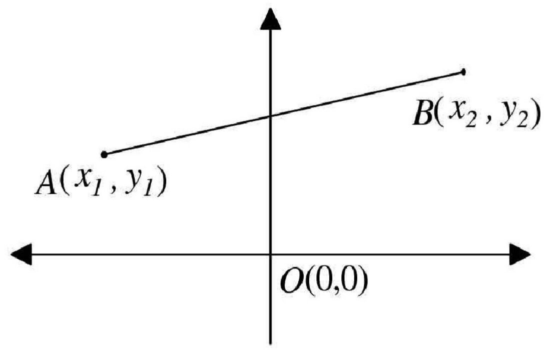
Exercise 4.1 (Solutions) Page 185

Calculus and Analytic Geometry, MATHEMATICS 12

Distance Formula

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in a plane and d be a distance between A and B then



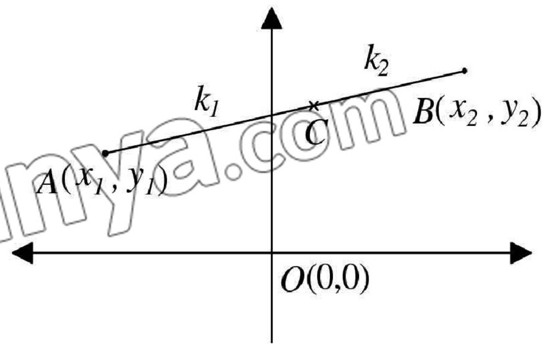
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

or $$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

See proof on book at page181

Ratio Formula

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in a plane. The coordinates of the point C dividing the line segment AB in the ratio $k_1 : k_2$ are



$$\left(\frac{k_1 x_2 + k_2 x_1}{k_1 + k_2}, \frac{k_1 y_2 + k_2 y_1}{k_1 + k_2} \right)$$

See proof on book at page 182

If C be the midpoint of AB i.e. $k_1 : k_2 = 1:1$ then coordinate of C becomes

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Question # 1

Describe the location in the plane of the point $P(x, y)$ for which

(i) $x > 0$ (ii) $x > 0$ and $y > 0$ (iii) $x = 0$

(iv) $y = 0$ (v) $x < 0$ and $y \geq 0$ (vi) $x = y$

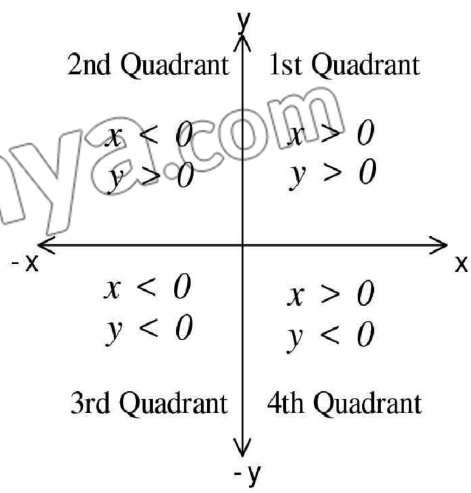
(vii) $|x| = -|y|$ (viii) $|x| \geq 3$ (ix) $x > 2$ and $y = 2$

(x) x and y have opposite signs.

Solution

(i) $x > 0$

Right half plane



(ii) $x > 0$ and $y > 0$

The 1[st] quadrant.

(iii) $x = 0$

y-axis

(iv) $y = 0$

x-axis

(v) $x<0$ and $y\geq 0$

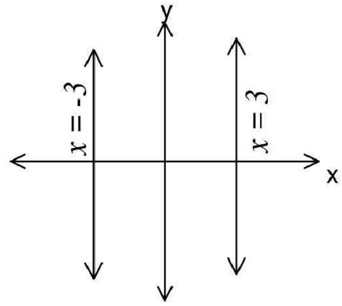
2nd quadrant & negative x-axis

(vi) $x=y$

It is a line bisecting 1st and 3rd quadrant.

(vii) $|x|=-|y|$

A positive value can't equal to a negative value, except number zero, so origin, $(0,0)$, is the only point which satisfies $|x|=-|y|$



(viii) $|x|\geq 3$

$\Rightarrow \pm x \geq 3 \quad \Rightarrow x\geq 3$ or $-x\geq 3$

$\Rightarrow x\geq 3$ or $x\leq -3$

which is the set of points lying on right side of the line $x=3$ and the points lying on left side of the line $x=-3$.

(ix) $x>2$ and $y=2$

The set of all points on the line $y=2$ for which $x>2$.

(x) x and y have opposite signs.

It is the set of points lying in 2nd and 4th quadrant.

Question # 2

Find each of the following

(i)the distance between the two given points

(ii)Midpoint of the line segment joining the two points

(a) $A(3,1):B(-2,-4)$ (b) $A(-8,3);B(2,-1)$ (c) $A\left(-\sqrt{5},-\frac{1}{3}\right);B\left(-3\sqrt{5},5\right)$

Solution

(a) $A(3,1)$; $B(-2,-4)$

(i) $|AB|=\sqrt{(-2-3)^2+(-4-1)^2} =\sqrt{(-5)^2+(-5)^2}$

$=\sqrt{25+25} =\sqrt{50} =\sqrt{25\times 2} =5\sqrt{2}$

(ii) Midpoint of AB = $\left(\frac{3-2}{2},\frac{1-4}{2}\right)=\left(\frac{1}{2},\frac{-3}{2}\right)$

> Review:
> The midpoint of $A(x_1,y_1)$ and $B(x_2,y_2)$ is $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$.

(b) $A(-8,3)$; $B(2,-1)$

Do yourself as above.

(c) $A\left(-\sqrt{5},-\frac{1}{3}\right)$; $B\left(-3\sqrt{5},5\right)$

(i) $$|AB|=\sqrt{\left(-3\sqrt{5}+\sqrt{5}\right)^2+\left(5+\frac{1}{3}\right)^2}=\sqrt{\left(2\sqrt{5}\right)^2+\left(\frac{16}{3}\right)^2}$$

$$=\sqrt{20+\frac{256}{9}}=\sqrt{\frac{436}{9}}=\sqrt{\frac{4\times109}{9}}=\frac{2\sqrt{109}}{3}$$

(ii) Midpoint of $AB=\left(\frac{-\sqrt{5}-3\sqrt{5}}{2},\frac{-\frac{1}{3}+5}{2}\right)=\left(\frac{-4\sqrt{5}}{2},\frac{\frac{14}{3}}{2}\right)=\left(-2\sqrt{5},\frac{7}{3}\right)$

Question # 3

Which of the following points are at a distance of 15 units from the origin?

(a) $\left(\sqrt{176},7\right)$ (b) $(10,-10)$ (c) $(1,15)$ (d) $\left(\frac{15}{2},\frac{15}{2}\right)$

Solution

(a) Distance of $\left(\sqrt{176},7\right)$ from origin $=\sqrt{\left(\sqrt{176}-0\right)^2+(7-0)^2}$

$$=\sqrt{(176)+(49)}$$

$$=\sqrt{(176)+(49)}=\sqrt{225}=15$$

$\Rightarrow$ the point $\left(\sqrt{176},7\right)$ is at 15 unit away from origin.

(b) Distance of $(10,-10)$ from origin $=\sqrt{(10-0)^2+(-10-0)^2}$

$$=\sqrt{100+100}=\sqrt{200}$$

$$=\sqrt{100\times2}=10\sqrt{2}\neq15$$

$\Rightarrow$ the point $(10,-10)$ is not at distance of 15 unit from origin.

(c) *Do yourself as above*

(d) Distance of $\left(\frac{15}{2},\frac{15}{2}\right)$ from origin $=\sqrt{\left(\frac{15}{2}-0\right)^2+\left(\frac{15}{2}-0\right)^2}$

$$=\sqrt{\frac{225}{4}+\frac{225}{4}}=\sqrt{\frac{225}{2}}=\frac{15}{\sqrt{2}}\neq15$$

Hence the point $\left(\frac{15}{2},\frac{15}{2}\right)$ is not at distance of 15 unit from origin.

Question # 4

Show that

(i) the point $A(0,2)$, $B(\sqrt{3},-1)$ and $C(0,-2)$ are vertices of a right triangle.

(ii) the point $A(3,1)$, $B(-2,-3)$ and $C(2,2)$ are vertices of an isosceles triangle.

(iii) the point $A(3,1)$, $B(-2,-3)$ and $C(2,2)$ and $\mathrm{D}(4,-5)$ are vertices of a parallelogram. Is the parallelogram a square?

Solution

(i) Given: $A(0,2)$, $B(\sqrt{3},-1)$ and $C(0,-2)$

$$|AB|=\sqrt{\left(\sqrt{3}-0\right)^2+(-1-2)^2}=\sqrt{\left(\sqrt{3}\right)^2+(-3)^2}$$

$$=\sqrt{3+9}=\sqrt{12} \qquad \Rightarrow |AB|^2=12$$

$$|BC|=\sqrt{\left(0-\sqrt{3}\right)^2+(-2+1)^2}=\sqrt{\left(-\sqrt{3}\right)^2+(-1)^2}$$

$$=\sqrt{3+1}=\sqrt{4}=2 \qquad \Rightarrow |BC|^2=4$$

$$|CA|=\sqrt{(0-0)^2+(2+2)^2}=\sqrt{0+(4)^2}$$

$$=\sqrt{16}=4 \qquad \Rightarrow |CA|^2=16$$

$\because$ $|AB|^2+|BC|^2=12+4=16=|CA|^2$

$\therefore$ by Pythagoras theorem A, B & C are vertices of a right triangle.

(ii) Given: $A(3,1)$, $B(-2,-3)$ and $C(2,2)$

$$|AB|=\sqrt{(-2-3))^2+(-3-1)^2}=\sqrt{(-5)^2+(-4)^2} \quad =\sqrt{25+16}=\sqrt{41}$$

$$|BC|=\sqrt{(2-(-2))^2+(2-(-3))^2}=\sqrt{(4)^2+(5)^2} \quad =\sqrt{16+25}=\sqrt{41}$$

$$|CA|=\sqrt{(3-2)^2+(1-2)^2}=\sqrt{(1)^2+(-1)^2}$$

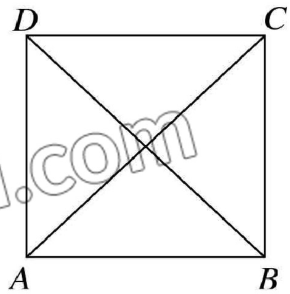
$$=\sqrt{1+1}=\sqrt{2}$$

$\because$ $|AB|=|BC|$ $\Rightarrow$ A, B & C are vertices of an isosceles triangle.

(iii) Given: $A(5,2)$, $B(-2,3)$ & $C(-3,-4)$ and $D(4,-5)$

$$|AB|=\sqrt{(-2-5)^2+(3-2)^2}=\sqrt{(-7)^2+(1)^2}$$

$$=\sqrt{49+1}=\sqrt{50}=5\sqrt{2}$$



$$|BC|=\sqrt{(-3+2)^2+(-4-3)^2}=\sqrt{(-1)^2+(-7)^2}$$

$$=\sqrt{1+49}=\sqrt{50}=5\sqrt{2}$$

$$|CD|=\sqrt{(4+3)^2+(-5+4)^2}=\sqrt{(7)^2+(-1)^2}$$

$$=\sqrt{49+1}=\sqrt{50}=5\sqrt{2}$$

$$|DA|=\sqrt{(5-4)^2+(2+5)^2}=\sqrt{(1)^2+(7)^2}$$

$$=\sqrt{1+49}=\sqrt{50}=5\sqrt{2}$$

$\because$ $|AB|=|CD|$ and $|BC|=|DA|$ $\Rightarrow$ A, B, C and D are vertices of parallelogram.

Now $|AC| = \sqrt{(-3-5)^2 + (-4-2)^2} = \sqrt{(-8)^2 + (-6)^2}$

$= \sqrt{64+36} = \sqrt{100} = 10$

$|BD| = \sqrt{(4+2)^2 + (-5-3)^2} = \sqrt{(6)^2 + (-8)^2}$

$= \sqrt{36+64} = \sqrt{100} = 10$

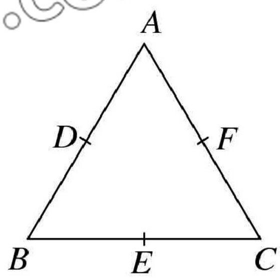
Since all sides are equals and also both diagonals are equal therefore A, B, C, D are vertices of a square.

Question # 5

The midpoints of the sides of a triangle are $(1,-1)$,$(-4,-3)$ and $(-1,1)$. Find coordinates of the vertices of the triangle.

Solution

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of triangle ABC, and let $D(1,-1)$, $E(-4,-3)$ and $F(-1,1)$ are midpoints of sides AB, BC and CA respectively. Then



$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = (1,-1)$$

$\Rightarrow x_1 + x_2 = 2 \ldots$ (i) and $y_1 + y_2 = -2 \ldots$ (ii)

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right) = (-4,-3)$$

$\Rightarrow x_2 + x_3 = -8 \ldots$ (iii) and $y_2 + y_3 = -6 \ldots$ (iv)

$$\left(\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2}\right) = (-1,1)$$

$\Rightarrow x_1 + x_3 = -2 \ldots$ (v) , and $y_1 + y_3 = 2 \ldots$ (vi)

Subtracting (i) and (iii)

$x_1 + x_2 = 2$

$\underline{\ \ _- x_2 \underset{-}{+} x_3 = -8} \ \ldots$ (vii)

$x_1 - x_3 = 10$

Subtracting (ii) and (iv)

$y_1 + y_2 = -2$

$\underline{\ \ _- y_2 \underset{-}{+} y_3 = \underset{+}{-}6} \ \ldots$ (viii)

$y_1 - y_3 = 4$

Adding (v) and (vii)

$x_1 + x_3 = -2$

$\underline{x_1 - x_3 = 10}$

$2x_1 = 8 \Rightarrow \boxed{x_1 = 4}$

Putting value of x_1 in (i)

$4 + x_2 = 2$

$\Rightarrow x_2 = 2 - 4 \Rightarrow \boxed{x_2 = -2}$

Putting value of x_1 in (v)

$4 + x_3 = -2$

$\Rightarrow x_3 = -2 - 4 \Rightarrow \boxed{x_3 = -6}$

Adding (vi) and (viii)

$y_1 + y_3 = 2$

$\underline{y_1 - y_3 = 4}$

$2y_1 = 6 \Rightarrow \boxed{y_1 = 3}$

Putting value of y_1 in (ii)

$3 + y_2 = -2$

$\Rightarrow y_2 = -2 - 3 \Rightarrow \boxed{y_2 = -5}$

Putting value of y_1 in (v)

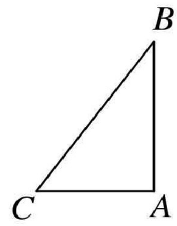
$3 + y_3 = 2$

$\Rightarrow y_3 = 2 - 3 \Rightarrow \boxed{y_3 = -1}$

Hence vertices of triangle are $(4,3),(-2,-5)$ & $(-6,-1)$.

Question # 6

Find h such that the point $A(\sqrt{3},-1)$, $B(0,2)$ and $C(h,-2)$ are vertices of a right angle with right angle at the vertex A.



Solution

Since ABC is a right triangle therefore by Pythagoras theorem

$$|AB|^2+|CA|^2=|BC|^2$$

$$\Rightarrow \left[\left(0-\sqrt{3}\right)^2+(2+1)^2\right]+\left[\left(\sqrt{3}-h\right)^2+(-1+2)^2\right]=(h-0)^2+(-2-2)^2$$

$$\Rightarrow [3+9]+\left[3-2\sqrt{3}h+h^2+1\right]=h^2+16$$

$$\Rightarrow 12+4-2\sqrt{3}\,h+h^2=h^2+16$$

$$\Rightarrow -2\sqrt{3}\,h=h^2+16-12-4-h^2 \quad \Rightarrow -2\sqrt{3}\,h=0 \quad \Rightarrow \boxed{h=0}.$$

Question # 7

Find h such that $A(-1,h),B(3,2)$ and $C(7,3)$ are collinear.

Solution

Points (x_1,y_1), (x_2,y_2) and (x_3,y_3) are collinear if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$$

Since given points are collinear therefore

$$\begin{vmatrix} -1 & h & 1 \\ 3 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix}=0$$

$$\Rightarrow -1(2-3)-h(3-7)+1(9-14)=0 \quad \Rightarrow -1(-1)-h(-4)+1(-5)=0$$

$$\Rightarrow 1+4h-5=0 \quad \Rightarrow 4h-4=0 \quad \Rightarrow 4h=4 \quad \Rightarrow \boxed{h=1}$$

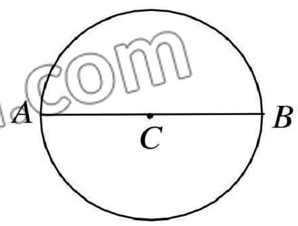
Question # 8

The points $A(-5,-2)$ and $B(5,-4)$ are end of a diameter of a circle. Find the centre and radius of the circle.

Solution

The centre of the circle is mid point of AB

i.e. centre 'C' $=\left(\dfrac{-5+5}{2},\dfrac{-2-4}{2}\right)=\left(\dfrac{0}{2},\dfrac{-6}{2}\right)=(0,-3)$



Now radius $=|AC|$

$$=\sqrt{(0+5)^2+(-3+2)^2}$$

$$=\sqrt{25+1}=\sqrt{26}$$

Question # 9

Find h such that the points $A(h,1)$, $B(2,7)$ and $C(-6,-7)$ are vertices of a right triangle with right angle at the vertex A

Solution

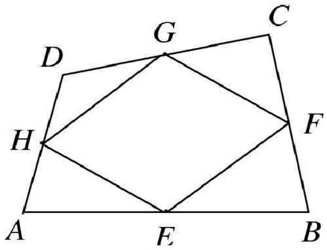
Do yourself as Question # 6

Hint: you will get a equation $h^2+4h-60=0$

Solve this quadratic equation to get two values of h.

Question # 10

A quadrilateral has the points $A(9,3), B(-7,7), C(-3,-7)$ and $D(-5,5)$ as its vertices. Find the midpoints of its sides. Show that the figure formed by joining the midpoints consecutively is a parallelogram.



