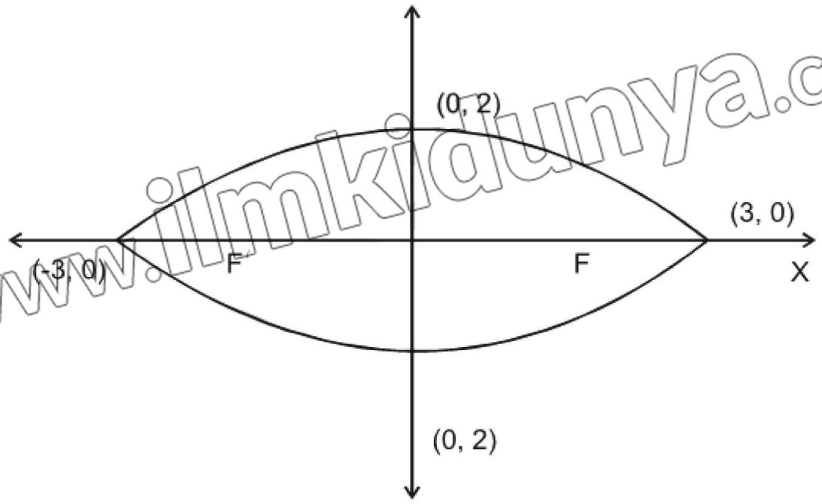
$$4b^4 - b^2 - 60 = 0$$

$$b^2 = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-60)}}{2(4)}$$

$$b^2 = \frac{1 \pm \sqrt{961}}{8}$$

$$= \frac{1 \pm 31}{8}$$

$$b^2 = \frac{1+31}{8}, \quad \frac{1-31}{8}$$



$$b^2 = \frac{32}{8}, \quad b^2 = \frac{-30}{8} \quad \text{(solution not possible)}$$

$$b^2 = 4$$

Required equation of Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

i.e; $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Vertices $(\pm a, 0) = (\pm 3, 0)$

Co-vertices $(0, \pm b) = (0, \pm 2)$

(vi) Vertices (0, ± 5), eccentricity = $\frac{3}{5}$

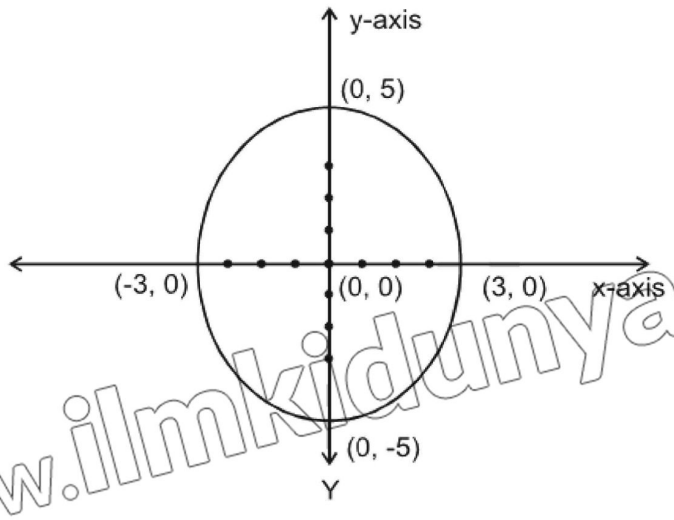
Solution:

Vertices (0, ± a) = (0, ± 5)

a = 5 e = $\frac{3}{5}$

ae = $5 \times \frac{3}{5}$

=> c = 3



We know that

$c^2 = a^2 - b^2$

$(3)^2 = (5)^2 - b^2$

$9 = 25 - b^2$

$b^2 = 16$

Required equation of Ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

Co-vertices

(± b, 0) = (± 4, 0)

Foci = (0, ± c) = (0, ± 3)

(vii) Centre (0, 0) focus (0, – 3), vertex (0, 4)
(Lahore Board 2011)

Solution:

$(0, -c) = (0, -3)$ $\quad a = 4$

$c = 3$

We know that $e^2 = \dfrac{a^2 - b^2}{a^2}$

$(ae)^2 = a^2 - b^2$

$c^2 = a^2 - b^2$

$(3)^2 = 16 - b^2$

$9 = 16 - b^2$

$b^2 = 16 - 9 \quad \Rightarrow \quad b^2 = 7$

Required equation of ellipse is $\dfrac{y^2}{a^2} + \dfrac{x^2}{b^2} = 1$

$\Rightarrow \dfrac{y^2}{16} + \dfrac{x^2}{7} = 1$

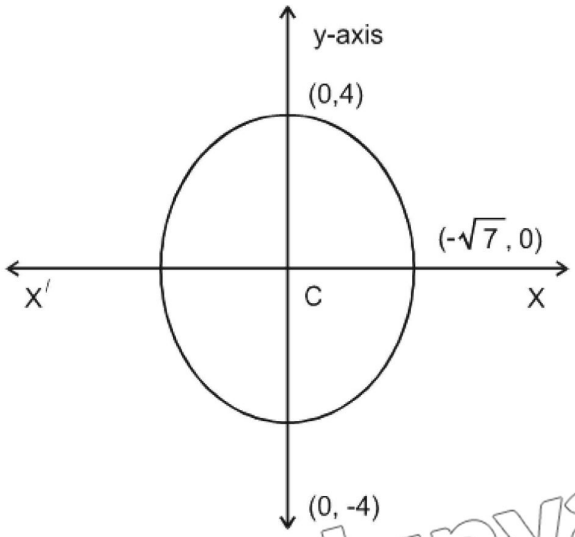
Its vertices are $(0, \pm a) = (0, \pm 4)$

Covertices $(\pm b, 0) = (\pm\sqrt{7}, 0)$

Coordinates of foci $= (0, \pm ae)$

$= (0, \pm 4\frac{3}{4})$

$= (0, \pm 3)$



(viii) Centre (2, 2) major axis parallel to y-axis and of length 8 units, minor axis parallel to x-axis and of length 6 units.

Solution:

Center (2, 2) also $2a = 8 \Rightarrow a = 4$

$2b = 6 \Rightarrow b = 3$

Equation of ellipse is $\dfrac{(y-k)^2}{a^2} + \dfrac{(x-h)^2}{b^2} = 1$

$\Rightarrow \dfrac{(y-2)^2}{16} + \dfrac{(x-2)^2}{9} = 1$

Its vertices are $(0, \pm a) = (0, \pm 2)$

and $e^2 = \dfrac{a^2-b^2}{a^2} = \dfrac{16-9}{16} = \dfrac{7}{16} \Rightarrow e = \dfrac{\sqrt{7}}{4}$

Coordinates of foci $= (\pm ae, 0) = (\pm \dfrac{4\sqrt{7}}{4}, 0) = (\pm\sqrt{7}, 0)$

$(y-2, x-2) = (\pm\sqrt{7}, 0)$

$y = 2 \pm \sqrt{7}$, $x = 2$

Thus foci are $(2, 2+\sqrt{7})$ & $(2, 2-\sqrt{7})$

For vertices, we have $x - 2 = 0$ $y - 2 = \pm a$

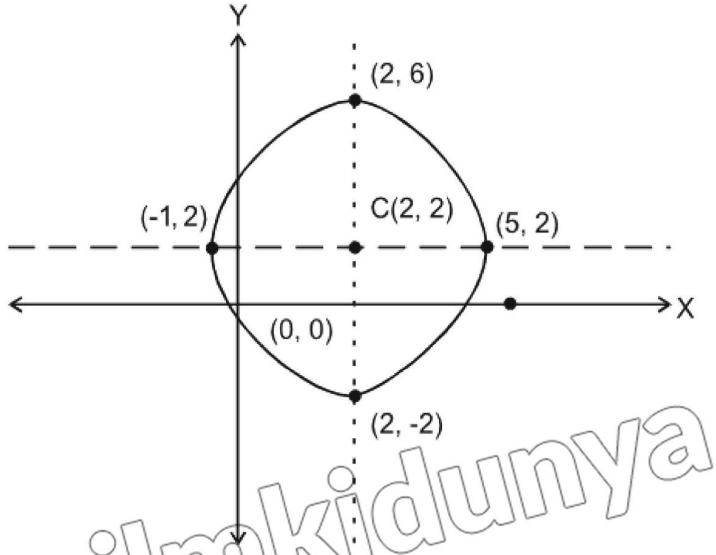
$x = 2$ $y = 2 \pm 4 \Rightarrow 6, -2$

So vertices are $(2, 6)$, $(2, -2)$

Next we have $x - 2 = \pm b$, $y - 2 = 0$

$x = 2 \pm 3$, $y = 2$

So covertices are $(5, 2)$ & $(-1, 2)$



(ix) Center (0, 0) symmetric with respect to both the axes and passing through the points (2, 3) and (6, 1).

Solution:

We know that equation of ellipse with center (0, 0) is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since it passes through the points (2, 3) & (6, 1)

$$\frac{4^2}{a^2} + \frac{9^2}{b^2} = 1 \quad \text{(I)} \qquad \frac{36}{a^2} + \frac{1}{b^2} = 1 \quad \text{(II)}$$

Subtracting

$$\begin{array}{rcl} 4b^2 + 9a^2 & = & a^2b^2 \\ -36b^2 \pm a^2 & = & -a^2b^2 \\ \hline -32b^2 + 8a^2 & = & 0 \end{array}$$

$$8a^2 = 32b^2$$

$$a^2 = 4b^2 \qquad \text{Put in (I)}$$

$$\frac{4}{4b^2} + \frac{9}{b^2} = 1$$

$$\frac{1+9}{b^2} = 1$$

$$\boxed{10 = b^2} \qquad \boxed{a^2 = 40}$$

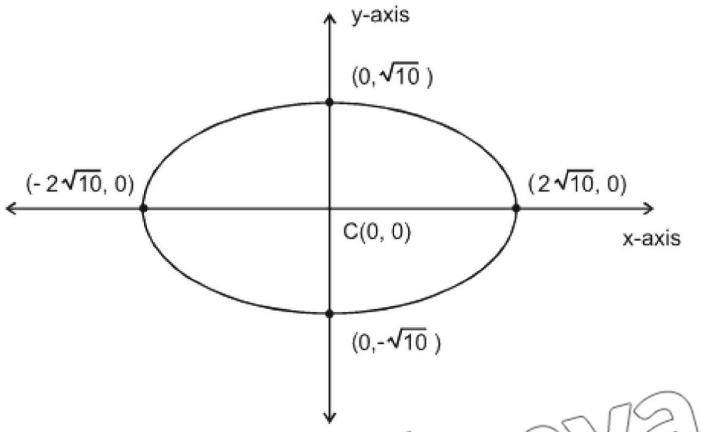
Required equation of ellipse is

$$\frac{x^2}{40} + \frac{y^2}{10} = 1$$

Vertices are $(\pm a, 0) = (\pm\sqrt{40}, 0) = (\pm 2\sqrt{10}, 0)$

Covertices $(0, \pm b) = (0, \pm\sqrt{10})$

Foci $(\pm\sqrt{30}, 0)$



(x) Center (0, 0) major axis horizontal, the points (3, 1) (4, 0) lie on the graph.

Solution:

We know that equation of Ellipse with center (0, 0) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since it passes though the points (3, 1) & (4, 0)

For (3, 1)

$$\frac{9}{a^2} + \frac{1}{b^2} = 1 \qquad (1)$$

For (4, 0)

$$\frac{16}{a^2} + \frac{0}{b^2} = 1 \qquad (2)$$

$\boxed{a^2 = 16}$ Put in (1)

$$\frac{9}{16} + \frac{1}{b^2} = 1$$

$$\frac{1}{b^2} = 1 - \frac{9}{16} = \frac{7}{16} \qquad \boxed{b^2 = \frac{16}{7}}$$

Required equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

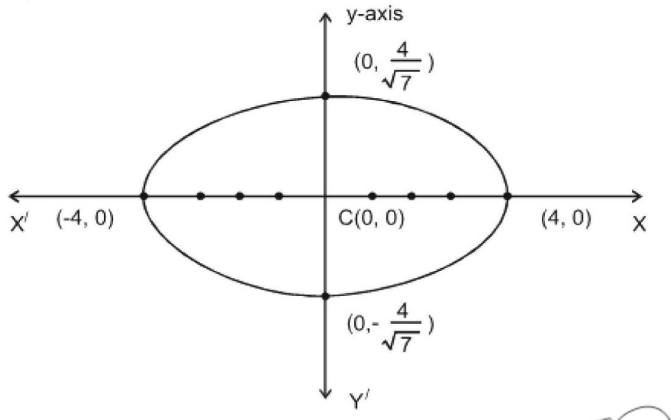
$$\frac{x^2}{16} + \frac{y^2}{16} = 1$$

$$\frac{x^2}{16} + \frac{7y^2}{16} = 1$$

Vertices $(\pm a, 0) = (\pm 4, 0)$

Covertices $(0, \pm b) = (0, \pm \frac{4}{\sqrt{7}})$

Foci $(\pm 4\sqrt{\frac{6}{7}}, 0)$



Q.2: Find the center, foci, eccentricity, vertices and directrix of the ellipse whose equation is given.

(i) $x^2 + 4y^2 = 16$ (Lahore Board 2009 (Supply))

Solution:

$$x^2 + 4y^2 = 16$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

Here $a^2 = 16$ $b^2 = 4$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{16-4}{16} = \frac{12}{16} = \frac{3}{4}$$

$$e = \frac{\sqrt{3}}{2}$$

Foci are $= (\pm c, 0) = (\pm 2\sqrt{3}, 0)$

Vertices are $= (\pm a, 0) = (\pm 4, 0)$

Directrix are $x = \pm \frac{a}{e}$ => $x = \pm \frac{4}{\frac{\sqrt{3}}{2}} = \pm \frac{8}{\sqrt{3}}$

Clearly center of ellipse is (0, 0)

(ii) $9x^2 + y^2 = 18$

Solution:

$$9x^2 + y^2 = 18$$

$$\frac{x^2}{2} + \frac{y^2}{18} = 1$$

$a^2 = 18$, $b^2 = 2$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{18-2}{18} = \frac{16}{18} = \frac{8}{9} \Rightarrow e = \frac{2\sqrt{2}}{3}$$

foci are $= (0, \pm c) = (0, \pm 4)$

vertices are $= (\pm a, 0) = (\pm 3\sqrt{2}, 0)$

directrix are $= y = \pm \frac{a}{e} = \pm \frac{3\sqrt{2}}{\frac{2\sqrt{2}}{3}} = \pm \frac{9}{2}$

Clearly center is (0, 0)

(iii) $\mathbf{25x^2 + 9y^2 = 225}$

Solution:

$$\frac{25x^2}{225} + \frac{9y^2}{225} = 1$$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1 \qquad \text{(I)}$$

$$a^2 = 25 \quad \& \quad b^2 = 9$$

Eccentricity

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{25 - 9}{25} = \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

foci $= (0, \pm c) = (0, \pm 4)$

vertices $= (0, \pm a) = (0, \pm 5)$

Center $= (0, 0)$

Directrix $y = \pm \frac{a}{e} = \pm \frac{5}{\frac{4}{5}} = \pm \frac{25}{4}$

(iv) $\mathbf{\frac{(2x-1)^2}{4} + \frac{(y+2)^2}{16} = 1}$

Solution:

Let $2x - 1 = X$, $y + 2 = Y$

Given equation becomes

$$\frac{X^2}{4} + \frac{Y^2}{16} = 1 \qquad \text{Here } a^2 = 16 \ , \ b^2 = 4$$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{16 - 4}{16} = \frac{12}{16} = \frac{3}{4}$$

Eccentricity

$$e = \frac{\sqrt{3}}{2}$$

Center:- For center put $X = 0$, $Y = 0$

$2x - 1 = 0$, $y + 2 = 0$

$x = \frac{1}{2}$, $y = -2$

Required center is $(\frac{1}{2}, -2)$

Foci $(0, \pm ae) = \left(0, \pm 4\left(\frac{\sqrt{3}}{2}\right)\right)$

$(X, Y) = (0, \pm 2\sqrt{3})$

$(2x - 1, y + 2) = (0, \pm 2\sqrt{3})$

$2x - 1 = 0 \qquad y + 2 = \pm 2\sqrt{3}$

$x = \frac{1}{2} \qquad y = -2 \pm 2\sqrt{3}$

Required foci are $(\frac{1}{2}, -2 \pm 2\sqrt{3})$

Vertices $(0, \pm a) = (0, \pm 4)$

$(X, Y) = (0, \pm 4)$

$(2x - 1, y + 2) = (0, \pm 4)$

$2x - 1 = 0 \qquad y + 2 = \pm 4$

$x = \frac{1}{2} \qquad y = -2 \pm 4$

$y = -6, \ 2$

Vertices are $\left(\frac{1}{2}, -6\right), \left(\frac{1}{2}, 2\right)$

Directrices $Y = \pm \frac{a}{e}$

$y + 2 = \pm \frac{4}{\frac{\sqrt{3}}{2}}$

$y + 2 = \pm \frac{8}{\sqrt{3}} \Rightarrow y = -2 \pm \frac{8}{\sqrt{3}}$

(v) $\mathbf{x^2 + 16x + 4y^2 - 16y + 76 = 0}$

Solution:

$x^2 + 16x + 4(y^2 - 4y) = -76$

$(x^2 + 16x + 64) + 4(y^2 - 4y + 4) = -76 + 64 + 16$

$(x + 8)^2 + 4(y - 2)^2 = 4$

$\frac{(x + 8)^2}{4} + \frac{4(y - 2)^2}{4} = \frac{4}{4}$

$\frac{(x + 8)^2}{4} + (y - 2)^2 = 1$

Let

$x + 8 = X, \qquad y - 2 = Y$

$$\frac{X^2}{4} + \frac{Y^2}{1} = 1 \quad \text{(an ellipse)}$$

$a^2 = 4$, $b^2 = 1$ $\qquad e^2 = \frac{a^2 - b^2}{a^2} = \frac{4-1}{4} = \frac{3}{4}$

$\Rightarrow e = \frac{\sqrt{3}}{2}$

For the center Put $X = 0$, $Y = 0$

$x + 8 = 0$, $y - 2 = 0$

$x = -8$, $y = 2$

Required centre $(-8, 2)$

Foci $= (\pm ae, 0)$

$(X, Y) = (\pm 2\left(\frac{\sqrt{3}}{2}\right), 0)$

$(x + 8, y - 2) = (\pm\sqrt{3}, 0)$

$x + 8 = \pm\sqrt{3}$ $\quad y - 2 = 0$

$x = -8 \pm \sqrt{3}$ $\quad y = 2$

Required foci are $(-8 \pm \sqrt{3}, 2)$

Vertices are $= (\pm a, 0)$

$(X, Y) = (\pm a, 0)$

$(x + 8, y - 2) = (\pm 2, 0)$

$x + 8 = \pm 2$, $y - 2 = 0$

$x = -8 \pm 2$ $\quad y = 2$

Required vertices are $(-6, 2)$ & $(-10, 2)$

Directrix $X = \pm\frac{a}{e}$

$$x + 8 = \pm\frac{2}{\frac{\sqrt{3}}{2}} = \pm\frac{4}{\sqrt{3}}$$

$$x = -8 \pm \frac{4}{\sqrt{3}}$$

(vi) $\mathbf{25x^2 + 4y^2 - 250x - 16y + 541 = 0}$

Solution:

$25x^2 - 250x + 4y^2 + 16y = -541$

$25(x^2 - 10x) + 4(y^2 - 4y) = -541$

$25(x^2 - 10x + 25) + 4(y^2 - 4y + 4) = -541 + 625 + 16$

$$25(x-5)^2 + 4(y-2)^2 = 100$$

Dividing both sides by 100

$$\frac{(x-5)^2}{4} + \frac{(y-2)^2}{25} = 1$$

Let

$x - 5 = X$, $y - 2 = Y$ above equation

Becomes

$$\frac{X^2}{4} + \frac{Y^2}{25} = 1 \quad \text{(an Ellipse)}$$

$a^2 = 25$, $b^2 = 4$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{25-4}{25} = \frac{21}{25}$$

$$e = \frac{\sqrt{21}}{5}$$

For Center Put $X = 0$, $Y = 0$

$(x-5, y-2) = (0, 0)$

$x - 5 = 0$, $y - 2 = 0$

$x = 5$ $y = 2$

Center is (5, 2)

$$\text{Foci} = (0, \pm ae) = (0, \pm 5\frac{\sqrt{21}}{5})$$

$(X, Y) = (0, \pm\sqrt{21})$

$(x-5, y-2) = (0, \pm\sqrt{21})$

$x - 5 = 0$, $y - 2 = \pm\sqrt{21}$

$x = 5$, $y = 2 \pm \sqrt{21}$

Foci are $(5, 2 \pm \sqrt{21})$

Vertices $(0, \pm a) = (0, \pm 5)$

$(X, Y) = (0, \pm 5)$

$(x-5, y-2) = (0, \pm 5)$

$x - 5 = 0$, $y - 2 = \pm 5$

$x = 5$, $y = 2 \pm 5$

$y = 7, -3$

Vertices are (5, 7) & (5, – 3)

Directrix are

$$Y = \pm \frac{a}{e}$$

$$y - 2 = \pm \frac{5}{\frac{\sqrt{21}}{5}}$$

$$y = 2 \pm \frac{25}{\sqrt{21}}$$

Q.3: Let a be a + ve number and 0 < c < a. Let F(– c, 0) & F′(c > 0) be two given points. Prove that the locus of points p(x, y) such that |PF| + |PF′| = 2a is an ellipse.

