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$$7x^2 + 8x - 11 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-8 \pm \sqrt{8^2 - 4(7)(-11)}}{2(7)}$$

UNIT 1

QUADRATIC EQUATIONS

Quadratic Equation (U.B)

(LHF 2014, 16, GRV 2014, FSD 2016, 17, MTN 2013, BWP 2015, D.G.K 2016)

"A polynomial equation, which contains the square of the unknown (variable) quantity, but no higher power, is called a quadratic equation, or an equation of degree two is called quadratic equation".

i.e., $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$.

For example: $5x^2 + 2x + 1 = 0$, $x^2 - 1 = 0$ etc.

General or Standard Form of Quadratic Equation (U.B)

Standard form of quadratic equation in one variable x is: $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$. Here a is the coefficient of x^2 , b is coefficient of x and c is constant term.

Pure Quadratic Equation (U.B)

A quadratic equation in which coefficient of 'x' is zero is called pure quadratic equation.

OR

In standard form of quadratic equation $ax^2 + bx + c = 0$, if $b = 0$, then it is called pure quadratic equation. i.e. $ax^2 + c = 0$

For example: $x^2 - 1 = 0$

Linear Equation (U.B)

An equation of degree one is called linear equation, or if $a = 0$ in standard form of quadratic equation $ax^2 + bx + c = 0$ then it reduces to linear equation. i.e. $bx + c = 0$ where $b, c \in R$ and $b \neq 0$. For example $3x + 2 = 0$

Methods to Solve a Quadratic

Equation (K.B)

(LHR 2017, GRW 2016, 17, SWL 2016, SGD 2013, 14, 15, MTN 2015, 17, RWP, 2016, D.G.K 2014, 17)

There are three methods to solve a quadratic equation.

- (i) Factorization method
- (ii) Completing square method
- (iii) Using quadratic formula

Note (K.B)

- For factorization of $ax^2 + bx + c = 0$, we make two factors r and s of ac , such that $r + s = b$ and $rs = ac$.
- Cancelling of x on both sides of an equation (for example $5x^2 = 30x$) means the loss of one root. i.e. $x = 0$
- For convenience, in the method of completing square, we make the coefficient of x^2 equal to 1.

Example 1: (Page # 2) (FSD 2015) (A.B)

Solve the quadratic equation

$3x^2 - 6x = x + 20$ by factorization.

Solution:

$$3x^2 - 6x = x + 20$$

$$3x^2 - 6x - x - 20 = 0$$

$$3x^2 - 7x - 20 = 0$$

$$\therefore -12 + 5 = -7, -12 \times 5 = -60$$

$$3x^2 - 12x + 5x - 20 = 0$$

$$3x(x - 4) + 5(x - 4) = 0$$

$$\Rightarrow (x - 4)(3x + 5) = 0$$

Either $x-4=0$ or $3x+5=0$,

$$\Rightarrow x=4 \text{ or } 3x=-5$$

$$\Rightarrow x=-\frac{5}{3}$$

$x=-\frac{5}{3}, 4$ are the solutions of the given equation

Thus, the solution set is $\left\{-\frac{5}{3}, 4\right\}$

Example 2: (Page # 4)

(A.B)

Solve the equation $2x^2 - 5x - 3 = 0$ by completing square.

Solution:

$$2x^2 - 5x - 3 = 0$$

Dividing each term by 2

$$x^2 - \frac{5}{2}x - \frac{3}{2} = 0$$

Now adding $\left(-\frac{5}{4}\right)^2$ on both sides

$$x^2 - \frac{5}{2}x + \left(-\frac{5}{4}\right)^2 = \frac{3}{2} + \left(-\frac{5}{4}\right)^2$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{3}{2} + \frac{25}{16} = \frac{24+25}{16}$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{49}{16}$$

Taking square root on both sides

$$\sqrt{\left(x - \frac{5}{4}\right)^2} = \pm \sqrt{\frac{49}{16}}$$

$$x - \frac{5}{4} = \pm \frac{7}{4}$$

Either $x - \frac{5}{4} = \frac{7}{4}$ or $x - \frac{5}{4} = -\frac{7}{4}$

$$x = \frac{7}{4} + \frac{5}{4} \text{ or } x = -\frac{7}{4} + \frac{5}{4}$$

$$= \frac{7+5}{4} \text{ or } = \frac{-7+5}{4}$$

$$= \frac{12}{4} \text{ or } = \frac{-2}{4}$$

$$x = 3 \text{ or } x = -\frac{1}{2}$$

$\therefore x = -\frac{1}{2}, 3$ are the roots of the given equation

Thus, the solution set is $\left\{-\frac{1}{2}, 3\right\}$

Exercise 1.1

Q.1 Write the following quadratic equations in the standard form and point out pure quadratic equations.

(SGD 2015, 17, RWP 2016) (A.B)

(i) $(x+7)(x-3) = -7$

Solution:

$$(x+7)(x-3) = -7$$

$$x^2 + 7x - 3x - 21 = -7$$

$$x^2 + 4x - 21 + 7 = 0$$

$$x^2 + 4x - 14 = 0$$

Which is the required standard form of quadratic equation.

(ii) $\frac{x^2+4}{3} - \frac{x}{7} = 1$ (A.B)

Solution:

$$\frac{x^2+4}{3} - \frac{x}{7} = 1$$

$$\frac{x^2+4}{3} - \frac{x}{7} - 1 = 0$$

$$\frac{7(x^2+4) - 3(x) - 1(21)}{21} = 0$$

$$\frac{7x^2 + 28 - 3x - 21}{21} = 0$$

$$7x^2 - 3x + 7 = 0$$

Which is the required standard form of quadratic equation.

(iii) $\frac{x}{x+1} + \frac{x+1}{x} = 6$ (A.B)
(BWP 2014, D.G.K 2014, 15)

Solution:

$$\frac{x}{x+1} + \frac{x+1}{x} = 6$$

$$\frac{x}{x+1} + \frac{x+1}{x} - 6 = 0$$

$$\frac{x(x) + (x+1)^2 - 6(x)(x+1)}{(x+1)(x)} = 0$$

$$\frac{x^2 + (x^2 + 2x + 1) - 6x(x+1)}{(x+1)(x)} = 0$$

$$x^2 + x^2 + 2x + 1 - 6x(x+1) = 0$$

$$2x^2 + 2x + 1 - 6x^2 - 6x = 0$$

$$-4x^2 - 4x + 1 = 0$$

$$-(4x^2 + 4x - 1) = 0$$

$$4x^2 + 4x - 1 = 0$$

Which is the required standard form of quadratic equation.

(iv) $\frac{x+4}{x-2} - \frac{x-2}{x} + 4 = 0$ (A.B)

Solution:

$$\frac{x+4}{x-2} - \frac{x-2}{x} + 4 = 0$$

Multiplying both sides by $x(x-2)$, we get

$$x(x+4) - (x-2)(x-2) + 4x(x-2) = 0$$

$$x^2 + 4x - (x^2 - 4x + 4) + 4x^2 - 8x = 0$$

$$x^2 + 4x - x^2 + 4x - 4 + 4x^2 - 8x = 0$$

$$4x^2 + 8x - 8x - 4 = 0$$

$$4x^2 - 4 = 0$$

$$4(x^2 - 1) = 0$$

$$x^2 - 1 = 0 \quad \because 4 \neq 0$$

is the required standard form and it is a pure quadratic equation.

(v) $\frac{x+3}{x+4} - \frac{x-5}{x} = 1$ (A.B)

Solution:

$$\frac{x+3}{x+4} - \frac{x-5}{x} = 1$$

$$\frac{x(x+3) - (x+4)(x-5)}{(x+4)(x)} = 1$$

$$\frac{x^2 + 3x - (x^2 - 5x + 4x - 20)}{(x+4)(x)} = 1$$

$$\frac{x^2 + 3x - x^2 + 5x - 4x + 20}{(x+4)(x)} = 1$$

$$\frac{4x + 20}{(x+4)(x)} = 1$$

$$4x + 20 = x(x+4)$$

$$4x + 20 - x(x+4) = 0$$

$$4x + 20 - x^2 - 4x = 0$$

$$-x^2 + 20 = 0 \quad \text{Or} \quad x^2 - 20 = 0$$

Is the required standard form of quadratic equation and it is a pure quadratic.

(vi) $\frac{x+1}{x+2} + \frac{x+2}{x+3} = \frac{25}{12}$ (A.B)

Solution:

$$\frac{x+1}{x+2} + \frac{x+2}{x+3} = \frac{25}{12}$$

$$\frac{(x+1)(x+3) + (x+2)^2}{(x+2)(x+3)} = \frac{25}{12}$$

$$\frac{x^2 + x + 3x + 3 + x^2 + 4x + 4}{(x+2)(x+3)} = \frac{25}{12}$$

$$\frac{2x^2 + 8x + 7}{(x+2)(x+3)} = \frac{25}{12}$$

$$\frac{12(2x^2 + 8x + 7) - [25(x+3)(x+2)]}{12(x+2)(x+3)} = 0$$

$$24x^2 + 96x + 84 - [25(x^2 + 5x + 6)] = 0$$

$$24x^2 + 96x + 84 - [25x^2 + 125x + 150] = 0$$

$$24x^2 + 96x + 84 - 25x^2 - 125x - 150 = 0$$

$$-x^2 - 29x - 66 = 0$$

$$\text{Or} \quad x^2 + 29x + 66 = 0$$

Is the required standard form of quadratic equation.

Q.2 Solve by factorization:

(i) $x^2 - x - 20 = 0$ **(A.B)**
(LHR 2014, 15, FSD 2016, SGD 2016, SWL 2016, RWP 2016)

Solution:

$$x^2 - x - 20 = 0$$

$$x^2 - 5x + 4x - 20 = 0$$

$$\therefore -5 + 4 = -1, -5 \times 4 = -20$$

$$x(x-5) + 4(x-5) = 0$$

Either

$$x-5=0 \text{ or } x+4=0$$

$$x=5 \text{ or } x=-4$$

$$\therefore \text{Solution Set} = \{5, -4\}$$

(ii) $3y^2 = y(y-5)$ **(A.B)**
(FSD 2015, BWP 2017)

Solution:

$$3y^2 = y(y-5)$$

$$3y^2 = y^2 - 5y$$

$$3y^2 - y^2 + 5y = 0$$

$$2y^2 + 5y = 0$$

$$y(2y+5) = 0$$

Either

$$y=0 \text{ or } 2y+5=0$$

$$y=0 \text{ or } 2y=-5$$

$$y = -\frac{5}{2}$$

$$\therefore \text{Solution Set} = \left\{0, -\frac{5}{2}\right\}$$

Note

The cancelling of y on both sides of $3y^2 = y(y-5)$ means the loss of 1 root i.e. $y=0$

(iii) $4-32x=17x^2$ **(K.B, A.B)**

Solution:

$$4-32x=17x^2$$

$$17x^2+32x-4=0$$

$$17x^2+34x-2x-4=0$$

$$17x(x+2)-2(x+2)=0$$

$$(x+2)(17x-2)=0$$

$$x+2=0 \text{ or } 17x-2=0$$

$$x=-2 \text{ or } 17x=2$$

$$x = \frac{2}{17}$$

$$\therefore \text{Solution Set} = \left\{-2, \frac{2}{17}\right\}$$

(iv) $x^2 - 11x = 152$ **(A.B)**

Solution:

$$x^2 - 11x = 152$$

$$x^2 - 11x - 152 = 0$$

$$x^2 - 19x + 8x - 152 = 0$$

$$x(x-19) + 8(x-19) = 0$$

$$(x-19)(x+8) = 0$$

Either

$$x-19=0 \text{ or } x+8=0$$

$$x=19 \text{ or } x=-8$$

$$\therefore \text{Solution Set} = \{-8, 19\}$$

(v) $\frac{x+1}{x} + \frac{x}{x+1} = \frac{25}{12}$ **(A.B)**

(GRW 2014, 17, BWP 2016)

Solution:

$$\frac{x+1}{x} + \frac{x}{x+1} = \frac{25}{12}$$

$$\frac{(x+1)^2 + (x)^2}{(x+1)(x)} = \frac{25}{12}$$

$$\frac{x^2 + 2x + 1 + x^2}{(x+1)(x)} = \frac{25}{12}$$

$$\frac{2x^2 + 2x + 1}{(x+1)(x)} = \frac{25}{12}$$

$$12(2x^2 + 2x + 1) = 25(x)(x+1)$$

$$24x^2 + 24x + 12 = 25x(x+1)$$

$$24x^2 + 24x + 12 = 25x^2 + 25x$$

$$24x^2 + 24x + 12 - 25x^2 - 25x = 0$$

$$-x^2 - x + 12 = 0$$

$$x^2 + x - 12 = 0$$

$$x^2 + 4x - 3x - 12 = 0$$

$$x(x+4) - 3(x+4) = 0$$

$$(x+4)(x-3) = 0$$

Either

$$x+4=0 \text{ or } x-3=0$$

$$x=-4 \text{ or } x=3$$

$$\therefore \text{Solution Set} = \{-4, 3\}$$

(vi) $\frac{2}{x-9} = \frac{1}{x-3} - \frac{1}{x-4}$ (A.B)
(LHR 2014, FSD 2017, RWP 2017, BWP 2016, D.G.K 2016)

Solution:

$$\begin{aligned}\frac{2}{x-9} &= \frac{1}{x-3} - \frac{1}{x-4} \\ \frac{2}{x-9} &= \frac{1(x-4) - 1(x-3)}{(x-3)(x-4)} \\ \frac{2}{x-9} &= \frac{x-4-x+3}{x^2-4x-3x+12} \\ \frac{2}{x-9} &= \frac{-1}{x^2-7x+12}\end{aligned}$$

$$2(x^2-7x+12) = -1(x-9)$$

$$2x^2-14x+24 = -x+9$$

$$2x^2-14x+x+24-9=0$$

$$2x^2-13x+15=0$$

$$2x^2-10x-3x+15=0$$

$$2x(x-5)-3(x-5)=0$$

$$(x-5)(2x-3)=0$$

Either

$$x-5=0 \quad \text{or} \quad 2x-3=0$$

$$x=5 \quad \text{or} \quad 2x=3$$

$$x = \frac{3}{2}$$

$$\therefore \text{Solution Set} = \left\{ \frac{3}{2}, 5 \right\}$$

Q.3 Solve the following equations by completing square

(i) $7x^2 + 2x - 1 = 0$ (A.B)

Solution:

$$7x^2 + 2x - 1 = 0$$

$$7x^2 + 2x = 1$$

Divide by '7' on both sides

$$\frac{7x^2}{7} + \frac{2x}{7} = \frac{1}{7}$$

$$x^2 + \frac{2x}{7} = \frac{1}{7}$$

$$\text{Adding } \left(\frac{1}{7}\right)^2 \text{ on both sides}$$

$$x^2 + \frac{2x}{7} + \left(\frac{1}{7}\right)^2 = \frac{1}{7} + \left(\frac{1}{7}\right)^2$$

$$\left(x + \frac{1}{7}\right)^2 = \frac{1}{7} + \frac{1}{49}$$

$$\left(x + \frac{1}{7}\right)^2 = \frac{1(7)+1}{49}$$

$$\left(x + \frac{1}{7}\right)^2 = \frac{7+1}{49}$$

$$\left(x + \frac{1}{7}\right)^2 = \frac{8}{49}$$

Taking square root on both sides

$$\sqrt{\left(x + \frac{1}{7}\right)^2} = \sqrt{\frac{8}{49}}$$

$$x + \frac{1}{7} = \pm \frac{\sqrt{8}}{7}$$

$$x = -\frac{1}{7} \pm \frac{\sqrt{8}}{7}$$

$$x = \frac{-1 \pm \sqrt{8}}{7}$$

$$\text{Or } x = \frac{-1 \pm 2\sqrt{2}}{7}$$

$$\therefore \text{Solution Set} = \left\{ \frac{-1 \pm 2\sqrt{2}}{7} \right\}$$

(ii) $ax^2 + 4x - a = 0$, where $a \neq 0$ (A.B)

Solution:

$$ax^2 + 4x - a = 0$$

$$ax^2 + 4x = a$$

Divide by 'a'

$$\frac{ax^2}{a} + \frac{4x}{a} = \frac{a}{a}$$

$$x^2 + \frac{4x}{a} = 1$$

$$\text{Adding } \left(\frac{2}{a}\right)^2 \text{ on both sides}$$

$$x^2 + \frac{4x}{a} + \left(\frac{2}{a}\right)^2 = 1 + \left(\frac{2}{a}\right)^2$$

$$\left(x + \frac{2}{a}\right)^2 = 1 + \frac{4}{a^2}$$

$$\left(x + \frac{2}{a}\right)^2 = \frac{a^2 + 4}{a^2}$$

Taking square root on both sides

$$\sqrt{\left(x + \frac{2}{a}\right)^2} = \sqrt{\frac{a^2 + 4}{a^2}}$$

$$x + \frac{2}{a} = \pm \frac{\sqrt{a^2 + 4}}{a}$$

$$x = -\frac{2}{a} \pm \frac{\sqrt{a^2 + 4}}{a}$$

$$x = \frac{-2 \pm \sqrt{a^2 + 4}}{a}$$

$$\therefore \text{Solution Set} = \left\{ \frac{-2 \pm \sqrt{a^2 + 4}}{a} \right\}$$

(iii) $11x^2 - 34x + 3 = 0$

(A.B)

(FSD 2018)

Solution:

$$11x^2 - 34x + 3 = 0$$

$$11x^2 - 34x = -3$$

Divide by '11' on both sides

$$\frac{11x^2}{11} - \frac{34x}{11} = \frac{-3}{11}$$

$$x^2 - \frac{34x}{11} = -\frac{3}{11}$$

Adding $\left(\frac{17}{11}\right)^2$ on both sides

$$x^2 - \frac{34}{11}x + \left(\frac{17}{11}\right)^2 = \frac{-3}{11} + \left(\frac{17}{11}\right)^2$$

$$\left(x - \frac{17}{11}\right)^2 = \frac{-3}{11} + \frac{289}{121}$$

$$\left(x - \frac{17}{11}\right)^2 = \frac{-3(11) + 289}{121}$$

$$\left(x - \frac{17}{11}\right)^2 = \frac{-33 + 289}{121}$$

$$\left(x - \frac{17}{11}\right)^2 = \frac{256}{121}$$

Taking square root on both sides

$$\sqrt{\left(x - \frac{17}{11}\right)^2} = \sqrt{\frac{256}{121}}$$

$$x - \frac{17}{11} = \pm \frac{16}{11}$$

$$x = \frac{17}{11} \pm \frac{16}{11}$$

Either

$$x = \frac{16}{11} + \frac{17}{11} \quad \text{or} \quad x = \frac{-16}{11} + \frac{17}{11}$$

$$x = \frac{16+17}{11} \quad \text{or} \quad x = \frac{-16+17}{11}$$

$$x = \frac{33}{11} \quad \text{or} \quad x = \frac{1}{11}$$

$$x = 3$$

$$\therefore \text{Solution Set} = \left\{ \frac{1}{11}, 3 \right\}$$

(iv) $lx^2 + mx + n = 0$ (FSD 2015) **(A.B)**

Solution:

$$lx^2 + mx + n = 0$$

$$lx^2 + mx = -n$$

Divide by 'l' on both sides

$$\frac{lx^2}{l} + \frac{mx}{l} = \frac{-n}{l}$$

$$x^2 + \frac{mx}{l} = \frac{-n}{l}$$

Adding $\left(\frac{m}{2l}\right)^2$ on both sides

$$x^2 + \frac{m}{l}x + \left(\frac{m}{2l}\right)^2 = \left(\frac{m}{2l}\right)^2 - \frac{n}{l}$$

$$\left(x + \frac{m}{2l}\right)^2 = \frac{m^2}{4l^2} - \frac{n}{l}$$

$$\left(x + \frac{m}{2l}\right)^2 = \frac{m^2 - 4ln}{4l^2}$$

Taking square roots on both sides

$$\sqrt{\left(x + \frac{m}{2l}\right)^2} = \pm \sqrt{\frac{m^2 - 4ln}{4l^2}}$$

$$x + \frac{m}{2l} = \pm \frac{\sqrt{m^2 - 4ln}}{2l}$$

$$x = -\frac{m}{2l} \pm \frac{\sqrt{m^2 - 4ln}}{2l}$$

$$\text{Or } x = -\frac{m}{2l} \pm \frac{\sqrt{m^2 - 4ln}}{2l}$$

$$\therefore \text{Solution Set} = \left\{ \frac{-m \pm \sqrt{m^2 - 4ln}}{2l} \right\}$$

(v) $3x^2 + 7x = 0$ **(A.B)**

Solution:

$$3x^2 + 7x = 0$$

Dividing by 3

$$x^2 + \frac{7}{3}x = 0$$

$$\therefore \frac{1}{2}(\text{Coefficient of } x) = \frac{1}{2} \times \frac{7}{3} = \frac{7}{6}$$

Adding $\left(\frac{7}{6}\right)^2$ on both sides, we get

$$x^2 + \frac{7}{3}x + \left(\frac{7}{6}\right)^2 = 0 + \left(\frac{7}{6}\right)^2$$

$$(x)^2 + 2(x)\left(\frac{7}{6}\right) + \left(\frac{7}{6}\right)^2 = \left(\frac{7}{6}\right)^2$$

$$\left(x + \frac{7}{6}\right)^2 = \frac{49}{36}$$

Taking square roots on both sides

$$\sqrt{\left(x + \frac{7}{6}\right)^2} = \pm \sqrt{\frac{49}{36}}$$

$$x + \frac{7}{6} = \pm \frac{7}{6}$$

Either

$$x = -\frac{7}{6} + \frac{7}{6} \quad \text{or} \quad x = -\frac{7}{6} - \frac{7}{6}$$

$$x = 0$$

$$x = \frac{-7-7}{6}$$

$$x = \frac{-14}{6}$$

$$x = -\frac{7}{3}$$

$$\therefore \text{Solution Set} = \left\{ 0, -\frac{7}{3} \right\}$$

(vi) **Given:**

$$x^2 - 2x - 195 = 0$$

(A.B)

Solution:

$$x^2 - 2x - 195 = 0$$

$$x^2 - 2x = 195$$

Adding $(1)^2$ on both sides

$$x^2 - 2x + (1)^2 = 195 + (1)^2$$

$$(x-1)^2 = 195 + 1$$

$$(x-1)^2 = 196$$

Taking square root on both sides

$$\sqrt{(x-1)^2} = \sqrt{196}$$

$$x-1 = \pm 14$$

Either

$$x-1 = 14 \quad \text{or} \quad x-1 = -14$$

$$x = 14 + 1 \quad x = -14 + 1$$

$$x = 15 \quad x = -13$$

$$\therefore \text{Solution Set} = \{-13, 15\}$$

(vii) $-x^2 + \frac{15}{2} = \frac{7}{2}x$ **(A.E)**

Solution:

$$-x^2 + \frac{15}{2} - \frac{7}{2}x = 0$$

$$x^2 + \frac{7}{2}x - \frac{15}{2} = 0$$

$$x^2 + \frac{7}{2}x = \frac{15}{2}$$

$$\therefore \frac{1}{2}(\text{Coefficient of } x) = \frac{1}{2} \left(\frac{7}{2} \right) = \frac{7}{4}$$

Adding $\left(\frac{7}{4}\right)^2$ on both sides

$$x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = \left(\frac{15}{2}\right) + \left(\frac{7}{4}\right)^2$$

$$(x)^2 + 2(x)\left(\frac{7}{4}\right) + \left(\frac{7}{4}\right)^2 = \frac{15}{2} + \frac{49}{16}$$

$$\left(x + \frac{7}{4}\right)^2 = \frac{120 + 49}{16} \\ = \frac{169}{16}$$

Taking square root on both sides

$$x + \frac{7}{4} = \pm \sqrt{\frac{169}{16}}$$

$$x + \frac{7}{4} = \pm \frac{13}{4}$$

$$x = -\frac{7}{4} \pm \frac{13}{4}$$

Either

$$x = -\frac{7}{4} + \frac{13}{4} \quad \text{or} \quad x = -\frac{7}{4} - \frac{13}{4}$$

$$= \frac{-7+13}{4} \quad \text{or} \quad x = \frac{-7-13}{4}$$

$$= \frac{6}{4} \quad \text{or} \quad = -\frac{20}{4}$$

$$x = \frac{3}{2} \quad \text{or} \quad x = -5$$

$$\therefore \text{Solution Set} = \left\{-5, \frac{3}{2}\right\}$$

(viii) $x^2 + 17x + \frac{33}{4} = 0$

(A.B)

Solution:

$$x^2 + 17x + \frac{33}{4} = 0$$

$$x^2 + 17x = -\frac{33}{4}$$

Adding $\left(\frac{17}{2}\right)^2$ on both sides

$$x^2 + 17x + \left(\frac{17}{2}\right)^2 = \frac{-33}{4} + \left(\frac{17}{2}\right)^2$$

$$\left(x + \frac{17}{2}\right)^2 = \frac{-33}{4} + \frac{289}{4}$$

$$\left(x + \frac{17}{2}\right)^2 = \frac{-33 + 289}{4}$$

$$\left(x + \frac{17}{2}\right)^2 = \frac{256}{4}$$

Taking square root on both sides

$$\sqrt{\left(x + \frac{17}{2}\right)^2} = \sqrt{\frac{256}{4}}$$

$$x + \frac{17}{2} = \pm \frac{16}{2}$$

$$x = \pm \frac{16}{2} - \frac{17}{2}$$

Either

$$x = \frac{16}{2} - \frac{17}{2} \quad \text{or} \quad x = \frac{-16}{2} - \frac{17}{2}$$

$$x = \frac{16-17}{2} \quad \text{or} \quad x = \frac{-16-17}{2}$$

$$x = \frac{-1}{2} \quad \text{or} \quad x = \frac{-33}{2}$$

$$\therefore \text{Solution Set} = \left\{-\frac{1}{2}, -\frac{33}{2}\right\}$$

(ix) $4 - \frac{8}{3x+1} = \frac{3x^2+5}{3x+1}$

(A.B + K.B + U.B)

Solution:

$$4 - \frac{8}{3x+1} = \frac{3x^2+5}{3x+1}$$

$$4 = \frac{3x^2+5}{3x+1} + \frac{8}{3x+1}$$

$$\text{Or } \frac{3x^2+5}{3x+1} + \frac{8}{3x+1} = 4$$

$$\frac{3x^2+5+8}{3x+1} = 4$$

$$\frac{3x^2 + 13}{3x + 1} = 4$$

$$3x^2 + 13 = 4(3x + 1)$$

$$3x^2 + 13 = 12x + 4$$

$$3x^2 - 12x = 4 - 13$$

$$3x^2 - 12x = -9$$

Divide by 3 on both sides

$$x^2 - 4x = -3$$

Adding $(2)^2$ on both side

$$x^2 - 4x + (2)^2 = -3 + (2)^2$$

$$(x - 2)^2 = -3 + 4$$

$$(x - 2)^2 = 1$$

Taking square root on both sides

$$\sqrt{(x - 2)^2} = \sqrt{1}$$

$$x - 2 = \pm 1$$

$$x = \pm 1 + 2$$

Either

$$x = 1 + 2 \quad \text{or} \quad x = -1 + 2$$

$$x = 3 \quad \text{or} \quad x = 1$$

$$\therefore \text{Solution Set} = \{1, 3\}$$

$$(x) \quad 7(x + 2a)^2 + 3a^2 = 5a(7x + 23a)$$

(A.B + K.B + U.B)

Solution:

$$7(x + 2a)^2 + 3a^2 = 5a(7x + 23a)$$

$$7(x^2 + 4ax + 4a^2) + 3a^2 = 35ax + 115a^2$$

$$7x^2 + 28ax + 28a^2 + 3a^2 = 35ax + 115a^2$$

$$7x^2 + 28ax + 31a^2 - 35ax - 115a^2 = 0$$

$$7x^2 - 7ax - 84a^2 = 0$$

$$7x^2 - 84a^2 - 7ax = 0$$

$$7(x^2 - ax - 12a^2) = 0$$

$$x^2 - ax - 12a^2 = 0 \quad \because 7 \neq 0$$

$$x^2 - ax = 12a^2$$

Adding $\left(\frac{a}{2}\right)^2$ on both sides

$$x^2 - ax + \left(\frac{a}{2}\right)^2 = 12a^2 + \left(\frac{a}{2}\right)^2$$

$$\left(x - \frac{a}{2}\right)^2 = 12a^2 + \frac{a^2}{4}$$

$$\left(x - \frac{a}{2}\right)^2 = \frac{4(12a^2) + a^2}{4}$$

$$\left(x - \frac{a}{2}\right)^2 = \frac{48a^2 + a^2}{4}$$

$$\left(x - \frac{a}{2}\right)^2 = \frac{49a^2}{4}$$

Taking square root on both sides

$$\sqrt{\left(x - \frac{a}{2}\right)^2} = \sqrt{\frac{49a^2}{4}}$$

$$x - \frac{a}{2} = \pm \frac{7a}{2}$$

$$x = \pm \frac{7a}{2} + \frac{a}{2}$$

Either

$$x = \frac{7a}{2} + \frac{a}{2} \quad \text{or} \quad x = \frac{-7a}{2} + \frac{a}{2}$$

$$= \frac{7a + a}{2} = \frac{-7a + a}{2}$$

$$= \frac{8a}{2} = \frac{-6a}{2}$$

$$x = 4a \quad \text{or} \quad x = -3a$$

$$\therefore \text{Solution Set} = \{-3a, 4a\}$$

Quadratic Formula

(U.B)

For a standard quadratic equation

$$ax^2 + bx + c = 0 \quad \text{where } a \neq 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use of Quadratic Formula

(U.B)

The quadratic formula is useful tool for solving all those equations which can or cannot be factorized.

Derivation of Quadratic Formula by using Completing Square Method

(K.B + U.B + A.P)

The quadratic equation in standard form is

$$ax^2 + bx + c = 0, \quad a \neq 0$$

Shifting constant term c to the right, we have

$$ax^2 + bx = -c$$

Dividing each term by a

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Adding $\left(\frac{b}{2a}\right)^2$ on both sides

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\text{Or } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking square root on both sides

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\text{Or } x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$ is known as "Quadratic Formula".

Example 1: (Page # 6)**(A.B)**

Solve the quadratic equation $2 + 9x = 5x^2$ by using quadratic formula.

Solution:

$$2 + 9x = 5x^2$$

$$\Rightarrow 5x^2 - 9x - 2 = 0$$

By comparing given equation with standard quadratic

equation $ax^2 + bx + c = 0$, we have

$$a = 5, b = -9, c = -2$$

Quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Putting the values in quadratic formula

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(5)(-2)}}{2(5)}$$

$$x = \frac{9 \pm \sqrt{81 + 40}}{10}$$

$$= \frac{9 \pm \sqrt{121}}{10}$$

$$= \frac{9 \pm 11}{10}$$

Either

$$x = \frac{9+11}{10} \quad \text{or} \quad x = \frac{9-11}{10}$$

$$x = \frac{20}{10} = 2 \quad \text{or} \quad x = \frac{-2}{10} = -\frac{1}{5}$$

$\therefore 2, -\frac{1}{5}$ are the roots of the given equation.

Thus, the solution set is $\left\{-\frac{1}{5}, 2\right\}$

Exercise 1.2

Q.1 Solve the following equations using quadratic formula:

(i) $2 - x^2 = 7x$ **(A.B)**

Solution:

$$2 - x^2 = 7x$$

$$0 = x^2 + 7x - 2$$

$$\Rightarrow x^2 + 7x - 2 = 0$$

By comparing given equation with

$ax^2 + bx + c$, we get

$$a = 1, b = 7, c = -2$$

Putting values of a, b, c in Quadratic

Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{(7)^2 - 4(1)(-2)}}{2 \times 1}$$

$$= \frac{-7 \pm \sqrt{49 + 8}}{2} = \frac{-7 \pm \sqrt{57}}{2}$$

$$\therefore \text{Solution Set} = \left\{ \frac{-7 \pm \sqrt{57}}{2} \right\}$$

(ii) $5x^2 + 8x + 1 = 0$

(A.B)

Solution:

$$5x^2 + 8x + 1 = 0$$

Here $a = 5, b = 8, c = 1$

Putting the values in Quadratic Formula

$$\begin{aligned}
 x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{-8 \pm \sqrt{(8)^2 - 4 \times 5 \times 1}}{2 \times 5} \\
 &= \frac{-8 \pm \sqrt{64 - 20}}{10} = \frac{-8 \pm \sqrt{44}}{10} \\
 &= \frac{-8 \pm \sqrt{4 \times 11}}{10} = \frac{-8 \pm 2\sqrt{11}}{10} \\
 &= \cancel{2} \frac{(-4 \pm \sqrt{11})}{\cancel{10}_5} \\
 &= \frac{-4 \pm \sqrt{11}}{5}
 \end{aligned}$$

$$\therefore \text{Solution Set} = \left\{ \frac{-4 \pm \sqrt{11}}{5} \right\}$$

(iii) $\sqrt{3}x^2 + x = 4\sqrt{3}$

(A.B + U.B + K.B)

Solution:

$$\sqrt{3}x^2 + x = 4\sqrt{3}$$

$$\sqrt{3}x^2 + x - 4\sqrt{3} = 0$$

Here $a = \sqrt{3}, b = 1, c = -4\sqrt{3}$

Putting in Quadratic Formula

$$\begin{aligned}
 x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{-1 \pm \sqrt{(1)^2 - 4 \times \sqrt{3} \times (-4\sqrt{3})}}{2 \times \sqrt{3}} \\
 &= \frac{-1 \pm \sqrt{1 + 16 \times (\sqrt{3})^2}}{2\sqrt{3}} \\
 &= \frac{-1 \pm \sqrt{1 + 48}}{2\sqrt{3}}
 \end{aligned}$$

$$x = \frac{-1 \pm \sqrt{49}}{2\sqrt{3}}$$

$$x = \frac{-1 \pm 7}{2\sqrt{3}}$$

Either

$$x = \frac{-1 + 7}{2\sqrt{3}} \quad \text{or} \quad x = \frac{-1 - 7}{2\sqrt{3}}$$

$$x = \frac{\cancel{6}^3}{\cancel{2}\sqrt{3}} \quad x = \frac{\cancel{8}^4}{\cancel{2}\sqrt{3}}$$

$$x = \frac{3}{\sqrt{3}} \quad x = -\frac{4}{\sqrt{3}}$$

$$x = \frac{\sqrt{3} \times \sqrt{3}}{\sqrt{3}} \quad \text{or} \quad x = -\frac{4}{\sqrt{3}}$$

$$x = \sqrt{3}$$

$$\therefore \text{Solution of Set} = \left\{ \sqrt{3}, -\frac{4}{\sqrt{3}} \right\}$$

(iv) $4x^2 - 14 = 3x$

(A.B)

Solution:

$$4x^2 - 14 = 3x$$

$$4x^2 - 3x - 14 = 0$$

Here $a = 4, b = -3, c = -14$

Putting values of a, b, c in Quadratic Formula

$$\begin{aligned}
 x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 4 \times (-14)}}{2 \times 4}
 \end{aligned}$$

$$x = \frac{3 \pm \sqrt{9 + 224}}{8}$$

$$x = \frac{3 \pm \sqrt{233}}{8}$$

$$\therefore \text{Solution Set} = \left\{ \frac{3 \pm \sqrt{233}}{8} \right\}$$

(v) $6x^2 - 3 - 7x = 0$

(A.B)

Solution:

$$6x^2 - 3 - 7x = 0$$

$$6x^2 - 7x - 3 = 0$$

Here $a = 6$, $b = -7$, $c = -3$ Putting values of a, b, c in Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(6)(-3)}}{2 \times 6}$$

$$x = \frac{7 \pm \sqrt{49 + 72}}{12}$$

$$x = \frac{7 \pm \sqrt{121}}{12}$$

$$x = \frac{7 \pm 11}{12}$$

Either

$$x = \frac{7+11}{12} \quad \text{or} \quad x = \frac{7-11}{12}$$

$$x = \frac{18}{12}, \quad x = \frac{-4}{12}$$

$$x = \frac{3}{2}, \quad x = \frac{-1}{3}$$

$$\therefore \text{Solution Set} = \left\{ \frac{3}{2}, \frac{-1}{3} \right\}$$

(vi) $3x^2 + 8x + 2 = 0$

(A.B)

Solution:

$$3x^2 + 8x + 2 = 0$$

Here

$$a = 3, \quad b = 8, \quad c = 2$$

Putting values of a, b, c in Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$x = \frac{-8 \pm \sqrt{64 - 24}}{6}$$

$$x = \frac{-8 \pm \sqrt{40}}{6}$$

$$x = \frac{-8 \pm 2\sqrt{10}}{6}$$

$$x = \frac{-4 \pm \sqrt{10}}{3}$$

$$x = \frac{-4 \pm \sqrt{10}}{3}$$

$$\therefore \text{Solution Set} = \left\{ \frac{-4 \pm \sqrt{10}}{3} \right\}$$

(vii) $\frac{3}{x-6} - \frac{4}{x-5} = 1$ (A.B + K.B)

Solution:

$$\frac{3}{x-6} - \frac{4}{x-5} = 1$$

Multiplying both sides by

$$(x-6)(x-5)$$

$$3(x-5) - 4(x-6) = 1(x-6)(x-5)$$

$$3x - 15 - 4x + 24 = x^2 - 6x - 5x + 30$$

$$-x + 9 = x^2 - 11x + 30$$

$$0 = x^2 - 11x + 30 + x - 9$$

$$0 = x^2 - 10x + 21$$

$$x^2 - 10x + 21 = 0$$

Here

$$a = 1, \quad b = -10, \quad c = 21$$

Putting values of a, b, c in Quadratic

Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4 \times 1 \times 21}}{2 \times 1}$$

$$x = \frac{10 \pm \sqrt{100 - 84}}{2}$$

$$x = \frac{10 \pm \sqrt{16}}{2}$$

$$x = \frac{10 \pm 4}{2}$$

Either

$$x = \frac{10+4}{2} \quad \text{or} \quad x = \frac{10-4}{2}$$

$$x = \frac{14}{2}, \quad x = \frac{6}{2}$$

$$x = 7, \quad x = 3$$

$$\therefore \text{Solution Set} = \{3, 7\}$$

(viii) $\frac{x+2}{x-1} - \frac{4-x}{2x} = 2 \frac{1}{3}$ (A.B + K.B)

Solution:

$$\frac{x+2}{x-1} - \frac{4-x}{2x} = 2 \frac{1}{3}$$

$$\frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}$$

Multiplying throughout by $3(2x)(x-1)$

$$3(2x)(x+2) - 3(x-1)(4-x) = 7(2x)(x-1)$$

$$6x(x+2) + 3(x-1)(x-4) = 14x(x-1)$$

$$6x^2 + 12x + 3(x^2 - 5x + 4) = 14x^2 - 14x$$

$$6x^2 + 12x + 3x^2 - 15x + 12 = 14x^2 - 14x$$

$$9x^2 - 3x + 12 = 14x^2 - 14x$$

$$0 = 14x^2 - 14x - 9x^2 + 3x - 12$$

$$0 = 5x^2 - 11x - 12$$

$$5x^2 - 11x - 12 = 0$$

Here

$$a = 5, \quad b = -11, \quad c = -12$$

Putting values of a, b, c in Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4 \times 5 \times (-12)}}{2 \times 5}$$

$$x = \frac{11 \pm \sqrt{121 + 240}}{10}$$

$$x = \frac{11 \pm \sqrt{361}}{10}$$

$$x = \frac{11 \pm 19}{10}$$

Either

$$x = \frac{11+19}{10} \quad \text{or} \quad x = \frac{11-19}{10}$$

$$x = \frac{30}{10}, \quad x = \frac{-8}{10}$$

$$x = 3, \quad x = -\frac{4}{5}$$

$$\therefore \text{Solution Set} = \left\{ -\frac{4}{5}, 3 \right\}$$

(ix) $\frac{a}{x-b} + \frac{b}{x-a} = 2$ (A.B + K.B)

(FSD 2016)

Solution:

$$\frac{a}{x-b} + \frac{b}{x-a} = 2$$

Multiplying throughout by $(x-a)(x-b)$

$$a(x-a) + b(x-b) = 2(x-a)(x-b)$$

$$ax - a^2 + bx - b^2 = 2(x^2 - ax - bx + ab)$$

$$ax + bx - a^2 - b^2 = 2x^2 - 2ax - 2bx + 2ab$$

$$0 = 2x^2 - 2ax - 2bx + 2ab - ax - bx + a^2 + b^2$$

$$0 = 2x^2 - 3ax - 3bx + a^2 + b^2 + 2ab$$

$$2x^2 - 3(a+b)x + (a+b)^2 = 0$$

Here

$$A = 2, \quad B = -3(a+b), \quad C = (a+b)^2$$

Putting values of A, B, C in Quadratic Formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$x = \frac{-[-3(a+b)] \pm \sqrt{[-3(a+b)]^2 - 4 \times 2(a+b)^2}}{2 \times 2}$$

$$x = \frac{3(a+b) \pm \sqrt{9(a+b)^2 - 8(a+b)^2}}{4}$$

$$x = \frac{3(a+b) \pm \sqrt{(a+b)^2}}{4}$$

$$x = \frac{3(a+b) \pm (a+b)}{4}$$

Either

$$x = \frac{3(a+b) + (a+b)}{4} \quad \text{or} \quad x = \frac{3(a+b) - (a+b)}{4}$$

$$x = \frac{A(a+b)}{A}, \quad x = \frac{Z(a+b)}{2A}$$

$$x = a+b, \quad x = \frac{a+b}{2}$$

$$\therefore \text{Solution Set} = \left\{ a+b, \frac{a+b}{2} \right\}$$

$$(x) \quad -(l+m) - lx^2 + (2l+m)x = 0$$

(A.B + K.B + U.B)

Solution:

$$-(l+m) - lx^2 + (2l+m)x = 0$$

$$-lx^2 + (2l+m)x - (l+m) = 0$$

$$lx^2 - (2l+m)x + (l+m) = 0$$

$$\text{Here } a=l, \quad b=-(2l+m), \quad c=l+m$$

Putting the values of a, b, c in
Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-[-(2l+m)] \pm \sqrt{[-(2l+m)]^2 - 4 \times l \times (l+m)}}{2 \times l}$$

$$x = \frac{(2l+m) \pm \sqrt{(2l+m)^2 - 4l(l+m)}}{2l}$$

$$x = \frac{(2l+m) \pm \sqrt{4l^2 + m^2 + 4lm - 4lm}}{2l}$$

$$x = \frac{(2l+m) \pm \sqrt{m^2}}{2l}$$

$$x = \frac{2l+m \pm m}{2l}$$

$$x = \frac{2l+m+m}{2l}, \quad x = \frac{2l+m-m}{2l}$$

$$x = \frac{2l+2m}{2l}, \quad x = \frac{2l}{2l}$$

$$x = \frac{Z(l+m)}{Zl}, \quad x = 1$$

$$x = \frac{l+m}{l}$$

$$\therefore \text{Solution Set} = \left\{ 1, \frac{l+m}{l} \right\}$$

Reciprocal Equation (U.B)

(LHR 2015, 16, 17, FSD 2017, MTN 2015, 17, RWP 2017, BWP 2017, D.G.K 2013, 17)

An equation, which remains unchanged when 'x' is replaced by $\frac{1}{x}$ is called reciprocal equation.

$$\text{e.g. } a \left(x^2 + \frac{1}{x^2} \right) + b \left(x + \frac{1}{x} \right) + c = 0$$

$$ax^4 + bx^3 + cx^2 + bx + a = 0$$

Exponential Equation (U.B)

(LHR 2015, GRW 2017 SWL 2017, SGD 2017, D.G.K 2015, MTN 2016)

An equation, in which a variable or an algebraic expression occurs in exponent is called exponential equation.

$$\text{e.g. } 2^x + 64.2^{-x} - 20 = 0$$

Or

$$3^{2x+2} = 12.3^x - 3 \quad \text{etc.}$$

Equations Reducible to Quadratic Form (K.B + A.B + U.B)

Type (i)

The Equations of the type

$$ax^4 + bx^2 + c = 0$$

Example 1: (Page # 8)

Solve the equation $x^4 - 13x^2 + 36 = 0$

Solution:

$$x^4 - 13x^2 + 36 = 0$$

$$(x^2)^2 - 13x^2 + 36 = 0 \rightarrow (i)$$

$$\text{Let } x^2 = y \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow$$

$$y^2 - 13y + 36 = 0$$

$$y^2 - 9y - 4y + 36 = 0$$

$$y(y-9) - 4(y-9) = 0$$

$$(y-9)(y-4) = 0$$

Either

$$y-9=0 \quad \text{or} \quad y-4=0$$

$$\Rightarrow y=9 \quad \text{or} \quad y=4$$

Putting the value of y in equation (ii)

$$\Rightarrow x^2=9 \quad \text{or} \quad x^2=4$$

Taking square root on both sides

$$\Rightarrow x = \pm 3 \quad \text{or} \quad x = \pm 2$$

$$\therefore \text{Solution Set} = \{\pm 3, \pm 2\}$$

Type (ii)