Solution

Given: $A(9,3)$, $B(-7,7)$, $C(-3,-7)$ and $D(5,-5)$

Let E, F, G and H be the mid-points of sides of quadrilateral

Coordinate of E $= \left(\frac{9-7}{2},\frac{3+7}{2}\right) = \left(\frac{2}{2},\frac{10}{2}\right) = (1,5)$

Coordinate of F $= \left(\frac{-7-3}{2},\frac{7-7}{2}\right) = \left(\frac{-10}{2},\frac{0}{2}\right) = (-5,0)$

Coordinate of G $= \left(\frac{-3+5}{2},\frac{-7-5}{2}\right) = \left(\frac{2}{2},\frac{-12}{2}\right) = (1,-6)$

Coordinate of H $= \left(\frac{9+5}{2},\frac{3-5}{2}\right) = \left(\frac{14}{2},\frac{-2}{2}\right) = (7,-1)$

Now $|EF| = \sqrt{(-5-1)^2+(0-5)^2} = \sqrt{36+25} = \sqrt{61}$

$|FG| = \sqrt{(1+5)^2+(-6-0)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$

$|GH| = \sqrt{(7-1)^2+(-1+6)^2} = \sqrt{36+25} = \sqrt{61}$

$|HE| = \sqrt{(1-7)^2+(5+1)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$

Since $|EF| = |GH|$ and $|FG| = |HE|$

Therefore $EFGH$ is a parallelogram.

Question # 11

Find h such that the quadrilateral with vertices $A(-3,0), B(1,-2,)C(5,0)$ and $D(1,h)$ is parallelogram. Is it a square?

Solution

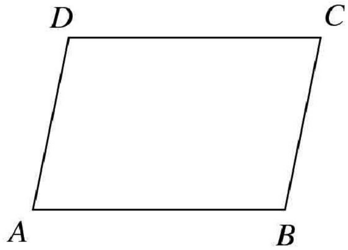
Given: $A(-3,0)$, $B(1,-2)$, $C(5,0)$, $D(1,h)$

Quadrilateral $ABCD$ is a parallelogram if

$|AB| = |CD|$ & $|BC| = |AD|$

when $|AB| = |CD|$

$\Rightarrow \sqrt{(1+3)^2+(-2-0)^2} = \sqrt{(1-5)^2+(h-0)^2}$



$\Rightarrow \sqrt{16+4} = \sqrt{16+h^2} \quad \Rightarrow \sqrt{20} = \sqrt{16+h^2}$

On squaring

$20 = 16 + h^2 \quad \Rightarrow h^2 = 20 - 16 \quad \Rightarrow h^2 = 4 \quad \Rightarrow h = \pm 2$

When $h = 2$, then $D(1,h) = D(1,2)$

Then $|AB| = \sqrt{(1+3)^2 + (-2-0)^2} = \sqrt{16+4} = \sqrt{20}$

$|BC| = \sqrt{(5-1)^2 + (0+2)^2} = \sqrt{16+4} = \sqrt{20}$

$|CA| = \sqrt{(1-5)^2 + (2-0)^2} = \sqrt{16+4} = \sqrt{20}$

$|DA| = \sqrt{(-3-1)^2 + (-0-2)^2} = \sqrt{16+4} = \sqrt{20}$

Now for diagonals

$|AC| = \sqrt{(5+3)^2 + (0-0)^2} = \sqrt{64+0} = 8$

$|BD| = \sqrt{(1-1)^2 + (2+2)^2} = \sqrt{0+16} = 4$

Since all sides are equal but diagonals $|AC| \neq |BD|$

Therefore $ABCD$ is not a square.

Now when $h = -2$, then $D(1,h) = D(1,-2)$ but we also have $B(1,-2)$

i.e. B and D represents the same point, which can not happened in quadrilateral so we can not take $h = -2$.

Question # 12

If two vertices of an equilateral triangle are $A(-3,0)$ and $B(3,0)$, find the third vertex. How many of these triangles are possible?

Solution

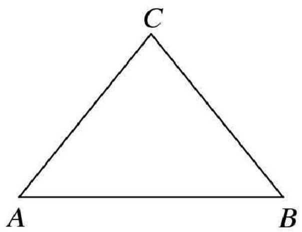
Given: $A(-3,0)$, $B(3,0)$

Let $C(x,y)$ be a third vertex of an equilateral triangle ABC.

Then $|AB| = |BC| = |CA|$

$\Rightarrow \sqrt{(3+3)^2 + (0-0)^2} = \sqrt{(x-3)^2 + (y-0)^2} = \sqrt{(x+3)^2 + (y-0)^2}$

$\Rightarrow \sqrt{36+0} = \sqrt{x^2 - 6x + 9 + y^2} = \sqrt{x^2 + 6x + 9 + y^2}$



On squaring

$36 = x^2 + y^2 - 6x + 9 = x^2 + y^2 + 6x + 9 \quad \ldots\ldots\ldots\ldots(i)$

From equation (i)

$x^2 + y^2 - 6x + 9 = x^2 + y^2 + 6x + 9$

$\Rightarrow x^2 + y^2 - 6x + 9 - x^2 - y^2 - 6x - 9 = 0$

$\Rightarrow -12x = 0 \quad \Rightarrow x = 0$

Again from equation (i)

$36 = x^2 + y^2 - 6x + 9$

$\Rightarrow 36 = (0)^2 + y^2 - 6(0) + 9 \quad \because x = 0$

$\Rightarrow 36 = y^2 + 9 \quad \Rightarrow y^2 = 36 - 9 = 27 \quad \Rightarrow y = \pm 3\sqrt{3}$

so coordinate of C is $(0, 3\sqrt{3})$ or $(0, -3\sqrt{3})$.

And hence two triangle can be formed with vertices $A(-3,0), B(3,0), C\left(0,3\sqrt{3}\right)$ and $A(-3,0), B(3,0), C\left(0,-3\sqrt{3}\right)$.

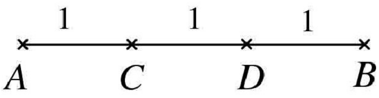
Question # 13

Find the points trisecting the join of $A(-1,4)$ and $B(6,2)$.

Solution

Given: $A(-1,4)$, $B(6,2)$

Let C and D be points trisecting A and B



Then $AC:CB = 1:2$

So coordinate of $C = \left(\dfrac{1(6)+2(-1)}{1+2}, \dfrac{1(2)+2(4)}{1+2}\right)$

$= \left(\dfrac{6-2}{3}, \dfrac{2+8}{3}\right) = \left(\dfrac{4}{3}, \dfrac{10}{3}\right)$

Also $AD:DB = 2:1$

So coordinate of $D = \left(\dfrac{2(6)+1(-1)}{2+1}, \dfrac{2(2)+1(4)}{2+1}\right)$

$= \left(\dfrac{12-1}{3}, \dfrac{4+4}{3}\right) = \left(\dfrac{11}{3}, \dfrac{8}{3}\right)$

Hence $\left(\dfrac{4}{3}, \dfrac{10}{3}\right)$ and $\left(\dfrac{11}{3}, \dfrac{8}{3}\right)$ are points trisecting A and B.

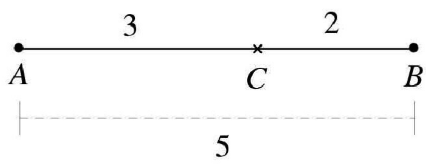
Question # 14

Find the point three-fifth of the way along the line segment from $A(-5,8)$ to $B(5,3)$.

Solution

Given: $A(-5,8)$, $B(5,3)$

Let $C(x, y)$ be a required point



$\because\ AC:CB = 3:2$

$\therefore$ Co-ordinate of $C = \left(\dfrac{3(5)+2(-5)}{3+2}, \dfrac{3(3)+2(8)}{3+2}\right)$

$= \left(\dfrac{15-10}{5}, \dfrac{9+16}{5}\right) = \left(\dfrac{5}{5}, \dfrac{25}{5}\right) = (1, 5)$

Question # 15

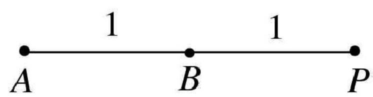
Find the point P on the joint of $A(1,4)$ and $B(5,6)$ that is twice as far from A as B is from A and lies

(i) on the same side of A as B does. (ii) on the opposite side of A as B does.

Solution

Given: $A(1,4)$, $B(5,6)$

(i) Let $P(x, y)$ be required point, then



$AB : AP = 1:2$

$\Rightarrow AB : BP = 1:1$ i.e. B is midpoint of AP

Then $B(5,6) = \left(\dfrac{1+x}{2}, \dfrac{4+y}{2}\right)$

$\Rightarrow 5 = \dfrac{1+x}{2}$ and $6 = \dfrac{4+y}{2}$

$\Rightarrow 10 = 1+x$ and $12 = 4+y$

$\Rightarrow x = 10-1$, $y = 12-4$

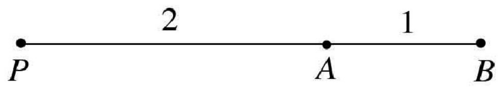
$=9$, $=8$

Hence $P(9,8)$ is required point.

(ii) Since $PA : AB = 2:1$

$\Rightarrow A(1,4) = \left(\dfrac{2(5)+1(x)}{2+1}, \dfrac{2(6)+1(y)}{2+1}\right)$

$= \left(\dfrac{10+x}{3}, \dfrac{12+y}{3}\right)$



$\Rightarrow 1 = \dfrac{10+x}{3}$ and $4 = \dfrac{12+y}{3}$

$\Rightarrow 3 = 10+x$ and $12 = 12+y$

$\Rightarrow x = 3-10$ and $y = 12-12$

$= -7$, $= 0$

Hence $P(-7,0)$ is required point.

Question # 16

Find the point which is equidistant from the points $A(5,3), B(2,-2)$ and $C(4,2)$. What is the radius of the circumcircle of the ΔABC ?

Solution

Given: $A(5,3)$, $B(-2,2)$ and $C(4,2)$

Let $D(x,y)$ be a point equidistance from A , B and C then

$$\left|\overline{DA}\right| = \left|\overline{DB}\right| = \left|\overline{DC}\right|$$

$$\Rightarrow \left|\overline{DA}\right|^2 = \left|\overline{DB}\right|^2 = \left|\overline{DC}\right|^2$$

$$\Rightarrow (x-5)^2+(y-3)^2 = (x+2)^2+(y-2)^2 = (x-4)^2+(y-2)^2 \text{ .... (i)}$$

From eq. (i)

$$(x-5)^2+(y-3)^2 = (x+2)^2+(y-2)^2$$

$$\Rightarrow x^2-10x+25+y^2-6y+9 = x^2+4x+4+y^2-4y+4$$

$$\Rightarrow x^2-10x+25+y^2-6y+9-x^2-4x-4-y^2+4y-4 = 0$$

$$\Rightarrow -14x-2y+26 = 0 \quad \Rightarrow 7x+y-13 = 0 \text{ .... (ii)}$$

Again from equation (i)

$$(x+2)^2+(y-2)^2 = (x-4)^2+(y-2)^2$$

$\Rightarrow\ x^2+4x+4+y^2-4y+4 = x^2-8x+16+y^2-4y+4$

$\Rightarrow\ 12x-12 = 0 \quad \Rightarrow\ 12x = 12 \quad \Rightarrow\ x = 1$

Put $x=1$ in eq. (ii)

$7(1)+y-13 = 0 \qquad \Rightarrow\ y-6 = 0 \qquad \Rightarrow\ y = 6$

Hence $(1,6)$ is required point.

Now radius of circumcircle $= \left|\overline{DA}\right|$

$= \sqrt{(5-1)^2+(3-6)^2} = \sqrt{16+9} = \sqrt{25} = 5$ units

Intersection of Median

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of triangle.

Intersection of median is called centroid of triangle and can be determined as

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$$

See proof at page 184

Centre of In-Circle (In-Centre)

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of triangle.

And $|AB| = c$, $|BC| = a$, $|CA| = b$

Then in-centre of triangle $= \left(\dfrac{ax_1+bx_2+cx_3}{a+b+c},\dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$

See proof at page 184

Question # 17

The points $(4,-2),(-2,4)$ and $(5,5)$ are the vertices of a triangle. Find in-centre of the triangle.

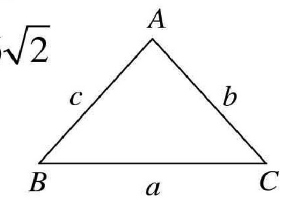
Solution

Let $A(4,-2)$, $B(-2,4)$, $C(5,5)$ are vertices of triangle then

$a = |BC| = \sqrt{(5+2)^2+(5-4)^2} = \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$

$b = |CA| = \sqrt{(4-5)^2+(-2-5)^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$

$c = |AB| = \sqrt{(-2-4)^2+(4+2)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$



Now

In-centre $= \left(\dfrac{ax_1+bx_2+cx_3}{a+b+c},\dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$

$= \left(\dfrac{5\sqrt{2}(4)+5\sqrt{2}(-2)+6\sqrt{2}(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}},\dfrac{5\sqrt{2}(-2)+5\sqrt{2}(4)+6\sqrt{2}(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}\right)$

$= \left(\dfrac{20\sqrt{2}-10\sqrt{2}+30\sqrt{2}}{16\sqrt{2}},\dfrac{-10\sqrt{2}+20\sqrt{2}+30\sqrt{2}}{16\sqrt{2}}\right)$

$= \left(\dfrac{40\sqrt{2}}{16\sqrt{2}},\dfrac{40\sqrt{2}}{16\sqrt{2}}\right) = \left(\dfrac{5}{2},\dfrac{5}{2}\right)$

Question # 18

Find the points that divide the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ into four equal parts.

Solution

Given: $A(x_1, y_1)$, $B(x_2, y_2)$

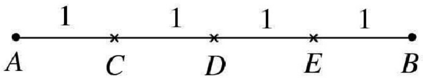
Let C, D and E are points dividing AB into four equal parts.

$\because\ AC : CB = 1:3$

$$\Rightarrow \text{ Co-ordinates of } C = \left(\frac{1(x_2)+3(x_1)}{1+3}, \frac{1(y_2)+3(y_1)}{1+3}\right) = \left(\frac{3x_1+x_2}{4}, \frac{3y_1+y_2}{4}\right)$$

Now $AD : DB = 2:2$

$= 1:1$ i.e. D is midpoint of AB.



$$\Rightarrow \text{ Co-ordinates of } D = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Now $AE : EB = 3:1$

$$\Rightarrow \text{ Co-ordinates of } E = \left(\frac{3(x_2)+1(x_1)}{3+1}, \frac{3(y_2)+1(y_1)}{3+1}\right) = \left(\frac{x_1+3x_2}{4}, \frac{y_1+3y_2}{4}\right)$$

Hence $\left(\frac{3x_1+x_2}{4}, \frac{3y_1+y_2}{4}\right)$, $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ and $\left(\frac{x_1+3x_2}{4}, \frac{y_1+3y_2}{4}\right)$ are the points dividing AB into four equal parts.

Question # 1

The two points P and O' are given in $xy-coordinates$ system. Find the $XY-coordinates$ of P referred to the translated axes $O'X$ and $O'Y$

(i) $P(3,2);O'(1,3)$

(ii) $P(-2,6);O'(-3,2)$

(iii) $P(-6,-8);O'(-4,-6)$

(iv) $P\left(\frac{3}{2},\frac{5}{2}\right);O'\left(-\frac{1}{2},\frac{7}{2}\right)$

Solution

(i) Since $P(x,y) = P(3,2)$

i.e. $x=3$ and $y=2$

$O'(h,k) = O'(1,3)$

i.e. $h=1$ and $k=3$

$\because\ X = x-h$

$= 3-1 = 2$

Also $Y = y-k$

$= 2-3 = -1$

Hence $(2,-1)$ is point P in $XY-$coordinates.

(ii) *Do yourself*

(iii) *Do yourself*

(iv) Since $P(x,y) = P\left(\frac{3}{2},\frac{5}{2}\right)$

i.e. $x = \frac{3}{2}$ and $y = \frac{5}{2}$

$O'(h,k) = O'\left(-\frac{1}{2},\frac{7}{2}\right)$

i.e. $h=-\frac{1}{2}$ and $k=\frac{7}{2}$

$\because\ X = x-h$

$= \frac{3}{2}-\left(-\frac{1}{2}\right) = 2$

And $Y = y-k$

$= \frac{5}{2}-\frac{7}{2} = -1$

Hence $(2,-1)$ are coordinates of P in XY-axes.

Question # 2

The $xy-coordinates$ axes are translated through the point O' whose coordinates are given in $xy-coordinates$. The coordinates of P are given in the $XY-coordinates$ system. Find the coordinates of P in $xy-coordinates$ system.

(i) $P(8,10);O'(3,4)$

(ii) $P(-5,-3);O'(-2,-6)$

(iii) $P\left(-\frac{3}{4},-\frac{7}{6}\right);O'\left(\frac{1}{4},-\frac{1}{6}\right)$

(iv) $P(4,-3);O'(-2,3)$

Solution

(i) $\because\ P(X,Y) = P(8,10)$

$\Rightarrow X=8$ and $Y=10$

$O'(h,k) = O'(3,4)$

$\Rightarrow h=3$ and $k=4$

$\because\ X = x-h$

$\Rightarrow 8 = x-3$

$\Rightarrow x = 8+3 \quad \Rightarrow \quad x = 11$

Also $Y = y-k$

$\Rightarrow 10 = y-4$

$\Rightarrow y = 10+4 \quad \Rightarrow \quad y = 14$

Hence $(11,14)$ are coordinates of P in xy-axes.

(ii) *Do yourself*

(iii) $\because\ P(X,Y) = P\left(-\frac{3}{4},-\frac{7}{6}\right)$

$\Rightarrow X=-\frac{3}{4}$ and $Y=-\frac{7}{6}$

$O'(h,k) = O'\left(\frac{1}{4},-\frac{1}{6}\right)$

$\Rightarrow h=\frac{1}{4}$ and $k=-\frac{1}{6}$

$\because\ X = x-h$

$\Rightarrow -\frac{3}{4} = x-\frac{1}{4}$

$\Rightarrow x = -\frac{3}{4}+\frac{1}{4} \quad \Rightarrow \quad x = -\frac{1}{2}$

Also $Y = y-k$

$\Rightarrow -\frac{7}{6} = y+\frac{1}{6}$

$\Rightarrow y = -\frac{7}{6}-\frac{1}{6} \Rightarrow y = -\frac{4}{3}$

Hence $\left(-\frac{1}{2},-\frac{4}{3}\right)$ is the required point.

(iv) *Do yourself*

Rotation of Axes

Let (x, y) be the coordinates of point P in xy-coordinate system. If the axes are rotated through at angle of θ and (X,Y) are coordinate of P in new XY-coordinate system then

$$X = x\cos\theta + y\sin\theta$$
$$Y = y\cos\theta - x\sin\theta$$

Question # 3

The $xy-coordinates$ axes are rotated about the origin through the indicated angle. The new axes are OX and OY. Find the $XY-coordinates$ of the point P with the given $xy-coordinates$.