Solution:

Given P(x, y) , F(– c, 0) , F′(c , 0)

$\therefore \quad |PF| + |PF| = 2a$

$$\sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

Squaring on both sides

$$[\sqrt{(x+c)^2 + y^2}\,]^2 = [2a - \sqrt{(x-c)^2 + y^2}\,]^2$$

$$(x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x+c)^2 + y^2}$$

$$x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 - 4a\sqrt{(x-c)^2 + y^2}$$

$$4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

(Dividing by 4)

$$(cx - a^2) = -a\sqrt{(x-c)^2 + y^2}$$

Again Squaring

$$(cx - a^2)^2 = [-a\sqrt{(x-c)^2 + y^2}\,]^2$$

$$c^2x^2 + a^4 - 2cxa^2 = a^2(x^2 + c^2 - 2cx + y^2)$$

$$c^2x^2 + a^4 - 2cxa^2 = a^2x^2 + a^2c^2 - 2cxa^2 + a^2y^2$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

Dividing throughout by $c^2 - a^2$

$$x^2 - \frac{a^2y^2}{c^2 - a^2} = a^2 \qquad (1)$$

We know that for ellipse $c^2 = a^2 - b^2 \Rightarrow c^2 - a^2 = -b^2$ put in (1)

$$x^2 - \frac{a^2y^2}{-b^2} = a^2$$

$$x^2 + \frac{a^2y^2}{b^2} = a^2$$

(Dividing throughout by a^2)

$$\frac{x^2}{a^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2}{a^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{which is an ellipse}$$

Q.4: Use problem 3 to find equation of the ellipse as locus of points P(x, y) such that the sum of the distances from P to the points (0, 0) & (1, 1) is 2.

Solution:

Given P(x, y), F(0, 0) & F′(1, 1) Also given that 2a = 2

For ellipse we know that

$$|PF| + |PF'| = 2a$$

$$\sqrt{(x-0)^2 + (y-0)^2} + \sqrt{(x-1)^2 + (y-1)^2} = 2$$

$$\sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + (y-1)^2} = 2$$

$$\sqrt{(x-1)^2 + (y-1)^2} = 2 - \sqrt{x^2 + y^2} \quad \text{Squaring}$$

$$(x-1)^2 + (y-1)^2 = 4 + (x^2 + y^2) - 4\sqrt{x^2 + y^2}$$

$$x^2 + 1 - 2x + y^2 + 1 - 2y = 4 + x^2 + y^2 - 4\sqrt{x^2 + y^2}$$

$$-2x - 2y - 2 = -4\sqrt{x^2 + y^2}$$

$$x + y + 1 = 2\sqrt{x^2 + y^2} \qquad \text{(Dividing both sides by 2)}$$

Again squaring

$$(x + y + 1)^2 = (2\sqrt{x^2 + y^2})^2$$

$$x^2 + y^2 + 1 + 2xy + 2y + 2x = 4(x^2 + y^2)$$

$$4x^2 + 4y^2 - x^2 - y^2 - 1 - 2xy - 2y - 2x = 0$$

$$3x^2 + 3y^2 - 2xy - 2x - 2y - 1 = 0 \qquad \text{required ellipse}$$

Q.5: Prove that latusrectim of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$

Solution:

The given ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

From the figure, the points A(c, h) & B(c, – h) lies on ellipse (1) therefore.

For A(c, h) equation (1) becomes

$$\frac{c^2}{a^2} + \frac{h^2}{b^2} = 1$$

$$\frac{h^2}{b^2} = 1 - \frac{c^2}{a^2}$$

$$\frac{h^2}{b^2} = \frac{a^2 - c^2}{a^2} \qquad (2)$$

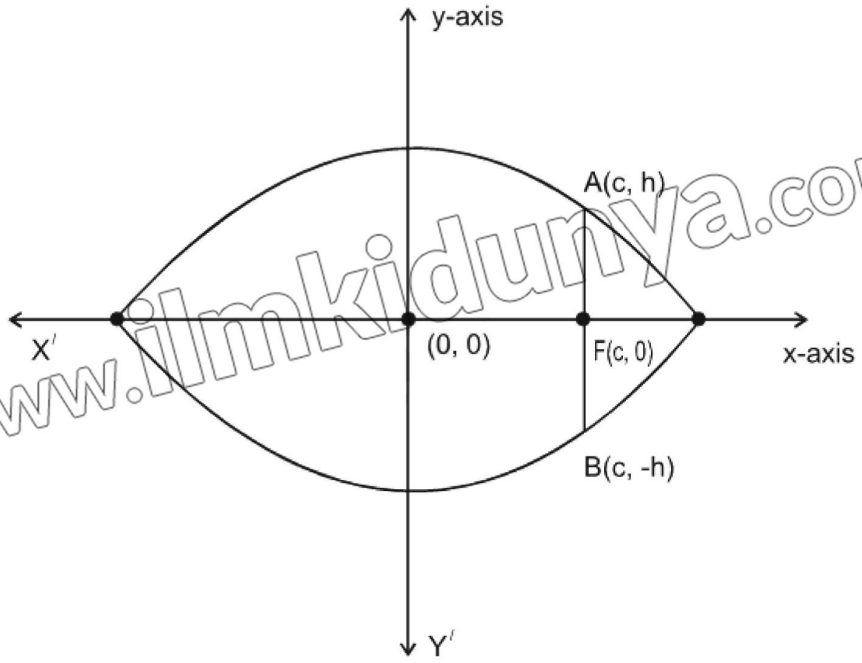
For ellipse, we know that

$$c^2 = a^2 - b^2$$

$$\Rightarrow \quad b^2 = a^2 - c^2 \qquad \text{Put in (2)}$$

$$\frac{h^2}{b^2} = \frac{b^2}{a^2}$$

$$h^2 = \frac{b^4}{a^2} \quad \Rightarrow \quad h = \pm\frac{b^2}{a}$$



Points A & B becomes

$$A(c, \frac{b^2}{a}) \text{ and } B(c, -\frac{b^2}{a})$$

Length of Latus rectum AB $= \sqrt{(c-c)^2 + \left(\frac{-b^2}{a} - \frac{b^2}{a}\right)^2}$

$$= \sqrt{\left(\frac{-2b^2}{a}\right)^2} = \sqrt{\frac{4b^4}{a^2}}$$

$$= \frac{2b^2}{a} \quad \text{Hence proved.}$$

Q.6: The major axis of an ellipse in standard form lies along the x-axis and has length $4\sqrt{2}$. The distance between the foci equals the length of minor axis. Write an equation of the ellipse.

Solution:

Since the major axis of the ellipse lies along x–axis so its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{.....} \quad (1)$$

By the given condition $2a = 4\sqrt{2}$

$\Rightarrow \quad a = 2\sqrt{2}$

We know that

Distance between foci = length of minor axis

$2c = 2b$

$\Rightarrow \quad c = b$

Since for ellipse $c^2 = a^2 - b^2$

$b^2 = a^2 - b^2$

$2b^2 = a^2 \quad \Rightarrow \quad b^2 = \frac{a^2}{2}$

$\Rightarrow \quad b^2 = \frac{(2\sqrt{2})^2}{2} = \frac{8}{2} = 4$

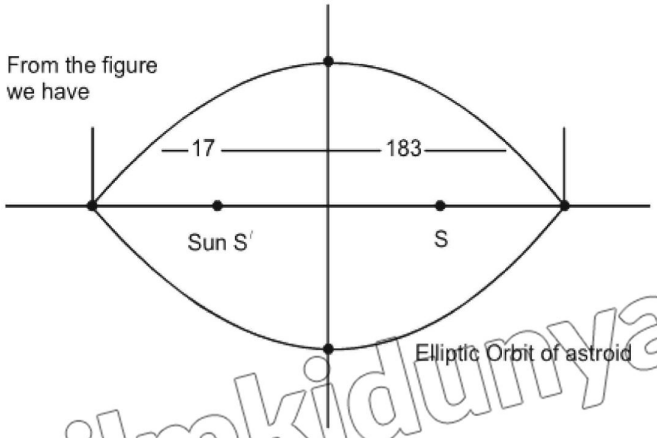
Putting values of a & b in (1)

$$\frac{x^2}{8} + \frac{y^2}{4} = 1 \qquad \text{Required equation of ellipse}$$

Q.7: An astroid has elliptic orbit with the sun at one focus. Its distance from the sun ranges from 17 million miles to 183 million miles. Write on equation of the orbit of the astroid.

Solution: (Lahore Board 2004)

From the figure we have



Greatest distance of astroid from the sun is $a + c = 183$ (1)

And least distance of astroid from the sun is

$a - c = 17$ (2)

Adding (1) & (2)

$$\begin{aligned} a + c &= 183 \\ a - c &= 17 \\ \hline 2a &= 200 \\ a &= 100 \quad \text{Put in (1)} \\ 100 + c &= 183 \\ c &= 83 \end{aligned}$$

For ellipse, we know that

$$c^2 = a^2 - b^2 \quad \Rightarrow \quad b^2 = a^2 - c^2$$

$$\Rightarrow \quad b^2 = (100)^2 - (83)^2$$

$$b^2 = 3111$$

Required equation of the orbit of astroid is

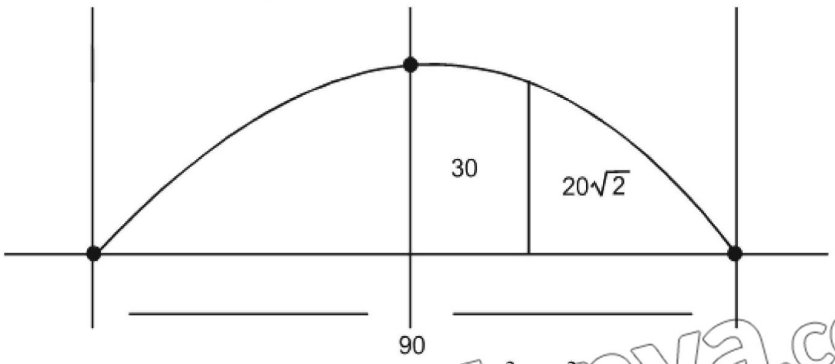
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{(100)^2} + \frac{y^2}{3111} = 1$$

$$\frac{x^2}{10000} + \frac{y^2}{3111} = 1 \quad \text{Ans.}$$

Q.8: An arch in the shape of semi ellipse is 90m wide at the base and 30 m high at the center. At what distance from the center is the arch $20\sqrt{2}$ m high?

Solution: (Lahore Board 2004)



Let the equation of the semi-elliptic are be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ..... (1)

Since the arch is 90m wide at the base

so $2a = 90 \Rightarrow a = 45$

Since arch is 30 m high at the center, so

$b = 30$, putting a & b in (1)

$$\frac{x^2}{(45)^2} + \frac{y^2}{(30)^2} = 1 \quad \text{.....(2)}$$

Let d be the required distance from the center when arch is $20\sqrt{2}$ m high.

Putting $x = d$, $y = 20\sqrt{2}$ in (2)

$$\frac{d^2}{(45)^2} + \frac{(20\sqrt{2})^2}{(30)^2} = 1$$

$$\frac{d^2}{2025} + \frac{800}{900} = 1$$

$$\frac{d^2}{2025} = 1 - \frac{800}{900}$$

$$d^2 = \frac{100}{900} \times 2025$$

$$d^2 = \frac{(45)^2}{(3)^2}$$

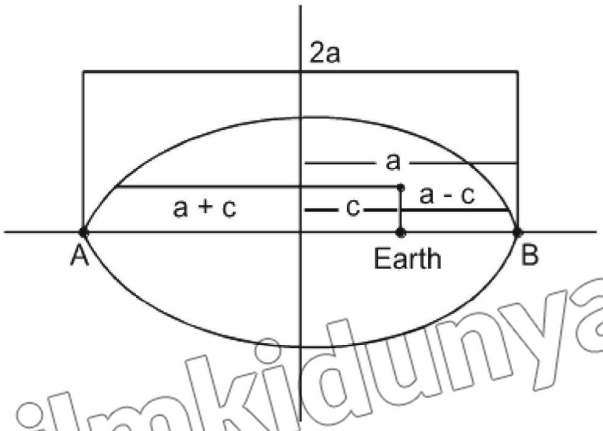
$$d = \frac{45}{3} = 15 \text{ meter}$$

Q.9: The moon orbits the earth in an elliptic path with earth at one focus. The major and minor axes of the orbit are 768, 806 km & 767, 746 km respectively. Find the greatest and least distances in astronomy called apogee & perigee) of the moon from the earth.

Solution:

By the given conditions

$2a = 768806 \Rightarrow a = 384403$



$2b = 767746 \Rightarrow b = 383874$

Since $\sqrt{c^2} = \sqrt{a^2 - b^2}$

$c = \sqrt{(384403)^2 - (383874)^2}$

$c = 20179$ km

The greatest distance

$a + c = 384403 + 20179$

$= 404582$ km

The least distance

$a - c = 384403 - 20179$

$= 364224$ km

Hyperbola

Let $e > 1$ and F be a fixed point and L be line not containing F. Also let P(x, y) be any point in the plane and |PM| be the perpendicular distance of P from L. The set of all the points P(x, y) such that $\frac{|PF|}{|PM|} = e > 1$ is called hyperbola.

Key points about Hyperbola

(1) Equation (Standard form)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(2) Eccentricity $e^2 = \frac{a^2+b^2}{a^2}$

$(ae)^2 = a^2 + b^2 \Rightarrow c^2 = a^2 + b^2$

(3) Foci $(\pm ae, 0)$

(4) Directrix $x = \pm\frac{a}{e}$

(5) Center $(0, 0)$

(6) Vertices $(\pm a, 0)$

(7) Covertices $(0, \pm b)$

(1) (Lahore Board 2009)

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

(2) Eccentricity $e^2 = \frac{a^2+b^2}{a^2}$

$(ae)^2 = a^2 + b^2$

$c^2 = a^2 + b^2$

(3) Foci $(0, \pm ae)$

(4) Directrix $y = \pm\frac{a}{e}$

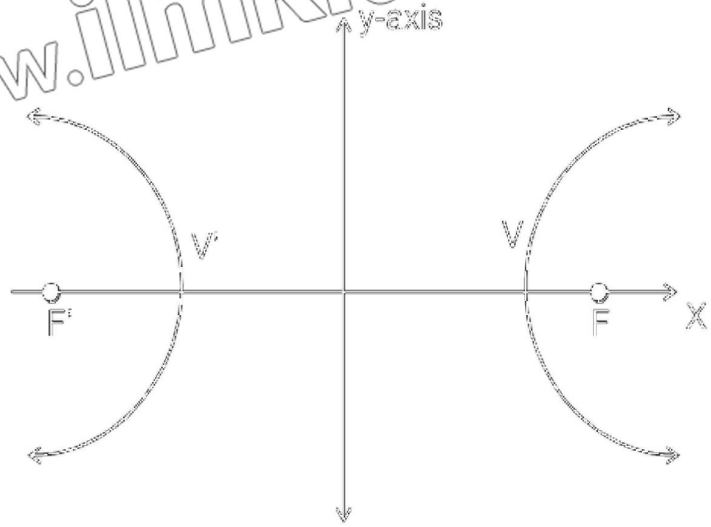
(5) Center $(0, 0)$

(6) Vertices $(0, \pm a)$

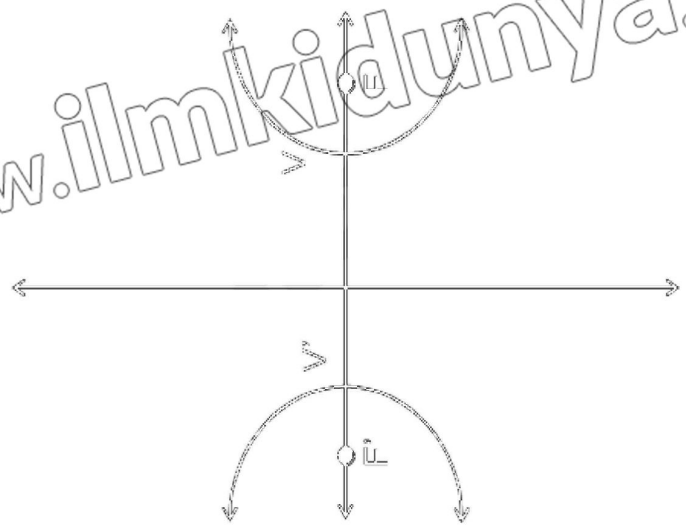
(7) Covertices $(\pm b, 0)$

Length of latus rectum is $\frac{2b^2}{a}$.

If center is not (0, 0) then equations become



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \& \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$



EXERCISE 6.6

Q.1: Find an equation of the hyperbola with given data. Sketch the graph of each.