The Equations of the type

$$ap(x) + \frac{b}{p(x)} = c$$

Example 2: (Page # 8)**(K.B + A.B)**Solve the equation $2(2x-1) + \frac{3}{2x-1} = 5$ **Solution:**

$$2(2x-1) + \frac{3}{2x-1} = 5 \rightarrow (i)$$

$$\text{Let } 2x-1 = y \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow$$

$$2y + \frac{3}{y} = 5$$

$$2y^2 + 3 = 5y$$

(by Mul both sides by y)

$$2y^2 - 5y + 3 = 0$$

$$2y^2 - 2y - 3y + 3 = 0$$

$$2y(y-1) - 3(y-1) = 0$$

$$(y-1)(2y-3) = 0$$

Either

$$y-1=0 \quad \text{or} \quad 2y-3=0$$

$$\Rightarrow y=1 \quad \text{or} \quad 2y=3$$

$$\Rightarrow y = \frac{3}{2}$$

Putting the values of y in equation (ii)

$$\Rightarrow 2x-1=1 \quad \text{or} \quad 2x-1=\frac{3}{2}$$

$$2x=1+1 \quad \text{or} \quad 2x=\frac{3}{2}+1$$

$$2x=2 \quad \text{or} \quad 2x=\frac{5}{2}$$

$$x=1 \quad \text{or} \quad x=\frac{5}{4}$$

$$\therefore \text{Solution Set} = \left\{ 1, \frac{5}{4} \right\}$$

Example 3:**(A.B + K.B)**

Reciprocal Equation of the type:

$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0 \quad \text{or}$$

$$ax^4 + bx^3 + cx^2 + bx + a = 0$$

Example 3: (Page # 9)**(A.B)**

Solve the equation

$$2x^4 - 5x^3 - 14x^2 - 5x + 2 = 0$$

Solution:

$$2x^4 - 5x^3 - 14x^2 - 5x + 2 = 0$$

$$2x^4 + 2 - 5x^3 - 5x - 14x^2 = 0$$

Dividing each term by x^2

$$2x^2 + \frac{2}{x^2} - 5x - \frac{5}{x} - 14 = 0$$

$$2\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) - 14 = 0 \rightarrow (i)$$

$$\text{Let } x + \frac{1}{x} = y \rightarrow (ii)$$

Squaring both sides

$$y = -2$$

$$x^2 + \frac{1}{x^2} + 2 = y^2$$

$$x^2 + \frac{1}{x^2} = y^2 - 2$$

Equation (i) $\Rightarrow$

$$2(y^2 - 2) - 5y - 14 = 0$$

$$2y^2 - 4 - 5y - 14 = 0$$

$$2y^2 - 5y - 18 = 0$$

$$2y^2 - 9y + 4y - 18 = 0$$

$$y(2y-9) + 2(2y-9) = 0$$

$$(2y-9)(y+2) = 0$$

Either

$$2y-9=0 \quad \text{or} \quad y+2=0$$

$$\Rightarrow 2y=9 \quad \text{or} \quad y=-2$$

$$\Rightarrow y = \frac{9}{2}$$

Putting the values of y in equation (ii)

$$\text{When } y = \frac{9}{2}$$

$$x + \frac{1}{x} = \frac{9}{2}$$

$$2x^2 + 2 = 9x$$

$$2x^2 - 9x + 2 = 0$$

By using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(2)(2)}}{2(2)}$$

$$x = \frac{9 \pm \sqrt{81-16}}{4}$$

$$x = \frac{9 \pm \sqrt{65}}{4}$$

$$\text{When } y = -2$$

$$x + \frac{1}{x} = -2$$

$$x^2 + 1 = -2x$$

$$x^2 + 2x + 1 = 0$$

$$(x+1)^2 = 0$$

$$\Rightarrow x+1 = 0$$

$$\text{Or } x = -1$$

$$\text{Thus, Solution Set} = \left\{ -1, \frac{9 \pm \sqrt{65}}{4} \right\}$$

Type (iv) (A.B + K.B + U.B)

Exponential Equation:

$$5y + \frac{5}{y} = 26 \quad \text{or} \quad 5^{1+x} + 5^{1-x} = 26$$

Example 4: (Page # 10)

Solve the equation $5^{1+x} + 5^{1-x} = 26$

Solution:

$$5^{1+x} + 5^{1-x} = 26$$

$$5^1 \cdot 5^x + 5^1 \cdot 5^{-x} = 26$$

$$5 \cdot 5^x + \frac{5}{5^x} = 26 \rightarrow (i)$$

$$\text{Let } 5^x = y \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow$$

$$5y + \frac{5}{y} = 26$$

$$5y^2 - 26y + 5 = 0$$

$$5y^2 - 26y + 5 = 0$$

$$5y^2 - 25y - y + 5 = 0$$

$$(y-5)(5y-1) = 0$$

$$(y-5)(5y-1) = 0$$

Either

$$y-5=0 \quad \text{or} \quad 5y-1=0$$

$$y = 5 \quad \text{or} \quad y = \frac{1}{5}$$

Putting the values of y in equation (ii),

$$\Rightarrow 5^x = 5 \quad \text{or} \quad 5^x = \frac{1}{5}$$

$$\Rightarrow 5^x = 5^1 \quad \text{or} \quad 5^x = 5^{-1}$$

$\therefore$ Bases are same

$$x = 1 \quad \text{or} \quad x = -1$$

Thus, Solution Set = $\{\pm 1\}$

Type (v) (A.B + U.B + K.B)

The Equation of the type of

$$(x+a)(x+b)(x+c)(x+d) = k \quad \text{where}$$

$$a+b=c+d$$

Example 5: (Page # 11)

Solve the equation

$$(x-1)(x+2)(x+8)(x+5) = 19$$

Solution:

$$(x-1)(x+2)(x+8)(x+5) = 19$$

$$(x-1)(x+8)(x+2)(x+5) = 19 \quad (\because -1+8=2+5)$$

$$(x^2 + 7x - 8)(x^2 + 7x + 10) - 19 = 0 \rightarrow (i)$$

$$\text{Let } x^2 + 7x = y \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow$$

$$(y-8)(y+10) - 19 = 0$$

$$y^2 + 10y - 8y - 80 - 19 = 0$$

$$y^2 + 2y - 99 = 0$$

$$y^2 + 11y - 9y - 99 = 0$$

$$y(y+11) - 9(y+11) = 0$$

$$(y+11)(y-9) = 0$$

Putting the values of y

$$(x^2 + 7x + 11)(x^2 + 7x - 9) = 0$$

Either

$$x^2 + 7x + 11 = 0 \quad \text{or} \quad x^2 + 7x - 9 = 0$$

$$\text{When } x^2 + 7x + 11 = 0$$

Solving by quadratic formula, we have

$$x = \frac{\sqrt{-7 \pm \sqrt{(-7)^2 - (1)(11)}}}{2(1)}$$

$$= \frac{-7 \pm \sqrt{49 - 44}}{2}$$

$$= \frac{-7 \pm \sqrt{5}}{2}$$

When $x^2 + 7x - 9 = 0$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(1)(-9)}}{2(1)}$$

$$= \frac{-7 \pm \sqrt{49 + 36}}{2}$$

$$= \frac{-7 \pm \sqrt{85}}{2}$$

$$\therefore \text{Solution Set} = \left\{ \frac{-7 \pm \sqrt{5}}{2}, \frac{-7 \pm \sqrt{85}}{2} \right\}$$

Exercise 1.3

Solve the following equations.

Q.1 $2x^4 - 11x^2 + 5 = 0$ **(A.B)**

Solution:

$$2x^4 - 11x^2 + 5 = 0$$

$$2(x^2)^2 - 11x^2 + 5 = 0 \rightarrow (i)$$

Let $x^2 = y$

Putting $x^2 = y$ in equation (i)

$$2y^2 - 11y + 5 = 0$$

$$2y^2 - 10y - y + 5 = 0$$

$$2y(y-5) - 1(y-5) = 0$$

$$(y-5)(2y-1) = 0$$

Either

$$y-5=0 \quad \text{or} \quad 2y-1=0$$

$$y=5 \quad \quad \quad 2y=1$$

$$y = \frac{1}{2}$$

Putting $y = x^2$

$$x^2 = 5 \quad \text{or} \quad x^2 = \frac{1}{2}$$

Taking square root on both sides

$$x = \pm\sqrt{5} \quad \quad \quad x = \pm\frac{1}{\sqrt{2}}$$

$$\therefore \text{Solution Set} = \left\{ \pm\sqrt{5}, \pm\frac{1}{\sqrt{2}} \right\}$$

Q.2 $2x^4 = 9x^2 - 4$ **(A.B)**

Solution:

$$2x^4 = 9x^2 - 4$$

$$2(x^2)^2 - 9x^2 + 4 = 0 \rightarrow (i)$$

Let $x^2 = y$

Putting $x^2 = y$ in equation (i)

$$2y^2 - 9y + 4 = 0$$

$$2y^2 - 8y - y + 4 = 0$$

$$2y(y-4) - 1(y-4) = 0$$

$$(y-4)(2y-1) = 0$$

Either

$$y-4=0 \quad \text{or} \quad 2y-1=0$$

$$y=4 \quad \quad \quad 2y=1$$

$$y = \frac{1}{2}$$

Putting $y = x^2$

$$x^2 = 4 \quad \text{or} \quad x^2 = \frac{1}{2}$$

Taking square root on both sides

$$x = \pm 2 \quad \quad \quad x = \pm \frac{1}{\sqrt{2}}$$

$$\therefore \text{Solution set} = \left\{ \pm 2, \pm \frac{1}{\sqrt{2}} \right\}$$

Q.3 $5x^{\frac{1}{2}} = 7x^{\frac{1}{4}} - 2$ **(FSD 2016)** **(A.B)**

Solution:

$$5x^{\frac{1}{2}} = 7x^{\frac{1}{4}} - 2$$

$$5(x^{\frac{1}{4}})^2 - 7x^{\frac{1}{4}} + 2 = 0 \rightarrow (i)$$

Let $x^{\frac{1}{4}} = y$

Putting $x^{\frac{1}{4}} = y$ in equation (i)

$$5y^2 - 7y + 2 = 0$$

$$5y^2 - 5y - 2y + 2 = 0$$

$$5y(y-1) - 2(y-1) = 0$$

$$(y-1)(5y-2) = 0$$

Either

$$y-1=0 \quad \text{or} \quad 5y-2=0$$

$$y=1 \quad \quad \text{or} \quad 5y=2$$

$$y = \frac{2}{5}$$

Putting $y = x^{1/4}$

$$x^{1/4} = 1 \quad \text{or} \quad x^{1/4} = \frac{2}{5}$$

$$x = (1)^4 \quad \left(x^{1/4} \right)^4 = \left(\frac{2}{5} \right)^4$$

$$x = 1 \quad x = \frac{16}{625}$$

$$\therefore \text{Solution Set} = \left\{ 1, \frac{16}{625} \right\}$$

Q.4 $x^{2/3} + 54 = 15x^{1/3}$ **(A.B)**

Solution:

$$x^{2/3} + 54 = 15x^{1/3}$$

$$\left(x^{1/3} \right)^2 - 15x^{1/3} + 54 = 0 \rightarrow (i)$$

Let $x^{1/3} = y$

Putting $x^{1/3} = y$ in equation (i)

$$y^2 - 15y + 54 = 0$$

$$y^2 - 9y - 6y + 54 = 0$$

$$y(y-9) - 6(y-9) = 0$$

$$(y-9)(y-6) = 0$$

Either

$$y-9=0 \quad \text{or} \quad y-6=0$$

$$y=9 \quad \text{or} \quad y=6$$

Putting $y = x^{1/3}$

$$x^{1/3} = 9 \quad \text{or} \quad x^{1/3} = 6$$

Taking cube on both sides

$$\left(x^{1/3} \right)^3 = (9)^3 \quad \text{or} \quad \left(x^{1/3} \right)^3 = (6)^3$$

$$x = 729 \quad \text{or} \quad x = 216$$

$$\therefore \text{Solution Set} = \{729, 216\}$$

Q.5 $3x^{-2} + 5 = 8x^{-1}$ (SWL 2014) **(A.B)**

Solution:

$$3x^{-2} + 5 = 8x^{-1}$$

$$3(x^{-1})^2 - 8x^{-1} + 5 = 0 \rightarrow (i)$$

Let $x^{-1} = y$

Putting $x^{-1} = y$ in equation (i)

$$3y^2 - 8y + 5 = 0$$

$$3y^2 - 3y - 5y + 5 = 0$$

$$3y(y-1) - 5(y-1) = 0$$

$$(y-1)(3y-5) = 0$$

Either

$$y-1=0 \quad \text{or} \quad 3y-5=0$$

$$y=1 \quad \text{or} \quad 3y=5$$

$$y = \frac{5}{3}$$

Putting $y = x^{-1}$

$$x^{-1} = 1, \quad x^{-1} = \frac{5}{3}$$

$$\frac{1}{x} = 1, \quad \frac{1}{x} = \frac{5}{3}$$

Taking reciprocal on both sides

$$\Rightarrow x=1 \quad x = \frac{3}{5}$$

$$\therefore \text{Solution Set} = \left\{ 1, \frac{3}{5} \right\}$$

Q.6 $(2x^2+1) + \frac{3}{2x^2+1} = 4$ **(A.B)**

Solution:

$$(2x^2+1) + \frac{3}{2x^2+1} = 4 \rightarrow (i)$$

Let $a = 2x^2 + 1$

Putting the value of 'a' in equation (i)

$$a + \frac{3}{a} - 4 = 0 \quad (\text{Mul both sides by } a)$$

$$a^2 + 3 - 4a = 0$$

$$a^2 - 4a + 3 = 0$$

$$a^2 - 3a - a + 3 = 0$$

$$a(a-3) - 1(a-3) = 0$$

$$(a-3)(a-1) = 0$$

Either

$$a-3=0 \quad \text{or} \quad a-1=0$$

$$a=3 \quad a=1$$

$$\text{Putting } a = 2x^2 + 1$$

$$2x^2 + 1 = 3 \quad \text{or} \quad 2x^2 + 1 = 1$$

$$2x^2 = 3 - 1 \quad 2x^2 = 1 - 1$$

$$2x^2 = 2 \quad 2x^2 = 0$$

$$x^2 = 1 \quad x^2 = 0$$

Taking square root on both sides

$$x = \pm 1 \quad x = 0$$

$$\therefore \text{Solution Set} = \{0, \pm 1\}$$

$$\text{Q.7} \quad \frac{x}{x-3} + 4\left(\frac{x-3}{x}\right) = 4 \quad (\text{A.B})$$

Solution:

$$\frac{x}{x-3} + 4\left(\frac{x-3}{x}\right) = 4 \rightarrow (i)$$

$$\text{If } y = \frac{x}{x-3} \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow y + \frac{4}{y} = 4$$

$$y^2 + 4 = 4y$$

$$y^2 - 4y + 4 = 0$$

$$(y^2) - 2(y)(2) + (2)^2 = 0$$

$$(y-2)^2 = 0$$

Taking square root on both sides

$$y - 2 = 0$$

$$y = 2$$

Putting value of y in equation (ii)

$$\frac{x}{x-3} = 2$$

$$x = 2(x-3)$$

$$x = 2x - 6$$

$$6 = 2x - x$$

$$6 = x$$

$$\text{Or } x = 6$$

$$\therefore \text{Solution Set} = \{6\}$$

$$\text{Q.8} \quad \frac{4x+1}{4x-1} + \frac{4x-1}{4x+1} = 2\frac{1}{6} \quad (\text{A.B})$$

Solution:

$$\frac{4x+1}{4x-1} + \frac{4x-1}{4x+1} = 2\frac{1}{6} \rightarrow (i)$$

$$\text{Let } a = \frac{4x+1}{4x-1}$$

Then,

$$\frac{1}{a} = \frac{4x-1}{4x+1}$$

Putting the values in equation (i)

$$a + \frac{1}{a} = \frac{13}{6}$$

$$\frac{a^2 + 1}{a} = \frac{13}{6}$$

$$6a^2 + 6 = 13a$$

$$6a^2 - 13a + 6 = 0$$

$$6a^2 - 9a - 4a + 6 = 0$$

$$3a(2a-3) - 2(2a-3) = 0$$

$$(2a-3)(3a-2) = 0$$

Either

$$2a - 3 = 0 \text{ or } 3a - 2 = 0$$

$$2a = 3 \quad 3a = 2$$

$$a = \frac{3}{2} \quad a = \frac{2}{3}$$

$$\text{Putting } a = \frac{4x+1}{4x-1}$$

$$\frac{4x+1}{4x-1} = \frac{3}{2} \quad \text{or} \quad \frac{4x+1}{4x-1} = \frac{2}{3}$$

$$2(4x+1) = 3(4x-1) \quad \text{or} \quad 3(4x+1) = 2(4x-1)$$

$$8x + 2 = 12x - 3$$

$$12x + 3 = 8x - 2$$

$$12x - 8x = 2 + 3$$

$$12x - 8x = -2 - 3$$

$$4x = 5$$

$$4x = -5$$

$$x = \frac{5}{4}$$

$$x = \frac{-5}{4}$$

$$\therefore \text{Solution Set} = \left\{ \pm \frac{5}{4} \right\}$$

Q.9 $\frac{x-a}{x+a} - \frac{x+a}{x-a} = \frac{7}{12}$

(A.B)

Solution:

$$\frac{x-a}{x+a} - \frac{x+a}{x-a} = \frac{7}{12} \rightarrow (i)$$

$$\text{Let } \frac{x-a}{x+a} = b$$

$$\frac{1}{b} = \frac{x+a}{x-a}$$

Putting the values in equation (i)

$$b - \frac{1}{b} = \frac{7}{12}$$

$$b - \frac{1}{b} - \frac{7}{12} = 0$$

Multiplying by 12b

$$12b^2 - 12 - 7b = 0$$

$$12b^2 - 7b - 12 = 0$$

$$12b^2 - 16b + 9b - 12 = 0$$

$$4b(3b-4) + 3(3b-4) = 0$$

$$(3b-4)(4b+3) = 0$$

Either

$$3b-4=0 \text{ or } 4b+3=0$$

$$3b=4 \quad 4b=-3$$

$$b = \frac{4}{3} \quad b = -\frac{3}{4}$$

$$\text{Put } b = \frac{x-a}{x+a}$$

$$\frac{x-a}{x+a} = -\frac{3}{4} \quad \frac{x-a}{x+a} = +\frac{4}{3}$$

$$4(x-a) = -3(x+a) \quad 3(x-a) = 4(x+a)$$

$$4x-4a = -3x-3a \quad 3x-3a = 4x+4a$$

$$4x+3x = -3a+4a \quad 3x-4x = 4a+3a$$

$$7x = a - x \quad 7a$$

$$x = \frac{a}{8} \quad x = -7a$$

$$\therefore \text{Solution Set} = \left\{ -7a, \frac{a}{8} \right\}$$

Q.10 $x^4 - 2x^3 - 2x^2 + 2x + 1 = 0$

(A.B + K.B)

Solution:

$$x^4 - 2x^3 - 2x^2 + 2x + 1 = 0$$

Divide by x^2

$$\frac{x^4}{x^2} - \frac{2x^3}{x^2} - \frac{2x^2}{x^2} + \frac{2x}{x^2} + \frac{1}{x^2} = 0$$

$$x^2 - 2x - 2 + \frac{2}{x} + \frac{1}{x^2} = 0$$

$$x^2 + \frac{1}{x^2} - 2x + \frac{2}{x} - 2 = 0$$

$$\left(x^2 + \frac{1}{x^2}\right) - 2\left(x - \frac{1}{x}\right) - 2 = 0 \rightarrow (i)$$

$$\text{Let } x - \frac{1}{x} = a \rightarrow (ii)$$

Taking square on both sides

$$\left(x - \frac{1}{x}\right)^2 = (a)^2$$

$$x^2 + \frac{1}{x^2} - 2 = a^2$$

$$x^2 + \frac{1}{x^2} = a^2 + 2$$

By putting values in equation (i)

$$(a^2 + 2) - 2a - 2 = 0$$

$$a^2 + 2 - 2a - 2 = 0$$

$$a^2 - 2a = 0$$

$$a(a-2) = 0$$

Either

$$a = 0 \text{ or } a - 2 = 0$$

$$a = 2$$

putting the value of a in equation (ii)when $a = 0$ when $a = 2$

$$x - \frac{1}{x} = 0 \rightarrow (iii) \quad x - \frac{1}{x} = 2 \rightarrow (iv)$$

$$\text{Equation (iii)} \Rightarrow x = \frac{1}{x}$$

$$x^2 = 1$$

Taking square root on both sides

$$x = \pm 1$$

$$\text{Equation (iv)} \Rightarrow$$

$$x^2 - 1 = 2x$$

$$x^2 - 2x - 1 = 0$$

Here

$$a = 1, b = -2, c = -1$$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4+4}}{2}$$

$$= \frac{2 \pm \sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

$$= \frac{2(1 \pm \sqrt{2})}{2}$$

$$= 1 \pm \sqrt{2}$$

$$\therefore \text{Solution Set} = \{1 \pm \sqrt{2}\}$$

Q.11 $2x^4 + x^3 - 6x^2 + x + 2 = 0$

(A.B + K.B)

Solution:

$$2x^4 + x^3 - 6x^2 + x + 2 = 0 \rightarrow (i)$$

Dividing by x^2 , we get

$$2x^2 + x - 6 + \frac{1}{x} + \frac{2}{x^2} = 0$$

$$2\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 6 = 0$$

$$\text{Let } x + \frac{1}{x} = y \rightarrow (ii)$$

Squaring both sides

$$\left(x + \frac{1}{x}\right)^2 = y^2$$

$$x^2 + \frac{1}{x^2} + 2 = y^2$$

$$x^2 + \frac{1}{x^2} = y^2 - 2$$

Putting the values in equation (i)

$$2(y^2 - 2) + y - 6 = 0$$

$$2y^2 - 4 + y - 6 = 0$$

$$2y^2 + y - 10 = 0$$

$$2y^2 + 5y - 4y - 10 = 0$$

$$y(2y + 5) - 2(2y + 5) = 0$$

$$(2y + 5)(y - 2) = 0$$

Either

$$2y + 5 = 0 \text{ or } y - 2 = 0$$

$$2y = -5, \quad y = 2$$

$$y = \frac{-5}{2}$$

Putting the values of y in equation (ii)

$$\text{When } y = \frac{-5}{2}$$

$$x + \frac{1}{x} = y$$

$$x + \frac{1}{x} = \frac{-5}{2}$$

$$\frac{x^2 + 1}{x} = \frac{-5}{2}$$

$$2x^2 + 2 = -5x$$

$$2x^2 + 5x + 2 = 0$$

$$2x^2 + 4x + x + 2 = 0$$

$$2x(x + 2) + 1(x + 2) = 0$$

$$(x + 2)(2x + 1) = 0$$

Either

$$x + 2 = 0 \text{ or } 2x + 1 = 0$$

$$x = -2 \quad 2x = -1$$

$$x = \frac{-1}{2}$$

$$\text{When } y = 2$$

$$x + \frac{1}{x} = y$$

$$x + \frac{1}{x} = 2$$

$$\frac{x^2 + 1}{x} = 2$$

$$x^2 + 1 = 2x$$

$$x^2 - 2x + 1 = 0$$

$$(x - 1)^2 = 0$$

Taking square root on both sides

$$x-1=0$$

$$x=1$$

$$\therefore \text{Solution Set} = \left\{1, -2, -\frac{1}{2}\right\}$$

Q.12 $4.2^{2x+1} - 9.2^x + 1 = 0$ (A.B + K.B)

Solution:

$$4.2^{2x+1} - 9.2^x + 1 = 0$$

$$4.2^{2x}.2 - 9.2^x + 1 = 0$$

$$8.(2^x)^2 - 9.2^x + 1 = 0 \rightarrow (i)$$

$$\text{Let } 2^x = y \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow$$

$$8y^2 - 9y + 1 = 0$$

$$8y^2 - 8y - y + 1 = 0$$

$$8y(y-1) - 1(y-1) = 0$$

$$(y-1)(8y-1) = 0$$

Either

$$y-1=0 \quad \text{or} \quad 8y-1=0$$

$$y=1 \quad \quad \quad 8y=1$$

$$y = \frac{1}{8}$$

Putting the value of y in equation (ii)

$$\text{when } y=1, \text{ when } y = \frac{1}{8}$$

$$2^x = y, \quad 2^x = y$$

$$2^x = 1, \quad 2^x = \frac{1}{8}$$

$$2^x = 2^0, \quad 2^x = \frac{1}{2^3}$$

$$2^x = 2^{-3}$$

$\therefore$ Bases are same

$$\Rightarrow x=0, \quad x=-3$$

$$\therefore \text{Solution Set} = \{0, -3\}$$

Q.13 $3^{2x+2} = 12.3^x - 3$ (A.B + K.B)

Solution:

$$3^{2x+2} = 12.3^x - 3$$

$$3^{2x+2} - 12.3^x + 3 = 0$$

$$3^{2x}.3^2 - 12.3^x + 3 = 0$$

$$9(3^x)^2 - 12.3^x + 3 = 0 \rightarrow (i)$$

$$\text{Let } 3^x = y \rightarrow (ii)$$

$$\text{Putting } 3^x = y \text{ in equation (i)}$$

$$9y^2 - 12y + 3 = 0$$

$$3y^2 - 4y - 3y + 3 = 0$$

$$9y(y-1) - 3(y-1) = 0$$

$$(y-1)(9y-3) = 0$$

Either

$$y-1=0 \quad \text{or} \quad 9y-3=0$$

$$y=1 \quad \quad \quad 9y=3$$

$$y = \frac{3}{9}$$

$$y = \frac{1}{3}$$

Putting the value of y in equation (ii)

$$\text{when } y=1 \quad \text{when } y = \frac{1}{3}$$

$$3^x = y \quad \quad \quad 3^x = y$$

$$3^x = 1 \quad \quad \quad 3^x = \frac{1}{3}$$

$$3^x = 3^0 \quad \quad \quad 3^x = \frac{1}{3^1}$$

$$3^x = 3^{-1}$$

$\therefore$ Bases are same

$$x=0 \quad \quad \quad x=-1$$

$$\therefore \text{Solution Set} = \{0, -1\}$$

Q.14 $2^x + 64 \times 2^{-x} - 20 = 0$ (A.B + K.B)

Solution:

$$2^x + 64 \times 2^{-x} - 20 = 0$$

$$2^x + 64 \times 2^{-x} - 20 = 0$$

$$2^x + \frac{64}{2^x} - 20 = 0 \rightarrow (i)$$

$$\text{Let } 2^x = y \rightarrow (ii)$$

$$\text{Put } 2^x = y \text{ in equation (i)}$$

$$y + \frac{64}{y} - 20 = 0$$

Multiplying throughout by y

$$y^2 + 64 - 20y = 0$$

$$y^2 - 20y + 64 = 0$$

$$y^2 - 4y - 16y + 64 = 0$$

$$y(y-4) - 16(y-4) = 0$$

$$(y-4)(y-16) = 0$$

Either

$$y-4=0 \quad \text{or} \quad y-16=0$$

$$y=4 \quad \text{or} \quad y=16$$

Putting the value of y in equation (ii)

$$\text{when } y=4 \quad \text{when } y=16$$

$$2^x = y \quad 2^x = y$$

$$2^x = 4 \quad 2^x = 16$$

$$2^x = 2^2 \quad 2^x = 2^4$$

$$x=2 \quad x=4$$

$$\therefore \text{Solution Set} = \{2, 4\}$$

Q.15 $(x+1)(x+3)(x-5)(x-7) = 192$

(A.B + K.B + U.B)

Solution:

$$(x+1)(x+3)(x-5)(x-7) = 192$$

$$\therefore (1-5 = -4 \quad 3-7 = -4)$$

$$(x+1)(x-5)(x+3)(x-7) = 192$$

$$(x^2 - 4x - 5)(x^2 - 4x - 21) = 192 \rightarrow (i)$$

$$\text{Let } x^2 - 4x = y \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow$$

$$(y-5)(y-21) = 192$$

$$y^2 - 26y + 105 = 192$$

$$y^2 - 26y + 105 - 192 = 0$$

$$y^2 - 26y - 87 = 0$$

$$y^2 - 29y + 3y - 87 = 0$$

$$y(y-29) + 3(y-29) = 0$$

$$(y-29)(y+3) = 0$$

Either

$$y-29=0 \quad \text{or} \quad y+3=0$$

$$y=29 \quad \text{or} \quad y=-3$$

Putting the values of y in equation (ii)

$$\text{when } y=29$$

$$x^2 - 4x = y$$

$$x^2 - 4x = 29$$

$$x^2 - 4x - 29 = 0$$

$$a=1, b=-4, c=-29$$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times (-29)}}{2 \times 1}$$

$$x = \frac{4 \pm \sqrt{16 + 116}}{2}$$

$$x = \frac{4 \pm \sqrt{132}}{2}$$

$$x = \frac{4 \pm \sqrt{4 \times 33}}{2}$$

$$x = \frac{4 \pm 2\sqrt{33}}{2} = 2 \left(\frac{2 \pm \sqrt{33}}{2} \right)$$

$$x = 2 \pm \sqrt{33}$$

$$\text{when } y = -3$$

$$x^2 - 4x = y$$

$$x^2 - 4x = -3$$

$$x^2 - 4x + 3 = 0$$

$$x^2 - 3x - x + 3 = 0$$

$$x(x-3) - 1(x-3) = 0$$

$$(x-3)(x-1) = 0$$

Either

$$x-3=0 \quad \text{or} \quad x-1=0$$

$$x=3 \quad , \quad x=1$$

$$\therefore \text{Solution Set} = \{1, 3, 2 \pm \sqrt{33}\}$$

Q.16 $(x-1)(x-2)(x-8)(x+5) + 360 = 0$

(U.B + K.B)

Solution:

$$(x-1)(x-2)(x-8)(x+5) + 360 = 0$$

$$(x^2 - 2x - x + 2)(x^2 - 3x + 5x - 40) + 360 = 0$$

$$(x^2 - 3x + 2)(x^2 - 3x - 40) + 360 = 0 \rightarrow (i)$$

$$\text{Let } x^2 - 3x = a \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow$$

$$(a+2)(a-40) + 360 = 0$$

$$a^2 - 40a + 2a - 80 + 360 = 0$$

$$a^2 - 38a + 280 = 0$$

$$a^2 - 28a - 10a + 280 = 0$$

$$a(a-28) - 10(a-28) = 0$$

$$(a-28)(a-10) = 0$$

Either

$$a - 28 = 0 \quad \text{or} \quad a - 10 = 0$$

$$a = 28 \quad a = 10$$

Putting the values of a in equation (ii)

When $a = 28$

$$x^2 - 3x = 28$$

$$x^2 - 3x - 28 = 0$$

$$x^2 - 7x + 4x - 28 = 0$$

$$x(x - 7) + 4(x - 7) = 0$$

$$(x - 7)(x + 4) = 0$$

Either

$$x - 7 = 0 \quad \text{or} \quad x + 4 = 0$$

$$x = 7 \quad x = -4$$

when $a = 10$

$$x^2 - 3x = 10$$

$$x^2 - 3x - 10 = 0$$

$$x^2 - 5x + 2x - 10 = 0$$

$$x(x - 5) + 2(x - 5) = 0$$

$$(x - 5)(x + 2) = 0$$

Either

$$x - 5 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 5 \quad x = -2$$

$$\therefore \text{Solution Set} = \{7, -4, 5, -2\}$$

Radical Equation (K.B)

(LHR 2016, GRW 2014, 17, BWP 2015, 17, MTN 2015, SWL 2016, SGD 2017, D.G.K 2016, 17)

An equation in which a variable or an algebraic expression occurs under radical sign is called radical equation.

e.g. $\sqrt{ax+b} = cx+d$, $2\sqrt{x-3} = 0$ etc.

Extraneous Root (K.B)

Value of variable obtained by solving the equation not satisfying it is called extraneous root.

Note (U.B + K.B)

- Roots of radical equation must be verified.
- Extraneous is introduced by either squaring the given equation or clearing it of fractions.

Type 1 (K.B)

Equation of the type:

$$\sqrt{ax+b} = cx+d$$

Example 1: (Page # 12)

Solve the equation $\sqrt{5x+7} = 2x+3$

Solution:

$$(\sqrt{5x+7})^2 = (2x+3)^2$$

$$5x+7 = 4x^2 + 12x+9$$

$$4x^2 + 9x + 2 = 0$$

$$x = \frac{-9 \pm \sqrt{(9)^2 - 4(4)(2)}}{2(4)}$$

$$= \frac{-9 \pm \sqrt{81-32}}{8}$$

$$= \frac{-9 \pm \sqrt{49}}{8}$$

$$= \frac{-9 \pm 7}{8}$$

Either

$$x = \frac{-9+7}{8} \quad \text{or} \quad x = \frac{-9-7}{8}$$

$$= \frac{-2}{8}$$

$$= \frac{-1}{4}$$

$$= \frac{-16}{8}$$

$$= -2$$

Checking:

Putting $x = -\frac{1}{4}$ in the equation (i),

we have

$$\sqrt{3\left(-\frac{1}{4}\right)+7} = 2\left(-\frac{1}{4}\right)+3$$

$$\sqrt{\frac{-3+28}{4}} = -\frac{1}{2}+3$$

$$\sqrt{\frac{25}{4}} = \frac{5}{2}$$

$$\frac{5}{2} = \frac{5}{2} \quad (\text{True})$$

Now putting $x = -2$ in equation (i)

$$\sqrt{3(-2)+7} = 2(-2)+3$$

$$\sqrt{1} = -1 \quad (\text{False})$$

Thus, the solution set is $\left\{-\frac{1}{4}\right\}$, -2

is an extraneous root.

Type 2

Equation of the type:

$$\sqrt{x+a} + \sqrt{x+b} = \sqrt{x+c}$$

Example 2: (Page # 13)

Solve the equation

$$\sqrt{x+3} + \sqrt{x+6} = \sqrt{x+11}$$

(A.B + K.B + U.B)**Solution:**

$$\sqrt{x+3} + \sqrt{x+6} = \sqrt{x+11}$$

Squaring both sides

$$x+3+x+6+2(\sqrt{x+3})(\sqrt{x+6})=x+11$$

$$2\sqrt{x^2+9x+18}=-x+2$$

Squaring both sides of the equation

(ii), we get

$$4(x^2+9x+18)=x^2-4x+4$$

$$3x^2+40x+68=0$$

Applying quadratic formula,

$$x = \frac{-40 \pm \sqrt{(40)^2 - 4(3)(68)}}{2(3)}$$

$$= \frac{-40 \pm \sqrt{1600 - 816}}{6}$$

$$= \frac{-40 \pm \sqrt{784}}{6}$$

$$= \frac{-40 \pm 28}{6}$$

Either

$$x = \frac{-40+28}{6} \quad \text{or} \quad x = \frac{-40-28}{6}$$

$$= \frac{-12}{6} \quad \text{or} \quad = \frac{-68}{6}$$

$$= -2 \quad \text{or} \quad = \frac{-34}{3}$$

CheckingPutting $x = -2$ in the equation (i)

$$\sqrt{-2+3} + \sqrt{-2+6} = \sqrt{-2+11}$$

$$\sqrt{1} + \sqrt{4} = \sqrt{9}$$

$$1+2=3$$

$$3=3 \quad (\text{True})$$

Now, putting $x = \frac{-34}{3}$ in the

equation (i)

$$\sqrt{\frac{-34}{3}+3} + \sqrt{\frac{-34}{3}+6} = \sqrt{\frac{-34}{3}+11}$$

$$\sqrt{\frac{-34+9}{3}} + \sqrt{\frac{-34+18}{3}} = \sqrt{\frac{-34+33}{3}}$$

$$\sqrt{\frac{-25}{3}} + \sqrt{\frac{-16}{3}} = \sqrt{\frac{-1}{3}}$$

$$\frac{5i}{\sqrt{3}} + \frac{4i}{\sqrt{3}} = \frac{i}{\sqrt{3}}$$

$$\frac{9i}{\sqrt{3}} = \frac{i}{\sqrt{3}} \quad (\text{False})$$

Thus, solution set is $\{-2\}$, $\frac{-34}{3}$ is

an extraneous root.

Type 3

Equation of the type:

$$\sqrt{x^2+px+m} + \sqrt{x^2+px+n} = q$$

Example 3: (Page # 14)

Solve the equation

$$\sqrt{x^2-3x+36} - \sqrt{x^2-3x+9} = 3$$

(A.B + K.B + U.B)**Solution:**

$$\sqrt{x^2-3x+36} - \sqrt{x^2-3x+9} = 3 \rightarrow (i)$$

$$\text{Let } x^2-3x = y \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow$$

$$\sqrt{y+36} - \sqrt{y+9} = 3$$

$$\Rightarrow x = 3$$

Squaring both sides

$$(\sqrt{y+36} - 3)^2 = (\sqrt{y+9})^2$$

$$(\sqrt{y+36})^2 + (3)^2 - 2(3)\sqrt{y+36} = y+9$$

$$y+36+9-6\sqrt{y+36} = y+9$$

$$36 = 6\sqrt{y+36}$$

$$6 = \sqrt{y+36}$$

Again squaring both sides

$$36 = y + 36$$

$$\Rightarrow y = 0$$

Putting the value of y in equation (ii)

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

Either

$$x = 0 \quad \text{or} \quad x - 3 = 0$$

$$\Rightarrow x = 3$$

On checking it is found that 0 and 3 are roots of the given equation.

Thus, solution set = {0, 3}

Exercise 1.4

Solve the following equations.

Q.1 $2x + 5 = \sqrt{7x + 16}$

(A.B)

Solution:

$$2x + 5 = \sqrt{7x + 16} \longrightarrow (i)$$

Taking square on both sides

$$(2x + 5)^2 = (\sqrt{7x + 16})^2$$

$$4x^2 + 20x + 25 = 7x + 16$$

$$4x^2 + 20x - 7x + 25 - 16 = 0$$

$$4x^2 + 13x + 9 = 0$$

$$4x^2 + 9x + 4x + 9 = 0$$

$$x(4x + 9) + 1(4x + 9) = 0$$

$$(4x + 9)(x + 1) = 0$$

Either

$$4x + 9 = 0 \text{ or } x + 1 = 0$$

$$4x = -9 \quad x = -1$$

$$x = \frac{-9}{4}$$

Check

Put $x = \frac{-9}{4}$ in equation (i)

$$2\left(\frac{-9}{4}\right) + 5 = \sqrt{7\left(\frac{-9}{4}\right) + 16}$$

$$\frac{-18}{4} + 5 = \sqrt{\frac{-63}{4} + 16}$$

$$\frac{-18 + 20}{4} = \sqrt{\frac{-63 + 64}{4}}$$

$$\frac{2}{4} = \sqrt{\frac{1}{4}}$$

$$\frac{1}{2} = \frac{1}{2} \text{ Satisfied}$$

Put $x = -1$ in equation (i)

$$2(-1) + 5 = \sqrt{7(-1) + 16}$$

$$-2 + 5 = \sqrt{-7 + 16}$$

$$3 = \sqrt{9}$$

$$3 = 3 \text{ Satisfied}$$

$$\therefore \text{Solution Set} = \left\{ \frac{-9}{4}, -1 \right\}$$

Q.2 $\sqrt{x + 3} = 3x - 1$ (FSD 2014) **(A.B)**

Solution:

$$\sqrt{x + 3} = 3x - 1 \longrightarrow (i)$$

Taking square on both sides

$$(\sqrt{x + 3})^2 = (3x - 1)^2$$

$$x + 3 = 9x^2 - 6x + 1$$

$$-9x^2 + x + 6x + 3 - 1 = 0$$

$$-9x^2 + 7x + 2 = 0$$

$$-(9x^2 - 7x - 2) = 0$$

$$9x^2 - 7x - 2 = 0$$

$$9x^2 - 9x + 2x - 2 = 0$$

$$9x(x - 1) + 2(x - 1) = 0$$

$$(x - 1)(9x + 2) = 0$$

Either

$$x - 1 = 0 \text{ or } 9x + 2 = 0$$

$$x = 1 \quad 9x = -2$$

$$x = \frac{-2}{9}$$

Check

Put $x = 1$ in equation (i)

$$\sqrt{(1) + 3} = 3(1) - 1$$

$$\sqrt{4} = 3 - 1$$

$$2 = 2 \text{ Satisfied}$$

Put $x = \frac{-2}{9}$ in equation (i)

$$\sqrt{\left(\frac{-2}{9}\right) + 3} = 3\left(\frac{-2}{9}\right) - 1$$

$$\sqrt{\frac{-2+27}{9}} = \frac{-6}{9} - 1$$

$$\sqrt{\frac{25}{9}} = \frac{-6-9}{9}$$

$$\frac{5}{3} = \frac{-15}{9}$$

$$\frac{5}{3} = \frac{-5}{3} \text{ Not Satisfied}$$

$\therefore$ Solution Set = {1},

$\frac{-2}{9}$ is extraneous root.

Q.3 $4x+3 = \sqrt{13x+14}$ (A.B)

Solution:

$$4x+3 = \sqrt{13x+14} \rightarrow (i)$$

Squaring both sides, we get

$$(4x+3)^2 = (\sqrt{13x+14})^2$$

$$(4x)^2 + 2(4x)(3) + (3)^2 = 13x+14$$

$$16x^2 + 24x + 9 - 13x - 14 = 0$$

$$16x^2 + 11x - 5 = 0$$

$$16x^2 + 16x - 5x - 5 = 0$$

$$16x(x+1) - 5(x+1) = 0$$

$$(x+1)(16x-5) = 0$$

Either

$$x+1 = 0$$

$$\text{or } 16x-5 = 0$$

$$x = -1$$

$$16x = 5$$

$$x = \frac{5}{16}$$

Check

Put $x = -1$ in equation (i)

$$4x = \sqrt{13x+14} - 3$$

$$4(-1) = \sqrt{13(-1)+14} - 3$$

$$-4 = \sqrt{13+14} - 3$$

$$-4 = \sqrt{27} - 3$$

$$-4 = 1 - 3$$

$$-4 = -2 \text{ (Not true)}$$

Put $x = \frac{5}{16}$ in equation (i)

$$4x = \sqrt{13x+14} - 3$$

$$4\left(\frac{5}{16}\right) = \sqrt{13\left(\frac{5}{16}\right)+14} - 3$$

$$\frac{5}{4} = \sqrt{\frac{65}{16}+14} - 3$$

$$\frac{5}{4} = \sqrt{\frac{65+224}{16}} - 3$$

$$\frac{5}{4} = \sqrt{\frac{289}{16}} - 3$$

$$\frac{5}{4} = \frac{17}{4} - 3$$

$$\frac{5}{4} = \frac{17-12}{4}$$

$$\frac{5}{4} = \frac{5}{4} \text{ True}$$

$\therefore$ Solution Set = $\left\{\frac{5}{16}\right\}$,

Extraneous root = -1

Q.4 $\sqrt{3x+100} - x = 4$ (A.B)

Solution:

$$\sqrt{3x+100} - x = 4 \rightarrow (i)$$

$$\sqrt{3x+100} = 4+x$$

Taking square on both sides

$$(\sqrt{3x+100})^2 = (4+x)^2$$

$$3x+100 = 16+8x+x^2$$

$$-x^2+3x-8x+100-16=0$$

$$-x^2-5x+84=0$$

$$-(x^2+5x-84)=0$$

$$\text{Or } x^2+5x-84=0$$

$$x^2+12x-7x-84=0$$

$$x(x+12)-7(x+12)=0$$

$$(x-7)(x+12)=0$$

Either

$$x-7=0 \text{ or } x+12=0$$

$$x=7$$

$$x=-12$$

Check

Put $x = 7$ in equation (i)

$$\sqrt{3(7)+100} - 7 = 4$$

$$\sqrt{21+100} - 7 = 4$$

$$\sqrt{121} - 7 = 4$$

$$11 - 7 = 4$$

$$4 = 4 \text{ Satisfied}$$

Put $x = -12$ in equation (i)

$$\sqrt{3(-12)+100} - (-12) = 4$$

$$\sqrt{-36+100} + 12 = 4$$

$$\sqrt{64} + 12 = 4$$

$$8 + 12 = 4$$

$$20 = 4 \text{ Not satisfied}$$

$\therefore$ **Solution Set** = $\{7\}$, -12 is **extraneous root**.