(i) $P(5,3);\theta = 45^\circ$

(ii) $P(3,-7);\theta = 30^\circ$

(iii) $P(11,-15);\theta = 60^\circ$

(iv) $P(15,10);\theta = \arctan\frac{1}{3}$

Solution

(i) $\because P(x,y) = P(5,3)$

$\Rightarrow x = 5 \;\&\; y = 3 \;,\; \theta = 45^\circ$

Since $X = x\cos\theta + y\sin\theta$

$= 5\cos 45^\circ + 3\sin 45^\circ$

$= 5\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(5+3)$

$= \frac{8}{\sqrt{2}} = \frac{4\times 2}{\sqrt{2}} = 4\sqrt{2}$

Now $Y = y\cos\theta - x\sin\theta$

$= 3\cos 45^\circ - 5\sin 45^\circ$

$= 3\left(\frac{1}{\sqrt{2}}\right) - 5\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(3-5)$

$= -2\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$

Hence the required point is $(4\sqrt{2},-\sqrt{2})$.

(ii) $\because P(x,y) = P(3,-7)$

$\Rightarrow x = 3 \;\&\; y = -7 \;,\; \theta = 30^\circ$

Since $X = x\cos\theta + y\sin\theta$

$= 3\cos 30^\circ - 7\sin 30^\circ$

$\Rightarrow X = 3\left(\frac{\sqrt{3}}{2}\right) - 7\left(\frac{1}{2}\right)$

$= \frac{3\sqrt{3}-7}{2}$

Now $Y = y\cos\theta - x\sin\theta$

$= -7\cos 30^\circ - 3\sin 30^\circ$

$= -7\left(\frac{\sqrt{3}}{2}\right) - 3\left(\frac{1}{2}\right) = \frac{-7\sqrt{3}-3}{2}$

Hence the required point is

$\left(\frac{3\sqrt{3}-7}{2},\frac{-7\sqrt{3}-3}{2}\right)$.

(iv) *Do yourself*

(iv) $\because P(x,y) = P(15,10)$

$\Rightarrow x = 15 \;\&\; y = 10$

Also $\theta = \tan^{-1}\left(\frac{1}{3}\right)$

$\Rightarrow \tan\theta = \frac{1}{3}$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{1/\sqrt{10}}{3/\sqrt{10}}$

$\tan\theta = \frac{y}{x} = \frac{1}{3}$

$\Rightarrow x = 3 \;,\; y = 1$

$r = \sqrt{x^2+y^2}$

$= \sqrt{3^2+1^2}$

$= \sqrt{10}$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{10}} \;,\; \cos\theta = \frac{3}{\sqrt{10}}$

Now $X = x\cos\theta + y\sin\theta$

$= 15\left(\frac{3}{\sqrt{10}}\right) + 10\left(\frac{1}{\sqrt{10}}\right)$

$= \frac{1}{\sqrt{10}}(45+10) = \frac{55}{\sqrt{10}}$

$Y = y\cos\theta - x\sin\theta$

$= 10\left(\frac{3}{\sqrt{10}}\right) - 15\left(\frac{1}{\sqrt{10}}\right)$

$= \frac{1}{\sqrt{10}}(30-15) = \frac{15}{\sqrt{10}}$

Hence the required point is $\left(\frac{55}{\sqrt{10}},\frac{15}{\sqrt{10}}\right)$.

(iv)

$\because P(x,y) = P(15,10)$

$\Rightarrow x=15 \quad \& \quad y=10$

Also $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{1/2}{\sqrt{3}/2}$

$\Rightarrow \sin\theta = \frac{1}{2} \quad , \quad \cos\theta = \frac{\sqrt{3}}{2}$

$\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}}$

$\Rightarrow x = \sqrt{3}\ ,\ y = 1$

$r = \sqrt{x^2 + y^2}$

$= \sqrt{\left(\sqrt{3}\right)^2 + 1^2}$

$= \sqrt{4} = 2$

Now $X = x\cos\theta + y\sin\theta$

$= 15\left(\frac{\sqrt{3}}{2}\right) + 10\left(\frac{1}{2}\right)$

$= \frac{15\sqrt{3}+10}{2}$

$Y = y\cos\theta - x\sin\theta$

$= 10\left(\frac{\sqrt{3}}{2}\right) - 15\left(\frac{1}{2}\right)$

$= \frac{10\sqrt{3}-15}{2}$

Hence the required point is

$\left(= \frac{15\sqrt{3}+10}{2}, \frac{10\sqrt{3}-15}{2}\right)$.

Question # 4

The $xy-coordinates$ axes are rotated about the origin through the indicated angle and the new axes are OX and OY , Find the $xy-coordinates$ of P with the given $XY-coordinates$.

(i) $P(-5,3);\theta = 30^\circ$

(ii) $P\left(-7\sqrt{2}, 5\sqrt{2}\right);\theta = 45^\circ$

Solution

(i) $\because P(X,Y)$

$\Rightarrow X = -5 \quad \& \quad Y$

Also $\theta = 30^\circ$

Therefore $\sin\theta = \frac{1}{2} \quad \& \quad \cos\theta = \frac{\sqrt{3}}{2}$

Now $X = x\cos\theta + y\sin\theta$

$\Rightarrow -5 = x\left(\frac{\sqrt{3}}{2}\right) + y\left(\frac{1}{2}\right)$

$\Rightarrow \sqrt{3}x + y = -10$ …….. (i)

Also $Y = y\cos\theta - x\sin\theta$

$\Rightarrow 3 = y\left(\frac{\sqrt{3}}{2}\right) - x\left(\frac{1}{2}\right)$

$\Rightarrow 6 = \sqrt{3}y - x$

$\Rightarrow x = \sqrt{3}y - 6$ ……… (ii)

Putting value of x in (i)

$\sqrt{3}\left(\sqrt{3}y - 6\right) + y = -10$

$\Rightarrow 3y - 6\sqrt{3} + y = -10$

$\Rightarrow 4y = -10 + 6\sqrt{3}$

$\Rightarrow y = \frac{-10+6\sqrt{3}}{4}$

$= \frac{-5+3\sqrt{3}}{2}$

Putting value of y in (ii)

$x = \sqrt{3}\left(\frac{-5+3\sqrt{3}}{2}\right) - 6$

$= \frac{-5\sqrt{3}+9}{2} - 6 = \frac{-5\sqrt{3}+9-12}{2}$

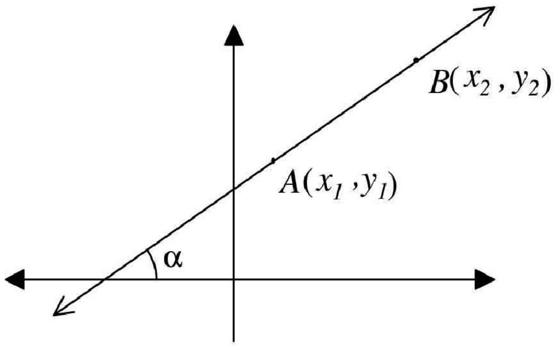
$= \frac{-5\sqrt{3}-3}{2}$

Hence $\left(\frac{-5\sqrt{3}-3}{2}, \frac{-5+3\sqrt{3}}{2}\right)$ is required point.

(ii) *Do yourself*

Inclination of a Line:

The angle $\alpha\ (0^\circ \le \alpha < 180^\circ)$ measure anti-clockwise from positive $x-$axis to the straight line l is called *inclination* of a line l.



Slope or Gradient of Line

The slope m of the line l is defined by:

$$m = \tan\alpha$$

If $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two distinct points on the line l then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

See proof on book at page: 191

Note: l is horizontal, iff $m = 0$ $(\because \alpha = 0^\circ)$

l is vertical, iff $m = \infty$ i.e. m is not defined. $(\because \alpha = 90^\circ)$

If slope of AB = slope of BC, then the points A, B and C are collinear i.e. lie on the same line.

Theorem

The two lines l_1 and l_2 with respective slopes m_1 and m_2 are

(i) Parallel iff $m_1 = m_2$

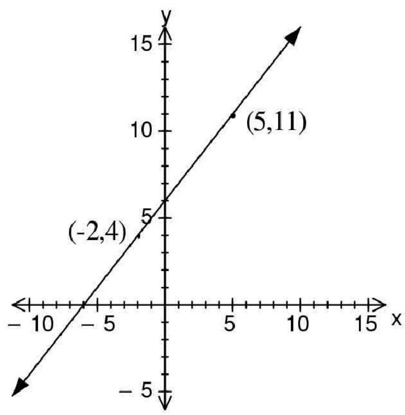
(ii) Perpendicular iff $m_1 m_2 = -1$ or $m_1 = -\dfrac{1}{m_2}$

Question # 1

Find the slope and inclination of the line joining the points:

(i) $(-2,4)$; $(5,11)$ (ii) $(3,-2)$; $(2,7)$

(iii) $(4,6)$; $(4,8)$



Solution

(i) $(-2,4)$; $(5,11)$

Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{11-4}{5+2} = \dfrac{7}{7} = 1$

Since $\tan\alpha = m = 1$

$\Rightarrow \alpha = \tan^{-1}(1) = 45^\circ$

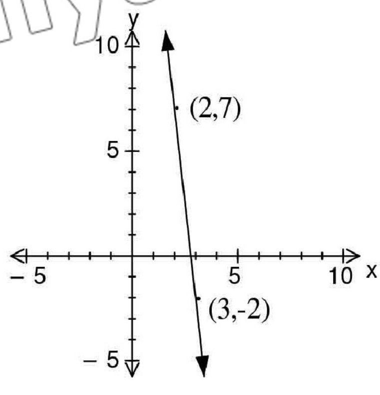
(ii) $(3,-2)$; $(2,7)$

Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7+2}{2-3} = \dfrac{9}{-1} = -9$

Since $\tan\alpha = m = -9$

$\Rightarrow -\tan\alpha = 9 \Rightarrow \tan(180-\alpha) = 9$

$\Rightarrow 180 - \alpha = \tan^{-1}(9)$



$\Rightarrow\ 180-\alpha = 83^\circ 40'$

$\Rightarrow\ \alpha = 180-83^\circ 40' = 96^\circ 20'$

(ii) $(4,6)$; $(4,8)$

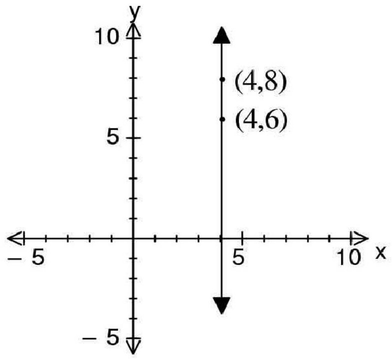
Slope $m=\dfrac{y_2-y_1}{x_2-x_1}$

$=\dfrac{8-6}{4-4}=\dfrac{2}{0}=\infty$

Since $\tan\alpha = m = \infty$

$\Rightarrow\ \alpha=\tan^{-1}(\infty)$

$=90^\circ$



Question # 2

In the triangle $A(8,6)$, $B(-4,2)$ and $C(-2,-6)$, find the slope of

(i) each side of the triangle (ii) each median of the triangle

(iii) each altitude of the triangle

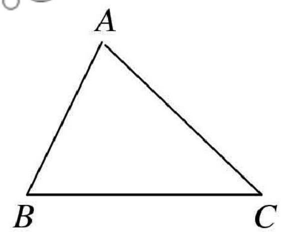
Solution

Since $A(8,6)$, $B(-4,2)$ and $C(-2,-6)$ are vertices of triangle therefore

(i) Slope of side $AB=\dfrac{2-6}{-4-8}=\dfrac{-4}{-12}=\dfrac{1}{3}$

Slope of side $BC=\dfrac{-6-2}{-2+4}=\dfrac{-8}{2}=-4$

Slope of side $CA=\dfrac{6+6}{8+2}=\dfrac{12}{10}=\dfrac{6}{5}$

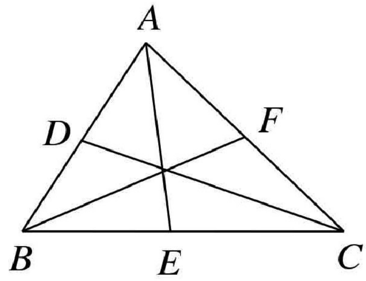


(ii) Let D, E and F are midpoints of sides AB, BC and CA respectively.
Then

Coordinate of $D=\left(\dfrac{8-4}{2},\dfrac{6+2}{2}\right)=\left(\dfrac{4}{2},\dfrac{8}{2}\right)=(2,4)$

Coordinate of $E=\left(\dfrac{-4-2}{2},\dfrac{2-6}{2}\right)=\left(\dfrac{-6}{2},\dfrac{-4}{2}\right)=(-3,-2)$

Coordinate of $F=\left(\dfrac{-2+8}{2},\dfrac{-6+6}{2}\right)=\left(\dfrac{6}{2},\dfrac{0}{2}\right)=(3,0)$



Hence Slope of median $AE=\dfrac{-2-6}{-3-8}=\dfrac{-8}{-11}=\dfrac{8}{11}$

Slope of median $BF=\dfrac{0-2}{3+4}=\dfrac{-2}{7}$

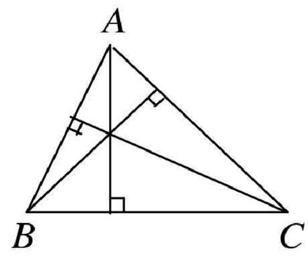
Slope of median $CD=\dfrac{4+6}{2+2}=\dfrac{10}{4}=\dfrac{5}{2}$

(iii) Since altitudes are perpendicular to the sides of a triangle therefore

Slope of altitude from vertex $A = \dfrac{-1}{\text{slope of side } BC} = \dfrac{-1}{-4} = \dfrac{1}{4}$

Slope of altitude from vertex $B = \dfrac{-1}{\text{slope of side } AC} = \dfrac{-1}{6/5} = -\dfrac{5}{6}$

Slope of altitude from vertex $C = \dfrac{-1}{\text{slope of side } AB} = \dfrac{-1}{1/3} = -3$



Question # 3
By means of slopes, show that the following points lie in the same line:
(a) $(-1,-3)$; $(1,5)$; $(2,9)$ (b) $(4,-5);(7,5);(10,15)$
(c) $(-4,6);(3,8);(10,10)$ (d) $(a,2b);(c,a+b);(2c-a,2a)$

Solution
(a) Let $A(-1,-3)$, $B(1,5)$ and $C(2,9)$ be given points

$$\text{Slope of } AB = \frac{5+3}{1+1} = \frac{8}{2} = 4$$
$$\text{Slope of } BC = \frac{9-5}{2-1} = \frac{4}{1} = 4$$

Since slope of $AB =$ slope of BC
Therefore A, B and C lie on the same line.

(b) *Do yourself as above*

(c) *Do yourself as above*

(d) Let $A(a,2b), B(c,a+b)$ and $C(2c-a,2a)$ be given points.

$$\text{Slope of } AB = \frac{(a+b)-2b}{c-a} = \frac{a-b}{c-a}$$
$$\text{Slope of } BC = \frac{2a-(a+b)}{(2c-a)-c} = \frac{2a-a-b}{2c-a-c} = \frac{a-b}{c-a}$$

Since slope of $AB =$ slope of BC
Therefore A, B and C lie on the same line.

Question # 4
Find k so that the line joining $A(7,3)$; $B(k,-6)$ and the line joining $C(-4,5)$; $D(-6,4)$ are (i) parallel (ii) perpendicular.

Solution
Since $A(7,3)$, $B(k,-6)$, $C(-4,5)$ and $D(-6,4)$

$$\text{Therefore slope of } AB = m_1 = \frac{-6-3}{k-7} = \frac{-9}{k-7}$$
$$\text{Slope of } CD = m_2 = \frac{4-5}{-6+4} = \frac{-1}{-2} = \frac{1}{2}$$

(i) If AB and CD are parallel then $m_1 = m_2$

$$\Rightarrow \frac{-9}{k-7} = \frac{1}{2} \qquad \Rightarrow -18 = k - 7$$

$$\Rightarrow k = -18 + 7 \qquad \Rightarrow \boxed{k = -11}$$

(ii) If AB and CD are perpendicular then $m_1 m_2 = -1$

$$\Rightarrow \left(\frac{-9}{k-7}\right)\left(\frac{1}{2}\right) = -1 \qquad \Rightarrow -9 = -2(k-7)$$

$$\Rightarrow 9 = 2k - 14 \qquad \Rightarrow 2k = 9 + 14 = 23$$

$$\Rightarrow \boxed{k = \frac{23}{2}}$$

Question # 5

Using slopes, show that the triangle with its vertices $A(6,1), B(2,7)$ and $C(-6,-7)$ is a right triangle.

Solution

Since $A(6,1), B(2,7)$ and $C(-6,-7)$ are vertices of triangle therefore

Slope of $\overline{AB}$ $= m_1 = \frac{7-1}{2-6} = \frac{6}{-4} = -\frac{3}{2}$

Slope of $\overline{BC}$ $= m_2 = \frac{-7-7}{-6-2} = \frac{-12}{-8} = \frac{7}{4}$

Slope of $\overline{CA}$ $= m_3 = \frac{1+7}{6+6} = \frac{8}{12} = \frac{2}{3}$

Since $m_1 m_3 = \left(-\frac{3}{2}\right)\left(\frac{2}{3}\right) = -1$

$\Rightarrow$ The triangle ABC is a right triangle with $m\angle A = 90^\circ$

> **REMEMBER**
> The symbols
> (*i*) $\|$ stands for 'parallel"
> (*ii*) $\nparallel$ stands for "not parallel"
> (*iii*) $\perp$ stands for "perpendicular"

Question # 6

The three points $A(7,-1), B(-2,2)$ and $C(1,4)$ are consecutive vertices of a parallelogram. Find the fourth vertex.

Solution

Let $D(a,b)$ be a fourth vertex of the parallelogram.

Slope of $\overline{AB} = \frac{2+1}{-2-7} = \frac{3}{-9} = -\frac{1}{3}$

Slope of $\overline{BC} = \frac{4-2}{1+2} = \frac{2}{3}$

Slope of $\overline{CD} = \frac{b-4}{a-1}$

Slope of $\overline{DA} = \frac{-1-b}{7-a}$

Since $ABCD$ is a parallelogram therefore

Slope of $\overline{AB}$ = Slope of $\overline{CD}$

$\Rightarrow -\dfrac{1}{3}=\dfrac{b-4}{a-1} \qquad \Rightarrow -(a-1)=3(b-4)$

$\Rightarrow -a+1-3b+12=0 \quad \Rightarrow -a-3b+13=0 \ldots$(i)

Also slope of $\overline{BC}=$ slope of $\overline{DA}$

$\Rightarrow \dfrac{2}{3}=\dfrac{-1-b}{7-a} \qquad \Rightarrow 2(7-a)=3(-1-b) \qquad \Rightarrow 14-2a=-3-3b$

$\Rightarrow 14-2a+3+3b=0 \quad \Rightarrow -2a+3b+17=0\ldots$ (ii)

Adding (i) and (ii)

$$\begin{array}{l} -a-3b+13=0 \\ -2a+3b+17=0 \\ \hline -3a \qquad +30=0 \end{array} \quad \Rightarrow 3a=30 \quad \Rightarrow \boxed{a=10}$$

Putting value of a in (i)

$-10-3b+13=0 \quad \Rightarrow -3b+3=0 \quad \Rightarrow 3b=3 \Rightarrow \boxed{b=1}$

Hence $D(10,1)$ is the fourth vertex of parallelogram.