(i) Center (0, 0) Focus (6, 0) , Vertex (4, 0)

Solution:

Given $(ae, 0) = (6, 0)$; $(a, 0) = (4, 0)$

$(c, 0) = (6, 0)$ $a = 4$

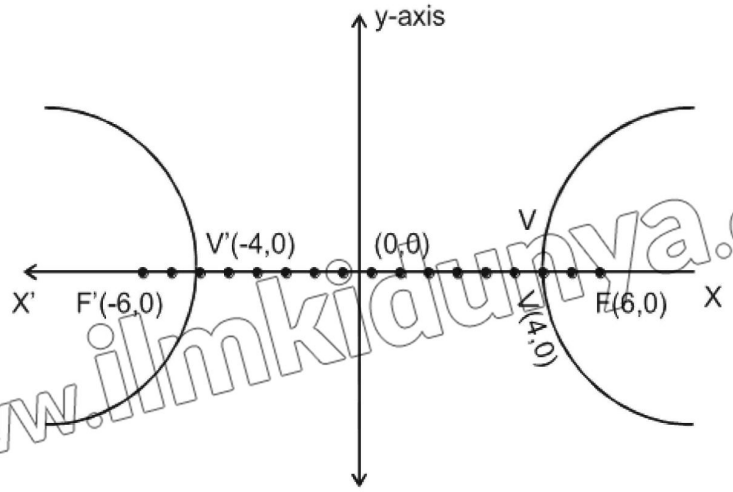
$c = 6$

For hyperbola

$$c^2 = a^2 + b^2$$
$$(6)^2 = (4)^2 + b^2$$
$$36 - 16 = b^2$$
$$20 = b^2$$

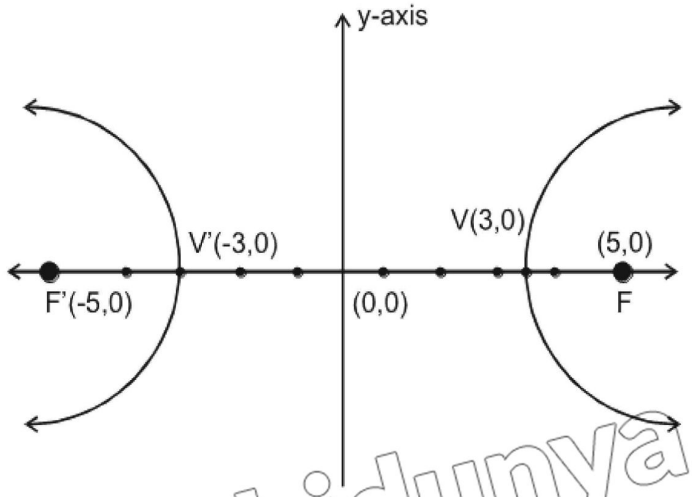
Required equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$



(ii) Foci (± 5, 0), vertex (3, 0)

Solution:



Foci (± 5, 0) Vertex (3, 0)

(± c, 0) = (± 5, 0) (a, 0) = (3, 0)

$\boxed{c = 5}$ $\boxed{a = 3}$

For hyperbola

$c^2 = a^2 + b^2$

$(5)^2 = (3)^2 + b^2$

$25 = 9 + b^2$

$16 = 9 + b^2$

$16 = b^2$

Required equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \qquad \text{Ans.}$$

(iii) Foci $(2 \pm 5\sqrt{2}, -7)$, Length of the transverse axis 10.

Solution:

$(\pm ae, 0) = (2 \pm 5\sqrt{2}, -7)$

Foci are $(2 + 5\sqrt{2}, -7)$ & F' $(2 - 5\sqrt{2}, -7)$

Also given $2a = 10$

$a = 5$

Center = mid point of foci

$$= \left(\frac{2 + 5\sqrt{2} + 2 - 5\sqrt{2}}{2}, \frac{-7-7}{2}\right)$$

Center $= (2, -7)$

We know that

c = distance between center and focus

$$= \sqrt{(2 - 2 - 5\sqrt{2})^2 + (-7 + 7)^2} = \sqrt{50}$$

For hyperbola

$c^2 = a^2 + b^2$

$(\sqrt{50})^2 = (5)^2 + b^2$

$50 - 25 = b^2$

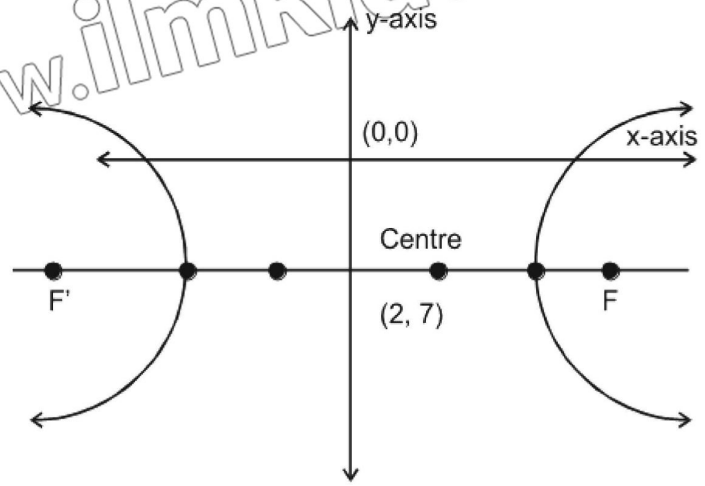
$25 = b^2$

$b^2 = 25$

with center $(2, -7)$ equation of hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{25} - \frac{(y+7)^2}{25} = 1 \qquad \text{Ans}$$



(iv) Foci $(0, \pm 6)$, e = 2 (Lahore Board 2011)

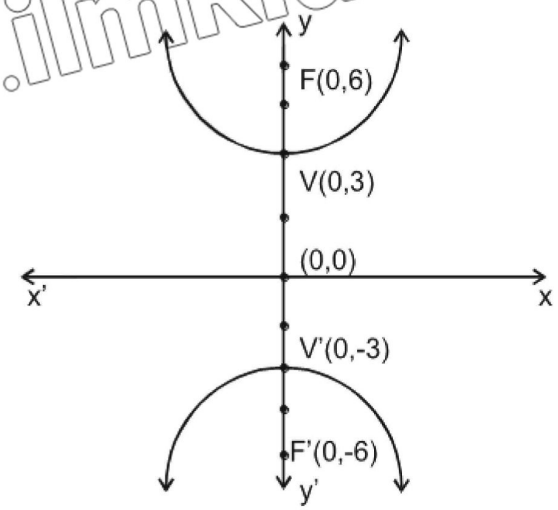
Solution:

Foci $(0, \pm c) = (0, \pm 6)$

$c = 6$, $e = 2$

$ae = c$

$a = \frac{c}{e} = \frac{6}{2} = \boxed{3 = a}$



For hyperbola

$c^2 = a^2 + b^2$

$(6)^2 = (3)^2 + b^2$

$36 - 9 = b^2$

$b^2 = 27$

Equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{9} - \frac{x^2}{27} = 1$$

(v) Foci $(0, \pm 9)$ directrix $y = \pm 4$

Solution:

Foci $(0, \pm ae) = (0, \pm 9)$, $y = \pm \frac{a}{e}$

$ae = 9$, $\pm 4 = \pm \frac{a}{e}$

=> $c = 9$, $e = \frac{a}{4}$

=> $a\left(\frac{a}{4}\right) = 9$

$a^2 = 36$

For hyperbola

$c^2 = a^2 + b^2$

$(9)^2 = 36 + b^2$

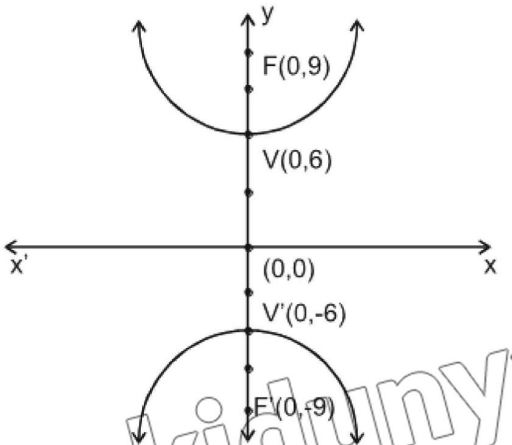
$81 - 36 = b^2$

$b^2 = 45$

Required equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{36} - \frac{x^2}{45} = 1$$



(vi) Centre (2, 2) horizontal transverse axis of length 6 & eccentricity e = 2.

Solution:

Centre (2, 2) & $2a = 6 \Rightarrow \boxed{a = 3}$ & $e = 2$

$c = ae = 3(2) = 6$

For hyperbola

$c^2 = a^2 + b^2$

$(6)^2 = (3)^2 + b^2$

$36 - 9 = b^2$

$b^2 = 27$

with centre (2, 2) required equation of hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{9} - \frac{(y-2)^2}{27} = 1$$

foci are $(\pm ae, 0) = (\pm 6, 0)$

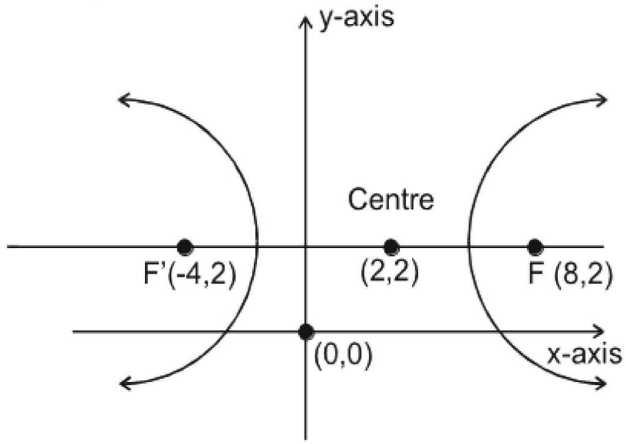
$(x - 2, y - 2) = (\pm 6, 0)$

$x - 2 = \pm 6 \qquad y - 2 = 0$

$x = \pm 6 + 2 \qquad y = 2$

$x = 8, -4 \qquad y = 2$

$\therefore$ F(8, 2) & (– 4, 2)



(vii) Vertices (2, ± 3) and (0, 5) lies on the curve.

Solution:

Center of the hyperbola = mid point of vertices

$$= \left(\frac{2+2}{2}, \frac{3-3}{2}\right)$$

Center = (2, 0)

Since $V = (2, 3)$ & $V' = (2, -3)$

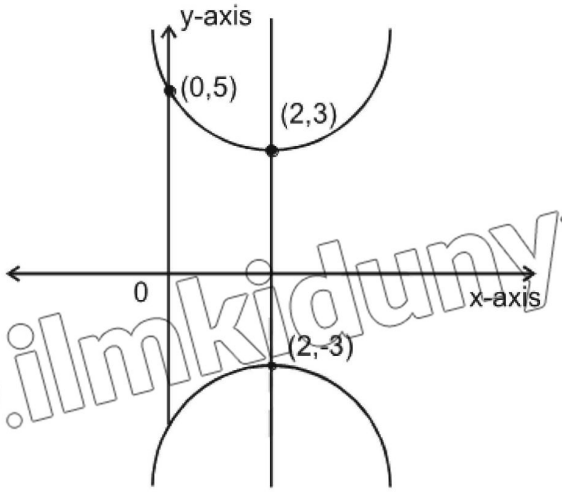
$2a = |VV'| = \sqrt{(2-2)^2 + (3+3)^2} = 6$

$\boxed{a = 3}$

With center (2, 0) equation of hyperbola is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-0)^2}{9} - \frac{(x-2)^2}{b^2} = 1 \qquad (1)$$



Since (0, 5) lies on (1)

$$\frac{25}{9} - \frac{4}{b^2} = 1$$

$$\frac{25}{9} - 1 = \frac{4}{b^2}$$

$$\frac{16}{9} = \frac{4}{b^2}$$

$b^2 = \frac{9}{4}$ (1) becomes $\frac{y^2}{9} - \frac{(x-2)^2}{\frac{9}{4}} = 1$ Ans.

(viii) Foci (5, – 2) (5, 4) & One vertex (5, 3)

Solution:

Center = mid point of foci

$= \left(\frac{5+5}{2}, \frac{-2+4}{2}\right)$

Center = (5, 1)

$c = $ distance between center & focus

$= \sqrt{(5-5)^2 + (1-4)^2}$

$= \sqrt{9}$

$= 3$

$\boxed{c = 3}$

$a = $ distance between vertex & center

$= \sqrt{(5-5)^2 + (3-1)^2}$

$= \sqrt{4}$

$= 2$

$\boxed{a = 2}$

For hyper bola

$c^2 = a^2 + b^2$

$(3)^2 = (2)^2 + b^2$

$9 - 4 = b^2$

$\boxed{b^2 = 5}$

with center (5, 1) required equation of hyperbola is

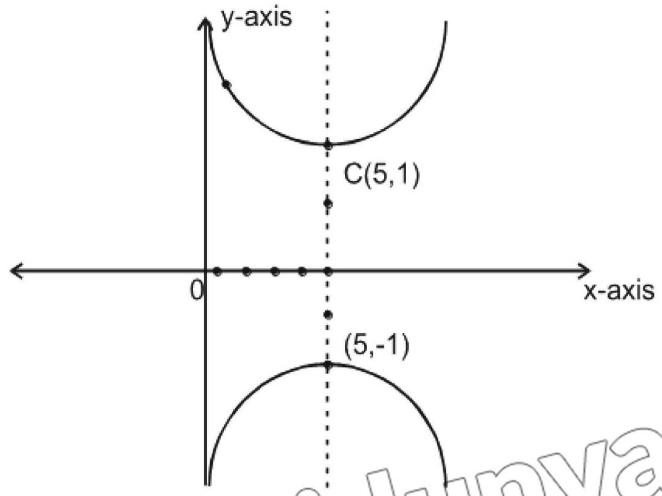
$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-1)^2}{4} - \frac{(x-5)^2}{5} = 1$$

The second vertex is

(5, 1 – 2)

(5, – 1)



Q.2: Find the center, foci, eccentricity, vertices and equations of directrices of each of the following. (i) $x^2 - y^2 = 9$

Solution:

$$\frac{x^2}{9} - \frac{y^2}{9} = 1$$

$\Rightarrow \quad a^2 = 9 \;,\; b^2 = 9$

$c^2 = a^2 + b^2$

$c^2 = 9 + 9 = 18$

$c^2 = 18 \Rightarrow \quad c = 3\sqrt{2}$

$$e^2 = \frac{a^2 + b^2}{a^2} = \frac{18}{9} = 2$$

$$\boxed{e = \sqrt{2}}$$

foci $= (\pm c, 0) = (\pm 3\sqrt{2}, 0)$

Vertices $= (\pm a, 0) = (\pm 3, 0)$

Directrix $x = \pm \frac{a}{e}$

Directrix $x = \pm \frac{3}{\sqrt{2}}$

Center is $(0, 0)$

(ii) $\mathbf{\frac{x^2}{4} - \frac{y^2}{9} = 1}$

Solution:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

$a^2 = 4 \qquad b^2 = 9$

$\therefore \quad c^2 = a^2 + b^2 = 4 + 9 = 13 \quad \Rightarrow \quad c = \sqrt{13}$

$\therefore \quad ae = c \quad \Rightarrow \quad e = \frac{c}{a}$

eccentricity $\boxed{e = \frac{\sqrt{13}}{2}}$

foci $= (\pm c, 0) = (\pm\sqrt{13}, 0)$

Vertices $= (\pm a, 0) = (\pm 2, 0)$

Directrix $x = \pm \frac{a}{e}$

$$x = \pm \frac{2}{\frac{\sqrt{13}}{2}}$$

$$x = \pm \frac{4}{\sqrt{13}}$$

Center is (0, 0)

(iii) $\mathbf{\frac{y^2}{16} - \frac{x^2}{9} = 1}$

Solution:

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

$a^2 = 16$, $b^2 = 9$

$\therefore$ $c^2 = a^2 + b^2$

$= 16 + 9$

$c^2 = 25$

eccentricity $ae = c$ => $e = \frac{c}{a}$ => $\frac{5}{4}$

$$\boxed{e = \frac{5}{4}}$$

foci $= (0, \pm c)$

$= (0, \pm 5)$

Vertices $= (0, \pm a) = (0, \pm 4)$

Directrix $y = \pm \frac{a}{e}$

Directrix $y = \pm \frac{4}{\frac{5}{4}} = \pm \frac{16}{5}$

Center is (0, 0)

(iv) $\mathbf{\frac{y^2}{4} - x^2 = 1}$

Solution:

$$\frac{y^2}{4} - \frac{x^2}{1} = 1$$

Center = (0, 0)

$a^2 = 4$, $b^2 = 1$

$c^2 = a^2 + b^2$

$c^2 = 4 + 1$

$c^2 = 5$

eccentricity $ae = c \Rightarrow e = \frac{c}{a} = \boxed{\frac{\sqrt{5}}{2} = e}$

foci $= (0, \pm c) = (0, \pm\sqrt{5})$

Vertices $= (0, \pm a) = (0, \pm 2)$

Directrix $y = \pm\frac{a}{e}$

Directrix $y = \pm\frac{2}{\frac{\sqrt{5}}{2}}$

$= \pm\frac{4}{\sqrt{5}}$

(v) $\frac{(x-1)^2}{2} - \frac{(y-1)^2}{9} = 1$

Solution:

Let $x - 1 = X$, $y - 1 = Y$

$\frac{X^2}{2} - \frac{Y^2}{2} = 1$

$a^2 = 2$, $b^2 = 9$ $c^2 = a^2 + b^2 = 2 + 9 = 11$

$ae = c$, $e = \frac{c}{a} = \frac{\sqrt{11}}{\sqrt{2}}$

Eccentricity $e = \sqrt{\frac{11}{2}}$

Coordinates of foci $(\pm c, 0)$

$(X, Y) = (\pm\sqrt{11}, 0)$

$(x - 1, y - 1) = (\pm\sqrt{11}, 0)$

$x - 1 = \pm\sqrt{11}$ $y - 1 = 0$

$x = 1 \pm\sqrt{11}$ $y = 1$

foci $(1 \pm \sqrt{11}, 1)$

For center Put $X = 0$, $Y = 0$

$x - 1 = 0$, $y - 1 = 0$

$\Rightarrow x = 1$, $y = 1$ $\boxed{\text{cenere} = (1, 1)}$

Vertices $(X, Y) = (\pm\sqrt{2}, 0)$

$(x-1, y-1) = (\pm\sqrt{2}, 0)$

$x - 1 = \pm\sqrt{2} = \quad y - 1 = 0$

$x = 1 \pm \sqrt{2} \quad y = 1$

Vertices are $\boxed{(1+\sqrt{2}, 1), \quad (1-\sqrt{2}, 1)}$

Equation of directrix

$$X = \pm\frac{a}{e}$$

$$x - 1 = \pm\frac{\sqrt{2}}{\frac{\sqrt{11}}{\sqrt{2}}}$$

$$x - 1 = \pm\frac{2}{\sqrt{11}}$$

$$\boxed{x = 1 \pm \frac{2}{\sqrt{11}}}$$

(vi) $\mathbf{\frac{(y+2)^2}{9} - \frac{(x-2)^2}{16} = 1}$

Solution:

Let $y + 2 = Y \quad , \quad x - 2 = X$

Given equation becomes

$$\frac{Y^2}{9} - \frac{X^2}{16} = 1$$

$a^2 = 9 \ , \ b^2 = 16 \qquad c^2 = a^2 + b^2 = 9 + 16 = 25$

$\Rightarrow \quad c = \pm 5$

$ae = c$

Eccentricity $\Rightarrow \quad e = \frac{c}{a} = \frac{5}{3}$

Foci $(X, Y) = (0, \pm c)$

$(x-2, y+2) = (0, \pm 5)$

$x - 2 = 0 \quad , \quad y + 2 = \pm 5$

$x = 2 \quad , \quad y = 2 \pm 5$

$y = -7, 3$

Foci $(2, -7) \ , \ (2, 3)$

For the center put $X = 0$ $Y = 0$

$x - 2 = 0$, $y + 2 = 0$

$x = 2$, $y = -2$

Center $(2, -2)$

Vertices $(X, Y) = (0, \pm a)$

$(x - 2, y + 2) = (0, \pm 3)$

$x - 2 = 0$ $y + 2 = \pm 3$

$x = 2$ $y = -2 \pm 3$

$y = -5, 1$

Vertices

$(2, -5)$ & $(2, 1)$

Equations of directrices are

$$Y = \pm \frac{a}{e}$$

$$y + 2 = \pm \frac{3}{\frac{5}{3}}$$

$$y = -2 \pm \frac{9}{5} \quad \text{Ans.}$$

(vii) $\mathbf{9x^2 - 12x - y^2 - 2y + 2 = 0}$

Solution:

$$9x^2 - 12x - y^2 - 2y = -2$$

$$9(x^2 - \frac{12}{9}x) - 1(y^2 + 2y) = -2$$

$$9(x^2 - \frac{4}{3}x + \frac{4}{9}) - 1(y^2 + 2y + 1) = -2 + 4 - 1$$

$$9(x - \frac{2}{3})^2 - 1(y + 1)^2 = 1$$

$$\frac{\left(x - \frac{2}{3}\right)^2}{\frac{1}{9}} - \frac{(y + 1)^2}{1} = 1 \qquad (1)$$

Let $x - \frac{2}{3} = X$, $y + 1 = Y$ Equation (1) becomes

$$\frac{X^2}{\frac{1}{9}} - \frac{Y^2}{1} = 1$$ which represents hyperbola

$a^2 = \frac{1}{9}$, $b^2 = 1$

$$e^2 = \frac{a^2 + b^2}{a^2} = \frac{\frac{1}{9} + 1}{\frac{1}{9}} = \frac{\frac{10}{9}}{\frac{1}{9}} = 10$$

$e = \sqrt{10}$ $\quad c^2 = \frac{10}{9} \Rightarrow c = \frac{\sqrt{10}}{3}$

Coordinates of foci $(X, Y) = (\pm C = 0)$

$(x - \frac{2}{3}, y + 1) = (\pm\frac{\sqrt{10}}{3}, 0)$

$x - \frac{2}{3} = \pm\frac{\sqrt{10}}{3}$ $\quad y + 1 = 0$

$x = \frac{2}{3} \pm \frac{\sqrt{10}}{3}$ $\quad y = -1$

$x = \frac{2 \pm \sqrt{10}}{3}$, $y = -1$

Foci $(\frac{2 \pm \sqrt{10}}{3}, -1)$

For center put $X = 0$, $Y = 0$

$x - \frac{2}{3} = 0$, $y + 1 = 0$

$x = \frac{2}{3}$ $\quad y = -1$ $\quad$ Center $(\frac{2}{3}, -1)$

Vertices $(\pm a, 0)$

$(X + Y) = (\pm\frac{1}{3}, 0)$

$(x - \frac{2}{3}, y + 1) = (\pm\frac{1}{3}, 0)$

$x - \frac{2}{3} = \pm\frac{1}{3}$, $y + 1 = 0$

$x = \frac{2}{3} \pm \frac{1}{3}$ $\quad y = -1$

Vertices $(1, -1)$, $(\frac{1}{3}, -1)$

Equation of directrices are

$X = \pm\frac{a}{e}$

$x - \frac{2}{3} = \pm\frac{1}{3} \div \sqrt{10}$

$x = \frac{2}{3} \pm \frac{1}{3\sqrt{10}}$

(viii) $\mathbf{4y^2 + 12y - x^2 + 4x + 1 = 0}$

Solution:

$$4(y^2 + 3y) - 1(x^2 - 4x) = -1$$

$$4(y^2 + 3y + \frac{9}{4}) - 1(x^2 - 4x + 4 = -1 + 9 - 4$$

$$4(y + \frac{3}{2})^2 - 1(x - 2)^2 = 4$$

$$\frac{\left(y + \frac{3}{2}\right)^2}{1} - \frac{(x-2)^2}{4} = 1 \qquad (1)$$

Let $y + \frac{3}{2} = Y$, $x - 2 = X$

Equation (1) becomes

$$\frac{Y^2}{1} - \frac{X^2}{4} = 1 \qquad \text{(Represents a hyperbola)}$$

$$a^2 = 1 \quad , \quad b^2 = 4$$

$\therefore$ $c^2 = a^2 + b^2 = 1 + 4 = 5 \Rightarrow c = \sqrt{5}$

Eccentricity $ae = c \Rightarrow e = \frac{c}{a} = \frac{\sqrt{5}}{1} = \boxed{\sqrt{5} = e}$

Foci $(X, Y) = (0, \pm c)$

$$(x - 2, y + \frac{3}{2}) = (0, \pm\sqrt{5})$$

$$x - 2 = 0 \qquad y + \frac{3}{2} = \pm\sqrt{5}$$

$$x = 2 \qquad y = \frac{-3}{2} \pm \sqrt{5}$$

Foci $(2, \frac{-3}{2} \pm \sqrt{5})$

For center Put $X = 0, Y = 0$

$$x - 2 = 0 \quad , \quad y + \frac{3}{2} = 0$$

$$x = 2 \qquad y = \frac{-3}{2}$$

Center $(2, \frac{-3}{2})$

Vertices $(X, Y) = (0, \pm a)$

$$(x-2,\quad y+\frac{3}{2}) = (0, \pm 1)$$

$$x-2 = 0 \quad , \quad y+\frac{3}{2} = \pm 1$$

$$x = 2 \quad , \quad y = \frac{-3}{2} \pm 1$$

$$y = \frac{-1}{2}, \frac{-5}{2}$$

Vertices are $(2, \frac{-1}{2})$, $(2, \frac{-5}{2})$

Equation of directrices are

$$Y = \pm \frac{a}{e}$$

$$y+\frac{3}{2} = \pm \frac{1}{\sqrt{5}}$$

$$y = \frac{-3}{2} \pm \frac{1}{\sqrt{5}} \quad \text{Ans}$$

(ix) $\mathbf{x^2 - y^2 + 8x - 2y - 10 = 0}$

Solution:

$$x^2 + 8x - y^2 - 2y = 10$$

$$(x^2 + 8x + 16) - (y^2 + 2y + 1) = 10 + 16 - 1$$

$$(x+4)^2 - (y+1)^2 = 25$$

$$\frac{(x+4)^2}{25} - \frac{(y+1)^2}{25} = 1 \qquad \text{(i)}$$

Let $x + 4 = X$, $y + 1 = Y$

(i) Becomes $\frac{X^2}{25} - \frac{Y^2}{25} = 1$ (which represents hyperbola)

$a^2 = 25$, $b^2 = 25$ $\quad c^2 = a^2 + b^2 = 50 \Rightarrow c = \sqrt{50}$

$ae = c$

Eccentricity $e = \frac{c}{a} = \frac{\sqrt{50}}{5} = \frac{5\sqrt{2}}{5} = \sqrt{2}$

Foci $= (\pm c, 0)$

$(X, Y) = (\pm 5\sqrt{2}, 0)$

$(x + 4, y + 1) = (\pm 5\sqrt{2}, 0)$

$x + 4 = \pm 5\sqrt{2}$, $y + 1 = 0$

$x = -4 \pm 5\sqrt{2}$ $\quad y = -1$ $\quad$ foci $(-4 \pm 5\sqrt{2}, -1)$

For center put $X = 0$, $Y = 0$

$x + 4 = 0$, $y + 1 = 0$ => $x = -4$, $y = -1$

Center $(-4, -1)$

Vertices $(X, Y) = (\pm a, 0)$

$(x + 4, y + 1) = (\pm 5, 0)$

$x + 4 = \pm 5$ $\quad$ $y + 1 = 0$

$x = -4 \pm 5$ $\quad$ $y = -1$

$= -9, 1$

Vertices $(-9, -1)$ & $(1, -1)$

Equations of directrices

$$X = \pm \frac{a}{e}$$

$$x + 4 = \pm \frac{5}{\sqrt{2}}$$

$$x = -4 \pm \frac{5}{\sqrt{2}} \quad \text{Ans.}$$

(x) $\mathbf{9x^2 - y^2 - 36x - 6y + 18 = 0}$

Solution:

$$9x^2 - 36x - y^2 - 6y = -18$$

$$9(x^2 - 4x) - 1(y^2 + 6y) = -18$$

$$9(x^2 - 4x + 4) - 1(y^2 + 6y + 9) = -18 + 36 - 9$$

$$9(x - 2)^2 - 1(y + 3)^2 = 9$$

$$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{1} = 1 \quad \text{(i)}$$

Let $x - 2 = X$, $y + 3 = Y$ Equation (i) becomes

$$\frac{X^2}{1} - \frac{Y^2}{9} = 1$$

$a^2 = 1$, $b^2 = 9$

$\therefore$ $c^2 = a^2 + b^2$ => $c^2 = 1 + 9 = 10$ $\quad$ $\boxed{c^2 = 10}$

$\therefore$ $ae = c$

Eccentricity $\therefore$ $e = \frac{c}{a} = \frac{\sqrt{10}}{1}$ $\quad$ $\boxed{e = \sqrt{10}}$

Foci $(\pm c, 0) = (\pm\sqrt{10}, 0)$

$(X, Y) = \pm\sqrt{10}, 0$

$(x - 2, y + 3) = (\pm\sqrt{10}, 0)$

$x - 2 = \pm\sqrt{10}$, $y + 3 = 0$

$x = 2 \pm \sqrt{10}$, $y = -3$

Foci $(2 \pm \sqrt{10}, -3)$

For center Put $X = 0$, $Y = 0$

$x - 2 = 0$ $y + 3 = 0$

$x = 2$, $y = -3$ $\boxed{\text{center } (2, -3)}$

Vertices $(X, Y) = (\pm a, 0)$

$(x - 2, y + 3) = (\pm 1, 0)$

$x - 2 = \pm 1$ $y + 3 = 0$

$x = 2 \pm 1$ $y = -3$

$x = 3, 1$, $y = -3$ $\boxed{\text{Vertices } (3, -3), (1, -3)}$

Equations of directrices

$$X = \pm \frac{a}{e}$$

$$x - 2 = \pm \frac{1}{\sqrt{10}}$$

$$x = 2 \pm \frac{1}{\sqrt{10}} \quad \text{Ans}$$

Q.3: Let $0<a<c$ and $F(-c, 0)$, $F'(c, 0)$ be two fixed points. Show that the set of points $P(x, y)$ such that $|PF| - |PF'| = \pm 2a$ is the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

Solution:

Given $F(-c, 0)$, $F'(c, 0)$ and $p(x, y)$

Also $|PF| - |PF'| = 2a$

$$\sqrt{(x + c)^2 + (y - 0)^2} - \sqrt{(x - c)^2 + (y - 0)^2} = 2a$$

$$\sqrt{(x + c)^2 + y^2} = 2a + \sqrt{(x - c)^2 + y^2}$$

Squaring on both sides

$$(\sqrt{(x + c)^2 + y^2})^2 = (2a + \sqrt{(x - c)^2 + y^2})^2$$

$$x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 + 4a\sqrt{(x - c)^2 + y^2}$$

$$4cx - 4a^2 = 4a^2\sqrt{x^2 + c^2 - 2cx + y^2}$$

$$4(cx - a^2) = 4a^2\sqrt{x^2 + c^2 - 2cx + y^2}$$

Again Squaring

$$c^2x^2 + a^4 - 2cxa^2 = a^2(x^2 + c^2 - 2cx + y^2)$$

$$c^2x^2 + a^4 - 2cxa^2 - a^2x^2 - a^2c^2 + 2a^2cx - a^2y^2 = 0$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

Dividing throughout by $c^2 - a^2$

$$\frac{x^2(c^2 - a^2)}{c^2 - a^2} - \frac{a^2y^2}{c^2 - a^2} = \frac{a^2(c^2 - a^2)}{c^2 - a^2}$$

$$x^2 - \frac{a^2y^2}{c^2 - a^2} = a^2$$

Dividing throughout by a^2 $\quad \frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$ Hence proved.

Q.4: Find an equation of the hyperbola with foci (– 5, –5) & (5, 5) vertices ($-3\sqrt{2}$, $-3\sqrt{2}$) and ($3\sqrt{2}$, $3\sqrt{2}$).

Solution:

Since $2a =$ distance between two vertices

$$2a = \sqrt{(3\sqrt{2} + 3\sqrt{2})^2 + (3\sqrt{2} + 3\sqrt{2})^2}$$

$$2a = \sqrt{(6\sqrt{2})^2 + (6\sqrt{2})^2} = \sqrt{72 + 72} = \sqrt{144} = 12$$

$$2a = 12 \quad \boxed{a = 6}$$

Also given F(–5, – 5) & F′(5, 5). Let P(x, y) be any point on the hyperbola such that $|PF| - |PF'| = 2a$

$$\sqrt{(x + 5)^2 + (y + 5)^2} - \sqrt{(x - 5)^2 + (y - 5)^2} = 12$$

Squaring $\sqrt{(x + 5)^2 + (y + 5)^2} = 12 + \sqrt{(x - 5)^2 + (y - 5)^2}$

$$x^2 + 25 + 10x + y^2 + 25 + 10y = x^2 + 25 - 10x + y^2 + 25 - 10y + 144 + 24\sqrt{(x - 5)^2 + (y - 5)^2}$$

$$20x + 20y - 144 = 24\sqrt{(x - 5)^2 + (y - 5)^2}$$

$$4(5x + 5y - 36) = 4[6\sqrt{(x - 5)^2 + (y - 5)^2}]$$

Again squaring on both sides

$$(5x + 5y - 36)^2 = 36(x^2 + 25 - 10x + y^2 + 25 - 10y)$$

$$25x^2 + 5y^2 + 1296 + 50xy - 360y - 360x = 36x^2 + 900 - 360x + 36y^2 + 900 - 360y$$

$$25x^2 + 25y^2 + 1296 + 50xy - 360y - 360x - 36x^2 - 900 + 360x - 36y^2 - 900 + 360y = 0$$

$$11x^2 - 50xy - 11y^2 + 504 = 0 \quad \text{Ans}$$

Q.5: For any point on hyperbola the difference of its distances from the points (2, 2) & (10, 2) is 6. Find an equation of hyperbola.

Solution:

Given $2a = 6$ $\quad F = (2, 2)$ & $F' = (10, 2)$

Let P(x, y) be any point on the hyperbola, such that

$|PF| - |PF'| = 2a$

$$\sqrt{(x-2)^2+(y-2)^2} - \sqrt{(x-10)^2+(y-2)^2} = 6$$

$$\sqrt{(x-2)^2+(y-2)^2} = 6 + \sqrt{(x-10)^2+(y-2)^2}$$

Squaring

$$x^2 + 4 - 4x + y^2 + 4 - 4y = 36 + x^2 + 100 - 20x + y^2 + 4 - 4y + 12\sqrt{(x-10)^2+(y-2)^2}$$

$$16x - 132 = 12\sqrt{(x-10)^2+(y-2)^2}$$

$$4(4x-33) = 12\sqrt{(x-10)^2+(y-2)^2}$$

$$4x - 33 = 3\sqrt{(x-10)^2+(y-2)^2}$$

Again squaring

$$16x^2 + 1089 - 264 = 9(x^2 + 100 - 20x + y^2 + 4 - 4y)$$

$$16x^2 + 1089 - 264 - 9x^2 + 900 + 180x - 9y^2 - 36 + 36y = 0$$

$$7x^2 - 9y^2 - 84x + 36y + 153 = 0 \qquad \text{Ans}$$

Q.6: Two listening posts hear the sound of an enemy gun. The difference in time is one second. If the listening posts are 1400 feet apart, write an equation of the hyperbola passing though the position of the enemy gun.

Solution:

Since the difference in time is one second, so let the listening posts F&F′ near the second of the enemy gm after t, t–1 seconds respectively. If P is position of enemy then $|PF| - |PF'| = 2a$ ..... (1)

Since posts are 1400 feet apart, so

$2c = 1400 \Rightarrow c = 700$

since distance = (Velocity) (time)

So $|PF| = (1080)\,t$ & $|PF'| = (1080)(t-1)$

Putting these values in (1)

$$(1080)t - (1080)(t-1) = 2a$$

$$1080t - 1080\,t + 1080 = 2a$$

$$a = \frac{1080}{2} = 540$$

For hyperbola, we know that $c^2 = a^2 + b^2$

$$(700)^2 = (540)^2 + b^2$$

$$b^2 = 490000 - 291600$$

$$b^2 = 198400$$

Required equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{x^2}{291600} - \frac{y^2}{198400} = 1 \qquad \text{Ans}$$

Tangents to Conics

(I) Parabola

Equations of Tangent in different forms

(i) Point form:

The equation of the tangent to the parabola

$y^2 = 4ax$ at the point (x, y) is $yy_1 = 2a(x + x_1)$

(ii) Slope form:

The equation of the tangent to the parabola

$y^2 = 4ax$ in the term of slope 'm' is

$$y = mx + \frac{a}{m} \qquad (\because \quad c = \frac{a}{m})$$

Note:

The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{1}{2}(x + x_1)$, y by $\frac{1}{2}(y + y_1)$ and xy by $\frac{xy_1 + yx_1}{2}$

(II) Ellipse

Equations of tangents in different forms.

(i) Point form:

Equations of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(ii) Slope form:

The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of slope m is

$$y = mx \pm \sqrt{a^2m^2 + b^2} \qquad (\because \quad c = \pm\sqrt{a^2m^2 + b^2})$$

(III) Hyperbola

Equations of tangent in different forms

(i) Point form:

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

(ii) Slope form:

The equation of tangent to hyperbola in terms of 'm' is

$$y = mx \pm \sqrt{a^2m^2 - b^2} \qquad (c = \pm\sqrt{a^2m^2 - b^2})$$

(IV) Circle

Equations of tangent in different forms

(i) Point form:

Equation of tangent to the circle at (x_1, y_1) is $xx_1 + yy_1 = a^2$

(ii) Slope form:

Equation of tangent is terms of slope 'm' is $y = mx \pm a\sqrt{1 + m^2}$

$(\therefore \quad c^2 = a^2 (1 + m^2)$

Equations of Normal

(i) Parobola $y^2 = 4ax$ is at (x_1, y_1)

$$y - y_1 = \frac{-y_1}{2a}(x - x_1)$$

(ii) Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

(iii) Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{xa^2}{x_1} + \frac{yb^2}{y_1} = a^2 + b^2$$

EXERCISE 6.7

Q.1: Find equations of tangent and normal to each of the following at the indicated point.