Q.5 $\sqrt{x+5} + \sqrt{x+21} = \sqrt{x+60}$

(A.B + U.B)

Solution:

$$\sqrt{x+5} + \sqrt{x+21} = \sqrt{x+60} \rightarrow (i)$$

Taking square on both sides

$$(\sqrt{x+5} + \sqrt{x+21})^2 = (\sqrt{x+60})^2$$

$$(\sqrt{x+5})^2 + (\sqrt{x+21})^2 + 2(\sqrt{x+5})(\sqrt{x+21}) = x+60$$

$$x+5+x+21+2\sqrt{(x+5)(x+21)} = x+60$$

$$2x+26+2\sqrt{x^2+26x+105} = x+60$$

$$2x-x+26-60 = -2\sqrt{x^2+26x+105}$$

$$x-34 = -2\sqrt{x^2+26x+105}$$

Again squaring both sides

$$(x-34)^2 = (-2\sqrt{x^2+26x+105})^2$$

$$x^2 - 68x + 1156 = 4(x^2 + 26x + 105)$$

$$x^2 - 68x + 1156 = 4x^2 + 104x + 420$$

$$x^2 - 4x^2 - 68x - 104x + 1156 - 420 = 0$$

$$-3x^2 - 172x + 736 = 0$$

$$-(3x^2 + 172x - 736) = 0$$

$$\text{Or } 3x^2 + 172x - 736 = 0$$

$$\text{As } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Here } a = 3, b = 172, c = -736$$

Putting the values

$$x = \frac{-172 \pm \sqrt{(172)^2 - 4(3)(-736)}}{2(3)}$$

$$= \frac{-172 \pm \sqrt{29584 + 8832}}{6}$$

$$= \frac{-172 \pm \sqrt{38416}}{6}$$

$$= \frac{-172 \pm 196}{6}$$

Either

$$x = \frac{-176-196}{6} \quad \text{or} \quad x = \frac{-176+196}{6}$$

$$= \frac{-368}{6} \quad x = \frac{24}{6}$$

$$x = \frac{-184}{3} \quad x = 4$$

Check

Put $x = \frac{-184}{3}$ in equation (i)

$$\sqrt{\frac{-184}{3}+5} + \sqrt{\frac{-184}{3}+21} = \sqrt{\frac{-184}{3}+60}$$

$$\sqrt{\frac{-184+15}{3}} + \sqrt{\frac{-184+63}{3}} = \sqrt{\frac{-184+180}{3}}$$

$$\sqrt{\frac{-169}{3}} + \sqrt{\frac{-121}{3}} = \sqrt{\frac{-4}{3}}$$

$$\frac{13}{\sqrt{3}}i + \frac{11}{\sqrt{3}}i = \frac{2}{3}i$$

$$\frac{24}{\sqrt{3}}i = \frac{2}{3}i$$

Not satisfied

Put $x = 4$ in equation (i)

$$\sqrt{4+5} + \sqrt{4+21} = \sqrt{4+60}$$

$$\sqrt{9} + \sqrt{25} = \sqrt{64}$$

$$3 + 5 = 8$$

$$8 = 8 \text{ Satisfied}$$

$$\therefore \text{ **Solution Set** } = \{4\}, \frac{-184}{3}$$

is extraneous root

Q.6 $\sqrt{x+1} + \sqrt{x-2} = \sqrt{x+6}$

(A.B + U.B)

Solution:

$$\sqrt{x+1} + \sqrt{x-2} = \sqrt{x+6} \rightarrow (i)$$

Taking square on both sides

$$(\sqrt{x+1} + \sqrt{x-2})^2 = (\sqrt{x+6})^2$$

$$(\sqrt{x+1})^2 + (\sqrt{x-2})^2 + 2(\sqrt{x+1})(\sqrt{x-2}) = x+6$$

$$x+1+x-2+2\sqrt{x^2-x-2} = x+6$$

$$2x-1+2\sqrt{x^2-x-2} = x+6$$

$$2x-x-1-6 = -2\sqrt{x^2-x-2}$$

$$x-7 = -2\sqrt{x^2-x-2}$$

Again taking square on both sides

$$(x-7)^2 = (-2\sqrt{x^2-x-2})^2$$

$$x^2 - 14x + 49 = 4(x^2 - x - 2)$$

$$x^2 - 14x + 49 = 4x^2 - 4x - 8$$

$$x^2 - 4x^2 - 14x + 4x + 49 + 8 = 0$$

$$-3x^2 - 10x + 57 = 0$$

$$-(3x^2 + 10x - 57) = 0$$

$$\text{Or } 3x^2 + 10x - 57 = 0$$

$$3x^2 + 19x - 9x - 57 = 0$$

$$x(3x+19) - 3(3x+19) = 0$$

$$(x-3)(3x+19) = 0$$

Either

$$x-3=0 \text{ or } 3x+19=0$$

$$x=3 \quad 3x=-19$$

$$x = \frac{-19}{3}$$

Check

Put $x=3$ in equation (i)

$$\sqrt{3+1} + \sqrt{3-2} = \sqrt{3+6}$$

$$\sqrt{4} + \sqrt{1} = \sqrt{9}$$

$$2+1=3$$

$$3=3 \text{ Satisfied}$$

Put $x = \frac{-19}{3}$ in equation (i)

$$\sqrt{\frac{-19}{3}+1} + \sqrt{\frac{-19}{3}-2} = \sqrt{\frac{-19}{3}+6}$$

$$\sqrt{\frac{-19+3}{3}} + \sqrt{\frac{-19-6}{3}} = \sqrt{\frac{-19+18}{3}}$$

$$\sqrt{\frac{-16}{3}} + \sqrt{\frac{-25}{3}} = \sqrt{\frac{-1}{3}}$$

$$\frac{4}{\sqrt{3}}i + \frac{5}{\sqrt{3}}i = \frac{1}{\sqrt{3}}i$$

$$\frac{4i+5i}{\sqrt{3}} = \frac{1}{\sqrt{3}}i$$

$$\frac{9}{\sqrt{3}}i = \frac{1}{\sqrt{3}}i, \text{ Not satisfied}$$

$$\therefore \text{Solution Set} = \left\{3\right\}, \frac{-19}{3} \text{ is}$$

extraneous root

Q.7 $\sqrt{11-x} - \sqrt{6-x} = \sqrt{27-x}$

(A.B + U.B)

Solution:

$$\sqrt{11-x} - \sqrt{6-x} = \sqrt{27-x} \rightarrow (i)$$

Squaring both sides, we get

$$(\sqrt{11-x} - \sqrt{6-x})^2 = (\sqrt{27-x})^2$$

$$(\sqrt{11-x})^2 + (\sqrt{6-x})^2 - 2(\sqrt{11-x})(\sqrt{6-x}) = 27-x$$

$$11-x+6-x-2\sqrt{(11-x)(6-x)} = 27-x$$

$$17-2x-2\sqrt{66-11x-6x+x^2} = 27-x$$

$$-2\sqrt{66-17x+x^2} = 27-x-17+2x$$

$$-2\sqrt{x^2-17x+66} = x+10$$

Again squaring both sides

$$(-2\sqrt{x^2-17x+66})^2 = (x+10)^2$$

$$4(x^2-17x+66) = x^2+20x+100$$

$$4x^2-68x+264-x^2-20x-100=0$$

$$3x^2-88x+164=0$$

$$a=3, \quad b=-88, \quad c=164$$

Using Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-88) \pm \sqrt{(-88)^2 - 4(3)(164)}}{2 \times 3}$$

$$x = \frac{88 \pm \sqrt{7744 - 1968}}{6}$$

$$x = \frac{88 \pm \sqrt{5776}}{6}$$

$$x = \frac{88 \pm 76}{6}$$

$$x = \frac{88+76}{6}, \quad x = \frac{88-76}{6}$$

$$x = \frac{164}{6}, \quad x = \frac{12}{6}$$

$$x = \frac{82}{3}, \quad x = 2$$

Check

When $x = 2$

$$\sqrt{11-x} - \sqrt{6-x} = \sqrt{27-x}$$

$$\sqrt{11-2} - \sqrt{6-2} = \sqrt{27-2}$$

$$\sqrt{9} - \sqrt{4} = \sqrt{25}$$

$$3 - 2 = 5$$

$$1 = 5 \text{ (Not true)}$$

Put $x = \frac{82}{3}$ in equation (i)

$$\sqrt{11-x} - \sqrt{6-x} = \sqrt{27-x}$$

$$\sqrt{11-\frac{82}{3}} - \sqrt{6-\frac{82}{3}} = \sqrt{27-\frac{82}{3}}$$

$$\sqrt{\frac{33-82}{3}} - \sqrt{\frac{18-82}{3}} = \sqrt{\frac{81-82}{3}}$$

$$\sqrt{\frac{-49}{3}} - \sqrt{\frac{-64}{3}} = \sqrt{\frac{-1}{3}}$$

$$\frac{7}{\sqrt{3}}i - \frac{8}{\sqrt{3}}i = \frac{1}{\sqrt{3}}i$$

$$\frac{7i-8i}{\sqrt{3}} = \frac{1}{\sqrt{3}}i$$

$$\frac{-i}{\sqrt{3}} = \frac{1}{\sqrt{3}}i \text{ (Not true)}$$

$\therefore$ **Solution Set** = { }

Extraneous roots = $\frac{82}{3}, 2$

Q.8 $\sqrt{4a+x} - \sqrt{a-x} = \sqrt{a}$

(A.13)

Solution:

$$\sqrt{4a+x} - \sqrt{a-x} = \sqrt{a} \rightarrow (i)$$

Taking sq. on both sides

$$(\sqrt{4a+x} - \sqrt{a-x})^2 = (\sqrt{a})^2$$

$$(\sqrt{4a+x})^2 + (\sqrt{a-x})^2 - 2(\sqrt{4a+x})(\sqrt{a-x}) = a$$

$$4a+x+a-x-2\sqrt{(4a+x)(a-x)} = a$$

$$5a-2\sqrt{4a^2-3ax-x^2} = a$$

$$5a-a = 2\sqrt{4a^2-3ax-x^2}$$

$$4a = 2\sqrt{4a^2-3ax-x^2}$$

Again taking square on both sides

$$(4a)^2 = (2\sqrt{4a^2-3ax-x^2})^2$$

$$16a^2 = 4(4a^2-3ax-x^2)$$

$$16a^2 = 16a^2 - 12ax - 4x^2$$

$$16a^2 - 16a^2 + 12ax + 4x^2 = 0$$

$$12ax + 4x^2 = 0$$

$$4x(3a+x) = 0$$

Either

$$4x = 0 \text{ or } 3a+x = 0$$

$$x = \frac{0}{4} \quad x = -3a$$

$$x = 0$$

Check

Put $x = 0$ in equation (i)

$$\sqrt{4a+0} - \sqrt{a-0} = \sqrt{a}$$

$$\sqrt{4a} - \sqrt{a} = \sqrt{a}$$

$$2\sqrt{a} - \sqrt{a} = \sqrt{a}$$

$$\sqrt{a} = \sqrt{a} \therefore \text{Satisfied}$$

Put $x = -3a$ in equation (i)

$$\sqrt{4a-3a} - \sqrt{a+3a} = \sqrt{a}$$

$$\sqrt{a} - \sqrt{4a} = \sqrt{a}$$

$$\sqrt{a} - 2\sqrt{a} = \sqrt{a}$$

$$-\sqrt{a} = \sqrt{a} \text{ Not satisfied}$$

$\therefore$ **Solution Set** = {0}, $-3a$

is extraneous root

Q.9 $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$

(A.B)

Solution:

$$\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1 \rightarrow (i)$$

$$\text{Let } x^2+x=y \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow \sqrt{y+1}-\sqrt{y-1}=1$$

Taking square on both sides

$$(\sqrt{y+1}-\sqrt{y-1})^2=(1)^2$$

$$(\sqrt{y+1})^2+(\sqrt{y-1})^2-2(\sqrt{y+1})(\sqrt{y-1})=1$$

$$y+1+y-1-2\sqrt{(y+1)(y-1)}=1$$

$$2y-2\sqrt{y^2-1}=1$$

$$2y-1=2\sqrt{y^2-1}$$

Again taking square on both sides

$$(2y-1)^2=(2\sqrt{y^2-1})^2$$

$$4y^2-4y+1=4(y^2-1)$$

$$4y^2-4y+1=4y^2-4$$

$$4y^2-4y^2-4y+1+4=0$$

$$-4y+5=0$$

$$-4y=-5$$

$$y=\frac{5}{4}$$

$$\text{Put } y=\frac{5}{4} \text{ in equation (ii)}$$

$$x^2+x=\frac{5}{4}$$

$$4(x^2+x)=5$$

$$4x^2+4x=5$$

$$4x^2+4x-5=0$$

$$\text{Here } a=4, b=4, c=-5$$

$$\text{As } x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Putting the values

$$x=\frac{-4 \pm \sqrt{(4)^2-4(4)(-5)}}{2(4)}$$

$$x=\frac{-4 \pm \sqrt{16+80}}{8}$$

$$x=\frac{-4 \pm \sqrt{96}}{8}$$

$$x=\frac{-4 \pm 4\sqrt{6}}{8}$$

$$x=\frac{-1 \pm \sqrt{6}}{2}$$

$$x=\frac{-1 \pm \sqrt{6}}{2}$$

On checking, we know that these values satisfy the equation.

$$\therefore \text{Solution Set} = \left\{ \frac{-1 \pm \sqrt{6}}{2} \right\}$$

Q.10 $\sqrt{x^2+3x+8}+\sqrt{x^2+3x+2}=3$

(A.B)

Solution:

$$\sqrt{x^2+3x+8}+\sqrt{x^2+3x+2}=3 \rightarrow (i)$$

$$\text{Let } x^2+3x=y \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow \sqrt{y+8}+\sqrt{y+2}=3$$

Taking square on both sides

$$(\sqrt{y+8}+\sqrt{y+2})^2=(3)^2$$

$$(\sqrt{y+8})^2+(\sqrt{y+2})^2+2(\sqrt{y+8})(\sqrt{y+2})=9$$

$$y+8+y+2+2\sqrt{(y+8)(y+2)}=9$$

$$2y+10+2\sqrt{y^2+10y+16}=9$$

$$2y+10-9=-2\sqrt{y^2+10y+16}$$

$$2y+1=-2\sqrt{y^2+10y+16}$$

Again taking square on both sides

$$(2y+1)^2=(-2\sqrt{y^2+10y+16})^2$$

$$4y^2+4y+1=4(y^2+10y+16)$$

$$4y^2+4y+1=4y^2+40y+64$$

$$4y^2-4y^2+4y-40y+1-64=0$$

$$-36y-63=0$$

$$-36y=63$$

$$y=\frac{63}{-36}$$

$$y=\frac{-7}{4}$$

Put $y = \frac{-7}{4}$ in equation (i)

$$x^2 + 3x = \frac{-7}{4}$$

$$4(x^2 + 3x) = -7$$

$$4x^2 + 12x = -7$$

$$4x^2 + 12x + 7 = 0$$

Here $a = 4$, $b = 12$, $c = 7$

$$\text{As } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Putting the values

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4(4)(7)}}{2(4)}$$

$$x = \frac{-12 \pm \sqrt{144 - 112}}{8}$$

$$x = \frac{-12 \pm \sqrt{32}}{8}$$

$$x = \frac{-12 \pm 4\sqrt{2}}{8}$$

$$x = 4 \frac{(-3 \pm \sqrt{2})}{8}$$

$$x = \frac{-3 \pm \sqrt{2}}{2}$$

On checking, we know that these values satisfy the equation.

$$\therefore \text{Solution Set} = \left\{ \frac{-3 \pm \sqrt{2}}{2} \right\}$$

Q.11 $\sqrt{x^2 + 3x + 9} + \sqrt{x^2 + 3x + 4} = 5$

(A.B)

Solution:

$$\sqrt{x^2 + 3x + 9} + \sqrt{x^2 + 3x + 4} = 5 \rightarrow (i)$$

$$\text{Let } x^2 + 3x = y \rightarrow (ii)$$

Put in equation (i) $\Rightarrow$

$$\sqrt{y+9} + \sqrt{y+4} = 5$$

Squaring on both sides, we get

$$(\sqrt{y+9} + \sqrt{y+4})^2 = (5)^2$$

$$(\sqrt{y+9})^2 + (\sqrt{y+4})^2 + 2\sqrt{y+9}\sqrt{y+4} = 25$$

$$y+9 + y+4 + 2\sqrt{(y+9)(y+4)} = 25$$

$$2y+13 + 2\sqrt{y^2 + 13y + 36} = 25$$

$$2\sqrt{y^2 + 13y + 36} = 25 - 13 - 2y$$

$$2\sqrt{y^2 + 13y + 36} = 12 - 2y$$

$$2\sqrt{y^2 + 13y + 36} = 2(6 - y)$$

$$\sqrt{y^2 + 13y + 36} = (6 - y)$$

Again squaring both sides.

$$(\sqrt{y^2 + 13y + 36})^2 = (6 - y)^2$$

$$y^2 + 13y + 36 = 36 + y^2 - 12y$$

$$13y + 12y = 36 - 36$$

$$25y = 0$$

$$y = 0$$

Put in equation (ii)

$$x^2 + 3x = y$$

$$x^2 + 3x = 0$$

$$x(x+3) = 0$$

Either

$$x = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = 0 \quad \text{or} \quad x = -3$$

Check

When $x = 0$

$$\sqrt{x^2 + 3x + 9} + \sqrt{x^2 + 3x + 4} = 5$$

$$\sqrt{0^2 + 3(0) + 9} + \sqrt{0^2 + 3(0) + 4} = 5$$

$$\sqrt{0+9} + \sqrt{0+4} = 5$$

$$\sqrt{9} + \sqrt{4} = 5$$

$$3 + 2 = 5$$

$$5 = 5 \text{ (True)}$$

When $x = -3$

$$\sqrt{x^2 + 3x + 9} + \sqrt{x^2 + 3x + 4} = 5$$

$$\sqrt{(-3)^2 + 3(-3) + 9} + \sqrt{(-3)^2 + 3(-3) + 4} = 5$$

$$\sqrt{9-9+9} + \sqrt{9-9+4} = 5$$

$$\sqrt{9} + \sqrt{4} = 5$$

$$3 + 2 = 5$$

$$5 = 5 \text{ (True)}$$

$$\therefore \text{Solution Set} = \{0, -3\}$$

Miscellaneous Exercise 1

Q.1 Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

- (i) Standard form of quadratic equation is; **(K.B)**
(GRW 2014, 16, MTN 2017, SGD 2015, 17, FSD 2018, RWP 2015)
(a) $bx + c = 0, b \neq 0$ (b) $ax^2 + bx + c = 0, a \neq 0$
(c) $ax^2 = bx, a \neq 0$ (d) $ax^2 = 0, a \neq 0$
- (ii) The number of terms in a standard quadratic equation $ax^2 + bx + c = 0$ is; **(K.B)**
(LHR 2015, BWP 2015, MTN 2015)
(a) 1 (b) 2
(c) 3 (d) 4
- (iii) The number of methods to solve a quadratic equation is; **(K.B)**
(GRW 2017, FSD 2014, 17, SWL 2015, 16, RWP 2016, 17, D.G.K 2014, 15, 17)
(a) 1 (b) 2
(c) 3 (d) 4
- (iv) The quadratic formula is; (FSD 2015, D.G.K 2014, SWL 2014, 17) **(A.B)**
(a) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (b) $x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$
(c) $x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$ (d) $x = \frac{b \pm \sqrt{b^2 + 4ac}}{2a}$
- (v) Two linear factors of $x^2 - 15x + 56$ are; **(A.B + U.B)**
(LHR 2014, FSD 2016, RWP 2017, D.G.K 2016)
(a) $(x - 7)$ and $(x + 8)$ (b) $(x + 7)$ and $(x - 8)$
(c) $(x - 7)$ and $(x - 8)$ (d) $(x + 7)$ and $(x + 8)$
- (vi) An equation, which remains unchanged when x is replaced by $\frac{1}{x}$ is called a/an; **(K.B)**
(a) Exponential equation (b) Reciprocal equation
(c) Radical equation (d) None of these
- (vii) An equation of the type $3^x + 3^{2-x} + 6 = 0$ is a/an; **(K.B)**
(GRW 2016, SGD 2014, 16, D.G.K 2015, 17)
(a) Exponential equation (b) Radical equation
(c) Reciprocal equation (d) None of these
- (viii) The solution set of equation $4x^2 - 16 = 0$ is; **(K.B + U.B + A.P)**
(a) $\{\pm 4\}$ (b) $\{4\}$
(c) $\{\pm 2\}$ (d) ± 2
- (ix) The equation of the form $2x^4 - 3x^3 + 7x^2 - 3x + 2 = 0$ is called a/an; **(A.B+K.B+U.B)**
(a) Reciprocal equation (b) Radical equation
(c) Exponential equation (d) None of these

ANSWER KEY

i.	ii.	iii.	iv.	v.	vi.	vii.	viii.	ix.
b	c	c	a	c	b	a	c	a

Q.2 Write short answers of the following questions. (A.B)

(i) Solve $x^2 + 2x - 2 = 0$

(LHR 2017, SWL 2017, SGD 2014, 17, RWF 2015)

Solution:

$$x^2 + 2x - 2 = 0$$

$$x^2 + 2x = 2$$

Adding $(1)^2$ on both sides

$$x^2 + 2x + (1)^2 = 2 + (1)^2$$

$$(x+1)^2 = 2+1$$

$$(x+1)^2 = 3$$

Taking sq. root on both sides

$$x+1 = \pm\sqrt{3}$$

$$x = -1 \pm \sqrt{3}$$

$$\therefore \text{Solution Set} = \{-1 \pm \sqrt{3}\}$$

(ii) Solve by factorization $5x^2 = 15x$

(LHR 2015, 16, GRW 2014, 16, 17, SWL 2016, 17, BWP 2014, 16, D.G.K 2017)

(A.B)

Solution:

$$5x^2 = 15x$$

$$5x^2 - 15x = 0$$

$$5x(x-3) = 0$$

Either

$$5x = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 0 \quad x = 3$$

$$\therefore \text{Solution Set} = \{0, 3\}$$

(iii) Write in standard form $\frac{1}{x+4} + \frac{1}{x-4} = 3$
(FSD 2017, SWL 2016, SGD 2015, BWP 2015, MTN 2014, D.G.K 2014) **(A.B)**

Solution:

$$\frac{1}{x+4} + \frac{1}{x-4} = 3$$

Multiply both sides by

$$(x+4)(x-4),$$

we get

$$x-4 + x+4 = 3(x+4)(x-4)$$

$$2x = 3(x^2 - 16)$$

$$2x = 3x^2 - 48$$

$$3x^2 - 2x - 48 = 0$$

Which is in standard form of quadratic equation.

(iv) Write the name of the methods for solving a quadratic equation. **(A.B)**

Solution:

Following are the three methods:

(a) Factorization

(b) Completing square

(c) Quadratic Formula

(v) Solve $\left(2x - \frac{1}{2}\right)^2 = \frac{9}{4}$

Solution:

$$\left(2x - \frac{1}{2}\right)^2 = \frac{9}{4}$$

Taking square root on both sides

$$\sqrt{\left(2x - \frac{1}{2}\right)^2} = \sqrt{\frac{9}{4}}$$

$$2x - \frac{1}{2} = \pm \frac{3}{2}$$

Either

$$2x - \frac{1}{2} = \frac{3}{2} \quad \text{or} \quad 2x - \frac{1}{2} = -\frac{3}{2}$$

$$2x = \frac{3}{2} + \frac{1}{2}, \quad 2x = -\frac{3}{2} + \frac{1}{2}$$

$$2x = \frac{3+1}{2}, \quad 2x = \frac{-3+1}{2}$$

$$2x = \frac{4}{2}, \quad 2x = \frac{-2}{2}$$

$$2x = 2, \quad 2x = -1$$

$$x = \frac{2}{2}, \quad x = -\frac{1}{2}$$

$$x = 1$$

$$\therefore \text{Solution Set} = \left\{-\frac{1}{2}, 1\right\}$$

(vi) Solve $\sqrt{3x+18} = x$ (A.B)

Solution:

$$\sqrt{3x+18} = x$$

Squaring both sides

$$(\sqrt{3x+18})^2 = (x)^2$$

$$3x+18 = x^2$$

$$x^2 - 3x - 18 = 0$$

$$x^2 - 6x + 3x - 18 = 0$$

$$x(x-6) + 3(x-6) = 0$$

$$(x-6)(x+3) = 0$$

Either

$$x-6=0 \quad \text{or} \quad x+3=0$$

$$x=6 \quad \text{or} \quad x=-3$$

Verification

When $x = 6$ When $x = -3$

$$\sqrt{3x+18} = x$$

$$\sqrt{3x+18} = x$$

$$\sqrt{3(6)+18} = 6$$

$$\sqrt{3(-3)+18} = -3$$

$$\sqrt{18+18} = 6$$

$$\sqrt{-9+18} = -3$$

$$\sqrt{36} = 6$$

$$\sqrt{9} = -3$$

$$6 = 6 \text{ (True)} \quad 3 = -3 \text{ (false)}$$

$$\therefore \text{Solution Set} = \{6\}$$

Extraneous root = -3

(vii) Define quadratic equation. (U.B)

Answer: See definition Page # 1

(viii) Define reciprocal equation. (U.B)

Answer: See definition Page # 14

(ix) Define exponential equation. (U.B)

Answer: See definition Page # 14

(x) Define radical equation. (U.B)

Answer: See definition Page # 24

Q.3 Fill in the blanks

- (i) The standard form of the quadratic equation is _____. **(K.B)**
- (ii) The number of methods to solve a quadratic equation are _____. **(K.B)**
- (iii) The name of the method to derive a quadratic formula is _____. **(K.B)**
- (iv) The solution of the equation $ax^2 + bx + c = 0$, $a \neq 0$ is _____. **(K.B + A.B)**
- (v) The solution set of $25x^2 - 1 = 0$ is _____. **(K.B)**
- (vi) An equation of the form $2^{2x} - 3 \cdot 2^x + 5 = 0$ is called a/an _____ equation. **(U.B)**
- (GRW 2014, 17, SGD 2016, 14, BWP 2016)**
- (vii) The solution set of the equation $x^2 - 9 = 0$ is _____. **(U.B)**
- (viii) An equation of the type $x^4 + x^3 + x^2 + x + 1 = 0$ called a/ an _____ equation. **(U.B)**
- (ix) A root of an equation, which do not satisfy the equation is called _____ root. **(U.B)**
- (x) An equation involving impression of the variable under _____ is called radical equation. **(U.B)**

ANSWER KEY

- (i) $ax^2 + bx + c = 0$
- (ii) 3
- (iii) Completing square
- (iv) $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- (v) $\left\{ \pm \frac{1}{5} \right\}$
- (vi) Exponential
- (vii) $\{\pm 3\}$
- (viii) Reciprocal
- (ix) Extraneous
- (x) Radical sign



CUT HERE

Unit-1**Quadratic Equations****SELF TEST****Time: 40 min****Marks: 25****Q.1 Four possible answers (A), (B), (C) & (D) to each question are given, mark the correct answer. (7×1=7)****1 A quadratic equation can be solved by:**

- (A) Factorization (B) Completing square
(C) Quadratic formula (D) All of these

2 In quadratic equation $ax^2 + bx + c = 0$, $a, b, c, \in$ _____

- (A) $\mathbb{N}$ (B) $\mathbb{Q}$
(C) $\mathbb{Z}$ (D) $\mathbb{R}$

3 The equation $2x^4 - 3x^3 + 7x^2 - 3x + 2 = 0$ is called a/an

- (A) Reciprocal equation (B) Radical equation
(C) Exponential equation (D) None of these

4 The solution set of the equation $4x^2 - 16 = 0$ is

- (A) $\{\pm 4\}$ (B) $\{4\}$
(C) $\{\pm 2\}$ (D) $\{-2\}$

5 Two linear factors of $x^2 - 15x + 56$ are:

- (A) $(x-7), (x-8)$ (B) $(x+7), (x+8)$
(C) $(x+7), (x-8)$ (D) $(x-7), (x+8)$

6 The quadratic formula is:

- (A) $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (B) $\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$
(C) $\frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$ (D) $\frac{b \pm \sqrt{b^2 + 4ac}}{2a}$

7 Solution set of $5x^2 = 30x$

- (A) $\{6\}$ (B) $\{-6\}$
(C) $\{0, 6\}$ (D) $\{25\}$

Q.2 Give Short Answers to following Questions.**(5×2=10)****(i) Define quadratic equation? Write the standard form of quadratic equation.****(ii) Find the solution set of the standard quadratic equation**

$$ax^2 + bx + c = 0 \text{ for } a=1, b=-3, c=-5.$$

(iii) Solve: $\sqrt{3x+18} = x$ **(iv) Solve: $2x^4 - 11x^2 + 5 = 0$** **(v) Solve: $\left(2x - \frac{1}{2}\right)^2 = \frac{9}{4}$** **Q.3 Answer the following Questions.****(4+4=8)****(a) Solve: $3^{2x+2} = 12 \cdot 3^x - 3$** **(b) Solve the equation by completing square method. $x^2 - 2x - 195 = 0$.****NOTE: Parents or guardians can conduct this test in their supervision in order to check the skill of students.**

UNIT 2

THEORY OF QUADRATIC EQUATIONS

The Discriminant $b^2 - 4ac$

Discriminant > 0
 $= 2 \text{ real roots}$
 $x = \frac{-b \pm \sqrt{\text{positive}}}{2a}$

Discriminant $= 0$
 $= 1 \text{ repeated root}$
 $x = \frac{-b \pm \sqrt{\text{zero}}}{2a}$

Discriminant < 0
 $= \text{no real roots}$
 $x = \frac{-b \pm \sqrt{\text{negative}}}{2a}$

Discriminant (EWP 2018) (U.B + K.B)

“For a standard quadratic equation $ax^2 + bx + c = 0$, the value of the expression $b^2 - 4ac$ is called discriminant.”

It is used to find the nature of roots without solving the equation.

Nature or Characteristics of the Roots (U.B + K.B)

Nature of a quadratic equation

$ax^2 + bx + c = 0$, when $a, b, c \in Q$ and $a \neq 0$ as:

- (i) If $b^2 - 4ac = 0$, then the roots are rational (real) and equal.
- (ii) If $b^2 - 4ac < 0$, then the roots are complex conjugate or imaginary.
- (iii) If $b^2 - 4ac > 0$, and is a perfect square, then the roots are rational (real) and unequal.
- (iv) If $b^2 - 4ac > 0$, and is not a perfect square, the roots are irrational (real) and unequal.

Note (K.B)

If given polynomial expression is a perfect square then discriminant is 0.

Example 2: (Page # 19)

Using discriminant, find the nature of the roots of the following equation and verify the result by solving the equation.

$$x^2 - 5x + 5 = 0$$

(LHR 2015, GRW 2016-17, SWI 2017, RWP 2015, L.G.K 2017)

Solution:

$$x^2 - 5x + 5 = 0$$

Here $a = 1$, $b = -5$, $c = 5$

$$\begin{aligned} \text{Discriminant} &= b^2 - 4ac \\ &= (-5)^2 - 4(1)(5) \end{aligned}$$

$$= 25 - 20 = 5$$

As discriminant > 0 but not perfect square, Roots are irrational (real) and unequal.

Verification:

Solving the equation by using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)}$$

$$x = \frac{5 \pm \sqrt{5}}{2}$$

Evidently, Roots are irrational (real) and unequal.

Example 2: (Page # 21)

Find k , if the roots of the equation $(k+3)x^2 - 2(k+1)x - (k+1) = 0$ are equal, if $k \neq -3$ (A.B)

Solution:

$$(k+3)x^2 - 2(k+1)x - (k+1) = 0$$

Here

$$a = k+3, b = -2(k+1), c = -(k+1)$$

As roots are equal, discriminant is zero

$$\Rightarrow \text{Disc} = b^2 - 4ac = 0$$

$$[-2(k+1)]^2 - 4(k+3)[-(k+1)] = 0$$

$$4(k+1)^2 + 4(k+3)(k+1) = 0$$

$$4(k+1)[(k+1) + (k+3)] = 0$$

$$4(k+1)(2k+4) = 0$$

Either

$$k+1 = 0 \quad \text{or} \quad 2k+4 = 0 \quad \because 4 \neq 0$$

$$k = -1 \quad \text{or} \quad 2k = -4$$

$$k = -2$$

Thus, roots will be equal if $k = -1, -2$

Exercise 2.1

Q.1 Find the discriminant of the following given quadratic equations:

Solution:

(i) $2x^2 + 3x - 1 = 0$ **(A.B)**
(GRW 2017, FSD 2016, MTN 2014, D.G.K 2015)

By comparing given equation with $ax^2 + bx + c = 0$, we get
 $a = 2, b = 3, c = -1$

$$\begin{aligned}\text{Disc} &= b^2 - 4ac \\ &= (3)^2 - 4(2)(-1) \\ &= 9 + 8 \\ &= 17\end{aligned}$$

(ii) $6x^2 - 8x + 3 = 0$ **(A.B)**
(LHR 2016, SWL 2016, D.G.K 2015, 17)

By comparing given equation with $ax^2 + bx + c = 0$, we get
 $a = 6, b = -8, c = 3$

$$\begin{aligned}\text{Disc} &= b^2 - 4ac \\ &= (-8)^2 - 4(6)(3) \\ &= 64 - 72 \\ &= -8\end{aligned}$$

(iii) $9x^2 - 30x + 25 = 0$ **(A.B)**
(LHR 2017, MTN 2015)

By comparing given equation with $ax^2 + bx + c = 0$, we get
 $a = 9, b = -30, c = 25$

$$\begin{aligned}\text{Disc} &= (-30)^2 - 4(9)(25) \\ &= 900 - 900 \\ &= 0\end{aligned}$$

(iv) $4x^2 - 7x - 2 = 0$ **(A.B)**
(GRW 2014, SGD 2017, MTN 2016)

By comparing given equation with $ax^2 + bx + c = 0$, we get
 $a = 4, b = -7, c = -2$

$$\begin{aligned}\text{Disc} &= b^2 - 4ac \\ &= (-7)^2 - 4(4)(-2) \\ &= 49 + 32 \\ &= 81\end{aligned}$$

Q.2 Find the nature of the roots of the following given quadratic equations and verify the result by solving the equations.

Solution:

(i) $x^2 - 23x + 120 = 0$ **(A.B)**

By comparing given equation with $ax^2 + bx + c = 0$, we get
 $a = 1, b = -23, c = 120$

$$\begin{aligned}\text{Disc} &= b^2 - 4ac \\ &= (-23)^2 - 4(1)(120) \\ &= 529 - 480 \\ &= 49 \\ &= 7^2\end{aligned}$$

Since disc > 0 and perfect square, roots are rational (real) and unequal.

Verification:

$$\begin{aligned}x^2 - 23x + 120 &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-23) \pm \sqrt{(-23)^2 - 4(1)(120)}}{2(1)} \\ &= \frac{23 \pm \sqrt{7^2}}{2}\end{aligned}$$

$$x = \frac{23 \pm 7}{2}$$

Either

$$x = \frac{23-7}{2} \quad \text{or} \quad x = \frac{23+7}{2}$$

$$= \frac{16}{2} \quad \text{or} \quad = \frac{30}{2}$$

$$x = 8 \quad \text{or} \quad x = 15$$

Hence roots are rational and unequal.

(ii) $2x^2 + 3x + 7 = 0$ **(A.B)**

By comparing given equation with $ax^2 + bx + c = 0$, we get
 $a = 2, b = 3, c = 7$

$$\begin{aligned}\text{Disc} &= b^2 - 4ac \\ &= (3)^2 - 4(2)(7) \\ &= 9 - 56 \\ &= -47\end{aligned}$$

Since disc < 0 , roots are complex and imaginary.

Verification:

$$2x^2 + 3x + 7 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{(3)^2 - 4(2)(7)}}{2(2)}$$

$$= \frac{-3 \pm \sqrt{9 - 56}}{4}$$

$$= \frac{-3 \pm \sqrt{-47}}{4}$$

$$= \frac{-3 \pm \sqrt{47}i}{4}$$

Hence roots are complex/imaginary and unequal.

(iii) $16x^2 - 24x + 9 = 0$ **(A.B)**

By comparing given equation with

$$ax^2 + bx + c = 0, \text{ we get}$$

$$a = 16, b = -24, c = 9$$

$$\begin{aligned} \text{Disc} &= b^2 - 4ac \\ &= (-24)^2 - 4(16)(9) \\ &= 576 - 576 \\ &= 0 \end{aligned}$$

Since disc = 0, roots are rational (real) and equal.

Verification:

$$16x^2 - 24x + 9 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-24) \pm \sqrt{(-24)^2 - 4(16)(9)}}{2(16)}$$

$$x = \frac{24 \pm \sqrt{0}}{32}$$

$$x = \frac{24 \pm 0}{32}$$

Either

$$x = \frac{24+0}{32} \quad \text{or} \quad x = \frac{24-0}{32}$$

$$x = \frac{24}{32} \quad x = \frac{24}{32}$$

Hence roots are rational and equal.

(iv) $3x^2 + 7x - 13 = 0$ **(A.B)**

By comparing given equation with

$$ax^2 + bx + c = 0, \text{ we get}$$

$$a = 3, b = 7, c = -13$$

$$\begin{aligned} \text{Disc} &= b^2 - 4ac \\ &= (7)^2 - 4(3)(-13) \\ &= 49 + 156 \\ &= 205 \end{aligned}$$

Since disc > 0, but not a perfect sq. roots are irrational and unequal.

Verification:

$$3x^2 + 7x - 13 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-13)}}{2(3)}$$

$$= \frac{-7 \pm \sqrt{49 + 156}}{6}$$

$$= \frac{-7 \pm \sqrt{205}}{6}$$

Either

$$x = \frac{-7 - \sqrt{205}}{6} \quad \text{or} \quad x = \frac{-7 + \sqrt{205}}{6}$$

Hence roots are irrational and unequal.

Q.3 For what value of k, the expression

$$k^2x^2 + 2(k+1)x + 4 \text{ is perfect}$$

square. (A.B + K.B)

Solution:

$$k^2x^2 + 2(k+1)x + 4 = 0$$

By comparing given equation with

$$ax^2 + bx + c = 0, \text{ we get}$$

$$a = k^2, b = 2(k+1), c = 4$$

$$\begin{aligned} \text{Discriminant} &= b^2 - 4ac \\ &= [2(k+1)]^2 - 4(k^2)(4) \\ &= 4(k+1)^2 - 16k^2 \\ &= 4(k^2 + 2k + 1) - 16k^2 \\ &= 4k^2 + 8k + 4 - 16k^2 \\ &= 4 + 8k - 12k^2 \\ &= 4(1 + 2k - 3k^2) \end{aligned}$$

As expression is a perfect square, the discriminant = 0

$$\Rightarrow 4(1+2k-3k^2) = 0$$

$$1+2k-3k^2 = 0 \quad \because 4 \neq 0$$

$$1+3k-k-3k^2 = 0$$

$$1(1+3k) - k(1+3k) = 0$$

$$(1+3k)(1-k) = 0$$

Either

$$1+3k = 0 \quad \text{or} \quad 1-k = 0$$

$$3k = -1 \quad \text{or} \quad k = 1$$

$$k = -\frac{1}{3}$$

Result

$$k = 1, -\frac{1}{3}$$

Q.4 Find the value of k , if the roots of the following equations are equal.

(A.B + K.B)

Solution:

(i) $(2k-1)x^2 + 3kx + 3 = 0$

Here $a = 2k-1, b = 3k, c = 3$

$$\text{Disc} = b^2 - 4ac$$

$$= (3k)^2 - 4(2k-1)(3)$$

$$= 9k^2 - 24k + 12$$

Since roots are equal,

$$\text{Disc} = 0$$

$$\Rightarrow 9k^2 - 24k + 12 = 0$$

$$3(3k^2 - 8k + 4) = 0$$

$$3k^2 - 8k + 4 = 0 \quad \because 3 \neq 0$$

$$3k^2 - 6k - 2k + 4 = 0$$

$$3k(k-2) - 2(k-2) = 0$$

$$(k-2)(3k-2) = 0$$

Either

$$k-2 = 0 \quad \text{or} \quad 3k-2 = 0$$

$$\Rightarrow k = 2 \quad \text{or} \quad 3k = 2$$

$$k = \frac{2}{3}$$

Result:

$$k = 2, \frac{2}{3}$$

(ii) $x^2 + 2(k+2)x + (3k+4) = 0$

Here $a = 1, b = 2(k+2), c = 3k+4$

$$\text{Disc} = b^2 - 4ac$$

$$= [2(k+2)]^2 - 4(1)(3k+4)$$

$$= 4(k+2)^2 - 4(3k+4)$$

$$= 4(k^2 + 4k + 4) - 4(3k+4)$$

$$= 4[k^2 + 4k + 4 - 3k - 4]$$

$$= 4[k^2 + k]$$

$$= 4k(k+1)$$

Since roots are equal, disc = 0

$$4k(k+1) = 0$$

Either

$$4k = 0 \quad \text{or} \quad k+1 = 0$$

$$k = 0 \quad \text{or} \quad k = -1$$

Result:

$$k = 0, -1$$

(iii) $(3k+2)x^2 - 5(k+1)x + (2k+3) = 0$

(A.B + K.B)

Here

$$a = 3k+2, b = -5(k+1), c = 2k+3$$

$$\text{Disc} = b^2 - 4ac$$

$$= [-5(k+1)]^2 - 4(3k+2)(2k+3)$$

$$= 25(k+1)^2 - 4(6k^2 + 9k + 4k + 6)$$

$$= 25(k^2 + 2k + 1) - 4(6k^2 + 13k + 6)$$

$$= 25k^2 + 50k + 25 - 24k^2 - 52k - 24$$

$$= k^2 - 2k + 1$$

Since roots are equal, disc = 0

$$\Rightarrow k^2 - 2k + 1 = 0$$

$$(k-1)^2 = 0$$

Taking square root on both sides

$$k-1 = 0$$

$$k = 1$$

Result:

$$k = 1$$

Q.5 Show that the equation

$$x^2 + (mx + c)^2 = a^2 \text{ has equal roots,}$$

if $c^2 = a^2(1 + m^2)$ **(A.B + K.B)**

Proof

(ESD 20.6)

$$x^2 + (mx + c)^2 = a^2$$

$$x^2 + m^2x^2 + 2mcx + c^2 = a^2$$

$$(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$$

Here

$$A = 1 + m^2, B = 2mc, C = c^2 - a^2$$

$$\text{Disc} = B^2 - 4AC$$

$$= (2mc)^2 - 4(1 + m^2)(c^2 - a^2)$$

$$= 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - a^2m^2)$$

$$= \cancel{4m^2c^2} - 4c^2 + 4a^2 - \cancel{4m^2c^2} + 4a^2m^2$$

$$= -4c^2 + 4a^2 + 4a^2m^2$$

$$\text{Disc} = -4[a^2(1 + m^2)] + 4a^2 + 4m^2$$

$$(\because c^2 = a^2(1 + m^2))$$

$$= -4[a^2 + a^2m^2] + 4a^2 + 4a^2m^2$$

$$= -4a^2 - 4a^2m^2 + 4a^2 + 4a^2m^2$$

$$= 0$$

$$\text{Disc} = 0$$

$\therefore$ Roots are equal

Hence roots are equal, if $c^2 = a^2(1 + m^2)$

Proved

Q.6 Find the condition that the roots of the equation $(mx + c)^2 - 4ax = 0$ are equal. **(A.B + K.B + U.B)**

Solution:

$$(mx + c)^2 - 4ax = 0$$

$$m^2x^2 + 2mcx - 4ax + c^2 = 0$$

$$m^2x^2 + (2mc - 4a)x + c^2 = 0$$

$$\text{Here } A = m^2, B = 2mc - 4a, C = c^2$$

$$\text{Disc} = B^2 - 4AC$$

$$= (2mc - 4a)^2 - 4(m^2)(c^2)$$

$$= 4m^2c^2 - 16amc + 16a^2 - 4m^2c^2$$

$$= -16amc + 16a^2$$

$$= -16a(mc - a)$$

Since roots are equal, disc = 0

$$\Rightarrow -16a(mc - a) = 0$$

Either

$$mc - a = 0 \quad \text{or} \quad -16a = 0$$

$$a = mc \quad \text{or} \quad a = 0$$

Roots of the given equation are equal

If $a = mc$ or $a = 0$

Q.7 If the roots of the equation.

$$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$$

are equal, then $a = 0$ or

$$a^3 + b^3 + c^3 = 3abc.$$

(A.B + K.B + U.B)

Proof

$$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$$

Here

$$A = c^2 - ab, B = -2(a^2 - bc), C = b^2 - ac$$

$$\text{Disc} = B^2 - 4AC$$

$$= [-2(a^2 - bc)]^2 - 4(c^2 - ab)(b^2 - ac)$$

$$= 4(a^2 - bc)^2 - 4(b^2c^2 - ac^3 - ab^3 + a^2bc)$$

$$= 4(a^4 - 2a^2bc + b^2c^2) - 4(b^2c^2 - ac^3 - ab^3 + a^2bc)$$

$$= 4(a^4 - 2a^2bc + b^2c^2 - b^2c^2 + ac^3 + ab^3 - a^2bc)$$

$$= 4(a^4 + ab^3 + ac^3 - 3a^2bc)$$

$$= 4a(a^3 + b^3 + c^3 - 3abc)$$

Since roots are equal, disc = 0

$$\Rightarrow 4a(a^3 + b^3 + c^3 - 3abc) = 0$$

Either

$$4a = 0 \quad \text{or} \quad a^3 + b^3 + c^3 - 3abc = 0$$

$$a = 0 \quad \text{or} \quad a^3 + b^3 + c^3 = 3abc$$

Roots are equal if either

$$a = 0 \quad \text{or} \quad a^3 + b^3 + c^3 = 3abc$$

Q.8 Show that the roots of the following equations are rational.

