

Exercise 1.1

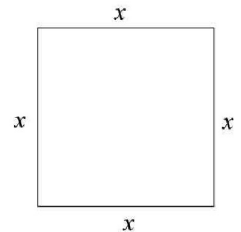
CALCULUS AND ANALYTIC GEOMETRY, MATHEMATICS 12

Function and Limits

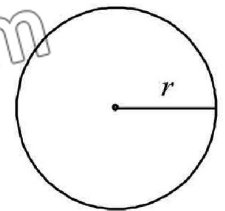
Concept of Functions:

Historically, the term function was first used by German mathematician Leibnitz (1646-1716) in 1673 to denote the dependence of one quantity on another e.g.

- 1) The area “A” of a square of side “x” is given by the formula $A=x^2$. As area depends on its side x , so we say that A is a function of x .



- 2) The area “A” of a circular disc of radius “r” is given by the formula $A=\pi r^2$. As area depends on its radius r , so we say that A is a function of r .



- 3) The volume “V” of a sphere of radius “r” is given by the formula $V=\frac{4}{3}\pi r^3$. As volume V of a sphere depends on its radius r , so we say that V is a function of r .

The Swiss mathematician, Leonard Euler conceived the idea of denoting function written as $y=f(x)$ and read as y is equal to f of x. $f(x)$ is called the value of f at x or image of x under f.

The variable x is called independent variable and the variable y is called dependent variable of f .

If x and y are real numbers then f is called real valued function of real numbers.

Domain of the function:

If the independent variable of a function is restricted to lie in some set, then this set is called the domain of the function e.g.

Dom of $f = \{0 \leq x \leq 5\}$

Range of the function:

The set of all possible values of $f(x)$ as x varies over the domain of f is called the range of f e.g. $y = 100 - 4x^2$

As x varies over the domain $[0,5]$ the values of $y = 100 - 4x^2$ vary between $y=0$ (when $x=5$) and $y = 100$ (when $x=0$)

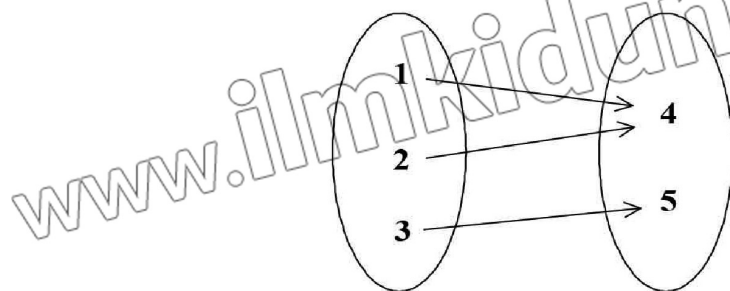
Range of $f = \{0 \leq y \leq 100\}$

Definition:

A function is a rule by which we relate two sets A and B (say) in such a way that each element of A is assigned with one and only one element of B. For example

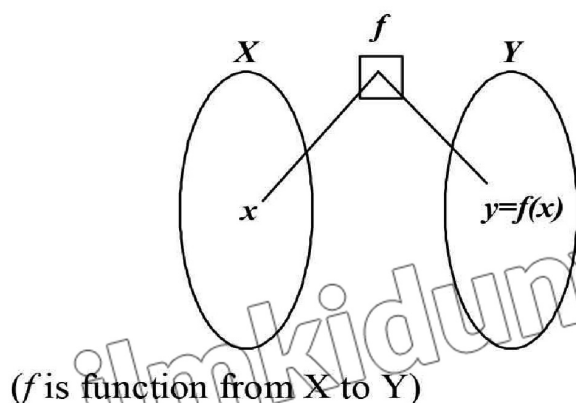
is a function from A to B.

its Domain = $\{1,2,3\}$ and Range = $\{4,5\}$



In general:

A function f from a set 'X' to a set 'Y' is a rule that assigns to each element x in X one and only one element y in Y. (a unique element y in Y)



If an element "y, of Y" is associated with an element "x, of X, then we write $y=f(x)$ & read as y" is equal to f of x. Here $f(x)$ is called image of f at x or value of f at x .

Or if a quantity y depends on a quantity x in such a way that each value of x determines exactly one value of y . Then we say that y is a function of x .

The set x is called Domain of f . The set of corresponding elements y in y is called Range of f . we say that y is a function of x .

Exercise 1.1

Q1. (a) Given that $f(x) = x^2 - x$

- i. $f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$
- ii. $f(0) = (0)^2 - (0) = 0$
- iii. $f(x-1) = (x-1)^2 - (x-1) = x^2 - 2x + 1 - x + 1 = x^2 - 3x + 2$
- iv. $f(x^2+4) = (x^2+4)^2 - (x^2+4) = x^4 + 8x^2 + 16 - x^2 - 4 = x^4 + 7x^2 + 12$

(b) Given that $f(x) = \sqrt{x+4}$

$$i) f(-2) = \sqrt{-2+4} = \sqrt{2}$$

$$ii) f(0) = \sqrt{0+4} = \sqrt{4} = 2$$

$$iii) f(x-1) = \sqrt{x-1+4} = \sqrt{x+3}$$

$$iv) f(x^2+4) = \sqrt{x^2+4+4} = \sqrt{x^2+8}$$

Q2. Given that

$$i) f(x) = 6x - 9$$

$$f(a+h) = 6(a+h) - 9 = 6a + 6h - 9$$

$$f(a) = 6a - 9$$

$$\text{Now } \frac{f(a+h) - f(a)}{h} = \frac{(6a + 6h - 9) - (6a - 9)}{h}$$

$$= \frac{6a + 6h - 9 - 6a + 9}{h} = \frac{6h}{h} = 6$$

$$ii) f(x) = \sin x \quad \text{given}$$

$$\therefore \sin \theta - \sin \phi = 2 \cos \left(\frac{\theta + \phi}{2} \right) \sin \left(\frac{\theta - \phi}{2} \right)$$

$$f(a+h) = \sin(a+h) \quad \text{and} \quad f(a) = \sin a$$

$$\text{Now } \frac{f(a+h) - f(a)}{h} = \frac{\sin(a+h) - \sin a}{h}$$

$$= \frac{1}{h} [\sin(a+h) - \sin a]$$

$$= \frac{1}{h} \left[2 \cos \left(\frac{a+h+a}{2} \right) \sin \left(\frac{a+h-a}{2} \right) \right] = \frac{1}{h} \left[2 \cos \left(\frac{2a+h}{2} \right) \sin \left(\frac{h}{2} \right) \right]$$

$$= \frac{1}{h} \left[2 \cos \left(\frac{2a}{2} + \frac{h}{2} \right) \sin \left(\frac{h}{2} \right) \right] = \frac{2}{h} \cos \left(a + \frac{h}{2} \right) \sin \left(\frac{h}{2} \right)$$

iii) Given that $f(x) = x^3 + 2x^2 - 1$

$$f(a+h) = (a+h)^3 + 2(a+h)^2 - 1 = a^3 + h^3 + 3ah(a+h) + 2(a^2 + 2ah + h^2) - 1$$

$$= a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1$$

$$f(a) = a^3 + 2a^2 - 1$$

Now $f(a+h) - f(a)$

$$= \frac{a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1 - (a^3 + 2a^2 - 1)}{h}$$

$$= \frac{1}{h} [a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1 - a^3 - 2a^2 + 1]$$

$$= \frac{1}{h} [h^3 + 3a^2h + 3ah^2 + 4ah + 2h^2] = \frac{h}{h} [h^2 + 3a^2 + 3ah + 4a + 2h]$$

$$= h^2 + 3a^2 + 3ah + 4a + 2h = h^2 + 3ah + 2h + 3a^2 + 4a = h^2 + (3a+2)h + 3a^2 + 4a$$

iv) Given that $f(x) = \cos x$

so $f(a+h) = \cos(a+h)$

and $f(a) = \cos a$

Now $\frac{f(a+h) - f(a)}{h}$

$$= \frac{\cos(a+h) - \cos a}{h} = \frac{1}{h} \left[-2 \sin \left(\frac{2a+h}{2} \right) \sin \left(\frac{h}{2} \right) \right] = \frac{-2}{h} \sin \left(a + \frac{h}{2} \right) \sin \left(\frac{h}{2} \right)$$

Q3. (a) If 'x' unit be the side of square.

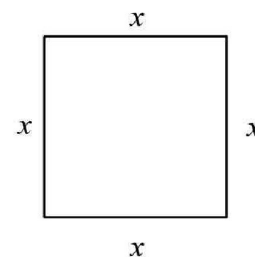
Then its perimeter $P = x + x + x + x = 4x$ (1)

$A = \text{Area} = x \cdot x = x^2$ (2)

From (2) $x = \sqrt{A}$ putting in (1)

$$P = 4\sqrt{A}$$

$\therefore$ P is expressed as Area



(b) Let x units be the radius of circle

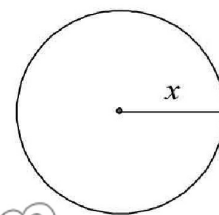
Then Area = $A = \pi x^2$ (1)

Circumference = $C = 2\pi x$ (2)

From (2) $x = \frac{C}{2\pi}$ Putting in (1)

$$A = \pi \left(\frac{C}{2\pi} \right)^2 = \pi \left(\frac{C^2}{4\pi^2} \right) = \frac{C^2}{4\pi}$$

$$A = \frac{C^2}{4\pi} \quad \therefore \text{Area is a function of Circumference}$$

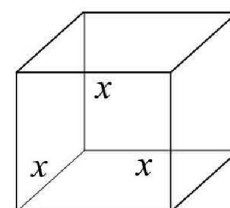


(c) Let x unit be each side of cube.

The Volume of Cube = $x \cdot x \cdot x = x^3$ (1)

Area of base = $A = x^2$ (2)

From (2) $x = \sqrt{A}$ Putting in (1)



$$V = (\sqrt{A})^3 = (A)^{\frac{3}{2}}$$

Q5. $f(x) = x^3 - ax^2 + bx + 1$

If $f(2) = -3$ and

$f(-1) = 0$

$$(2)^3 - a(2)^2 + b(2) + 1 = -3$$

$$(-1)^3 - a(-1)^2 + (-1) + 1 = 0$$

$$8 - 4a + 2b + 1 = -3$$

$$-1 - a - b + 1 = 0$$

$$9 - 4a + 2b = -3$$

$$-a - b = 0$$

$$12 - 4a + 2b = 0$$

$$a + b = 0 \quad \dots\dots\dots (2)$$

Dividing by -2

$$-6 + 2a - b = 0 \quad \dots\dots\dots (1)$$

Solving (1) and (2)

$$2a - b - 6 = 0$$

$$\frac{a + b}{3a - 6} = 0$$

$$3a - 6 = 0$$

$$a = 2 \quad \text{and} \quad (2) \Rightarrow b = -a \quad \Rightarrow \quad b = -2$$

Q6. $h(x) = 40 - 10x^2$

(a) $x = 1 \text{ sec}$

$$h(1) = 40 - 10(1)^2$$

$$= 30m$$



(b) $x = 1.5 \text{ sec}$

$$h(1.5) = 40 - 10(1.5)^2$$

$$= 40 - 10(2.25) = 40 - 22.5 = 17.5m$$

(c) $x = 1.7 \text{ sec}$

$$h(1.7) = 40 - 10(1.7)^2$$

$$= 40 - 10(2.89) = 40 - 28.9 = 11.1m$$

ii) Does the stone strike the ground = ?

$$h(x) = 0$$

$$40 - 10x^2 = 0$$

$$-10x^2 = -40 \Rightarrow x^2 = 4$$

$$x = \pm 2$$

Stone strike the ground after 2 sec.

Graphs of Function

Definition:

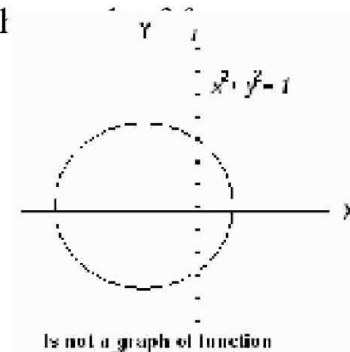
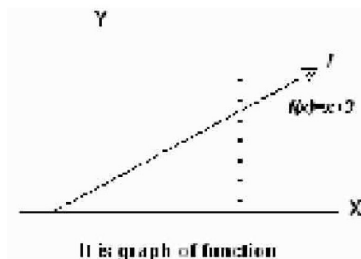
Ex # 1.1 – FSc Part 2

The graph of a function f is the graph of the equation $y = f(x)$. It consists of the points in the Cartesian plane whose co-ordinates (x, y) are input-output pairs for f .

Note that not every curve we draw in the graph of a function. A function f can have only one value $f(x)$ for each x in its domain.

Vertical Line Test

No vertical line can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function. Since some vertical lines intersect the circle twice. If 'a' is the domain of the function f , then the vertical line $x = a$ will intersect the graph in the single point $(a, f(a))$.



Types of Function

ALGEBRAIC FUNCTIONS

Those functions which are defined by algebraic expressions.