Question # 7

The points $A(-1,2), B(3,-1)$ and $C(6,3)$ are consecutive vertices of a rhombus. Find the fourth vertex and show that the diagonals of the rhombus are perpendicular to each other.

Solution

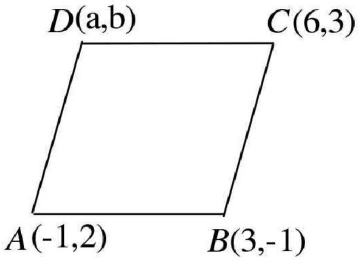
Let $D(a,b)$ be a fourth vertex of rhombus.

Slope of $\overline{AB}=\dfrac{-1-2}{3+1}=\dfrac{-3}{4}$

Slope of $\overline{BC}=\dfrac{3+1}{6-3}=\dfrac{4}{3}$

Slope of $\overline{CD}=\dfrac{b-3}{a-6}$

Slope of $\overline{DA}=\dfrac{2-b}{-1-a}$



Since $ABCD$ is a rhombus therefore

Slope of $\overline{AB}=$ Slope of $\overline{CD}$

$\Rightarrow -\dfrac{3}{4}=\dfrac{b-3}{a-6} \qquad \Rightarrow -3(a-6)=4(b-3)$

$\Rightarrow -3a+18=4b-12 \qquad \Rightarrow -3a+18-4b+12=0$

$\Rightarrow -3a-4b+30=0\ldots$ (i)

Also slope of $\overline{BC}=$ slope of $\overline{DA}$

$\Rightarrow \dfrac{4}{3}=\dfrac{2-b}{-1-a} \qquad \Rightarrow 4(-1-a)=3(2-b)$

$\Rightarrow -4-4a=6-3b \qquad \Rightarrow -4-4a-6+3b=0$

$\Rightarrow -4a+3b-10=0\ldots$ (ii)

$\times$ing eq. (i) by 3 and (ii) by 4 and adding.

$$\begin{array}{l} -9a \ -12b + 90 = 0 \\ -16a + 12b - 40 = 0 \\ \hline -25a \qquad\quad + 50 = 0 \end{array} \Rightarrow 25a = 50 \quad \Rightarrow \boxed{a = 2}$$

Putting value of a in (ii)

$$-4(2) + 3b - 10 = 0 \Rightarrow 3b - 18 = 0 \quad \Rightarrow 3b = 18 \Rightarrow \boxed{b = 6}$$

Hence $D(2,6)$ is the fourth vertex of rhombus.

Now slope of diagonal $\overline{AC} = \dfrac{3-2}{6+1} = \dfrac{1}{7}$

Slope of diagonal $\overline{BD} = \dfrac{b-(-1)}{a-3} = \dfrac{6+1}{2-3} = \dfrac{7}{-1} = -7$

Since

(Slope of $\overline{AC}$)(Slope of $\overline{BD}$) = $\left(\dfrac{1}{7}\right)(-7) = -1$

$\Rightarrow$ Diagonals of a rhombus are $\perp$ to each other.

Question # 8

Two pairs of points are given. Find whether the two lines determined by these points are:

(i) Parallel (ii) perpendicular (iii) none

(a) $(1,-2),(2,4) \, and \, (4,1),(-8,2)$ (b) $(-3,4),(6,2) \, and \, (4,5),(-2,-7)$

Solution

(a) Slope of line joining $(1,-2)$ and $(2,4) = m_1 = \dfrac{4+2}{2-1} = \dfrac{6}{1} = 6$

Slope of line joining $(4,1)$ and $(-8,2) = m_2 = \dfrac{2-1}{-8-4} = \dfrac{1}{-12}$

Since $m_1 \neq m_2$

Also $m_1 m_2 = 6 \cdot \dfrac{1}{-12} = -\dfrac{1}{2} \neq -1$

$\Rightarrow$ lines are neither parallel nor perpendicular.

(b) *Do yourself as above.*

Equation of Straight Line:

(i) **Slope-intercept form**

Equation of straight line with slope m and $y-$intercept c is given by:

$$\boxed{y = mx + c}$$

See proof on book at page 194

(ii) **Point-slope form**

Let m be a slope of line and $A(x_1, y_1)$ be a point lies on a line then equation of line is given by:

$$\boxed{y - y_1 = m(x - x_1)}$$

See proof on book at page 195

(iii) **Symmetric form**

Let α be an inclination of line and $A(x_1, y_1)$ be a point lies on a line then equation of line is given by:

$$\boxed{\frac{y-y_1}{\cos\alpha}=\frac{x-x_1}{\sin\alpha}}$$

See proof on book at page 195

(iv) **Two-points form**

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be points lie on a line then it's equation is given by:

$$\boxed{y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)} \quad \text{or} \quad \boxed{y-y_2=\frac{y_2-y_1}{x_2-x_1}(x-x_2)} \quad \text{or} \quad \boxed{\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0}$$

See proof on book at page 196

(v) **Two-intercept form**

When a line intersect $x-$axis at $x=a$ and $y-$axis at $y=b$ i.e. $x-$intercept $=a$ and $y-$intercept $=b$, then equation of line is given by:

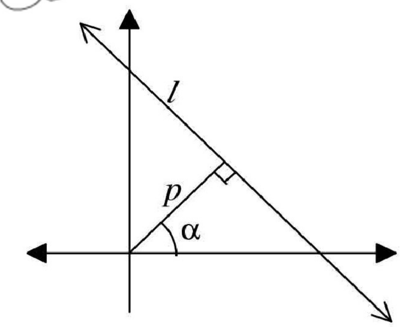
$$\boxed{\frac{x}{a}+\frac{y}{b}=1}$$

See proof on book at page 197

(vi) **Normal form**

Let p denoted length of perpendicular from the origin to the line and α is the angle of the perpendicular from +ive x-axis then equation of line is given by:

$$\boxed{x\cos\alpha+y\sin\alpha=p}$$



See proof on book at page 198

Question # 9

Find an equation of

(a) the horizontal line through $(7,-9)$

(b) the vertical line through $(-5,3)$

(c) the line bisecting the first and third quadrants.

(d) the line bisecting the second and fourth quadrants.

Solution

(a) Since slope of horizontal line $=m=0$

$$\& \ (x_1, y_1)=(7,-9)$$

therefore equation of line:

$$y-(-9)=0(x-7)$$

$$\Rightarrow \ y+9=0 \quad \text{Answer}$$

(b) Since slope of vertical line $m=\infty=\dfrac{1}{0}$

$$\& \ (x_1, y_1)=(-5,3)$$

therefore required equation of line

$$y-3=\infty\left(x-(-5)\right)$$

$$\Rightarrow\ y-3=\frac{1}{0}(x+5)\quad\Rightarrow\ 0(y-3)=1(x+5)$$

$$\Rightarrow\ x+5=0$$ Answer

(c) The line bisecting the first and third quadrant makes an angle of 45° with the $x-$axis therefore slope of line $=m=\tan 45^\circ=1$

Also it passes through origin $(0,0)$, so its equation

$$y-0=1(x-0)\quad\Rightarrow\ y=x$$

$$\Rightarrow\ x-y=0$$ Answer

(d) The line bisecting the second and fourth quadrant makes an angle of 135° with $x-$axis therefore slope of line $=m=\tan 135^\circ=-1$

Also it passes through origin $(0,0)$, so its equation

$$y-0=-1(x-0)\quad\Rightarrow\ y=-x$$

$$\Rightarrow\ x+y=0$$ Answer

Question # 10

Find an equation of the line

(a) through $A(-6,5)$ having slope 7

(b) through $(8,-3)$ having slope 0

(c) through $(-8,5)$ having slope undefined

(d) through $(-5,-3)$ and $(9,-1)$

(e) $y-intercept\ -7$ and $slope\ -5$

(f) $x-intercept:-3$ and $y-intercept:-4$

(g) $x-intercept:-9$ and $slope:-4$

Solution

(a) $\because\ (x_1,y_1)=(-6,5)$

and slope of line $=m=7$

so required equation

$$y-5=7\left(x-(-6)\right)$$

$$\Rightarrow\ y-5=7(x+6)\quad\Rightarrow\ y-5=7x+42$$

$$\Rightarrow\ 7x+42-y+5=0\quad\Rightarrow\ 7x-y+47=0$$ Answer

(b) *Do yourself as above.*

(c) $\because\ (x_1,y_1)=(-8,5)$

and slope of line $=m=\infty$

So required equation

$$y-5=\infty\left(x-(-8)\right)$$

$$\Rightarrow\ y-5=\frac{1}{0}(x+8)\quad\Rightarrow\ 0(y-5)=1(x+8)$$

$$\Rightarrow\ x+8=0$$ Answer

(d) The line through $(-5,-3)$ and $(9,-1)$ is

$$y-(-3)=\frac{-1-(-3)}{9-(-5)}(x-(-5)) \quad \Rightarrow \quad y+3=\frac{2}{14}(x+5)$$

$$\Rightarrow \quad y+3=\frac{1}{7}(x+5) \qquad \Rightarrow \quad 7y+21=x+5$$

$$\Rightarrow \quad x+5-7y-21=0 \qquad \Rightarrow \quad x-7y-16=0$$ Answer

(e) $\because$ $y-$ intercept $=-7$

$\Rightarrow$ $(0,-7)$ lies on a required line

Also slope $=m=-5$

So required equation

$y-(-7)=-5(x-0)$

$\Rightarrow$ $y+7=-5x$ $\quad \Rightarrow$ $5x+y+7=0$ Answer

(f) $\because$ $x-$intercept $=-9$

$\Rightarrow$ $(-9,0)$ lies on a required line

Also slope $=m=4$

Therefore required line

$$y-0=4(x+9)$$

$\Rightarrow$ $y=4x+9$ $\quad \Rightarrow$ $4x-y+9=0$ Answer

(g) $x-$intercept $=a=-3$

$y-$intercept $=b=4$

Using two-intercept form of equation line

$$\frac{x}{a}+\frac{y}{b}=1 \quad \Rightarrow \quad \frac{x}{-3}+\frac{y}{4}=0$$

$\Rightarrow$ $4x-3y=-12$ $\qquad$ $\times$ing by -12

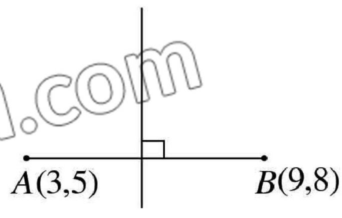
$\Rightarrow$ $4x-3y+12=0$ Answer

Question # 11

Find an equation of the perpendicular bisector of the segment joining the points $A(3,5)$ and $B(9,8)$

Solution

Given points $A(3,5)$ and $B(9,8)$



Midpoint of $\overline{AB}=\left(\frac{3+9}{2},\frac{5+8}{2}\right)=\left(\frac{12}{2},\frac{13}{2}\right)=\left(6,\frac{13}{2}\right)$

Slope of $\overline{AB}=m=\frac{8-5}{9-3}=\frac{3}{6}=\frac{1}{2}$

Slope of line $\perp$ to $\overline{AB}=-\frac{1}{m}=-\frac{1}{\frac{1}{2}}=-2$

Now equation of $\perp$ bisector having slope -2 through $\left(6,\frac{13}{2}\right)$

$\Rightarrow\ y-\dfrac{13}{2}=-2(x-6)$

$\Rightarrow\ y-\dfrac{13}{2}=-2x+12 \qquad \Rightarrow\ y-\dfrac{13}{2}+2x-12=0$

$\Rightarrow\ 2x+y-\dfrac{37}{2}=0 \qquad \Rightarrow\ 4x+2y-37=0$

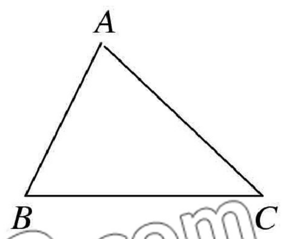
Question # 12

Find equations of the sides, altitudes and medians of the triangle whose vertices are $A(-3,2), B(5,4)$ and $C(3,-8)$.

Solution

Given vertices of triangle are $A(-3,2)$, $B(5,4)$ and $C(3,-8)$.

Equation of sides:



Slope of $\overline{AB}=m_1=\dfrac{4-2}{5-(-3)}=\dfrac{2}{8}=\dfrac{1}{4}$

Slope of $\overline{BC}=m_2=\dfrac{-8-4}{3-5}=\dfrac{-12}{-2}=6$

Slope of $\overline{CA}=m_3=\dfrac{2-(-8)}{-3-3}=\dfrac{10}{-6}=-\dfrac{5}{3}$

Now equation of side $\overline{AB}$ having slope $\dfrac{1}{4}$ passing through $A(-3,2)$

[You may take $B(5,4)$ instead of $A(-3,2)$]

$y-2=\dfrac{1}{4}(x-(-3)) \quad \Rightarrow\ 4y-8=x+3$

$\Rightarrow\ x+3-4y+8=0 \quad \Rightarrow\ \boxed{x-4y+11=0}$

Equation of side $\overline{BC}$ having slope 6 passing through $B(5,4)$.

$y-4=6(x-5) \qquad \Rightarrow\ y-4=6x-30$

$\Rightarrow\ 6x-30-y+4=0 \quad \Rightarrow\ \boxed{6x-y-26=0}$

Equation of side $\overline{CA}$ having slope $-\dfrac{5}{3}$ passing through $C(3,-8)$

$y-(-8)=-\dfrac{5}{3}(x-3) \qquad \Rightarrow\ 3(y+8)=-5(x-3)$

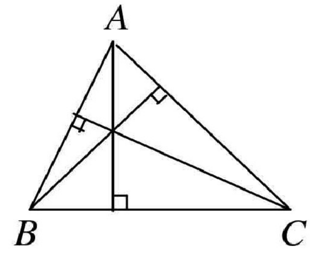
$\Rightarrow\ 3y+24=-5x+15 \qquad \Rightarrow\ 5x-15+3y+24=0$

$\Rightarrow\ \boxed{5x+3y+9=0}$

Equation of altitudes:

Since altitudes are perpendicular to the sides of triangle therefore

Slope of altitude on $\overline{AB}=-\dfrac{1}{m_1}=-\dfrac{1}{1/4}=-4$



Equation of altitude from $C(3,-8)$ having slope -4

$y+8=-4(x-3) \qquad \Rightarrow\ y+8=-4x+12$

$$\Rightarrow\ 4x-12+y+8=0 \quad \Rightarrow \boxed{4x+y-4=0}$$

Slope of altitude on $\overline{BC}=-\dfrac{1}{m_2}=-\dfrac{1}{6}$

Equation of altitude from $A(-3,2)$ having slope $-\dfrac{1}{6}$

$$y-2=-\frac{1}{6}(x+3) \quad \Rightarrow\ 6y-12=-x-3$$

$$\Rightarrow\ x+3+6y-12=0 \quad \Rightarrow \boxed{x+6y-9=0}$$

Slope of altitude on $\overline{CA}=-\dfrac{1}{m_3}=-\dfrac{1}{-5/3}=\dfrac{3}{5}$

Equation of altitude from $B(5,4)$ having slope $\dfrac{3}{5}$

$$y-4=\frac{3}{5}(x-5) \quad \Rightarrow\ 5y-20=3x-15$$

$$\Rightarrow\ 3x-15-5y+20=0 \quad \Rightarrow \boxed{3x-5y+5=0}$$

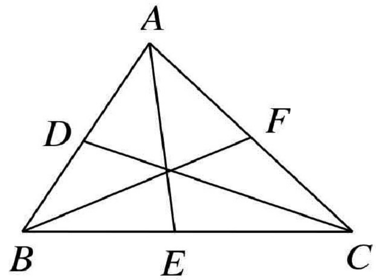
Equation of Medians:

Suppose D, E and F are midpoints of sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively.

Then coordinate of $D=\left(\dfrac{-3+5}{2},\dfrac{2+4}{2}\right)=\left(\dfrac{2}{2},\dfrac{6}{2}\right)=(1,3)$

Coordinate of $E=\left(\dfrac{5+3}{2},\dfrac{4-8}{2}\right)=\left(\dfrac{8}{2},\dfrac{-4}{2}\right)=(4,-2)$

Coordinate of $F=\left(\dfrac{3-3}{2},\dfrac{-8+2}{2}\right)=\left(\dfrac{0}{2},\dfrac{-6}{2}\right)=(0,-3)$



Equation of median $\overline{AE}$ by two-point form

$$y-2=\frac{-2-2}{4-(-3)}(x-(-3))$$

$$\Rightarrow\ y-2=\frac{-4}{7}(x+3) \quad \Rightarrow\ 7y-14=-4x-12$$

$$\Rightarrow\ 7y-14+4x+12=0 \quad \Rightarrow \boxed{4x+7y-2=0}$$

Equation of median $\overline{BF}$ by two-point form

$$y-4=\frac{-3-4}{0-5}(x-5)$$

$$\Rightarrow\ y-4=\frac{-7}{-5}(x-5) \quad \Rightarrow\ -5y+20=-7x+35$$

$$\Rightarrow\ -5y+20+7x-35=0 \quad \Rightarrow \boxed{7x-5y-15=0}$$

Equation of median $\overline{CD}$ by two-point form

$$y-(-8)=\frac{3-(-8)}{1-3}(x-3)$$

$$\Rightarrow\ y+8=\frac{11}{-2}(x-3) \quad \Rightarrow\ -2y-16=11x-33$$

$$\Rightarrow\ 11x-33+2y+16=0 \quad \Rightarrow\ \boxed{11x+2y-17=0}$$

Question # 13
Find an equation of the line through $(-4,-6)$ and perpendicular to the line having slope $\frac{-3}{2}$.

Solution

Here $(x_1, y_1)=(-4,-6)$

Slope of given line $=m=\frac{-3}{2}$

$\because$ required line is $\perp$ to given line

$\therefore$ slope of required line $=-\frac{1}{m}=-\frac{1}{-3/2}=\frac{2}{3}$

Now equation of line having slope $\frac{2}{3}$ passing through $(-4,-6)$

$$y-(-6)=\frac{2}{3}(x-(-4))$$

$$\Rightarrow\ 3(y+6)=2(x+4) \quad \Rightarrow\ 3y+18=2x+8$$

$$\Rightarrow\ 2x+8-3y-18=0 \quad \Rightarrow\ 2x-3y-10=0$$

Question # 14
Find an equation of the line through $(11,-5)$ and parallel to a line with slope -24.