Proof. (A.B + K.B + U.B)

(i) $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$

Here

$$A = a(b - c), B = b(c - a), C = c(a - b)$$

$$\text{Discriminant} = B^2 - 4AC$$

$$= [b(c - a)]^2 - 4[a(b - c)][c(a - b)]$$

$$= b^2(c - a)^2 - 4ac(b - c)(a - b)$$

$$\begin{aligned}
 &= b^2(c^2 - 2ac + a^2) - 4ac(ab - b^2 - ac + bc) \\
 &= b^2c^2 - 2ab^2c + a^2b^2 - 4a^2bc + 4ab^2c + 4a^2c^2 - 4abc^2 \\
 &= b^2c^2 + a^2b^2 + 4a^2c^2 + 2ab^2c - 4a^2bc - 4abc^2 \\
 &= (bc)^2 + (ab)^2 + (-2ac)^2 + 2(bc)(ab) \\
 &\quad + 2(ab)(-2ac) + 2(-2ac)(bc) \\
 \therefore a^2 + b^2 + c^2 - 2ab + 2bc + 2ca &= (a + b + c)^2 \\
 &= (bc + ab + 2ac)^2
 \end{aligned}$$

As discriminant > 0 and perfect square, roots are rational.

$$(ii) \quad (a + 2b)x^2 + 2(a + b + c)x + (a + 2c) = 0$$

Here

$$A = a + 2b, B = 2(a + b + c), C = a + 2c$$

$$\text{Disc} = B^2 - 4AC$$

$$\begin{aligned}
 &= [2(a + b + c)]^2 - 4(a + 2b)(a + 2c) \\
 &= 4(a + b + c)^2 - 4(a^2 + 2ac + 2ab + 4bc) \\
 &= 4(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) \\
 &\quad - 4(a^2 + 2ac + 2ab + 4bc) \\
 &= 4(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \\
 &\quad - a^2 - 2ac - 2ab - 4bc) \\
 &= 4(b^2 + c^2 - 2bc) \\
 &= 4(b - c)^2 \\
 &= [2(b - c)]^2 > 0
 \end{aligned}$$

Since disc is perfect square roots are rational.

Q.9 For all values of k , prove that the roots of the equation.

$$x^2 - 2\left(k + \frac{1}{k}\right)x + 3 = 0, k \neq 0 \text{ are real.}$$

Proof: (A.B + U.B)

Here

$$a = 1, b = -2\left(k + \frac{1}{k}\right), c = 3$$

$$\text{Discriminant} = b^2 - 4ac$$

$$= \left[-2\left(k + \frac{1}{k}\right)\right]^2 - 4(1)(3)$$

$$\begin{aligned}
 &= 4\left(k + \frac{1}{k}\right)^2 - 12 \\
 &= 4\left(k^2 + \frac{1}{k^2} + 2\right) - 12 \\
 &= 4\left[k^2 + \frac{1}{k^2} + 2 - 3\right] \\
 &= 4\left[k^2 + \frac{1}{k^2} - 1\right] \\
 &= 4\left[k^2 + \frac{1}{k^2} - 2 + 1\right] \\
 &= 4\left[\left(k - \frac{1}{k}\right)^2 + 1\right]
 \end{aligned}$$

> 0

As disc. > 0 , roots are real.

Q.10 Show that the roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$

are real. (A.B + U.B + K.B)

Proof:

$$(b - c)x^2 + (c - a)x + (a - b) = 0$$

$$\text{Here } A = b - c, B = c - a, C = a - b$$

$$\text{Disc} = B^2 - 4AC$$

$$\begin{aligned}
 &= (c - a)^2 - 4(b - c)(a - b) \\
 &= c^2 - 2ac + a^2 - 4(ab - b^2 - ac + bc) \\
 &= c^2 - 2ac + a^2 - 4ab + 4b^2 + 4ac - 4bc \\
 &= c^2 + a^2 + 4b^2 + 2ac - 4ab - 4bc \\
 &= (c)^2 + (a)^2 + (2b)^2 + 2(c)(a) + 2(a)(-2b) + 2(-2b)(c) \\
 \therefore a^2 + b^2 + c^2 + 2ab + 2bc + 2ca &= (a + b + c)^2 \\
 &= (c + a - 2b)^2 > 0
 \end{aligned}$$

Hence roots are real.

Derivation of Cube Roots of Unity

(A.B + K.B)

$$\text{Let } x = \sqrt[3]{1}$$

Taking cube on both sides

$$x^3 = 1$$

$$x^3 - 1 = 0$$

$$(x)^3 - (1)^3 = 0$$

$$(x-1)(x^2+x+1)=0$$

$$(x-1)(x^2+x+1)=0$$

Either

$$x-1=0 \quad \text{or} \quad x^2+x+1=0$$

$$x=1$$

Here $a=1, b=1, c=1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2} \quad \because \sqrt{-1} = i$$

Either

$$x = \frac{-1 + \sqrt{3}i}{2}, \quad x = \frac{-1 - \sqrt{3}i}{2}$$

$$= \omega$$

$$= \omega^2$$

$\therefore$ Cube root of unity are $1, \omega, \omega^2$

Note

(K.B)

We can write any complex cube root as ω (Omega), then other will be ω^2 .

Properties of Cube Root of Unity

(i) Proving that each complex cube root of unity is the square of other.

$$\text{i.e.} \left(\frac{-1 + \sqrt{-3}}{2} \right)^2 = \frac{-1 - \sqrt{-3}}{2}$$

$$\text{and} \left(\frac{-1 - \sqrt{-3}}{2} \right)^2 = \frac{-1 + \sqrt{-3}}{2}$$

(LHR 2014, GRW 2017, FSD 2016, 17, SGD 2015, 16, BWP 2017, MTN 2017)

(K.B + U.B + A.B)

Proof:

We have to prove

$$(i) \left(\frac{-1 + \sqrt{-3}}{2} \right)^2 = \frac{-1 - \sqrt{-3}}{2}$$

Consider

$$\left(\frac{-1 + \sqrt{-3}}{2} \right)^2$$

$$= \frac{(-1)^2 + (\sqrt{-3})^2 + 2(-1)(\sqrt{-3})}{4}$$

$$= \frac{1 + (-3) - 2\sqrt{-3}}{4}$$

$$= \frac{1 - 3 - 2\sqrt{-3}}{4}$$

$$= \frac{-2 - 2\sqrt{-3}}{4}$$

$$= \frac{2(-1 - \sqrt{-3})}{4}$$

$$= \frac{-1 - \sqrt{-3}}{2}$$

$$(ii) \left(\frac{-1 - \sqrt{-3}}{2} \right)^2 = \frac{-1 + \sqrt{-3}}{2}$$

Now consider

$$\left(\frac{-1 - \sqrt{-3}}{2} \right)^2$$

$$= \frac{(1)^2 + (\sqrt{-3})^2 - 2(-1)(-\sqrt{3})}{4}$$

$$= \frac{1 + (-3) + 2\sqrt{-3}}{4}$$

$$= \frac{1 - 3 + 2\sqrt{-3}}{4}$$

$$= \frac{-2 + 2\sqrt{-3}}{4}$$

$$= \frac{2(-1 + \sqrt{-3})}{4}$$

$$= \frac{-1 + \sqrt{-3}}{2}$$

Thus, each of the complex cube roots of unity is the square of the other.

Unit-2

Theory of Quadratic Equations

- (ii) Proving that product of three cube root of unity is one.

i.e. $1 \cdot \omega \cdot \omega^2 = 1$ **(K.B + U.B)**

Proof:

L.H.S = $1 \cdot \omega \cdot \omega^2$

By putting the values

$$= (1) \times \left(\frac{-1 + \sqrt{-3}}{2} \right) \left(\frac{-1 - \sqrt{-3}}{2} \right)$$

$$= \frac{(-1)^2 - (\sqrt{-3})^2}{4}$$

$$= \frac{1 - (-3)}{4}$$

$$= \frac{1 + 3}{4}$$

$$= \frac{4}{4}$$

$$= 1$$

$$= \text{R.H.S}$$

Proved

- (iii) Proving that sum of three cube roots of unit is zero.

i.e. $1 + \omega + \omega^2 = 0$

(LHR 2016, GRW 2017, FSD 2015, 17, SGD 2015, 16, BWP 2016, RWP 2015)

Proof:

L.H.S = $1 + \omega + \omega^2$

(By putting values)

$$= 1 + \frac{-1 + \sqrt{-3}}{2} + \frac{-1 - \sqrt{-3}}{2}$$

$$= \frac{2 + (-1) + \sqrt{-3} - 1 - \sqrt{-3}}{2}$$

$$= \frac{2 - 1 + \sqrt{-3} - 1 - \sqrt{-3}}{2}$$

$$= \frac{0}{2}$$

$$= 0$$

$$= \text{R.H.S}$$

Proved

Important Results

(K.B + U.B)

(i) $1 + \omega + \omega^2 = 0$

$$\Rightarrow 1 + \omega = -\omega^2$$

$$1 + \omega^2 = -\omega$$

$$\omega + \omega^2 = -1$$

(ii) $1 \cdot \omega \cdot \omega^2 = 1$

$$\Rightarrow \omega^3 = 1$$

(iii) $1 \cdot \omega \cdot \omega^2 = 1$

$$\omega \cdot \omega^2 = 1$$

$$\Rightarrow \omega = \frac{1}{\omega^2} \quad \text{or} \quad \omega^2 = \frac{1}{\omega}$$

Note: Complex cube roots are reciprocal of each other.

(iv) $\omega^4 = \omega^3 \cdot \omega = 1 \cdot \omega = \omega$

$$\omega^5 = \omega^3 \cdot \omega^2 = 1 \cdot \omega^2 = \omega^2$$

$$\omega^6 = (\omega^3)^2 = (1)^2 = 1 \quad \text{and so on}$$

$$\omega^{-16} = \frac{1}{\omega^{16}} = \frac{1}{\omega^{15} \cdot \omega} = \frac{1}{(\omega^3)^5 \cdot \omega} = \frac{1}{1 \cdot \omega} = \frac{1}{\omega} = \omega^2$$

Example 1: (Page # 25)

Evaluate: $(-1 + \sqrt{-3})^8 + (-1 - \sqrt{-3})^8$ **(A.B)**

Solution:

$$(-1 + \sqrt{-3})^8 + (-1 - \sqrt{-3})^8$$

$$= \left[2 \left(\frac{-1 + \sqrt{-3}}{2} \right) \right]^8 + \left[2 \left(\frac{-1 - \sqrt{-3}}{2} \right) \right]^8$$

$$= (2\omega)^8 + (2\omega^2)^8$$

$$= 2^8 \omega^8 + 2^8 \omega^{16}$$

$$= 2^8 (\omega^6 \cdot \omega^2 + \omega^{15} \cdot \omega)$$

$$= 256 ((\omega^3)^2 \cdot \omega^2 + (\omega^3)^5 \cdot \omega)$$

$$= 256 ((1)^2 \cdot \omega^2 + (1)^5 \cdot \omega) \because \omega^3 = 1$$

$$256 (\omega^2 + \omega)$$

$$= 256 (-1) \because 1 + \omega + \omega^2 = 0$$

$$= -256$$

Unit-2

Theory of Quadratic Equations

Example 2: (Page # 25)

(A.B)

To Prove

$$x^3 - y^3 = (x - y)(x - \omega y)(x - \omega^2 y)$$

Proof:

$$\begin{aligned} \text{R.H.S} &= (x - y)(x - \omega y)(x - \omega^2 y) \\ &= (x - y)(x^2 - \omega xy - \omega^2 xy + \omega^3 y^2) \\ &= (x - y)(x^2 - (\omega + \omega^2)xy + y^2) \\ &= (x - y)(x^2 - (-1)xy + y^2) \because 1 + \omega + \omega^2 = 0 \\ &= (x - y)(x^2 + xy + y^2) \\ &= x^3 - y^3 = \text{L.H.S} \end{aligned}$$

Exercise 2.2

Q.1 Find the cube roots of -1, 8, -27, 64.

(LHR 2015) (K.B + A.B)

Solution:

(i) Finding cube roots of -1

$$\text{Let } x = \sqrt[3]{-1}$$

Taking cube on both sides

$$x^3 = -1$$

$$x^3 + 1 = 0$$

$$\therefore a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$(x + 1)(x^2 - x + 1) = 0$$

Either

$$x + 1 = 0 \rightarrow (i) \text{ or } x^2 - x + 1 = 0 \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow$$

$$x = -1$$

$$\text{Equation (ii)} \Rightarrow$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1 - 4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

Either

$$x = \frac{1 - \sqrt{-3}}{2} \text{ or } x = \frac{1 + \sqrt{-3}}{2}$$

$$x = -\left(\frac{1 + \sqrt{-3}}{2}\right) \text{ or } x = -\left(\frac{1 - \sqrt{-3}}{2}\right)$$

$$= -\omega$$

$$= -\omega^2$$

$\therefore$ Cube roots of -1 are -1, $-\omega$, $-\omega^2$

(ii) Finding cube roots of 8

(K.B + A.B)

$$\text{Let } x = \sqrt[3]{8}$$

Taking cube on both sides

$$x^3 = 8$$

$$x^3 - 8 = 0$$

$$x^3 - 2^3 = 0$$

$$\therefore a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$(x - 2)(x^2 + 2x + 4) = 0$$

Either

$$x - 2 = 0 \rightarrow (i) \text{ or } x^2 + 2x + 4 = 0 \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow$$

$$x = 2$$

$$\text{Equation (ii)} \Rightarrow$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{-3}}{2}$$

Either

$$x = 2\left(\frac{-1 + \sqrt{-3}}{2}\right) \text{ or } x = 2\left(\frac{-1 - \sqrt{-3}}{2}\right)$$

$$= 2\omega$$

$$= 2\omega^2$$

$\therefore$ Cube roots of 8 are 2, 2ω , $2\omega^2$

(iii) Finding cube roots of -27

(SGT 2014) (K.B + A.B)

Let

$$x = \sqrt[3]{-27}$$

$$x^3 - 27 = 0$$

$$\therefore a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$x^3 + 3^3 = 0$$

$$(x + 3)(x^2 - 3x + 9) = 0$$

Either

$$x + 3 = 0 \rightarrow (i) \text{ or } x^2 - 3x + 9 = 0 \rightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow$$

$$x = -3$$

Equation (ii) $\Rightarrow$

$$\begin{aligned}
 x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)} \\
 &= \frac{3 \pm \sqrt{9-36}}{2} \\
 &= \frac{3 \pm \sqrt{-27}}{2} \\
 &= \frac{3 \pm 3\sqrt{-3}}{2} \\
 &= -3 \left(\frac{-1 \pm \sqrt{-3}}{2} \right)
 \end{aligned}$$

Either

$$x = -3 \left(\frac{-1 + \sqrt{-3}}{2} \right) \text{ or } x = -3 \left(\frac{-1 - \sqrt{-3}}{2} \right)$$

$$x = -3\omega \quad x = -3\omega^2$$

 $\therefore$ Cube roots of -27 are $-3, -3\omega, -3\omega^2$

(iv) Finding cube roots of 64.

(LHR 2015) (K.B + A.B)

Let

$$x = \sqrt[3]{64}$$

Taking cube on both sides

$$x^3 = 64$$

$$x^3 - 64 = 0$$

$$x^3 - 4^3 = 0$$

$$\therefore a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$(x-4)(x^2 + 4x + 16) = 0$$

Either

$$x - 4 = 0 \rightarrow (i) \text{ or } x^2 + 4x + 16 = 0 \rightarrow (ii)$$

Equation (i) $\Rightarrow$

$$x = 4$$

Equation (ii) $\Rightarrow$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(16)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16-64}}{2}$$

$$x = \frac{-4 \pm \sqrt{-48}}{2}$$

$$x = \frac{-4 \pm 4\sqrt{-3}}{2}$$

$$x = 4 \left(\frac{-1 \pm \sqrt{-3}}{2} \right)$$

Either

$$x = 4 \left(\frac{-1 + \sqrt{-3}}{2} \right) \text{ or } x = 4 \left(\frac{-1 - \sqrt{-3}}{2} \right)$$

$$x = 4\omega \quad \text{or} \quad x = 4\omega^2$$

 $\therefore$ Cube roots of 64 are $4, 4\omega, 4\omega^2$

Q.2 Evaluate: (K.B + A.B)

(FSD 2017, BWP 2016, RWP 2015, MTN 2014, 15, 17, D.G.K 2015, 17)

(i) $(1 - \omega - \omega^2)^7$

(ii) $(1 - 3\omega - 3\omega^2)^5$

(iii) $(9 + 4\omega + 4\omega^2)^3$

(iv) $(2 + 2\omega - 2\omega^2)(3 - 3\omega + 3\omega^2)$

(v) $(-1 + \sqrt{-3})^6 + (-1 - \sqrt{-3})^6$

(vi) $\left(\frac{1 + \sqrt{-3}}{2} \right)^9 + \left(\frac{1 - \sqrt{-3}}{2} \right)^9$

(vii) $\omega^{37} + \omega^{38} - 5$

(viii) $\omega^{-13} + \omega^{-17}$

Solution:

(i) $(1 - \omega - \omega^2)^7$

(GRW 2014, 16, 17, FSD 2016, BWP 2017, MTN 2017)

$$= [1 - (\omega + \omega^2)]$$

$$= [1 - (-1)]^7 \quad \because \omega + \omega^2 = -1$$

$$= (1 + 1)^7$$

$$= 2^7$$

$$= 128$$

(ii) $(1 - 3\omega - 3\omega^2)^5$

$$= [1 - 3(\omega + \omega^2)]^5$$

$$= [1 - 3(-1)]^5 \quad \because \omega + \omega^2 = -1$$

$$= (1 + 3)^5$$

$$= (4)^5$$

$$= 1024$$

Unit-2

Theory of Quadratic Equations

(iii) $(9 + 4\omega + 4\omega^2)^3$
(GRW 2014, RWP 2017, FSD 2017, BWP 2016)

$$\begin{aligned} &= [9 + 4(\omega + \omega^2)]^3 \\ &= [9 + 4(-1)]^3 \quad \because \omega + \omega^2 = -1 \\ &= (9 - 4)^3 \\ &= (5)^3 \\ &= 125 \end{aligned}$$

(iv) $(2 + 2\omega - 2\omega^2)(3 - 3\omega + 3\omega^2)$
(SWL 2017)

$$\begin{aligned} &= 2(1 + \omega - \omega^2)3(1 - \omega + \omega^2) \\ &= 6[(1 + \omega) - \omega^2][(1 + \omega^2) - \omega] \\ &\because 1 + \omega + \omega^2 = 0 \\ &= 6(-\omega^2 - \omega^2)(-\omega - \omega) \\ &= 6(-2\omega^2)(-2\omega) \\ &= 6(4\omega^3) \\ &= 24\omega^3 \\ &= 24(1) \quad \because \omega^3 = 1 \\ &= 24 \end{aligned}$$

(v) $(-1 + \sqrt{-3})^6 + (-1 - \sqrt{-3})^6$
 $\because \omega = \frac{-1 + \sqrt{-3}}{2}, \omega^2 = \frac{-1 - \sqrt{-3}}{2}$

$$\begin{aligned} &= (2\omega)^6 + (2\omega^2)^6 \\ &= 2^6 \omega^6 + 2^6 \omega^{12} \\ &= 2^6 (\omega^6 + \omega^{12}) \\ &= 64 \left[(\omega^3)^2 + (\omega^3)^4 \right] \\ &= 64 \left[(1)^2 + (1)^4 \right] \quad \because \omega^3 = 1 \\ &= 64(1 + 1) \\ &= 64(2) \\ &= 128 \end{aligned}$$

(vi) $\left(\frac{1 + \sqrt{-3}}{2}\right)^9 + \left(\frac{1 - \sqrt{-3}}{2}\right)^9$
 $\omega = \frac{-1 + \sqrt{-3}}{2}, \omega^2 = \frac{-1 - \sqrt{-3}}{2}$

$$\begin{aligned} &= (\omega)^9 + (\omega^2)^9 \\ &= (\omega)^9 + (\omega)^{18} \\ &= (\omega^3)^3 + (\omega^3)^6 \\ &= (1)^3 + (1)^6 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

(vii) $\omega^{37} + \omega^{38} - 5$
(LHR 2015, SWL 2016, MTN 2015, D.G.K 2016, 17)

$$\begin{aligned} &= \omega \cdot \omega^{36} + \omega^2 \cdot \omega^{36} - 5 \\ &= (\omega + \omega^2) \cdot \omega^{36} - 5 \\ &= (-1) \left[(\omega^3)^{12} \right] - 5 \\ &= -1 \left[(1)^{12} \right] - 5 \quad \because \omega^3 = 1 \\ &= -1(1) - 5 \\ &= -1 - 5 \\ &= -6 \end{aligned}$$

(viii) $\omega^{-13} + \omega^{-17}$

$$\begin{aligned} &= \frac{1}{\omega^{13}} + \frac{1}{\omega^{17}} \\ &= \frac{1}{\omega \cdot \omega^{12}} + \frac{1}{\omega^2 \cdot \omega^{15}} \\ &= \frac{1}{\omega (\omega^3)^4} + \frac{1}{\omega^2 (\omega^3)^5} \\ &= \frac{1}{\omega (1)^4} + \frac{1}{\omega^2 (1)^5} \quad \because \omega^3 = 1 \\ &= \frac{1}{\omega (1)} + \frac{1}{\omega^2 (1)} \\ &= \frac{1}{\omega} + \frac{1}{\omega^2} \\ &= \omega^2 + \omega \quad \because \omega \cdot \omega^2 = 1 \\ &= -1 \quad \because 1 + \omega + \omega^2 = 0 \end{aligned}$$

Unit-2

Theory of Quadratic Equations

Q.3 Prove that (K.B + U.B)

$$x^3 + y^3 = (x+y)(x+\omega y)(x+\omega^2 y)$$

(SGD 2015, BWP 2016)

Proof:

R.H.S

$$\begin{aligned} &= (x+y)(x+\omega y)(x+\omega^2 y) \\ &= (x+y)(x^2 + \omega^2 xy + \omega xy + \omega^3 y^2) \\ &= (x+y)[x^2 + (\omega^2 + \omega)xy + (1)y^2] \because \omega^3 = 1 \\ &= (x+y)[x^2 + (-1)xy + y^2] \because 1 + \omega + \omega^2 = 0 \\ &= (x+y)[x^2 - xy + y^2] \\ &= x^3 + y^3 \\ &= \text{L.H.S} \end{aligned}$$

Proved

Q.4 Prove that $x^3 + y^3 + z^3 - 3xyz$

$$= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$$

(K.B + U.B)

Proof:

R.H.S

$$\begin{aligned} &= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) \\ &= (x+y+z)(x^2 + \omega^2 xy + \omega xz + \omega xy + \omega^3 y^2 \\ &\quad + \omega^2 yz + \omega^2 xz + \omega^4 yz + \omega^3 z^2) \\ &= (x+y+z)(x^2 + \omega^3 y^2 + \omega^3 z^2 + \omega^2 xy + \omega xy \\ &\quad + \omega^2 yz + \omega^4 yz + \omega xz + \omega^2 xz) \\ &= (x+y+z)(x^2 + (1)y^2 + (1)z^2 + (\omega^2 + \omega)xy \\ &\quad + (\omega^2 + \omega^4)yz + (\omega + \omega^2)xz) \because \omega^3 = 1 \\ &= (x+y+z)[x^2 + y^2 + z^2 + (-1)xy + (\omega^2 + \omega)yz + (-1)xz] \\ &\quad \because 1 + \omega + \omega^2 = 0, \omega^4 = \omega, \omega^3 = \omega \\ &= (x+y+z)(x^2 + y^2 + z^2 - xy + (-1)yz - xz) \\ &= (x+y+z)(x^2 + y^2 + z^2 - xy - yz - xz) \\ &= x^3 + y^3 + z^3 - 3xyz \\ &= \text{L.H.S} \end{aligned}$$

Proved

Q.5 Prove that

$$(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8) \dots 2n$$

(K.B + A.P + U.B)

Proof:

L.H.S

$$\begin{aligned} &= (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8) \dots 2n \text{ factors} \\ &\because \omega^4 = \omega \times \omega^3 = \omega, \omega^8 = \omega^2 \times \omega^6 = \omega^2 \times (\omega^3)^2 = \omega^2 (1) = \omega^2 \\ &= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2) \dots 2n \text{ factors} \\ &= [(1+\omega)(1+\omega)(1+\omega) \dots n \text{ factors}] \\ &\quad [(1+\omega^2)(1+\omega^2)(1+\omega^2) \dots n \text{ factors}] \\ &= (1+\omega)^n (1+\omega^2)^n \\ &= [(1+\omega)(1+\omega^2)]^n \\ &= (1+\omega+\omega^2+\omega^3)^n \\ &= (0+1)^n \because 1+\omega+\omega^2=0, \omega^3=1 \\ &= (1)^n \\ &= 1 \\ &= \text{R.H.S} \end{aligned}$$

Relation between Roots and Co-efficient of a Quadratic Equation (K.B + U.B)

Roots of standard quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\text{If } \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Sum of roots

$$\begin{aligned} S &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b + \sqrt{b^2 - 4ac} + (-b) - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2b}{2a} \\ &= -\frac{b}{a} \\ \Rightarrow S &= -\frac{b}{a} \end{aligned}$$

Unit-2

Theory of Quadratic Equations

Product of roots

$$P = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} \right)$$

$$\therefore (a+b)(a-b) = a^2 - b^2$$

$$= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{(2a)^2}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$\Rightarrow P = \frac{c}{a}$$

Note (K.B + U.B)

(i) $S = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

(ii) $P = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$

Quadratic Equation with Given Roots

(K.B + U.B)

A quadratic equation whose roots are given can be obtained by using formula

$$x^2 - Sx + P = 0 \quad \text{Or}$$

$$x^2 - (\text{Sum of roots})x + \text{Product of roots} = 0$$

Example 1: (Page # 26) (A.B)

Without solving, find the sum and product of roots of the equation

$$3x^2 - 5x + 7 = 0$$

Solution:

$$3x^2 - 5x + 7 = 0$$

Here $a = 3, b = -5, c = 7$

$$\text{Sum of roots} = S = -\frac{b}{a}$$

$$= -\frac{-5}{3} = \frac{5}{3}$$

$$\text{Product of roots} = P = \frac{c}{a}$$

$$= \frac{7}{3}$$

To find unknown values involved in a given Quadratic Equation

Example 1: (Page # 27) (A.B)

Find the value of h , if the sum of roots is equal to 3-times the product of roots of the equation:

$$3x^2 + (9 - 6h)x + 5h = 0.$$

Solution:

$$3x^2 + (9 - 6h)x + 5h = 0$$

Here $a = 3, b = 9 - 6h, c = 5h$

Let α, β be the roots of given equation

$$\text{Then } \alpha + \beta = -\frac{b}{a} = -\frac{9 - 6h}{3} = \frac{6h - 9}{3}$$

$$\text{And } \alpha\beta = \frac{c}{a} = \frac{5h}{3}$$

According to given condition

$$\text{Sum of roots} = 3(\text{Product of roots})$$

$$\alpha + \beta = 3\alpha\beta$$

$$\frac{6h - 9}{3} = 3\left(\frac{5h}{3}\right)$$

$$6h - 9 = 15h$$

$$6h - 15h = 9$$

$$-9h = 9$$

$$h = -1$$

Exercise 2.3

Q.1 Without solving, find the sum and the product of the roots of the following quadratic equations.

(i) $x^2 - 5x + 3 = 0$ (MTN 2017) (A.B)

Here $a = 1, b = -5, c = 3$

$$\text{Sum of roots} = -\frac{b}{a}$$

$$= -\frac{(-5)}{1}$$

$$= 5$$

$$\text{Product of roots} = \frac{c}{a}$$

$$= \frac{3}{1}$$

$$= 3$$

Unit-2

Theory of Quadratic Equations

- (ii) $3x^2 + 7x - 11 = 0$ **(A.B)**
(LHR 2017, SWL 2017, SGD 2016, D.G.K 2014)

Here

$$a = 3, b = 7, c = -11$$

$$\text{Sum of roots} = -\frac{b}{a}$$

$$= -\frac{7}{3}$$

$$\text{Product of roots} = \frac{c}{a}$$

$$= \frac{-11}{3}$$

$$= -\frac{11}{3}$$

- (iii) $px^2 - qx + r = 0$ **(A.B)**
(LHR 2014, GRW 2014, SWL 2016, MTN 2017, SGD 2016)

Here $a = p, b = -q, c = r$

$$\text{Sum of roots} = -\frac{b}{a}$$

$$= -\frac{(-q)}{p}$$

$$= \frac{q}{p}$$

$$\text{Product of roots} = \frac{c}{a}$$

$$= \frac{r}{p}$$

- (iv) $(a+b)x^2 - ax + b = 0$ **(A.B)**
(BWP 2014, 17)

Here

$$A = a+b, B = -a, C = b$$

$$\text{Sum of roots} = -\frac{B}{A}$$

$$= -\frac{-a}{a+b}$$

$$= \frac{a}{a+b}$$

$$\text{Product of roots} = \frac{C}{A}$$

$$= \frac{b}{a+b}$$

- (v) $(l+m)x^2 + (m+n)x + n-1 = 0$

Here

$$a = l+m, b = m+n, c = n-1$$

$$\text{Sum of roots} = S = -\frac{b}{a}$$

$$= -\frac{m+n}{l+m}$$

$$\text{Product of roots} = P = \frac{c}{a}$$

$$= \frac{n-1}{l+m}$$

- (vi) $7x^2 - 5mx + 9n = 0$ **(A.B)**

Here

$$a = 7, b = -5m, c = 9n$$

$$\text{Sum of roots} = -\frac{b}{a}$$

$$= -\frac{-5m}{7}$$

$$= \frac{5m}{7}$$

$$\text{Product of roots} = \frac{c}{a}$$

$$= \frac{9n}{7}$$

Q.2 Find the value of k, if

- (i) **Sum of the roots of the equation**
 $2kx^2 - 3x + 4k = 0$ **is twice the**
product of the roots. (A.B)

Solution:

Let α, β be the roots of equation

$$2kx^2 - 3x + 4k = 0$$

Here

$$a = 2k, b = -3, c = 4k$$

$$S = \alpha + \beta = -\frac{b}{a}$$

$$= -\frac{-3}{2k}$$

$$= \frac{3}{2k}$$

$$P = \alpha\beta = \frac{c}{a}$$

$$= \frac{4k}{2k}$$

$$= 2$$

According to given condition

$$S = 2P$$

$$\frac{3}{2k} = 2 \times 2$$

$$\frac{3}{2k} = \frac{4}{1}$$

$$3 \times 1 = 4 \times 2k$$

$$3 = 8k$$

$$\frac{3}{8} = k$$

- (ii) **Sum of the roots of the equation**
 $x^2 + (3k - 7)x + 5k = 0$ is $\frac{3}{2}$ times the
product of the roots. (A.B)

Solution:

$$x^2 + (3k - 7)x + 5k = 0$$

Here, $a = 1$, $b = 3k - 7$, $c = 5k$

$$\begin{aligned} \text{Sum of roots} = S &= \frac{-b}{a} \\ &= \frac{3k - 7}{1} \\ &= -3k + 7 \end{aligned}$$

$$\begin{aligned} \text{Product of roots} = P &= \frac{c}{a} \\ &= \frac{5k}{1} \\ &= 5k \end{aligned}$$

According to given condition:

$$S = \frac{3}{2}P$$

$$-3k + 7 = \frac{3}{2}(5k)$$

$$2(-3k + 7) = 3(5k)$$

$$-6k + 14 = 15k$$

$$14 = 15k + 6k$$

$$14 = 21k$$

$$\frac{14}{21} = k$$

$$\frac{2}{3} = k$$

Or $k = \frac{2}{3}$

Q.3 Find k,

- (i) **If sum of squares of the roots of the equation $4x^2 + 3kx - 8 = 0$ is 2.**

Solution (FSD 2015) (A.B)

$$4x^2 + 3kx - 8 = 0$$

Here $a = 4$, $b = 3k$, $c = -8$

Let α , β be the roots

$$\begin{aligned} \text{Sum of roots} = \alpha + \beta &= -\frac{b}{a} \\ &= -\frac{3k}{4k} \end{aligned}$$

$$\Rightarrow \alpha + \beta = -\frac{3}{4}$$

$$\text{Product of roots} = \alpha\beta = \frac{c}{a}$$

$$= \frac{-8}{4k}$$

$$\Rightarrow \alpha\beta = \frac{-2}{k}$$

According to given condition

$$\alpha^2 + \beta^2 = 2$$

$$(\alpha + \beta)^2 - 2\alpha\beta = 2$$

Putting the values

$$\left(-\frac{3}{4}\right)^2 - 2\left(-\frac{2}{k}\right) = 2$$

$$\frac{9}{16} + \frac{4}{k} = 2$$

$$\frac{4}{k} = 2 - \frac{9}{16}$$

$$\frac{4}{k} = \frac{32 - 9}{16}$$

$$\frac{4}{k} = \frac{23}{16}$$

$$\Rightarrow \frac{k}{4} = \frac{16}{23}$$

$$k = \frac{64}{23}$$

Unit-2

Theory of Quadratic Equations

- (ii) Sum of the squares of the roots of the equation $x^2 - 2kx + (2k+1) = 0$ is 6. **(A.B)**

Solution:

Here

$$a=1, b=-2k, c=2k+1$$

Let, α, β be the roots of given equation.

Then

$$\begin{aligned}\text{Sum of roots} &= \alpha + \beta = -\frac{b}{a} \\ &= -\left(\frac{-2k}{1}\right) \\ &= 2k\end{aligned}$$

$$\begin{aligned}\text{Product of roots} &= \alpha\beta = \frac{c}{a} \\ &= \frac{2k+1}{1} \\ &= 2k+1\end{aligned}$$

According to given condition

$$\alpha^2 + \beta^2 = 6$$

$$\text{Or } (\alpha + \beta)^2 - 2\alpha\beta = 6$$

Putting the values

$$(2k)^2 - 2(2k+1) = 6$$

$$4k^2 - 4k - 2 - 6 = 0$$

$$4k^2 - 4k - 8 = 0$$

$$4(k^2 - k - 2) = 0$$

$$\text{Or } k^2 - k - 2 = 0$$

$$k^2 - 2k + k - 2 = 0$$

$$k(k-2) + 1(k-2) = 0$$

$$(k-2)(k+1) = 0$$

Either

$$k-2=0 \quad \text{or} \quad k+1=0$$

$$k+1=0$$

$$k = -1$$

Result

$$k = 2, -1$$

Q.4 Find p, if

- (i) The roots of the equation $x^2 - x + p^2 = 0$ differ by unity.

(FSD 2015) (A.B)

Let α, β be the roots of given equation.

$$\text{Here } a=1, b=-1, c=p^2$$

$$\begin{aligned}\text{Then } \alpha + \beta &= -\frac{b}{a} \\ &= -\left(\frac{-1}{1}\right) \\ &= 1\end{aligned}$$

$$\begin{aligned}\alpha\beta &= \frac{c}{a} \\ &= \frac{p^2}{1} \\ &= p^2\end{aligned}$$

According to given condition

$$\alpha - \beta = 1$$

Taking square on both sides

$$(\alpha - \beta)^2 = 1$$

$$\therefore (\alpha + \beta)^2 - (\alpha - \beta)^2 = 4\alpha\beta$$

$$(\alpha + \beta)^2 - 4\alpha\beta = 1$$

Putting the values

$$(1)^2 - 4p^2 = 1$$

$$1 - 4p^2 = 1$$

$$1 - 1 = 4p^2$$

$$0 = 4p^2$$

Or

$$p^2 = 0$$

By taking square root

$$p = 0$$

Result.

$$p = 0$$

- (ii) The roots of the equation $x^2 + 3x + P - 2 = 0$ differ by 2.

Solution: (A.B)

$$x^2 + 3x + P - 2 = 0$$

$$\text{Here } a=1, b=3, c=P-2$$

Let roots of given equation are α, β

$$\text{Then sum of roots} = \alpha + \beta = -\frac{b}{a}$$

$$= -\frac{3}{1}$$

$$\Rightarrow \alpha + \beta = -3$$

$$\text{Product of roots} = \alpha\beta = \frac{c}{a}$$

$$= \frac{P-2}{1}$$

According to given condition

$$\alpha - \beta = 2$$

Taking square of both sides

$$(\alpha - \beta)^2 = 4$$

$$(\alpha + \beta)^2 - 4\alpha\beta = 4$$

Putting the values

$$(-3)^2 - 4(P-2) = 4$$

$$9 - 4P + 8 = 4$$

$$9 + 8 - 4 = 4P$$

$$13 = 4P$$

$$\text{Or } P = \frac{13}{4}$$

$$\text{Result: } P = \frac{13}{4}$$

Q.5 Find m, if

(i) The roots of the equation

$$x^2 - 7x + 3m - 5 = 0 \text{ satisfy the}$$

$$\text{relation } 3\alpha + 2\beta = 4 \quad \textbf{(A.B)}$$

Solution:

$$x^2 - 7x + 3m - 5 = 0$$

$$\text{Here } a = 1, b = -7, c = 3m - 5$$

Let α, β be the roots of given equation

$$\text{Then sum of roots} = \alpha + \beta = -\frac{b}{a}$$

$$= -\frac{-7}{1}$$

$$\Rightarrow \alpha + \beta = 7 \rightarrow (i)$$

$$\text{Product of roots} = \alpha\beta = \frac{c}{a}$$

$$= \frac{3m-5}{1}$$

$$\Rightarrow \alpha\beta = 3m - 5 \rightarrow (ii)$$

According to given condition

$$3\alpha + 2\beta = 4 \rightarrow (iii)$$

Multiply equation (i) by '2'

$$2\alpha + 2\beta = 14 \rightarrow (iv)$$

Sub. Equation (ii) & (iv)

$$3\alpha + 2\beta = 4$$

$$\underline{2\alpha + 2\beta = 14}$$

$$\alpha = -10$$

Put in equation (i)

$$\alpha + \beta = 7$$

$$-10 + \beta = 7$$

$$\beta = 7 + 10$$

$$\beta = 17$$

Putting the values of α and β in equation (ii)

$$\alpha\beta = 3m - 5$$

$$-10(17) = 3m - 5$$

$$-170 + 5 = 3m$$

$$\frac{-165}{3} = m$$

$$\text{Or } m = -55$$

Result:

$$m = -55$$

(ii) The roots of the equation
 $x^2 + 7x + 3m - 5 = 0$ **satisfy the**
relation $3\alpha - 2\beta = 4$ **(A.B)**

Let α, β be the roots of given equation

Here

$$a = 1, b = 7, c = 3m - 5$$

Then

$$\alpha + \beta = -\frac{b}{a}$$

$$= -\frac{7}{1}$$

$$\alpha + \beta = -7 \rightarrow (i)$$

$$\alpha\beta = \frac{c}{a}$$

$$= \frac{3m-5}{1}$$

$$\alpha\beta = 3m - 5 \rightarrow (ii)$$

Also given

$$3\alpha - 2\beta = 4 \rightarrow (iii)$$

Multiply equation (i) by 2

$$2\alpha + 2\beta = -14 \longrightarrow (iv)$$

Adding equation (iii) and (iv)

$$3\alpha - 2\beta = 4$$

$$2\alpha + 2\beta = -14$$

$$5\alpha = -10$$

$$\alpha = -\frac{10}{5}$$

$$\Rightarrow \alpha = -2$$

Put in equation (i)

$$\alpha + \beta = -7$$

$$-2 + \beta = -7$$

$$\beta = -7 + 2$$

$$\beta = -5$$

Now putting the values in equation (ii)

$$\alpha\beta = 3m - 5$$

$$-2(-5) = 3m - 5$$

$$10 + 5 = 3m$$

$$15 = 3m$$

$$\frac{15}{3} = m$$

$$5 = m$$

Result:

$$m = 5$$

(iii) $3x^2 - 2x + 7m + 2 = 0$ **(A.B)**

Here $a = 3, b = -2, c = 7m + 2$

Let α, β be the roots of given equation

$$\begin{aligned} \text{Then sum of roots} &= \alpha + \beta = -\frac{b}{a} \\ &= -\frac{-2}{3} \end{aligned}$$

$$\Rightarrow \alpha + \beta = \frac{2}{3} \longrightarrow (i)$$

$$\begin{aligned} \text{Product of roots} &= \alpha\beta = \frac{c}{a} \\ &= \frac{7m + 2}{3} \longrightarrow (ii) \end{aligned}$$

Also given

$$7\alpha - 3\beta = 18 \longrightarrow (iii)$$

Multiply equation (i) by '3'

$$3\alpha + 3\beta = 2 \longrightarrow (iv)$$

Adding equation (iii) and (iv)

$$7\alpha - 3\beta = 18$$

$$3\alpha + 3\beta = 2$$

$$10\alpha = 20$$

$$\Rightarrow \alpha = 2$$

Put in equation (i)

$$\alpha + \beta = \frac{2}{3}$$

$$2 + \beta = \frac{2}{3}$$

$$\beta = \frac{2}{3} - 2$$

$$= \frac{2 - 6}{3}$$

$$\beta = \frac{-4}{3}$$

Now putting the values of α and β in equation (ii)

$$2\left(\frac{-4}{3}\right) = \frac{7m + 2}{3}$$

$$-8 = 7m + 2$$

$$-8 - 2 = 7m$$

$$-10 = 7m$$

$$\frac{-10}{7} = m$$

Result:

$$m = \frac{-10}{7}$$

Q.6 Find m, if sum and product of the roots of the following equations is equal to a given number λ .

(A.B)

(i) $(2m+3)x^2 + (7m-5)x + (3m-10) = 0$

Here

$$a = 2m + 3, b = 7m - 5 \text{ and } c = 3m - 10$$

Let α, β be the roots of given equation,

$$\begin{aligned} \text{Then } \alpha + \beta &= -\frac{b}{a} \\ &= -\frac{7m-5}{2m+3} \\ &= \frac{5-7m}{2m+3} \end{aligned}$$

$$\begin{aligned}\text{And } \alpha\beta &= \frac{c}{a} \\ &= \frac{3m-10}{2m+3}\end{aligned}$$

According to given condition

$$\alpha + \beta = \lambda = \alpha\beta$$

(By using transitive property)

$$\Rightarrow \alpha + \beta = \alpha\beta$$

Putting the values

$$\frac{5-7m}{2m+3} = \frac{3m-10}{2m+3}$$

Or

$$5-7m = 3m-10$$

(By using cancellation property)

$$-7m - 3m = -10 - 5$$

$$-10m = -15$$

$$m = \frac{-15}{-10}$$

$$\Rightarrow m = \frac{3}{2}$$

$$\text{Result: } m = \frac{3}{2}$$

$$(ii) \quad 4x^2 - (3+5m)x - (9m-17) = 0$$

(A.B)

Here

$$a = 4, b = -(3+5m), c = -(9m-17)$$

Let α, β be the roots

$$\begin{aligned}\text{Then } \alpha + \beta &= -\frac{b}{a} \\ &= -\frac{-(3+5m)}{4} \\ &= \frac{3+5m}{4}\end{aligned}$$

$$\begin{aligned}\text{And } \alpha\beta &= \frac{c}{a} \\ &= \frac{-(9m-17)}{4}\end{aligned}$$

According to given condition:

$$\alpha + \beta = \lambda \quad \text{and} \quad \alpha\beta = \lambda$$

$$\Rightarrow \alpha + \beta = \alpha\beta$$

$\therefore$ Transitive property of equality

Putting the values

$$\frac{3+5m}{4} = \frac{-(9m-17)}{4}$$

$$\Rightarrow 3+5m = -9m+17$$

$$5m + 9m = 17 - 3$$

$$14m = 14$$

$$\Rightarrow m = 1$$

Result: $m = 1$

Symmetric Function of the Roots of Quadratic Equation (K.B + U.B)

(MTN 2014, FSD 2014)

A function in which the roots involved are such that the value of the expression remains same, when roots are interchanged is called symmetric function. i.e. $f(\alpha, \beta) = f(\beta, \alpha)$

Some symmetric functions are:

$$\alpha^2 + \beta^2, \quad \alpha^3 + \beta^3, \quad \frac{1}{\alpha} + \frac{1}{\beta}$$

Example: (Page # 30) (K.B + U.B)

Verify that

$$\alpha^2 + \beta^2 + 2\alpha\beta \text{ is Symmetric}$$

Verification:

$$\text{Let } f(\alpha, \beta) = \alpha^2 + \beta^2 + 2\alpha\beta \rightarrow (i)$$

$$f(\beta, \alpha) = (\beta)^2 + (\alpha)^2 + 2\beta\alpha$$

$$= \beta^2 + \alpha^2 + 2\alpha\beta$$

$$= \alpha^2 + \beta^2 + 2\alpha\beta \rightarrow (ii)$$

From equation (i) and (ii) we get

$$f(\alpha, \beta) = f(\beta, \alpha)$$

Hence $\alpha^2 + \beta^2 + 2\alpha\beta$ is symmetric.