1) Polynomial Functions:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \text{ Is a}$$

Polynomial Function for all x where $a_0, a_1, a_2, \dots, a_n$ are real numbers, and exponents are non-negative integer. a_n is called leading coefft of $p(x)$ of degree n ,

Where $a_n \neq 0$

$\Rightarrow$ Degree of polynomial function is the maximum power of x in equation

$$P(x) = 2x^4 - 3x^3 + 2x - 1 \quad \text{degree} = 4$$

2) Linear Function: if the degree of polynomial fn is '1', is called linear function i.e. $p(x) = ax + b$

or $\Rightarrow$ Degree of polynomial function is one.

$$f(x) = ax + b \quad a \neq 0$$

$$\therefore y = 5x + b$$

3) Identity Function: For any set X , a function $I: X \rightarrow x$ of the form $y = x$ or $f(x) = x$. Domain and range of I is x . Note. $I(x) = ax + b$ be a linear fn if $a=1, b=0$ then $I(x)=x$ or $y=x$ is called identity fn

4) Constant Function:

$C: X \rightarrow y$ defined by $f: X \rightarrow y$ If $f(x)=c$, (const) then f is called constant fn

$$C(x) = a \quad \forall x \in X \text{ and } a \in y$$

$$\text{e.g. } C: R \rightarrow R$$

$$C(x) = 2 \text{ or } y = 2 \quad \forall x \in R$$

Ex # 1.1 – FSc Part 2

$$\text{eg } y=5$$

5) Rational Function:

$$R(x) = \frac{P(x)}{Q(x)}$$

Both $P(x)$ and $Q(x)$ are polynomial and $Q(x) \neq 0$

e.g. $R(x) = \frac{3x^2 + 4x + 1}{5x^3 + 2x^2 + 1}$

Domain of rational function is the set of all real numbers for which $Q(x) \neq 0$

6) Exponential Function:

A function in which the variable appears as exponent (power) is called an exponential function.

i) $y = a^x \therefore x \in R \quad a > 0$

ii) $y = e^x \therefore x \in R$ and $e = 2.178$

iii) $y = 2^x$ or $y = e^{xh}$

are some exponential functions.

7) Logarithmic Function:

If $x = a^y$ then $y = \log_a x \quad x > 0$

$\therefore a > 0 \quad a \neq 1$

'a' is called the base of Logarithmic function

Then $y = \log_a x$ is Logarithmic function of base 'a'

i) If base = 10 then $y = \log_{10} x$

is called common Logarithm of x

ii) If base = $e = 2.718$

$y = \log_e x = \ln x$ is called natural log

8) Hyperbolic Function:

We define as

i) $y = \sinh(x) = \frac{e^x - e^{-x}}{2}$

Sine hyperbolic function or hyperbolic sine function

Dom = $\{x / x \in R\}$ and Range = $\{y / y \in R\}$

ii) $y = \cosh(x) = \frac{e^x + e^{-x}}{2}$

is called hyperbolic cosine function $\Rightarrow x \in R, y \in [1, \infty)$

iii) $y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$ (iv) $y = \coth x = \frac{\cosh x}{\sinh x}$

v) $y = \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \quad x \in R$

vi) $y = \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \quad \text{Dom} = \{x \neq 0 : x \in R\}$

9) Inverse Hyperbolic Function: (Study in B.Sc level)

- i) $y = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ for $\forall x \in R$
- ii) $y = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ for $\forall x \in R$ and $x > 1$
- iii) $y = \tanh^{-1} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|$ $x \neq 1$ and $|x| < 1$
- iv) $y = \operatorname{sech}^{-1} x = \ln \left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{x} \right)$ $0 < x \leq 1$
- v) $y = \coth^{-1} x = \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right|$ $\because |x| > 1$
- vi) $y = \operatorname{cosech}^{-1} x = \ln \left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|} \right)$ $x \neq 0$

10) Trigonometric Function:

Functions	Domain(x)	Range(y)
i) $y = \sin x$	All real numbers $\because -\infty < x < \infty$	$-1 \leq y \leq 1$
ii) $y = \cos x$	All real numbers $\because -\infty < x < \infty$	$-1 \leq y \leq 1$
iii) $y = \tan x$	$x \in R - (2k+1)\frac{\pi}{2}$ $k \in Z$	$\because 'R'$ all real numbers
iv) $y = \cot x$	$x \in R - k\pi$ $k \in Z$	R
v) $y = \sec x$	$x \in R - (2k+1)\frac{\pi}{2}$ $k \in Z$	$R - (-1, 1)$ or $R - (-1 < y < 1)$
vi) $y = \operatorname{cosec} x$	$x \in R - (k\pi)$ $k \in Z$	$R - (-1 < y < 1)$

11) Inverse Trigonometric Functions:

Function	Dom(x)	Range(y)
$y = \sin^{-1} x \Leftrightarrow x = \sin y$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \cos^{-1} x \Leftrightarrow x = \cos y$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$

$$y = \tan^{-1} x \Leftrightarrow x = \tan y$$

$$x \in \mathbb{R}$$

$$\text{or } -\infty < x < \infty$$

$$-\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$y = \sec^{-1} x \Leftrightarrow x = \sec y$$

$$x \in \mathbb{R} - (-1, 1)$$

$$y \in [0, \pi] - \left\{ \frac{\pi}{2} \right\}$$

$$y = \operatorname{cosec}^{-1} x \Leftrightarrow x = \operatorname{cosec} y$$

$$x \in \mathbb{R} - (-1, 1)$$

$$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] - \{0\}$$

$$y = \cot^{-1} x \Leftrightarrow x = \cot y$$

$$x \in \mathbb{R}$$

$$0 < y < \pi$$

12) Explicit Function:

If y is easily expressed in terms of x , then y is called an explicit function of x .

$$\Rightarrow y = f(x) \quad \text{e.g.} \quad y = x^3 + x + 1 \quad \text{etc.}$$

13) Implicit Function:

If x and y are so mixed up and y cannot be expressed in term of the independent variable x , Then y is called an implicit function of x . It can be written as.

$$f(x, y) = 0$$

$$\text{e.g.} \quad x^2 + xy + y^2 = 2 \quad \text{etc.}$$

14) Parametric Function:

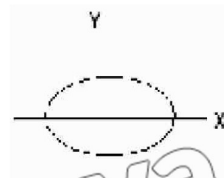
For a function $y = f(x)$ if both x & y are expressed in another variable say 't' or θ which is called a parameter of the given curve. Such as:

i) $x = at^2$ Parametric parabola
 $y = 2at$

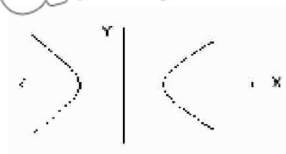
ii) $x = a \cos t$ Parametric equation of circle $y^2 = 4a$
 $y = a \sin t$
 $x^2 + y^2 = a^2$



iii) $x = a \cos \theta$ Parametric equation of Ellipse
 $y = b \sin \theta$
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



vi) $x = a \sec \theta$ Parametric equation of hyperbola
 $y = b \tan \theta$
 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Exercise 1.1

Q7. Parabola $\Rightarrow y^2 = 4ax$ (1)

$x = at^2$ (i)

$y = 2at$ (ii)

To eliminating t from (ii) $t = \frac{y}{2a}$ putting (i)

$$x = a\left(\frac{y}{2a}\right)^2 \Rightarrow x = a\left(\frac{y^2}{4a^2}\right) \Rightarrow x = \frac{y^2}{4a}$$

$\Rightarrow y^2 = 4ax$ which is same as (1)

which is equation of parabola.

ii) $x = a \cos \theta$, $y = b \sin \theta$

$\Rightarrow \frac{x}{a} = \cos \theta$ (i) and $\frac{y}{b} = \sin \theta$ (ii) To eliminating θ from (i) and (ii)

Squaring and adding (i) and (ii)

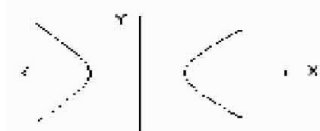
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{represent a Ellipse}$$

iii) $x = a \sec \theta$, $y = b \tan \theta$

$\frac{x}{a} = \sec \theta$ (i) $\frac{y}{b} = \tan \theta$ (ii)

Squaring and Subtracting (i) and (ii)

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \sec^2 \theta - \tan^2 \theta \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \tan^2 \theta - \tan^2 \theta \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Which is equation of hyperbola

Q8. (i) $\sinh 2x = 2 \sinh x \cosh x$

$$R.H.S = 2 \sinh x \cosh x = 2 \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) = 2 \left(\frac{e^{2x} - e^{-2x}}{4} \right) = \frac{e^{2x} - e^{-2x}}{2}$$

$= \sinh 2x = L.H.S$

ii) $\sec^2 hx = 1 - \tan^2 hx$

$$\begin{aligned} R.H.S. &= 1 - \tan^2 hx = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)^2 = 1 - \left(\frac{e^{2x} + e^{-2x} - 2}{e^{2x} + e^{-2x} + 2} \right)^2 \\ &= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} - 2}{e^{2x} + e^{-2x} + 2} = \frac{4}{(e^x + e^{-x})^2} = \frac{1}{\left(\frac{e^x + e^{-x}}{2} \right)^2} \end{aligned}$$

$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x = L.H.S$$

$$\text{iii) } \operatorname{cosech}^2 x = \coth^2 x - 1$$

$$\begin{aligned} R.H.S &= \coth^2 x - 1 = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} \right)^2 - 1 = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x - e^{-x})^2} = \frac{(e^{2x} + e^{-2x} + 2) - (e^{2x} + e^{-2x} - 2)}{(e^x - e^{-x})^2} \\ &= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x - e^{-x})^2} = \frac{4}{(e^x - e^{-x})^2} = \frac{1}{\left(\frac{e^x - e^{-x}}{2} \right)^2} = \frac{1}{\sinh^2 x} = \operatorname{cosech}^2 x = L.H.S \end{aligned}$$

$$Q9. \quad f(x) = x^3 + x$$

replace x by $-x$

$$f(-x) = (-x)^3 + (-x) = -x^3 - x = -[x^3 + x] = -f(x)$$

$$\Rightarrow f(x) = x^3 + x \text{ is odd function}$$

$$\text{ii) } f(x) = (x+2)^2$$

replace x by $-x$

$$f(-x) = (-x+2)^2 \neq \pm f(x)$$

$$f(x) = (x+2)^2 \text{ is neither even nor odd}$$

$$\text{iii) } f(x) = x\sqrt{x^2 + 5}$$

replace x by $-x$

$$f(-x) = (-x)\sqrt{(-x)^2 + 5} = -[x\sqrt{x^2 + 5}] = -f(x) \quad f(x) \text{ is odd function.}$$

$$\text{iv) } f(x) = \frac{x-1}{x+1}$$

replace x by $-x$

$$f(-x) = \frac{-x-1}{-x+1} = \frac{-(x+1)}{-(x-1)} = \frac{x+1}{x-1} \neq \pm f(x)$$

$$f(x) \text{ is neither even nor odd function.}$$

$$\text{v) } f(x) = x^{\frac{2}{3}} + 6$$

replace x by $-x$

$$f(-x) = (-x)^{\frac{2}{3}} + 6 = \left[(-x)^2 \right]^{\frac{1}{3}} + 6 = x^{\frac{2}{3}} + 6 = f(x)$$

$$f(x) \text{ is an even function.}$$

$$= \frac{x+1}{x-1} \neq \pm f(x)$$

$\therefore f(x)$ is neither even nor odd function.

(v) $f(x) = x^{2/3} + 6$

$$\begin{aligned} f(-x) &= (-x)^{2/3} + 6 \\ &= [(-x)^2]^{1/3} + 6 \\ &= (x^2)^{1/3} + 6 \\ &= x^{2/3} + 6 \\ &= f(x) \end{aligned}$$

$\therefore f(x)$ is an even function.

(vi) $f(x) = \frac{x^3 - x}{x^2 + 1}$

$$\begin{aligned} f(-x) &= \frac{(-x)^3 - (-x)}{(-x)^2 + 1} \\ &= \frac{-x^3 + x}{x^2 + 1} \\ &= \frac{-(x^3 - x)}{x^2 + 1} \\ &= -f(x) \end{aligned}$$

$\therefore f(x)$ is an odd function.

Composition of Functions:

Let f be a function from set X to set Y and g be a function from set Y to set Z . The composition of f and g is a function, denoted by $g \circ f$, from X to Z and is defined by.

$$(g \circ f)(x) = g(f(x)) = gf(x) \text{ for all } x \in X$$

Inverse of a Function:

Let f be one-one function from X onto Y . The inverse function of f , denoted by f^{-1} , is a function from Y onto X and is defined by.

$$x = f^{-1}(y), \forall y \in Y \text{ if and only if } y = f(x), \forall x \in X$$

EXERCISE 1.2

Q.1 The real valued functions f and g are defined below. Find

(a) $f \circ g(x)$ (b) $g \circ f(x)$ (c) $f \circ f(x)$ (d) $g \circ g(x)$

(i) $f(x) = 2x + 1$; $g(x) = \frac{3}{x-1}$, $x \neq 1$

$$(ii) \quad f(x) = \sqrt{x+1} ; g(x) = \frac{1}{x^2}, \quad x \neq 0$$

$$(iii) \quad f(x) = \frac{1}{\sqrt{x-1}} ; x \neq 1 ; g(x) = (x^2+1)^2$$

$$(iv) \quad f(x) = 3x^4 - 2x^2 ; g(x) = \frac{2}{\sqrt{x}}, \quad x \neq 0$$

Solution:

$$(i) \quad f(x) = 2x + 1 ; g(x) = \frac{3}{x-1}, \quad x \neq 1$$

$$\begin{aligned} (a) \quad fog(x) &= f(g(x)) \\ &= f\left(\frac{3}{x-1}\right) \\ &= 2\left(\frac{3}{x-1}\right) + 1 \\ &= \frac{6}{x-1} + 1 \\ &= \frac{6+x-1}{x-1} \\ &= \frac{x+5}{x-1} \quad \text{Ans.} \end{aligned}$$

$$\begin{aligned} (b) \quad gof(x) &= g(f(x)) \\ &= g(2x+1) \\ &= \frac{3}{2x+1-1} = \frac{3}{2x} \quad \text{Ans.} \end{aligned}$$

$$\begin{aligned} (c) \quad fof(x) &= f(f(x)) \\ &= f(2x+1) \\ &= 2(2x+1) + 1 \\ &= 4x + 2 + 1 \\ &= 4x + 3 \quad \text{Ans.} \end{aligned}$$

$$\begin{aligned} (d) \quad gog(x) &= g(g(x)) \\ &= g\left(\frac{3}{x-1}\right) \\ &= \frac{3}{\frac{3}{x-1}-1} \end{aligned}$$

$$= \frac{3}{3 - (x-1)}$$

$$= \frac{3(x-1)}{3-x+1}$$

$$= \frac{3(x-1)}{4-x} \quad \text{Ans.}$$

(ii) $f(x) = \sqrt{x+1}$; $g(x) = \frac{1}{x^2}$, $x \neq 0$

(a) $f \circ g(x) = f(g(x))$
 $= f\left(\frac{1}{x^2}\right)$
 $= \sqrt{\frac{1}{x^2} + 1}$
 $= \sqrt{\frac{1+x^2}{x^2}} = \frac{\sqrt{1+x^2}}{x} \quad \text{Ans.}$