Solution

Here $(x_1, y_1)=(11,-5)$

Slope of given line $=m=-24$

$\because$ required line is $||$ to given line

$\therefore$ slope of required line $=m=-24$

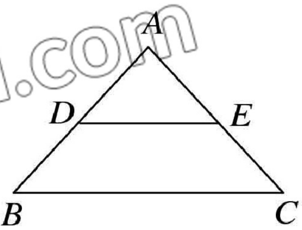
Now equation of line having slope -24 passing through $(11,-5)$

$$y-(-5)=-24(x-11)$$

$$\Rightarrow\ y+5=-24x+264 \quad \Rightarrow\ 24x-264+y+5=0$$

$$\Rightarrow\ 24x+y-259=0$$

Question # 15
The points $A(-1,2), B(6,3)$ and $C(2,-4)$ are vertices of a triangle. Show that the line joining the midpoint D of AB and the midpoint E of AC is parallel to BC and $DE=\frac{1}{2}BC$.



Solution Given vertices $A(-1,2)$, $B(6,3)$ and $C(2,-4)$

Since D and E are midpoints of sides $\overline{AB}$ and $\overline{AC}$ respectively.

Therefore coordinate of $D=\left(\dfrac{-1+6}{2},\dfrac{2+3}{2}\right)=\left(\dfrac{5}{2},\dfrac{5}{2}\right)$

Coordinate of $E=\left(\dfrac{-1+2}{2},\dfrac{2-4}{2}\right)=\left(\dfrac{1}{2},\dfrac{-2}{2}\right)=\left(\dfrac{1}{2},-1\right)$

Now slope of $\overline{DE}=\dfrac{-1-\frac{5}{2}}{\frac{1}{2}-\frac{5}{2}}=\dfrac{-\frac{7}{2}}{-\frac{4}{2}}=\dfrac{7}{4}$

slope of $\overline{BC}=\dfrac{-4-3}{2-6}=\dfrac{-7}{-4}=\dfrac{7}{4}$

Since slope of $\overline{DE}=$ slope of $\overline{BC}$

Therefore $\overline{DE}$ is parallel to $\overline{BC}$.

Now $\left|\overline{DE}\right|=\sqrt{\left(\dfrac{1}{2}-\dfrac{5}{2}\right)^2+\left(-1-\dfrac{5}{2}\right)^2}=\sqrt{\left(-\dfrac{4}{2}\right)^2+\left(-\dfrac{7}{2}\right)^2}$

$=\sqrt{4+\dfrac{49}{4}}=\sqrt{\dfrac{65}{4}}=\dfrac{\sqrt{65}}{2}$ ………... (i)

$\left|\overline{BC}\right|=\sqrt{(2-6)^2+(-4-3)^2}=\sqrt{(-4)^2+(-7)^2}$

$=\sqrt{16+49}=\sqrt{65}$ ………... (ii)

From (i) and (ii)

$$\left|\overline{DE}\right|=\frac{1}{2}\left|\overline{BC}\right|$$

Question # 16

A milkman can sell 560 litres of milk at $Rs12.50$ per litre and 700 litres of milk at $Rs12.00$ per litre. Assuming the graph of the sale price and the milk sold to be a straight line, find the number of litres of milk that the milkman can sell at $Rs12.25$ per litre.

Solution

Let l denotes the number of litres of milk and p denotes the price of milk,

Then $(l_1,p_1)=(560,12.50)$ & $(l_2,p_2)=(700,12.00)$

Since graph of sale price and milk sold is a straight line

Therefore, from two point form, it's equation

$$p-p_1=\frac{p_2-p_1}{l_2-l_1}(l-l_1)$$

$$\Rightarrow\ p-12.50=\frac{12.00-12.50}{700-560}(l-560)$$

$$\Rightarrow\ p-12.50=\frac{-0.50}{140}(l-560)$$

$$\Rightarrow\ 140p-1750=-0.50l+280$$

$$\Rightarrow\ 140p-1750+0.50l-280=0$$

$$\Rightarrow\ 0.50l+140p-2030=0$$

ALTERNATIVE
You may use determinant form of two-point form to find an equation of line.

$$\begin{vmatrix} l & p & 1\\ l_1 & p_1 & 1\\ l_2 & p_2 & 1\end{vmatrix}=0$$

If $p=12.25$

$$\Rightarrow 0.50l + 140(12.25) - 2030 = 0$$

$$\Rightarrow 0.50l + 1715 - 2030 = 0 \quad \Rightarrow 0.50l - 315 = 0$$

$$\Rightarrow 0.50l = 315 \quad \Rightarrow l = \frac{315}{0.50} = 630$$

Hence milkman can sell 630 litres milk at Rs. 12.25 per litre.

Question # 17

The population of Pakistan to the nearest million was 60 million in 1961 and 95 million in 1981. Using t as the number of years after 1961, Find an equation of the line that gives the population in terms of t. Use this equation to find the population in

(a) 1947 (b) 1997

Solution

Let p denotes population of Pakistan in million and t denotes year after 1961,

Then $(p_1, t_1) = (60, 1961)$ and $(p_2, t_2) = (95, 1981)$

Equation of line by two point form:

$$t - t_1 = \frac{t_2 - t_1}{p_2 - p_1}(p - p_1)$$

$$\Rightarrow t - 1961 = \frac{1981 - 1961}{95 - 60}(p - 60)$$

$$\Rightarrow t - 1961 = \frac{20}{35}(p - 60) \quad \Rightarrow t - 1961 = \frac{4}{7}(p - 60)$$

$$\Rightarrow 7t - 13727 = 4p - 240 \quad \Rightarrow 7t - 13727 + 240 = 4p$$

$$\Rightarrow 4p = 7t - 13487 \quad \Rightarrow p = \frac{7}{4}t - \frac{13487}{4} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

This is the required equation which gives population in term of t.

(a) Put $t = 1947$ in eq. (i)

$$p = \frac{7}{4}(1947) - \frac{13487}{4} = 3407.25 - 3371.75 = 35.5$$

Hence population in 1947 is 35.5 millions.

(b) Put $t = 1997$ in eq. (i)

$$p = \frac{7}{4}(1997) - \frac{13487}{4} = 3494.75 - 3371.75 = 123$$

Hence population in 1997 is 123 millions.

Question # 18

A house was purchased for $Rs1$ million in 1980. It is worth $Rs4$ million in 1996. Assuming that the value increased by the same amount each year, find an equation that gives the value of the house after t years of the date of purchase. What was the value in 1990?

Solution

Let p denotes purchase price of house in millions and t denotes year then

$$(p_1, t_1) = (1, 1980) \quad \text{and} \quad (p_2, t_2) = (4, 1996)$$

Equation of line by two point form:

$$t - t_1 = \frac{t_2 - t_1}{p_2 - p_1}(p - p_1)$$

$$\Rightarrow t - 1980 = \frac{1996 - 1980}{4 - 1}(p - 1)$$

$$\Rightarrow t - 1980 = \frac{16}{3}(p - 1)$$

$$\Rightarrow 3t - 5940 = 16p - 16$$

$$\Rightarrow 3t - 5940 + 16 = 16p \qquad \Rightarrow 16p = 3t - 5924$$

$$\Rightarrow p = \frac{3}{16}t - \frac{5924}{16} \qquad \Rightarrow p = \frac{3}{16}t - \frac{1481}{4} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

ALTERNATIVE
You may use determinant form of two-point form to find an equation of line.

$$\begin{vmatrix} p & t & 1 \\ p_1 & t_1 & 1 \\ p_2 & t_2 & 1 \end{vmatrix} = 0$$

This is the required equation which gives value of house in term of t.

Put $t = 1990$ in eq. (i)

$$p = \frac{3}{16}(1990) - \frac{1481}{4} = 373.125 - 370.25 = 2.875$$

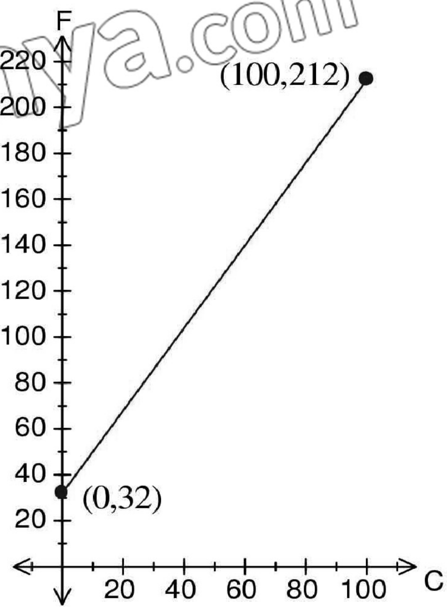
Hence value of house in 1990 is 2.875 millions.

Question # 19

Plot the Celsius (C) and Fahrenheit (F) temperature scales on the horizontal axis and the vertical axis respectively. Draw the line joining the freezing point and the boiling point of water. Find an equation giving F temperature in term of C.

Solution

Since freezing point of water $= 0^\circ C = 32^\circ F$
and boiling point of water $= 100^\circ C = 212^\circ F$
therefore we have points $(C_1, F_1) = (0, 32)$ and $(C_2, F_2) = (100, 212)$



Take scale 10ss = 20C and 10ss = 20F on x-axis and y-axis respectively to draw graph.

Equation of line by two point form

$$F - F_1 = \frac{F_2 - F_1}{C_2 - C_1}(C - C_1)$$

$$\Rightarrow F - 32 = \frac{212 - 32}{100 - 0}(C - 0)$$

$$\Rightarrow F - 32 = \frac{180}{100}C$$

$$\Rightarrow F = \frac{9}{5}C + 32$$

Question # 20

The average entry test score of engineering candidates was in the year 1998 while the score was 564 in 2002. Assuming that the relationship between time and score is linear, find the average score for 2006.

Solution

Let s denotes entry test score and y denotes year.

Then we have $(s_1, y_1) = (592, 1998)$ and $(s_2, y_2) = (564, 2002)$

By two point form of equation of line

$$y - y_1 = \frac{y_2 - y_1}{s_2 - s_1}(s - s_1)$$

$$\Rightarrow\ y - 1998 = \frac{2002 - 1998}{564 - 592}(s - 592) \quad \Rightarrow\ y - 1998 = \frac{4}{-28}(s - 592)$$

$$\Rightarrow\ y - 1998 = -\frac{1}{7}(s - 592) \quad \Rightarrow\ 7y - 13986 = -s + 592$$

$$\Rightarrow\ 7y - 13986 + s - 592 = 0 \quad \Rightarrow\ s + 7y - 14578 = 0$$

Put $y = 2006$ in (i)

$$s + 7(2006) - 14578 = 0 \quad \Rightarrow\ s + 14042 - 14578 = 0$$

$$\Rightarrow\ s - 536 = 0 \quad \Rightarrow\ s = 536$$

Hence in 2006 the average score will be 536.

Question # 21

Convert each of the following equation into

(i) Slope intercept form (ii) Two-intercept form (iii) Normal form

(a) $2x - 4y + 11 = 0$ (b) $4x + 7y - 2 = 0$ (c) $15y - 8x + 3 = 0$

Also find the length of the perpendicular from $(0,0)$ to each line.

Solution

(a)

(i) **- Slope-intercept form**

$$\because\ 2x - 4y + 11 = 0$$

$$\Rightarrow\ 4y = 2x + 11 \quad \Rightarrow\ y = \frac{2x + 11}{4}$$

$$\Rightarrow\ y = \frac{1}{2}x + \frac{11}{4}$$

is the intercept form of equation of line with $m = \frac{1}{2}$ and $c = \frac{11}{4}$

(ii) **- Two-intercept form**

$$\because\ 2x - 4y + 11 = 0 \quad \Rightarrow\ 2x - 4y = -11$$

$$\Rightarrow\ \frac{2}{-11}x - \frac{4}{-11}y = 1 \quad \Rightarrow\ \frac{x}{-11/2} + \frac{y}{11/4} = 1$$

is the two-point form of equation of line with $a = -\frac{11}{2}$ and $b = \frac{11}{4}$.

(iii) **- Normal form**

$$\because\ 2x - 4y + 11 = 0 \quad \Rightarrow\ 2x - 4y = -11$$

Dividing above equation by $\sqrt{(2)^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$

$$\frac{2x}{2\sqrt{5}} - \frac{4y}{2\sqrt{5}} = \frac{-11}{2\sqrt{5}} \quad \Rightarrow\ \frac{x}{\sqrt{5}} - \frac{2y}{\sqrt{5}} = \frac{-11}{2\sqrt{5}}$$

Point of intersection of lines

Let $l_1:\ a_1x+b_1y+c_1=0$

$l_2:\ a_2x+b_2y+c_2=0$ be non-parallel lines.

Let $P(x_1,y_1)$ be the point of intersection of l_1 and l_2. Then

$$a_1x_1+b_1y_1+c_1=0............(i)$$
$$a_2x_1+b_2y_1+c_2=0............(ii)$$

Solving (i) and (ii) simultaneously, we have

$$\frac{x_1}{b_1c_2-b_2c_1}=\frac{-y_1}{a_1c_2-a_2c_1}=\frac{1}{a_1b_2-a_2b_1}$$

$$\Rightarrow \frac{x_1}{b_1c_2-b_2c_1}=\frac{1}{a_1b_2-a_2b_1} \text{ and } \frac{-y_1}{a_1c_2-a_2c_1}=\frac{1}{a_1b_2-a_2b_1}$$

$$\Rightarrow x_1=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1} \text{ and } y_1=-\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$$

Hence $\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},-\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}\right)$ is the point of intersection of l_1 and l_2.

Equation of line passing through the point of intersection.

Let $l_1:\ a_1x+b_1y+c_1=0$

$l_2:\ a_2x+b_2y+c_2=0$

Then equation of line passing through the point of intersection of l_1 and l_2 is

$$l_1+k\,l_2=0\ ,\ \text{where } k \text{ is constant.}$$

i.e. $a_1x+b_1y+c_{11}+k\left(a_2x+b_2y+c_2\right)=0$

Question # 1

Find the point of intersection of the lines

(i) $x-2y+1=0 \quad 2x-y+2=0$ (ii) $3x+y+12=0 \quad x+2y-1=0$

(iii) $x+4y-12=0 \quad x-3y+3=0$

Solution

(i) $l_1:\ x-2y+1=0$

$l_2:\ 2x-y+2=0$

Slope of $l_1=m_1=-\dfrac{1}{-2}=\dfrac{1}{2}$

Slope of $l_2=m_2=-\dfrac{2}{-1}=2$

$\because\ m_1\neq m_2$, therefore lines are intersecting.

Now if (x,y) is the point of intersection of l_1 and l_2 then

$$\frac{x}{(-2)(2)-(-1)(1)} = \frac{-y}{(1)(2)-(2)(1)} = \frac{1}{(1)(-1)-(2)(-2)}$$

$$\Rightarrow \frac{x}{-4+1} = \frac{-y}{2-2} = \frac{1}{-1+4}$$

$$\Rightarrow \frac{x}{-3} = \frac{-y}{0} = \frac{1}{3}$$

$$\Rightarrow \frac{x}{-3} = \frac{1}{3} \quad \text{and} \quad \frac{-y}{0} = \frac{1}{3}$$

$$\Rightarrow x = \frac{-3}{3} \quad \text{and} \quad y = -\frac{0}{3}$$

$$\Rightarrow x = -1 \quad \text{and} \quad y = 0$$

Hence $(-1,0)$ is the point of intersection.

(ii) $l_1: \ 3x + y + 12 = 0$

$l_2: \ x + 2y - 1 = 0$

Slope of $l_1 = m_1 = -\frac{3}{1} = -3$

Slope of $l_2 = m_2 = -\frac{1}{2}$

$\because \ m_1 \neq m_2$, therefore lines are intersecting.

Now if (x, y) is the point of intersection of l_1 and l_2 then

$$\frac{x}{-1-24} = \frac{-y}{-3-12} = \frac{1}{6-1}$$

$$\Rightarrow \frac{x}{-25} = \frac{-y}{-15} = \frac{1}{5}$$

$$\Rightarrow \frac{x}{-25} = \frac{1}{5} \quad \text{and} \quad \frac{-y}{-15} = \frac{1}{5}$$

$$\Rightarrow x = \frac{-25}{5} = -5 \quad \text{and} \quad y = \frac{15}{5} = 3$$

Hence $(-5,3)$ is the point of intersection.

(iii) *Do yourself as above.*

Question # 2

Find an equation of the line through

(i) the point $(2,-9)$ and the intersection of the lines $2x + 5y - 8 = 0$ and $3x - 4y - 6 = 0$

(ii) the intersection of the lines $x - y - 4 = 0$ $\quad 7x + y + 20 = 0$ $\quad 6x + y - 14 = 0$

(a) Parallel (ii) Perpendicular

to the line $6x + y - 14 = 0$

(iii) through the intersection of the lines $x + 2y + 3 = 0$, $\ 3x + 4y + 7 = 0$

And making equal intercepts on the axes.

Solution

(i) Let $l_1:\ 2x+5y-8=0$

$l_2:\ 3x-4y-6=0$

Equation of line passing through point of intersection of l_1 and l_2 is

$$2x+5y-8+k(3x-4y-6)=0 \ldots \text{ (i)}$$

Since $(2,-9)$ lies on (i) therefore put $x=2$ and $y=-9$ in (i)

$$2(2)+5(-9)-8+k(3(2)-4(-9)-6)=0$$

$$\Rightarrow 4-45-8+k(6+36-6)=0$$

$$\Rightarrow -49+36k=0$$

$$\Rightarrow 36k=49 \qquad \Rightarrow k=\frac{49}{36}$$

Putting value of k in (i)

$$2x+5y-8+\frac{49}{36}(3x-4y-6)=0$$

$$\Rightarrow 72x+180y-288+49(3x-4y-6)=0 \qquad \text{×ing by 36}$$

$$\Rightarrow 72x+180y-288+147x-196y-294=0$$

$\Rightarrow 219x-16y-582=0$ is the required equation.