Example: (Page # 30) (A.B + U.B)

Find the value of $\alpha^3 + \beta^3 + 3\alpha\beta$, if

$\alpha = 2, \beta = 1$. Also find the value of

$\alpha^3 + \beta^3 + 3\alpha\beta$ if $\alpha = 1, \beta = 2$.

Solution:

When $\alpha = 2, \beta = 1$

$$\begin{aligned}\alpha^3 + \beta^3 + 3\alpha\beta &= (2)^3 + (1)^3 + 3(2)(1) \\ &= 8 + 1 + 6 = 15\end{aligned}$$

When $\alpha = 1, \beta = 2$

$$\begin{aligned}\alpha^3 + \beta^3 + 3\alpha\beta &= (1)^3 + (2)^3 + 3(1)(2) \\ &= 1 + 8 + 6 = 15\end{aligned}$$

Unit-2

Theory of Quadratic Equations

Note

Expression $\alpha^3 + \beta^3 + 3\alpha\beta$ represents a symmetric function.

Exercise 2.4

Q.1 If α, β are roots of the equation $x^2 + px + q = 0$, then evaluate

(i) $\alpha^2 + \beta^2$

(ii) $\alpha^3\beta + \alpha\beta^3$

(iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(i) Solution: (A.B + U.B)

$$x^2 + px + q = 0$$

Here $a = 1, b = p, c = q$

Roots of given equation are α, β

Then

$$\alpha + \beta = -\frac{b}{a}$$

$$= -\frac{p}{1}$$

$$= -p$$

$$\alpha\beta = \frac{c}{a}$$

$$= \frac{q}{1}$$

$$= q$$

(i) Now $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$\alpha^2 + \beta^2 = (-p)^2 - 2(q)$$

$$= p^2 - 2q$$

(ii) $\alpha^3\beta + \alpha\beta^3 = \alpha\beta(\alpha^2 + \beta^2)$ **(A.B)**

$$= \alpha\beta[(\alpha + \beta)^2 - 2\alpha\beta]$$

$$= q[(-p)^2 - 2(q)]$$

$$\Rightarrow \alpha^3\beta + \alpha\beta^3 = q(p^2 - 2q)$$

(iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$

(LHR 2015) (K.E + U.B)

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(-p)^2 - 2(q)}{q}$$

$$\Rightarrow \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{p^2 - 2q}{q}$$

Q.2 If α, β are the roots of the equation $4x^2 - 5x + 6 = 0$, then find the value of

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$ **(K.B + A.B)**

(ii) $\alpha^2\beta^2$ **(A.B + U.B)**

(iii) $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$ **(U.B + A.B)**

(iv) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ **(K.B + U.B)**

(LHR 2016, GRW 2014, SWL 2016, MTN 216, SGD 2015, D.G.K 2014)

Solution:

$$4x^2 - 5x + 6 = 0$$

Here $a = 4, b = -5, c = 6$

Since α, β be the roots of the given equation

Then

$$\alpha + \beta = -\frac{b}{a}$$

$$= -\frac{(-5)}{4}$$

$$= \frac{5}{4}$$

$$\alpha\beta = \frac{c}{a}$$

$$= \frac{6}{4}$$

$$= \frac{3}{2}$$

(i) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$

$$= \frac{\alpha + \beta}{\alpha\beta}$$

$$= \frac{5}{\frac{3}{2}}$$

$$= \frac{4}{3}$$

$$= \frac{5}{4} \times \frac{2}{3}$$

$$\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} = \frac{5}{6}$$

Unit-2

Theory of Quadratic Equations

(ii) $\alpha^2\beta^2 = (\alpha\beta)^2$

$$= \left(\frac{3}{2}\right)^2$$

$$\Rightarrow \alpha^2\beta^2 = \frac{9}{4}$$

(iii) $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{\beta + \alpha}{\alpha^2\beta^2}$

$$= \frac{\alpha + \beta}{(\alpha\beta)^2}$$

$$= \frac{5}{\left(\frac{3}{2}\right)^2}$$

$$= \frac{5}{\frac{9}{4}}$$

$$= \frac{5}{4} \times \frac{4}{9}$$

$$\Rightarrow \frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{5}{9}$$

(iv) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta}$

$$= \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta}$$

Putting the values

$$= \frac{\left(\frac{5}{4}\right)^3 - 3\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)}{\frac{3}{2}}$$

$$= \frac{\frac{125}{64} - \frac{45}{8}}{\frac{3}{2}}$$

$$= \frac{\frac{125 - 360}{64} \times \frac{2}{3}}$$

$$= \frac{-235}{64} \times \frac{2}{3}$$

$$\Rightarrow \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = -\frac{235}{96}$$

Q.3 If α, β are the roots of the equation $lx^2 + mx + n = 0$ ($l \neq 0$), then find the values of

(i) $\alpha^3\beta^2 + \alpha^2\beta^3$ **(A.B + U.B)**

(ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ **(K.B + U.B)**

Solution:

$$lx^2 + mx + n = 0$$

Roots of given equation are α, β

$$\text{Then } \alpha + \beta = -\frac{b}{a} = -\frac{m}{l}$$

$$\alpha\beta = \frac{c}{a}$$

$$= \frac{n}{l}$$

(i) Now $\alpha^3\beta^2 + \alpha^2\beta^3 = \alpha^2\beta^2(\alpha + \beta)$

$$= (\alpha\beta)^2(\alpha + \beta)$$

$$= \left(\frac{n}{l}\right)^2 \times \left(-\frac{m}{l}\right)$$

$$= \frac{n^2}{l^2} \times \left(-\frac{m}{l}\right)$$

$$\Rightarrow \alpha^3\beta^2 + \alpha^2\beta^3 = -\frac{mn^2}{l^3}$$

(ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}$

(FSD 2017, SWL 2017, BWP 2014, D.G.K 2017)

$$= \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2}$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$= \frac{\left(-\frac{m}{l}\right)^2 - 2\left(\frac{n}{l}\right)}{\left(\frac{n}{l}\right)^2}$$

$$= \frac{\frac{m^2}{l^2} - \frac{2n}{l}}{\frac{n^2}{l^2}}$$

$$= \frac{\frac{m^2 - 2ln}{l^2}}{\frac{n^2}{l^2}}$$

$$= \frac{m^2 - 2ln}{l^2} \times \frac{l^2}{n^2}$$

$$\Rightarrow \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{m^2 - 2ln}{n^2}$$

Exercise 2.5

Q.1 Write the quadratic equations having following roots.

- (a) 1, 5 **(K.B + A.B)**
 (b) 4, 9 **(K.B + A.B)**
 (c) -2, 3 **(K.B + A.B)**
 (d) 0, -3 **(K.B + A.B)**
 (e) 2, -6 **(K.B + A.B)**
 (f) -1, -7 **(K.B + A.B)**
 (g) $1+i, 1-i$ **(K.B + A.B)**
 (h) $3+\sqrt{2}, 3-\sqrt{2}$ **(K.B + A.B)**

Solution:

- (a) Roots of required equation are 1, 5
 Then sum of roots = $S = 1 + 5 = 6$
 And product of roots = $P = 1 \times 5 = 5$
 $\therefore$ Required quadratic equation is:
 $x^2 - Sx + P = 0$
 $x^2 - 6x + 5 = 0$
- (b) **(FSD 2016, 17, RWP 2017, RWP 2017)**
 Roots of required equation are 4, 9
 Then sum of roots = $S = 4 + 9 = 13$
 And product of roots = $P = 4 \times 9 = 36$
 $\therefore$ Required quadratic equation is:
 $x^2 - Sx + P = 0$
 $x^2 - 13x + 36 = 0$
- (c) **(LHR 2014, 16, GRW 2016, 17, SGD 2017, D.G.K 2017)**
 Roots of required equation are -2, 3
 Then sum of roots = $S = -2 + 3 = 1$
 And product of roots = $P = -2(3) = -6$
 $\therefore$ Required quadratic equation is:
 $x^2 - Sx + P = 0$
 $x^2 - 1x + (-6) = 0$ **(K.B + A.B)**
 $x^2 - x - 6 = 0$
- (d) **(SGD 2014, EWP 2017)**
 Roots of required equation are 0, -3
 Then sum of roots = $S = 0 + (-3) = -3$
 And product of roots = $P = 0(-3) = 0$
 $\therefore$ Required quadratic equation is:
 $x^2 - Sx + P = 0$
 $x^2 - (-3)x + 0 = 0$ **(K.B + A.B)**
 $x^2 + 3x = 0$

- (e) **(LHR 2014, 16, GRW 2016, 17, SGD 2017, D.G.K 2017)**

Roots of required equation are 2, -6

$$\text{Sum of roots} = S = 2 + (-6) = -4$$

$$\text{Product of roots} = P = 2(-6) = -12$$

$\therefore$ Required quadratic equation is:

$$x^2 - Sx + P = 0$$

$$x^2 - (-4)x + (-12) = 0$$

$$x^2 + 4x - 12 = 0$$
 (K.B + A.B)

- (f) **(LHR 2015, 17, RWP 2016)**

Roots of required equation are -1, -7

$$\text{Sum of roots} = S = -1 + (-7) = -8$$

$$\text{Product of roots} = P = -1(-7) = 7$$

$\therefore$ Required quadratic equation is:

$$x^2 - Sx + P = 0$$

$$x^2 - (-8)x + 7 = 0$$
 (K.B + A.B)

$$x^2 + 8x + 7 = 0$$

- (g) **(D.G.K 2014, SGD 2017)**

$$1+i, 1-i$$
 (K.B + A.B)

Roots of the required equation are

$$1+i, 1-i$$

$$\text{Sum of roots} = S = (1+i) + (1-i)$$

$$= 1+i+1-i$$

$$= 2$$

$$\text{Product of roots} = P = (1+i)(1-i)$$

$$= (1)^2 - (i)^2$$

$$= 1 - i^2$$

$$= 1 - (-1)$$

$$= 1+1$$

$$= 2$$

Required quadratic equation is

$$x^2 - Sx + P = 0$$

$$x^2 - 2x + 2 = 0$$

- (h) Roots of required equation are

$$3+\sqrt{2}, 3-\sqrt{2}$$
 (K.B + A.B)

$$\text{Sum of roots} = (3+\sqrt{2}) + (3-\sqrt{2})$$

$$= 3+\sqrt{2}+3-\sqrt{2}$$

$$= 6$$

$$\text{Product of roots} = (3+\sqrt{2})(3-\sqrt{2})$$

$$= (3)^2 - (\sqrt{2})^2$$

$$= 9 - 2$$

$$= 7$$

∴ Required quadratic equation is

$$x^2 - Sx + P = 0$$

$$x^2 - 6x + 7 = 0$$

Q.2 If α, β are the roots of the equation $x^2 - 3x + 6 = 0$. Form equations whose roots are

(a) $2\alpha + 1, 2\beta + 1$

(b) α^2, β^2

(c) $\frac{1}{\alpha}, \frac{1}{\beta}$

(d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

(e) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

Solution:

(K.B + A.B)

$$x^2 - 3x + 6 = 0$$

Here $a = 1, b = -3, c = 6$

Roots of given equations are α, β

$$\text{Then } \alpha + \beta = -\frac{b}{a}$$

$$= -\frac{-3}{1}$$

$$= 3$$

$$\alpha\beta = \frac{c}{a}$$

$$= \frac{6}{1}$$

$$= 6$$

$$= 6$$

(a) Roots of required equation are $2\alpha + 1, 2\beta + 1$

(FSD 2015) (K.B + A.B)

$$\text{Sum of roots} = S = (2\alpha + 1) + (2\beta + 1)$$

$$= 2\alpha + 1 + 2\beta + 1$$

$$= 2\alpha + 2\beta + 2$$

$$= 2(\alpha + \beta) + 2$$

$$= 2(3) + 2$$

$$= 6 + 2$$

$$= 8$$

$$\text{Product of roots} = P = (2\alpha + 1)(2\beta + 1)$$

$$= 4\alpha\beta + 2\alpha + 2\beta + 1$$

$$= 4(6) + 6 + 1$$

$$= 24 + 7 = 31$$

∴ Required quadratic equation is:

$$x^2 - Sx + P = 0$$

$$x^2 - 8x + 31 = 0$$

(b) Roots of required equation are α^2, β^2

$$\text{Sum of roots} = S = \alpha^2 + \beta^2$$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (3)^2 - 2(6)$$

$$= 9 - 12$$

$$= -3 \quad \textbf{(K.B + A.B)}$$

$$\text{Product of roots} = P = \alpha^2\beta^2$$

$$= (\alpha\beta)^2$$

$$= (6)^2$$

$$= 36$$

∴ Required quadratic equation is:

$$x^2 - Sx + P = 0$$

$$x^2 - (-3)x + 36 = 0$$

$$x^2 + 3x + 36 = 0$$

(c) Roots of required equation are $\frac{1}{\alpha}, \frac{1}{\beta}$

(K.B + A.B)

$$\text{Sum of roots} = S = \frac{1}{\alpha} + \frac{1}{\beta}$$

$$= \frac{\beta + \alpha}{\alpha\beta}$$

$$= \frac{\alpha + \beta}{\alpha\beta}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2}$$

$$\text{Product of roots} = P = \frac{1}{\alpha} \cdot \frac{1}{\beta}$$

$$= \frac{1}{\alpha\beta}$$

$$= \frac{1}{6}$$

∴ Required quadratic equation is:

$$x^2 - Sx + P = 0$$

$$x^2 - \frac{1}{2}x + \frac{1}{6} = 0$$

Multiply by '6'

$$6x^2 - 3x + 1 = 0$$

(d) Roots of required equation are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$
(K.B + A.B)

$$S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(3)^2 - 2(6)}{6}$$

$$= \frac{9-12}{6} = \frac{-3}{6}$$

$$\Rightarrow S = -\frac{1}{2}$$

$$P = \frac{\alpha}{\beta} \times \frac{\beta}{\alpha}$$

$$\Rightarrow P = 1$$

Required quadratic equation is

$$x^2 - Sx + p = 0$$

Or $x^2 - \left(-\frac{1}{2}\right)x + 1 = 0$

$$x^2 + \frac{1}{2}x + 1 = 0 \quad (\text{Multiplying by 2})$$

$$2x^2 + x + 2 = 0$$

(e) Roots of required equation,

$$\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta} \quad (\text{K.B + A.B})$$

$$S = (\alpha + \beta) + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

$$= (\alpha + \beta) + \left(\frac{\beta + \alpha}{\alpha\beta}\right)$$

$$= (\alpha + \beta) + \left(\frac{\alpha + \beta}{\alpha\beta}\right)$$

$$= 3 + \frac{3}{6} = \frac{18+3}{6}$$

$$= \frac{21}{6} = \frac{7}{2}$$

$$P = (\alpha + \beta) \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

$$= (\alpha + \beta) \left(\frac{\beta + \alpha}{\alpha\beta}\right)$$

$$= 3 \left(\frac{3}{6}\right)$$

$$= (3) \left(\frac{1}{2}\right)$$

$$P = \frac{3}{2}$$

Required equation is

$$x^2 - Sx + P = 0$$

$$x^2 - \frac{7}{2}x + \frac{3}{2} = 0$$

Multiplying by (2)

$$2x^2 - 7x + 3 = 0$$

Q.3 If α, β are the roots of the equation $x^2 + px + q = 0$ form equations whose roots are:

(a) α^2, β^2 (FSD 2015)

(b) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Solution:

$$x^2 + px + q = 0$$

Here

$$a = 1, b = p, c = q$$

Roots of given equation are α, β

$$\text{Then } \alpha + \beta = -\frac{b}{a} = -p$$

$$\alpha\beta = \frac{c}{a} = q$$

(a) Roots of required equation are α^2, β^2
(K.B + A.B)

$$\text{Sum of roots} = S = \alpha^2 + \beta^2$$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (-p)^2 - 2q$$

$$= p^2 - 2q$$

$$\begin{aligned}
 \text{Product of roots} = P &= \alpha^2 \beta^2 \\
 &= (\alpha\beta)^2 \\
 &= (q)^2 \\
 &= q^2
 \end{aligned}$$

∴ Required quadratic equation is

$$x^2 - Sx + P = 0$$

$$x^2 - (p^2 - 2q)x + q^2 = 0$$

(b) Roots of required equation are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$
(K.B + A.B)

$$\begin{aligned}
 \text{Sum of roots} = S &= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \\
 &= \frac{\alpha^2 + \beta^2}{\alpha\beta} \\
 &= \frac{p^2 - 2q}{q}
 \end{aligned}$$

$$\begin{aligned}
 \text{Product of roots} = P &= \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} \\
 &= 1
 \end{aligned}$$

∴ Required quadratic equation is:

$$x^2 - Sx + P = 0$$

$$\Rightarrow x^2 - \frac{p^2 - 2q}{q}x + 1 = 0$$

Multiply by 'q'

$$qx^2 - (p^2 - 2q)x + q = 0$$

Synthetic Division (K.B)

It is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial.

It is a short cut of long division.

Example 3: (Page # 35) (A.B)

Use synthetic division, divide the polynomial $P(x) = 5x^4 + x^3 - 3x^2 - 3x - 2$ by $x - 2$.

Solution:

$$P(x) = 5x^4 + x^3 + 0x^2 - 3x - 2$$

$$\text{Here } x - a = x - 2 \Rightarrow x = 2$$

$$\begin{array}{r|rrrrr}
 2 & 5 & 1 & 0 & -3 & -2 \\
 & \downarrow & 10 & 22 & 44 & 82 \\
 \hline
 & 5 & 11 & 22 & 41 & 82
 \end{array}$$

$$\therefore Q(x) = 5x^3 + 11x^2 + 22x + 41$$

$$R = 82$$

Example 3: (Page # 36) (A.B)

Using synthetic division, find the value of h . If the zero of polynomial $p(x) = 3x^2 + 4x - 7h$ is 1.

Solution:

$$p(x) = 3x^2 + 4x - 7h \text{ and its zero is 1.}$$

Then by the synthetic division.

$$\begin{array}{r|rrr}
 1 & 3 & 4 & -7h \\
 & & 3 & 7 \\
 \hline
 & 3 & 7 & 7-7h
 \end{array}$$

$$\text{Remainder} = 7 - 7h$$

Since 1 is the zero of the polynomial, therefore,

$$\text{Remainder} = 0, \text{ that is}$$

$$7 - 7h = 0$$

$$7 = 7h$$

$$\Rightarrow h = 1$$

Example 4: (Page # 36) (A.B)

Using synthetic division, find the values of l and m , if $x - 1$ and $x + 1$ are the factors of the polynomial

$$P(x) = x^3 + 3lx^2 + mx - 1$$

Solution:

Since $x - 1$ and $x + 1$ are the factors of $P(x) = x^3 + 3lx^2 + mx - 1$

Therefore, 1 and -1 are zeros of polynomial $P(x)$.

Now by synthetic division

$$\begin{array}{r|rrrr}
 1 & 1 & 3l & m & -1 \\
 & \downarrow & -1 & -3l+1 & 3l+m+1 \\
 \hline
 & 1 & 3l-1 & 3l+m+1 & 3l+m
 \end{array}$$

Since 1 is the zero of polynomial, therefore, remainder is zero, that is,

$$3l + m = 0 \rightarrow (i)$$

And

$$\begin{array}{r|rrrr}
 -1 & 1 & 3l & m & -1 \\
 & \downarrow & -1 & -3l+1 & 3l-m-1 \\
 \hline
 & 1 & 3l-1 & 3l+m+1 & 3l-m-2
 \end{array}$$

Since 1 is the zero of polynomial, therefore, remainder is zero, that is, $3l - m - 2 = 0 \rightarrow (ii)$

Adding equations (i) and (ii), we get

$$6l - 2 = 0$$

$$6l = 2 \Rightarrow l = \frac{2}{6} = \frac{1}{3}$$

Unit-2

Theory of Quadratic Equations

Put the value of l in equation (i)

$$3\left(\frac{1}{3}\right) + m = 0 \text{ or}$$

$$1 + m = 0 \Rightarrow m = -1$$

$$\text{Thus } l = \frac{1}{3} \text{ and } m = -1$$

Example 6: (Page # 38) (A.B)

By synthetic division, solve the equation $x^4 - 49x^2 + 36x + 252 = 0$ having roots -2 and 6 .

Solution:

Since -2 and 6 are the roots of the given equation $x^4 - 49x^2 + 36x + 252 = 0$.

Then by synthetic division, we get

	1	0	-49	36	252	
-2	↓	-2	4	90	-252	
	1	-2	-45	126	0	
6		6	24	-126		
	1	4	-21	0		

$\therefore$ The depressed equation is

$$x^2 + 4x - 21 = 0$$

$$x^2 + 7x - 3x - 21 = 0$$

$$x(x+7) - 3(x+7) = 0$$

$$(x+7)(x-3) = 0$$

$$\text{Either } x+7=0 \quad \text{or} \quad x-3=0$$

$$x = -7 \quad \text{or} \quad x = 3$$

Thus $-2, -7$ and 3 are the roots of the given equation.

Exercise 2.6

Q.1 Use synthetic division to find the quotient and the remainder, when

(i) $(x^2 + 7x - 1) \div (x + 1)$

(ii) $(4x^3 - 5x + 15) \div (x + 3)$

(iii) $(x^3 + x^2 - 3x + 2) \div (x - 2)$

Solution:

(i) $P(x) = x^2 + 7x - 1$ (FSL 2016, 17) (A.B)

	1	7	-1
-1	↓	-1	-6
	1	6	-7

$$\therefore Q(x) = x + 6$$

$$R = -7$$

(ii) $P(x) = 4x^3 - 5x + 15$ (A.B)

(SWL 2016, SSD 2017, MTN 2016)

$$= 4x^3 + 0x^2 - 5x + 15$$

	4	0	-5	15
-3	↓	-12	36	-93
	4	-12	31	-78

$$\therefore Q(x) = 4x^2 - 12x + 31$$

$$R = -78$$

(iii) $P(x) = x^3 + x^2 - 3x + 2$ (A.B)

(GRW 2017, FSD 2015, MTN 2017, D.G.K 2015)

	1	1	-3	2
2	↓	2	6	6
	1	3	3	8

$$\therefore Q(x) = x^2 + 3x + 3$$

$$R = 8$$

Q.2 Find the value of h using synthetic division, if (A.B)

(i) 3 is the zero of the polynomial $2x^3 - 3hx^2 + 9$

(ii) 1 is the zero of the polynomial $x^3 - 2hx^2 + 11$

(iii) -1 is the zero of the polynomial $2x^3 + 5hx - 23$

Solution:

(i) $P(x) = 2x^3 - 3hx^2 + 9$

(SWL 2014) (A.B)

$$= 2x^3 - 3hx^2 + 0x + 9$$

	2	-3h	0	9
3	↓	6	-9h	-27 + 54
	2	-3h - 6	-9h + 18	-27h + 63

Since 3 is the zero of given polynomial,

$$R = 0$$

$$\Rightarrow -27h + 63 = 0$$

$$-27h = -63$$

$$h = \frac{-63}{-27}$$

$$\Rightarrow h = \frac{7}{3}$$

Unit-2

Theory of Quadratic Equations

(ii) $P(x) = x^3 - 2hx^2 + 11$ (A.B)

$$= x^3 - 2hx^2 + 0x + 11$$

1	1	-2h	0	11
1	↓	1	-2h+1	-2h+1
	1	-2h+1	-2h+1	<u>-2h+1</u>

Since 1 is zero of given polynomial, $P = 0$

$$\Rightarrow -2h + 12 = 0$$

$$-2h = -12$$

$$h = \frac{-12}{-2}$$

$$h = 6$$

Result:

$$h = 6$$

(iii) $P(x) = 2x^3 + 5hx - 23$ (A.B)

$$= 2x^3 + 0x^2 + 5hx - 23$$

-1	2	0	5h	-23
-1	↓	-2	2	-5h-2
	1	-2h+1	-2h+1	<u>-5h-25</u>

Since 1 is zero of given polynomial, $R = 0$

$$\Rightarrow -5h - 25 = 0$$

$$-5h = 25$$

$$h = -\frac{25}{5} = -5$$

Result:

$$h = -5$$

Q.3 Use synthetic division to find the values of l and m , if (A.B)

(i) $(x + 3)$ and $(x - 2)$ are the factors of the polynomial $x^3 + 4x^2 + 2lx + m$

(ii) $(x - 1)$ and $(x + 1)$ are the factors of the polynomial $x^3 - 3lx^2 + 2mx + 6$

Solution:

(i) $P(x) = x^3 + 4x^2 + 2lx + m$ (A.B)

-3	1	4	2l	m
-3	↓	-3	-3	-6-9
	1	1	2l-3	<u>-6l+m+9</u>
2	↓	2	6	
	1	3	<u>2l+3</u>	

Since $x - 2$ is a factor, $R = 0$

$$\Rightarrow 2l + 3 = 0$$

$$2l = -3$$

$$\Rightarrow l = -\frac{3}{2}$$

Also $x + 3$ is a factor, $R = 0$

$$\Rightarrow -6l + m + 9 = 0$$

$$-6\left(-\frac{3}{2}\right) + m + 9 = 0 \quad \therefore l = -\frac{3}{2}$$

$$9 + m + 9 = 0$$

$$m + 18 = 0$$

$$\Rightarrow m = -18$$

Result

$$l = -\frac{3}{2}, \quad m = -18$$

(ii) $P(x) = x^3 - 3lx^2 + 2mx + 6$ (A.B)

1	1	-3l	2m	6
1	↓	1	-3l+1	-3l+2m+1
	1	-3l+1	-3l+2m+1	<u>-3l+2m+7</u>
-1	↓	-1	3l	
	1	-3l	<u>2m+1</u>	

Since $x + 1$ is a factor, $R = 0$

$$\Rightarrow 2m + 1 = 0$$

$$2m = -1$$

$$\Rightarrow m = -\frac{1}{2}$$

Also $x - 1$ is a factor, $R = 0$

$$-3l + 2m + 7 = 0$$

$$-3l + 2\left(-\frac{1}{2}\right) + 7 = 0 \quad \therefore m = -\frac{1}{2}$$

$$-3l - 1 + 7 = 0$$

$$-3l + 6 = 0$$

$$-3l = -6$$

$$\Rightarrow l = 2$$

Result

$$l = 2, \quad m = -\frac{1}{2}$$

Q.4 Solve by using synthetic division, if

(i) 2 is the root of the equation

$$x^3 - 28x + 48 = 0$$

(ii) 3 is the root of the equation

$$2x^3 - 3x^2 - 11x + 6 = 0$$

(iii) -1 is the root of the equation

$$4x^3 - x^2 - 11x - 6 = 0$$

Unit-2

Theory of Quadratic Equations

Solution:

(i) $P(x) = x^3 - 28x + 48$ **(A.B)**

$$= x^3 + 0x^2 - 28x + 48$$

2	1	0	-28	48
	↓	2	4	-48
	1	2	-24	0

∴ Depressed equation is.

$$x^2 - 2x - 24 = 0$$

$$x^2 + 6x - 4x - 24 = 0$$

$$x(x+6) - 4(x+6) = 0$$

$$(x+6)(x-4) = 0$$

Either

$$x+6=0 \quad \text{or} \quad x-4=0$$

$$x=-6 \quad \quad \quad x=4$$

Thus 2, 4 and -6 are the roots of the given equation.

$$\therefore \text{Solution Set} = \{2, 4, -6\}$$

(ii) $P(x) = 2x^3 - 3x^2 - 11x + 6$ **(A.B)**

3	2	-3	-11	6
	↓	4	9	-6
	2	3	-2	0

∴ Depressed equation is:

$$2x^2 + 3x - 2 = 0$$

$$2x^2 + 4x - x - 2 = 0$$

$$2x(x+2) - 1(x+2) = 0$$

$$(x+2)(2x-1) = 0$$

Either

$$x+2=0 \quad \text{Or} \quad 2x-1=0$$

$$x=-2$$

$$2x=1$$

$$x=\frac{1}{2}$$

$$\therefore \text{Solution Set} = \left\{3, -2, \frac{1}{2}\right\}$$

(iii) $P(x) = 4x^3 - x^2 - 11x - 6$ **(A.B)**

-1	4	-1	-11	-6
	↓	-4	5	6
	4	-5	-6	0

∴ Depressed equation is

$$4x^2 - 5x - 6 = 0$$

$$4x^2 - 8x + 3x - 6 = 0$$

$$4x(x-2) + 3(x-2) = 0$$

$$(x-2)(4x+3) = 0$$

Either

$$x-2=0$$

$$x=2$$

or

$$4x+3=0$$

$$4x=-3$$

$$x=-\frac{3}{4}$$

Thus $-1, 2, -\frac{3}{4}$ are the roots of the given equation.

$$\therefore \text{Solution Set} = \left\{-1, 2, -\frac{3}{4}\right\}$$

Q.5 Solve by using synthetic division, if

(i) 1 and 3 are the roots of the equation $x^4 - 10x^2 + 9 = 0$

(ii) 3 and -4 are the roots of the equation

$$x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$$

Solution: (A.B)

(i) $P(x) = x^4 + 0x^3 - 10x^2 + 0x + 9$

1	1	0	-10	0	9
		1	1	-9	-9
3	1	1	-9	-9	0
		3	12	9	
	1	4	3	0	

∴ Depressed equation is

$$x^2 + 4x + 3 = 0$$

$$x^2 + 3x + x + 3 = 0$$

$$x(x+3) + 1(x+3) = 0$$

$$(x+3)(x+1) = 0$$

Either

$$x+3=0$$

$$x=-3$$

or

$$x+1=0$$

$$x=-1$$

$$\therefore \text{Solution Set} = \{\pm 3, \pm 1\}$$

(ii) $P(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$

3	1	2	-13	-14	24
	↓	3	15	6	-24
-4	1	5	2	-8	0
	↓	-4	-4	8	
	1	1	-2	0	

∴ Depressed equation is:

$$x^2 + 2x - x - 2 = 0$$

$$x(x+2) - 1(x+2) = 0$$

$$x(x+2) - 1(x+2) = 0$$

$$(x+2)(x-1) = 0$$

Either

$$x+2=0 \quad \text{or} \quad x-1=0$$

$$x=-2 \quad \text{or} \quad x=1$$

Thus -2, -4, 1 and 3 are the roots of the given equation.

$$\therefore \text{Solution Set} = \{-2, -4, 1, 3\}$$

Simultaneous Equation (K.B)

A system of equations having a common solution is called a system of simultaneous equations.

For example $x+2y=3$, $2x-y=1$ having same solution (1,1).

Solution Set (K.B)

The set of all the ordered pairs (x, y) , which satisfies the system of equations is called the solution set of the system.

Ordered Pair (K.B)

An ordered pair of real numbers x and y is a pair (x, y) in which elements are written in specific order.

For example: (x, y) is an ordered pair in which first elements (abscissa) is x and second element (ordinate) is y .

Note (K.B)

$(x, y) \neq (y, x)$. For example (2,3) and (3,2) are two different ordered pairs.

Example 1 (Page # 39) (A.B)

Solve the system of equations

$$3x + y = 4 \quad \text{and} \quad 3x^2 + y^2 = 52$$

Solution:

The given equations are

$$3x + y = 4 \rightarrow (i)$$

$$3x^2 + y^2 = 52 \rightarrow (ii)$$

$$\text{From equation (i) } y = 4 - 3x \rightarrow (iii)$$

Put value of y in equation (ii)

$$3x^2 + (4 - 3x)^2 = 52$$

$$3x^2 + 16 - 24x + 9x^2 - 52 = 0$$

$$12x^2 - 24x - 36 = 0$$

$$12(x^2 - 2x - 3) = 0$$

$$x^2 - 2x - 3 = 0 \quad \because 12 \neq 0$$

By factorization

$$x^2 - 3x + x - 3 = 0$$

$$x(x-3) + 1(x-3) = 0$$

$$(x-3)(x+1) = 0$$

Either

$$x-3=0 \quad \text{or} \quad x+1=0$$

$$x=3 \quad \text{or} \quad x=-1$$

Put the values of x in equation (iii)

$$\text{When } x=3 \quad \text{when } x=-1$$

$$y=4-3x \quad y=4-3x$$

$$y=4-3(3) \quad y=4-3(-1)$$

$$=4-9 \quad =4+3$$

$$y=-5 \quad y=7$$

$\therefore$ ordered pairs are $(3, -5)$ and $(-1, 7)$

Thus, the solution set is $\{(3, -5), (-1, 7)\}$

Example 2 (Page # 40) (A.B)

Solve the equations

$$x^2 + y^2 + 2x = 8 \quad \text{and} \quad (x-1)^2 + (y+1)^2 = 8$$

Solution:

The given equations are

$$x^2 + y^2 + 2x = 8 \rightarrow (i)$$

$$(x-1)^2 + (y+1)^2 = 8 \rightarrow (ii)$$

From equation (i), we get

$$x^2 - 2x + 1 + y^2 + 2y + 1 = 8$$

$$\text{Or } x^2 + y^2 - 2x + 2y = 6 \rightarrow (iii)$$

Subtracting equation (iii) from equation (i) we have

$$4x - 2y = 2 \quad \text{or} \quad 2x - y = 1$$

$$\Rightarrow y = 2x - 1$$

Put the value of y in equation (ii)

$$(x-1)^2 + (2x-1+1)^2 = 8$$

$$x^2 - 2x + 1 + 4x^2 - 8 = 0$$

$$5x^2 - 2x - 7 = 0$$

$$5x^2 - 7x + 5x - 7 = 0$$

$$x(5x-7) + 1(5x-7) = 0$$

$$(5x-7)(x+1) = 0$$

Either

$$5x-7=0 \quad \text{or} \quad x+1=0$$

$$5x=7 \quad \text{or} \quad x=-1$$

$$\Rightarrow x = \frac{7}{5}$$

Now putting the values of x in equation (iv), we have

$$\text{When } x = \frac{7}{5} \quad \text{when } x = -1$$

$$y = 2\left(\frac{7}{5}\right) - 1 \quad y = 2(-1) - 1$$

$$y = \frac{14}{5} - 1 = \frac{14-5}{5} = \frac{9}{5}$$

$$y = -3$$

Thus, the solution set is

$$\left\{(-1, -3), \left(\frac{7}{5}, \frac{9}{5}\right)\right\}$$

Example 3 (Page # 41)

(A.B)

Solve the equations

$$x^2 + y^2 = 7 \text{ and } 2x^2 + 3y^2 = 18$$

Solution:

Given equations are

$$x^2 + y^2 = 7 \quad \text{(i)}$$

$$2x^2 + 3y^2 = 18 \quad \text{(ii)}$$

Multiply equation (i) with 3

$$3x^2 + 3y^2 = 21 \quad \text{(iii)}$$

Subtracting equations (ii) from (iii)

$$x^2 = 3 \Rightarrow x = \pm\sqrt{3}$$

When $x = \sqrt{3}$ then from equation (i)

$$x^2 + y^2 = 7 \text{ or}$$

$$3 + y^2 = 7 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

When $x = -\sqrt{3}$, then $y = \pm 2$

Thus, the required solution set is

$$\{(\pm\sqrt{3}, \pm 2)\}.$$

Example 4 (Page # 41)

(A.B)

Solve the equations

$$x^2 + y^2 = 20 \quad \text{(i)}$$

$$6x^2 + xy - y^2 = 0 \quad \text{(ii)}$$

The equation (ii) can be written as

$$y^2 - xy - 6x^2 = 0$$

$$\Rightarrow y = \frac{-(-x) \pm \sqrt{(-x)^2 - 4 \times 1 \times (-6x^2)}}{2 \times 1}$$

$$= \frac{x \pm \sqrt{x^2 + 24x^2}}{2} = \frac{x \pm \sqrt{25x^2}}{2}$$

$$= \frac{x \pm 5x}{2}$$

$$\text{We have } y = \frac{x+5x}{2} = \frac{6x}{2} = 3x$$

$$\text{Or } y = \frac{x-5x}{2} = \frac{-4x}{2} = -2x$$

Substituting $y = 3x$ in the equation (i), we get

$$x^2 + (3x)^2 = 20$$

$$x^2 + 9x^2 = 20$$

$$\Rightarrow 10x^2 = 20$$

$$\Rightarrow x^2 = 4 \Rightarrow x = \pm\sqrt{2}$$

$$\text{When } x = \sqrt{2}, y = 3(\sqrt{2}) = 3\sqrt{2} \text{ and}$$

$$\text{when } x = -\sqrt{2}, y = 3(-\sqrt{2}) = -3\sqrt{2}$$

Substituting $y = -2x$ in the equation (i), we have

$$x^2 + (-2x)^2 = 20 \text{ or } x^2 + 4x^2 = 20$$

$$\Rightarrow 5x^2 = 20 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$$\text{When } x = 2 \Rightarrow y = -2(2) = -4$$

$$\text{when } x = -2 \Rightarrow y = -2(-2) = 4$$

Thus, the solution is

$$\{(\sqrt{2}, 3\sqrt{2}), (-\sqrt{2}, -3\sqrt{2}), (2, -4), (-2, 4)\}.$$

Example 5 (Page # 42)

(A.B)

Solve the equations

$$x^2 + y^2 = 40 \text{ and } 3x^2 - 2xy - y^2 = 80$$

Solution:

Given equations are

$$x^2 + y^2 = 40 \rightarrow \text{(i)}$$

$$3x^2 - 2xy - y^2 = 80 \rightarrow (ii)$$

Multiplying equation (i) by 2, we have

$$2x^2 + 2y^2 = 80 \rightarrow (iii)$$

Subtracting the equation (iii) from equation (ii), we get

$$x^2 - 2xy - 3y^2 = 0 \rightarrow (iv)$$

The equation (iv) can be written as

$$x^2 - 3xy + xy - 3y^2 = 0$$

$$x(x - 3y) + y(x - 3y) = 0$$

$$(x - 3y)(x + y) = 0$$

Either $x - 3y$ or $x + y = 0$

$$\Rightarrow x = 3y \rightarrow (v) \text{ or } x = -y \rightarrow (vi)$$

Put in equation (i),

$$\text{Equation (i)} \Rightarrow (3y)^2 + y^2 = 40$$

$$10y^2 = 40$$

$$y^2 = 4$$

$$y = \pm 2$$

$$y = 2$$

$$y = -2$$

$$\text{eq. (v)} \Rightarrow$$

$$x = 3(2) \quad x = 3(-2)$$

$$x = 6 \quad x = -6$$

$$\text{Equation (i)} \Rightarrow (-y)^2 + y^2 = 40$$

$$2y^2 = 40$$

$$y^2 = 20$$

$$y = \pm 2\sqrt{5}$$

$$y = 2\sqrt{5}$$

$$y = -2\sqrt{5}$$

$$\text{eq. (vi)} \Rightarrow$$

$$x = -(2\sqrt{5}) \quad x = -(-2\sqrt{5})$$

$$x = -2\sqrt{5} \quad x = 2\sqrt{5}$$

$\therefore$ The solution set is

$$\{(6, 2), (-6, -2), (2\sqrt{5}, -2\sqrt{5}), (-2\sqrt{5}, 2\sqrt{5})\}$$

Exercise 2.7

Solve the following simultaneous equations.