(i) $y^2 = 4ax$ at $(at^2, 2at)$

Solution:

Equation of tangent at $(at^2, 2at)$ is

$$yy_1 = 2a(x + x_1)$$
$$y(2at) = 2a(x + at^2)$$
$$2ayt = 2ax + 2a^2t^2$$
$$2ayt = 2a(x + at^2)$$
$$yt = x + at^2$$

And equation of normal at $(at^2, 2at)$ is

$$y - y_1 = \frac{-y_1}{2a}(x - x_1)$$

$$y - 2at = \frac{-2at}{2a}(x - at^2)$$

$y - 2at = -tx + at^3$

$tx + y - 2at - at^3 = 0$

(ii) $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ **at (a cos θ, b sin θ)**

Solution:

Equation of tangent $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

$$\frac{x(a\cos\theta)}{a^2} + \frac{y(b\sin\theta)}{b^2} = 1$$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

and equation of normal

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

$$\frac{a^2x}{a\cos\theta} - \frac{b^2y}{b\sin\theta} = a^2 - b^2$$

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad ax\sec\theta - by\,\mathrm{cosec}\theta = a^2 - b^2$$

(iii) $\mathbf{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ **at (a secθ, b tanθ)**

Solution:

Equation of tangent

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

$$\frac{x\,a\sec\theta}{a^2} - \frac{y\,b\tan\theta}{b^2} = 1$$

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

And equation of normal

$$\frac{xa^2}{x_1} + \frac{yb^2}{y_1} = a^2 + b^2$$

$$\frac{xa^2}{a\sec\theta} + \frac{yb^2}{b\tan\theta} = a^2 + b^2$$

$$\frac{xa}{\sec\theta} + \frac{yb}{\tan\theta} = a^2 + b^2$$

OR $x\,a\cos\theta + y\,b\cot\theta = a^2 + b^2$

Q.2: Write equation of the tangent to the given conic at the indicated point

(i) $3x^2 = -16y$ at the points whose ordinate is -3

Solution:

$3x^2 = -16y$ ..... (1)

Put $y = -3$ in

$3x^2 = -16(-3)$

$3x^2 = 48 \Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

Hence points are

$(4, -3)$ & $(-4, -3)$

Now diff. (1) w.r.t 'x'

$$6x = -16\frac{dy}{dx}$$

$$\frac{6x}{-16} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-3}{8}x$$

$$m = \text{Slope} = \frac{dy}{dx}\Big|_{(4,-3)} = \frac{-3}{8}(4) = \frac{-3}{2}$$

$$\text{Also } m = \frac{dy}{dx}\Big|_{(-4,-3)} = \frac{-3}{8}(4) = \frac{3}{2}$$

$\therefore$ Equation of tangent at $(4, -3)$ is

$y - y_1 = m(x - x_1)$

$y + 3 = \frac{-3}{2}(x - 4)$

$2y + 6 = -3x + 12$

$3x + 2y = 6$

$3x + 2y - 6 = 0$

Equation of tangent at $(-4, -3)$ is

$y - y_1 = m(x - x_1)$

$y + 3 = \frac{+3}{2}(x + 4)$

$2y + 6 = 3x + 12$

$3x - 2y = -6$

$3x - 2y + 6 = 0$

(ii) $3x^2 - 7y^2 = 20$ at points where $y = -1$.

Solution:

$3x^2 - 7y^2 = 20$ ..... (1)

Put $y = -1$ in (1)

$3x^2 - 7(-1)^2 = 20$

$3x^2 = 20 + 7$

$3x^2 = 27 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

Thus the required points on the conic are $(3, -1)$ & $(-3, -1)$

Now diff (1) w.r.t. 'x' we have

$$6x - 14y\frac{dy}{dx} = 0$$

$$14\frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{6x}{14y} = \frac{3x}{7y}$$

Now $m = \text{Slope} = \frac{dy}{dx}\Big|_{(3,-1)} = \frac{9}{-7}$ Also $m = \frac{dy}{dx}\Big|_{(-3,-1)} = \frac{9}{7}$

Therefore equation of tangent at (3, – 1) is

$$y - y_1 = m(x - x_1)$$
$$y + 1 = \frac{-9}{7}(x - 3)$$
$$7y + 7 = -9x + 27$$
$$9x + 7y = 20$$
$$9x + 7y - 20 = 0 \quad \text{Ans}$$

Equation of tangent at (– 3, –1)

$$y - y_1 = m(x - x_1)$$
$$y + 1 = \frac{9}{7}(x + 3)$$
$$7y + 7 = 9x + 27$$
$$9x - 7y = -20$$
$$9x - 7y + 20 = 0 \quad \text{Ans}$$

(iii) $3x^2 - 7y^2 + 2x - y - 48 = 0$, at point where $x = 4$

Solution:

$$3x^2 - 7y^2 + 2x - y - 48 = 0 \qquad \text{..... (1)}$$

Put $x = 4$ in (1)

$$3(4)^2 - 7y^2 + 2(4) - y - 48 = 0$$
$$48 - 7y^2 + 8 - y - 48 = 0$$
$$\Rightarrow \quad -7y^2 - y + 8 = 0 \quad \Rightarrow \quad 7y^2 + y - 8 = 0$$
$$7y^2 + 8y - 7y - 8 = 0$$
$$y(7y + 8) - 1(7y + 8) = 0$$
$$(7y + 8)(y - 1) = 0$$

Either

$$7y + 8 = 0 \quad , \quad y - 1 = 0$$
$$y = \frac{-8}{7} \quad , \quad y = 1$$

Therefore, required points on the conic are $(4, \frac{-8}{7})$ & (4, 1)

Now diff. (1) w.r.t. 'x' $\quad 6x - 14y\frac{dy}{dx} + 2 - \frac{dy}{dx} = 0$

$$(-14y - 1)\frac{dy}{dx} = -6x - 2$$

$$\frac{dy}{dx} = \frac{6x+2}{14y+1}$$

$$m = \frac{dy}{dx}\Big|_{(4,1)} = \frac{6(4)+2}{14(1)+1} = \frac{26}{15} \quad \text{Also } m = \frac{dy}{dx}\Big|_{(4,-\frac{8}{7})} = \frac{6(4)+2}{14\left(\frac{-8}{7}\right)+1} = \frac{26}{-15}$$

Equation of tangent at (4, 1) is

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{26}{15}(x - 4)$

$15y - 15 = 26x - 104$

$26x - 15y - 89 = 0$ Ans

Equation of tangent at $(4, \frac{-8}{7})$ is

$y - y_1 = m(x - x_1)$

$y + \frac{8}{7} = \frac{-26}{15}(x - 4)$

$105y - 120 = -182x + 728$

$182x + 105y - 608 = 0$ Ans

Q.3: Find equations of the tangents to each of the following through the given point

(i) $x^2 + y^2 = 25$, through (7, – 1)

Solution:

$x^2 + y^2 = 25 \Rightarrow r = 5$

We know that condition of tangency for the circle is

$c^2 = r^2(1 + m^2)$

$c^2 = 25(1 + m^2)$

$\Rightarrow c = \pm 5\sqrt{1 + m^2}$

Let the required equation of tangent be

$y = mx + c$ .... (1) Putting value of C in (1)

$y = mx \pm 5\sqrt{1 + m^2}$ .... (2)

Since tangent line passes through point (7, – 1), therefore

$-1 = 7m \pm 5\sqrt{1 + m^2}$

$\pm 5\sqrt{1 + m^2} = 7m + 1$ Squaring

$25(1 + m^2) = (7m + 1)^2$

$25 + 25m^2 = 49m^2 + 1 + 14m$

$-24m^2 - 14m + 24 = 0$

$12m^2 + 7m - 12 = 0$

$12m^2 + 16m - 9m - 12 = 0$

$4m(3m + 4) - 3(3m + 4) = 0$

$(3m + 4)(4m - 3) = 0$

$m = \frac{-4}{3}$ $m = \frac{3}{4}$

with $m = \frac{-4}{3}$ ..... (2) becomes

$$y = -\frac{4}{3}x \pm 5\sqrt{1+\frac{16}{9}}$$

$$= -\frac{4}{3}x \pm 5\ \frac{5}{3}$$

$$3y = -4x \pm 25$$

$$4x + 3y \pm 25 = 0$$

with $m = \frac{3}{4}$ ..... (2) becomes

$$y = \frac{3x}{4} \pm 5\sqrt{1+\frac{9}{16}}$$

$$= \frac{3x}{4} \pm \frac{25}{4}$$

$$4y = 3x \pm 25$$

$$3x - 4y \pm 25 = 0$$

(ii) $y^2 = 12x$ through (1, 4)

Solution:

$$y^2 = 12x$$

As standard form is

$$y^2 = 4ax$$

$$4a = 12 \quad \Rightarrow \quad a = 3$$

Let $y = mx + c$ ..... (1) be the required equation of tangent. For Parabola we know that condition of tangency is $c = \frac{a}{m} = \frac{3}{m}$ put in (1)

$$y = mx + \frac{3}{m} \text{ ..... (2)}$$

Since tangent line passes through point (1, 4)

So (2) becomes

$$4 = m + \frac{3}{m} \quad \Rightarrow \quad 4m = m^2 + 3$$

$$m^2 - 4m + 3 = 0$$

$$(m-1)(m-3) = 0$$

$m = 1$, $m = 3$ Put in (1)

$y = x + 3$ & $y = 3x + \frac{3}{m}$

$x - y + 3 = 0$ $y = 3x + \frac{3}{3}$

$y = 3x + 1$

$3x - y + 1 = 0$ Ans

(iii) $x^2 - 2y^2 = 2$ through (1, – 2)

Solution:

$$x^2 - 2y^2 = 2$$

$$\frac{x^2}{2} - \frac{y^2}{1} = 1$$

$\Rightarrow$ $a^2 = 2$, $b^2 = 1$

For hyperbola, we know that condition of tangent is

$c^2 = a^2m^2 - b^2$

$\Rightarrow \quad c^2 = 2m^2 - 1 \quad \Rightarrow \quad c = \pm\sqrt{2m^2 - 1}$

Let $y = mx + c$ be tangent to the given hyperbola then $y = mx \pm \sqrt{2m^2 - 1}$ (1)

Since (1) passes through (1, – 2) (1) becomes

$-2 = m \pm \sqrt{2m^2 - 1}$

$-2 - m = \pm\sqrt{2m^2 - 1}$ Squaring

$4 + m^2 + 4m = 2m^2 - 1$

$2m^2 - 1 - m^2 - 4m - 4 = 0$

$m^2 - 4m - 5 = 0$

$\Rightarrow \quad (m - 5)(m + 1) = 0$

$\Rightarrow \quad m = 5, \quad m = -1$

Putting values of m in (1) we get

$y = 5x \pm \sqrt{2(25) - 1}$, $y = -x \pm \sqrt{2 - 1}$

$y = 5x \pm \sqrt{49}$, $= -x \pm 1$

$y = 5x \pm 7$, $y + x \pm 1 = 0$

$5x - y \pm 7 = 0$ Ans

Q.4: Find equations of normal to the Parabola $y^2 = 8x$, which are parallel to the line $2x + 3y = 10$.

Solution:

$y^2 = 8x$ ..... (1)

Diff. (1) w.r.t. 'x'

$2y\dfrac{dy}{dx} = 8$

$\dfrac{dy}{dx} = \dfrac{8}{2y} = \dfrac{4}{y}$

$m_1 = \dfrac{dy}{dx} = \dfrac{4}{y}$

$m_1 = \text{Slope of normal} = \dfrac{-y}{4}$

$2x + 3y = 10$ ..... (2)

$m_2 = \text{Slope of line}$

$= \dfrac{-\text{coeff of } x}{\text{coeff of } y}$

$= -\dfrac{2}{3}$

Since normal and given line are Parallel

$m_1 = m_2$

$\dfrac{-y}{4} = \dfrac{-2}{3} \quad \Rightarrow \quad y = \dfrac{8}{3}$ Put in (1)

$\left(\dfrac{8}{3}\right)^2 = 8x$

$$\frac{64}{9 \times 8} = x \quad \Rightarrow \quad x = \frac{8}{9}$$

Required point $\left(\frac{8}{9}, \frac{8}{3}\right)$

with $y = \frac{8}{3}$, m_1 become

$$m_1 = -\frac{8}{3} \times \frac{1}{4} = \frac{-2}{3}$$

Required equation of normal at $\left(\frac{8}{9}, \frac{8}{3}\right)$ is

$$y - y_1 = m(x - x_1)$$

$$y - \frac{8}{3} = \frac{-2}{3}\left(x - \frac{8}{9}\right)$$

$$3y - 8 = -2\left(\frac{9x - 8}{9}\right)$$

$$27y - 72 = -18x + 16$$

$$18x + 27y - 88 = 0$$

Q.5: Find equations of tangents to the ellipse $\frac{x^2}{4} + y^2 = 1$, which are parallel to the line $2x - 4y + 5 = 0$.

Solution:

$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \qquad 2x - 4y + 5 = 0$$

$$\Rightarrow \quad a^2 = 4, \quad b^2 = 1 \qquad m = \frac{-\text{coeff of } x}{\text{coeff of } y} = \frac{-2}{-4} = \frac{1}{2}$$

We know that condition of tangent for ellipse is

$$c^2 = a^2m^2 + b^2$$

$$c^2 = 4m^2 + 1$$

$$c = \pm\sqrt{4m^2 + 1}$$

Since tangent is parallel to line $2x - 4y + 5 = 0$

$\therefore$ Slope is also $m = \frac{1}{2}$

$$c = \pm\sqrt{4\frac{1}{4} + 1} = \pm\sqrt{2}$$

Let the equation of required tangent by

$$y = mx + c$$

$y = \frac{1}{2}x \pm \sqrt{2}$

$2y = x \pm 2\sqrt{2}$

$x - 2y \pm 2\sqrt{2} = 0$ Ans

Q.6: Find equations of the tangents to the conics $9x^2 - 4y^2 = 36$ Parallel to $5x - 2y + 7 = 0$.

Solution:

$9x^2 - 4y^2 = 36$

$\frac{x^2}{4} - \frac{y^2}{9} = 1$ (Dividing by 36)

=> $a^2 = 4$, $b^2 = 9$

$5x - 2y + 7 = 0$

$m =$ slope of line $= \frac{5}{2}$

For hyperbola, we know that

$c^2 = a^2m^2 - b^2$

$c^2 = 4m^2 - 9$

Since tangent and given line are parallel so their slopes are same. Thus $m = \frac{5}{2}$

$c^2 = 4\left(\frac{25}{4}\right) - 9$ $c^2 = 16$ => $c = \pm 4$

Let $y = mx + c$ be the required equation of the tangent then $y = \frac{5}{2}x \pm 4$

$2y = 5x \pm 8$

$5x - 2y \pm 8 = 0$ Ans.

Q.7: Find equations of common tangents to the given conics.

(i) $x^2 = 80y$ & $x^2 + y^2 = 81$

Solution:

$x^2 = 80y$ .... (1) $x^2 + y^2 = 81$ ..... (2)

Let $y = mx + c$ (3) be the required common tangent. Let a be radius of circle then (2) becomes $a^2 = 81$ Put in (1)

$x^2 = 80(mx + c)$

$x^2 - 80mx - 80c = 0$

For equal roots, we know that Disc = 0

$b^2 - 4ac = 0$

$(-80m)^2 - 4(1)(-80c) = 0$

$80(80m^2 + 4c) = 0$

$80\,m^2 + 4c = 0 \qquad c = -20m^2$

Condition of tangency for circle is $c^2 = a^2(1 + m^2)$ (4)

$(-20m^2)^2 = 81(1 + m^2)$

$400\,m^4 = 81 + 81m^2$

$400\,m^4 - 81m^2 - 81 = 0$

By Quadratic Formula

$$m^2 = \frac{-(-81) \pm \sqrt{(-81)^2 - 4(400)(-81)}}{2(400)}$$

$$= \frac{81 \pm \sqrt{136161}}{800} = \frac{9}{16}$$

$$m = \pm\frac{3}{4}$$

$$\therefore \quad c = -20\left(\frac{9}{16}\right) = \frac{-45}{4}$$

Putting values of m & c in y = mx + c

$$y = \pm\frac{3}{4}x - \frac{45}{4}$$

$4y = \pm 3x - 45$

$\pm 3x - 4y - 45 = 0$ Ans.

(ii) $y^2 = 16x$ & $x^2 = 2y$

Solution:

$y^2 = 16x$ .... (1) $\qquad x^2 = 2y$ .... (2)

$y^2 = 4ax$

$4a = 16$

$\boxed{a = 4}$

We know that condition of tangency for Parabola is $c = \frac{a}{m}$

$$c = \frac{4}{m}$$

Let $y = mx + c$ ..... (3) be required tangent

then $y = mx + \frac{4}{m}$ Putting value of y in (2)

$x^2 = 2(mx + \frac{4}{m}) \Rightarrow mx^2 = 2m^2x + 8$

$mx^2 - 2m^2x - 8 = 0$

For equal roots, we know that Disc = 0

i.e; $b^2 - 4ac = 0$

$(-2m^2)^2 - 4(m)(-8) = 0$

$4m^4 + 32m = 0$

$4m(m^3 + 8) = 0$

$m = 0 \quad , \quad m^3 = -8 \quad , \quad m = -2$

Equation of tangent is

$y = mx + c$

$y = -2x + \frac{4}{-2}$

$y = -2x - 2$

$\boxed{2x + y + 2 = 0}$ Ans.

Q.8: Find the points of intersection of the given conics.

(i) $\frac{x^2}{18} + \frac{y^2}{8} = 1 \quad \& \quad \frac{x^2}{3} - \frac{y^2}{3} = 1$

Solution:

$\frac{x^2}{18} + \frac{y^2}{8} = 1 \quad \& \quad \frac{x^2}{3} - \frac{y^2}{3} = 1$

$8x^2 + 18y^2 = 144 \qquad x^2 - y^2 = 3$ ..... (2)

$4x^2 + 9y^2 = 72$ ... (1) (Dividing by 2)

Multiplying Eq. (2) by 9 & add in (1)

$$\begin{array}{l} 9x^2 - 9y^2 = 27 \\ \underline{4x^2 + 9y^2 = 72} \\ 13x^2 = 99 \end{array}$$

$x^2 = \frac{99}{13} \quad \Rightarrow \quad x = \pm\sqrt{\frac{99}{13}}$

Put in (2)

$\frac{99}{13} - y^2 = 3$

$\frac{99}{13} - 3 = y^2$

$\frac{99 - 39}{13} = y^2$

$y^2 = \frac{60}{13} \quad \Rightarrow \quad y = \pm\sqrt{\frac{60}{13}}$

Points of intersection are $\left(\pm\sqrt{\frac{99}{13}}\ ,\ \pm\sqrt{\frac{60}{13}}\right)$ Ans.

(ii) $x^2 + y^2 = 8$ & $x^2 - y^2 = 1$

Solution:

$x^2 + y^2 = 8$ .... (1) $\quad x^2 - y^2 = 1$ .... (2)

Adding (1) & (2)

$$\begin{aligned} x^2 + y^2 &= 8 \\ x^2 - y^2 &= 1 \\ \hline 2x^2 &= 9 \end{aligned}$$

$\Rightarrow x^2 = \frac{9}{2} \Rightarrow x = \pm\frac{3}{\sqrt{2}}$

Put in (1) $\frac{9}{2} + y^2 = 8$

$y^2 = 8 - \frac{9}{2}$

$y^2 = \frac{16-9}{2} = \frac{7}{2}$

$y = \pm\sqrt{\frac{7}{2}}$

Hence points of intersection are $\left(\pm\frac{3}{\sqrt{2}}, \pm\sqrt{\frac{7}{2}}\right)$ Ans

(iii) $3x^2 - 4y^2 = 12$ & $3y^2 - 2x^2 = 7$

Solution:

$3x^2 - 4y^2 = 12$ ..... (1)

$3y^2 - 2x^2 = 7$ ..... (2)

Multiplying equation (1) by (2) & (2) by 3 and adding

$$\begin{aligned} 6x^2 - 8y^2 &= 24 \\ -6x^2 + 9y^2 &= 21 \\ \hline y^2 &= 45 \end{aligned}$$

$\Rightarrow y = \pm\sqrt{45}$

Put in (2)

$-2x^2 + 3(45) = 7$

$-2x^2 + 135 = 7$

$135 - 7 = 2x^2$

$128 = 2x^2$

$x^2 = 64 \Rightarrow x = \pm 8$

Hence points of intersection are

$(\pm 8\ ,\ \pm\sqrt{45})$ Ans.