(b) $g \circ f(x) = g(f(x))$
 $= g(\sqrt{x+1})$
 $= \frac{1}{(\sqrt{x+1})^2} = \frac{1}{x+1} \quad \text{Ans.}$

(c) $f \circ f(x) = f(f(x))$
 $= f(\sqrt{x+1})$
 $= \sqrt{\sqrt{x+1} + 1} \quad \text{Ans.}$

(d) $g \circ g(x) = g(g(x))$
 $= g\left(\frac{1}{x^2}\right)$
 $= \frac{1}{\left(\frac{1}{x^2}\right)^2} = \frac{1}{\frac{1}{x^4}} = x^4 \quad \text{Ans.}$

(iii) $f(x) = \frac{1}{\sqrt{x-1}}$; $x \neq 1$; $g(x) = (x^2+1)^2$

(a) $f \circ g(x) = f(g(x))$
 $= f((x^2+1)^2)$
 $= \frac{1}{\sqrt{(x^2+1)^2 - 1}}$

$$= \frac{1}{\sqrt{x^4 + 1 + 2x^2 - 1}}$$

$$= \frac{1}{\sqrt{x^2(x^2 + 2)}} = \frac{1}{x\sqrt{x^2 + 2}} \quad \text{Ans.}$$

(b) $g \circ f(x) = g(f(x))$

$$= g\left(\frac{1}{\sqrt{x-1}}\right)$$

$$= \left[\left(\frac{1}{\sqrt{x-1}}\right)^2 + 1\right]^2$$

$$= \left(\frac{1}{x-1} + 1\right)^2 = \left(\frac{1+x-1}{x-1}\right)^2$$

$$= \left(\frac{x}{x-1}\right)^2 \quad \text{Ans.}$$

(c) $f \circ f(x) = f(f(x))$

$$= f\left(\frac{1}{\sqrt{x-1}}\right)$$

$$= \frac{1}{\sqrt{\frac{1}{\sqrt{x-1}} - 1}}$$

$$= \frac{1}{\sqrt{\frac{1 - \sqrt{x-1}}{\sqrt{x-1}}}}} = \sqrt{\frac{\sqrt{x-1}}{1 - \sqrt{x-1}}} \quad \text{Ans.}$$

(d) $g \circ g(x) = g(g(x))$

$$= g((x^2 + 1)^2)$$

$$= [\{(x^2 + 1)^2\}^2 + 1]^2$$

$$= [(x^2 + 1)^4 + 1]^2 \quad \text{Ans.}$$

(iv) $f(x) = 3x^4 - 2x^2$; $g(x) = \frac{2}{\sqrt{x}}$, $x \neq 0$

(a) $f \circ g(x) = f(g(x))$

$$= f\left(\frac{2}{\sqrt{x}}\right)$$

$$= 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2$$

$$\begin{aligned}
 &= 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right) \\
 &= \frac{48}{x^2} - \frac{8}{x} \\
 &= \frac{48 - 8x}{x^2} \\
 &= \frac{8(6 - x)}{x^2} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \text{gof}(x) &= g(f(x)) \\
 &= g(3x^4 - 2x^2) \\
 &= \frac{2}{\sqrt{3x^4 - 2x^2}} \\
 &= \frac{2}{\sqrt{x^2(3x^2 - 2)}} = \frac{2}{x\sqrt{3x^2 - 2}} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \text{fof}(x) &= f(f(x)) \\
 &= f(3x^4 - 2x^2) \\
 &= 3(3x^4 - 2x^2)^4 - 2(3x^4 - 2x^2)^2 \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \text{gog}(x) &= g(g(x)) \\
 &= g\left(\frac{2}{\sqrt{x}}\right) \\
 &= \frac{2}{\sqrt{2/\sqrt{x}}} \\
 &= 2\sqrt{\frac{\sqrt{x}}{2}} \\
 &= \sqrt{2} \times \sqrt{2} \frac{\sqrt{\sqrt{x}}}{\sqrt{2}} \\
 &= \sqrt{2}\sqrt{x} \quad \text{Ans.}
 \end{aligned}$$

Q.2 For the real valued function, f defined below, find:

(a) $f^{-1}(x)$

(b) $f^{-1}(-1)$ and verify $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(i) $f(x) = -2x + 8$ (Lahore Board 2007, 2009) (ii) $f(x) = 3x^3 + 7$

(iii) $f(x) = (-x + 9)^3$ (iv) $f(x) = \frac{2x+1}{x-1}, x > 1$

Solution:

(i) $f(x) = -2x + 8$

(a) Since $y = f(x)$

$$\Rightarrow x = f^{-1}(y)$$

Now,

$$f(x) = -2x + 8$$

$$y = -2x + 8$$

$$2x = 8 - y$$

$$x = \frac{8 - y}{2}$$

$$f^{-1}(y) = \frac{8 - y}{2}$$

Replacing y by x

$$f^{-1}(x) = \frac{8 - x}{2}$$

Replacing y by x .

$$f^{-1}(x) = \frac{8 - x}{2}$$

(b) But, $x = -1$

$$f^{-1}(-1) = \frac{8 - (-1)}{2} = \frac{8 + 1}{2} = \frac{9}{2}$$

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{8 - x}{2}\right) \\ &= -2\left(\frac{8 - x}{2}\right) + 8 \\ &= -8 + x + 8 \\ &= x \end{aligned}$$

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(-2x + 8) \\ &= \frac{8 - (-2x + 8)}{2} \\ &= \frac{8 + 2x - 8}{2} \\ &= \frac{2x}{2} = x \end{aligned}$$

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x \quad \text{Hence proved.}$$

(ii) $f(x) = 3x^3 + 7$

(a) Since $y = f(x)$
 $\Rightarrow x = f^{-1}(y)$

Now

$$f(x) = 3x^3 + 7$$

$$y = 3x^3 + 7$$

$$3x^3 = y - 7$$

$$x^3 = \frac{y-7}{3}$$

$$x = \left(\frac{y-7}{3}\right)^{\frac{1}{3}}$$

$$f^{-1}(y) = \left(\frac{y-7}{3}\right)^{\frac{1}{3}}$$

Replacing y by x

$$f^{-1}(x) = \left(\frac{x-7}{3}\right)^{\frac{1}{3}}$$

(b) Put $x = -1$
 $f^{-1}(-1) = \left(\frac{-1-7}{3}\right)^{\frac{1}{3}}$

$$= \left(\frac{-8}{3}\right)^{\frac{1}{3}}$$

$$\begin{aligned} f(f^{-1}(x)) &= f\left[\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right] \\ &= 3\left[\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right]^3 + 7 \\ &= 3\left(\frac{x-7}{3}\right) + 7 \\ &= x - 7 + 7 = x \end{aligned}$$

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(3x^3 + 7) \\ &= \left(\frac{3x^3 + 7 - 7}{3}\right)^{\frac{1}{3}} \end{aligned}$$

$$= \left(\frac{3x^3}{3} \right)^{\frac{1}{3}}$$

$$= (x^3)^{\frac{1}{3}} = x$$

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x \quad \text{Hence proved.}$$

(iii) $f(x) = (-x + 9)^3$

(a) Since $y = f(x)$
 $x = f^{-1}(y)$

Now

$$f(x) = (-x + 9)^3$$

$$y = (-x + 9)^3$$

$$y^{\frac{1}{3}} = -x + 9$$

$$x = 9 - y^{\frac{1}{3}}$$

Replacing y by x

$$f^{-1}(x) = 9 - x^{\frac{1}{3}}$$

(b) Put $x = -1$

$$f^{-1}(-1) = 9 - (-1)^{\frac{1}{3}}$$

$$f(f^{-1}(x)) = f(9 - x^{\frac{1}{3}})$$

$$= [-(9 - x^{\frac{1}{3}}) + 9]^3$$

$$= (-9 + x^{\frac{1}{3}} + 9)^3$$

$$= \left(\frac{1}{x^{\frac{1}{3}}} \right)^3 = x$$

$$f^{-1}(f(x)) = f^{-1}((-x + 9)^3)$$

$$= 9 - [(-x + 9)^3]^{\frac{1}{3}}$$

$$= 9 - (-x + 9)$$

$$= 9 + x - 9$$

$$= x$$

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x \quad \text{Hence proved.}$$

$$(iv) \quad f(x) = \frac{2x+1}{x-1}, \quad x > 1$$

$$(a) \quad \text{Since } y = f(x) \\ x = f^{-1}(y)$$

Now

$$f(x) = \frac{2x+1}{x-1}$$

$$y = \frac{2x+1}{x-1}$$

$$y(x-1) = 2x+1$$

$$yx - y = 2x+1$$

$$yx - 2x = 1+y$$

$$x(y-2) = y+1$$

$$x = \frac{y+1}{y-2}$$

$$f^{-1}(y) = \frac{y+1}{y-2}$$

Replacing y by x

$$f^{-1}(x) = \frac{x+1}{x-2}$$

$$(b) \quad \text{Put } x = -1$$

$$f^{-1}(-1) = \frac{-1+1}{-1-2}$$

$$= \frac{0}{-3} = 0$$

$$f(f^{-1}(x)) = f\left(\frac{x+1}{x-2}\right)$$

$$= \frac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2}-1}$$

$$= \frac{2(x+1)+(x-2)}{x+1-(x-2)}$$

$$\begin{aligned}
 &= \frac{2x+2+x-2}{x+1-x+2} \\
 &= \frac{3x}{3} = x \\
 f^{-1}(f(x)) &= f^{-1}\left(\frac{2x+1}{x-1}\right) \\
 &= \frac{\frac{2x+1}{x-1}+1}{\frac{2x+1}{x-1}-2} \\
 &= \frac{\frac{2x+1+x-1}{x-1}}{\frac{2x+1-2(x-1)}{x-1}} \\
 &= \frac{3x}{2x+1-2x+2} \\
 &= \frac{3x}{3} = x
 \end{aligned}$$

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x \quad \text{Hence proved.}$$

Q.3 Without finding the inverse, state the domain and range of f^{-1} .

(i) $f(x) = \sqrt{x+2}$

(ii) $f(x) = \frac{x-1}{x-4}, x \neq 4$

(iii) $f(x) = \frac{1}{x+3}, x \neq -3$

(iv) $f(x) = (x-5)^2, x \geq 5$

Solution:

(i) $f(x) = \sqrt{x+2}$

Domain of $f(x)$ = $[-2, +\infty)$

Range of $f(x)$ = $[0, +\infty)$

Domain of $f^{-1}(x)$ = Range of $f(x)$ = $[0, +\infty)$

Range of $f^{-1}(x)$ = Domain of $f(x)$ = $[-2, +\infty)$

(ii) $f(x) = \frac{x-1}{x-4}, x \neq 4$

Domain of $f(x)$ = $\mathbb{R} - \{4\}$

Range of $f(x)$ = $\mathbb{R} - \{1\}$

Domain of $f^{-1}(x)$ = Range of $f(x)$ = $\mathbb{R} - \{1\}$

Range of $f^{-1}(x)$ = Domain of $f(x)$ = $\mathbb{R} - \{4\}$

$$(iii) \quad f(x) = \frac{1}{x+3}, x \neq -3$$

$$\text{Domain of } f(x) = \mathbb{R} - \{-3\}$$

$$\text{Range of } f(x) = \mathbb{R} - \{0\}$$

$$\text{Domain of } f^{-1}(x) = \text{Range of } f(x) = \mathbb{R} - \{0\}$$

$$\text{Range of } f^{-1}(x) = \text{Domain of } f(x) = \mathbb{R} - \{-3\}$$

$$(iv) \quad f(x) = (x-5)^2, x \geq 5 \quad (\text{Gujranwala Board 2007})$$

$$\text{Domain of } f(x) = [5, +\infty)$$

$$\text{Range of } f(x) = [0, +\infty)$$

$$\text{Domain of } f^{-1}(x) = \text{Range of } f(x) = [0, +\infty)$$

$$\text{Range of } f^{-1}(x) = \text{Domain of } f(x) = [5, +\infty)$$

Limit of a Function:

Let a function $f(x)$ be defined in an open interval near the number 'a' (need not at a) if, as x approaches 'a' from both left and right side of 'a', $f(x)$ approaches a specific number 'L' then 'L', is called the limit of $f(x)$ as x approaches a symbolically it is written as.

$$\lim_{x \rightarrow a} f(x) = L \text{ read as "Limit of } f(x) \text{ as } x \rightarrow a, \text{ is } L"$$

Theorems on Limits of Functions:

Let f and g be two functions, for which $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then

Theorem 1: The limit of the sum of two functions is equal to the sum of their limits.

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) + g(x)] &= \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) \\ &= L + M \end{aligned}$$

Theorem 2: The limit of the difference of two functions is equal to the difference of their limits.