Q.1 $x + y = 5$ **(A.3)**

$$x^2 - 2y - 14 = 0$$

Solution:

$$x + y = 5 \rightarrow (i)$$

$$x^2 - 2y - 14 = 0 \rightarrow (ii)$$

From equation (i)

$$y = 5 - x \rightarrow (iii)$$

Put in equation (ii)

$$x^2 - 2y - 14 = 0$$

$$x^2 - 2(5 - x) - 14 = 0$$

$$x^2 - 10 + 2x - 14 = 0$$

$$x^2 + 2x - 24 = 0$$

$$x^2 + 6x - 4x - 24 = 0$$

$$x(x + 6) - 4(x + 6) = 0$$

$$(x + 6)(x - 4) = 0$$

Either

$$x + 6 = 0 \quad \text{or} \quad x - 4 = 0$$

$$\Rightarrow x = -6 \quad x = 4$$

Put in equation (iii)

$$y = 5 - x \quad y = 5 - x$$

$$= 5 - (-6) \quad = 5 - 4$$

$$= 5 + 6 \quad = 11$$

$$= 11$$

$$\therefore \text{Solution Set} = \{(-6, 11), (4, 1)\}$$

Q.2 $3x - 2y = 1$ **(A.B)**

$$x^2 + xy - y^2 = 1$$

Solution:

$$3x - 2y = 1 \rightarrow (i)$$

$$\text{and } x^2 + xy - y^2 = 1 \rightarrow (ii)$$

From equation (i)

$$3x - 2y = 1$$

$$3x = 2y + 1$$

$$x = \frac{2y + 1}{3} \rightarrow (iii)$$

Now put in equation (ii)

$$x^2 + xy - y^2 = 1$$

$$\left(\frac{1+2y}{3}\right)^2 + \left(\frac{1+2y}{3}\right)y - y^2 = 1$$

$$\frac{1+4y+4y^2}{9} + \frac{y+2y^2}{3} - y^2 = 1$$

Multiplying both sides by 1 CM

We get

$$1+4y+4y^2+3(y+2y^2)-9y^2=9$$

$$1+4y+4y^2+3y+6y^2-9y^2-9=0$$

$$y^2+7y-8=0$$

$$y^2+8y-y-8=0$$

$$\Rightarrow y(y+8)-1(y+8)=0$$

$$(y+8)(y-1)=0$$

Either

$$y+8=0 \quad \text{or} \quad y-1=0$$

$$y=-8 \quad \quad \quad y=1$$

Putting values in equation (iii)

When $y = -8$

$$x = \frac{1+2y}{3}$$

$$x = \frac{1+2(-8)}{3}$$

$$x = \frac{1-16}{3}$$

$$x = \frac{-15}{3}$$

$$x = -5$$

When $y = 1$

$$x = \frac{1+2y}{3}$$

$$x = \frac{1+2(1)}{3}$$

$$x = \frac{3}{3}$$

$$x = 1$$

$$\therefore \text{Solution Set} = \{(-5, -8), (1, 1)\}$$

Q.3 $x - y = 7$

(A.B)

$$\frac{2}{x} - \frac{5}{y} = 2$$

Solution:

$$x - y = 7 \rightarrow (i)$$

$$\frac{2}{x} - \frac{5}{y} = 2 \rightarrow (ii)$$

From equation (ii)

$$\frac{2}{x} - \frac{5}{y} = 2$$

Multiple by xy

$$2y - 5x = 2xy$$

$$2y - 5x - 2xy = 0$$

$$2y - 2xy - 5x = 0 \rightarrow (iii)$$

From equation (i)

$$x = 7 + y \rightarrow (iv)$$

Put in equation (iii)

$$2y - 2(7+y)(y) - 5(7+y) = 0$$

$$2y - 14y - 2y^2 - 5y - 35 = 0$$

$$-2y^2 - 17y - 35 = 0$$

$$-(2y^2 + 17y + 35) = 0$$

$$\Rightarrow 2y^2 + 17y + 35 = 0$$

$$2y^2 + 10y + 7y + 35 = 0$$

$$2y(y+5) + 7(y+5) = 0$$

$$(y+5)(2y+7) = 0$$

Either

$$y+5=0 \quad \text{or} \quad 2y+7=0$$

$$y = -5 \quad \quad \quad y = -\frac{7}{2}$$

Now put in equation (i)

when $y = -5$

$$x - (-5) = 7$$

$$x + 5 = 7$$

$$x = 7 - 5$$

$$x = 2$$

when

$$y = -\frac{7}{2}$$

$$x - \left(\frac{-7}{2}\right) = 7$$

$$x + \frac{7}{2} = 7$$

$$x = 7 - \frac{7}{2}$$

$$x = \frac{14 - 7}{2}$$

$$x = \frac{7}{2}$$

$$\therefore \text{Solution Set} = \left\{ (2, -5), \left(\frac{7}{2}, -\frac{7}{2}\right) \right\}$$

Q.4 $x + y = a - b$ **(A.B)**

$$\frac{a}{x} - \frac{b}{y} = 2$$

Solution:

$$x + y = a - b \rightarrow (i)$$

$$\frac{a}{x} - \frac{b}{y} = 2 \rightarrow (ii)$$

From equation (ii)

$$\frac{a}{x} - \frac{b}{y} = 2$$

Multiply by 'xy'

$$ay - bx = 2xy \rightarrow (iii)$$

From equation (i)

$$\Rightarrow x + y = a - b$$

$$y = a - b - x \rightarrow (iv)$$

Put in equation (iii)

$$a(a - b - x) - bx = 2x(a - b - x)$$

$$a^2 - ab - ax - bx = 2ax - 2bx - 2x^2$$

$$2x^2 - ax - 2ax - bx + 2bx + a^2 - ab = 0$$

$$2x^2 - 3ax + bx + a^2 - ab = 0$$

$$2x^2 - (3a - b)x + (a^2 - ab) = 0$$

$$\text{Here } A = 2, B = -(3a - b), C = a^2 - ab$$

Using quadratic formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$x = \frac{-[-(3a - b)] \pm \sqrt{[-(3a - b)]^2 - 4(2)(a^2 - ab)}}{2(2)}$$

$$x = \frac{3a - b \pm \sqrt{(3a - b)^2 - 8(a^2 - ab)}}{4}$$

$$= \frac{3a - b \pm \sqrt{9a^2 - 6ab + b^2 - 8a^2 + 8ab}}{4}$$

$$= \frac{3a - b \pm \sqrt{a^2 + 2ab + b^2}}{4}$$

$$= \frac{3a - b \pm \sqrt{(a + b)^2}}{4}$$

$$= \frac{3a - b \pm (a + b)}{4}$$

Either

$$x = \frac{3a - b + a + b}{4} \quad \text{or} \quad x = \frac{3a - b - a - b}{4}$$

$$= \frac{4a}{4}$$

$$x = a$$

$$= \frac{2a - 2b}{4}$$

$$x = \frac{a - b}{2}$$

Put in equation (iv)

$$y = a - b - a$$

$$y = -b$$

$$y = a - b - \frac{a - b}{2}$$

$$y = \frac{2a - 2b - (a - b)}{2}$$

$$y = \frac{a - b}{2}$$

$$\therefore \text{Solution Set} = \left\{ (a, -b), \left(\frac{a - b}{2}, \frac{a - b}{2}\right) \right\}$$

Q.5 $x^2 + (y - 1)^2 = 10$ **(A.B)**

$$x^2 - y^2 + 4x = 1$$

Solution:

$$x^2 + (y - 1)^2 = 10 \rightarrow (i)$$

$$x^2 + y^2 + 4x = 1 \rightarrow (ii)$$

Equation (i) $\Rightarrow$

$$x^2 + y^2 - 2y + 1 = 10$$

$$x^2 + y^2 - 2y = 9 \rightarrow (iii)$$

Subtract equation (ii) and (iii)

$$x^2 + y^2 + 4x = 1$$

$$\underline{-x^2 + y^2} \quad \mp 2y = -9$$

$$4x + 2y = -8$$

$$2(2x + y) = -8$$

$$2x + y = -4$$

$$y = -4 - 2x \rightarrow (i)$$

Put in equation (ii)

$$x^2 + (-4 - 2x)^2 + 4x = 1$$

$$x^2 + [-(4 + 2x)]^2 + 4x = 1$$

$$x^2 + (4 + 2x)^2 + 4x = 1$$

$$x^2 + 16 + 16x + 4x^2 + 4x - 1 = 0$$

$$5x^2 + 20x + 15 = 0$$

$$5(x^2 + 4x + 3) = 0$$

$$\Rightarrow x^2 + 4x + 3 = 0 \quad \because 5 \neq 0$$

$$x^2 + 3x + x + 3 = 0$$

$$x(x + 3) + 1(x + 3) = 0$$

$$(x + 3)(x + 1) = 0$$

Either

$$x + 3 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = -3 \quad \quad \quad x = -1$$

Put in equation (iv)

$$y = -4 - 2(-3) \quad y = -4 - 2(-1)$$

$$y = -4 + 6 \quad y = -4 + 2$$

$$y = 2 \quad y = -2$$

$$\therefore \text{Solution Set} = \{(-3, 2), (-1, -2)\}$$

Q.6 $(x+1)^2 + (y+1)^2 = 5$ **(A.B)**

$$(x+2)^2 + y^2 - 5$$

Solution:

$$(x+1)^2 + (y+1)^2 = 5 \rightarrow (i)$$

$$(x+2)^2 + y^2 - 5 \rightarrow (ii)$$

From equation (i) $\Rightarrow$

$$x^2 + 2x + 1 + y^2 + 2y + 1 = 5$$

$$x^2 + y^2 + 2x + 2y = 3$$

$$x^2 + y^2 + 2x + 2y = 3 \rightarrow (iii)$$

$$(x+2)^2 + y^2 = 5$$

$$x^2 + 4x + 4 + y^2 = 5$$

$$x^2 + y^2 + 4x = 1 \rightarrow (iv)$$

Sub equation (iii) and (iv)

$$x^2 + y^2 + 2x + 2y = -3$$

$$\underline{-x^2 + y^2 + 4x} \quad = -1$$

$$-2x + 2y = 2$$

$$-x + y = 1$$

$$y = x + 1 \rightarrow (v)$$

Put in equation (iv)

$$x^2 + (x+1)^2 + 4x = 1$$

$$x^2 + x^2 + 2x + 1 + 4x - 1$$

$$2x^2 + 6x = 0$$

$$2x(x + 3) = 0$$

Either

$$2x = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = 0 \quad \quad \quad x = -3$$

Put in equation (v)

When $x = 0$

$$y = 1 + 0$$

$$y = 1$$

When $x = -3$

$$y = 1 - 3$$

$$y = -2$$

$$\therefore \text{Solution Set} = \{(0, 1), (-3, -2)\}$$

Q.7 $x^2 + 2y^2 = 22$ **(A.B)**

$$5x^2 + y^2 = 29$$

Solution:

$$x^2 + 2y^2 = 22 \rightarrow (i)$$

$$5x^2 + y^2 = 29 \rightarrow (ii)$$

Multiply equation (ii) by '2'

$$10x^2 + 2y^2 = 58 \rightarrow (iii)$$

Subtract equation (i) and (iii)

$$10x^2 + 2y^2 = 58$$

$$\underline{-x^2 + 2y^2} \quad = -22$$

$$9x^2 = 36$$

$$x^2 = 4 \quad (\text{Div both sides by 9})$$

Taking square root on both sides

$$x = \pm 2$$

Put $x^2 = 4$ in equation (ii)

$$5(4) + y^2 = 29$$

$$20 + y^2 = 29$$

$$y^2 = 29 - 20$$

$$y^2 = 9$$

Taking square root

$$y = \pm 3$$

$$\therefore \text{Solution Set} = \{(\pm 2, \pm 3)\}$$

Q.8

$$4x^2 - 5y^2 = 6$$

(A.B)

$$3x^2 + y^2 = 14$$

Solution:

$$4x^2 - 5y^2 = 6 \rightarrow (i)$$

$$3x^2 + y^2 = 14 \rightarrow (ii)$$

Multiply equation (ii) by 5

$$15x^2 + 5y^2 = 70 \rightarrow (iii)$$

Adding equation (i) and (iii)

$$4x^2 - 5y^2 = 6$$

$$15x^2 + 5y^2 = 70$$

$$19x^2 = 76$$

$$x^2 = \frac{76}{19}$$

$$x^2 = 4$$

Taking square root

$$x = \pm 2$$

Put $x^2 = 4$ in equation (ii)

$$3(4) + y^2 = 14$$

$$y^2 = 14 - 12$$

$$y^2 = 2$$

Taking square root

$$y = \pm \sqrt{2}$$

$$\therefore \text{Solution Set} = \{(\pm 2, \pm \sqrt{2})\}$$

Q.9

$$7x^2 - 3y^2 = 4$$

(A.B)

$$2x^2 + 5y^2 = 7$$

Solution:

$$7x^2 - 3y^2 = 4 \rightarrow (i)$$

$$2x^2 + 5y^2 = 7 \rightarrow (ii)$$

Multiply equation (i) by '5'

$$35x^2 - 15y^2 = 20 \rightarrow (iii)$$

Multiply equation (ii) by '3'

$$6x^2 + 15y^2 = 21 \rightarrow (iv)$$

Add equation (iii) and (iv)

$$35x^2 - 15y^2 = 20$$

$$6x^2 + 15y^2 = 21$$

$$41x^2 = 41$$

$$x^2 = 1$$

On taking square root, we get

$$x = \pm 1$$

Put $x^2 = 1$ in equation (ii)

$$2x^2 + 5y^2 = 7$$

$$2(1) + 5y^2 = 7$$

$$2 + 5y^2 = 7$$

$$5y^2 = 5$$

$$y^2 = \frac{5}{5}$$

$$y^2 = 1$$

On taking the square root, we get

$$y = \pm 1$$

$$\therefore \text{Solution Set} = \{(\pm 1, \pm 1)\}$$

Q.10

$$3x^2 - y^2 = 3$$

(A.B)

$$x^2 + 4xy - 5y^2 = 0$$

Solution:

$$3x^2 - y^2 = 3 \rightarrow (i)$$

$$x^2 + 4xy - 5y^2 = 0 \rightarrow (ii)$$

Equation (ii) $\Rightarrow$

$$x^2 + 4xy - 5y^2 = 0$$

$$x^2 + 5xy - xy - 5y^2 = 0$$

$$x(x + 5y) - y(x + 5y) = 0$$

$$(x + 5y)(x - y) = 0$$

Either

$$x + 5y = 0$$

or

$$x - y = 0$$

$$x = -5y \rightarrow (iii)$$

$$x = y \rightarrow (iv)$$

Put $x = -5y$ in equation (i)

$$x^2 + 2y^2 = 3$$

$$(-5y)^2 + 2y^2 = 3$$

$$25y^2 + 2y^2 = 3$$

$$27y^2 = 3$$

$$y^2 = \frac{3}{27}$$

$$y^2 = \frac{1}{9}$$

On taking square root, we get

$$y = \pm \frac{1}{3}$$

Putting in equation (iii)

When

$$y = \frac{1}{3}$$

$$x = -5\left(\frac{1}{3}\right)$$

$$x = -\frac{5}{3}$$

When $y = -\frac{1}{3}$

$$x = -5\left(-\frac{1}{3}\right)$$

$$x = \frac{5}{3}$$

Put $x = y$ in equation (i)

$$x^2 + 2y^2 = 3$$

$$y^2 + 2y^2 = 3$$

$$3y^2 = 3$$

$$y^2 = 1$$

On taking square root, we get

$$y = \pm 1$$

Put in equation (iv)

When $y = 1$

$$x = 1$$

When $y = -1$

$$x = -1$$

∴ **Solution Set**

$$= \left\{ (1, 1), (-1, -1), \left(-\frac{5}{3}, \frac{1}{3}\right), \left(\frac{5}{3}, -\frac{1}{3}\right) \right\}$$

Q.11 $3x^2 - y^2 = 26$

(A.B)

$$3x^2 - 5xy - 12y^2 = 0$$

Solution:

$$3x^2 - y^2 = 26 \rightarrow (i)$$

$$3x^2 - 5xy - 12y^2 = 0 \rightarrow (ii)$$

$$\text{Equation (ii)} \Rightarrow 3x^2 - 5xy - 12y^2 = 0$$

$$3x^2 - 9xy + 4xy - 12y^2 = 0$$

$$3x(x - 3y) + 4y(x - 3y) = 0$$

$$(x - 3y)(3x + 4y) = 0$$

Either

$$x - 3y = 0 \rightarrow (a) \text{ or } 3x + 4y = 0 \rightarrow (b)$$

$$\text{Equation (a)} \Rightarrow x = 3y \rightarrow (iii)$$

Put in equation (i)

$$3(3y)^2 - y^2 = 26$$

$$3(9y^2) - y^2 = 26$$

$$27y^2 - y^2 = 26$$

$$26y^2 = 26$$

$$y^2 = 1$$

$$\Rightarrow y = \pm 1$$

Put in equation (iii)

When $y = -1$

$$x = 3(-1)$$

$$x = -3$$

When $y = 1$

$$x = 3(1)$$

$$x = 3$$

$$\text{Equation (b)} \Rightarrow 3x + 4y = 0$$

$$3x = -4y$$

$$x = \frac{-4}{3}y \rightarrow (iv)$$

Put in equation (i)

$$3\left(\frac{-4}{3}y\right)^2 - y^2 = 26$$

$$\left(\frac{16}{3}y^2\right) - y^2 = 26$$

$$\frac{16y^2 - 3y^2}{3} = 26$$

$$13y^2 = 26 \times 3$$

$$y^2 = 6$$

$$\Rightarrow y = \pm\sqrt{6}$$

Put in equation (iv)

When $y = -\sqrt{6}$

$$x = \frac{-4}{3}(-\sqrt{6})$$

$$x = \frac{4\sqrt{6}}{3}$$

When $y = \sqrt{6}$

$$x = \frac{-4}{3}(\sqrt{6})$$

$$x = \frac{-4\sqrt{6}}{3}$$

$\therefore$ **Solution Set**

$$= \left\{ (-3, -1), (3, 1), \left(\frac{-4\sqrt{6}}{3}, \sqrt{6} \right), \left(\frac{4\sqrt{6}}{3}, -\sqrt{6} \right) \right\}$$

Q.12 $x^2 + xy = 5$

(A.B)

$$y^2 + xy = 3$$

Solution:

$$x^2 + xy = 5 \rightarrow (i)$$

$$y^2 + xy = 3 \rightarrow (ii)$$

Multiply equation (i) by '3' and equation (ii) '5'

$$3x^2 + 3xy = 15 \rightarrow (iii)$$

$$5y^2 + 5xy = 15 \rightarrow (iv)$$

Subtraction equation (iii) and (iv)

$$3x^2 + 3xy = 15$$

$$5y^2 + 5xy = 15$$

$$3x^2 - 2xy - 5y^2 = 0 \rightarrow (v)$$

Equation (v)

$$3x^2 - 5xy + 3xy - 5y^2 = 0$$

$$x(3x - 5y) + y(3x - 5y) = 0$$

$$(3x - 5y)(x + y) = 0$$

Either

$$3x - 5y = 0 \quad \text{or} \quad x + y = 0$$

$$x = \frac{5y}{3} \rightarrow (vi) \quad x = -y \rightarrow (vii)$$

Put $x = \frac{5y}{3}$ in equation (i)

$$\left(\frac{5y}{3} \right)^2 + \left(\frac{5y}{3} \right)y = 5$$

$$\frac{25}{9}y^2 + \frac{5y^2}{3} = 5$$

$$\frac{25y^2 + 15y^2}{9} = 5$$

$$25y^2 + 15y^2 = 45$$

$$40y^2 = 45$$

$$y^2 = \frac{9}{8}$$

On taking square root, we get.

$$y = \pm \frac{3}{2\sqrt{2}}$$

Put $y = \frac{3}{2\sqrt{2}}$ in equation (vi)

$$x = \frac{5}{3} \times \frac{3}{2\sqrt{2}}$$

$$x = \frac{5}{2\sqrt{2}}$$

Now put $y = \frac{-3}{2\sqrt{2}}$ in equation (vi)

$$x = \frac{5}{3} \left(\frac{-3}{2\sqrt{2}} \right)$$

$$x = \frac{-5}{2\sqrt{2}}$$

Now put $x = -y$ in equation (i)

$$(-y)^2 + (-y)(y) = 5$$

$$y^2 - y^2 = 5$$

$$0 = 5$$

$$\text{But } 0 \neq 5$$

$$\therefore \text{Solution Set} = \left\{ \left(\frac{5}{2\sqrt{2}}, \frac{3}{2\sqrt{2}} \right), \left(\frac{-5}{2\sqrt{2}}, \frac{-3}{2\sqrt{2}} \right) \right\}$$

Alternate Method

$$x^2 + xy = 5 \rightarrow (i)$$

$$y^2 + xy = 3 \rightarrow (ii)$$

$$eq(i) \Rightarrow$$

$$x(x + y) = 5 \rightarrow (iii)$$

$$eq(ii) \Rightarrow$$

$$y(x + y) = 3 \rightarrow (iv)$$

Dividing equation (iii) and (iv)

$$\frac{x(x+y)}{y(x+y)} = \frac{5}{3}$$

$$\frac{x}{y} = \frac{5}{3}$$

$$x = \frac{5y}{3}$$

Put $x = \frac{5y}{3}$ in equation (i)

$$\left(\frac{5y}{3}\right)^2 + \left(\frac{5y}{3}\right)y = 5$$

$$\frac{25}{9}y^2 + \frac{5y^2}{3} = 5$$

$$\frac{25y^2 + 15y^2}{9} = 5$$

$$25y^2 + 15y^2 = 45$$

$$40y^2 = 45$$

$$y^2 = \frac{9}{8}$$

On taking square root, we get.

$$y = \pm \frac{3}{2\sqrt{2}}$$

Put $y = \frac{3}{2\sqrt{2}}$ in equation (vi)

$$x = \frac{5}{3} \times \frac{3}{2\sqrt{2}}$$

$$x = \frac{5}{2\sqrt{2}}$$

Now put $y = \frac{-3}{2\sqrt{2}}$ in equation (vi)

$$x = \frac{5}{3} \left(\frac{-3}{2\sqrt{2}}\right)$$

$$x = \frac{-5}{2\sqrt{2}}$$

$$\therefore \text{Solution Set} = \left\{ \left(\frac{5}{2\sqrt{2}}, \frac{3}{2\sqrt{2}} \right), \left(\frac{-5}{2\sqrt{2}}, \frac{-3}{2\sqrt{2}} \right) \right\}$$

Q.13 $x^2 - 2xy = 7$

(A.B)

$$xy + 3y^2 = 2$$

Solution:

$$x^2 - 2xy = 7 \rightarrow (i)$$

$$xy + 3y^2 = 2 \rightarrow (ii)$$

Multiply equation (i) by '2' and equation (ii) by '7'

$$2x^2 - 4xy = 14 \rightarrow (iii)$$

$$7xy + 21y^2 = 14 \rightarrow (iv)$$

Subtract equation (iii) and (iv)

$$2x^2 - 4xy = 14$$

$$\underline{-7xy + 21y^2 = -14}$$

$$2x^2 - 11xy - 21y^2 = 0$$

$$2x^2 - 14xy + 3xy - 21y^2 = 0$$

$$2x(x - 7y) + 3y(x - 7y) = 0$$

$$(x - 7y)(2x + 3y) = 0$$

Either

$$x - 7y = 0$$

or $2x + 3y = 0$

$$x = 7y \rightarrow (v)$$

Put in equation (ii)

$$(7y)y + 3y^2 = 2$$

$$7y^2 + 3y^2 = 2$$

$$10y^2 = 2$$

$$5y^2 = 1$$

$$y^2 = \frac{1}{5}$$

$$y = \pm \frac{1}{\sqrt{5}}$$

Put in equation (v)

$$\text{when } y = -\frac{1}{\sqrt{5}}$$

$$x = 7\left(-\frac{1}{\sqrt{5}}\right)$$

$$x = \frac{-3}{2}y \rightarrow (vi)$$

Put in equation (ii)

$$\left(\frac{-3}{2}y\right)y + 3y^2 = 2$$

$$\frac{-3y^2 + 6y^2}{2} = 2$$

$$3y^2 = 4$$

$$y^2 = \frac{4}{3}$$

$$y = \pm \frac{2}{\sqrt{3}}$$

Put in equation (vi)

$$\text{when } y = \frac{-2}{\sqrt{3}}$$

$$x = \frac{-3}{2}\left(-\frac{2}{\sqrt{3}}\right)$$

$$x = \frac{3}{\sqrt{3}}$$

$$x = \frac{-7}{\sqrt{5}}$$

when

$$y = \frac{1}{\sqrt{5}}$$

$$x = 7\left(\frac{1}{\sqrt{5}}\right)$$

$$x = \frac{7}{\sqrt{5}}$$

∴ Solution Set

$$= \left\{ \left(\frac{7}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right), \left(\frac{-7}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right), \left(-\sqrt{3}, \frac{2}{\sqrt{3}} \right), \left(\sqrt{3}, \frac{-2}{3} \right) \right\}$$

Problems Leading to Quadratic Equations (A.B + K.B + U.B)

Example 1: (Page # 43)

Three less than a certain number multiplied by 9 less than twice the number is 104. Find the number.

Solution:

Let the required number = x

Then, three less than the number = $x - 3$

And, 9 less than twice the number = $2x - 9$

According to given condition

$$(x - 3)(2x - 9) = 104$$

$$2x^2 - 9x - 6x + 27 - 104 = 0$$

$$2x^2 - 15x - 77 = 0$$

$$2x^2 - 22x + 7x - 77 = 0$$

$$2x(x - 11) + 7(x - 11) = 0$$

$$(x - 11)(2x + 7) = 0$$

Either

$$x - 11 = 0 \text{ or } 2x + 7 = 0$$

$$\Rightarrow x = 11 \text{ or } x = -\frac{7}{2}$$

Result:

Thus, required number is either 11 or $-\frac{7}{2}$

$$x = \sqrt{3}$$

$$\text{when } y = \frac{2}{\sqrt{3}}$$

$$x = -\frac{3}{2}\left(\frac{2}{\sqrt{3}}\right)$$

$$x = -\sqrt{3}$$

Example 2: (Page # 44)

(A.B)

The length of rectangle is 4cm more than its breadth. If the area of rectangle is 45cm^2 . Find its sides.

Solution:

Let breadth of rectangle = x

Then, length of rectangle = $x + 4$

Area of rectangle = 45cm^2

According to given condition

$$x(x + 4) = 45$$

$$x^2 + 4x - 45 = 0$$

$$x^2 + 9x - 5x - 45 = 0$$

$$x(x + 9) - 5(x + 9) = 0$$

$$(x + 9)(x - 5) = 0$$

Either

$$x + 9 = 0 \text{ or } x - 5 = 0$$

$$\Rightarrow x = -9 \text{ or } x = 5$$

(Neglecting -ve value)

$$\therefore x + 4 = 5 + 4 = 9$$

Result:

Thus, the breadth is 5cm and length is 9cm

Exercise 2.8

Q.1 The product of two positive consecutive numbers is 182. Find the numbers. (A.B)

Solution:

Let two positive consecutive numbers are $x, x + 1$

According to given condition:

$$x(x + 1) = 182$$

$$x^2 + x - 182 = 0$$

$$x^2 + 14x - 13x - 182 = 0$$

$$x(x + 14) - 13(x + 14) = 0$$

$$(x + 14)(x - 13) = 0$$

Either

$$x + 14 = 0 \text{ or } x - 13 = 0$$

$$x = -14 \text{ or } x = 13$$

(Ignore negative value)

Therefore,

$$x = 13$$

$$\Rightarrow x + 1 = 13 + 1 = 14$$

Result:

Thus, required Numbers are 13 and 14.

Unit-2

Theory of Quadratic Equations

- Q.2** The sum of squares of three positive consecutive numbers is 77. Find the numbers. **(A.B + K.B)**
(SWL 2015)

Solution:

Let three consecutive numbers are

$$x, x+1, x+2$$

According to given condition:

$$x^2 + (x+1)^2 + (x+2)^2 = 77$$

$$x^2 + x^2 + 2x + 1 + x^2 + 4x + 4 - 77 = 0$$

$$3x^2 + 6x - 72 = 0$$

$$3(x^2 + 2x - 24) = 0$$

$$x^2 + 2x - 24 = 0$$

$$x^2 + 6x - 4x - 24 = 0$$

$$x(x+6) - 4(x-6) = 0$$

$$(x+6)(x-4) = 0$$

Either

$$x+6=0 \quad \text{or} \quad x-4=0$$

$$x=-6 \quad \quad \quad x=4$$

(Ignore negative value)

$$\text{Therefore, } x=4$$

$$\Rightarrow x+1=4+1=5$$

$$\& \quad x+2=4+2=6$$

Result:

Thus required numbers are 4, 5 and 6.

- Q.3** The sum of five times a number and the square of the numbers is 204. **(A.B + K.B)**

Solution:

Let required number = x

Five times of the number = $5x$

According to given condition:

$$x^2 + 5x = 204$$

$$x^2 + 5x - 204 = 0$$

$$x^2 + 17x - 12x - 204 = 0$$

$$x(x+17) - 12(x+17) = 0$$

$$(x+17)(x-12) = 0$$

Either

$$x+17=0 \quad \text{or} \quad x-12=0$$

$$x=-17 \quad \quad \quad x=12$$

Result:

Thus required number is either -17 or 12.

- Q.4** The product of five less than three times a certain number and one less than four times the number is 7. Find the number. **(A.B + K.B)**

Solution:

Let the required number is x

Five less than three times the

$$\text{number} = 3x - 5$$

One less than four times a

$$\text{number} = 4x - 1$$

According to given condition

$$(3x-5)(4x-1) = 7$$

$$12x^2 - 3x - 20x + 5 - 7 = 0$$

$$12x^2 - 23x - 2 = 0$$

$$12x^2 - 24x + x - 2 = 0$$

$$12x(x-2) + 1(x-2) = 0$$

$$12x(x-2) + 1(x-2) = 0$$

$$(x-2)(12x+1) = 0$$

Either

$$x-2=0 \quad \text{or} \quad 12x+1=0$$

$$x=2 \quad \quad \quad 12x=-1$$

$$x = \frac{-1}{12}$$

Result:

Thus, required number is either 2

$$\text{or } -\frac{1}{12}.$$

- Q.5** The difference of a number and its reciprocal is $\frac{15}{4}$. Find the number. **(A.B + K.B)**

Solution:

Let, required number is x

$$\text{Reciprocal of the number} = \frac{1}{x}$$

$$\text{Difference of the numbers} = \frac{15}{4}$$

According to given condition

$$x - \frac{1}{x} = \frac{15}{4}$$

$$\frac{x^2 - 1}{x} = \frac{15}{4}$$

By cross multiplication

$$\begin{aligned}
 4x^2 - 4 &= 15x \\
 4x^2 - 15x - 4 &= 0 \\
 4x^2 - 16x + x - 4 &= 0 \\
 4x(x-4) + 1(x-4) &= 0 \\
 (x-4)(4x+1) &= 0
 \end{aligned}$$

Either

$$\begin{aligned}
 x-4 &= 0 & \text{or} & & 4x+1 &= 0 \\
 x &= 4 & & & 4x &= -1 \\
 & & & & x &= \frac{-1}{4}
 \end{aligned}$$

Result:Thus, required number is either 4 or $\frac{-1}{4}$

Q.6 The sum of a number of two digits of a positive integral number is 65 and the number is 9 times the sum of its digits. Find the number.

(A.B + K.B)**Solution:**Let unit's digit = x And ten's digit = y $\therefore$ Required number = $10y + x$

According to given condition (I)

$$x^2 + y^2 = 65 \longrightarrow (i)$$

According to condition (II)

$$10y + x = 9(x + y)$$

$$10y + x = 9x + 9y$$

$$10y - 9y = 9x - x$$

$$y = 8x \longrightarrow (ii)$$

Put in equation (i)

$$x^2 + (8x)^2 = 65$$

$$x^2 + 64x^2 = 65$$

$$65x^2 = 65$$

$$x^2 = 1$$

Taking positive square root

$$x = 1$$

Put in equation (ii)

$$y = 8(1)$$

$$y = 8$$

$$\begin{aligned}
 \text{Required number} &= 10y + x \\
 &= 10(8) + 1 \\
 &= 80 + 1 \\
 &= 81
 \end{aligned}$$

Result:

Thus, required number is 81

Q.7 The sum of the co-ordinates of a point is 9 and sum of their squares is 45. Find the co-ordinates of the point. **(A.B + K.B)**

Solution:Let required point is (x, y)

According to condition I

$$x + y = 9 \longrightarrow (i)$$

According to condition II

$$x^2 + y^2 = 45 \longrightarrow (ii)$$

Equation (i) $\Rightarrow$

$$x = 9 - y \longrightarrow (iii)$$

Put in equation (ii)

$$(9 - y)^2 + y^2 = 45$$

$$81 - 18y + y^2 + y^2 = 45$$

$$2y^2 - 18y + 81 - 45 = 0$$

$$2y^2 - 18y + 36 = 0$$

$$2(y^2 - 9y + 18) = 0$$

$$y^2 - 9y + 18 = 0$$

$$y^2 - 6y - 3y + 18 = 0$$

$$y(y - 6) - 3(y - 6) = 0$$

$$(y - 6)(y - 3) = 0$$

Either

$$y - 6 = 0$$

$$y = 6$$

or

$$y - 3 = 0$$

$$y = 3$$

Put in equation (iii)

$$x = 9 - 6$$

$$x = 3$$

$$x = 9 - (3)$$

$$x = 9 - 3$$

$$x = 6$$

Result:

Thus, required point is either

$$(3, 6) \text{ or } (6, 3)$$

Unit-2

Theory of Quadratic Equations

Q.8 Find two integers whose sum is 9 and the difference of their squares is also 9. **(A.B + K.B)**

Solution:

Let two integer are x and y

According to condition I

$$x + y = 9 \longrightarrow (i)$$

According to condition (i)

$$x^2 - y^2 = 9 \longrightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow y = 9 - x \longrightarrow (iii)$$

Put in equation (ii)

$$x^2 - (9 - x)^2 = 9$$

$$x^2 - (81 - 18x + x^2) = 9$$

$$x^2 - 81 + 18x - x^2 = 9$$

$$18x = 9 + 81$$

$$\Rightarrow x = 5$$

Put in equation (iii)

$$y = 9 - 5 = 4$$

Result:

Thus, required numbers are 5 and 4.

Q.9 Find two integers whose difference is 4 and whose squares differ by 72. **(A.B + K.B)**

Solution:

Let the integers are x and y

According to condition-I

$$x - y = 4 \longrightarrow (i)$$

According to condition-II

$$x^2 - y^2 = 72 \longrightarrow (ii)$$

$$\text{Equation (i)} \Rightarrow x = y + 4 \longrightarrow (iii)$$

Put in equation (ii)

$$(y + 4)^2 - y^2 = 72$$

$$y^2 + 8y + 16 - y^2 = 72$$

$$8y = 72 - 16$$

$$8y = 56$$

$$\Rightarrow y = 7$$

Put in equation (iii)

$$x = 7 + 4$$

$$x = 11$$

Result:

Thus, required integers are 11 and 7.

Q.10 Find the dimensions of a rectangle, whose perimeter is 80cm and its area is 375cm². **(K.B + A.B)**

Solution:

Let length of rectangle = x cm

And width of rectangle = y cm

According to condition-I

$$2(x + y) = 80$$

$$\therefore \text{perimeter} = 2(\text{length} + \text{width})$$

$$x + y = 40 \longrightarrow (i)$$

$$\therefore \text{Area} = \text{length} \times \text{width}$$

According to condition-II

$$xy = 375 \longrightarrow (ii)$$

From equation (i)

$$y = 40 - x \longrightarrow (iii)$$

Put in (ii)

$$x(40 - x) = 375$$

$$40x - x^2 - 375 = 0$$

$$-x^2 - 40 - 375 = 0$$

$$x^2 - 40x + 375 = 0$$

$$x^2 - 25x - 15x + 375 = 0$$

$$x(x - 25) - 15(x - 25) = 0$$

$$(x - 25)(x - 15) = 0$$

$$\text{Either } x - 25 = 0 \quad \text{or} \quad x - 15 = 0$$

$$x = 25$$

$$x = 15$$

Put in equation (iii)

$$y = 40 - 25$$

$$y = 40 - 15$$

$$y = 15$$

$$y = 25$$

Result:

Dimension of rectangle are either 25cm by 15cm or 15cm by 25cm.

Miscellaneous Exercise 2

Q.1 Multiple Choice Questions

Four possible answers are given for the following question. Tick (✓) the correct answer.

- (i) If α, β are the roots of $3x^2 + 5x - 2 = 0$, then $\alpha + \beta$ is; **(K.B + A.B)**
(LHR 2017, SWL 2014, MTN 2015, 17, SGD 2015, 17, RWP 2016, D.G.K 2016)

- (a) $\frac{5}{3}$ (b) $\frac{3}{5}$
(c) $\frac{-5}{3}$ (d) $\frac{-2}{3}$

- (ii) If α, β are the roots of $7x^2 - x + 4 = 0$, then $\alpha\beta$ is; **(K.B + U.B)**
(LHR 2014, GRW 2014, 15, FSD 2016, BWP 2016)

- (a) $\frac{-1}{7}$ (b) $\frac{4}{7}$
(c) $\frac{7}{4}$ (d) $\frac{-4}{7}$

- (iii) Roots of the equation $4x^2 - 5x + 2 = 0$ are; **(K.B + A.B)**

- (a) Irrational (b) Imaginary
(c) Rational (d) None of these

- (iv) Cube roots of -1 are; **(K.B + U.B)**
(LHR 2017, GRW 2017, SWL 2017, MTN 2014, 17, SGD 2015, 16, D.G.K 2017)

- (a) $-1, -\omega, -\omega^2$ (b) $-1, \omega, -\omega^2$
(c) $-1, -\omega, \omega^2$ (d) $1, -\omega, -\omega^2$

- (v) Sum of the cube roots of unity is; **(K.B + A.B)**

- (a) 0 (b) 1
(c) -1 (d) 3

- (vi) Product of cube roots of unity is; **(K.B + A.B)**
(LHR 2016, GRW 2014, 16, SGD 2015, 17, BWP 2016, 17, RWP 2017)

- (a) 0 (b) 1
(c) -1 (d) 3

- (vii) If $b^2 - 4ac < 0$, then the roots of $ax^2 + bx + c = 0$ are; (GRW 2014) **(K.B + A.B)**

- (a) Irrational (b) Rational
(c) Imaginary (d) None of these

- (viii) If $b^2 - 4ac > 0$, but not a perfect square then roots of $ax^2 + bx + c = 0$ are; **(K.B)**
(LHR 2014, BWP 2017)

- (a) Imaginary (b) Rational
(c) Irrational (d) None of these

- (ix) $\frac{1}{\alpha} + \frac{1}{\beta}$ is equal to; **(K.B + U.B)**

(LHR 2014, 15, GRW 2016, FSD 2017, BWP 2017, RWP 2016, SGD 2017)

- (a) $\frac{1}{\alpha}$ (b) $\frac{1}{\alpha} - \frac{1}{\beta}$
(c) $\frac{\alpha - \beta}{\alpha\beta}$ (d) $\frac{\alpha + \beta}{\alpha\beta}$

Unit-2

Theory of Quadratic Equations

- (x) $\alpha^2 + \beta^2$ is equal to; **(U.B + A.B)**
(LHR 2014, 15, GRW 2014, 17, FSD 2016, BWP 2015, RWP 2016, 17)
- (a) $\alpha^2 - \beta^2$ (b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$
(c) $(\alpha + \beta)^2 - 2\alpha\beta$ (d) $\alpha + \beta$
- (xi) Two square roots of unity are; **(U.B + A.B)**
(LHR 2015, 16, GRW 2014, FSD 2015, 16, MTN 2016, SGD 2016, D.G.K 2015, 16, 17)
- (a) $1, -1$ (b) $1, \omega$
(c) $1, -\omega$ (d) ω, ω^2
- (xii) Roots of the equation $4x^2 - 4x + 1 = 0$ are; **(U.B + A.B)**
(LHR 2015, GRW 2017, FSD 2016, BWP 2015, MTN 2017, SGD 2016)
- (a) Real equal (b) Real unequal
(c) Imaginary (d) Irrational
- (xiii) If α, β are the roots of $px^2 + qx + r = 0$, then sum of the roots 2α and 2β is; **(K.B)**
- (a) $\frac{-q}{p}$ (b) $\frac{r}{p}$
(c) $\frac{-2q}{p}$ (d) $-\frac{q}{2p}$
- (xiv) If α, β are the roots of $x^2 - x - 1 = 0$, then product of the roots 2α and 2β is; **(U.B)**
(FSD 2014, 17, BWP 2016, D.G.K 2015, 16, 17)
- (a) -2 (b) 2
(c) 4 (d) -4
- (xv) The nature of the roots of equation $ax^2 + bx + c = 0$ is determined by; **(A.B)**
(GRW 2016, SWL 2015, 2017, MTN 2015)
- (a) Sum of the roots (b) Product of the roots
(c) Synthetic division (d) Discriminant
- (xvi) The discriminant of $ax^2 + bx + c = 0$ is; **(K.B + A.B)**
(LHR 2016, FSD 2017, SWL 2016, 17, RWP 2014, 16, SGD 2016, MTN 2015, D.G.K 2016)
- (a) $b^2 - 4ac$ (b) $b^2 + 4ac$
(c) $-b^2 + 4ac$ (d) $-b^2 - 4ac$

ANSWER KEY

(i)	c	(v)	a	(ix)	d	(xiii)	c
(ii)	b	(vi)	b	(x)	c	(xiv)	d
(iii)	b	(vii)	c	(xi)	a	(xv)	d
(iv)	a	(viii)	c	(xii)	a	(xvi)	a

Unit-2

Theory of Quadratic Equations

Q.2

- (i) Discuss the nature of the roots of the following equations. **(A.B)**

Solution:

(a) $x^2 + 3x + 5 = 0$

Here $a = 1, b = 3, c = 5$

$$\begin{aligned}\text{Disc} &= b^2 - 4ac \\ &= (3)^2 - 4(1)(5) \\ &= 9 - 20 \\ &= -11 \\ &< 0\end{aligned}$$

∴ Roots are complex conjugate or imaginary.

(b) $2x^2 - 7x + 3 = 0$ **(A.B)**
(GRW 2016, SGD 2014, RWP 2017, D.G.K 2016)

Here $a = 2, b = -7, c = 3$

$$\begin{aligned}\text{Disc} &= b^2 - 4ac \\ &= (-7)^2 - 4(2)(3) \\ &= 49 - 24 \\ &= 25\end{aligned}$$

Since disc > 0 and perfect square roots are rational and unequal.

(c) $x^2 + 6x - 1 = 0$ **(A.B)**

Here $a = 1, b = 6, c = -1$

$$\begin{aligned}\text{Disc} &= b^2 - 4ac \\ &= (6)^2 - 4(1)(-1) \\ &= 36 + 4 \\ &= 40\end{aligned}$$

Since Disc. > 0 and not a perfect square roots are irrational and unequal.

(d) $16x^2 - 8x + 1 = 0$ (FSD 2017) **(A.B)**

Here $a = 16, b = -8, c = 1$

$$\begin{aligned}\text{Disc.} &= b^2 - 4ac \\ &= (-8)^2 - 4(16)(1) \\ &= 64 - 64 \\ &= 0\end{aligned}$$

Since, Disc. = 0, roots are rational and equal.

(ii) Find ω^2 , if $\omega = \frac{-1 + \sqrt{-3}}{2}$ **(A.B)**
(GRW 2017, SGD 2014, RWP 2016)

Solution:

Here

$$\omega = \frac{-1 + \sqrt{-3}}{2}$$

Square both sides

$$(\omega)^2 = \left(\frac{-1 + \sqrt{-3}}{2} \right)^2$$

$$\omega^2 = \frac{(-1)^2 + (\sqrt{-3})^2 + 2(-1)(\sqrt{-3})}{(2)^2}$$

$$\omega^2 = \frac{1 + (-3) - 2\sqrt{-3}}{4}$$

$$\omega^2 = \frac{-2 - 2\sqrt{-3}}{4}$$

$$\omega^2 = 2 \left(\frac{-1 - \sqrt{-3}}{4} \right)$$

$$\omega^2 = \frac{-1 - \sqrt{-3}}{2}$$

- (iii) Prove that the sum of all cube roots of unity is zero. **(A.B + K.B)**

Ans. See property of cube roots Page # 45

- (iv) Find the product of complex cube roots of unity. **(A.B + K.B)**

Ans. See property of cube roots Page # 44

- (v) Show that:

$$x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y)$$

(A.B + K.B)

Ans. See Exe-2.2 Q.3 Page # 47

- (vi) Evaluate: $\omega^{37} + \omega^{38} + 1$

(A.B + K.B)

Solution:

$$\begin{aligned}\omega^{37} + \omega^{38} + 1 &= \omega^{36} \cdot \omega + \omega^{36} \cdot \omega^2 + 1 \\ &= (\omega^3)^{12} \cdot \omega + (\omega^3)^{12} \cdot \omega^2 + 1 \\ &= (1)^{12} \cdot \omega + (1)^{12} \cdot \omega^2 + 1 \\ &= \omega + \omega^2 + 1 \\ &= 0\end{aligned}$$

$$\therefore \omega^{37} + \omega^{38} + 1 = 0$$

- (vii) Evaluate $(1 - \omega + \omega^2)^6$ **(A.B + K.B)**

Solution:

$$\begin{aligned}(1 - \omega + \omega^2)^6 &= [1 + \omega^2 - \omega]^6 \\ &= (-\omega - \omega)^6 \\ &= (-2\omega)^6 \\ &= (-2)^6 \omega^6 \\ &= 64(\omega^3)^2 \\ &= 64(1)^2 \\ &= 64(1) \\ &= 64\end{aligned}$$

- (viii) If ω is cube root of unity, form an equation whose roots are 3ω and $3\omega^2$. **(A.B + K.B)**

Solution:

Roots of required equation are 3ω and $3\omega^2$.

$$\begin{aligned}\text{Sum of roots} = S &= 3\omega + 3\omega^2 \\ &= 3(\omega + \omega^2) \\ &= 3(-1) \\ &= -3\end{aligned}$$

$$\begin{aligned}\text{Product of roots} = P &= (3\omega)(3\omega^2) \\ &= 9\omega^3 \\ &= 9(1) \\ &= 9\end{aligned}$$

$\therefore$ Required quadratic equation is

$$x^2 - Sx + P = 0$$

$$x^2 - (-3)x + 9 = 0$$

$$x^2 + 3x + 9 = 0$$

- (ix) Use synthetic division, find the remainder and quotient when $(x^3 + 3x^2 + 2) \div (x - 2)$. **(A.B)**

Solution:

$$P(x) = x^3 + 3x^2 + 2$$

$$= x^3 + 3x^2 + 0x + 2$$

2	1	3	0	2
	↓	2	10	20
	1	5	10	22

$\therefore$ Remainder = 22

$$Q(x) = x^2 + 5x + 10$$

- (x) Use synthetic division, show that $x - 2$ is the factor $x^3 + x^2 - 7x + 2$. **(A.B + K.B)**

Solution:

$$P(x) = x^3 + x^2 - 7x + 2$$

2	1	1	-7	2
	↓	2	6	-2
	1	3	-1	0

Since remainder is zero, $x - 2$ is a factor of given polynomial.