(ii) Let $l_1:\ x-y-4=0$

$l_2:\ 7x+y+20=0$

$l_3:\ 6x+y-14=0$

Let l_4 be a line passing through point of intersection of l_1 and l_2, then

$$l_4:\ l_1+kl_2=0$$

$$\Rightarrow x-y-4+k(7x+y+20)=0 \ldots \quad \text{(i)}$$

$$\Rightarrow (1+7k)x+(-1+k)y+(-4+20k)=0$$

$$\text{Slope of } l_4=m_1=-\frac{1+7k}{-1+k}$$

$$\text{Slope of } l_3=m_2=-\frac{6}{1}=-6$$

(a) If l_3 and l_4 are parallel then

$$m_1=m_2$$

$$\Rightarrow -\frac{1+7k}{-1+k}=-6$$

$$\Rightarrow 1+7k=6(-1+k) \qquad \Rightarrow 1+7k=-6+6k$$

$$\Rightarrow 7k-6k=-6-1 \qquad \Rightarrow k=-7$$

Putting value of k in (i)

$$x-y-4-7(7x+y+20)=0$$

$$\Rightarrow x-y-4-49x-7y-140=0$$

$$\Rightarrow\ -48x-8y-144=0$$
$$\Rightarrow\ 6x+y+18=0$$
is the required equation

(b) If l_3 and l_4 are $\perp$ then

$$m_1m_2=-1$$
$$\Rightarrow\ \left(-\frac{1+7k}{-1+k}\right)(-6)=-1$$
$$\Rightarrow\ 6(1+7k)=-(-1+k) \qquad \Rightarrow\ 6+42k=1-k$$
$$\Rightarrow\ 42k+k=1-6 \qquad \Rightarrow\ 43k=-5 \qquad \Rightarrow\ k=-\frac{5}{43}$$

Putting in (i) we have

$$x-y-4-\frac{5}{43}(7x+y+20)=0$$
$$\Rightarrow\ 43x-43y-172-5(7x+y+20)=0$$
$$\Rightarrow\ 43x-43y-172-35x-5y-100=0$$
$$\Rightarrow\ 8x-48y-272=0$$
$$\Rightarrow\ x-6y-34=0 \text{ is the required equation.}$$

(iii) Suppose $l_1:\ x+2y+3=0$

$l_2:\ 3x+4y+7=0$

Equation of line passing through the intersection of l_1 and l_2 is given by:

$$x+2y+3+k(3x+4y+7)=0 \ \ldots\ldots\ldots\ldots \text{(i)}$$
$$\Rightarrow\ (1+3k)x+(2+4k)y+(3+7k)=0$$
$$\Rightarrow\ (1+3k)x+(2+4k)y=-(3+7k)$$
$$\Rightarrow\ \frac{(1+3k)x}{-(3+7k)}+\frac{(2+4k)y}{-(3+7k)}=1$$
$$\Rightarrow\ \frac{x}{-(3+7k)/(1+3k)}+\frac{y}{-(3+7k)/(2+4k)}=1$$

Which is two-intercept form of equation of line with

$$x-\text{intercept}=\frac{-(3+7k)}{(1+3k)} \quad \text{and} \quad y-\text{intercept}=\frac{-(3+7k)}{(2+4k)}$$

We have given

$$x-\text{intercept} = y-\text{intercept}$$
$$\Rightarrow\ \frac{-(3+7k)}{(1+3k)}=\frac{-(3+7k)}{(2+4k)}$$
$$\Rightarrow\ \frac{1}{(1+3k)}=\frac{1}{(2+4k)} \qquad \Rightarrow\ (2+4k) = (1+3k)$$
$$\Rightarrow\ 4k-3k = 1-2 \quad \Rightarrow\ k=-1$$

Putting value of k in (i)

$$x+2y+3-1(3x+4y+7)=0$$
$$\Rightarrow\ x+2y+3-3x-4y-7=0\ \Rightarrow\ -2x-2y-4=0$$
$$\Rightarrow\ x+y+2=0$$
is the required equation.

Question # 3

Find an equation of the line through the intersection of $16x-10y-33=0$; $12x+14y+29=0$ and the intersection of $x-y+4=0$; $x-7y+2=0$

Solution

Let $l_1:\ 16x-10y-33=0$

$l_2:\ 12x+14y+29=0$

$l_3:\ x-y+4=0$

$l_4:\ x-7y+2=0$

For point of intersection of l_1 and l_2

$$\frac{x}{-290+462}=\frac{-y}{464+396}=\frac{1}{224+120}$$
$$\Rightarrow\ \frac{x}{172}=\frac{-y}{860}=\frac{1}{334}$$
$$\Rightarrow\ \frac{x}{172}=\frac{1}{334}\quad\text{and}\quad\frac{-y}{860}=\frac{1}{334}$$
$$\Rightarrow\ x=\frac{172}{334}=\frac{1}{2}\quad\text{and}\quad y=-\frac{860}{334}=-\frac{5}{2}$$
$\Rightarrow\ \left(\frac{1}{2},-\frac{5}{2}\right)$ is a point of intersection of l_1 and l_2.

For point of intersection of l_3 and l_4.

$$\frac{x}{-2+28}=\frac{-y}{2-4}=\frac{1}{-7+1}$$
$$\Rightarrow\ \frac{x}{26}=\frac{-y}{-2}=\frac{1}{-6}$$
$$\Rightarrow\ \frac{x}{26}=\frac{1}{-6}\quad\text{and}\quad\frac{-y}{-2}=\frac{1}{-6}$$
$$\Rightarrow\ x=\frac{26}{-6}=-\frac{13}{3}\quad\text{and}\quad y=\frac{2}{-6}=-\frac{1}{3}$$
$\Rightarrow\ \left(-\frac{13}{3},-\frac{1}{3}\right)$ is a point of intersection of l_3 and l_4.

Now equation of line passing through $\left(\frac{1}{2},-\frac{5}{2}\right)$ and $\left(-\frac{13}{3},-\frac{1}{3}\right)$

$$y+\frac{5}{2}=\frac{-1/3+5/2}{-13/3-1/2}\left(x-\frac{1}{2}\right)$$

$$\Rightarrow\ y+\frac{5}{2}=\frac{\frac{13}{6}}{-\frac{29}{6}}\left(x-\frac{1}{2}\right)\quad\Rightarrow\ y+\frac{5}{2}=-\frac{13}{29}\left(x-\frac{1}{2}\right)$$

$$\Rightarrow\ 29y+\frac{145}{2}=-13x+\frac{13}{2}\qquad\Rightarrow\ 13x-\frac{13}{2}+29y+\frac{145}{2}=0$$

$$\Rightarrow\ 13x+29y+66=0$$

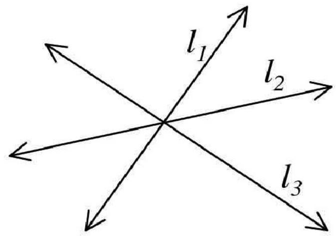
is the required equation.

Three Concurrent Lines

Suppose $l_1:\ a_1x+b_1y+c_1=0$

$l_2:\ a_2x+b_2y+c_2=0$

$l_3:\ a_3x+b_3y+c_3=0$



If l_1, l_2 and l_3 are concurrent (intersect at one point) then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

See proof on book at page 208

Question # 4

Find the condition that the lines $y=m_1x+c_1$; $y=m_2x+c_2$ and $y=m_3x+c_3$ are concurrent.

Solution

Assume that $l_1:\ y=m_1x+c_1$

$\Rightarrow\ m_1x-y+c_1=0$

$l_2:\ y=m_2x+c_2$

$\Rightarrow\ m_2x-y+c_2=0$

$l_3:\ y=m_3x+c_3$

$\Rightarrow\ m_3x-y+c_3=0$

If l_1, l_2 and l_3 are concurrent then

$$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix}=0$$

$$\Rightarrow\ \begin{vmatrix} m_1 & -1 & c_1 \\ m_2-m_1 & 0 & c_2-c_1 \\ m_3-m_1 & 0 & c_3-c_1 \end{vmatrix}=0 \qquad \text{by}\ \begin{matrix} R_2-R_1 \\ R_3-R_1 \end{matrix}$$

Expanding by C_2

$$-(-1)\left[(m_2-m_1)(c_3-c_1)-(m_3-m_1)(c_2-c_1)\right]+0-0=0$$

$$\Rightarrow\ \left[(m_2-m_1)(c_3-c_1)-(m_3-m_1)(c_2-c_1)\right]=0$$

$$\Rightarrow\ (m_2-m_1)(c_3-c_1)=(m_3-m_1)(c_2-c_1)$$

is the required condition.

Question # 5

Determine the value of P such that the lines $2x-3y-1=0$, $3x-y-5=0$ and $3x+py+8=0$ meet at a point .

Solution

Let $l_1:\ 2x-3y-1=0$

$l_2:\ 3x-y-5=0$

$l_3:\ 3x+py+8=0$

Since l_1, l_2 and l_3 meets at a point i.e. concurrent therefore

$$\begin{vmatrix} 2 & -3 & -1 \\ 3 & -1 & -5 \\ 3 & p & 8 \end{vmatrix} = 0$$

$\Rightarrow\ 2(-8+5p)+3(24+15)-1(3p+3)=0$

$\Rightarrow\ -16+10p+72+45-3p-3=0$

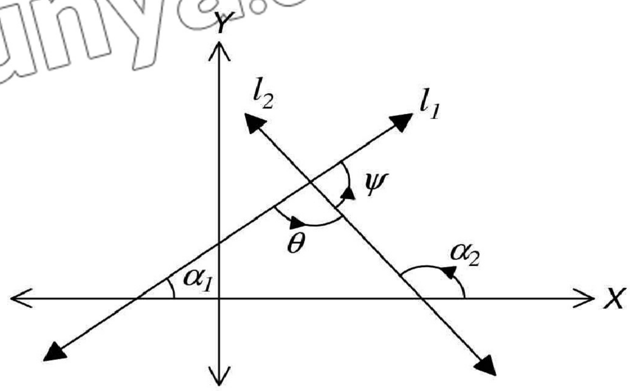
$\Rightarrow\ 7p+98=0 \quad \Rightarrow\ 7p=-98$

$\Rightarrow\ p=-\dfrac{98}{7} \quad \Rightarrow\ \boxed{p=-14}$

Angle between lines

Let l_1 and l_2 be two lines. If α_1 and α_2 be inclinations and m_1 and m_2 be slopes of lines l_1 and l_2 respectively. Let θ be a angle from line l_1 to l_2 then θ is given by

$$\tan\theta = \frac{m_2-m_1}{1+m_1m_2}$$



See proof on book at page 219

Question # 6

Show that the lines $4x-3y-8=0$, $3x-4y-6=0$ and $x-y-2=0$ are concurrent and the third-line bisects the angle formed by the first two lines.

Solution

Let $l_1:\ 4x-3y-8=0$

$l_2:\ 3x-4y-6=0$

$l_3:\ x-y-2=0$

To check l_1, l_2 and l_3 are concurrent, let

$$\begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix}$$

$=4(8-6)+3(-6+6)-8(-3+4)$

$=4(2)+3(0)-8(1)$

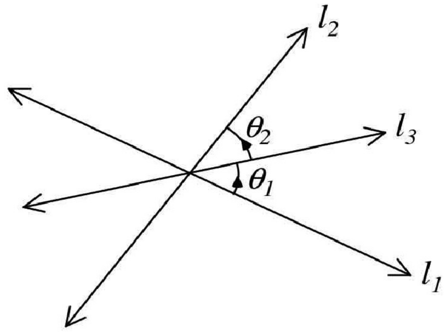
$=8+0-8=0$

Hence l_1, l_2 and l_3 are concurrent.

Slope of $l_1 = m_1 = -\dfrac{4}{-3} = \dfrac{4}{3}$

Slope of $l_2 = m_2 = -\dfrac{3}{-4} = \dfrac{3}{4}$

Slope of $l_3 = m_3 = -\dfrac{1}{-1} = 1$



Now let θ_1 be angle from l_1 to l_3 and θ_2 be a angle from l_3 to l_2. Then

$$\tan\theta_1 = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{1 - 4/3}{1 + (1)\left(4/3\right)} = \frac{-1/3}{7/3} = -\frac{1}{7} \quad \text{............ } (i)$$

And $$\tan\theta_2 = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{3/4 - 1}{1 + \left(3/4\right)(1)} = \frac{-1/4}{7/4} = -\frac{1}{7} \quad \text{............ } (ii)$$

From (i) and (ii)

$$\tan\theta_1 = \tan\theta_2 \quad \Rightarrow \theta_1 = \theta_2$$

$\Rightarrow$ l_3 bisect the angle formed by the first two lines.

Question # 7

The vertices of a triangle are $A(-2,3)$, $B(-4,1)$ and $C(3,5)$. Find coordinates of the
(i) centroid (ii) orthocentre (iii) circumcentre of the triangle
Are these three points are collinear?

Solution

Given vertices of triangles are $A(-2,3)$, $B(-4,1)$ and $C(3,5)$.

(i) Centroid of triangle is the intersection of medians and is given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

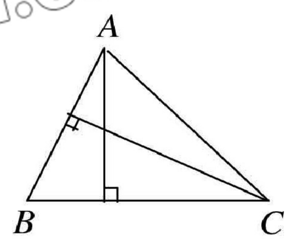
$$= \left(\frac{-2 - 4 + 3}{3}, \frac{3 + 1 + 5}{3}\right) = \left(\frac{-3}{3}, \frac{9}{3}\right) = (-1,3)$$

Hence $(-1,3)$ is the centroid of the triangle.

(ii) Orthocentre is the point of intersection of altitudes.

Slope of $\overline{AB} = m_1 = \dfrac{1-3}{-4+2} = \dfrac{-2}{-2} = 1$

Slope of $\overline{BC} = m_2 = \dfrac{5-1}{3+4} = \dfrac{4}{7}$



Since altitudes are $\perp$ to sides therefore

Slope of altitude on $\overline{AB} = -\dfrac{1}{m_1} = -\dfrac{1}{1} = -1$

Slope of altitude on $\overline{BC} = -\frac{1}{m_2} = -\frac{1}{4/7} = -\frac{7}{4}$

Equation of altitude on $\overline{AB}$ with slope -1 from $C(3,5)$

$$y-5=-1(x-3)$$

$$\Rightarrow\ y-5=-x+3 \quad \Rightarrow\ x-3+y-5=0$$

$$\Rightarrow\ x+y-8=0 \ \ldots\ldots\ldots\ldots \text{(i)}$$

Now equation of altitude on $\overline{BC}$ with slope $-\frac{7}{4}$ from $A(-2,3)$

$$y-3=-\frac{7}{4}(x+2)$$

$$\Rightarrow\ 4y-12=-7x-14 \quad \Rightarrow\ 7x+14+4y-12=0$$

$$\Rightarrow\ 7x+4y+2=0 \ \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

For point of intersection of (i) and (ii)

$$\frac{x}{2+32}=\frac{-y}{2+56}=\frac{1}{4-7}$$

$$\Rightarrow\ \frac{x}{34}=\frac{-y}{58}=\frac{1}{-3}$$

$$\Rightarrow\ \frac{x}{34}=\frac{1}{-3} \quad \text{and} \quad \frac{-y}{58}=\frac{1}{-3}$$

$$\Rightarrow\ x=-\frac{34}{3} \quad \text{and} \quad y=-\frac{58}{-3}=\frac{58}{3}$$

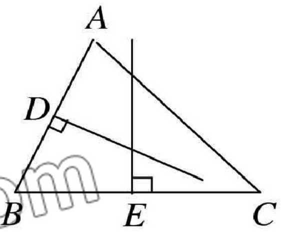
Hence $\left(-\frac{34}{3},\frac{58}{3}\right)$ is orthocentre of triangle ABC.

(iii) Circumcentre of the triangle is the point of intersection of perpendicular bisector.

Let D and E are midpoints of side $\overline{AB}$ and $\overline{BC}$ respectively.

Then coordinate of $D=\left(\frac{-4-2}{2},\frac{1+3}{2}\right)=\left(\frac{-6}{2},\frac{4}{2}\right)=(-3,2)$

Coordinate of $E=\left(\frac{-4+3}{2},\frac{1+5}{2}\right)=\left(\frac{-1}{2},\frac{6}{2}\right)=\left(-\frac{1}{2},3\right)$



Slope of $\overline{AB}=m_1=\frac{1-3}{-4+2}=\frac{-2}{-2}=1$

Slope of $\overline{BC}=m_2=\frac{5-1}{3+4}=\frac{4}{7}$

Slope of $\perp$ bisector on $\overline{AB}=-\frac{1}{m_1}=-\frac{1}{1}=-1$

Slope of $\perp$ bisector on $\overline{BC}=-\frac{1}{m_2}=-\frac{1}{4/7}=-\frac{7}{4}$

Now equation of ⊥ bisector having slope -1 through $D(-3,2)$

$$y-2=-1(x+3)$$
$$\Rightarrow y-2=-x-3 \quad \Rightarrow x+3+y-2=0$$
$$\Rightarrow x+y+1=0 \text{ ………….. (iii)}$$

Now equation of ⊥ bisector having slope $-\frac{7}{4}$ through $E\left(-\frac{1}{2},3\right)$

$$y-3=-\frac{7}{4}\left(x+\frac{1}{2}\right) \quad \Rightarrow 4y-12=-7x-\frac{7}{2}$$
$$\Rightarrow 7x+\frac{7}{2}+4y-12=0 \quad \Rightarrow 7x+4y-\frac{17}{2}=0$$
$$\Rightarrow 14x+8y-17=0 \text{ ……………. (iv)}$$

For point of intersection of (iii) and (iv)

$$\frac{x}{-17-8}=\frac{-y}{-17-14}=\frac{1}{8-14}$$
$$\Rightarrow \frac{x}{-25}=\frac{-y}{-31}=\frac{1}{-6}$$
$$\Rightarrow \frac{x}{-25}=\frac{1}{-6} \quad \text{and} \quad \frac{-y}{-31}=\frac{1}{-6}$$
$$\Rightarrow x=\frac{-25}{-6}=\frac{25}{6} \quad \text{and} \quad y=-\frac{31}{6}$$

Hence $\left(\frac{25}{6},-\frac{31}{6}\right)$ is the circumcentre of the triangle.

Now to check $(-1,3)$, $\left(-\frac{34}{3},\frac{58}{3}\right)$ and $\left(\frac{25}{6},-\frac{31}{6}\right)$ are collinear, let

$$\begin{vmatrix} -1 & 3 & 1 \\ -34/3 & 58/3 & 1 \\ 25/6 & -31/6 & 1 \end{vmatrix}$$
$$=-1\left(\frac{58}{3}+\frac{31}{6}\right)-3\left(-\frac{34}{3}-\frac{25}{6}\right)+1\left(\frac{1054}{18}-\frac{1450}{18}\right)$$
$$=-1\left(\frac{49}{2}\right)-3\left(-\frac{31}{2}\right)+1(-22)$$
$$=-\frac{49}{2}+\frac{93}{2}-22=0$$

Hence centroid, orthocentre and circumcentre of triangle are collinear.

Question # 8

Check whether the lines $4x-3y-8=0$; $3x-4y-6=0$; $x-y-2=0$ are concurrent. If so, find the point where they meet.

Solution

Let $l_1:\ 4x-3y-8=0$

$l_2:\ 3x-4y-6=0$

$l_3:\ x-y-2=0$

To check lines are concurrent, let

$$\begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix}$$

$$=4(8-6)+3(-6+6)-8(-3+4)$$

$$=4(2)+3(0)-8(1)\quad =8+0-8\ =0$$

Hence l_1 , l_2 and l_3 are concurrent.