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) - g(x)] &= \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) \\ &= L - M \end{aligned}$$

Theorem 3: If K is any real numbers, then.

$$\lim_{x \rightarrow a} [kf(x)] = K \lim_{x \rightarrow a} f(x) = kL$$

Theorem 4: The limit of the product of the functions is equal to the product of their limits.

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = [\lim_{x \rightarrow a} f(x)] [\lim_{x \rightarrow a} g(x)] = LM$$

Theorem 5: The limit of the quotient of the functions is equal to the quotient of their limits provided the limit of the denominator is non-zero.

$$\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}, \quad g(x) \neq 0, M \neq 0$$

Theorem 6: Limit of $[f(x)]^n$, where n is an integer.

$$\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n = L^n$$

The Sandwich Theorem:

Let f , g and h be functions such that $f(x) \leq g(x) \leq h(x)$ for all number x in some open interval containing “C”, except possibly at C itself.

If, $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} h(x) = L$, then $\lim_{x \rightarrow c} g(x) = L$

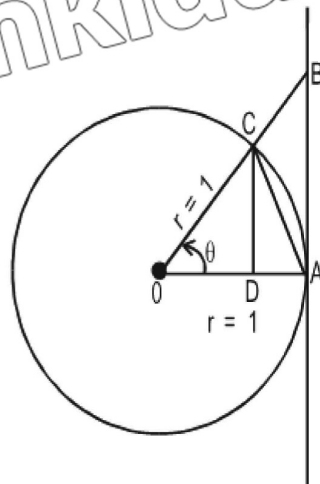
Prove that

If θ is measured in radian, then

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Proof:

Take θ a positive acute central angle of a circle with radius $r = 1$. OAB represents the sector of the circle.



$$|OA| = |OC| = 1 \quad (\text{radii of unit circle})$$

From right angle $\triangle ODC$

$$\sin \theta = \frac{|DC|}{|OC|} = |DC| \quad (\because |OC| = 1)$$

From right angle $\triangle OAB$

$$\tan \theta = \frac{|AB|}{|OA|} = |AB| \quad (\because |OA| = 1)$$

In terms of θ , the areas are expressed as

$$\text{Area of } \triangle OAC = \frac{1}{2} |OA| |CD| = \frac{1}{2} (1) \sin \theta = \frac{1}{2} \sin \theta$$

$$\text{Area of sector OAC} = \frac{1}{2} r^2 \theta = \frac{1}{2} (1)(\theta) = \frac{1}{2} \theta$$

$$\text{Area of } \triangle OAB = \frac{1}{2} |OA| |AB| = \frac{1}{2} (1) \tan \theta = \frac{1}{2} \tan \theta$$

From figure

Area of $\triangle OAB >$ Area of sector OAC $>$ Area of $\triangle OAC$

$$\frac{1}{2} \tan \theta > \frac{1}{2} \theta > \frac{1}{2} \sin \theta$$

$$\frac{1}{2} \frac{\sin \theta}{\cos \theta} > \frac{\theta}{2} > \frac{\sin \theta}{2}$$

As $\sin \theta$ is positive, so on division by $\frac{1}{2} \sin \theta$, we get.

$$\frac{1}{\cos \theta} > \frac{\theta}{\sin \theta} > 1 \quad (0 < \theta < \pi/2)$$

i.e.

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

When, $\theta \rightarrow 0$, $\cos \theta \rightarrow 1$

Since $\frac{\sin \theta}{\theta}$ is sandwiched between 1 and a quantity approaching 1 itself.

So by the sandwich theorem it must also approach 1.

i.e.

$$\boxed{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1}$$

Theorem: Prove that

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Proof:

Taking

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!} \left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{1}{n}\right)^3 + \dots \\ &= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \dots \end{aligned}$$

Taking $\lim_{n \rightarrow +\infty}$ on both sides.

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots \\ &= 1 + 1 + 0.5 + 0.166667 + 0.0416667 + \dots \end{aligned}$$

As approximate value of e is = 2.718281.....

$$\therefore \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^2 = e$$

Deduction:

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} = e$$

We know that,

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Put $x = \frac{1}{n}$ then $\frac{1}{x} = n$

As $n \rightarrow +\infty$, $x \rightarrow 0$

$$\therefore \lim_{n \rightarrow +\infty} (1 + x)^{1/x} = e$$

Theorem:

Prove that:

$$\lim_{x \rightarrow a} \frac{a^x - 1}{x} = \log_e a$$

Proof:

Taking,

$$\lim_{x \rightarrow a} \frac{a^x - 1}{x}$$

Let $a^x - 1 = y$

$$a^x = 1 + y$$

$$x = \log_a (1 + y)$$

As, $x \rightarrow a$, $y \rightarrow 0$

$$\lim_{x \rightarrow a} \frac{a^x - 1}{x} = \lim_{y \rightarrow 0} \frac{y}{\log_a (1 + y)}$$

$$= \lim_{y \rightarrow 0} \frac{1}{\frac{1}{y} \log_a (1 + y)} = \lim_{y \rightarrow 0} \frac{1}{\log_a (1 + y)^{\frac{1}{y}}}$$

$$= \frac{1}{\log_a e} \because \lim_{y \rightarrow 0} (1 + y)^{1/y} = e$$

$$= \log_a a$$

Deduction

$$\lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) = \log_e e = 1$$

We know that

$$\lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right) = \log_e a$$

Put $a = e$

$$\lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) = \log_e e = 1$$

Important results to remember

- (i) $\lim_{x \rightarrow +\infty} (e^x) = \infty$ (ii) $\lim_{x \rightarrow -\infty} (e^x) = \lim_{x \rightarrow -\infty} \left(\frac{1}{e^{-x}} \right) = 0$
- (iii) $\lim_{x \rightarrow \pm\infty} \left(\frac{a}{x} \right) = 0$, where a is any real number.

EXERCISE 1.3

Q.1 Evaluate each limit by using theorems of limits.

(i) $\lim_{x \rightarrow 3} (2x + 4)$

(ii) $\lim_{x \rightarrow 1} (3x^2 - 2x + 4)$

(iii) $\lim_{x \rightarrow 3} \sqrt{x^2 + x + 4}$

(iv) $\lim_{x \rightarrow 2} x\sqrt{x^2 - 4}$

(v) $\lim_{x \rightarrow 2} (\sqrt{x^3 + 1} - \sqrt{x^2 + 5})$

(iv) $\lim_{x \rightarrow 2} \frac{2x^3 + 5x}{3x - 2}$

Solution:

(i) $\lim_{x \rightarrow 3} (2x + 4) = \lim_{x \rightarrow 3} (2x) + \lim_{x \rightarrow 3} (4)$
 $= 2 \lim_{x \rightarrow 3} x + 4$

$$= 2(3) + 4 = 6 + 4 = 10 \quad \text{Ans.}$$

(ii) $\lim_{x \rightarrow 1} (3x^2 - 2x + 4) = \lim_{x \rightarrow 1} (3x^2) - \lim_{x \rightarrow 1} (2x) + \lim_{x \rightarrow 1} (4)$

$$= 3 \lim_{x \rightarrow 1} x^2 - 2 \lim_{x \rightarrow 1} x + 4$$

$$= 3(1)^2 - 2(1) + 4$$

$$= 3 - 2 + 4$$

$$= 5 \quad \text{Ans.}$$

(iii) $\lim_{x \rightarrow 3} \sqrt{x^2 + x + 4} = [\lim_{x \rightarrow 3} (x^2 + x + 4)]^{1/2}$

Important Limits

- I. $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$, where n is integer and $a > 0$.
- II. $\lim_{x \rightarrow 0} \frac{\sqrt{x+a} - \sqrt{a}}{x} = \frac{1}{2\sqrt{a}}$.
- III. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$.
- IV. $\lim_{x \rightarrow \infty} (1 + x)^{\frac{1}{x}} = e$.
- V. $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$, where $a > 0$.
- VI. $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \ln e = 1$.
- VII. If θ is measured in radian, then $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$.

Question # 1

- (i) $\lim_{x \rightarrow 3} (2x + 4) = \lim_{x \rightarrow 3} (2x) + \lim_{x \rightarrow 3} (4) = 2\lim_{x \rightarrow 3} (x) + 4 = 2(3) + 4 = 10$.
- (ii) $\lim_{x \rightarrow 1} (3x^2 - 2x + 4) = 3(1)^2 - 2(1) + 4 = 3 - 2 + 4 = 5$.
- (iii) $\lim_{x \rightarrow 3} \sqrt{x^2 + x + 4} = \sqrt{(3)^2 + (3) + 4} = \sqrt{9 + 3 + 4} = \sqrt{16} = 4$.
- (iv) $\lim_{x \rightarrow 2} x\sqrt{x^2 - 4} = 2\sqrt{2^2 - 4} = 0$.
- (v) $\begin{aligned} \lim_{x \rightarrow 2} (\sqrt{x^3 + 1} - \sqrt{x^2 + 5}) &= \lim_{x \rightarrow 2} (\sqrt{x^3 + 1}) - \lim_{x \rightarrow 2} (\sqrt{x^2 + 5}) \\ &= (\sqrt{(2)^3 + 1}) - (\sqrt{(2)^2 + 5}) \\ &= \sqrt{8 + 1} - \sqrt{4 + 5} = \sqrt{9} - \sqrt{9} = 0. \end{aligned}$
- (vi) $\lim_{x \rightarrow -2} \frac{2x^3 + 5x}{3x - 2} = \frac{2(-2)^3 + 5(-2)}{3(-2) - 2} = \frac{-16 - 10}{-6 - 2} = \frac{-26}{-8} = \frac{13}{4}$.

Question # 2

- (i) $\begin{aligned} \lim_{x \rightarrow -1} \frac{x^3 - x}{x + 1} &= \lim_{x \rightarrow -1} \frac{x(x^2 - 1)}{x + 1} = \lim_{x \rightarrow -1} \frac{x(x + 1)(x - 1)}{x + 1} \\ &= \lim_{x \rightarrow -1} x(x - 1) = (-1)(-1 - 1) = 2 \end{aligned}$
- (ii) $\lim_{x \rightarrow 0} \left(\frac{3x^3 + 4x}{x^2 + x} \right) = \lim_{x \rightarrow 0} \frac{x(3x^2 + 4)}{x(x + 1)} = \lim_{x \rightarrow 0} \frac{3x^2 + 4}{x + 1} = \frac{3(0) + 4}{0 + 1} = 4$.

$$\begin{aligned}
 \text{(iii)} \quad & \lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 + x - 6} \\
 &= \lim_{x \rightarrow 2} \frac{x^3 - (2)^3}{x^2 + 3x - 2x - 6} = \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4)}{x(x+3) - 2(x+3)} \\
 &= \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4)}{(x+3)(x-2)} = \lim_{x \rightarrow 2} \frac{(x^2 + 2x + 4)}{(x+3)} \\
 &= \frac{(2)^2 + 2(2) + 4}{(2+3)} = \frac{12}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & \lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 3x - 1}{x^3 - x} \\
 &= \lim_{x \rightarrow 1} \frac{(x-1)^3}{x(x^2 - 1)} = \lim_{x \rightarrow 1} \frac{(x-1)^3}{x(x-1)(x+1)} \\
 &= \lim_{x \rightarrow 1} \frac{(x-1)^2}{x(x+1)} = \lim_{x \rightarrow 1} \frac{(1-1)^2}{(1)(1+1)} = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad & \lim_{x \rightarrow -1} \left(\frac{x^3 + x^2}{x^2 - 1} \right) = \lim_{x \rightarrow -1} \frac{x^2(x+1)}{(x+1)(x-1)} = \lim_{x \rightarrow -1} \frac{x^2}{(x-1)} \\
 &= \frac{(-1)^2}{(-1-1)} = -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad & \lim_{x \rightarrow 4} \frac{2x^2 - 32}{x^3 - 4x^2} = \lim_{x \rightarrow 4} \frac{2(x^2 - 16)}{x^2(x-4)} \\
 &= \lim_{x \rightarrow 4} \frac{2(x+4)(x-4)}{x^2(x-4)} = \lim_{x \rightarrow 4} \frac{2(x+4)}{x^2} \\
 &= \frac{2(4+4)}{4^2} = \frac{16}{16} = 1.
 \end{aligned}$$

$$\begin{aligned}
 \text{(vii)} \quad & \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} = \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} \left(\frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}} \right) \\
 &= \lim_{x \rightarrow 2} \frac{(\sqrt{x})^2 - (\sqrt{2})^2}{(x-2)(\sqrt{x} + \sqrt{2})} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(\sqrt{x} + \sqrt{2})} \\
 &= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(viii)} \quad & \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
 &= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ix)} \quad & \lim_{x \rightarrow a} \frac{x^n - a^n}{x^m - a^m} \\
 &= \lim_{x \rightarrow a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1})}{(x-a)(x^{m-1} + x^{m-2}a + x^{m-3}a^2 + \dots + a^{m-1})} \\
 &= \lim_{x \rightarrow a} \frac{(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1})}{(x^{m-1} + x^{m-2}a + x^{m-3}a^2 + \dots + a^{m-1})} \\
 &= \frac{a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \dots + a^{n-1}}{a^{m-1} + a^{m-2}a + a^{m-3}a^2 + \dots + a^{m-1}} \\
 &= \frac{a^{n-1} + a^{n-1} + a^{n-1} + \dots + a^{n-1} \text{ (n terms)}}{a^{m-1} + a^{m-1} + a^{m-1} + \dots + a^{m-1} \text{ (m terms)}} \\
 &= \frac{na^{n-1}}{ma^{m-1}} = \frac{n}{m} a^{n-1-m+1} = \frac{n}{m} a^{n-m}
 \end{aligned}$$