- (xi) Find the sum and product of the roots of the equation $2Px^2 + 3qx - 4r = 0$. **(A.B + K.B)**

Solution:

$$2Px^2 + 3qx - 4r = 0$$

$$\text{Here } a = 2P, b = 3q, c = -4r$$

$$\text{Sum of roots} = \frac{-b}{a}$$

$$\begin{aligned}&= -\frac{3q}{2P} \\ \text{Product of roots} &= \frac{c}{a} \\ &= \frac{-4r}{2P} \\ &= -\frac{2r}{P}\end{aligned}$$

- (xii) Find $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ when α, β are of the roots of the equation $x^2 - 4x + 3 = 0$. **(A.B + K.B)**

Solution:

$$x^2 - 4x + 3 = 0$$

$$\text{Here } a = 1, b = -4, c = 3$$

Let roots of given equation are α, β

$$\begin{aligned}\text{Then sum of roots} = \alpha + \beta &= -\frac{b}{a} \\ &= -\frac{-4}{1} \\ &= 4\end{aligned}$$

$$\begin{aligned}\text{Product of roots} = \alpha\beta &= \frac{c}{a} \\ \alpha\beta &= \frac{3}{1} \\ \alpha\beta &= 3\end{aligned}$$

Consider

$$\begin{aligned}\frac{1}{\alpha^2} + \frac{1}{\beta^2} &= \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} \\ &= \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} \\ &= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} \\ &= \frac{(4)^2 - 2(3)}{(3)^2} \\ &= \frac{16 - 6}{9} \\ \Rightarrow \frac{1}{\alpha^2} + \frac{1}{\beta^2} &= \frac{10}{9}\end{aligned}$$

(xiii) If α, β are the roots of $4x^2 - 3x + 6 = 0$, find (A.B)

(a) $\alpha^2 + \beta^2$

(b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(c) $\alpha - \beta$

Solution:

$$4x^2 - 3x + 6 = 0$$

Roots of given equation are α, β

$$\alpha + \beta = -\frac{-3}{4}$$

$$= \frac{3}{4}$$

$$\alpha\beta = \frac{6}{4}$$

$$= \frac{3}{2}$$

(a) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ (A.B)

$$= \left(\frac{3}{4}\right)^2 - 2\left(\frac{3}{2}\right)$$

$$= \frac{9}{16} - 3$$

$$= \frac{9 - 48}{16}$$

$$\Rightarrow \alpha^2 + \beta^2 = \frac{-39}{16}$$

(b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$ (A.B)

$$= \frac{-39/16}{3/2}$$

$$= \frac{-39}{16} \times \frac{2}{3}$$

$$\Rightarrow \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{-13}{8}$$

(c) $\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$ (A.B)

$$= \sqrt{\left(\frac{3}{4}\right)^2 - 4\left(\frac{3}{2}\right)}$$

$$= \sqrt{\left(\frac{3}{4}\right)^2 - 4\left(\frac{3}{2}\right)}$$

$$= \sqrt{\frac{9}{16} - 6}$$

$$= \sqrt{\frac{9 - 96}{16}}$$

$$= \sqrt{\frac{-87}{16}}$$

$$= \frac{\sqrt{-87}}{4}$$

$$= \frac{\sqrt{-87}}{4}$$

$$= \frac{\sqrt{-87}}{4}$$

$$= \frac{\sqrt{-87}}{4}$$

$$= \frac{\sqrt{-87}}{4}$$

(xiv) If α, β are the roots of

$x^2 - 5x + 7 = 0$, find an equation whose roots are

(a) $-\alpha, -\beta$

(b) $2\alpha, 2\beta$

Solution:

$$x^2 - 5x + 7 = 0$$

Roots of given equation are α, β

$$\alpha + \beta = -\frac{-5}{1}$$

$$= 5$$

$$\alpha\beta = \frac{7}{1}$$

$$\alpha\beta = 7$$

(a) Roots of required equation are

$-\alpha, -\beta$ (A.B)

$$\text{Sum of roots} = S = -\alpha + (-\beta)$$

$$= -(\alpha + \beta)$$

$$= -5$$

$$\text{Prod. Of roots} = P = (-\alpha)(-\beta)$$

$$= \alpha\beta$$

$$= 7$$

$\therefore$ Required equation is:

$$x^2 - Sx + P = 0$$

$$x^2 - (-5)x + 7 = 0$$

$$x^2 + 5x + 7 = 0$$

(b) Roots of required equation are

$2\alpha, 2\beta$ (A.B)

$$\text{Sum of roots} = S = 2\alpha + 2\beta$$

$$= 2(\alpha + \beta)$$

$$= 2(5)$$

$$= 10$$

$$\text{Prod. Of roots} = P = (2\alpha)(2\beta)$$

$$= 4\alpha\beta$$

$$= 4(7)$$

$$= 28$$

$\therefore$ Required quadratic equation is:

$$x^2 - Sx + P = 0$$

$$x^2 - 10x + 28 = 0$$

Unit-2

Theory of Quadratic Equations

Q.3 Fill in the blanks

- (i) The discriminant of $ax^2 + bx + c = 0$ is _____. (K.B)
- (ii) If $b^2 - 4ac = 0$, then roots of $ax^2 + bx + c = 0$ are _____. (K.B)
- (iii) If $b^2 - 4ac > 0$, then the roots of $ax^2 + bx + c = 0$ are _____. (K.B)
- (iv) If $b^2 - 4ac < 0$, then the roots of $ax^2 + bx + c = 0$ are _____. (K.B)
- (v) If $b^2 - 4ac < 0$ and perfect square, then the roots of $ax^2 + bx + c = 0$ are _____. (K.B)
- (vi) If $b^2 - 4ac < 0$ and not a perfect square, then roots of $ax^2 + bx + c = 0$ are _____. (K.B)
- (vii) If α, β are the roots of $ax^2 + bx + c = 0$, then sum of the roots is _____. (K.B)
- (viii) If α, β are the roots of $ax^2 + bx + c = 0$, then product of the roots is _____. (K.B)
- (ix) If α, β are the roots of $7x^2 - 5x + 3 = 0$, then sum of the roots is _____. (K.B)
- (x) If α, β are the roots of $5x^2 + 3x - 9 = 0$, then product of the roots is _____. (K.B)
- (xi) For a quadratic equation $ax^2 + bx + c = 0$, $\frac{1}{\alpha\beta}$ is equal to _____. (K.B)
- (xii) Cube roots of unity are _____. (K.B)
- (xiii) Under usual notation sum of the roots of unity is _____. (K.B)
- (xiv) If $1, \omega, \omega^2$ are the cube roots of unity, then ω^{-7} is equal to _____. (K.B)
- (xv) If α, β are the roots of the quadratic equation, then the quadratic equation is written as _____. (K.B)
- (xvi) If 2ω and $2\omega^2$ are the roots of an equation, then equation is _____. (K.B)

ANSWER KEY

- | | |
|------------------------|--|
| (i) $b^2 - 4ac$ | (x) $\frac{-9}{5}$ |
| (ii) Equal | (xi) $\frac{a}{c}$ |
| (iii) Real | (xii) $1, \omega, \omega^2$ |
| (iv) Imaginary | (xiii) Zero |
| (v) Rational | (xiv) ω^2 |
| (vi) Irrational (real) | (xv) $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ |
| (vii) $-\frac{b}{a}$ | (xvi) $x^2 + 2x + 4 = 0$ |
| (viii) $\frac{c}{a}$ | |
| (ix) $\frac{5}{7}$ | |



Unit-2

Theory of Quadratic Equations

SELF TEST

Time: 40 min

Marks: 25

Q.1 Four possible answers (A), (B), (C) & (D) to each question are given, mark the correct answer. (7×1=7)

1 $\omega^{-7} = \underline{\hspace{2cm}}$ is:

(A) ω^2

(B) ω

(C) 1

(D) 0

2 Which is not a symmetric function?

(A) $\alpha^2 - \beta^2$

(B) $\alpha^2 + \beta^2$

(C) $\alpha^3 + \beta^3$

(D) $\frac{1}{\alpha} + \frac{1}{\beta}$

3 If $\frac{3}{2}, \frac{1}{2}$ are the roots of a quadratic equation, then required quadratic equation is:

(A) $2x^2 + 2x + 3 = 0$

(B) $4x^2 + 8x + 3 = 0$

(C) $x^2 + 4x + 3 = 0$

(D) $4x^2 - 8x + 3 = 0$

4 If roots of a quadric equation $x^2 + qx + p = 0$ are the additive inverse of each other, then:

(A) $p = 0, q = 0$

(B) $p = 0$

(C) $q = 0$

(D) $p = 1, q = 1$

5 What will be the remainder if $4x^3 - 5x + 15$ is divided by $x + 3$?

(A) 78

(B) 139

(C) -78

(D) 125

6 Cube roots of -1 are:

(A) $-1, -\omega, -\omega^2$

(B) $-1, \omega, -\omega^2$

(C) $-1, -\omega, \omega^2$

(D) $1, \omega, \omega^2$

7 Roots of the equation $4x^2 - 5x + 2 = 0$ are:

(A) Irrational

(B) Imaginary

(C) Rational

(D) None

Q.2 Give Short Answers to following Questions.

(5×2=10)

(i) Evaluate: $(1 - 3\omega - 3\omega^2)^5$.

(ii) Prove that each complex cube root of unity is reciprocal of the other.

(iii) Show that the roots of the equation $(p + q)x^2 - px - q = 0$ are rational.

(iv) If ω is a cube root of unity, form an equation whose roots are 2ω and $2\omega^2$.

(v) Use synthetic division to find the quotient and the remainder when the polynomial $x^4 - 10x^3 - 2x$ is divided by $x + 3$.

Q.3 Answer the following Questions.

(4+4=8)

(a) Prove that $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

(b) Find two integers whose sum is 9 and the difference of their squares is also 9.

NOTE: Parents or guardians can conduct this test in their supervision in order to check the skill of students.



Ratio

(K.B)

(FSD 2014, MTN 2015, SGD 2015, RWP 2014, 15, D.G.I. 2015)

Ratio is a relation between two quantities of same kind. It tells what part one quantity of an other or how much time one quantity of other.

For any two quantities a and b of the same kind, it is represented as, $a:b = \frac{a}{b}$, where

$b \neq 0$.

For example, if a hockey team wins 4 games and losses 5, then the ratio of the games won to lost is $4:5$ or $\frac{4}{5}$.

Antecedent

(K.B + U.B)

In ratio $a:b$, the first term a is called antecedent.

For example: In $2:3$, 2 is antecedent.

Consequent

(K.B + U.B)

In the ratio $a:b$, the second term b is called consequent.

For example: In $2:3$, 3 is consequent.

Proportion

(SGD 2017)

(K.B + U.B)

The statement of equality of two ratios is called proportion. If two ratios $a:b$ and $c:d$ are equal then, we can write it as $a:b::c:d$ or $a:b = c:d$, where quantities a and d are called extremes, while b and c are called means proportions.

Note

(K.B + U.B)

- Ratio has no unit.
- The order of elements in ratio is important. i.e. $a:b \neq b:a$
- Product of extremes = Product of means

Example 1: (Page # 50)

(K.B + U.B)

Find the ratio of

- 200gm to 700gm
- 1km to 600m

UNIT

3

VARIATIONS

Solution:

- Ratio of 200gm to 700gm (A.B)

$$200:700 = \frac{200}{700} = \frac{2}{7} = 2:7$$

Where, $2:7$ is the simplest (lowest) form of the ratio $200:700$.

- Ratio of 1 km to 600 m (A.B)

Since $1\text{km} = 1000\text{m}$

$$\begin{aligned} \text{Then } 1\text{km}:600\text{m} &= 1000\text{m}:600\text{m} \\ &= 1000:600 \\ &= 10:6 \\ &= 5:3 \end{aligned}$$

Example 2: (Page # 50)

(A.B)

Find a , if the ratios $a+3:7+a$ and $4:5$ are equal.

Solution:

Since the ratios $a+3:7+a$ and $4:5$ are equal.

$$\frac{a+3}{7+a} = \frac{4}{5} \quad \therefore \text{ in fraction form}$$

$$5(a+3) = 4(7+a)$$

$$5a+15 = 28+4a$$

$$5a-4a = 28-15$$

$$a = 13$$

Thus, given ratios will be equal if $a = 13$.

Example 3: (Page # 51)

(A.B)

If 2 is added in each number of the ratio $3:4$, we get a new ratio $5:6$. Find the numbers.

Solution:

Ratio of two numbers is $3:4$.

Multiply each number of the ratio with x . Then the numbers be $3x, 4x$.

Now according to the given condition

$$\frac{3x+2}{4x+2} = \frac{5}{6}$$

$$6(3x+2) = 5(4x+2) \Rightarrow 18x+12 = 20x+10$$

$$18x - 20x = 10 - 12 \Rightarrow -2x = -2 \Rightarrow x = 1$$

Therefore

$$3x = 3(1) = 3$$

$$4x = 4(1) = 4$$

Thus, the required numbers are 3, 4

Example 4: (Page # 51) (A.B)

Find the ratio $3a + 4b : 5a + 7b$

if $a : b = 5 : 3$

Solution:

Give that $a : b = 5 : 3$ or $\frac{a}{b} = \frac{5}{3}$

$$\text{Now } 3a + 4b : 5a + 7b = \frac{3a + 4b}{5a + 7b}$$

(Dividing numerator and denominator by b)

$$= \frac{\frac{3a + 4b}{b}}{\frac{5a + 7b}{b}} = \frac{3\left(\frac{a}{b}\right) + 4\left(\frac{b}{b}\right)}{5\left(\frac{a}{b}\right) + 7\left(\frac{b}{b}\right)}$$

$$= \frac{3\left(\frac{5}{3}\right) + 4(1)}{5\left(\frac{5}{3}\right) + 7(1)} \quad \left(\because \frac{a}{b} = \frac{5}{3}\right)$$

$$= \frac{\frac{15}{3} + 4}{\frac{25}{3} + 7} = \frac{\frac{15 + 12}{3}}{\frac{25 + 21}{3}} = \frac{27}{46}$$

Hence, $3a + 4b : 5a + 7b = 27 : 46$

Example 6: (Page # 52) (LHR 2014) (A.B)

Find the cost of 15kg of sugar, if 7kg of sugar costs 560 rupees.

Solution:

Let the cost of 15kg of sugar be x rupees.

Then in proportion form

$$15\text{kg} : 7\text{kg} :: \text{Rs. } x : \text{Rs. } 560$$

$$15 : 7 = x : 560$$

Product of extremes = Product of means

$$15 \times 560 = 7x$$

$$7x = 15 \times 560$$

$$x = \frac{15 \times 560}{7} = 15(10) = 1200$$

Thus, Cost of 15 kg sugar is Rs. 1200.

Exercise 3.1

Q.1 Expressing as a ratio and as a fraction.

Solution:

(i) Here (GRW 2016, FSD 2017) **(A.B)**

$$a = \text{Rs. } 750$$

$$b = \text{Rs. } 1250$$

Now,

$$a : b = 750 : 1250 \quad (\div \text{ by } 10)$$

$$= 75 : 125 \quad (\div \text{ by } 5)$$

$$= 15 : 25 \quad (\div \text{ by } 5)$$

$$\Rightarrow a : b = 3 : 5$$

$\therefore$ The fractional form of this expression is $\frac{3}{5}$

(ii) Here **(A.B)**

$$a = 450 \text{ cm}$$

$$b = 3\text{m} = 3 \times 100\text{cm} = 300 \text{ cm}$$

Now,

$$a : b = 450 : 300 \quad (\div \text{ by } 10)$$

$$= 45 : 30 \quad (\div \text{ by } 5)$$

$$\Rightarrow a : b = 3 : 2 \quad (\div \text{ by } 15)$$

$\therefore$ The fractional form of this expression is $\frac{3}{2}$

(iii) Here **(A.B)**

$$a = 4\text{kg} = 4 \times 1000\text{gm} = 4000\text{gm}$$

$$b = 2\text{kg } 750\text{ gm} = [(2 \times 1000) + 750]\text{gm}$$

$$= 2750\text{gm}$$

Now

$$a : b = 4000 : 2750$$

$$= 400 : 275 \quad (\div \text{ by } 10)$$

$$= 80 : 55 \quad (\div \text{ by } 5)$$

$$\Rightarrow a : b = 16 : 11 \quad (\div \text{ by } 5)$$

The fractional form of this expression is $\frac{16}{11}$

$$\begin{aligned} \text{(iv)} \quad a &= 27 \text{ min } 30 \text{ sec} = (27 \times 60 + 30) \text{ sec} \\ &= (1620 + 30) \text{ sec} \\ &= 1650 \text{ sec} \end{aligned}$$

$$\begin{aligned} b &= 1 \text{ hour} = 1 \times 60 \times 60 \text{ sec} \\ &= 3600 \text{ sec} \end{aligned}$$

$$\begin{aligned} \Rightarrow a : b &= 1650 \text{ sec} : 3600 \text{ sec} \\ &= 1650 : 3600 \\ &= 165 : 360 \quad (\div \text{ by } 10) \\ &= 33 : 72 \quad (\div \text{ by } 3) \\ &= 11 : 24 \quad (\div \text{ by } 3) \end{aligned}$$

The fractional form of this expression is $\frac{11}{24}$

(v) Here **(A.B)**
(SWL 2015, BWP 2016, D.G.K 2014)

$$a = 75^\circ, b = 225^\circ$$

$$\text{Now, } a : b = 75 : 225 \quad (\div 25)$$

$$= 3 : 9 \quad (\div 3)$$

$$\Rightarrow a : b = 1 : 3$$

The fractional form of this

expression is $\frac{1}{3}$.

Q.2 In a class of 60 students, 25 students are girls and remaining students are boys. Computer the ratio of

Given (A.B + K.B)

Total students in the class = 60

Number of girls in the class = 25

Number of boys in the class = $60 - 25 = 35$

Required

(i) Ratio of boys to total students

(ii) Ratio of boy to girls.

Solution:

(i) **(A.B)**

Now,

$$\begin{aligned} \text{Boys : Total students} &= 35 : 60 \\ &= 7 : 12 \quad (\div 5) \end{aligned}$$

(ii) Here, **(A.B)**

$$\begin{aligned} \text{Boys : Girls} &= 35 : 25 \\ &= 7 : 5 \quad (\div \text{ by } 5) \end{aligned}$$

Q.3 If $3(4x-5y) = 2x-7y$, find the ratios $x : y$. **(A.B)**

Given

$$3(4x-5y) = 2x-7y$$

Required

$$x : y = ?$$

(LHR 2015)

(MTN 2016)

Solution:

Here

$$3(4x-5y) = 2x-7y$$

$$12x-15y = 2x-7y$$

$$12x-2x = 15y-7y$$

$$10x = 8y$$

$$\frac{x}{y} = \frac{8}{10}$$

$$\frac{x}{y} = \frac{4}{5}$$

$$\Rightarrow x : y = 4 : 5$$

Result:

$$x : y = 4 : 5$$

Q.4 Find the value of p , if the ratios $2p+5:3p+4$ and $3:4$ are equal.

Find value of ' p ' **(A.B)**

(GRW 2015, SWL 2016, RWP 2015, 17)

Solution:

According to given condition.

$$2p+5:3p+4 = 3:4$$

$$\frac{2p+5}{3p+4} = \frac{3}{4}$$

By cross multiplication

$$4(2p+5) = 3(3p+4)$$

$$8p+20 = 9p+12$$

$$8p-9p = 12-20$$

$$-p = -8$$

$$\Rightarrow p = 8$$

Result:

$$p = 8$$

Q.5 If the ratios $3x+1:6+4x$ and $2:5$ are equal. Find the value of x .

Solution: **(D.G.K 2015) (A.B + K.B)**

Here

$$3x+1:6+4x = 2:5$$

$$\Rightarrow \frac{3x+1}{6+4x} = \frac{2}{5}$$

By cross multiplication, we get

$$5(3x+1) = 2(6+4x)$$

$$15x+5 = 12+8x$$

$$15x-8x = 12-5$$

$$7x = 7$$

$$x = \frac{7}{7}$$

$$x = 1$$

Result:

$$x = 1$$

Q.6 Two numbers are in the ratio 5:8. If 9 is added to each number, we get a new ratio 8:11. Find the numbers. (FSD 2016) **(A.B + K.B)**

Solution:

Ratio between two numbers = 5:8

Let required numbers are 5x, 8x

According to given condition

$$5x+9 : 8x+9 = 8:11$$

$$\frac{5x+9}{8x+9} = \frac{8}{11}$$

By cross multiplication

$$11(5x+9) = 8(8x+9)$$

$$55x+99 = 64x+72$$

$$55x-64x = 72-99$$

$$-9x = -27$$

$$x = \frac{-27}{-9}$$

$$x = 3$$

Now

$$5x = 5(3) = 15$$

And

$$8x = 8(3) = 24$$

Result:

∴ Required numbers are 15 and 24

Q.7 If 10 is added in each number of the ratio 4:13, we get a new ratio 1:2. What are the numbers? **(A.B + K.B)**

Solution:

Ratio between two numbers = 4:13

Let, the two numbers be 4x & 13x.

According to the given condition;

$$4x+10 : 13x+10 = 1:2$$

$$\frac{4x+10}{13x+10} = \frac{1}{2}$$

$$2(4x+10) = 1(13x+10)$$

$$8x+20 = 13x+10$$

$$20-10 = 13x-8x$$

$$10 = 5x$$

$$\frac{10}{5} = x$$

$$x = 2$$

$$\Rightarrow \therefore 4x = 4(2) = 8$$

$$\Rightarrow 13x = 13(2) = 26$$

Result:

∴ Required two numbers are 8 and 26.

Q.8 Find the cost of 8kg of mangoes, if 5kg of mangoes cost Rs. 250.

Solution: **(A.B + K.B)**

Weight of mangoes = 5 kg

Cost of mangoes of 5kg = Rs.250

Now, weight of mangoes = 8kg

Here

weight: weight :: cost : cost

$$5:8 :: 250:x$$

Product of extreme = product of means

$$5x = 250 \times 8$$

$$5x = 2000$$

$$x = \frac{2000}{5}$$

$$x = 400$$

Result:

∴ Cost of 8kg of mangoes is Rs. 400.

Q.9 If $a:b = 7:6$, find the value of $3a+5b:7b-5a$. **(A.B + K.B)**

Solution: (FSD 2015)

Here

$$a:b = 7:6$$

$$\Rightarrow \frac{a}{b} = \frac{7}{6}$$

$$\text{or } a = \frac{7}{6}b$$

Consider

$$\begin{aligned} 3a+5b:7b-5a &= 3\left(\frac{7}{6}b\right)+5b:7b-5\left(\frac{7}{6}b\right) \\ &= \frac{21b+30b}{6} : \frac{42b-35b}{6} \end{aligned}$$

$$= \frac{51b}{6} : \frac{7b}{6}$$

$$= 51b : 7b \quad (\times \text{ by } 6)$$

$$\Rightarrow 3a + 5b : 7b - 5a = 51 : 7 \quad (\div \text{ by } b)$$

Result:

$$3a + 5b : 7b - 5a = 51 : 7$$

Q.10 Complete the following:**(i) (A.B + K.B)****Given Data:**

$$\frac{24}{7} = \frac{6}{x}$$

Required

$$4x = ?$$

Solution:

Consider,

$$\frac{24}{7} = \frac{6}{x}$$

By cross multiplication, we get;

$$24(x) = 6(7)$$

$$24x = 6(7)$$

$$\frac{24x}{6} = 7$$

$$4x = 7$$

Result:

$$4x = 7$$

(ii) (A.B + K.B)**Given Data:**

$$\frac{5a}{3x} = \frac{15b}{y}$$

Required

$$ay = ?$$

Solution:

Consider,

$$\frac{5a}{3x} = \frac{15b}{y}$$

By cross multiplication, we get

$$5a(y) = 15b(3x)$$

$$5ay = 45bx$$

$$ay = \frac{45}{5} bx$$

$$ay = 9bx$$

Result

$$ay = 9bx$$

(iii) (SWL 2014) (A.B + K.B)**Given Data:**

$$\frac{9pq}{2lm} = \frac{18p}{5m}$$

Required

$$5q = ?$$

Solution:

Consider,

$$\frac{9pq}{2lm} = \frac{18p}{5m}$$

By cross multiplication, we get

$$9pq(5m) = 18p(2lm)$$

$$45mpq = 36mpl$$

$$\frac{45mpq}{36mp} = l$$

$$\frac{5q}{4} = l$$

$$5q = 4l$$

Result:

$$5q = 4l$$

Q.11 Find x in the following proportions.**(i) (A.B + K.B)****(GRW 2014, SWL 2016, MTN 2017, RWP 2017)****Given Data:**

$$3x - 2 : 4 :: 2x + 3 : 7$$

Required

$$x = ?$$

Solution:

$$3x - 2 : 4 :: 2x + 3 : 7$$

Product of extremes = product of means

$$7(3x - 2) = 4(2x + 3)$$

$$21x - 14 = 8x + 12$$

$$21x - 8x = 12 + 14$$

$$13x = 26$$

$$x = \frac{26}{13}$$

$$x = 2$$

Result:

$$x = 2$$

(ii) (A.B + K.B)

$$\frac{3x-1}{7} : \frac{3}{5} :: \frac{2x}{3} : \frac{7}{5}$$

Product of extremes = Product of means

$$\frac{3}{5} \left(\frac{2x}{3} \right) = \frac{7}{5} \left(\frac{3x-1}{7} \right)$$

$$\frac{2x}{3} = \frac{3x-1}{3}$$

$$2x = 3x - 1$$

$$1 = 3x - 2x$$

$$\Rightarrow x = 1$$

$$(iii) \quad \frac{x-3}{2} : \frac{5}{x-1} :: \frac{x-1}{3} : \frac{4}{x+4}$$

(GRW 2014) (A.B + K.B)

Product of extremes = Product of means

$$\left(\frac{x-3}{2}\right)\left(\frac{4}{x+4}\right) = \left(\frac{5}{x-1}\right)\left(\frac{x-1}{3}\right)$$

$$(x-3)\left(\frac{2}{x+4}\right) = \frac{5}{3}$$

By cross multiplication

$$6(x-3) = 5(x+4)$$

$$6x - 5x = 20 + 18$$

$$x = 38$$

(iv) Here (A.B + K.B)

$$p^2 + pq + q^2 : x :: \frac{p^3 - q^3}{p+q} : (p-q)^2$$

$\therefore$ Product of extremes = Product of means

$$(p^2 + pq + q^2)(p-q)^2 = x \left(\frac{p^3 - q^3}{p+q} \right)$$

$$x \left(\frac{p^3 - q^3}{p+q} \right) = (p^2 + pq + q^2)(p-q)^2$$

$$x = (p^2 + pq + q^2)(p-q)^2 \times \frac{(p+q)}{p^3 - q^3}$$

$$= \frac{(p^2 + pq + q^2)(p-q)(p-q)(p+q)}{(p-q)(p^2 + pq + q^2)}$$

$$= (p-q)(p+q)$$

$$\Rightarrow x = p^2 - q^2$$

(v) Here (BWP 2014) (A.B + K.B)

$$8-x : 11-x :: 16-x : 25-x$$

$\therefore$ Product of extremes = Product of means

$$(8-x)(25-x) = (11-x)(16-x)$$

$$200 - 8x - 25x + x^2 = 176 - 11x - 16x + x^2$$

$$200 - 33x = 176 - 27x$$

$$27x - 33x = 176 - 200$$

$$-6x = -24$$

$$x = \frac{-24}{-6}$$

$$\Rightarrow x = 4$$

Variation (K.B)

Change in one quantity due to change in other quantity(s) is called variation.

Types of Variations (K.B)

There are two types of variations.

(i) Direct variation

(ii) Inverse variation

Direct Variation (K.B)

(LHR 2014, 16, 17, GRW 2014, 16, 17, BWP 2015, 16, SWL 2016, 17, SGD 2017)

A relation of two quantities, such that increase in one quantity causes the increase in the other or decrease in one quantity causes the decrease in other is called direct variation.

For any two quantities x and y , it is written as: $y \propto x$ or $y = kx$ where $k \neq 0$ is constant of proportionality.

For example: radius and circumference or faster the speed longer the distance etc.

Example 1: (Page # 53) (A.B)

Find the relation between distance d of a body falling from rest varies directly as the square of the time t , neglecting air resistance. Find k , if $d=16$ feet for $t=1$ sec. Also derive a relation between d and t .

Solution:

Since d is the distance of the body falling from rest in time t .

Then under the given condition

$$d \propto t^2$$

$$\text{i.e., } d = kt^2$$

Since $d=16$ feet and $t=1$ sec

Then equation (i) becomes

$$16 = k(1)^2$$

$$\text{i.e., } k = 16$$

$$\text{put in eq. (i) } d = 16t^2$$

Which is a relationship between the distance d and time t .

Example 3: (Page # 54) (K.B + A.B)

Give that A varies directly as the square of r and $A = \frac{1782}{7} \text{ cm}^2$, when $r = 9 \text{ cm}$. If $r = 14 \text{ cm}$, then find A.

Solution:

Since A varies directly as square of r

$$\therefore A \propto r^2$$

$$\text{or } A = kr^2$$

$$\frac{1782}{7} = k(9)^2$$

$$\frac{1782}{7 \times 81} = k$$

$$\text{or } k = \frac{22}{7}$$

Put $k = \frac{22}{7}$ and $r = 14 \text{ cm}$ in eq. (i)

$$A = \frac{22}{7}(14)^2 = \frac{22}{7} \times 14 \times 14$$

$$= 616 \text{ cm}^2$$

Inverse Variation: (K.B)
(LHR 2014, GRW 2017, SWL 2015, 16)

“A relation between two quantities, such that increase in one quantity causes the decrease in the other or vice versa is called inverse variation”. For any two quantities x and y , it is

$$\text{written as: } y \propto \frac{1}{x} \text{ or } y = \frac{k}{x} \quad \text{or}$$

$yx = k$ where $k \neq 0$ is constant of proportionality.

For example:

Increase in worker will decrease the days or
Increase the speed will decrease the time.

Example 2: (Page # 55) (K.B + A.B)

If y varies inversely as x^2 and $y = 16$, when $x = 5$, so find x , when $y = 100$.

Solution:

Since y varies inversely as x^2 , therefore

$$y \propto \frac{1}{x^2} \Rightarrow y = \frac{k}{x^2}$$

$$k = x^2 y \quad \rightarrow (i)$$

Put $x = 5$ and $y = 16$ in (i)

$$k = (5)^2 \times 16$$

$$k = 400$$

Now put $k = 400$ and $y = 100$ in (i)

$$400 = 100x^2 \quad \text{or} \quad x^2 = \frac{400}{100} = 4$$

$$\Rightarrow x = \pm 2$$

Exercise 3.2**Q.1****Given Data:**

y varies directly as x

$y = 8$ when $x = 2$

Required

(i) y in terms of x

(ii) $y = ?$ when $x = 5$

(iii) $x = ?$ when $y = 28$

Solution:

Here $y \propto x$

$$y = kx \rightarrow (i)$$

For value of k

Put $y = 8$, $x = 2$

$$(8) = k(2)$$

$$\Rightarrow k = 4$$

(i) y in terms of x : **(A.B + K.B)**

Put $k = 4$ in equation (i)

$$y = 4x$$

(ii) **For value of y : (A.B + K.B)**

Put $x = 5$ and $k = 4$, we get

$$y = (4)(5)$$

$$y = 20$$

(iii) **For value of x : (A.B + K.B)**

Put k and y in the equation (i), we get

$$28 = 4x$$

$$\Rightarrow x = 7$$

Result

(i) $y = 4x$

(ii) $y = 20$ when $x = 5$

(iii) $x = 7$ when $y = 28$

Q.2

Given Data:

$y \propto x$

$y = 7$ when $x = 3$

Required(i) y in term of x (ii) (a) $x = ?$ when $y = 35$ **(A.B)**(b) $y = ?$ when $x = 18$ **(A.B)****Solution:**

Here $x \propto y$ or $y \propto x$

$y = kx \rightarrow (i)$

For value of k

Put $y = 7, x = 3$

$7 = k(3)$

$\Rightarrow k = \frac{7}{3}$

(i) y in terms of x **(A.B)**

$y = kx$

$\Rightarrow y = \frac{7}{3}x$

(ii) **For values of x :** **(A.B)**Put $y = 35$ in equation (i), we get

$35 = \frac{7}{3}x$

$\frac{35 \times 3}{7} = x$

$\Rightarrow x = 15$

For value of y Put $x = 18$ in equation (i), we get;

$y = \frac{7}{3}(18)$

$y = 7(6)$

$y = 42$

Result

(i) $y = \frac{7}{3}x$

(ii) (a) $x = 15$ (b) $y = 42$ Q.3 **(A.B + U.B)****Given Data:**

$R \propto T$

$R = 5$ when $T = 8$

RequiredEquation connecting R and T .

$R = ?$ when $T = 64$

$T = ?$ when $R = 20$

Solution:

Here, $R \propto T$

$R = kT$

For value of k

Put $R = 5$ and $T = 8$

$5 = 8(k)$

$\frac{5}{8} = k$

The equation connecting R and T is:

$R = \frac{5}{8}T \rightarrow (i)$

For value of R By putting $T = 64$ in equation (i), we get;

$R = \frac{5}{8}(64)$

$R = 5(8)$

$R = 40$

For value of T By putting $R = 20$ in equation (i), we get;

$20 = \frac{5}{8}T$

$20 \times \frac{8}{5} = T$

$\Rightarrow T = 32$

Result

$R = \frac{5}{8}T$

$R = 40$ when $T = 64$

$T = 32$ when $R = 20$

Q.4

Given Data:

$R \propto T^2$

$R = 3$ when $T = 3$

Required

$R = ?$ when $T = 6$

Solution:

$R \propto T^2$

$R = kT^2 \rightarrow (i)$

For value of k Put $R = 3, T = 3$ in equation (i), we get

$3 = k(3)^2$

$$8 = k(9)$$

$$\frac{8}{9} = k$$

$$\Rightarrow k = \frac{8}{9}$$

$$\therefore R = \frac{8}{9} T^2 \rightarrow (ii)$$

For value of R

Put $T = 6$ in equation (ii), we get

$$R = \frac{8}{9} (6)^2$$

$$R = \frac{8}{9} (36)$$

$$R = 8(4)$$

$$R = 32$$

Result:

$$R = 32 \text{ when } T = 6$$

Q.5

(A.B)

Given Data:

$$V \propto R^3$$

$$V = 5 \text{ when } R = 3$$

Required data:

$$R = ? \text{ when } V = 625$$

Solution:

$$\text{Here, } V \propto R^3$$

$$V = kR^3 \rightarrow (i)$$

For value of k

Put $V = 5$ and $R = 3$ in equation (i), we get

$$5 = k(3)^3$$

$$5 = 27k$$

$$k = \frac{5}{27}$$

$$V = \frac{5}{27} R^3$$

For value of R

$$625 = \frac{5}{27} R^3$$

$$\frac{625 \times 27}{5} = R^3$$

$$125 \times 27 = R^3$$

$$3375 = R^3$$

Taking cube root on both sides

$$\Rightarrow \sqrt[3]{R^3} = \sqrt[3]{3375}$$

$$R = 15$$

Result:

$$R = 15 \text{ when } V = 625$$

Q.6

(A.B)

Given Data:

$$w \propto u^3$$

$$w = 81 \text{ when } u = 3$$

Required data:

$$w = ? \text{ when } u = 5$$

Solution

$$\text{Here, } w \propto u^3$$

$$w = ku^3 \rightarrow (i)$$

For value of k

Put $w = 81, u = 3$ in equation (i), we get

$$(81) = k(3)^3$$

$$81 = k(27)$$

$$\frac{81}{27} = k$$

$$k = 3$$

$$\therefore w = 3u^3 \rightarrow (ii)$$

Put $u = 5$ in equation (ii), we get

$$w = 3(5)^3$$

$$w = 3(125)$$

$$w = 375$$

Result

$$w = 375 \text{ when } u = 5$$

Q.7

(A.B)

Given Data:

$$y \propto \frac{1}{x}$$

$$y = 7 \text{ when } x = 2$$

Required data:

$$y = ? \text{ when } x = 126$$

Solution:

$$\text{Here, } y \propto \frac{1}{x}$$

$$y = k \frac{1}{x} \rightarrow (i)$$

For value of kPut $y = 7$ and $x = 2$ in equation (i), we get

$$7 = \frac{k}{2}$$

$$\Rightarrow k = 7(2)$$

$$k = 14$$

$$\therefore y = \frac{14}{x} \rightarrow (i)$$

For value of yPut $x = 126$ in equation (ii), we get

$$y = \frac{14}{126}$$

$$y = \frac{1}{9}$$

Result:

$$y = \frac{1}{9} \text{ when } x = 126$$

Q.8**(A.B + U.B)****Given Data:**

$$y \propto \frac{1}{x}$$

$$y = 4 \text{ when } x = 3$$

Required data:

$$x = ? \text{ when } y = 24$$

Solution:

$$\text{Here, } y \propto \frac{1}{x}$$

$$y = \frac{k}{x} \rightarrow (i)$$

For value of kPut $y = 4$ and $x = 3$ in equation (i), we get

$$4 = \frac{k}{3}$$

$$\Rightarrow k = 12$$

$$\therefore y = \frac{12}{x} \rightarrow (ii)$$

For value of xPut $y = 24$ in equation (ii), we get

$$24 = \frac{12}{x}$$

$$\Rightarrow x = \frac{12}{24}$$

$$x = \frac{1}{2}$$

Result:

$$x = \frac{1}{2} \text{ when } y = 24$$

Q.9**(A.B + U.B)****Given Data:**

$$w \propto \frac{1}{z}$$

$$w = 5 \text{ when } z = 7$$

Required data:

$$w = ? \text{ when } z = \frac{175}{4}$$

Solution:

$$\text{Here, } w \propto \frac{1}{z}$$

$$w = \frac{k}{z} \rightarrow (i)$$

For value of kPut $w = 5$ and $z = 7$ in equation (i), we get

$$5 = \frac{k}{7}$$

$$\Rightarrow k = 35$$

$$\therefore w = \frac{35}{z} \rightarrow (ii)$$

For value of wPut $z = \frac{175}{4}$ in equation (ii), we get

$$w = \frac{35}{\left(\frac{175}{4}\right)}$$

$$w = 35 \div \frac{175}{4}$$

$$w = 35 \times \frac{4}{175}$$

$$w = \frac{4}{5}$$

Result:

$$w = \frac{4}{5} \text{ when } z = \frac{175}{4}$$

Q.10

(A.B + U.B)

Given Data:

$$A \propto \frac{1}{r^2}$$

$$A = 2 \text{ when } r = 3$$

Required data:

$$r = ? \text{ when } A = 72$$

Solution:

$$A \propto \frac{1}{r^2}$$

$$A = \frac{k}{r^2}$$

For value of k

$$A = \frac{k}{r^2}$$

$$\text{Put } A = 2 \text{ and } r = 3$$

$$2 = \frac{k}{(3)^2}$$

$$18 = k$$

Or $k = 18$

$$A = \frac{18}{k}$$

For value of r

$$A = \frac{k}{r^2}$$

$$\text{Put } A = 72 \text{ and } k = 18$$

$$72 = \frac{18}{r^2}$$

$$r^2 = \frac{18}{72}$$

$$r^2 = \frac{1}{4}$$

Taking square root

$$r = \pm \frac{1}{2}$$

Result:

$$r = \pm \frac{1}{2} \text{ when } A = 72$$

Q.11

(A.B + U.B)

Given Data:

$$a \propto \frac{1}{b^2}$$

$$a = 3 \text{ when } b = 4$$

Required data:

$$a = ? \text{ when } b = 8$$

Solution:

$$a \propto \frac{1}{b^2}$$

$$a = \frac{k}{b^2} \rightarrow (i)$$

For value of kPut the value $a = 3$ and $b = 4$ in eq (i), we get

$$3 = \frac{k}{(4)^2}$$

$$3(16) = k$$

$$\Rightarrow k = 48$$

$$\therefore a = \frac{48}{b^2} \rightarrow (ii)$$

For value of aPut $b = 8$ in equation (ii), we get

$$a = \frac{48}{8^2}$$

$$a = \frac{48}{64}$$

$$a = \frac{3}{4}$$

Result:

$$a = \frac{3}{4} \text{ when } b = 8$$

Q.12

(A.B + U.B)

Given data:

$$V \propto \frac{1}{r^3}$$

$$V = 5 \text{ when } r = 3$$

Required data:

$$V = ? \text{ when } r = 6$$

$$r = ? \text{ when } V = 320$$

Solution:

$$V \propto \frac{1}{r^3}$$

$$V = \frac{k}{r^3} \rightarrow (i)$$

For value of kPut $V = 5$ and $r = 3$ in equation (i), we get

$$5 = \frac{k}{3^3}$$

$$5(27) = k$$

$$k = 135$$

$$\therefore V = \frac{135}{r^3} \rightarrow (ii)$$

For value of V

Put $r = 6$ in equation (ii), we get

$$V = \frac{135}{6^3}$$

$$V = \frac{135}{216}$$

$$V = \frac{5}{8}$$

For value of r

Put $V = 320$ in equation (ii), we get

$$320 = \frac{135}{r^3}$$

$$r^3 = \frac{27}{64}$$

Taking cube root on both sides, we get

$$\sqrt[3]{r^3} = \sqrt[3]{\frac{27}{64}}$$

$$r = \frac{3}{4}$$

Result

$$V = \frac{5}{8} \text{ when } r = 6$$

$$r = \frac{3}{4} \text{ when } V = 320$$

Q.13 (SGD 2014) **(A.B + U.B)**

Given

$$m \propto \frac{1}{n^3} \text{ and } m = 2 \text{ when } n = 4$$

Required data:

$$(i) m = ? \quad \text{When } n = 6$$

$$(ii) n = ? \quad \text{when } m = 432$$

Solution

$$m \propto \frac{1}{n^3}$$

$$m = k \times \frac{1}{n^3}$$

$$m = \frac{k}{n^3} \rightarrow (i)$$

For value of k

Put $m = 2, n = 4$ in equation (i)

$$2 = \frac{k}{(4)^3}$$

$$2 \times 64 = k$$

$$k = 2 \times 64$$

$$k = 128$$

$$m = \frac{128}{n^3}$$

For value of m

Put $k = 128, n = 6$

$$m = \frac{128}{(6)^3} \quad \therefore n = 6$$

$$m = \frac{128}{216}$$

$$m = \frac{16}{27}$$

For value of n

$$m = \frac{k}{n^3}$$

$$432 = \frac{128}{n^3} \quad \therefore m = 432$$

$$432(n^3) = 128$$

$$n^3 = \frac{128}{432}$$

$$n^3 = \frac{64}{216}$$

$$n^3 = \frac{32}{108}$$

$$n^3 = \frac{16}{54}$$

$$n = \frac{8}{27}$$

Taking cube root on both sides

$$n = \frac{2}{3}$$

Result:

$$m = \frac{16}{27} \text{ when } n = 6$$

$$n = \frac{2}{3} \text{ when } m = 432$$

Finding 3rd, 4th, Mean and Continued Proportion**Third Proportional (K.B + U.B)**

For three quantities a, b and c are related as, $a:b::b:c$, then c is called the third proportional.

$$\text{i.e. } c = \frac{b^2}{a}$$

Example 1: (Page # 56) (A.B)

Find a third proportional of $x+y$ and $x^2 - y^2$

Solution:

Let c be the third proportional,

Then $x+y : x^2 - y^2 :: x^2 - y^2 : c$

$$c(x+y) = (x^2 - y^2)(x^2 - y^2)$$

$$c = \frac{(x^2 - y^2)(x^2 - y^2)}{x+y}$$

$$= \frac{(x^2 - y^2)(x-y)(x+y)}{x+y}$$

$$= (x^2 - y^2)(x-y)$$

$$\Rightarrow c = (x+y)(x-y)^2$$

Mean Proportional (K.B)

If three quantities a, b and c are related as $a:b::b:c$, then b is called mean proportional. i.e. $b^2 = ac$

Continued Proportion (K.B)

If three quantities a, b and c are related as $a:b::b:c$ where a is first, b is mean and c is the third proportional, then a, b and c are in continued proportion.

$$\text{i.e. } b^2 = ac$$

Example 4: (Page # 57) (A.B)

(GRV 2016, FSD 2015, 17, SGD 2015)

Find p, if 12, p and 3 are in continued proportion.