For point of concurrency, we find intersection of l_1 and l_2 (You may choose any two lines)

$$\frac{x}{18-32}=\frac{-y}{-24+24}=\frac{1}{-16+9}$$

$$\Rightarrow\ \frac{x}{-14}=\frac{-y}{0}=\frac{1}{-7}$$

$$\Rightarrow\ \frac{x}{-14}=\frac{1}{-7}\ \text{ and }\ \frac{-y}{0}=\frac{1}{-7}$$

$$\Rightarrow\ x=\frac{-14}{-7}=2\quad \text{and}\quad y=-\frac{0}{-7}=0$$

Hence $(2,0)$ is the point of concurrency.

Question # 9

Find the coordinates of the vertices of the triangle formed by the lines $x-2y-6=0;\quad 3x-y+3=0;\quad 2x+y-4=0$. Also find measures of the angles of the triangle.

Solution

Let $l_1:\ x-2y-6=0$

$l_2:\ 3x-y+3=0$

$l_3:\ 2x+y-4=0$

For point of intersection of l_1 and l_2

$$\frac{x}{-6-6}=\frac{-y}{3+18}=\frac{1}{-1+6}$$

$$\Rightarrow\ \frac{x}{-12}=\frac{-y}{21}=\frac{1}{5}$$

$$\Rightarrow\ \frac{x}{-12}=\frac{1}{5}\ \text{ and }\ \frac{-y}{21}=\frac{1}{5}$$

$$\Rightarrow\ x=-\frac{12}{5}\quad \text{and}\quad y=-\frac{21}{5}$$

$\Rightarrow \left(-\frac{12}{5},-\frac{21}{5}\right)$ is the point of intersection of l_1 and l_2.

For point of intersection of l_2 and l_3.

$$\frac{x}{4-3}=\frac{-y}{-12-6}=\frac{1}{3+2}$$

$$\Rightarrow \frac{x}{1}=\frac{-y}{-18}=\frac{1}{5}$$

$$\Rightarrow \frac{x}{1}=\frac{1}{5} \quad \text{and} \quad \frac{-y}{-18}=\frac{1}{5}$$

$$\Rightarrow x=\frac{1}{5} \quad \text{and} \quad y=\frac{18}{5}$$

$\Rightarrow \left(\frac{1}{5},\frac{18}{5}\right)$ is the point of intersection of l_2 and l_3.

Now for point of intersection of l_1 and l_3

$$\frac{x}{8+6}=\frac{-y}{-4+12}=\frac{1}{1+4}$$

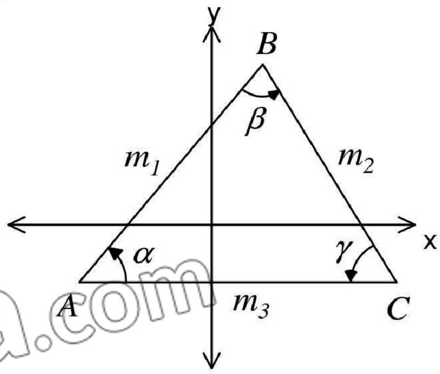
$$\Rightarrow \frac{x}{14}=\frac{-y}{8}=\frac{1}{5}$$

$$\Rightarrow \frac{x}{14}=\frac{1}{5} \quad \text{and} \quad \frac{-y}{8}=\frac{1}{5}$$

$$\Rightarrow x=\frac{14}{5} \quad \text{and} \quad y=-\frac{8}{5}$$

$\Rightarrow \left(\frac{14}{5},-\frac{8}{5}\right)$ is the point of intersection of l_1 and l_3.

Hence $\left(-\frac{12}{5},-\frac{21}{5}\right)$, $\left(\frac{1}{5},\frac{18}{5}\right)$ and $\left(\frac{14}{5},-\frac{8}{5}\right)$ are vertices of triangle made by l_1, l_2 and l_3. We say these vertices as A, B and C respectively.



Slope of side $AB = m_1 = \dfrac{18/5 + 21/5}{1/5 + 12/5} = \dfrac{39/5}{13/5} = \dfrac{39}{13} = 3$

Slope of side $BC = m_2 = \dfrac{-8/5 - 18/5}{14/5 - 1/5} = \dfrac{-26/5}{13/5} = -\dfrac{26}{13} = -2$

Slope of side $CA = m_3 = \dfrac{-21/5 + 8/5}{-12/5 - 14/5} = \dfrac{-13/5}{-26/5} = \dfrac{13}{26} = \dfrac{1}{2}$

Let α, β and γ denotes angles of triangle at vertices A, B and C respectively. Then

$$\tan\alpha = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{3 - \frac{1}{2}}{1 + (3)\left(\frac{1}{2}\right)} = \frac{\frac{5}{2}}{\frac{5}{2}} = 1$$

$$\Rightarrow \alpha = \tan^{-1}(1) \quad \Rightarrow \boxed{\alpha = 45^\circ}$$

Now $$\tan\beta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{-2-3}{1 + (-3)(3)} = \frac{-5}{-5} = 1$$

$$\Rightarrow \beta = \tan^{-1}(1) \quad \Rightarrow \boxed{\beta = 45^\circ}$$

Now $$\tan\gamma = \frac{m_3 - m_2}{1 + m_3 m_2} = \frac{\frac{1}{2} + 2}{1 + \left(\frac{1}{2}\right)(-2)} = \frac{\frac{5}{2}}{0} = \infty$$

$$\Rightarrow \gamma = \tan^{-1}(\infty) \quad \Rightarrow \boxed{\gamma = 90^\circ}.$$

Question # 10

Find the angle measured from the line l_1 to the line l_2 where

(a) l_1 : joining $(2,7)$ and $(7,10)$
l_2 : joining $(1,1)$ and $(-5,3)$

(b) l_1 : joining $(3,-1)$ and $(5,7)$
l_2 : joining $(2,4)$ and $(-8,2)$

(c) l_1 : joining $(1,-7)$ and $(6,-4)$
l_2 : joining $(-1,2)$ and $(-6,-1)$

(d) l_1 : joining $(-9,-1)$ and $(3,-5)$
l_2 : joining $(2,7)$ and $(-6,-7)$

Solution

(a) Since l_1 : joining $(2,7)$ and $(7,10)$

Therefore slope of $l_1 = m_1 = \frac{10-7}{7-2} = \frac{3}{5}$

Also l_2 : joining $(1,1)$ and $(-5,3)$

Therefore slope of $l_2 = m_2 = \frac{3-1}{-5-1} = \frac{2}{-6} = -\frac{1}{3}$

Let θ be a angle from l_1 to l_2 then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-\frac{1}{3} - \frac{3}{5}}{1 + \left(\frac{3}{5}\right)\left(-\frac{1}{3}\right)}$$

$$= \frac{-\frac{14}{15}}{1 - \frac{1}{5}} = \frac{-\frac{14}{15}}{\frac{4}{5}} = -\frac{14}{15} \times \frac{5}{4} = -\frac{7}{6}$$

$$\Rightarrow -\tan\theta = \frac{7}{6} \quad \Rightarrow \tan(180 - \theta) = \frac{7}{6}$$

$\because \tan(180 - \theta) = -\tan\theta$

$$\Rightarrow\ 180-\theta=\tan^{-1}\left(\frac{7}{6}\right)=49.4$$

$\Rightarrow\ \theta=180-49.4 \Rightarrow \boxed{\theta=130.6^\circ}$

Now acute angle between lines = $180-130.6=49.4^\circ$

(b) *Do yourself as above.*

(c) Since l_1: joining $(1,-7)$ and $(6,-4)$

Therefore slope of $l_1=m_1=\dfrac{-4+7}{6-1}=\dfrac{3}{5}$

Also l_2: joining $(-1,2)$ and $(-6,-1)$

Therefore slope of $l_2=m_2=\dfrac{-1-2}{-6+1}=\dfrac{-3}{-5}=\dfrac{3}{5}$

Let θ be a angle from l_1 to l_2 then

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}=\frac{3/5-3/5}{1+(3/5)(3/5)}=\frac{0}{1+9/25}=0$$

$\Rightarrow\ \theta=\tan^{-1}(0) \Rightarrow \boxed{\theta=0^\circ}$

Also acute angle between lines = 0°

(d) Since l_1: joining $(-9,-1)$ and $(3,-5)$

Therefore slope of $l_1=m_1=\dfrac{-5+1}{3+9}=\dfrac{-4}{12}=-\dfrac{1}{3}$

Also l_2: joining $(2,7)$ and $(-6,-7)$

Therefore slope of $l_2=m_2=\dfrac{-7-7}{-6-2}=\dfrac{-14}{-8}=\dfrac{7}{4}$

Let θ be a angle from l_1 to l_2 then

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}=\frac{7/4-(-1/3)}{1+(7/4)(-1/3)}$$

$$=\frac{7/4+1/3}{1-7/12}=\frac{25/12}{5/12}=\frac{25}{12}\times\frac{12}{5}=5$$

$\Rightarrow\ \theta=\tan^{-1}(5) \Rightarrow \boxed{\theta=78.69^\circ}$

Also acute angle between lines $=78.69^\circ$

Question # 11

Find the interior angle of the triangle whose vertices are

(a) $A(-2,11)$, $B(-6,-3)$, $C(4,-9)$

(b) $A(6,1)$, $B(2,7)$, $C(-6,-7)$

(c) $A(2,-5)$, $B(-4,-3)$,$C(-1,5)$

(d) $A(2,8)$, $B(-5,4)$ and $C(4,-9)$

Solution

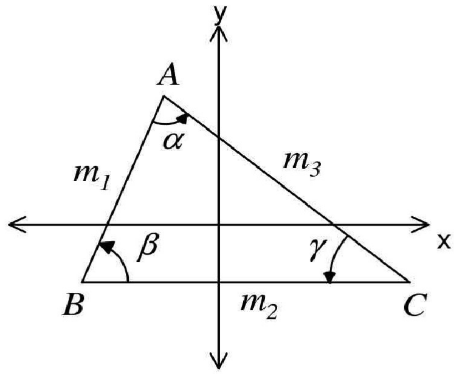
(a) Given vertices $A(-2,11)$, $B(-6,-3)$ and $C(4,-9)$

Let m_1, m_2 and m_3 denotes the slopes of side AB, BC and CA respectively. Then

$$m_1 = \frac{-3-11}{-6+2} = \frac{-14}{-4} = \frac{7}{2}$$

$$m_2 = \frac{-9+3}{4+6} = \frac{-6}{10} = -\frac{3}{5}$$

$$m_3 = \frac{11+9}{-2-4} = \frac{20}{-6} = -\frac{10}{3}$$



Let α, β and γ denotes angles of triangle at vertex A, B and C respectively. Then

$$\tan\alpha = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{-10/3 - 7/2}{1 + \left(-10/3\right)\left(7/2\right)}$$

$$= \frac{-41/6}{1 - 35/3} = \frac{-41/6}{-32/3} = \frac{41}{6} \times \frac{3}{32} = \frac{41}{64}$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{41}{64}\right) \qquad \Rightarrow \boxed{\alpha = 32.64^\circ}$$

$$\tan\beta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{7/2 - \left(-3/5\right)}{1 + \left(7/2\right)\left(-3/5\right)}$$

$$= \frac{7/2 + 3/5}{1 - 21/10} = \frac{41/10}{-11/10} = -\frac{41}{10} \times \frac{10}{11} = -\frac{41}{11}$$

$$\Rightarrow -\tan\beta = \frac{41}{11} \qquad \Rightarrow \tan(180 - \beta) = \frac{41}{11} \qquad \because \tan(180 - \theta) = -\tan\theta$$

$$\Rightarrow 180 - \beta = \tan^{-1}\left(\frac{41}{11}\right) = 74.98$$

$$\Rightarrow \beta = 180 - 74.98 \qquad \Rightarrow \boxed{\beta = 105.02}$$

$$\tan\gamma = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{-3/5 - \left(-10/3\right)}{1 + \left(-3/5\right)\left(-10/3\right)}$$

$$= \frac{-3/5 + 10/3}{1 + 2} = \frac{41/15}{3} = \frac{41}{15 \times 3} = \frac{41}{45}$$

$$\Rightarrow \gamma = \tan^{-1}\left(\frac{41}{45}\right) \quad \Rightarrow \boxed{\gamma = 42.34^\circ}$$

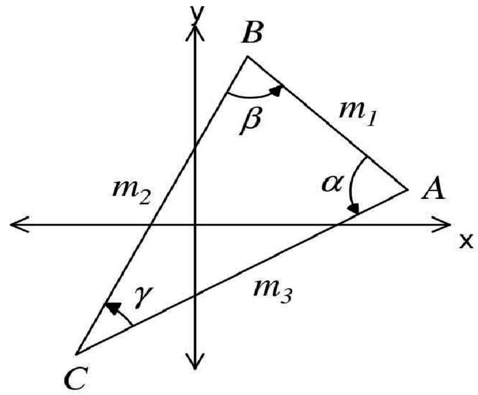
(b) Given vertices $A(6,1)$, $B(2,7)$ and $C(-6,-7)$

Let m_1, m_2 and m_3 denotes the slopes of side AB, BC and CA respectively. Then

$$m_1 = \frac{7-1}{2-6} = \frac{6}{-4} = -\frac{3}{2}$$

$$m_2 = \frac{-7-7}{-6-2} = \frac{-14}{-8} = \frac{7}{4}$$

$$m_3 = \frac{1+7}{6+6} = \frac{8}{12} = \frac{2}{3}$$



Let α, β and γ denotes angles of triangle at vertex A, B and C respectively. Then

$$\tan\alpha = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{\frac{2}{3} - \left(-\frac{3}{2}\right)}{1 + \left(\frac{2}{3}\right)\left(-\frac{3}{2}\right)}$$

$$= \frac{\frac{2}{3} + \frac{3}{2}}{1-1} = \frac{\frac{13}{6}}{0} = \infty$$

$$\Rightarrow \alpha = \tan^{-1}(\infty) \quad \Rightarrow \boxed{\alpha = 90^\circ}$$

$$\tan\beta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\frac{3}{2} - \frac{7}{4}}{1 + \left(-\frac{3}{2}\right)\left(\frac{7}{4}\right)}$$

$$= \frac{-\frac{13}{4}}{1 - \frac{21}{8}} = \frac{-\frac{13}{4}}{-\frac{13}{8}} = \frac{13}{4} \times \frac{8}{13} = 2$$

$$\Rightarrow \beta = \tan^{-1}(2) \quad \Rightarrow \boxed{\beta = 63.43^\circ}$$

$$\tan\gamma = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{\frac{7}{4} - \frac{2}{3}}{1 + \left(\frac{7}{4}\right)\left(\frac{2}{3}\right)}$$

$$= \frac{\frac{13}{12}}{1 + \frac{7}{6}} = \frac{\frac{13}{12}}{\frac{13}{6}} = \frac{13}{12} \times \frac{6}{13} = \frac{1}{2}$$

$$\Rightarrow \gamma = \tan^{-1}\left(\frac{1}{2}\right) \quad \Rightarrow \boxed{\gamma = 26.57^\circ}$$

(c) *Do yourself as above.*

(d) *Do yourself as above.*

Question # 12

Find the interior angles of the quadrilateral whose vertices are $A(5,2)$, $B(-2,3)$, $C(-3,-4)$ and $D(4,-5)$

Solution Given vertices are $A(5,2)$, $B(-2,3)$, $C(-3,-4)$ and $D(4,-5)$

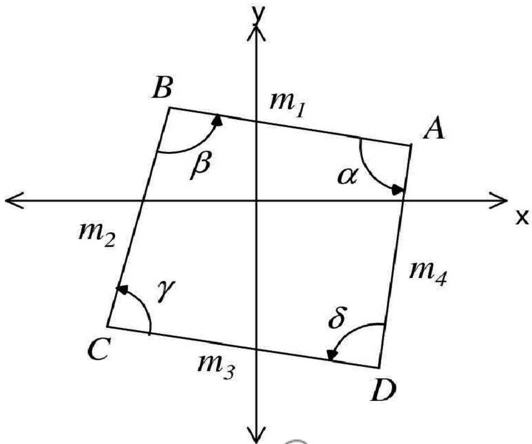
Let m_1, m_2, m_3 and m_4 be slopes of side AB, BC, CD and DA. Then

$$m_1 = \frac{3-2}{-2-5} = \frac{1}{-7}$$

$$m_2 = \frac{-4-3}{-3+2} = \frac{-7}{-1} = 7$$

$$m_3 = \frac{-5+4}{4+3} = \frac{-1}{7}$$

$$m_4 = \frac{2+5}{5-4} = \frac{7}{1} = 7$$



Now suppose α, β, γ and δ are angels of quadrilateral at vertices A, B, C and D respectively. Then

$$\tan\alpha = \frac{m_4 - m_1}{1 + m_4 m_1} = \frac{7 - \left(-\frac{1}{7}\right)}{1 + (7)\left(-\frac{1}{7}\right)} = \frac{7 + \frac{1}{7}}{1-1} = \frac{\frac{50}{7}}{0} = \infty$$

$$\Rightarrow \alpha = \tan^{-1}(\infty) \qquad \Rightarrow \boxed{\alpha = 90^\circ}$$

$$\tan\beta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\frac{1}{7} - 7}{1 + \left(-\frac{1}{7}\right)(7)} = \frac{-\frac{50}{7}}{0} = \infty$$

$$\Rightarrow \beta = \tan^{-1}(\infty) \qquad \Rightarrow \boxed{\beta = 90^\circ}$$

$$\tan\gamma = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{7 - \left(-\frac{1}{7}\right)}{1 + (7)\left(-\frac{1}{7}\right)} = \frac{7 + \frac{1}{7}}{1-1} = \frac{\frac{50}{7}}{0} = \infty$$

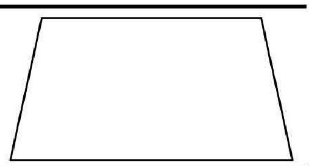
$$\Rightarrow \gamma = \tan^{-1}(\infty) \qquad \Rightarrow \boxed{\gamma = 90^\circ}$$

$$\tan\delta = \frac{m_3 - m_4}{1 + m_3 m_4} = \frac{-\frac{1}{7} - 7}{1 + \left(-\frac{1}{7}\right)(7)} = \frac{-\frac{50}{7}}{0} = \infty$$

$$\Rightarrow \delta = \tan^{-1}(\infty) \qquad \Rightarrow \boxed{\delta = 90^\circ}$$

Trapezium

If any two opposite sides of the quadrilateral are parallel then it is called *trapezium*.