Law of Sine

If θ is measured in radian, then $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

See proof on book at page 25

Question # 3

$$\text{(i)} \quad \lim_{x \rightarrow 0} \frac{\sin 7x}{x}$$

$$\text{Put } t = 7x \Rightarrow \frac{t}{7} = x$$

When $x \rightarrow 0$ then $t \rightarrow 0$, so

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sin 7x}{x} &= \lim_{t \rightarrow 0} \frac{\sin t}{t/7} \\
 &= 7 \lim_{t \rightarrow 0} \frac{\sin t}{t} = 7(1) = 7 \quad \text{By law of sine.}
 \end{aligned}$$

$$\text{(ii)} \quad \lim_{x \rightarrow 0} \frac{\sin x^\circ}{x}$$

$$\text{Since } 180^\circ = \pi \text{ rad} \Rightarrow 1^\circ = \frac{\pi}{180} \text{ rad} \Rightarrow x^\circ = \frac{x\pi}{180} \text{ rad}$$

$$\text{So } \lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} = \lim_{x \rightarrow 0} \frac{\sin \frac{x\pi}{180}}{x}$$

Now put $\frac{\pi x}{180} = t$ i.e. $x = \frac{180t}{\pi}$

When $x \rightarrow 0$ then $t \rightarrow 0$, so

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{x} &= \lim_{x \rightarrow 0} \frac{\sin t}{\frac{180t}{\pi}} \\ &= \frac{\pi}{180} \lim_{x \rightarrow 0} \frac{\sin t}{t} = \frac{\pi}{180} (1) = \frac{\pi}{180} \quad \text{by law of sine} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\sin \theta (1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\sin \theta (1 + \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{(1 + \cos \theta)} = \frac{\sin(0)}{1 + \cos(0)} = \frac{0}{1 + 1} = 0 \end{aligned}$$

$$\text{(iv)} \quad \lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x}$$

Put $t = \pi - x \Rightarrow x = \pi - t$

When $x \rightarrow \pi$ then $t \rightarrow 0$, so

$$\begin{aligned} \lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x} &= \lim_{t \rightarrow 0} \frac{\sin(\pi - t)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\sin t}{t} \quad \because \sin(\pi - t) = \sin\left(2 \cdot \frac{\pi}{2} - t\right) = \sin t \\ &= 1 \quad \text{By law of sine.} \end{aligned}$$

$$\begin{aligned} \text{(v)} \quad \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} &= \lim_{x \rightarrow 0} \sin ax \cdot \frac{1}{\sin bx} \\ &= \lim_{x \rightarrow 0} \sin ax \cdot \left(\frac{ax}{ax}\right) \frac{1}{\sin bx \cdot \left(\frac{bx}{bx}\right)} = \lim_{x \rightarrow 0} \frac{\sin ax}{ax} \cdot ax \frac{1}{\frac{\sin bx}{bx} \cdot bx} \\ &= \frac{a}{b} \lim_{x \rightarrow 0} \frac{\sin ax}{ax} \cdot \frac{1}{\lim_{x \rightarrow 0} \frac{\sin bx}{bx}} = \frac{a}{b} \cdot (1) \cdot \frac{1}{(1)} = \frac{a}{b} \quad \text{by law of sine} \end{aligned}$$

$$\begin{aligned} \text{(vi)} \quad \lim_{x \rightarrow 0} \frac{x}{\tan x} &= \lim_{x \rightarrow 0} \frac{x}{\frac{\sin x}{\cos x}} = \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot \cos x \\ &= \lim_{x \rightarrow 0} \frac{1}{\frac{\sin x}{x}} \cdot \cos x = \frac{1}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} \cdot \lim_{x \rightarrow 0} \cos x = \frac{1}{1} \cdot 1 = 1 \end{aligned}$$

$$\begin{aligned} \text{(vii)} \quad \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \quad \because \sin^2 x = \frac{1 - \cos 2x}{2} \\ &\quad \therefore 2 \sin^2 x = 1 - \cos 2x \end{aligned}$$

$$= 2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 = 2 \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^2 = 2(1)^2 = 2$$

(vii) *Do yourself by rationalizing*

$$\begin{aligned} \text{(viii)} \quad \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \sin \theta \\ &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \sin \theta = (1) \cdot (0) = 0 \end{aligned}$$

$$\begin{aligned} \text{(x)} \quad \lim_{x \rightarrow 0} \frac{\sec x - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x} - \cos x}{x} = \lim_{x \rightarrow 0} \frac{\frac{1 - \cos^2 x}{\cos x}}{x} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x \cos x} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x \cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = (1) \frac{\sin(0)}{\cos(0)} = (1) \cdot \frac{0}{1} = 0 \end{aligned}$$

$$\begin{aligned} \text{(xi)} \quad \lim_{x \rightarrow 0} \frac{1 - \cos p\theta}{1 - \cos q\theta} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{p\theta}{2}}{2 \sin^2 \frac{q\theta}{2}} \quad \because \sin^2 \frac{x}{2} = \frac{1 - \cos x}{2} \end{aligned}$$

$$= \lim_{x \rightarrow 0} \sin^2 \frac{p\theta}{2} \cdot \frac{1}{\sin^2 \frac{q\theta}{2}} = \lim_{x \rightarrow 0} \sin^2 \frac{p\theta}{2} \cdot \frac{\left(\frac{p\theta}{2}\right)^2}{\left(\frac{p\theta}{2}\right)^2} \cdot \frac{1}{\sin^2 \frac{q\theta}{2} \cdot \frac{\left(\frac{q\theta}{2}\right)^2}{\left(\frac{q\theta}{2}\right)^2}}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 \frac{p\theta}{2}}{\left(\frac{p\theta}{2}\right)^2} \cdot \frac{\left(\frac{p\theta}{2}\right)^2}{\frac{\sin^2 \frac{q\theta}{2}}{\left(\frac{q\theta}{2}\right)^2} \cdot \left(\frac{q\theta}{2}\right)^2} = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{p\theta}{2}}{\frac{p\theta}{2}} \right)^2 \cdot \frac{1}{\left(\frac{\sin \frac{q\theta}{2}}{\frac{q\theta}{2}} \right)^2} \cdot \frac{p^2 \theta^2 / 4}{q^2 \theta^2 / 4}$$

$$= \frac{p^2}{q^2} \left(\lim_{x \rightarrow 0} \frac{\sin \frac{p\theta}{2}}{\frac{p\theta}{2}} \right)^2 \cdot \frac{1}{\left(\lim_{x \rightarrow 0} \frac{\sin \frac{q\theta}{2}}{\frac{q\theta}{2}} \right)^2} = \frac{p^2}{q^2} (1)^2 \cdot \frac{1}{(1)^2} = \frac{p^2}{q^2}$$

$$\begin{aligned}
 \text{(xii)} \quad & \lim_{\theta \rightarrow 0} \frac{\tan \theta - \sin \theta}{\sin^3 \theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta}{\cos \theta} - \sin \theta}{\sin^3 \theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta - \sin \theta \cos \theta}{\sin^3 \theta \cos \theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{\sin \theta (1 - \cos \theta)}{\sin^3 \theta \cos \theta} = \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin^2 \theta \cos \theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin^2 \theta \cos \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\sin^2 \theta \cos \theta (1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\sin^2 \theta \cos \theta (1 + \cos \theta)} \\
 &= \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta (1 + \cos \theta)} = \lim_{x \rightarrow 0} \frac{1}{\cos \theta (1 + \cos \theta)} \\
 &= \frac{1}{\cos(1)(1 + \cos(1))} = \frac{1}{1 \cdot (1 + 1)} = \frac{1}{2}
 \end{aligned}$$

Note:

$$\text{a)} \quad \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\text{b)} \quad \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e \quad \text{where} \quad e = 2.718281...$$

See proof of (a) and (b) on book at page 23

$$\text{c)} \quad \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a \text{ or } \ln a$$

Proof:

Put $y = a^x - 1$ (i)

When $x \rightarrow 0$ then $y \rightarrow 0$

Also from (i) $1 + y = a^x$

Taking log on both sides

$$\ln(1 + y) = \ln a^x \Rightarrow \ln(1 + y) = x \ln a$$

$$\therefore \ln x^m = m \ln x$$

$$\Rightarrow x = \frac{\ln(1 + y)}{\ln a}$$

$$\text{Now} \quad \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \lim_{y \rightarrow 0} \frac{y}{\ln(1 + y)}$$

$$\begin{aligned}
 &= \lim_{y \rightarrow 0} \frac{\ln a}{y \ln a} = \lim_{y \rightarrow 0} \frac{\ln a}{\frac{1}{y} \ln(1 + y)} \\
 &= \lim_{y \rightarrow 0} \frac{\ln a}{\frac{1}{y} \ln(1 + y)}
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{y \rightarrow 0} \frac{\ln a}{\ln(1+y)^{\frac{1}{y}}} = \frac{\ln a}{\lim_{y \rightarrow 0} \ln(1+y)^{\frac{1}{y}}} \quad \because \ln x^m = m \ln x \\
 &= \frac{\ln a}{\ln \left(\lim_{y \rightarrow 0} (1+y)^{\frac{1}{y}} \right)} = \frac{\ln a}{\ln(e)} \quad \because \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \\
 &= \frac{\ln a}{1} = \ln a \quad \because \ln e = 1
 \end{aligned}$$

Question # 4

$$\begin{aligned}
 \text{(i)} \quad & \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n} \right)^{2n} = \left[\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n} \right)^n \right]^2 = e^2 \\
 \text{(ii)} \quad & \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n} \right)^{\frac{n}{2}} = \left[\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n} \right)^n \right]^{\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e} \\
 \text{(iii)} \quad & \lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n} \right)^n = \left[\lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n} \right)^{-n} \right]^{-1} = e^{-1} = \frac{1}{e} \\
 \text{(iv)} \quad & \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{3n} \right)^n \\
 &= \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{3n} \right)^{\frac{3n}{3}} = \left[\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{3n} \right)^{3n} \right]^{\frac{1}{3}} = e^{\frac{1}{3}} \\
 \text{(v)} \quad & \lim_{n \rightarrow +\infty} \left(1 + \frac{4}{n} \right)^n = \lim_{n \rightarrow +\infty} \left(1 + \frac{4}{n} \right)^{\frac{4n}{4}} = \left[\lim_{n \rightarrow +\infty} \left(1 + \frac{4}{n} \right)^{\frac{n}{4}} \right]^4 = e^4. \\
 \text{(vi)} \quad & \lim_{x \rightarrow 0} (1+3x)^{\frac{2}{x}} \\
 &= \lim_{x \rightarrow 0} (1+3x)^{\frac{6}{3x}} = \left[\lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3x}} \right]^6 = e^6 \\
 \text{(vii)} \quad & \lim_{x \rightarrow 0} (1+2x^2)^{\frac{1}{x^2}} = \lim_{x \rightarrow 0} (1+2x^2)^{\frac{2}{2x^2}} = \left[\lim_{x \rightarrow 0} (1+2x^2)^{\frac{1}{2x^2}} \right]^2 = e^2 \\
 \text{(viii)} \quad & \lim_{h \rightarrow 0} (1-2h)^{\frac{1}{h}} = \lim_{h \rightarrow 0} (1-2h)^{\frac{-2}{-2h}} = \left[\lim_{h \rightarrow 0} (1-2h)^{\frac{1}{-2h}} \right]^{-2} = e^{-2} = \frac{1}{e^2} \\
 \text{(ix)} \quad & \lim_{x \rightarrow \infty} \left(\frac{x}{1+x} \right)^x \\
 &= \lim_{x \rightarrow \infty} \left(\frac{1+x}{x} \right)^{-x} = \lim_{x \rightarrow \infty} \left(\frac{1}{x} + \frac{x}{x} \right)^{-x} = \lim_{x \rightarrow \infty} \left(\frac{1}{x} + 1 \right)^{-x}
 \end{aligned}$$

$$= \left[\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x \right]^{-1} = e^{-1} = \frac{1}{e}$$

$$(x) \quad \lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}; \quad x < 0$$

Put $x = -t$ where $t > 0$

When $x \rightarrow 0$ then $t \rightarrow 0$, so

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} &= \lim_{t \rightarrow 0} \frac{e^{-1/t} - 1}{e^{-1/t} + 1} = \frac{e^{-1/0} - 1}{e^{-1/0} + 1} \\ &= \frac{e^{-\infty} - 1}{e^{-\infty} + 1} = \frac{0 - 1}{0 + 1} \quad \because e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0 \\ &= -1 \end{aligned}$$

$$(xi) \quad \lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}; \quad x > 0$$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{e^{1/x} \left(1 - \frac{1}{e^{1/x}} \right)}{e^{1/x} \left(1 + \frac{1}{e^{1/x}} \right)} = \lim_{x \rightarrow 0} \frac{\left(1 - \frac{1}{e^{1/x}} \right)}{\left(1 + \frac{1}{e^{1/x}} \right)} \\ &= \frac{1 - \frac{1}{e^{1/0}}}{1 + \frac{1}{e^{1/0}}} = \frac{1 - \frac{1}{e^{\infty}}}{1 + \frac{1}{e^{\infty}}} = \frac{1 - \frac{1}{\infty}}{1 + \frac{1}{\infty}} = \frac{1 - 0}{1 + 0} = 1 \end{aligned}$$

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Book: *Exercise 1.3 (Page 27)*

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Exercise 1.4 (Solutions)

CALCULUS AND ANALYTIC GEOMETRY, MATHEMATICS 12

Question # 1:

(i) $f(x) = 2x^2 + x - 5$ $c = 1$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x^2 + x - 5) = 2(1)^2 + 1 - 5 = 2 + 1 - 5 = -2$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2x^2 + x - 5) = 2(1)^2 + 1 - 5 = 2 + 1 - 5 = -2$$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = -2 \quad \therefore \lim_{x \rightarrow 1} f(x) = -2$$

(ii) $f(x) = \frac{x^2 - 9}{x - 3}$ $C = -3$

$$\lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^-} \frac{x^2 - 9}{x - 3} = \frac{\lim_{x \rightarrow -3^-} (x^2 - 9)}{\lim_{x \rightarrow -3^-} (x - 3)} = \frac{(-3)^2 - 9}{-3 - 3} = \frac{9 - 9}{-6} = \frac{0}{-6} = 0$$

$$\text{Now } \lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} \frac{x^2 - 9}{x - 3} = \frac{\lim_{x \rightarrow -3^+} (x^2 - 9)}{\lim_{x \rightarrow -3^+} (x - 3)} = \frac{(-3)^2 - 9}{-3 - 3} = \frac{9 - 9}{-6} = \frac{0}{-6} = 0$$

$$\Rightarrow \lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^+} f(x) = 0 \quad \therefore \lim_{x \rightarrow -3} f(x) = 0$$

(iii) $f(x) = |x - 5|$ $C = 5$

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^-} |x - 5| \quad |x - 5| = \pm(x - 5)$$

$-(x - 5)$	$+(x - 5)$
$-\infty$	$+\infty$
5	

$$= \lim_{x \rightarrow 5^-} [-(x - 5)] = -\lim_{x \rightarrow 5^-} (x - 5) = -(5 - 5) = 0$$

$$\lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+} |x - 5| = \lim_{x \rightarrow 5^+} (x - 5) = 5 - 5 = 0$$

$$\Rightarrow \lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^+} f(x) = 0$$

$$\lim_{x \rightarrow 5} f(x) = 0$$

Question # 2:

Discuss the continuity of $f(x)$ at $x = c$

(i) $f(x) = \begin{cases} 2x + 5 & \text{if } x \leq 2 \\ 4x + 1 & \text{if } x > 2 \end{cases}$ $c = 2$

We have to discuss the continuity of $f(x)$ at $x = 2$

(a) $f(2) = 2(2) + 5 = 4 + 5 = 9$ (1)

(b) $\lim_{x \rightarrow 2} f(x) = ?$

$f(x) = 2x + 5$	$f(x) = 4x + 1$
$-\infty$	$+\infty$
2	

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x + 5) = 2(2) + 5 = 4 + 5 = 9$$

and $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (4x + 1) = 4(2) + 1 = 8 + 1 = 9$

$$\therefore \lim_{x \rightarrow 2} f(x) = 9 \quad \text{.....(2)}$$

(c) from (1) and (2) we get

$$\lim_{x \rightarrow 2} f(x) = f(2)$$

$\therefore f(x)$ is continuous at $x = 2$

$$(ii) \quad f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases} \quad c = 2$$

$$\text{if } c = 2 \quad f(c) = f(2)$$

is not defined so given function is discontinuous

(ii) Correction

$$f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases}$$

$c = 1$ (correction)

$$\begin{array}{ccc} & f(1) = 4 & \\ -\infty & 1 & +\infty \\ f(x) = 3x-1 & & f(x) = 2x \end{array}$$

$$(a) \quad f(1) = 4 \quad (\text{given})$$

$$(b) \quad \lim_{x \rightarrow 1} f(x) = ?$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x-1) = 3(1)-1 = 2$$

$$\text{and } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2x) = 2(1) = 2$$

$$\Rightarrow \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} f(x) = 2$$

$$\therefore \lim_{x \rightarrow 1} f(x) = 2 \quad \dots\dots\dots(2)$$

(c) From (1) and (2) we get

$$\lim_{x \rightarrow 1} f(x) \neq f(1)$$

$\therefore f(x)$ is discontinuous at $x = 1$

$$(iii) \quad f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 2x & \text{if } x > 1 \end{cases} \quad c = 1$$

(a) $f(1)$ is not defined

$\therefore f(x)$ is discontinuous at $x = 1$

Question # 3:

Given that

$$f(x) = \begin{cases} 3x & \text{if } x \leq -2 \\ x^2 - 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$$

$$\begin{array}{ccc} -\infty & & +\infty \\ f(x) = 3x & -2 & f(x) = x^2 - 1 & 2 & f(x) = 3 \end{array}$$

(i) We check continuity at $x = 2$

$$(a) \quad f(2) = 3 \quad \dots\dots\dots(1) \quad (\text{given})$$

$$(b) \quad \lim_{x \rightarrow 2} f(x) = ?$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^2 - 1) = (2)^2 - 1 = 4 - 1 = 3$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (3) = 3$$

$$\Rightarrow \quad \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 3$$

$$\therefore \quad \lim_{x \rightarrow 2} f(x) = 3 \quad \dots\dots\dots(2)$$

(c) From (1) and (2), we get

$$\lim_{x \rightarrow 2} f(x) = f(2)$$

$$\therefore \quad f(x) \text{ is continuous at } x = 2$$

(ii) At $x = -2$

$$(a) \quad f(-2) = 3(-2) = -6 \quad \dots\dots\dots(1)$$

$$(b) \quad \lim_{x \rightarrow -2} f(x) = ?$$

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} (3x) = 3(-2) = -6$$

$$\text{and} \quad \lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} (x^2 - 1) = (-2)^2 - 1 = 4 - 1 = 3$$

$$\Rightarrow \quad \lim_{x \rightarrow -2^-} f(x) \neq \lim_{x \rightarrow -2^+} f(x) \quad \Rightarrow \quad \lim_{x \rightarrow -2} f(x) \text{ does not exist}$$

$$\therefore \quad f(x) \text{ is discontinuous at } x = -2$$

Question # 4:

Given that

$$f(x) = \begin{cases} x+2 & x \leq -1 \\ c+2 & x > -1 \end{cases}$$

$$c = ?$$

$$\begin{array}{ccc} -\infty & & +\infty \\ \hline f(x) = x+2 & -1 & f(x) = c+2 \end{array}$$

$$\therefore \quad \lim_{x \rightarrow -1} f(x) \text{ exists}$$

$$\therefore \quad \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x)$$

$$\Rightarrow \quad \lim_{x \rightarrow -1^-} (x+2) = \lim_{x \rightarrow -1^+} (c+2)$$

$$\Rightarrow \quad -1+2 = c+2$$

$$\Rightarrow \quad 1 = c+2$$

$$\Rightarrow \quad c = 1-2 \quad \Rightarrow \quad c = -1$$

Question # 5:

(i)

$$f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x+9 & \text{if } x > 3 \end{cases}$$

$$\text{here} \quad f(3) = n \quad (\text{given})$$

$$\therefore \quad f(x) \text{ is continuous at } x = 3$$

$$\therefore \quad \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$$

$$\Rightarrow \quad \lim_{x \rightarrow 3^-} (mx) = \lim_{x \rightarrow 3^+} (-2x+9) = n$$

$$\Rightarrow (m)(3) = -2(3) + 9 = n$$

$$\Rightarrow 3m = -6 + 9 = n$$

$$\Rightarrow 3m = 3 = n$$

$$\Rightarrow 3m = 3 \quad , \quad n = 3$$

$$\Rightarrow m = 1 \quad , \quad n = 3$$

$$(ii) \quad f(x) = \begin{cases} mx & \text{if } x < 4 \\ x^2 & \text{if } x \geq 4 \end{cases}$$

$$\text{here } f(4) = (4)^2 = 16$$

$$\therefore f(x) \text{ is continuous at } x = 4$$

$$\therefore \lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^+} f(x) = f(4)$$

$$\Rightarrow \lim_{x \rightarrow 4} (mx) = \lim_{x \rightarrow 4} (x^2) = 16$$

$$\Rightarrow 4m = (4)^2 = 16$$

$$\Rightarrow 4m = 16 = 16 \quad \Rightarrow \quad 4m = 16$$

$$\Rightarrow m = 4$$

Question # 6:

Given that

$$f(x) = \begin{cases} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} & x \neq 2 \\ K & x = 2 \end{cases}$$

$$K = ?$$

$$\text{here } f(2) = K \text{ given}$$

$$\therefore f(x) \text{ is continuous at } x = 2$$

$$\therefore \lim_{x \rightarrow 2} f(x) = f(2)$$

$$\lim_{x \rightarrow 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} = K \quad \Rightarrow \quad \lim_{x \rightarrow 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \cdot \frac{\sqrt{2x+5} + \sqrt{x+7}}{\sqrt{2x+5} + \sqrt{x+7}} = K$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{(\sqrt{2x+5})^2 - (\sqrt{x+7})^2}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = K \quad \Rightarrow \quad \frac{(2x+5) - (x+7)}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = K$$

$$\Rightarrow \frac{2x+5-x-7}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = K \quad \Rightarrow \quad \frac{(x-2)}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = K$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{1}{\sqrt{2x+5} + \sqrt{x+7}} = K \quad \Rightarrow \quad \frac{1}{\lim_{x \rightarrow 2} [\sqrt{2x+5} + \sqrt{x+7}]} = K$$

$$\Rightarrow \frac{1}{\sqrt{2(2)+5} + \sqrt{2+7}} = K \quad \Rightarrow \quad \frac{1}{\sqrt{9} + \sqrt{9}} = K$$

$$\Rightarrow \frac{1}{3+3} = K$$

$$\Rightarrow K = \frac{1}{6}$$

EXERCISE 1.5

Q.1 Draw the graphs of the following equations.

(i) $x^2 + y^2 = 9$

(ii) $\frac{x^2}{16} + \frac{y^2}{4} = 1$

(iii) $y = e^{2x}$

(iv) $y = 3^x$

Solution:

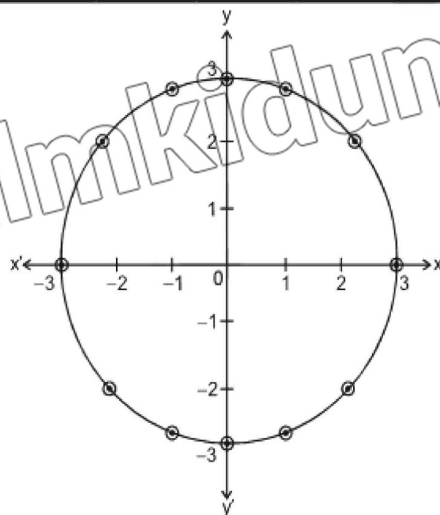
(i) $x^2 + y^2 = 9$

$$y^2 = 9 - x^2$$

$$y = \pm \sqrt{9 - x^2}$$

Its domain is $-3 \leq x \leq 3$

x	-3	-2	-1	0	1	2	3
$y = \pm \sqrt{9 - x^2}$	0	± 2.2	± 2.8	± 3	± 2.8	± 2.2	0



(ii) $\frac{x^2}{16} + \frac{y^2}{4} = 1$

$$\frac{y^2}{4} = 1 - \frac{x^2}{16}$$

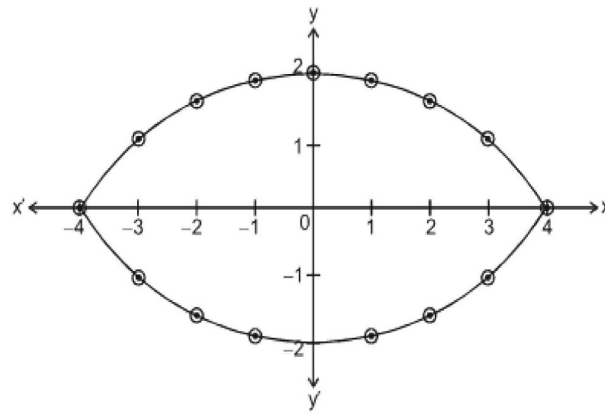
$$y^2 = 4 \left(\frac{16 - x^2}{16} \right)$$

$$y^2 = \frac{16 - x^2}{4}$$

$$y = \pm \frac{\sqrt{16 - x^2}}{2}$$

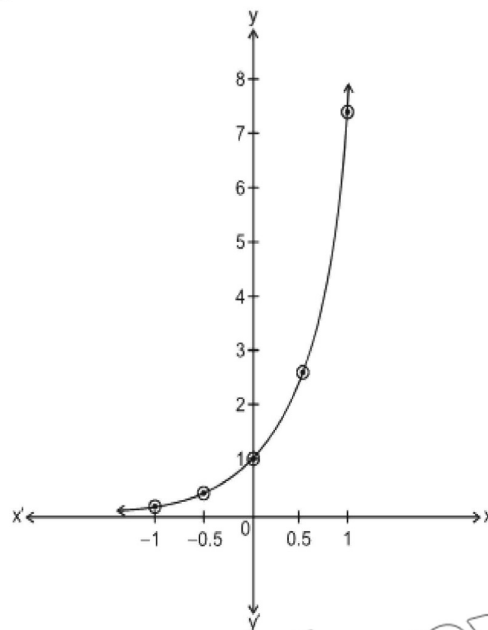
Its domain is $-4 \leq x \leq 4$.

x	-4	-3	-2	-1	0	1	2	3	4
$y = \pm \sqrt{9 - x^2}$	0	± 1.3	± 1.7	± 1.9	± 2	± 1.9	± 1.7	± 1.3	0