(iv) $\mathbf{3x^2 + 5y^2 = 60}$ **and** $\mathbf{9x^2 + y^2 = 124}$

Solution:

$3x^2 + 5y^2 = 60$ ..... (1) $\qquad 9x^2 + y^2 = 124$ ..... (2)

Multiplying (1) by (3) & Subtracting from (2)

$$\begin{array}{rcl} 9x^2 + y^2 & = & 124 \\ -9x^2 \pm 15y^2 & = & -180 \\ \hline -14y^2 & = & -56 \end{array}$$

$y^2 = 4 \qquad \Rightarrow \qquad y = \pm 2$

Put in (1)

$9x^2 + 4 = 124$

$9x^2 = 120$

$x^2 = \frac{120}{9} = \frac{40}{3} \qquad x = \pm\sqrt{\frac{40}{3}}$

Hence points of intersection are $\left(\pm\sqrt{\frac{40}{3}}, \pm 2\right)$

EXERCISE 6.8

1. Find an equation of each of the following with respect to new parallel axes obtained by shifting the origin to the indicated points.

(i) $x^2 + 16y - 16 = 0,\ O'(0, 1)$

Solution. $x^2 + 16y - 16 = 0 \quad \ldots (1), \quad O'(0, 1) = (h, k)$

Equations of transformation are

$$x = X + h = X + 0 = X, \quad y = Y + k = Y + 1$$

Substituting these values of x, y into (1), we have

$$X^2 + 16(Y + 1) - 16 = 0$$

or $\quad X^2 + 16Y + 16 - 16 = 0 \quad \Rightarrow \quad X^2 + 16Y = 0$

is the required transformed equation.

(ii) $4x^2 + y^2 + 16x - 10y + 37 = 0,\ O'(-2, 5)$

Solution. $4x^2 + y^2 + 16x - 10y + 37 = 0 \quad \ldots (1), \quad O'(-2, 5) = (h, k)$

Equations of transformation are

$$x = X + h = X - 2, \quad y = Y + k = Y + 5$$

Substituting these values of x, y into (1), we have

$$4(X-2)^2 + (Y+5)^2 + 16(X-2) - 10(Y+5) + 37 = 0$$

or $\quad 4(X^2 - 4X + 4) + (Y^2 + 10Y + 25) + 16(X-2) - 10(Y+5) + 37 = 0$

or $\quad 4X^2 - 16X + 16 + Y^2 + 10Y + 25 + 16X - 32 - 10Y - 50 + 37 = 0$

or $\quad 4X^2 + Y^2 - 4 = 0$ is the required transformed equation.

(iii) $\quad 9x^2 + 4y^2 + 18x - 16y - 11 = 0,\ O'(-1, 2)$

Solution. $9x^2 + 4y^2 + 18x - 16y - 11 = 0 \quad \ldots (1), \qquad O'(-1, 2) = (h, k)$

Equations of transformation are

$$x = X + h = X - 1, \quad y = Y + k = Y + 2$$

Putting these values of x and y into (1), we have

$$9(X-1)^2 + 4(Y+2)^2 + 18(X-1) - 16(Y+2) - 11 = 0$$

or $\quad 9(X^2 - 2X + 1) + 4(Y^2 + 4Y + 4) + 18(X-1) - 16(Y+2) - 11 = 0$

or $\quad 9X^2 - 18X + 9 + 4Y^2 + 16Y + 16 + 18X - 18 - 16Y - 32 - 11 = 0$

or $\quad 9X^2 + 4Y^2 - 36 = 0$ is the required transformed equation.

(iv) $\quad x^2 - y^2 + 4x + 8y - 11 = 0,\ O'(-2, 4)$

Solution. $x^2 - y^2 + 4x + 8y - 11 = 0 \quad \ldots (1), \qquad O'(-2, 4) = (h, k)$

Equations of transformation are

$$x = X + h = X - 2, \quad y = Y + k = Y + 4$$

Substituting these values of x and y into (1), we have

$$(X-2)^2 - (Y+4)^2 + 4(X-2) + 8(Y+4) - 11 = 0$$

or $\quad X^2 - 4X + 4 - (Y^2 + 8Y + 16) + 4X - 8 + 8Y + 32 - 11 = 0$

or $\quad X^2 - 4X + 4 - Y^2 - 8Y - 16 + 4X - 8 + 8Y + 32 - 11 = 0$

or $\quad X^2 - Y^2 + 1 = 0$ is the required transformed equation.

(v) $\quad 9x^2 - 4y^2 + 36x + 8y - 4 = 0,\ O'(-2, 1)$

Solution. $9x^2 - 4y^2 + 36x + 8y - 4 = 0 \quad \ldots (1) \quad O'(-2, 1) = (h, k)$

Equations of transformation are

$$x = X + h = X - 2, \quad y = Y + k = Y + 1$$

Substituting these values of x and y into (1), we have

$$9(X-2)^2 - 4(Y+1)^2 + 36(X-2) + 8(Y+1) - 4 = 0$$

or $\quad 9(X^2 - 4X + 4) - 4(Y^2 + 2Y + 1) + 36(X-2) + 8(Y+1) - 4 = 0$

or $\quad 9X^2 - 36X + 36 - 4Y^2 - 8Y - 4 + 36X - 72 + 8Y + 8 - 4 = 0$

or $9X^2 - 4Y^2 - 36 = 0$ is the required transformed equation.

2. Find coordinates of the new origin (axes remaining parallel) so that first degree terms are removed from the transformed equation of each of the following. Also find the transformed equation.

(i) $3x^2 - 2y^2 + 24x + 12y + 24 = 0$

Solution. $3x^2 - 2y^2 + 24x + 12y + 24 = 0$ ... (1)

Let the coordinates of the new origin be $O'(h, k)$.

Equations of transformation are

$$x = X + h, \quad y = Y + k$$

Substituting these values of x, y into (1), we get

$$3(X+h)^2 - 2(Y+k)^2 + 24(X+h) + 12(Y+k) + 24 = 0$$

or $3(X^2 + 2Xh + h^2) - 2(Y^2 + 2kY + k^2) + 24(X+h) + 12(Y+k) + 24 = 0$

or $3X^2 + 6hX + 3h^2 - 2Y^2 - 4kY - 2k^2 + 24X + 24h + 12Y + 12k + 24 = 0$

or $3X^2 - 2Y^2 + X(6h + 24) + Y(-4k + 12) + 3h^2 + 24h - 2k^2 + 12k + 24 = 0$ ...(2)

To remove first degree terms, we have

$6h + 24 = 0 \Rightarrow h = -4$

and $-4k + 12 = 0 \Rightarrow k = 3$

New origin is $O'(-4, 3)$. Putting $h = -4$ and $k = 3$ into (2), the transformed equation is

$$3X^2 - 2Y^2 + X[6(-4) + 24] + Y[-4(3) + 12] + 3(-4)^2 + 24(-4) - 2(3)^2 + 12(3) + 24 = 0$$

or $3X^2 - 2Y^2 + X(-24 + 24) + Y(-12 + 12) + 3(16) - 96 - 2(9) + 36 + 24 = 0$

or $3X^2 - 2Y^2 + X(0) + Y(0) + 48 - 96 - 18 + 36 + 24 = 0$

or $3X^2 - 2Y^2 - 6 = 0$

which is the required transformed equation with new origin at $(-4, 3)$

(ii) $25x^2 + 9y^2 + 50x - 36y - 164 = 0$

Solution. $25x^2 + 9y^2 + 50x - 36y - 164 = 0$ ... (1)

Let the coordinates of the new origin be $O'(h, k)$. Equations of transformed are

$$x = X + h, \quad y = Y + k$$

Substituting these values of x, y into (1), we get

$$25(X+h)^2 + 9(Y+k)^2 + 50(X+h) - 36(Y+k) - 164 = 0$$

or $\quad 25(X^2 + 2Xh + h^2) + 9(Y^2 + 2Yk + k^2) + 50X + 50h - 36Y - 36k - 164 = 0$

or $\quad 25X^2 + 50Xh + 25h^2 + 9Y^2 + 18Yk + 9k^2 + 50X + 50h - 36Y - 36k - 164 = 0$

or $25X^2 + 9Y^2 + (50h + 50)X + (18k - 36)Y + 25h^2 + 9k^2 + 50h - 36k - 164 = 0$ ...(2)

To reove the first degree terms, we put

$$50h + 50 = 0 \Rightarrow 50h = -50 \Rightarrow h = -1$$

and $\quad 18k - 36 = 0 \Rightarrow 18k = 36 \Rightarrow k = 2$

$\therefore$ New origin is $O'(-1, 2)$.

Putting $h = -1$, $k = 2$ into (2), the transformed equation is

$$25X^2 + 9Y^2 + (-50 + 50)X + (36 - 36)Y + 25(-1)^2 + 9(2)^2 + 50(-1) - 36(2) - 164 = 0$$

or $\quad 25X^2 + 9Y^2 + (0)X + (0)Y + 25(1) + 9(4) - 50 - 72 - 164 = 0$

or $\quad 25X^2 + 9Y^2 + 25 + 36 - 50 - 72 - 164 = 0$

or $\quad 25X^2 + 9Y^2 - 225 = 0$

which is the required equation with new origin at $(-1, 2)$.

(iii) $\quad x^2 - y^2 - 6x + 2y + 7 = 0$

Solution. $x^2 - y^2 - 6x + 2y + 7 = 0 \quad \ldots (1)$

Let the coordinates of the new origin be $O'(h, k)$. Equations of transformation are

$$x = X + h, \quad y = Y + k$$

Substituting these values of x, y into (1), we get

$$(X+h)^2 - (Y+k)^2 - 6(X+h) + 2(Y+k) + 7 = 0$$

or $\quad X^2 + 2Xh + h^2 - (Y^2 + 2Yk + k^2) - 6X - 6h + 2Y + 2k + 7 = 0$

or $\quad X^2 - Y^2 + X(2h - 6) + Y(-2k + 2) + h^2 - k^2 - 6h + 2k + 7 = 0 \quad \ldots (2)$

To remoe the first gdegree terms, we have

$$2h - 6 = 0 \Rightarrow h = 3$$

and $\quad -2k + 2 = 0 \Rightarrow k = 1$

New origin is $O'(3, 1)$. Putting $h = 3, k = 1$ into (2) the transformed equation is

$$X^2 - Y^2 + X(0) + Y(0) + (3)^2 - (1)^2 - 6(3) + 2(1) + 7 = 0$$

or $X^2 - Y^2 + 9 - 1 - 18 + 2 + 7 = 0$

or $X^2 - Y^2 - 1 = 0$

which is the required equation with new origin at (3, 1).

3. In each of the following, find an equation referred to the new axes obtained by rotation of axes about the origin through the given angle:

(i) $xy = 1, \ \theta = 45^\circ$

Solution. Here $\theta = 45^\circ$. Equations of transformation are

$$x = X\cos 45^\circ - Y\sin 45^\circ = X \cdot \frac{1}{\sqrt{2}} - Y \cdot \frac{1}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}}$$

$$y = X\sin 45^\circ + Y\cos 45^\circ = X \cdot \frac{1}{\sqrt{2}} + Y \cdot \frac{1}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}}$$

Substituting these values for x, y into the given equation, we have

$$\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) = 1$$

or $\dfrac{X^2 - Y^2}{2} = 1$ or $X^2 - Y^2 = 2$

is the required transformed equation

(ii) $7x^2 - 8xy + y^2 - 9 = 0, \ \theta = \tan^{-1} 2$

Solution. Here $\theta = \tan^{-1} 2 \Rightarrow \tan\theta = 2 = \dfrac{2}{1}$

$\Rightarrow$ base $= 2, \perp = 1$ and hypotenuse $= \sqrt{4+1} = \sqrt{5}$

$\therefore \quad \sin\theta = \dfrac{2}{\sqrt{5}}$ and $\cos\theta = \dfrac{1}{\sqrt{5}}$

Equations of transformation are

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned}\right\}$$

Using the above values

$$x = X \cdot \frac{1}{\sqrt{5}} - Y \cdot \frac{2}{\sqrt{5}} = \frac{X - 2Y}{\sqrt{5}}$$

$$y = X \cdot \frac{2}{\sqrt{5}} + Y \cdot \frac{1}{\sqrt{5}} = \frac{2X + Y}{\sqrt{5}}$$

Substituting these values for x, y into the given equation, we have

$$7\left(\frac{X-2Y}{\sqrt{5}}\right)^2 - 8\left(\frac{X-2Y}{\sqrt{5}}\right)\left(\frac{2X+Y}{\sqrt{5}}\right) + \left(\frac{2X+Y}{\sqrt{5}}\right)^2 - 9 = 0$$

By multiplying by 5, we get

$$7\left(\frac{X^2-4XY+4Y^2}{5}\right) - 8\left(\frac{2X^2-3XY \quad 2Y^2}{5}\right) + \left(\frac{4X^2+4XY+Y^2}{5}\right) - 9 = 0$$

or $\quad 7X^2 - 28XY + 28Y^2 - 16X^2 + 24XY + 16Y^2 + 4X^2 + 4XY + Y^2 - 45 = 0$

or $\quad -5X^2 + 45Y^2 - 45 = 0$

or $\quad X^2 - 9Y^2 + 9 = 0$

which is the required transformed equation.

(iii) $\quad 9x^2 + 12xy + 4y^2 - x - y = 0 \;,\; \theta = \tan^{-1}\dfrac{2}{3}$

Solution. Here $\theta = \tan^{-1}\dfrac{2}{3} \Rightarrow \tan\theta = \dfrac{2}{3}$

$\Rightarrow$ base $= 3$, $\perp = 2$ and hypotenuse $= \sqrt{9+4} = \sqrt{13}$

$\therefore \quad \sin\theta = \dfrac{2}{\sqrt{13}}$ and $\cos\theta = \dfrac{3}{\sqrt{13}}$

Equations of transformation are

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned}\right\}$$

Putting values, we have

$$x = X\cdot\frac{3}{\sqrt{13}} - Y\cdot\frac{2}{\sqrt{13}} = \frac{3X-2Y}{\sqrt{13}}$$

$$y = X\cdot\frac{2}{\sqrt{13}} + Y\cdot\frac{3}{\sqrt{13}} = \frac{2X+3Y}{\sqrt{13}}$$

Substituting these values for x, y in to the given equations, we have

$$9\left(\frac{3X-2Y}{\sqrt{13}}\right)^2 + 12\left(\frac{3X-2Y}{\sqrt{13}}\right)\left(\frac{2X+3Y}{\sqrt{13}}\right) + 4\left(\frac{2X+3Y}{\sqrt{13}}\right)^2 - \left(\frac{3X-2Y}{\sqrt{13}}\right) - \left(\frac{3X+2Y}{\sqrt{13}}\right) = 0$$

or $\quad 9\left(\dfrac{9X^2 - 12XY + 4Y^2}{13}\right) + 12\left(\dfrac{6X^2 + 5XY - 6Y^2}{13}\right)$

$$+ 4\left(\frac{4X^2 + 12XY + 9Y^2}{13}\right) - \left(\frac{3X - 2Y}{\sqrt{13}}\right) - \left(\frac{2X + 3Y}{\sqrt{13}}\right) = 0$$

or $\quad 81X^2 - 108XY + 36Y^2 + 72X^2 + 60XY - 72Y^2 + 16X^2 + 48XY + 36Y^2$

$$- 3\sqrt{13}\,X + 2\sqrt{13}\,Y - 2\sqrt{13}\,X - 3\sqrt{13}\,Y = 0$$

or $\quad 169X^2 - 5\sqrt{13}\,X - \sqrt{13}\,Y = 0$

or $\quad 13\sqrt{13}\,X^2 - 5X - Y = 0$

is the required transformed equation

(iv) $\quad x^2 - xy + y^2 - 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0,\ \theta = 45°$

Solution. Here $\theta = 45°$

Equations of transformation are

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned}\right\}$$

Putting values, ewe have

$$x = X\cos 45° - Y\sin 45° = X\cdot\frac{1}{\sqrt{2}} - Y\cdot\frac{1}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}}$$

$$y = X\sin 45° + Y\cos 45° = X\cdot\frac{1}{\sqrt{2}} + Y\cdot\frac{1}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}}$$

Substituting these expressions for x, y into the given equation, we have

$$\left(\frac{X - Y}{\sqrt{2}}\right)^2 - 2\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) + \left(\frac{X + Y}{\sqrt{2}}\right)^2 - 2\sqrt{2}\left(\frac{X - Y}{\sqrt{2}}\right)$$

$$- 2\sqrt{2}\left(\frac{X + Y}{\sqrt{2}}\right) + 2 = 0$$

or $$\left(\frac{X^2 - 2XY + Y^2}{2}\right) - 2\left(\frac{X^2 - Y^2}{2}\right) + \left(\frac{X^2 + 2XY + Y^2}{2}\right)$$

$$- 2(X - Y) - 2(X + Y) + 2 = 0$$

(by $\times 2$): $X^2 - 2XY + Y^2 - 2X^2 + 2Y^2 + X^2 + 2XY + Y^2 - 4X + 4Y - 4X - 4Y + 4 = 0$

or $\quad 4Y^2 - 8X + 4 = 0$ or $Y^2 - 2X + 1 = 0$

which is the required transformed equation.

4. **Find measure of the angle through which the axes be rotated so that the product term XY is removed from the transformed equation. Also find the transformed equation.**

(i) $2x^2 + 6xy + 10y^2 - 11 = 0$

Solution. Let the axes be rotated through an angle θ.

Equations of transformation are

$$x = X\cos\theta - Y\sin\theta \quad , \quad y = X\sin\theta + Y\cos\theta$$

Substituting into the given equation, we get

$$2(X\cos\theta - Y\sin\theta)^2 + 6(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$$
$$+ 10(X\sin\theta + Y\cos\theta)^2 - 11 = 0$$

or $$2(X^2\cos^2\theta - 2XY\sin\theta\cos\theta + Y^2\sin^2\theta)$$
$$+ 6(X^2\sin\theta\cos\theta - Y^2\sin\theta\cos\theta + XY\cos^2\theta - XY\sin^2\theta)$$
$$+ 10(X^2\sin^2\theta + 2XY\sin\theta\cos\theta + Y^2\cos^2\theta) - 11 = 0$$

or $$X^2(2\cos^2\theta + 6\sin\theta\cos\theta + 10\sin^2\theta)$$
$$+ XY(-4\sin\theta\cos\theta + 6\cos^2\theta - 6\sin^2\theta + 20\sin\theta\cos\theta)$$
$$+ Y^2(2\sin^2\theta - 6\sin\theta\cos\theta + 10\cos^2\theta) - 11 = 0 \qquad \text{(i)}$$

Since this equation is to be free from the product term, i.e.;

$$-4\sin\theta\cos\theta + 6\cos^2\theta - 6\sin^2\theta + 20\sin\theta\cos\theta = 0$$

or $$-6\sin^2\theta + 16\sin\theta\cos\theta + 6\cos^2\theta = 0$$

or $$3\tan^2\theta - 8\tan\theta - 3 = 0 \qquad \{\text{by} \div \cos^2\theta\}$$

$$\therefore \quad \tan\theta = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(-3)}}{2(3)} = \frac{8 \pm \sqrt{64+36}}{6}$$

$$= \frac{8 \pm \sqrt{100}}{6} = \frac{8 \pm 10}{6} = \frac{8+10}{6}, \frac{8-10}{6} = \frac{18}{6}, \frac{-2}{6}$$

$$= 3, \frac{-1}{3} \Rightarrow \tan\theta = 3 \text{ (as } \theta \text{ is in the first quadrant)}$$

$$\Rightarrow \text{base} = 1, \perp = 3 \text{ and hypotenuse} = \sqrt{1+9} = \sqrt{10}$$

$$\therefore \quad \sin\theta = \frac{3}{\sqrt{10}} \text{ and } \cos\theta = \frac{1}{\sqrt{10}}$$

Substituting $\sin\theta = \frac{3}{\sqrt{10}}$ and $\cos\theta = \frac{1}{\sqrt{10}}$ in (i), then

$$X^2\left(\left(2\left(\frac{1}{\sqrt{10}}\right)\right)^2 + 6\left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right) + 10\left(\frac{3}{\sqrt{10}}\right)^2\right)$$
$$+ XY\left(-4\left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right) + 6\left(\frac{1}{\sqrt{10}}\right)^2 - 6\left(\frac{3}{\sqrt{10}}\right)^2 + 20\left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right)\right)$$

$$+ Y^2\left(2\left(\frac{3}{\sqrt{10}}\right)^2 - 6\left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right) + 10\left(\frac{1}{\sqrt{10}}\right)^2\right) - 11 = 0$$

or $$X^2\left(\frac{2}{10} + \frac{18}{10} + \frac{90}{10}\right) + XY\left(-\frac{12}{10} + \frac{6}{10} - \frac{54}{10} + \frac{60}{10}\right)$$

$$+ Y^2\left(\frac{18}{10} - \frac{18}{10} + \frac{10}{10}\right) - 1 = 0$$

or $$11X^2 + Y^2 - 11 = 0$$

(ii) $$xy + 4x - 3y - 10 = 0 \qquad \ldots (1)$$

Solution. Let the axes be rotated through an angle θ.