Question # 13

Show that the points $A(-1,-1)$, $B(-3,0)$, $C(3,7)$ and $D(1,8)$ are the vertices of the rhombus and find its interior angle.

Solution Given vertices are $A(-1,-1)$, $B(-3,0)$, $C(3,7)$ and $D(1,8)$

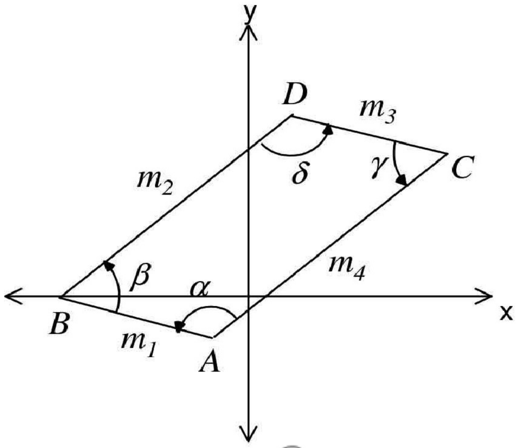
Let m_1, m_2, m_3 and m_4 be slopes of side $\overline{AB},\overline{BD},\overline{DC}$ and $\overline{CA}$. Then

$$m_1 = \frac{0+1}{-3+1} = \frac{1}{-2}$$

$$m_2 = \frac{8-0}{1+3} = \frac{8}{4} = 2$$

$$m_3 = \frac{7-8}{3-1} = \frac{-1}{2}$$

$$m_4 = \frac{-1-7}{-1-3} = \frac{-8}{-4} = 2$$



Since $m_2 = m_4$ or $m_1 = m_3$

Hence A, B, C and D are vertices of trapezium.

Now suppose α, β, γ and δ are angels of quadrilateral at vertices A, B, C and D respectively. Then

Now do yourself as above in Question # 12

Question # 14

Find the area of the region bounded by the triangle, whose sides are

$$7x - y + 10 = 0;\ \ 10x + y - 41 = 0;\ \ 3x + 2y + 3 = 0$$

Solution

Let $l_1:\ 7x - y - 10 = 0$

$l_2:\ 10x + y - 41 = 0$

$l_3:\ 3x + 2y + 3 = 0$

For intersection of l_1 and l_2

$$\frac{x}{41+10} = \frac{-y}{-287+100} = \frac{1}{7+10}$$

$$\Rightarrow \frac{x}{51} = \frac{-y}{-187} = \frac{1}{17}$$

$$\Rightarrow \frac{x}{51} = \frac{1}{17} \quad \text{and} \quad \frac{y}{187} = \frac{1}{17}$$

$$\Rightarrow x = \frac{51}{17} = 3 \quad \text{and} \quad y = \frac{187}{17} = 11$$

$\Rightarrow (3,11)$ is the point of intersection of l_1 and l_2.

Now for point of intersection of l_2 and l_3

$$\frac{x}{3+82} = \frac{-y}{30+123} = \frac{1}{20-3}$$

$$\Rightarrow \frac{x}{85} = \frac{-y}{153} = \frac{1}{17}$$

$$\Rightarrow \frac{x}{85} = \frac{-y}{153} = \frac{1}{17}$$

$$\Rightarrow \frac{x}{85} = \frac{1}{17} \quad \text{and} \quad \frac{-y}{153} = \frac{1}{17}$$

$$\Rightarrow x = \frac{85}{17} = 5 \quad \text{and} \quad y = -\frac{153}{17} = -9$$

$\Rightarrow (5,-9)$ is the point of intersection of l_2 and l_3.

For point of intersection of l_1 and l_3

$$\frac{x}{-3+20} = \frac{-y}{21+30} = \frac{1}{14+3}$$

$$\Rightarrow \frac{x}{17} = \frac{-y}{51} = \frac{1}{17}$$

$$\Rightarrow \frac{x}{17} = \frac{1}{17} \quad \text{and} \quad \frac{-y}{51} = \frac{1}{17}$$

$$\Rightarrow x = \frac{17}{17} = 1 \quad \text{and} \quad y = -\frac{51}{17} = 3$$

$\Rightarrow (1,-3)$ is the point of intersection of l_1 and l_3.

Now area of triangle having vertices $(3,11)$, $(5,-9)$ and $(1,-3)$ is given by:

$$\frac{1}{2}\begin{vmatrix} 3 & 11 & 1 \\ 5 & -9 & 1 \\ 1 & -3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|3(-9+3) - 11(5-1) + 1(-15+9)\right|$$

$$= \frac{1}{2}\left|3(-6) - 11(4) + 1(-6)\right| = \frac{1}{2}\left|-18-44-6\right|$$

$$= \frac{1}{2}\left|-68\right| = \frac{1}{2}(68) = 34 \text{ sq. unit}$$

Question # 15

The vertices of a triangle are $A(-2,3)$, $B(-4,1)$ and $C(3,5)$. Find the centre of the circum centre of the triangle?

Solution *Same Question # 7(c)*

Question # 16

Express the given system of equations in matrix form. Find in each case whether in lines are concurrent.

(a) $x+3y-2=0$; $2x-y+4=0$; $x-11y+14=0$

(b) $2x+3y+4=0$; $x-2y-3=0$; $3x+y-8=0$

(c) $3x-4y-2=0$; $x+2y-4=0$; $3x-2y+5=0$

Solution

(a)
$$x+3y-2=0$$
$$2x-y+4=0$$
$$x-11y+14=0$$

In matrix form

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Coefficient matrix of the system is

$$A=\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$$\Rightarrow \ |A| = 1(-14+44)-3(28-4)-2(-22+1)$$
$$=1(30)-3(24)-2(-21)$$
$$=30-72+42=0$$

Hence given lines are concurrent.

(b)
$$2x+3y+4=0$$
$$x-2y-3=0$$
$$3x+y-8=0$$

In matrix form

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Coefficient matrix of the system is

$$A=\begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{bmatrix}$$

$$\Rightarrow \ |A| = 2(16+3)-3(-8+9)+4(1+6)$$
$$=2(19)-3(1)+4(7) \quad =38-3+28 \quad =63\neq 0$$

Hence given lines are not concurrent.

(c) *Do yourself as above*

Question # 17

Find a system of linear equations corresponding to the given matrix form .Check whether the lines responded by the system are concurrent.

$$(a)\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (b)\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution

(a)

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x+0-1 \\ 2x+0+1 \\ 0-y+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x-1 \\ 2x+1 \\ -y+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equating the elements

$$x-1=0$$
$$2x+1=0$$
$$-y+2=0$$

are the required equation of lines.

Coefficients matrix of the system

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow \det A = 1(0+1) - 0 - 1(-2-0)$$
$$= 1+2 = 3 \neq 0$$

Hence system is not concurrent.

(b) *Do yourself as above.*

Homogenous 2nd Degree Equation

Every homogenous second degree equation

$$ax^2 + 2hxy + by^2 = 0$$

represents straight lines through the origin.

Consider the equations are $y = m_1x$ and $y = m_2x$

$\Rightarrow m_1x - y = 0$ and $m_2x - y = 0$

Taking product

$$(m_1x - y)(m_2x - y) = 0$$

$$\Rightarrow m_1m_2x^2 - m_1xy - m_2xy + y^2 = 0$$

$$\Rightarrow m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0 \text{ .... (i)}$$

Also we have

$$ax^2 + 2hxy + by^2 = 0$$

$$\Rightarrow \frac{a}{b}x^2 + \frac{2h}{b}xy + y^2 = 0 \qquad \div\text{ing by } b$$

$$\Rightarrow \frac{a}{b}x^2 - \left(-\frac{2h}{b}\right)xy + y^2 = 0$$

Comparing it with (i), we have

$$\boxed{m_1m_2 = \frac{a}{b}} \quad \text{and} \quad \boxed{m_1 + m_2 = -\frac{2h}{b}}$$

Let θ be the angles between the lines then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$= \frac{\sqrt{(m_1 - m_2)^2}}{1 + m_1m_2} = \frac{\sqrt{m_1^2 + m_2^2 - 2m_1m_2}}{1 + m_1m_1}$$

$$= \frac{\sqrt{m_1^2 + m_2^2 + 2m_1m_2 - 4m_1m_2}}{1 + m_1m_1} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_1}$$

$$= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4\frac{a}{b}}}{1 + \frac{a}{b}} = \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}} = \frac{\sqrt{\frac{4h^2 - 4ab}{b^2}}}{\frac{b + a}{b}}$$

$$\Rightarrow \tan\theta = \frac{\sqrt{4(h^2 - ab)}}{b + a} \quad \Rightarrow \quad \boxed{\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}}$$

Find the lines represented by each of the following and also find measure of the angle between them (problem 1-6)

Question # 1

$10x^2 - 23xy - 5y^2 = 0$

Solution

$$10x^2 - 23xy - 5y^2 = 0 \;\ldots\ldots\ldots\ldots\; \text{(i)}$$

$$\Rightarrow \; 10x^2 - 25xy + 2xy - 5y^2 = 0$$

$$\Rightarrow \; 5x(2x-5y) + y(2x-5y) = 0 \;\Rightarrow\; (2x-5y)(5x+y) = 0$$

$$\Rightarrow \; 2x - 5y = 0 \quad \text{and} \quad 5x + y = 0$$

are the required lines.

Comparing eq. (i) with

$$ax^2 + 2hxy + by^2 = 0$$

So $a = 10$, $2h = -23 \Rightarrow h = -\dfrac{23}{2}$, $b = -5$

Let θ be angle between lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\left(-\frac{23}{2}\right)^2 - (10)(-5)}}{10-5} = \frac{2\sqrt{\frac{529}{4} + 50}}{5}$$

$$= \frac{2\sqrt{729/4}}{5} = \frac{2\left(27/2\right)}{5} = \frac{27}{5}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{27}{5}\right) = 79^\circ 31'$$

Hence acute angle between the lines $= 79^\circ 31'$

Question # 2

$3x^2 + 7xy + 2y^2 = 0$

Solution *Do yourself as above*

Question # 3

$9x^2 + 24xy + 16y^2 = 0$

Solution *Do yourself as above*

Question # 4

$2x^2 + 3xy - 5y^2 = 0$

Solution $2x^2 + 3xy - 5y^2 = 0 \;\ldots\ldots\ldots\ldots\; \text{(i)}$

$$\Rightarrow \; 2x^2 + 5xy - 2xy - 5y^2 = 0$$

$$\Rightarrow \; x(2x+5y) - y(2x+5y) = 0$$

$$\Rightarrow \; (2x+5y)(x-y) = 0$$

$$\Rightarrow \; 2x + 5y = 0 \;\text{ and }\; x - y = 0$$

are the required lines.

Comparing eq. (i) with

$$ax^2 + 2hxy + by^2 = 0$$

$$\Rightarrow \quad a = 2 \quad , \quad 2h = 3 \;\Rightarrow\; h = \frac{3}{2} \quad , \quad b = -5$$

Let θ be angle between lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\left(\frac{3}{2}\right)^2 - (2)(-5)}}{2-5} = \frac{2\sqrt{\frac{9}{4}+10}}{-3}$$

$$= -\frac{2\sqrt{49/4}}{3} = -\frac{2\left(7/2\right)}{3} = -\frac{7}{3}$$

$$\Rightarrow \quad -\tan\theta = \frac{7}{3}$$

$$\Rightarrow \quad \tan(180-\theta) = \frac{7}{3} \qquad \because \; \tan(180-\theta) = -\tan\theta$$

$$\Rightarrow \quad 180-\theta = \tan^{-1}\left(\frac{7}{3}\right) \quad \Rightarrow \quad 180-\theta = 66^\circ 48'$$

$$\Rightarrow \quad \theta = 180 - 66^\circ 48' = 113^\circ 12'$$

Hence acute angle between the lines $= 180 - 113^\circ 12' = 66^\circ 48'$

Question # 5

$$6x^2 - 19xy + 15y^2 = 0$$

Solution

Do yourself as above

Question # 6

$$x^2 - 2xy\sec\alpha + y^2 = 0$$

Solution

$$x^2 - 2xy\sec\alpha + y^2 = 0 \; \ldots\ldots\ldots\; \text{(i)}$$

$\div$ing by y^2

$$\frac{x^2}{y^2} - \frac{2xy\sec\alpha}{y^2} + \frac{y^2}{y^2} = 0$$

$$\Rightarrow \quad \left(\frac{x}{y}\right)^2 - 2\sec\alpha\left(\frac{x}{y}\right) + 1 = 0$$

This is quadric equation in $\frac{x}{y}$ with $a = 1$, $b = -2\sec\alpha$, $c = 1$

$$\frac{x}{y} = \frac{2\sec\alpha \pm \sqrt{(-2\sec\alpha)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2\sec\alpha \pm \sqrt{4\sec^2\alpha - 4}}{2(1)} = \frac{2\sec\alpha \pm \sqrt{4(\sec^2\alpha - 1)}}{2}$$

$$= \frac{2\sec\alpha \pm \sqrt{4\tan^2\alpha}}{2} \qquad \because\ 1+\tan^2\alpha = \sec^2\alpha$$

$$= \frac{2\sec\alpha \pm 2\tan\alpha}{2}$$

$$\Rightarrow \frac{x}{y} = \sec\alpha \pm \tan\alpha$$

$$= \frac{1}{\cos\alpha} \pm \frac{\sin\alpha}{\cos\alpha} = \frac{1 \pm \sin\alpha}{\cos\alpha}$$

$$\Rightarrow \frac{x}{y} = \frac{1+\sin\alpha}{\cos\alpha} \quad \text{and} \quad \frac{x}{y} = \frac{1-\sin\alpha}{\cos\alpha}$$

$$\Rightarrow x\cos\alpha = (1+\sin\alpha)y \quad \text{and} \quad x\cos\alpha = (1-\sin\alpha)y$$

$$\Rightarrow x\cos\alpha - (1+\sin\alpha)y = 0 \quad \text{and} \quad x\cos\alpha - (1-\sin\alpha)y = 0$$

These are required equations of lines.
Now comparing (i) with

$$ax^2 + 2hxy + by^2 = 0$$

$$a = 1 \ , \quad 2h = -2\sec\alpha \ \Rightarrow h = -\sec\alpha \ , \quad b = 1b$$

If θ is angle between lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\sec^2\alpha - (1)(1)}}{1+1} = \frac{2\sqrt{\sec^2\alpha - 1}}{2} = \sqrt{\tan^2\alpha}$$

$$\Rightarrow \tan\theta = \tan\alpha \quad \Rightarrow \theta = \alpha.$$

Note:
If one wish to get the answer similar to given at the end of textbook, then follow the solution as follows after getting:

$$x\cos\alpha - (1+\sin\alpha)y = 0 \quad \text{and} \quad x\cos\alpha - (1-\sin\alpha)y = 0$$

Multiplying equation at left with $\frac{1-\sin\alpha}{\cos\alpha}$ and equation at right with $\frac{1+\sin\alpha}{\cos\alpha}$ to get

$$x(1-\sin\alpha) - \frac{(1-\sin^2\alpha)}{\cos\alpha}y = 0 \quad \text{and} \quad x(1+\sin\alpha) - \frac{(1-\sin^2\alpha)}{\cos\alpha}y = 0$$

$$\Rightarrow x(1-\sin\alpha) - y\cos\alpha = 0 \quad \text{and} \quad x(1+\sin\alpha) - y\cos\alpha = 0$$

Question # 7
Find a joint equation of the lines through the origin and perpendicular to the lines:
$x^2 - 2xy\tan\alpha - y^2 = 0$

Solution Given: $x^2 - 2xy\tan\alpha - y^2 = 0$

Suppose m_1 and m_2 are slopes of given lines then

$$m_1 + m_2 = -\frac{2h}{b}$$
$$= -\frac{-2\tan\alpha}{-1}$$
$$\Rightarrow m_1 + m_2 = -2\tan\alpha$$

$a = 1$,
$2h = -2\tan\alpha$
$\Rightarrow h = -\tan\alpha$
$b = -1$

$$\& \quad m_1 m_2 = \frac{a}{b} = \frac{1}{-1} \Rightarrow m_1 m_2 = -1$$

Now slopes of lines $\perp$ar to given lines are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, then their equations are

$$y = -\frac{1}{m_1}x \quad \& \quad y = -\frac{1}{m_2}x \quad \text{(Passing through origin)}$$
$$\Rightarrow m_1 y = -x \quad \& \quad m_2 y = -x$$
$$\Rightarrow x + m_1 y = 0 \quad \& \quad x + m_2 y = 0$$

Their joint equation:

$$(x + m_1 y)(x + m_2 y) = 0$$
$$\Rightarrow x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$$
$$\Rightarrow x^2 + (-2\tan\alpha)xy + (-1)y^2 = 0$$
$$\Rightarrow x^2 - 2xy\tan\alpha - y^2 = 0$$

Question # 8

Find a joint equation of the lines through the origin and perpendicular to the lines:
$ax^2 + 2hxy + by^2 = 0$

Solution

Given: $ax^2 + 2hxy + by^2 = 0$

Suppose m_1 and m_2 are slopes of given lines then

$$m_1 + m_2 = -\frac{2h}{b} \quad \& \quad m_1 m_2 = \frac{a}{b}$$

Now slopes of lines $\perp$ar to given lines are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, then their equations are

$$y = -\frac{1}{m_1}x \quad \& \quad y = -\frac{1}{m_2}x \quad \text{(Passing through origin)}$$
$$\Rightarrow m_1 y = -x \quad \& \quad m_2 y = -x$$
$$\Rightarrow x + m_1 y = 0 \quad \& \quad x + m_2 y = 0$$

Their joint equation:

$$(x + m_1 y)(x + m_2 y) = 0$$
$$\Rightarrow x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$$
$$\Rightarrow x^2 + \left(-\frac{2h}{b}\right)xy + \left(\frac{a}{b}\right)y^2 = 0$$
$$\Rightarrow bx^2 - 2hxy + ay^2 = 0$$

Question # 9

Find the area of the region bounded by:

$$10x^2 - xy - 21y^2 = 0 \quad \text{and} \quad x + y + 1 = 0$$

Solution

$$10x^2 - xy - 21y^2 = 0 \quad , \quad x + y + 1 = 0$$

$$\Rightarrow 10x^2 - 15xy + 14xy - 21y^2 = 0$$

$$\Rightarrow 5x(2x - 3y) + 7y(2x - 3y) = 0$$

$$\Rightarrow (2x - 3y)(5x + 7y) = 0$$

$$\Rightarrow 2x - 3y = 0 \quad \text{or} \quad 5x + 7y = 0$$

So we have equation of lines

$l_1 :\ 2x - 3y = 0$ ………… (i)

$l_2 :\ 5x + 7y = 0$ ………. (ii)

$l_3 :\ x + y + 1 = 0$ ……… (iii)

Now do yourself as Q # 14 (Ex. 4.4)