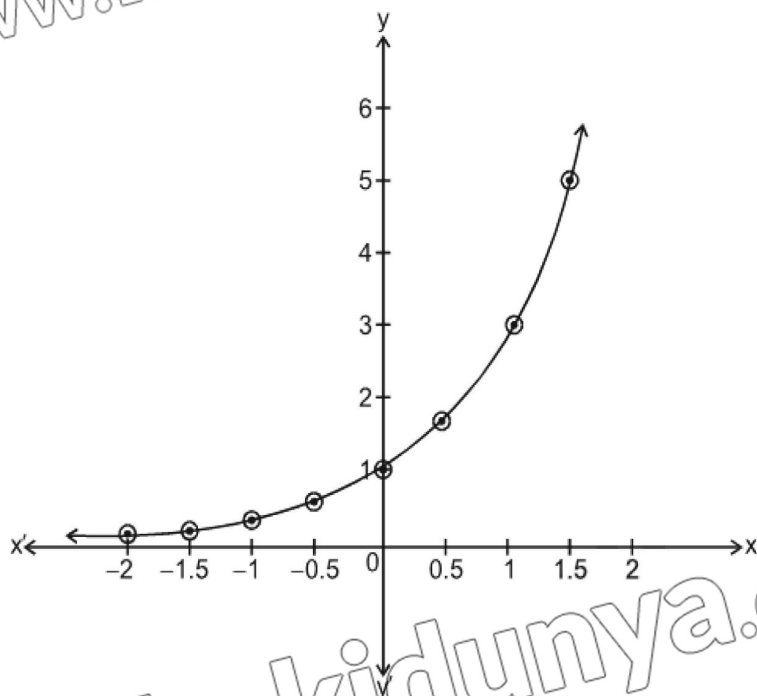
(iii) $y = e^{2x}$

x	-1	-0.5	0	0.5	1
$y = e^{2x}$	0.1	0.4	1	2.7	7.4



(iv) $y = 3^x$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5
$y = 3^x$	0.1	0.2	0.3	0.6	1	1.7	3	5.2



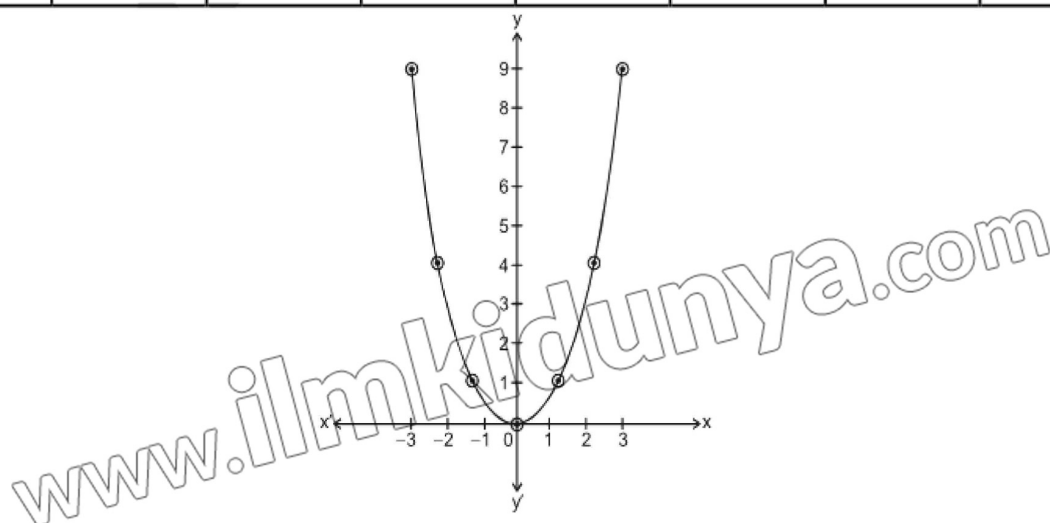
Q.2 Graph the curves that has the parametric equations given below.

- (i) $x = t$, $y = t^2$, $-3 \leq t \leq 3$ where 't' is a parameter
 (ii) $x = t - 1$, $y = 2t - 1$, $-1 < t < 5$ where 't' is a parameter
 (iii) $x = \sec\theta$, $y = \tan\theta$ where ' θ ' is a parameter

Solution:

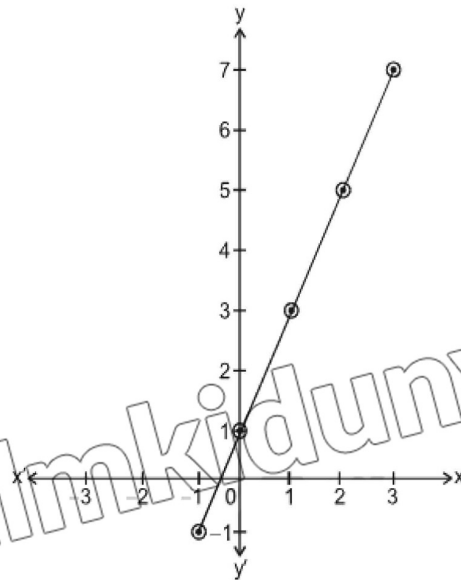
- (i) $x = t$, $y = t^2$, $-3 \leq t \leq 3$ where 't' is a parameter

t	-3	-2	-1	0	1	2	3
$x = t$	-3	-2	-1	0	1	2	3
$y = t^2$	9	4	1	0	1	4	9



(ii) $x = t - 1$, $y = 2t - 1$, $-1 < t < 5$ where 't' is a parameter

t	0	1	2	3	4
$x = t - 1$	-1	0	1	2	3
$y = 2t - 1$	-1	1	3	5	7



(iii) $x = \sec\theta$, $y = \tan\theta$ where 'θ' is a parameter

$$x^2 = \sec^2\theta, \quad y^2 = \tan^2\theta$$

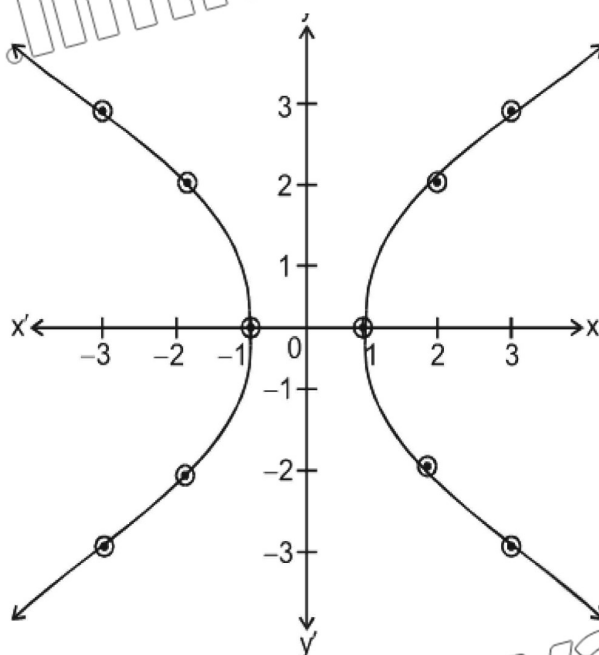
$$x^2 - y^2 = \sec^2\theta - \tan^2\theta$$

$$x^2 - y^2 = 1 \quad (\because 1 + \tan^2\theta = \sec^2\theta \Rightarrow 1 = \sec^2\theta - \tan^2\theta)$$

$$y^2 = x^2 - 1$$

$$y = \pm \sqrt{x^2 - 1}$$

x	-3	-2	-1	1	2	3
$y = \sqrt{x^2 - 1}$	± 2.8	± 1.7	0	0	± 1.7	± 2.8



Q.3 Draw the graphs of the functions defined below and find whether they are continuous.

(i) $y = \begin{cases} x-1 & \text{if } x < 3 \\ 2x+1 & \text{if } x \geq 3 \end{cases}$

(ii) $y = \frac{x^2-4}{x-2}, \quad x \neq 2$

(iii) $y = \begin{cases} x+3 & , \quad x \neq 3 \\ 2 & , \quad x = 3 \end{cases}$

(iv) $y = \frac{x^2-16}{x-4}, \quad x \neq 4$

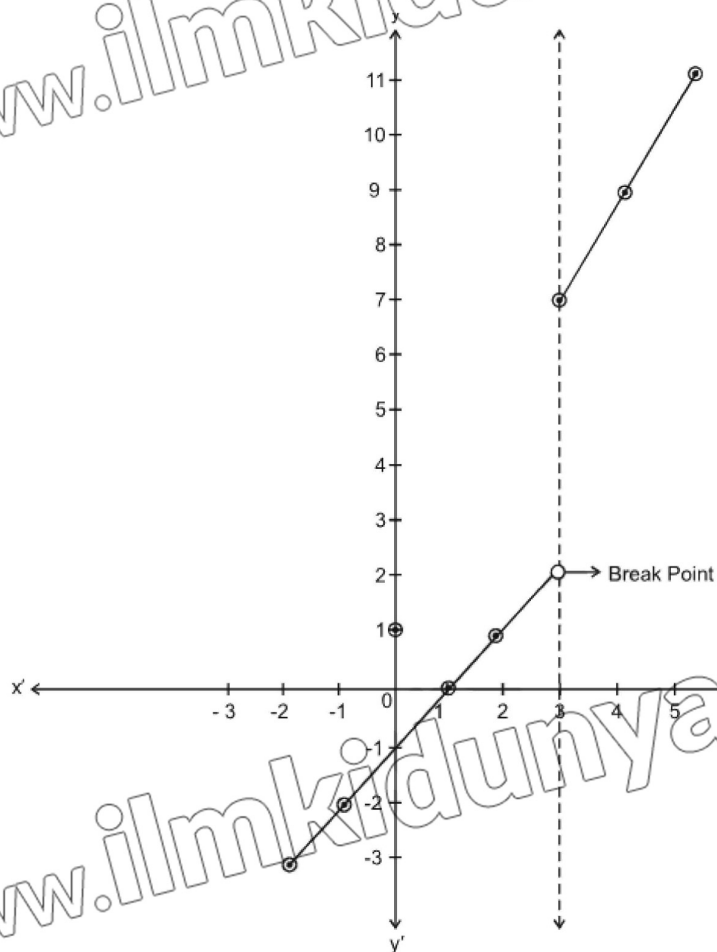
Solution:

(i) $y = \begin{cases} x-1 & \text{if } x < 3 \\ 2x+1 & \text{if } x \geq 3 \end{cases}$
 $y = x-1, \quad x < 3$

x	-2	-1	0	1	2
y = x - 1	-3	-2	-1	0	1

$y = 2x+1, \quad x \geq 3$

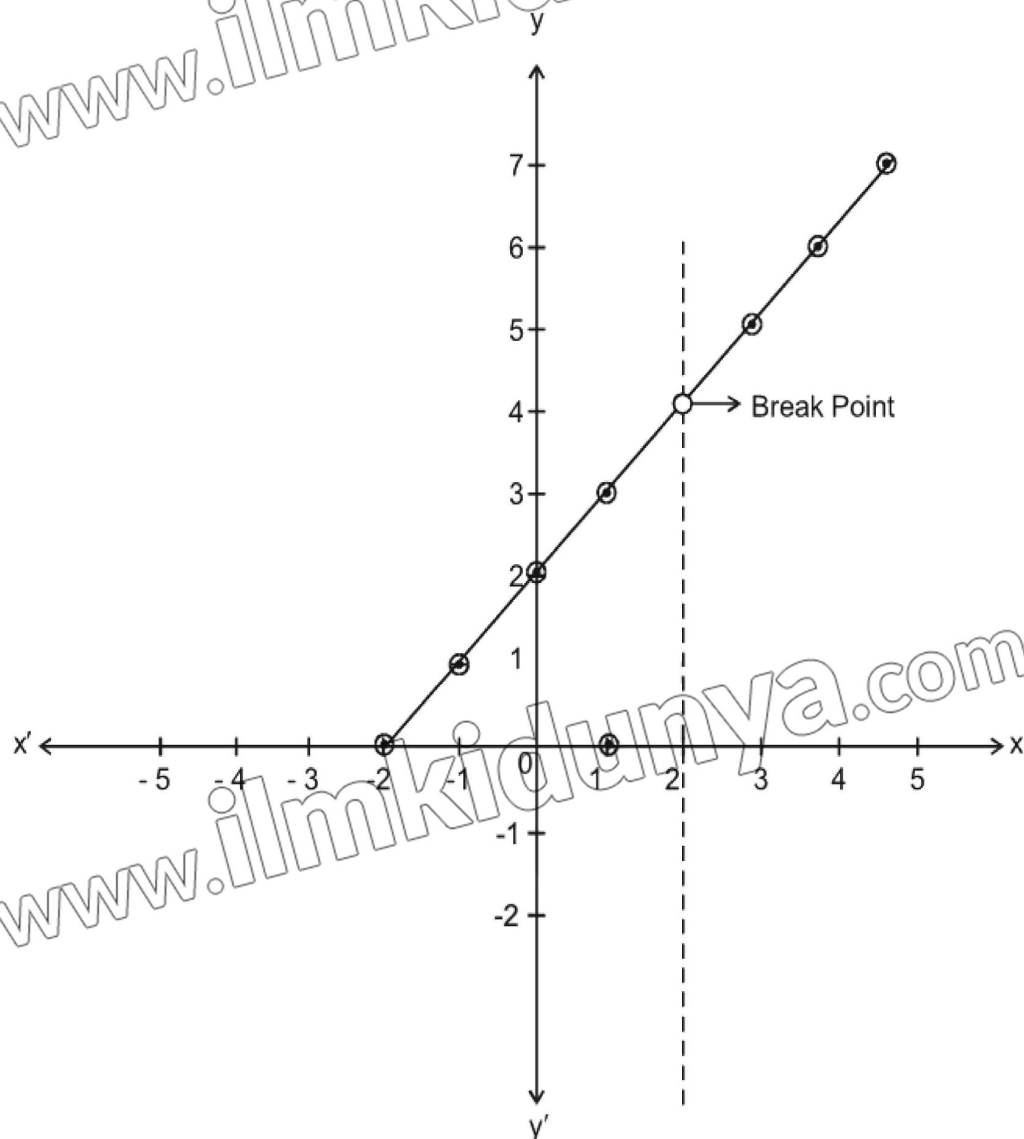
x	3	4	5
y = 2x + 1	7	9	11



Since there is a break in a graph. So this function is not continuous.