Equation of transformation are

$$x = X\cos\theta - Y\sin\theta \; ; \; y = X\sin\theta + Y\cos\theta$$

Substituting into the given equation (1), we have

$$(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + 4(X\cos\theta - Y\sin\theta)$$
$$+ XY\cos^2\theta - XY\sin^2\theta - 3(X\sin\theta + Y\cos\theta) - 10 = 0$$

$$X^2\sin\theta\cos\theta + XY\cos^2\theta - XY\sin^2\theta - Y^2\sin\theta\cos\theta + 4X\cos\theta$$
$$- 4Y\sin\theta - 3X\sin\theta - 3Y\cos\theta - 10 = 0$$

or $$X^2\sin\theta\cos\theta + XY(\cos^2\theta - \sin^2\theta) - Y^2\sin\theta\cos\theta + X(4\cos\theta - 3\sin\theta)$$
$$- Y(4\sin\theta + 3\cos\theta) - 10 = 0 \qquad \ldots (2)$$

Since this equation is to be free from the product term XY

i.e., $$\cos^2\theta - \sin^2\theta = 0$$

or $$\cos^2\theta = \sin^2\theta \Rightarrow \tan^2\theta = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45^\circ$$

Thus axes be rotated through an angle of 45°. So that XY term is removed from the transformed equation.

Setting $\theta = 45^\circ$ in (i), the transformed equation is

$$X^2\sin 45^\circ\cos 45^\circ + XY(\cos^2 45^\circ - \sin 2\,45^\circ) - Y^2\sin 45^\circ\cos 45^\circ$$
$$+ X(4\cos 45^\circ - 3\sin 45^\circ) - Y(4\sin 45^\circ + 3\cos 45^\circ) - 10 = 0$$

$$X^2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + XY(0) - Y^2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + X\left(4 \cdot \frac{1}{\sqrt{2}} - 3 \cdot \frac{1}{\sqrt{2}}\right)$$
$$- Y\left(4 \cdot \frac{1}{\sqrt{2}} + 3 \cdot \frac{1}{\sqrt{2}}\right) - 10 = 0$$

or $$X^2 \cdot \frac{1}{2} - Y^2 \cdot \frac{1}{2} + X \cdot \frac{1}{\sqrt{2}} - Y \cdot \frac{7}{\sqrt{2}} - 10 = 0$$

or $\quad X^2 - Y^2 + \sqrt{2}\,X - 7\sqrt{2}\,Y - 20 = 0$

is the requied equation.

(iii) $\quad 5x^2 - 6xy + 5y^2 - 8 = 0 \qquad \ldots (1)$

Solution. Let the axes be rotated through an angle θ.

Equation of transformation are

$$x = X\cos\theta - Y\sin\theta \;;\; y = X\sin\theta + Y\cos\theta$$

Substituting into the given equation, we get

$$5(X\cos\theta - Y\sin\theta)^2 - 6(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$$
$$+ 5(X\sin\theta + Y\cos\theta)^2 - 8 = 0$$

or $\quad 5(X^2\cos^2\theta - 2XY\sin\theta\cos\theta + Y^2\sin^2\theta) - 6(X^2\sin\theta\cos\theta$
$$+ XY\cos^2\theta - XY\sin^2\theta - Y^2\sin\theta\cos\theta)$$
$$+ 5(X^2\sin^2\theta + 2XY\sin\theta\cos\theta + Y^2\cos^2\theta) - 8 = 0$$

or $\quad X^2(5\cos^2\theta - 6\sin\theta\cos\theta + 5\sin^2\theta) + XY(-10\sin\theta\cos\theta - 6\cos^2\theta$
$$+ 6\sin^2\theta + 10\sin\theta\cos\theta) + Y^2(5\sin^2\theta + 6\sin\theta\cos\theta + 5\cos^2\theta) - 8 = 0$$

Since this equation is to be free from the product term XY

i.e., $\quad -6\cos^2\theta - 6\sin^2\theta = 0$

or $\quad \sin^2\theta = \cos^2\theta \quad \Rightarrow \tan^2\theta = 1 \quad \Rightarrow \quad \tan\theta = 1$

$\Rightarrow \quad \theta = 45^\circ$ (as θ is in the first quadrant)

Setting $\theta = 45^\circ$ in (1), the transformed equation is

$$X^2\left(5\cdot\left(\frac{1}{\sqrt{2}}\right)^2 - 6\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + 5\cdot\left(\frac{1}{\sqrt{2}}\right)^2\right)$$
$$+ XY\left(-10\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} - 6\cdot\left(\frac{1}{\sqrt{2}}\right)^2 + 6\cdot\left(\frac{1}{\sqrt{2}}\right)^2 + 10\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\right)$$
$$+ Y^2\left(5\cdot\left(\frac{1}{\sqrt{2}}\right)^2 + 6\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + 5\cdot\left(\frac{1}{\sqrt{2}}\right)^2\right) - 8 = 0$$

or $\quad X^2\left(\frac{5}{2} - \frac{6}{2} + \frac{5}{2}\right) + XY(0) + Y^2\left(\frac{5}{2} + \frac{6}{2} + \frac{5}{2}\right) - 8 = 0$

or $\quad 2X^2 + 8Y^2 - 8 = 0 \quad \Rightarrow \quad X^2 + 4Y^2 - 4 = 0$

which is the required equation.

EXERCISE 6.9

1. By a rotation of axes, eliminate the xy-term in each of the following equations. Identify the conic and find its elements.

(i) $4x^2 - 4xy + y^2 - 6 = 0$

Solution. $4x^2 - 4xy + y^2 - 6 = 0$ ... (I)

Here $a = 4$, $b = 1$, $2h = -4$ the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-4}{4-1} = -\frac{4}{3} \Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = -\frac{4}{3}$$

$$6\tan\theta = 4\tan^2\theta - 4 \Rightarrow 4\tan^2\theta - 6\tan\theta - 4 = 0$$

$$2\tan^2\theta - 3\tan q - 2 = 0$$

$$\tan\theta = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-2)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+16}}{4} = \frac{3 \pm \sqrt{25}}{4} = \frac{3 \pm 5}{4}$$

$$= 2, -\frac{1}{2} \implies \tan\theta = 2 \text{ (as } \theta \text{ is in the first quadrant)}$$

Now $\tan\theta = 2 = \frac{2}{1} \implies$ base $= 1$, $\perp = 2$, hypotenuse $= \sqrt{4+1} = \sqrt{5}$

$\therefore \quad \sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = \frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y \\ y &= X\sin\theta + Y\cos\theta = \frac{2}{\sqrt{5}}X + \frac{1}{\sqrt{5}}Y \end{aligned}\right\} \quad \ldots (2)$$

Substituting these expressions for x and y into (1), we get

$$4\left(\frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y\right)^2 - 4\left(\frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y\right)\left(\frac{2}{\sqrt{5}}X + \frac{1}{\sqrt{5}}Y\right) + \left(\frac{2}{\sqrt{5}}X + \frac{2}{\sqrt{5}}Y\right)^2 - 6 = 0$$

$$4\left(\frac{1}{5}X^2 - \frac{4}{5}XY + \frac{4}{5}Y^2\right) - 4\left(\frac{2}{5}X^2 - \frac{3}{5}XY + \frac{2}{5}Y^2\right) + \left(\frac{4}{5}X^2 - \frac{4}{5}XY + \frac{1}{5}Y^2\right) - 6 = 0$$

$$\left(\frac{4}{5} - \frac{8}{5} + \frac{4}{5}\right)X^2 + \left(-\frac{16}{5} + \frac{12}{5} + \frac{4}{5}\right)XY + \left(\frac{16}{5} + \frac{8}{5} + \frac{1}{5}\right)Y^2 - 6 = 0$$

$$25Y^2 - 30 = 0 \implies Y^2 = \frac{6}{5} \implies Y = \pm\sqrt{\frac{6}{5}}$$

represents a pair of lines. To find their equations in xy - plane, we have

From (2), we have

$$X - 2Y = \sqrt{5}\,x \quad (3)$$

$$2X + Y = \sqrt{5}\,y \quad (4)$$

Multiplying (3) by 2, we get

$$2X - 4Y = 2\sqrt{5}\,x \quad (5)$$

Subtracting equation (5) from (6), we get

$$5Y = \sqrt{5}\,y - 2\sqrt{5}\,x \implies Y = \frac{\sqrt{5}}{5}(y - 2x) = -\frac{1}{\sqrt{5}}(2x - y)$$

$$\pm\sqrt{\frac{6}{5}} = -\frac{1}{\sqrt{5}}(2x - y) \implies \pm\sqrt{6} = -(2x - y)$$

$$2x - y \pm \sqrt{6} = 0 \implies 2x - y + \sqrt{6} = 0\ ,\ 2x - y - \sqrt{6} = 0$$

(ii) Identify : $x^2 - 2xy + y^2 - 8x - 8y = 0$

Solution. $x^2 - 2xy + y^2 - 8x - 8y = 0$ ... (1)

Here $a = 1$, $b = 1$, $2h = -2$ the angle θ though which axes be rotated is given by

$$\tan 2\theta = \frac{2h}{a - b} = \frac{-2}{1 - 1} = \infty \implies 2\theta = 90^\circ \implies \theta = 45^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X\,.\,\frac{1}{\sqrt{2}} - Y\,.\,\frac{1}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X\,.\,\frac{1}{\sqrt{2}} + Y\,.\,\frac{1}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}} \end{aligned}\right] \ldots (2)$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X - Y}{\sqrt{2}}\right)^2 - 2\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) + \left(\frac{X + Y}{\sqrt{2}}\right)^2 - 8\left(\frac{X - Y}{\sqrt{2}}\right) - 8\left(\frac{X + Y}{\sqrt{2}}\right) = 0$$

$$\frac{1}{2}(X^2 - 2XY + Y^2) - \frac{2}{2}(X^2 - Y^2) + \frac{1}{2}(X^2 + 2X.Y + Y^2)$$

$$- \frac{8}{\sqrt{2}}(X - Y) - \frac{8}{\sqrt{2}}(X + Y) = 0$$

$$X^2 - 2XY + Y^2 - 2X^2 + 2Y^2 + X^2 + 2XY + Y^2 - 8\sqrt{2}\,X$$

$$+ 8\sqrt{2}\,Y - 8\sqrt{2}\,X - 8\sqrt{2}\,Y = 0$$

$$4Y^2 - 16\sqrt{2}\,X = 0 \implies Y^2 = 4\sqrt{2}\,X \qquad (3)$$

which represents a parabola. In xy-plane, we have

From 2i), we have

$$X - Y = \sqrt{2}\,x \qquad (4)$$

and $$X + Y = \sqrt{2}\,y \qquad (5)$$

Adding (3) and (4), we get

$$2X = \sqrt{2}\,(x + y) \quad \Rightarrow \quad X = \frac{1}{\sqrt{2}}\,(x + y)$$

Put the value of X in (4), we get

$$\frac{1}{\sqrt{2}}\,(x + y) - Y = \sqrt{2}\,x \quad \Rightarrow \quad Y = -\sqrt{2}\,x + \frac{1}{\sqrt{2}}\,(x + y)$$

$$= \frac{-2x + x + y}{\sqrt{2}} = \frac{1}{\sqrt{2}}\,(y - x)$$

Thus $X = \frac{1}{\sqrt{2}}\,(x + y)$ and $Y = \frac{1}{\sqrt{2}}\,(y - x)$

Elements of the parabola are

Focus of (3) is $Y = 0$, $X = \sqrt{2}$

i.e., $\frac{1}{\sqrt{2}}\,(x + y) = \sqrt{2}$ and $\frac{1}{\sqrt{2}}\,(y - x) = 0$

$x + y = 2$ and $y - x = 0$

Adding: $x + y = 2$

$\underline{-x + y = 0}$

$2y = 2 \quad \Rightarrow \quad y = 1$

Put $y = 1$ in $x + y = 2$, we get

$x + 1 = 2 \quad \Rightarrow \quad x = 1$

$(1, 1)$ is the focus of (1)

Vertex of (3) is $X = 0$, $Y = 0$

i.e., $\frac{1}{\sqrt{2}}\,(x + y) = 0 \quad \Rightarrow \quad x + y = 0$

and $\frac{1}{\sqrt{2}}\,(y - x) = 0 \quad \Rightarrow \quad -x + y = 0$

Solving, we get $x = 0$, $y = 0$

Vertex: $(0, 0)$ is the vertex of (1).

Axis: $Y = 0$ i.e., $\frac{1}{\sqrt{2}}\,(y - x) = 0 \quad \Rightarrow \quad x - y = 0$

Equation of directrix of (3) is

$$X = -\sqrt{2} \quad \Rightarrow \quad \frac{x + y}{\sqrt{2}} = -\sqrt{2} \quad \Rightarrow \quad \frac{x + y}{\sqrt{2}} + \sqrt{2} = 0$$

$x + y + 2 = 0$ is the directrix in xy-coodinate system.

(iii) Identify : $x^2 + 2xy + y^2 + 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0$

Solution. $x^2 + 2xy + y^2 + 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0 \qquad \ldots (1)$

Here $a = 1,\ b = 1,\ 2h = 2$ the angle θ through which axes be rotated to given by

$$\tan 2q = \frac{2h}{a-b} = \frac{2}{1-1} = \frac{2}{0} \implies 2\theta = 90^\circ \implies \theta = 45^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X\cdot\frac{1}{\sqrt{2}} - Y\cdot\frac{1}{\sqrt{2}} = \frac{X-Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X\cdot\frac{1}{\sqrt{2}} + Y\cdot\frac{1}{\sqrt{2}} = \frac{X+Y}{\sqrt{2}} \end{aligned}\right\} \ldots(2)$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 + 2\sqrt{2}\left(\frac{X-Y}{\sqrt{2}}\right) - 2\sqrt{2}\left(\frac{X+Y}{\sqrt{2}}\right) + 2 = 0$$

$$\frac{1}{2}(X^2 - 2XY + Y^2) + \frac{2}{2}(X^2 - Y^2) + \frac{1}{2}(X^2 + 2XY + Y^2) + 2(X-Y) - 2(X+Y) + 2 = 0$$

$$X^2 - 2XY + Y^2 + 2X^2 - 2Y^2 + X^2 + 2XY + Y^2 + 4X - 4Y - 4X - 4Y + 4 = 0$$

$$4X^2 - 8Y + 4 = 0 \implies X^2 - 2Y + 1 = 0$$

$$X^2 = 2\left(Y - \frac{1}{2}\right) \qquad \ldots (3)$$

Which represents a parabola.

From (ii), we have

$$X - Y = \sqrt{2}\,x \qquad \ldots (4)$$

and $$X + Y = \sqrt{2}\,y \qquad \ldots (5)$$

Adding (4) and (5), we get

$$2X = \sqrt{2}\,x + \sqrt{2}\,y \implies 2X = \sqrt{2}\,(x+y) \implies X = \frac{1}{\sqrt{2}}(x+y)$$

Put the value of X in (4), we get

$$\frac{1}{\sqrt{2}}(x+y) - Y = \sqrt{2}\,x \implies Y = \frac{1}{\sqrt{2}}(x+y) - \sqrt{2}\,x$$

$$= \frac{x+y-2x}{\sqrt{2}} = \frac{1}{\sqrt{2}}(y-x)$$

Thus $X = \frac{1}{\sqrt{2}}(x+y)$ and $Y = \frac{1}{\sqrt{2}}(y-x)$

Elements of parabola are

Focus of (3) is $X = 0,\ Y - \frac{1}{2} = \frac{1}{2} \implies Y = 1$

i.e., $\frac{1}{\sqrt{2}}(x+y) = 0$ and $\frac{1}{\sqrt{2}}(y-x) = 1$

i.e., $x + y = 0$ and $y - x = \sqrt{2}$

Adding: $x + y = 0$

$\underline{-x + y = \sqrt{2}}$

$2y = \sqrt{2} \implies y = \frac{1}{\sqrt{2}}$

Put $y = \frac{1}{\sqrt{2}}$ in $x + y = 0 \implies x = -y = -\frac{1}{\sqrt{2}}$

Focus: $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is the focus of (1)

Vertex of (3) is $X = 0,\ Y - \frac{1}{2} = 0 \implies Y = \frac{1}{2}$

i.e., $\frac{1}{\sqrt{2}}(x+y) = 0 \implies x + y = 0$

and $\frac{1}{\sqrt{2}}(y-x) = \frac{1}{2} \implies y - x = \frac{1}{\sqrt{2}}$

Solving, we get $x = -\frac{1}{2\sqrt{2}},\ y = \frac{1}{2\sqrt{2}}$

Vertex $\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ is the vertex of (1)

Axis $X = 0$ i.e., $\frac{1}{\sqrt{2}}(x+y) = 0$

$\implies x + y = 0$

Equation of directrix of (3) is

$$Y - \frac{1}{2} = -\frac{1}{2} \Rightarrow \frac{y-x}{\sqrt{2}} = 0 \Rightarrow y - x = 0 \Rightarrow x - y = 0$$

is the directrix in xy–coordinate system.

(iv) $x^2 + xy + y^2 - 4 = 0$

Solution. $x^2 + xy + y^2 - 4 = 0$ ... (1)

Here $a = 1,\ b = 1,\ 2h = 1$ the angle θ through which axes be rotated is given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{1}{1-1} = \frac{1}{0} = \infty \Rightarrow 2\theta = 90^\circ \Rightarrow \theta = 45^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X \cdot \frac{1}{\sqrt{2}} - Y \cdot \frac{1}{\sqrt{2}} = \frac{X-Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X \cdot \frac{1}{\sqrt{2}} + Y \cdot \frac{1}{\sqrt{2}} = \frac{X+Y}{\sqrt{2}} \end{aligned}\right] \quad \text{(ii)}$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 + 4 = 0$$

$$\left(\frac{X^2 - 2XY + Y^2}{2}\right) + \left(\frac{X^2 - Y^2}{2}\right) + \left(\frac{X^2 + 2XY + Y^2}{2}\right) - 4 = 0$$

$$X^2 - 2XY + Y^2 + X^2 - Y^2 + X^2 + 2XY + Y^2 - 8 = 0$$

$$3X^2 + Y^2 = 8$$

$$\frac{X^2}{8/3} + \frac{Y^2}{8} = 1 \quad \text{... (3)}$$

Which represents an ellipse.

From (2), we have

$$X - Y = \sqrt{2}\,x \quad \text{... (4)}$$

$$X + Y = \sqrt{2}\,y \quad \text{... (5)}$$

Adding (4) and (5) $2X = \sqrt{2}\,x + \sqrt{2}\,y \Rightarrow X = \frac{1}{\sqrt{2}}(x + y)$

Subtracting (iv) and (v):

$$-2Y = \sqrt{2}\,x - \sqrt{2}\,y \Rightarrow Y = \frac{1}{\sqrt{2}}(y - x)$$

Elements of ellipse are

Centre of (3) is $X = 0$, $Y = 0$

$$\frac{1}{\sqrt{2}}(x+y) = 0 \quad \Rightarrow \quad x + y = 0$$

and $$\frac{1}{\sqrt{2}}(y-x) = 0 \quad \Rightarrow \quad -x + y = 0 \quad \Rightarrow \quad x = 0 \ , \ y = 0$$

Hence $C(0, 0)$ is the centre of (1)

Vertices of (3) are: $X = 0$, $Y = \pm 2\sqrt{2}$

$$X = 0 \Rightarrow \frac{1}{\sqrt{2}}(x+y) = 0 \quad \Rightarrow \quad x + y = 0$$

and $$Y = \pm\sqrt{2} \Rightarrow \frac{1}{\sqrt{2}}(y-x) = \pm 2\sqrt{2} \quad \Rightarrow \quad -x + y = \pm 4$$

and

$\Rightarrow \quad x + y = 0$	$x + y = 0$
$-x + y = 4$	$-x + y = -4$
Adding: $2y = 4 \Rightarrow y = 2$	Adding: $2y = -4 \Rightarrow y = -2$
$\Rightarrow \quad x = -y = -2$	$x = -y = -(-2) = 2$
$(-2, 2)$	$(2, -2)$

$(-2, 2)$, $(2, -2)$, as vertices of (1)

Equation of major axis: $X = 0 \Rightarrow x + y = 0$

Equation of minor axis: $Y = 0 \Rightarrow x - y = 0$

Ecentricity: $$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{8 - \frac{8}{3}}}{2\sqrt{2}} = \frac{4}{2\sqrt{6}} = \frac{2}{\sqrt{6}}$$

Foci of (3) are $X = 0$, $Y = \pm\sqrt{8}\left(\frac{2}{\sqrt{6}}\right)$

i.e., $$\frac{1}{\sqrt{2}}(x+y) = 0 \ , \ -\frac{1}{\sqrt{2}}(x-y) = \pm\sqrt{8}\left(\frac{2}{\sqrt{6}}\right)$$

$\Rightarrow \quad x + y = 0,$	$x + y = 0$
$-x + y = \frac{2\sqrt{8}}{\sqrt{3}}$	$-x + y = \frac{-2\sqrt{8}}{\sqrt{3}}$
Adding: $2y = \frac{2\sqrt{8}}{\sqrt{3}} \Rightarrow y = \frac{2\sqrt{2}}{\sqrt{3}}$	Adding: $2y = -\frac{2\sqrt{8}}{3} \Rightarrow y = \frac{-2\sqrt{2}}{3}$

$x + y = 0 \Rightarrow x = -y = -\dfrac{2\sqrt{2}}{3}$

$x + y = 0$

$\Rightarrow x = -y = -\left(\dfrac{-2\sqrt{2}}{3}\right) = \dfrac{2\sqrt{2}}{3}$

Hence $\left(\dfrac{-2\sqrt{2}}{3}, \dfrac{2\sqrt{2}}{3}\right)$ and $\left(\dfrac{2\sqrt{2}}{3}, \dfrac{-2\sqrt{2}}{3}\right)$ are the foci of (1).

(v) $7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0$

Solution. $7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0$ ... (1)

Here $a = 7,\ b = 13,\ 2h = -6\sqrt{3}$, the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a - b} = \frac{-6\sqrt{3}}{7 - 13} = \frac{-6\sqrt{3}}{-6} = \sqrt{3}$$

$\Rightarrow \quad 2\theta = 60^\circ \quad \Rightarrow \quad \theta = 30^\circ$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 30^\circ - Y\sin 30^\circ = X\cdot\frac{\sqrt{3}}{2} - Y\cdot\frac{1}{2} = \frac{\sqrt{3}X - Y}{2} \\ y &= X\sin 30^\circ + Y\cos 30^\circ = X\cdot\frac{1}{2} + Y\,\frac{\sqrt{3}}{2} = \frac{X + \sqrt{3}\,Y}{2} \end{aligned}\right] \quad \text{... (2)}$$

Substituting these expressions for x and y into (1), we get

$$7\left(\frac{\sqrt{3}\,X - Y}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}\,X - Y}{2}\right)\left(\frac{X + \sqrt{3}\,Y}{2}\right) + 13\left(\frac{\sqrt{3}\,X + Y}{2}\right)^2 - 16 = 0$$

$$7\left(\frac{3X^2 - 2\sqrt{3}\,XY + Y^2}{4}\right) - 6\sqrt{3}\left(\frac{\sqrt{3}\,X^2 + 2XY - \sqrt{3}\,Y^2}{4}\right) + 13\left(\frac{X^2 + 2\sqrt{3}\,Y + 3Y^2}{4}\right) - 16 = 0$$

$$\frac{(21X^2 - 14\sqrt{3}\,XY + 7Y^2)}{4} - \frac{(18X^2 + 12\sqrt{3}\,XY - 18Y^2)}{4} + \frac{(13X^2 + 26\sqrt{3}\,XY + 39Y^2)}{4} - 16 = 0$$

$$21X^2 - 14\sqrt{3}\,XY + 7Y^2 - 18Y^2 - 12\sqrt{3}\,XY + 18Y^2 + 13X^2 + 26\sqrt{3}\,XY + 39Y^2 - 64 = 0$$

$$16X^2 + 64Y^2 = 64$$

$$\Rightarrow \quad \frac{X^2}{4} + \frac{Y^2}{1} = 1 \qquad \ldots (3)$$

Which represents an ellipse.

From (ii), we have

$$\sqrt{3}\ X - Y = 2x \qquad \ldots (4)$$

$$X + \sqrt{3}\ Y = 2y \qquad \ldots (5)$$

Multiplying (iv) by $\sqrt{3}$, we get

$$3X - \sqrt{3}\ Y = 2\sqrt{3}\ x \qquad \ldots (6)$$

Adding (5) and (6), we get

$$4X = 2y + 2\sqrt{3}\ x \quad \Rightarrow \quad X = \frac{1}{2}(\sqrt{3}\ x + y)$$

Multiplying (5) by $\sqrt{3}$, we get

$$\sqrt{3}\ X + 3Y = 2\sqrt{3}\ Y \qquad \ldots (7)$$

Subtracting (7) from (4), we get

$$-4Y = 2\sqrt{3}\ y - 2x \quad \Rightarrow \quad Y = \frac{1}{2}(x - \sqrt{3}\ y)$$

Thus $X = \frac{1}{2}(\sqrt{3}\ x + y)$ and $Y = \frac{1}{2}(x - \sqrt{3}\ y)$

Elements of ellipse are

Centre of (3) is $X = 0$, $Y = 0$

$$X = 0 \quad \left(\frac{1}{2}(\sqrt{3}\ x + y) = 0\right. \quad \left(\sqrt{3}\ x + y = 0\right.$$

$$Y = 0 \quad \left(\frac{1}{2}(x - \sqrt{3}\ y) = 0\right. \quad \left(x - \sqrt{3}\ y = 0\right.$$

Solving these equations, we get $x = 0$, $y = 0$

Hence, $C\ (0, 0)$ centre of (1).

Vertices of (3) are $X = \pm a = \pm 2$ and $Y = 0$

$$X = \pm 2 \quad \Rightarrow \quad \frac{1}{2}(\sqrt{3}\ x + y) = \pm 2 \quad \Rightarrow \quad \sqrt{3}\ x + y = \pm 4$$

$$Y = 0 \quad \Rightarrow \quad \frac{1}{2}(x - \sqrt{3}\ y) = 0 \quad \Rightarrow \quad x - \sqrt{3}\ y = 0$$

$$\sqrt{3}\ x + y = 4 \qquad \ldots (4)$$

$$x - \sqrt{3}\ y = 0 \qquad \ldots (5)$$

Multiplying (4) by $\sqrt{3}$ and adding these equations, we get

$$\sqrt{3}\ x + y = -4 \qquad \ldots (6)$$

$$x - \sqrt{3}\ y = 0 \qquad \ldots (7)$$

Multiplying (6) by $\sqrt{3}$ and adding these equations, we get

$4x = 4\sqrt{3} \Rightarrow x = \sqrt{3}$

(5) $\Rightarrow \sqrt{3}\,y = x = \sqrt{3} \Rightarrow y = 1$

$4x = -4\sqrt{3} \Rightarrow x = -\sqrt{3}$

(7) $\Rightarrow \sqrt{3}\,y = x = -\sqrt{3} \Rightarrow y = -1$

$(\sqrt{3}, 1), (-\sqrt{3}, -1)$, as vertices of (1)

Ecentricity: $e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{\sqrt{4-1}}{2} = \dfrac{\sqrt{3}}{2}$

Foci of (3) are: $X = \pm\sqrt{3}$, $Y = 0$

$X = \pm\sqrt{3} \Rightarrow \frac{1}{2}(\sqrt{3}\,x + y) = \pm\sqrt{3} \Rightarrow \sqrt{3}\,x + y = \pm 2\sqrt{3}$

$Y = 0 \Rightarrow \frac{1}{2}(\sqrt{3}\,y - x) = 0 \Rightarrow -x + \sqrt{3}\,y = 0$

$\sqrt{3}\,x + y = 2\sqrt{3}$ ... (8)

$-x + \sqrt{3}\,y = 0$ ... (9)

Multiplying (9) by $\sqrt{3}$ and adding these equations, we get

$4y = 2\sqrt{3} \Rightarrow y = \dfrac{\sqrt{3}}{2}$

(9) $\Rightarrow x$ \r(3) $y =$ \r(3) . \f(\r(3),2) $= \sqrt{3}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{3}{2}$

$\left(\dfrac{3}{2}, \dfrac{\sqrt{3}}{2}\right)$

$\sqrt{3}\,x + y = -2\sqrt{3}$ ... (10)

$-x + \sqrt{3}\,y = 0$ ... (11)

Multiplying (11) by $\sqrt{3}$ and adding these equations, we get

$4y = -2\sqrt{3} \Rightarrow y = \dfrac{-\sqrt{3}}{2}$

(11) $\Rightarrow x =$ \r(3) y \f(3,2)

$\left(\dfrac{-3}{2}, \dfrac{-\sqrt{3}}{2}\right)$

Hence $\left(\dfrac{3}{2}, \dfrac{\sqrt{3}}{2}\right)$ and $\left(\dfrac{-3}{2}, \dfrac{-\sqrt{3}}{2}\right)$, as foci of (1).

Equation of major axis: $Y = 0 \Rightarrow \frac{1}{2}\left(\sqrt{3}\,y - x\right) = 0 \Rightarrow x - \sqrt{3}y = 0$

Equation of minor axis: $X = 0 \Rightarrow \frac{1}{2}\left(\sqrt{3}\,x + y\right) = 0 \Rightarrow \sqrt{3}x + y = 0.$

(vi) Identify. $4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$

Solution. $4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$ ... (1)

Here $a = 4$, $b = 7$, $2h = -4$, the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-4}{4-7} = \frac{-4}{-3} = \frac{4}{3}$$

$$\frac{2\tan q}{1-\tan^2 q} = \frac{4}{3} \implies 6\tan\theta = 4 - 4\tan^2\theta$$

$$4\tan^2\theta + 6\tan\theta - 4 = 0 \implies 2\tan^2\theta + 3\tan\theta - 2 = 0$$

$$\tan\theta = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(-2)}}{2(2)} = \frac{-3 \pm \sqrt{9+16}}{4}$$

$$= \frac{-3 \pm \sqrt{25}}{4} = \frac{-3 \pm 5}{4} = -2, \frac{1}{2} \implies \tan\theta = \frac{1}{2}$$

Now $\tan\theta = \frac{1}{2} \implies$ base $= 2$, $\perp = 1$, so hypotenuse $= \sqrt{4+1} = \sqrt{5}$

$$\therefore \quad \sin\theta = \frac{1}{\sqrt{5}} \text{ and } \cos\theta = \frac{2}{\sqrt{5}}$$

Equations of transformations become

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = X.\frac{2}{\sqrt{5}} - Y.\frac{1}{\sqrt{5}} = \frac{2X-Y}{\sqrt{5}} \\ y &= X\sin\theta + Y\sin\theta = X.\frac{1}{\sqrt{5}} + Y.\frac{2}{\sqrt{5}} = \frac{X+2Y}{\sqrt{5}} \end{aligned}\right\} \quad \ldots(2)$$

Substituting these expressions for x and y into (i), we get

$$4\left(\frac{2X-Y}{\sqrt{5}}\right)^2 - 4\left(\frac{2X-Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right) + 7\left(\frac{X+2Y}{\sqrt{5}}\right)^2$$

$$+ 12\left(\frac{2X-Y}{\sqrt{5}}\right) + 6\left(\frac{X+2Y}{\sqrt{5}}\right) - 9 = 0$$

$$\left(\frac{4X^2 - 4XY + Y^2}{5}\right) - 4\left(\frac{2X^2 + 3XY - 2Y^2}{5}\right) + 7\left(\frac{X^2 + 4XY + 4Y^2}{5}\right)$$

$$+ \frac{24X - 12Y}{\sqrt{5}} + \frac{6X + 12Y}{\sqrt{5}} - 9 = 0$$

$$16X^2 - 16XY + 4Y^2 - 8X^2 - 12XY + 8Y^2 + 7X^2 + 28XY + 28Y^2$$

$$+ \sqrt{5}(24X - 12Y) + \sqrt{5}(6X + 12Y) - 45 = 0$$

$$16X^2 - 16XY + 4Y^2 - 8X^2 - 12XY + 8Y^2 + 7X^2 + 28XY + 28Y^2 + 24\sqrt{5}X$$

$$- 12\sqrt{5}Y + 6\sqrt{5}X + 12\sqrt{5}Y - 45 = 0$$

$$15X^2 + 40Y^2 + 30\sqrt{5}X - 45 = 0 \implies 3X^2 + 8Y^2 + 6\sqrt{5}X - 9 = 0$$

$$3(X^2 + 2\sqrt{5}X) + 8Y^2 = 9 \implies 3(X^2 + 2\sqrt{5}X + (\sqrt{5})^2) + 8Y^2 = 9 + 15$$

$$3(X + \sqrt{5})^2 + 8Y^2 = 24 \implies \frac{(X+\sqrt{5})^2}{8} + \frac{Y^2}{3} = 1 \qquad (3)$$

which represents an ellipse.

From (2), we have

$$2X - Y = \sqrt{5}\, x \qquad \ldots (4)$$

$$X + 2Y = \sqrt{5}\, y \qquad \ldots (5)$$

Multiplying (5) by 2 and subtracting from (4), we get

$$5Y = 2\sqrt{5}\, y - \sqrt{5}\, x \implies Y = \frac{1}{\sqrt{5}}(-x + 2Y)$$

Put $Y = \frac{1}{\sqrt{5}}(-x + 2y)$ in (5), we get

$$X + \frac{2}{\sqrt{5}}(-x + 2y) = \sqrt{5}\, y \implies X = \sqrt{5}\, y - \frac{2}{\sqrt{5}}(-x + 2y)$$

$$= \sqrt{5}\, y + \frac{2}{\sqrt{5}}\, x - \frac{4}{\sqrt{5}}\, y$$

$$= \frac{1}{\sqrt{5}}(2x + y)$$

Thus $X = \frac{1}{\sqrt{5}}(2x + y)$ and $Y = \frac{1}{\sqrt{5}}(-x + 2y)$

For centre of (3) $X + \sqrt{5} = 0,\ Y = 0 \implies X = -\sqrt{5},\ Y = 0$

$$X = -\sqrt{5} \implies \frac{1}{\sqrt{5}}(2x + y) = -\sqrt{5} \implies 2x + y = -5 \qquad (7)$$

$$Y = 0 \implies \frac{1}{\sqrt{5}}(-x + 2y) = 0 \implies -x + 2y = 0 \qquad (8)$$

Multiplying equation (8) by 2, we get

$$-2x + 4y = 0 \qquad (9)$$

Adding equation (7) and (9), we get

$$5y = -5 \implies y = -1$$

Equation (8) $\implies x = 2y = 2(-1) = -2$

Hence $C\ (-2, -1)$ is the centre of (1)

Vertices of (3) are $X + \sqrt{5} = \pm\sqrt{8},\ Y = 0$

$$X + \sqrt{5} = \sqrt{8},\ Y = 0$$

$$X + \sqrt{5} = \sqrt{8} \implies \frac{1}{\sqrt{5}}(2x + y) + \sqrt{5} = \sqrt{8}$$

$2x + y + 5 = \sqrt{40}$

$2x + y = -5 + \sqrt{40}$ ... (10)

$Y = 0 \implies -x + 2y = 0$ ... (11)

Multiplying (11) by 2

$-2x + 4y = 0$ ... (12)

Adding equation (10) and equation (12), we get

$$5y = -5 + \sqrt{40} \implies y = -1 + \sqrt{\frac{8}{5}}$$

$$(12) \implies x = 2y = 2\left(-1 + \sqrt{\frac{8}{5}}\right) = - + \sqrt{\frac{32}{5}}$$

Similarly, solving $X + \sqrt{5} = -\sqrt{8}$, and $Y = 0$, we get

$$x = -2 - \sqrt{\frac{32}{5}}, \quad y = -1 - \sqrt{\frac{8}{5}}$$

$$\therefore \left(-2 + \sqrt{\frac{32}{5}}, -1 + \sqrt{\frac{8}{5}}\right), \left(-2 - \sqrt{\frac{32}{2}}, -1 - \sqrt{\frac{8}{5}}\right)$$

are the vertices of (1)

Eccentricity:- $e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{\sqrt{8-3}}{\sqrt{8}} = \dfrac{\sqrt{5}}{\sqrt{8}} = \sqrt{\dfrac{5}{8}}$

Foci of (3) are $X + \sqrt{5} = \pm\sqrt{5}$, $Y = 0$

$X + \sqrt{5} = \sqrt{5}$ and $Y = 0$

$X + \sqrt{5} = \sqrt{5} \implies X = 0$

$\dfrac{2x + y}{\sqrt{5}} = 0 \implies 2x + y = 0$ ... (13)

$Y = 0 \implies x + 2y = 0$ ... (14)

Multiplying equation (14) by 2, we get

$-2x + 4y = 0$ ... (15)

Adding (13) and (15), we get

$5y = 0 \implies y = 0$

Equation (14) $\implies x = 2y = 2(0) = 0$

Similarly, solving $X + \sqrt{5} = -\sqrt{5}$ and $Y = 0$

We get $x = -4$, $y = -2$

Thus $(0, 0)$, $(-4, -2)$ are the foci of (1).