$$\begin{aligned}
 \text{(ii)} \quad y &= \frac{x^2 - 4}{x - 2}, \quad x \neq 2 \\
 &= \frac{(x + 2)(x - 2)}{x - 2}, \quad x \neq 2 \\
 y &= x + 2, \quad x \neq 2
 \end{aligned}$$

x	-3	-2	-1	0	1	3	4	5
y	-1	0	1	2	3	5	6	7



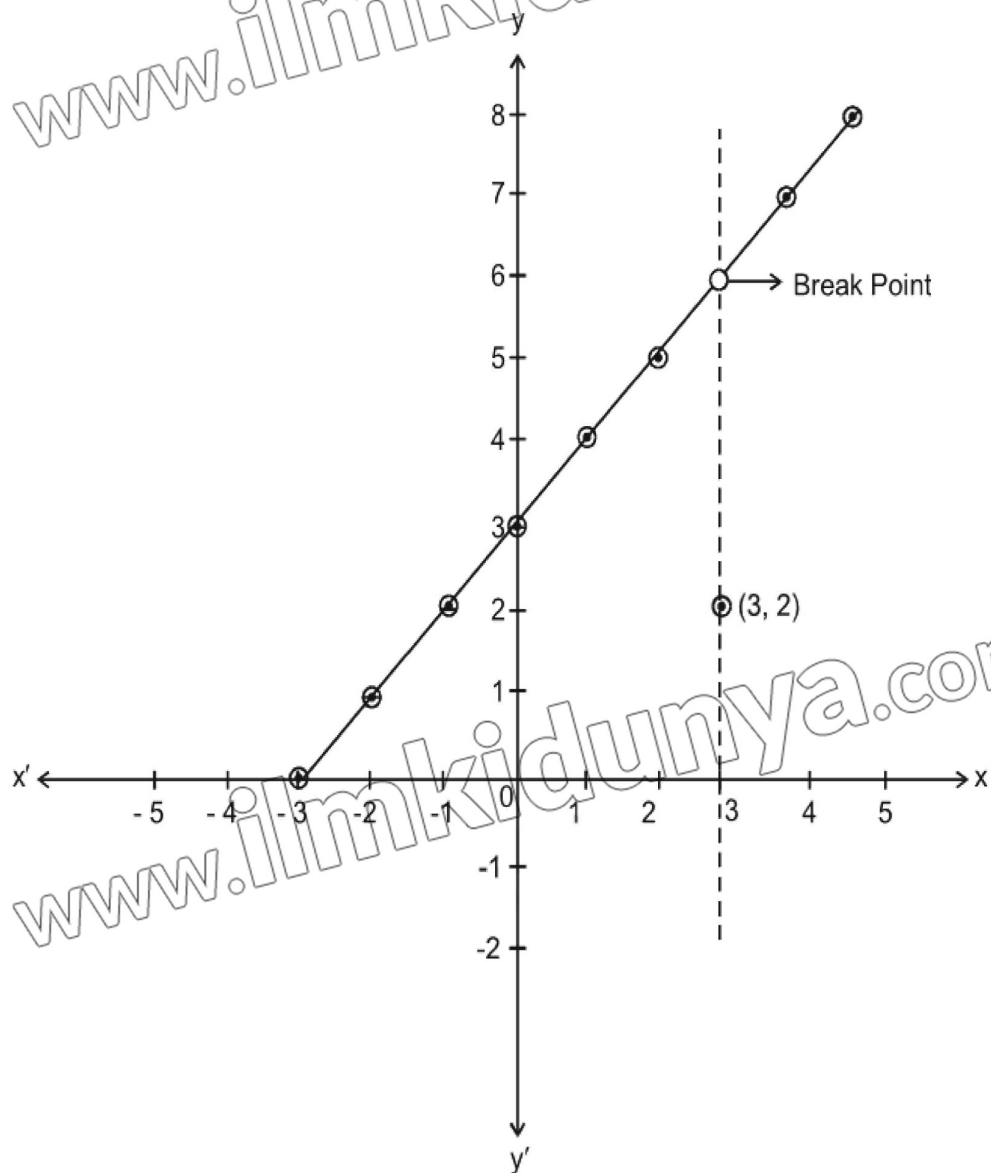
Since there is a break in a graph so this function is not continuous.

(iii) $y = \begin{cases} x + 3 & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$

$y = x + 3$ if $x \neq 3$

x	-3	-2	-1	0	1	3	4	5
y	0	1	2	3	4	5	7	8

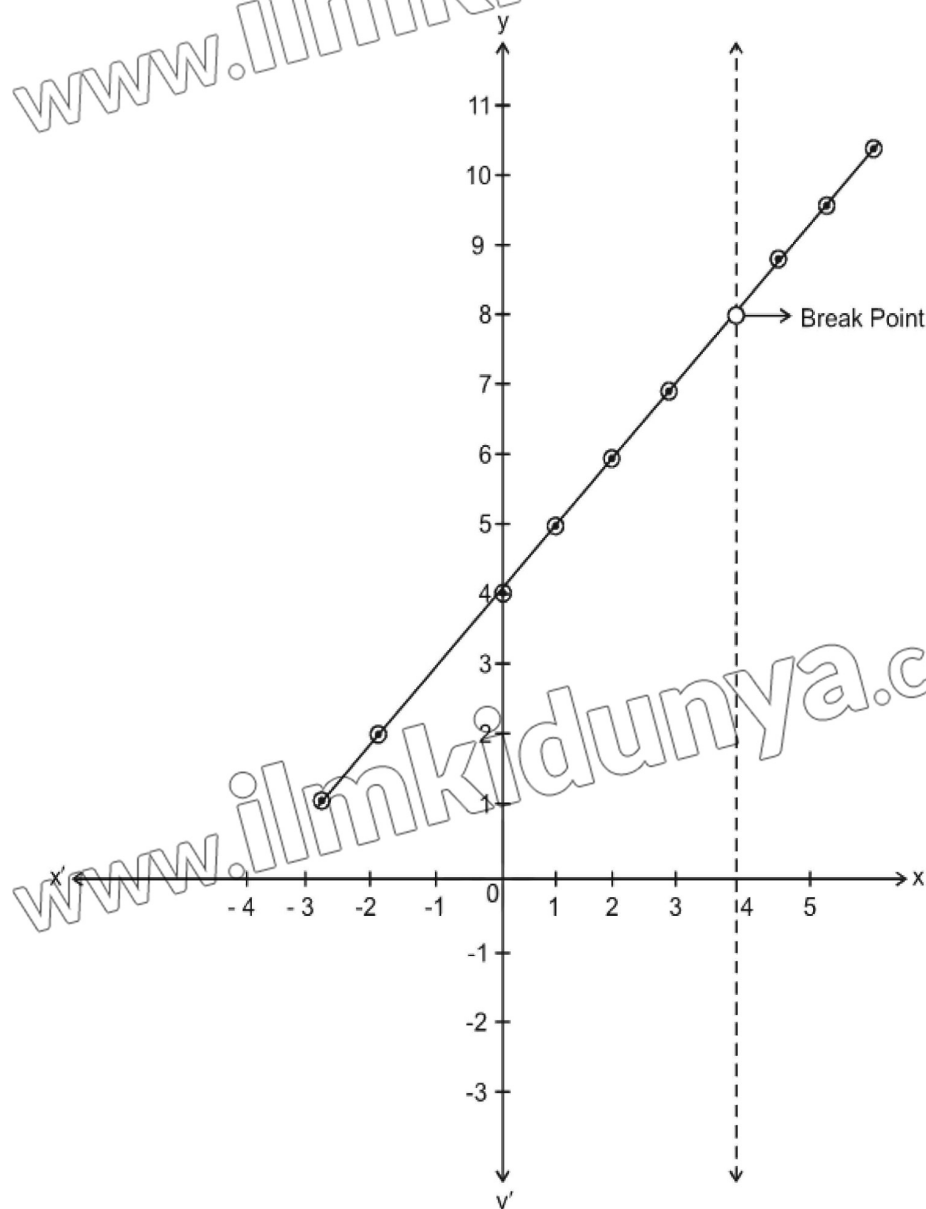
$y = 2$ if $x = 3$



Since there is a break in a graph. So this function is not continuous at $x = 3$.

$$\begin{aligned} \text{(iv)} \quad y &= \frac{x^2 - 16}{x - 4}, \quad x \neq 4 \\ &= \frac{(x + 4)(x - 4)}{x - 4}, \quad x \neq 4 \end{aligned}$$

x	-3	-2	-1	0	1	2	3	5	6
y	1	2	3	4	5	6	7	9	10



Since there is a break in a graph. So this function is not continuous at $x = 4$.

Q.4 Find the graphical solution of the following equations.

(i) $x = \sin 2x$

(ii) $\frac{x}{2} = \cos x$

(iii) $2x = \tan x$

Solution:

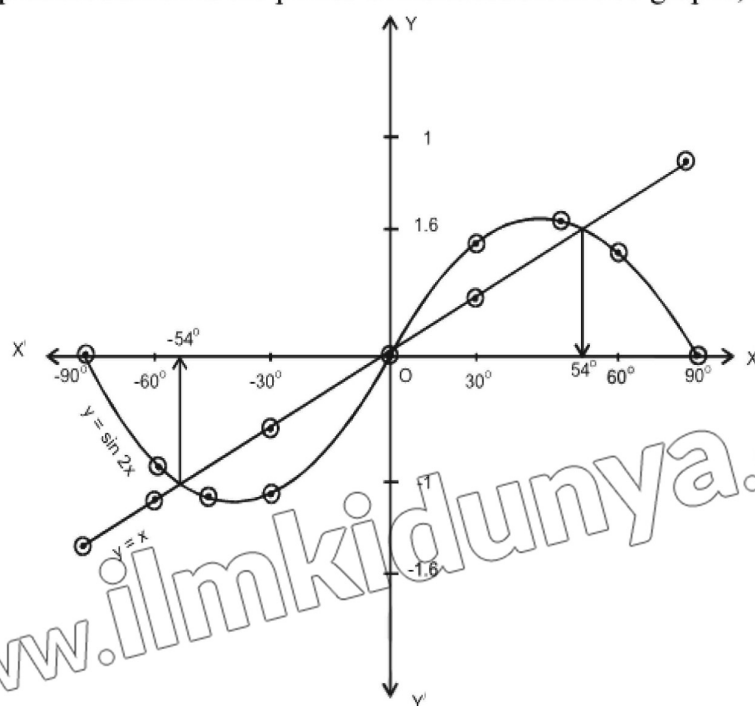
(i) Let $y = x = \sin 2x$

Therefore $y = x$ and $y = \sin 2x$

x	-90°	-60°	-30°	0°	30°	60°	90°
$y = x$	$-\pi/2 = -1.6$	$-\pi/3 = -1.05$	$-\pi/6 = -0.52$	0	$\pi/6 = 0.52$	$\pi/3 = 1.05$	$\pi/2 = 1.6$

x	-90°	-60°	-30°	0°	30°	60°	90°
$y = \sin 2x$	0	-0.87	-0.87	0	0.87	0.87	0

The graphical solution is the points of intersection of two graphs, i.e. $x = 0^\circ, 54^\circ$



(ii) Let $y = \frac{x}{2} = \cos x$

Therefore $y = \frac{x}{2}$ and $y = \cos x$

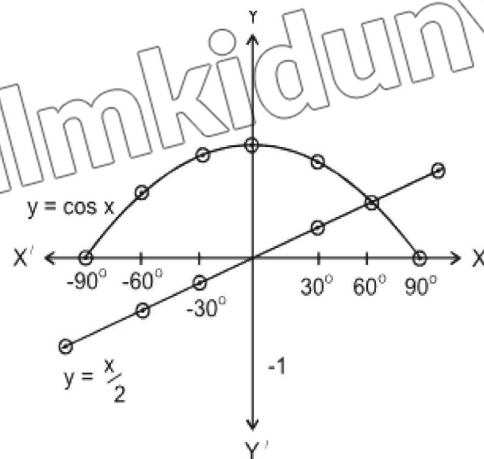
$$y = \frac{x}{2}$$

x	-90°	-60°	-30°	0°	30°	60°	90°
$y = \frac{x}{2}$	$-\pi/4$ $= -0.79$	$-\pi/6$ $= -0.52$	$-\pi/12$ $= -0.26$	0	$\pi/6$ $= 0.26$	$\pi/6$ $= 0.52$	$\pi/4$ $= 0.79$

$$y = \cos x$$

x	-90°	-60°	-30°	0°	30°	60°	90°
$y = \cos x$	0	0.5	0.87	1	0.87	0.5	0

The graphical solution is the point on x-axis, which is just below the point of intersection of two graphs. Hence $x = 60^\circ$.



(iii) Let $y = 2x = \tan x$
 Therefore $y = 2x$ and $y = \tan x$
 $y = 2x$

x	-90°	-60°	-30°	0°	30°	60°	90°
$y = 2x$	$-\pi = -3.14$	$-2\pi/3 = -2.09$	$-\pi/3 = -1.05$	0	$\pi/3 = 1.05$	$2\pi/3 = 2.09$	$\pi = 3.14$

$y = \tan x$

x	-90°	-60°	-30°	0°	30°	60°	90°
$y = \tan x$	∞	-1.73	-0.58	0	0.58	1.73	∞

The graphical solution is the point of intersection of two graphs, i.e. $x = 0^\circ$.