Solution:

Since 12, p and 3 are in continued proportion.

$$\therefore 12:p::p:3$$

$\therefore$ Product of extremes = product of means

$$p \cdot p = (12)(3)$$

$$\Rightarrow p^2 = 36$$

$$\text{Thus, } p = \pm 6$$

Example 3: (Page # 57)**(A.B)**

Find the mean proportional of $9p^6q^4$ and r^8

Solution:

Let m be the mean proportional.

$$\text{Then } 9p^6q^4 : m :: m : r^8$$

$$\text{Or } m \cdot m = 9p^6q^4(r^8)$$

$$m^2 = 9p^6q^4r^8$$

$$m = \pm \sqrt{9p^6q^4r^8}$$

$$\Rightarrow m = \pm 3p^3q^2r^4$$

Fourth Proportional (K.B + U.B)

If four quantities a, b, c and d are related as $a:b::c:d$, then d is called fourth proportional.

$$\text{i.e. } d = \frac{bc}{a}$$

Example 2: (Page # 57)**(K.B)**

Find fourth proportional of $a^3 - b^3$, $a + b$, and $a^2 + ab + b^2$

Solution:

Let x be the fourth proportional,

Then

$$(a^3 - b^3) : (a + b) :: (a^2 + ab + b^2) : x$$

$\therefore$ Product of extremes = Product of means

$$x(a^3 - b^3) = (a + b)(a^2 + ab + b^2)$$

$$x = \frac{(a + b)(a^2 + ab + b^2)}{a^3 - b^3}$$

$$= \frac{(a + b)(a^2 + ab + b^2)}{(a - b)(a^2 + ab + b^2)}$$

$$x = \frac{a + b}{a - b}$$

Exercise 3.3

Q.1 Find a third proportional to

(i) 6, 12 (A.B)

(SGD 2017, RWP 2017, D.G.K 2016, 17)

Let, the third proportional = a

According to the given condition;

$$6:12::12:a$$

 $\therefore$ Product of means = product of extremes

$$12(12) = 6a$$

$$144 = 6a$$

$$\frac{144}{6} = a$$

$$24 = a$$

$$\Rightarrow a = 24$$

 $\therefore$ Third proportional is 24(ii) $a^3, 3a^2$ (A.B)

(MTN 2014, 16, RWP 2017, D.G.K 2014)

Let, third proportional = x

According to the given condition;

$$a^3:3a^2::3a^2:x$$

 $\therefore$ Product of extremes = Product of means

$$(a^3)x = (3a^2)(3a^2)$$

$$a^3x = 9a^4$$

$$x = \frac{9a^4}{a^3}$$

$$x = 9a$$

 $\therefore$ Third proportional is $9a$ (iii) $a^2 - b^2, a - b$ (A.B) (LHR 2015,

GRW 2014, 16, SWL 2016, BWP 2015)

Let, third proportional = x

According to the given condition;

$$a^2 - b^2 : a - b :: a - b : x$$

 $\therefore$ Product of extremes = product of means

$$(a^2 - b^2)(x) = (a - b)^2$$

$$(a - b)(a + b)(x) = (a - b)^2$$

$$(a + b)x = a - b$$

$$x = \frac{a - b}{a + b}$$

 $\therefore$ Third proportional is $\frac{a - b}{a + b}$ (iv) $(x - y)^2, (x^3 - y^3)$ (A.B)

(FSD 2015, GRW 2016)

Let, third proportional = a

According to the given condition,

$$(x - y)^2 : (x^3 - y^3) :: (x^3 - y^3) : a$$

 $\therefore$ Product of extreme = product of mean

$$(x - y)^2 a = (x^3 - y^3)(x^3 - y^3)$$

$$a(x - y)(x - y)$$

$$= (x - y)(x^2 + xy + y^2)(x - y)(x^2 + xy + y^2)$$

$$a = (x^2 + xy + y^2)^2$$

 $\therefore$ Third proportional is $(x^2 + xy + y^2)^2$ (v) $(x + y)^2, x^2 - xy - 2y^2$ (A.B)Let third proportional = a

According to given condition

$$(x + y)^2 : x^2 - xy - 2y^2 :: x^2 - xy - 2y^2 : a$$

 $\therefore$ Product of extremes = Product of means

$$a(x + y)^2 = (x^2 - xy - 2y^2)(x^2 - xy - 2y^2)$$

$$a(x + y)^2 = (x^2 - xy - 2y^2)^2$$

$$a(x + y)^2 = (x^2 - 2xy + xy - 2y^2)^2$$

$$a(x + y)^2 = [x(x - 2y) + y(x - 2y)]^2$$

$$= [(x - 2y)(x + y)]^2$$

$$a(x + y)^2 = (x - 2y)^2(x + y)^2$$

$$a = (x - 2y)^2$$

 $\therefore$ Third proportional = $(x - 2y)^2$ (vi) $\frac{p^2 - q^2}{p^3 + q^3}, \frac{p - q}{p^2 - pq + q^2}$ (A.B)Let, the third proportional = a

According to the given condition;

$$\frac{p^2 - q^2}{p^3 + q^3} : \left(\frac{p - q}{p^2 - pq + q^2} \right) :: \left(\frac{p - q}{p^2 - pq + q^2} \right) : a$$

 $\therefore$ Product of extremes = Product of means

$$a \left(\frac{p^2 - q^2}{p^3 + q^3} \right) = \left(\frac{p - q}{p^2 - pq + q^2} \right) \left(\frac{p - q}{p^2 - pq + q^2} \right)$$

$$a = \left(\frac{p - q}{p^2 - pq + q^2} \right) \left(\frac{p - q}{p^2 - pq + q^2} \right) \left(\frac{p^3 + q^3}{p^2 - q^2} \right)$$

$$a = \left(\frac{p - q}{p^2 - pq + q^2} \right) \left(\frac{p - q}{p^2 - pq + q^2} \right) \left(\frac{(p + q)(p^2 - pq + q^2)}{(p - q)(p + q)} \right)$$

$$a = \frac{p - q}{p^2 - pq + q^2}$$

 $\therefore$ Third proportional is $\frac{p - q}{p^2 - pq + q^2}$

Q.2 Find a fourth proportional to**(i) 5, 8, 15 (A.B)****(LHR 2014, GRW 2017, BWP 2016)**Let, the fourth proportional = x

According to the given condition:

$$5:8::15:x$$

 $\therefore$ Product of mean = product of extreme

$$(5)x = 8(15)$$

$$5x = 120$$

$$x = \frac{120}{5}$$

$$x = 24$$

 $\therefore$ **Fourth proportional is 24****(ii) $4x^4, 2x^3, 18x^5$ (A.B)****(SWL 2015, BWP 2016)**Let, the fourth proportional = a

According to the given condition:

$$4x^4:2x^3::18x^5:a$$

 $\therefore$ Product of means = product of extremes

$$(18x^5)(2x^3) = (4x^4)(a)$$

$$\frac{36(x^8)}{4x^4} = a$$

$$\frac{18x^{8-4}}{2} = a$$

$$9x^4 = a$$

$$\Rightarrow a = 9x^4$$

 $\therefore$ **Fourth proportional is $9x^4$** **(iii) $15a^5b^6, 10a^2b^5, 21a^3b^3$ (A.B)****(FSD 2017, D.G.K 2015, 17)**Let, the third proportional = x

According to the given condition;

$$15a^5b^6:10a^2b^5::21a^3b^3:x$$

 $\therefore$ Product of extremes = product of means

$$(15a^5b^6)(x) = (10a^2b^5)(21a^3b^3)$$

$$x = \frac{10 \times 21 \times a^5 \times b^8}{15a^5b^6}$$

$$x = \frac{2 \times 21 \times b^2}{3}$$

$$x = 2 \times 7 \times b^2$$

$$x = 14b^2$$

 $\therefore$ **Fourth proportional is $14b^2$** **(iv) $x^2 - 11x + 24, x - 3, 5x^4 - 40x^3$ (A.B)**Let, Fourth proportional is a

According to the given condition;

$$x^2 - 11x + 24:(x-3)::5x^4 - 40x^3:a$$

 $\therefore$ Product of extremes = Product of means

$$(x^2 - 11x + 24)(a) = (x-3)(5x^4 - 40x^3)$$

$$(x^2 - 8x - 3x + 24)a = (x-3)(5x^4 - 40x^3)$$

$$[x(x-8) - 3(x-8)]a = 5x^3(x-3)(x-8)$$

$$(x-3)(x-8)a = 5x^3(x-3)(x-8)$$

$$a = 5x^3$$

 $\therefore$ **Fourth proportional is $5x^3$** **(v) $p^3 + q^3, p^2 - q^2, p^2 - pq + q^2$** Let, Fourth proportional = a

According to given condition:

$$p^3 + q^3:p^2 - q^2::p^2 - pq + q^2:a$$

 $\therefore$ Product of extremes = Product of means

$$(p^3 + q^3)a = (p^2 - q^2)(p^2 - pq + q^2)$$

$$(p+q)(p^2 - pq + q^2)a$$

$$= (p+q)(p-q)(p^2 - pq + q^2)$$

$$a = p - q$$

 $\therefore$ **Fourth proportional = $p - q$** **(vi) $(p^2 - q^2)(p^2 + pq + q^2), p^3 + q^3, p^3 - q^3$** **(A.B)**Let, the fourth proportional = x

According to the given condition;

$$(p^2 - q^2)(p^2 + pq + q^2):p^3 + q^3::p^3 - q^3:x$$

 $\therefore$ Product of extremes = Product of means

$$(p^2 - q^2)(p^2 + pq + q^2)(x) = (p^3 + q^3)(p^3 - q^3)$$

$$(p+q)(p-q)(p^2 + pq + q^2)(x)$$

$$= (p+q)(p^2 - pq + q^2)(p-q)(p^2 + pq + q^2)$$

$$x = p^2 - pq + q^2$$

 $\therefore$ **Fourth proportional is $p^2 - pq + q^2$** **Q.3 Find a mean proportional between****(i) 20, 45 (A.B)****(LHR 2016, GRW 2014, D.G.K 2016)**Let, the mean proportional = x

According to the given condition;

$$20:x::x:45$$

 $\therefore$ Product of means = Product of extremes

$$(x)(x) = (20)(45)$$

$$x^2 = 900$$

Taking square root on both sides

$$\sqrt{x^2} = \sqrt{900}$$

$$x = \pm 30$$

 $\therefore$ **The mean proportional is ± 30**

(ii) $20x^3y^3, 5x^7y$ (A.B)Let, mean proportional = a

According to given condition

$$20x^3y^3 : a :: a : 5x^7y$$

 $\therefore$ Product of extremes = Product of means

$$(20x^3y^3)(5x^7y) = a \cdot a$$

$$100x^{10}y^6 = a^2$$

$$a^2 = 100x^{10}y^6$$

Taking Sq. root on both sides

$$\sqrt{a^2} = \sqrt{100x^{10}y^6}$$

$$\Rightarrow a = \pm 10x^5y^3$$

 $\therefore$ Mean proportional = $\pm 10x^5y^3$ (iii) $15p^4qv^3, 135q^5r^7$ (A.B)Let, the mean proportional = a

According to the given condition,

$$15p^4qv^3 : a :: a : 135q^5r^7$$

 $\therefore$ Product of means = Product of extremes

$$a^2 = (15p^4qv^3)(135q^5r^7)$$

$$a^2 = 2025p^4q^6r^{10}$$

Taking square root on both sides,

$$\sqrt{a^2} = \sqrt{2025p^4q^6r^{10}}$$

$$a = \pm 45p^2q^3r^5$$

 $\therefore$ The mean proportional is $\pm 45p^2q^3r^5$ (iv) $x^2 - y^2 = \frac{x-y}{x+y}$ (GRW 2015) (A.B)Let, the mean proportional = a

According to given condition;

$$x^2 - y^2 : a :: a : \frac{x-y}{x+y}$$

 $\therefore$ Product of means = Product of extremes

$$a^2 = (x^2 - y^2) \left(\frac{x-y}{x+y} \right)$$

$$a^2 = (x-y)(x+y) \left(\frac{x-y}{x+y} \right)$$

$$a^2 = (x-y)^2$$

Taking square root on both sides;

$$\sqrt{a^2} = \sqrt{(x-y)^2}$$

$$a = \pm(x-y)$$

 $\therefore$ The mean proportional is $\pm(x-y)$

Q.4 Find the value of the letter involved in the following continued proportions.

(i) 5, p , 45 (A.B)

According to given condition

$$5 : p :: p : 45$$

 $\therefore$ Product of means = product of extremes

$$p^2 = 5 \times 45$$

$$p^2 = 225$$

Taking square root

$$p = \pm 15$$

Result

$$p = \pm 15$$

(ii) 8, x , 18 (A.B)

According to condition

$$8 : x :: x : 18$$

 $\therefore$ Product of means = Product of extremes

$$x^2 = 8 \times 18$$

$$x^2 = 144$$

Taking square root on both sides

$$x = \pm \sqrt{144}$$

$$\Rightarrow x = \pm 12$$

Result:

$$x = \pm 12$$

(iii) 12, $3p-6$, 27 (A.B)

According to given condition

$$12 : 3p-6 :: 3p-6 : 27$$

 $\therefore$ Product of extremes = Product of means

$$12 \times 27 = (3p-6)(3p-6)$$

$$324 = (3p-6)^2$$

$$(3p-6)^2 = 324$$

Taking Sq. root on both sides

$$3p-6 = \pm 18$$

Either

$$3p-6 = -18 \quad \text{or}$$

$$3p-6 = 18$$

$$3p = 6-18$$

$$3p = 6+18$$

$$3p = -12$$

$$3p = 24$$

$$p = \frac{-12}{3}$$

$$p = \frac{24}{3}$$

$$p = -4$$

$$p = 8$$

Result:

$$p = -4, 8$$

(iv) $7, m-3, 28$ According to given condition: **(A.B)**

(GRWP 2015, 17, FSD 2017, MTN 2017, BWP 2015)

$$7 : m-3 :: m-3 : 28$$

$\therefore$ Product of means = Product of extremes

$$(m-3)^2 = 7 \times 28$$

$$(m-3)^2 = 196$$

Taking square root on both sides

$$m-3 = \pm 14$$

Either

$$m-3 = -14 \quad \text{or} \quad m-3 = 14$$

$$m = -14 + 3 \quad m = 14 + 3$$

$$\Rightarrow m = -11 \quad m = 17$$

Result:

$$m = -11, 17$$

Theorems on Proportions

(K.B + U.B + A.B)

(LHR 2014, SWL 2016, BWP 2015, 16, MTN 2015)

❖ **Invertendo Theorem.**

If $a : b = c : d$ then $b : a = d : c$

❖ **Alternendo Theorem.**

If $a : b = c : d$ then $a : c = b : d$

❖ **Componendo Theorem**

If $a : b = c : d$ then

$$(i) \quad a + b : b = c + d : d$$

$$(ii) \quad a : a + b = c : c + d$$

❖ **Dividendo Theorem**

If $a : b = c : d$ then

$$(i) \quad a - b : b = c - d : d$$

$$(ii) \quad a : a - b = c : c - d$$

❖ **Componendo-Dividendo Theorem**

If $a : b = c : d$ then

$$(i) \quad a + b : a - b = c + d : c - d$$

$$(ii) \quad a - b : a + b = c - d : c + d$$

Example 5: (Page # 60)

(A.B)

Given:

$$m : n = p : q$$

To Prove:

$$3m + 7n : 3m - 7n = 3p + 7q : 3p - 7q$$

Solution:

$$m : n = p : q$$

$$\frac{m}{n} = \frac{p}{q}$$

Multiplying both sides by $\frac{3}{7}$, we get

$$\frac{3m}{7n} = \frac{3p}{7q}$$

Then using componendo-dividendo theorem

$$\frac{3m + 7h}{3m - 7n} = \frac{3p + 7q}{3p - 7q}$$

$$\frac{3m + 7n}{3m - 7n} = \frac{3p + 7q}{3p - 7q}$$

Thus, $3m + 7n : 3m - 7n = 3p + 7q : 3p - 7q$

Example 7: (Page # 60)

(A.B)

Using theorem of componendo-dividendo, find the value of

$$\frac{m+3p}{m-3p} + \frac{m+2p}{m-2p}, \text{ if } m = \frac{6pq}{p+q}$$

Solution:

$$\text{Since } m = \frac{6pq}{p+q}$$

$$\text{Or } m = \frac{(3p)(2q)}{p+q} \rightarrow (i)$$

$$\therefore \frac{m}{3p} = \frac{2q}{p+q}$$

By componendo-dividendo theorem

$$\frac{m+3p}{m-3p} = \frac{2q+(p+q)}{2q-(p+q)} = \frac{2q+p+q}{2q-p-q}$$

$$\frac{m+3p}{m-3p} = \frac{p+3q}{q-p} \rightarrow (ii)$$

Again from eq. (i), we have

$$\frac{m}{2q} = \frac{3p}{q-p}$$

By componendo-dividendo theorem

$$\frac{m+3p}{m-3p} = \frac{2q+(p+q)}{2q-(p+q)} = \frac{2q+p+q}{2q-p-q}$$

$$\frac{m+2p}{m-2p} = \frac{4p+q}{2p-q} \rightarrow (iii)$$

Adding (ii) and (iii)

$$\frac{m+3p}{m-3p} + \frac{m+2p}{m-2p} = \frac{p+3q}{q-p} + \frac{4p+q}{2p-q}$$

$$= -\frac{p+3q}{p-q} + \frac{4p+q}{2p-q}$$

$$= \frac{-(p+3q)(2p-q) + (p-q)(4p+q)}{(p-q)(2p-q)}$$

$$= \frac{-2p^2 - 5pq + 3q^2 + 4p^2 - 3pq - q^2}{(p-q)(2p-q)}$$

$$= \frac{-2p^2 - 8pq + 2q^2}{(p-q)(2p-q)} = \frac{-2(p^2 - 4pq - q^2)}{(p-q)(2p-q)}$$

Example 8: (Page # 61)**(A.B)**

Using theorem of componendo-dividendo, solve the equation

$$\frac{\sqrt{x+3} + \sqrt{x-3}}{\sqrt{x+3} - \sqrt{x-3}} = \frac{4}{3}$$

Solution:

Given equation is $\frac{\sqrt{x+3} + \sqrt{x-3}}{\sqrt{x+3} - \sqrt{x-3}} = \frac{4}{3}$

By componendo-dividendo theorem

$$\frac{\sqrt{x+3} + \sqrt{x-3} + \sqrt{x+3} - \sqrt{x-3}}{\sqrt{x+3} + \sqrt{x-3} - \sqrt{x+3} + \sqrt{x-3}} = \frac{4+3}{4-3}$$

$$\frac{2\sqrt{x+3}}{2\sqrt{x-3}} = \frac{7}{1} \Rightarrow \sqrt{\frac{x+3}{x-3}} = 7$$

Squaring both sides

$$\frac{x+3}{x-3} = 49$$

$$x+3 = 49(x-3)$$

$$x+3 = 49x-147$$

$$x-49x = -147-3$$

$$-48x = -150$$

$$\Rightarrow x = \frac{150}{48} = \frac{25}{8}$$

Therefore, Solution Set = $\left\{\frac{25}{8}\right\}$

Exercise 3.4**Q.1****(i) Given (A.B)**

$$\frac{4a+5b}{4a-5b} = \frac{4c+5d}{4c-5d}$$

To prove

$$a:b = c:d$$

Proof

Here

$$\frac{4a+5b}{4a-5b} = \frac{4c+5d}{4c-5d}$$

By applying componendo-dividendo property

$$\frac{(4a+5b) + (4a-5b)}{(4a+5b) - (4a-5b)} = \frac{(4c+5d) + (4c-5d)}{(4c+5d) - (4c-5d)}$$

$$\frac{8a}{10b} = \frac{8c}{10d}$$

(by using cancellation prop)

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow a:b = c:d$$

Proved**(ii)****Given****(A.B)**

$$\frac{2a+9b}{2a-9b} = \frac{2c+9d}{2c-9d}$$

To prove

$$a:b = c:d$$

Proof

Here

$$\frac{2a+9b}{2a-9b} = \frac{2c+9d}{2c-9d}$$

By applying componendo-dividendo prop

$$\frac{(2a+9b) + (2a-9b)}{(2a+9b) - (2a-9b)} = \frac{(2c+9d) + (2c-9d)}{(2c+9d) - (2c-9d)}$$

$$\frac{4a}{18b} = \frac{4c}{18d}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow a:b = c:d$$

Proved**(iii)****Given****(A.B)**

$$\frac{ac^2 + bd^2}{ac^2 - bd^2} = \frac{c^3 + d^3}{c^3 - d^3}$$

To prove

$$a:b = c:d$$

Proof

Here

$$\frac{ac^2 + bd^2}{ac^2 - bd^2} = \frac{c^3 + d^3}{c^3 - d^3}$$

By applying componendo-dividendo prop

$$\frac{(ac^2 + bd^2) + (ac^2 - bd^2)}{(ac^2 + bd^2) - (ac^2 - bd^2)} = \frac{(c^3 + d^3) + (c^3 - d^3)}{(c^3 + d^3) - (c^3 - d^3)}$$

$$\frac{ac^2 + bd^2 + ac^2 - bd^2}{ac^2 + bd^2 - ac^2 + bd^2} = \frac{c^3 + d^3 + c^3 - d^3}{c^3 + d^3 - c^3 + d^3}$$

$$\frac{2ac^2}{2bd^2} = \frac{c^3}{d^3}$$

$$\frac{ac^2}{bd^2} = \frac{c^3}{d^3}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow a:b = c:d$$

Hence Proved**(iv) Given**

$$\frac{a^2c + b^2d}{a^2c - b^2d} = \frac{ac^2 + bd^2}{ac^2 - bd^2}$$

To prove

$$a:b = c:d$$

Proof

Here

$$\frac{a^2c + b^2d}{a^2c - b^2d} = \frac{ac^2 + bd^2}{ac^2 - bd^2}$$

By applying componendo-dividendo prop

$$\frac{(a^2c + b^2d) + (a^2c - b^2d)}{(a^2c + b^2d) - (a^2c - b^2d)} = \frac{(ac^2 + bd^2) + (ac^2 - bd^2)}{(ac^2 + bd^2) - (ac^2 - bd^2)}$$

$$\frac{a^2c + b^2d + a^2c - b^2d}{a^2c + b^2d - a^2c + b^2d} = \frac{ac^2 + bd^2 + ac^2 - bd^2}{ac^2 + bd^2 - ac^2 + bd^2}$$

$$\frac{2a^2c}{2b^2d} = \frac{2ac^2}{2bd^2}$$

$$\frac{a^2c}{b^2d} = \frac{ac^2}{bd^2}$$

Multiplying by $\frac{bd}{ac}$ on both sides

$$\frac{a^2c}{b^2d} \times \frac{bd}{ac} = \frac{ac^2}{bd^2} \times \frac{bd}{ac}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow a:b = c:d$$

Hence Proved**(v) Given**

$$Pa + pq : pq - qb = pc + qd : pc - qa$$

To prove

$$a:b = c:d$$

Proof

Here

$$Pa + pq : pq - qb = pc + qd : pc - qa$$

$$\frac{pa + qb}{pa - qb} = \frac{pc + qd}{pc - qd}$$

$$\text{By applying componendo-dividendo prop}$$

$$\frac{(pa + qb) + (pa - qb)}{(pa + qb) - (pa - qb)} = \frac{(pc + qd) + (pc - qd)}{(pc + qd) - (pc - qd)}$$

$$\frac{pa + qb + pa - qb}{pa + qb - pa + qb} = \frac{pc + qd + pc - qd}{pc + qd - pc + qd}$$

$$\frac{2pa}{2qb} = \frac{2pc}{2qd}$$

$$\frac{pa}{qb} = \frac{pc}{qd}$$

$$\text{Multiply by } \frac{q}{p}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow a:b = c:d$$

Hence Proved**(vi) Given****(A.B)**

$$\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$$

To prove

$$a:b = c:d$$

Proof

Here

$$\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$$

By applying componendo-dividendo

$$\text{prop } \frac{(a+b+c+d) + (a+b-c-d)}{(a+b+c+d) - (a+b-c-d)} = \frac{(a-b+c-d) + (a-b-c+d)}{(a-b+c-d) - (a-b-c+d)}$$

$$\frac{a+b+c+d + a+b-c-d}{a+b+c+d - a-b-c-d} = \frac{a-b+c-d + a-b-c+d}{a-b+c-d - a-b-c+d}$$

$$= \frac{2a+2b}{2c+2d} = \frac{2a-2b}{2c-2d}$$

$$\frac{2a+2b}{2c+2d} = \frac{2a-2b}{2c-2d}$$

By applying alternendo property

$$\frac{2a+2b}{2a-2b} = \frac{2c+2d}{2c-2d}$$

Again applying componendo-dividendo prop

$$\frac{(2a+2b)+(2a-2b)}{(2a+2b)-(2a-2b)} = \frac{(2c+2d)+(2c-2d)}{(2c+2d)-(2c-2d)}$$

$$\frac{2a+2\cancel{b}+2a-2\cancel{b}}{2\cancel{a}+2b-2\cancel{a}+2b} = \frac{2c+2\cancel{d}+2c-2\cancel{d}}{2\cancel{c}+2d-2\cancel{c}+2d}$$

$$\frac{4a}{4b} = \frac{4c}{4d}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow a:b = c:d$$

Hence Proved

(vii) Given (A.B)

$$\frac{2a+3b+2c+3d}{2a+3b-2c-3d} = \frac{2a-3b+2c-3d}{2a-3b-2c+3d}$$

To prove

$$a:b = c:d$$

Proof

Here

$$\frac{2a+3b+2c+3d}{2a+3b-2c-3d} = \frac{2a-3b+2c-3d}{2a-3b-2c+3d}$$

By applying componendo-dividendo prop

$$\begin{aligned} \frac{(2a+3b+2c+3d)+(2a+3b-2c-3d)}{(2a+3b+2c+3d)-(2a+3b-2c-3d)} &= \frac{(2a-3b+2c-3d)+(2a-3b-2c+3d)}{(2a-3b+2c-3d)-(2a-3b-2c+3d)} \\ &= \frac{(2a+3b+2c+3d+2a+3b-2c-3d)}{(2a+3b+2c+3d-2a-3b+2c+3d)} \end{aligned}$$

$$\begin{aligned} &= \frac{4a+9b}{4c+9d} = \frac{4a-9b}{4c-9d} \end{aligned}$$

By applying alternendo property

$$\frac{4a+9b}{4a-9b} = \frac{4c+9d}{4c-9d}$$

Again applying componendo-dividendo prop

$$\frac{(4a+9b)+(4a-9b)}{(4a+9b)-(4a-9b)} = \frac{(4c+9d)+(4c-9d)}{(4c+9d)-(4c-9d)}$$

$$\frac{4a+9b+4a-9b}{4a+9b-4a+9b} = \frac{4c+9d+4c-9d}{4c+9d-4c+9d}$$

$$\frac{8a}{18b} = \frac{8c}{18d}$$

Multiply by $\frac{18}{8}$ on both sides

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow a:b = c:d$$

Hence Proved

(viii) Given (A.B)

$$\frac{a^2+b^2}{a^2-b^2} = \frac{ac+bd}{ac-bd}$$

To prove

$$a:b = c:d$$

Proof

Here

$$\frac{a^2+b^2}{a^2-b^2} = \frac{ac+bd}{ac-bd}$$

By applying componendo-dividendo prop

$$\frac{(a^2+b^2)+(a^2-b^2)}{(a^2+b^2)-(a^2-b^2)} = \frac{(ac+bd)+(ac-bd)}{(ac+bd)-(ac-bd)}$$

$$\frac{a^2+b^2+a^2-b^2}{a^2+b^2-a^2+b^2} = \frac{ac+bd+ac-bd}{ac+bd-ac+bd}$$

$$\frac{2a^2}{2b^2} = \frac{2ac}{2bd}$$

$$\frac{a^2}{b^2} = \frac{ac}{bd}$$

(Multiply both sides by $\frac{b}{a}$)

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow a:b = c:d$$

Hence Proved

Q.2 Using theorem of componendo-dividendo, find the value of

(i) Given (A.B)

$$x = \frac{4yz}{y+z}$$

Required

$$\frac{x+2y}{x-2y} + \frac{x+2z}{x-2z} = ?$$

Solution:

Here

$$x = \frac{4yz}{y+z}$$

Dividing by $2y$

$$\frac{x}{2y} = \frac{4yz}{2y(y+z)}$$

$$\frac{x}{2y} = \frac{2z}{y+z}$$

By applying componendo dividendo prop

$$\frac{x+2y}{x-2y} = \frac{2z+(y+z)}{2z-(y+z)}$$

$$= \frac{2z+y+z}{2z-y-z}$$

$$\frac{x+2y}{x-2y} = \frac{3z+y}{z-y} \rightarrow (i)$$

Again consider

$$x = \frac{4yz}{y+z}$$

Dividing by $2z$

$$\frac{x}{2z} = \frac{4yz}{2z(y+z)}$$

$$= \frac{2y}{y+z}$$

By applying componendo dividendo prop

$$\frac{x+2z}{x-2z} = \frac{2y+(y+z)}{2y-(y+z)}$$

$$= \frac{2y+y+z}{2y-y-z}$$

$$= \frac{3y+z}{y-z} \rightarrow (ii)$$

Adding equation (i) and (ii)

$$\begin{aligned} \frac{x+2y}{x-2y} + \frac{x+2z}{x-2z} &= \frac{3z+y}{z-y} + \frac{3y+z}{y-z} \\ &= \frac{3z+y}{z-y} + \frac{3y+z}{-(z-y)} \\ &= \frac{3z+y}{z-y} - \frac{3y+z}{z-y} \end{aligned}$$

$$= \frac{3z+y-(3y+z)}{z-y}$$

$$= \frac{3z+y-3y-z}{z-y}$$

$$= \frac{2z-2y}{z-y}$$

$$= \frac{2(z-y)}{(z-y)}$$

$$\Rightarrow \frac{x+2y}{x-2y} + \frac{x+2z}{x-2z} = 2$$

(ii) Given:**(A.B)**

$$m = \frac{10np}{n+p}$$

Required

$$\frac{m+5n}{m-5n} + \frac{m+5p}{m-5p} = ?$$

Solution:

Here

$$m = \frac{10np}{n+p}$$

Dividing by $5n$

$$\frac{m}{5n} = \frac{10np}{(5n)(n+p)}$$

$$\frac{m}{5n} = \frac{2p}{n+p}$$

By applying componendo dividendo property

$$\frac{m+5n}{m-5n} = \frac{2p+(n+p)}{2p-(n+p)}$$

$$\frac{m+5n}{m-5n} = \frac{2p+n+p}{2p-n-p}$$

$$\frac{m+5n}{m-5n} = \frac{3p+n}{p-n} \rightarrow (i)$$

Again consider

$$m = \frac{10np}{n+p}$$

Dividing by $5p$

$$\frac{m}{5p} = \frac{10np}{5p(n+p)}$$

$$\frac{m}{5p} = \frac{2n}{n+p}$$

Applying componendo – dividendo prop

$$\frac{m+5p}{m-5p} = \frac{2n+(n+p)}{2n-(n+p)}$$

$$\frac{m+5p}{m-5p} = \frac{2n+n+p}{2n-n-p}$$

$$\frac{m+5p}{m-5p} = \frac{3n+p}{n-p} \rightarrow (i)$$

Adding (i) and (ii)

$$\frac{m+5n}{m-5n} + \frac{m+5p}{m-5p} = \frac{3p+n}{p-n} + \frac{3n+p}{n-p}$$

$$= \frac{3p+n}{p-n} + \frac{3n+p}{-(p-n)}$$

$$= \frac{3p+n}{p-n} - \frac{3n+p}{p-n}$$

$$= \frac{3p+n-(3n+p)}{p-n}$$

$$= \frac{3p+n-3n-p}{p-n}$$

$$= \frac{2p-2n}{p-n}$$

$$= \frac{2(p-n)}{p-n}$$

$$= 2$$

Hence

$$\frac{m+5n}{m-5n} + \frac{m+5p}{m-5p} = 2$$

(iii) Given

(A.B)

$$x = \frac{12ab}{a-b}$$

Required

$$\frac{x-6a}{x+6a} - \frac{x+6b}{x-6b} = 5$$

Solution

Here

$$x = \frac{12ab}{a-b}$$

Dividing by 6a

$$\frac{x}{6a} = \frac{2b}{a-b}$$

Applying componendo – dividendo prop

$$\frac{x+6a}{x-6a} = \frac{2b+(a-b)}{2b-(a-b)}$$

$$\frac{x+6a}{x-6a} = \frac{2b+a-b}{2b-a+b}$$

$$\frac{x+6a}{x-6a} = \frac{b+a}{3b-a}$$

By invertendo theorem

$$\frac{x-6a}{x+6a} = \frac{3b-a}{a+b} \rightarrow (i)$$

Again consider

$$x = \frac{12ab}{a-b}$$

Dividing by 6b

$$\frac{x}{6b} = \frac{12ab}{6b(a-b)}$$

$$\frac{x}{6b} = \frac{2a}{a-b}$$

Applying componendo – Dividendo prop

$$\frac{x+6b}{x-6b} = \frac{2a+(a-b)}{2a-(a-b)}$$

$$\frac{x+6b}{x-6b} = \frac{2a+a-b}{2a-a+b}$$

$$\frac{x+6b}{x-6b} = \frac{3a-b}{a+b} \rightarrow (ii)$$

Subtracting (i) and (ii)

$$\frac{x-6a}{x+6a} - \frac{x+6b}{x-6b} = \frac{3b-a}{a+b} - \frac{3a-b}{a-b}$$

$$= \frac{4b-4a}{a+b}$$

$$= \frac{4(b-a)}{a+b}$$

Hence

$$\frac{x-6a}{x+6a} - \frac{x+6b}{x-6b} = \frac{4(b-a)}{a+b}$$

(iv) Given (A.B)

$$x = \frac{3yz}{y-z}$$

Required

$$\frac{x-3y}{x+3y} - \frac{x+3z}{x-3z}$$

Solution:

Here

$$x = \frac{3yz}{y-z}$$

Divided by 3y

$$\frac{x}{3y} = \frac{3yz}{3y(y-z)}$$

$$\frac{x}{3y} = \frac{z}{y-z}$$

By applying componendo dividendo prop

$$\frac{x+3y}{x-3y} = \frac{z+(y-z)}{z-(y-z)}$$

$$= \frac{z+y-z}{z-y+z}$$

$$= \frac{y}{2z-y}$$

Applying invertendo prop

$$\frac{x-3y}{x+3y} = \frac{2z-y}{y} \rightarrow (i)$$

Again consider

$$x = \frac{3yz}{y-z}$$

Divided by 3z

$$\frac{x}{3z} = \frac{3yz}{3z(y-z)}$$

$$= \frac{y}{y-z}$$

By applying componendo-dividendo prop

$$\frac{x+3z}{x-3z} = \frac{y+(y-z)}{y-(y-z)}$$

$$= \frac{y+y-z}{y-y+z}$$

$$= \frac{2y-z}{z} \rightarrow (i)$$

Sub.equation (i) and (ii)

$$\begin{aligned} \frac{x-3y}{x+3y} - \frac{x+3z}{x-3z} &= \frac{2z-y}{y} - \frac{2y-z}{z} \\ &= \frac{z(2z-y) - y(2y-z)}{yz} \\ &= \frac{2z^2 - yz - 2y^2 + yz}{yz} \\ &= \frac{2z^2 - 2y^2}{yz} \end{aligned}$$

$$\Rightarrow \frac{x-3y}{x+3y} - \frac{x+3z}{x-3z} = \frac{2(z^2 - y^2)}{yz}$$

(v) Given (A.B)

$$s = \frac{6pq}{p-q}$$

To find value of

$$\frac{s-3p}{s+3p} + \frac{s+3q}{s-3q} = ?$$

Solution:

Here

$$s = \frac{6pq}{p-q}$$

Divided by 3p

$$\frac{s}{3p} = \frac{6pq}{3p(p-q)}$$

$$\frac{s}{3p} = \frac{2q}{p-q}$$

By applying componendo-dividendo prop (ii)

$$\frac{s-3p}{s+3p} = \frac{2q-(p-q)}{2q+(p-q)}$$

$$\frac{s-3p}{s+3p} = \frac{2q-p+q}{2q+p-q}$$

$$= \frac{3q-p}{p+q} \rightarrow (i)$$

Again consider

$$s = \frac{6pq}{p-q}$$

Divided by 3q

$$\frac{s}{3q} = \frac{6pq}{3q(p-q)}$$

$$= \frac{2p}{p-q}$$

By applying componendo-dividendo prop

$$\frac{s+3q}{s-3q} = \frac{2p+(p-q)}{2p-(p-q)}$$

$$= \frac{2p+p-q}{2p-p+q}$$

$$= \frac{3p-q}{p+q} \rightarrow \text{(ii)}$$

Adding equation (i) and (ii)

$$\frac{s-3p}{s+3p} + \frac{s+3q}{s-3q} = \frac{3q-p}{p+q} + \frac{3p-q}{p+q}$$

$$= \frac{3q-p+3p-q}{p+q}$$

$$= \frac{2p+2q}{p+q}$$

$$= 2 \frac{(p+q)}{p+q}$$

$$\Rightarrow \frac{s-3p}{s+3p} + \frac{s+3q}{s-3q} = 2$$

(vi) (FSD 2016) (A.B)

Solution:

Here

$$\frac{(x-2)^2 - (x-4)^2}{(x-2)^2 + (x-4)^2} = \frac{12}{13}$$

By applying invertendo property

$$\frac{(x-2)^2 + (x-4)^2}{(x-2)^2 - (x-4)^2} = \frac{13}{12}$$

By applying componendo-dividendo prop

$$\frac{[(x-2)^2 + (x-4)^2] + [(x-2)^2 - (x-4)^2]}{[(x-2)^2 + (x-4)^2] - [(x-2)^2 - (x-4)^2]} = \frac{13+12}{13-12}$$

$$\frac{(x-2)^2 + \cancel{(x-4)^2} + (x-2)^2 - \cancel{(x-4)^2}}{\cancel{(x-2)^2} + (x-4)^2 - \cancel{(x-2)^2} + (x-4)^2} = \frac{25}{1}$$

$$\frac{2(x-2)^2}{2(x-4)^2} = 25$$

$$\left(\frac{x-2}{x-4}\right)^2 = 25$$

Taking square root on both sides

$$\sqrt{\left(\frac{x-2}{x-4}\right)^2} = \sqrt{25}$$

$$\frac{x-2}{x-4} = \pm 5$$

Either

$$\frac{x-2}{x-4} = 5$$

$$x-2 = 5(x-4)$$

$$x-2 = 5x-20$$

$$x-5x = -20+2$$

$$-4x = -18$$

$$x = \frac{18}{4}$$

$$x = \frac{9}{2}$$

or

$$\frac{x-2}{x-4} = -5$$

$$x-2 = -5(x-4)$$

$$x-2 = -5x+20$$

$$x+5x = 20+2$$

$$6x = 22$$

$$x = \frac{11}{3}$$

$$x = \frac{11}{3}$$

$$\therefore \text{Solution Set} = \left\{ \frac{11}{3}, \frac{9}{2} \right\}$$

(vii) (SGD 2015) (A.B)

Solution:

$$\frac{\sqrt{x^2+2} + \sqrt{x^2-2}}{\sqrt{x^2+2} - \sqrt{x^2-2}} = \frac{2}{1}$$

By applying componendo-dividendo prop

$$\frac{(\sqrt{x^2+2} + \sqrt{x^2-2}) + (\sqrt{x^2+2} - \sqrt{x^2-2})}{(\sqrt{x^2+2} + \sqrt{x^2-2}) - (\sqrt{x^2+2} - \sqrt{x^2-2})} = \frac{2+1}{2-1}$$

$$\frac{\sqrt{x^2+2} + \sqrt{x^2-2} + \sqrt{x^2+2} - \sqrt{x^2-2}}{\sqrt{x^2+2} + \sqrt{x^2-2} - \sqrt{x^2+2} + \sqrt{x^2-2}} = \frac{3}{1}$$

$$\frac{2\sqrt{x^2+2}}{2\sqrt{x^2-2}} = 3$$

$$\sqrt{\frac{x^2+2}{x^2-2}} = 3$$

Squaring on both sides

$$\frac{x^2 + 2}{x^2 - 2} = 9$$

$$x^2 + 2 = 9(x^2 - 2)$$

$$x^2 + 2 = 9x^2 - 18$$

$$x^2 - 9x^2 = -18 - 2$$

$$-8x^2 = -20$$

$$x^2 = \frac{-20}{-8}$$

$$\Rightarrow x^2 = \frac{5}{2}$$

Taking square root on both sides

$$x = \pm \sqrt{\frac{5}{2}}$$

$$\therefore \text{Solution Set} = \left\{ \pm \sqrt{\frac{5}{2}} \right\}$$

(viii)

(A.B)

Solution:

$$\frac{\sqrt{x^2 + 8P^2} - \sqrt{x^2 - P^2}}{\sqrt{x^2 + 8P^2} + \sqrt{x^2 - P^2}} = \frac{1}{3}$$

By applying Invertendo Property

$$\frac{\sqrt{x^2 + 8P^2} + \sqrt{x^2 - P^2}}{\sqrt{x^2 + 8P^2} - \sqrt{x^2 - P^2}} = \frac{3}{1}$$

By applying componendo-dividendo prop

$$\frac{\sqrt{x^2 + 8P^2} + \sqrt{x^2 - P^2} + (\sqrt{x^2 + 8P^2} - \sqrt{x^2 - P^2})}{\sqrt{x^2 + 8P^2} + \sqrt{x^2 - P^2} - (\sqrt{x^2 + 8P^2} - \sqrt{x^2 - P^2})} = \frac{3+1}{3-1}$$

$$\frac{\sqrt{x^2 + 8P^2} + \sqrt{x^2 - P^2} + \sqrt{x^2 + 8P^2} - \sqrt{x^2 - P^2}}{\sqrt{x^2 + 8P^2} + \sqrt{x^2 - P^2} - \sqrt{x^2 + 8P^2} + \sqrt{x^2 - P^2}} = \frac{4}{2}$$

$$\frac{2\sqrt{x^2 + 8P^2}}{2\sqrt{x^2 - P^2}} = 2$$

Squaring on both sides

$$\left(\frac{\sqrt{x^2 + 8P^2}}{\sqrt{x^2 - P^2}} \right)^2 = (2)^2$$

$$\frac{x^2 + 8P^2}{x^2 - P^2} = 4$$

$$x^2 + 8P^2 = 4x^2 - 4P^2$$

$$x^2 - 4x^2 = -4P^2 - 8P^2$$

$$-3x^2 = -12P^2$$

$$3x^2 = 12P^2$$

$$x^2 = \frac{12P^2}{3}$$

$$x^2 = 4P^2$$

Taking square root on both sides

$$\sqrt{x^2} = \sqrt{4P^2}$$

$$x = \pm 2P$$

Either

$$x = 2P \quad \text{or} \quad x = -2P$$

$$\therefore \text{Solution Set} = \{2P, -2P\}$$

$$(ix) \quad \frac{(x+5)^3 - (x-3)^3}{(x+5)^3 + (x-3)^3} = \frac{13}{14} \quad (A.B)$$

Solution:

$$\frac{(x+5)^3 - (x-3)^3}{(x+5)^3 + (x-3)^3} = \frac{13}{14}$$

By applying invertendo property

$$\frac{(x+5)^3 + (x-3)^3}{(x+5)^3 - (x-3)^3} = \frac{14}{13}$$

Now by applying componendo-dividendo property

$$\frac{[(x+5)^3 + (x-3)^3] + [(x+5)^3 - (x-3)^3]}{[(x+5)^3 + (x-3)^3] - [(x+5)^3 - (x-3)^3]} = \frac{14+13}{14-13}$$

$$\frac{(x+5)^3 + (x-3)^3 + (x+5)^3 - (x-3)^3}{(x+5)^3 + (x-3)^3 - (x+5)^3 + (x-3)^3} = \frac{27}{1}$$

$$\frac{2(x+5)^3}{2(x-3)^3} = 27$$

$$\left(\frac{x+5}{x-3} \right)^3 = 27$$

Taking cube root on both sides

$$\sqrt[3]{\left(\frac{x+5}{x-3}\right)^3} = \sqrt[3]{27}$$

$$\frac{x+5}{x-3} = 3$$

$$x+5 = 3(x-3)$$

$$x+5 = 3x-9$$

$$x-3x = -9-5$$

$$-2x = -14$$

$$x = \frac{-14}{-2}$$

$$x = 7$$

$\therefore$ Solution Set = {7}

Joint Variation

(K.B)

A combination of direct and inverse variations of one or more than one variables forms joint variation.

Symbolically

If a variable y varies directly as x and varies inversely as z , then

$$y \propto x \text{ and } y \propto \frac{1}{z}$$

In joint variation,

$$y \propto \frac{x}{z}, \text{ or } y = \frac{kx}{z}; k \neq 0$$

Problems Related to Joint Variation

Example 2: (Page # 64)

(K.B)

p varies jointly as q and r^2 and inversely as s and t^2 , $p=40$, when $q=8$, $r=5$, $s=3$, $t=2$. Find p in terms of q , r , s and t . Also find the value of p when $q=-2$, $r=4$, $s=3$ and $t=-1$.

Solution:

Given that $p \propto \frac{qr^2}{st^2}$

$$p = k \frac{qr^2}{st^2} \quad (i)$$

Put $p = 40$, $q = 8$, $r = 5$, $s = 3$ and $t = 2$

$$40 = k \frac{(8)(5)^2}{3(2)^2}$$

$$\frac{40 \times 3 \times 4}{8 \times 25} = k$$

$$k = \frac{12}{5}$$

Then eq. (i) becomes

$$p = \frac{12}{5} \frac{qr^2}{st^2}$$

Now for $q=-2$, $r=4$, $s=3$ and $t=-1$, we have

$$p = \frac{12}{5} \frac{(-2)(4)^2}{(3)(-1)^2} = -\frac{128}{5}$$

Exercise 3.5

Q.1 Given

(K.B + A.B)

(FSD 2015, SWL 2015)

$$s \propto u^2 \text{ and } s \propto \frac{1}{v}$$

$$s = 7 \text{ when } u = 3, v = 2$$

To find

Value of $s = ?$ when $u = 6$, $v = 10$

Solution:

Here

$$s \propto u^2 \text{ and } s \propto \frac{1}{v}$$

In joint variation

$$s \propto \frac{u^2}{v}$$

$$\Rightarrow s = \frac{ku^2}{v} \rightarrow (i)$$

For value of k

Put $s = 7$, $u = 3$ and $v = 2$ in equation (i)

$$7 = \frac{k(3)^2}{2}$$

$$\frac{14}{9} = k$$

$$\therefore s = \frac{14u^2}{9v}$$

For value of s

Put $u = 6$, $v = 10$ and $k = \frac{14}{9}$ in equation (i)

$$s = \frac{14}{9} \times \frac{(6)^2}{10}$$

$$s = \frac{14}{9} \times \frac{36}{10}$$

$$s = \frac{28}{5}$$

Result

$$s = \frac{28}{5} \text{ when } u = 6, v = 10$$

Q.2 Given (K.B + A.E)

$$w \propto xy^2z$$

$$w = 5 \text{ when } x = 2, y = 3, z = 10$$

To find

$$w = ? \text{ when } x = 4, y = 7, z = 3$$

Solution:

$$\text{Here } w \propto xy^2z$$

$$w = kxy^2z \rightarrow (i)$$

For value of kPut $w = 5, x = 2, y = 3$ and $z = 10$ in eq (i)

$$5 = k(2)(3)^2(10)$$

$$5 = 180k$$

$$\frac{5}{180} = k$$

$$k = \frac{1}{36}$$

$$\therefore w = \frac{xy^2z}{36}$$

For value of wPut $x = 4, y = 7, z = 3$ and $k = \frac{1}{36}$ in eq (i)

$$w = \frac{1}{36}(4)(7)^2(3)$$

$$\Rightarrow w = \frac{49}{3}$$

Result

$$w = \frac{49}{3} \text{ when } x = 4, y = 7 \text{ and } z = 3$$

Q.3 Given (K.B + A.E)

$$y \propto x^3 \text{ and } y \propto \frac{1}{z^2t}$$

$$y = 16 \text{ when } x = 4, z = 2, t = 3$$

To find

$$y = ? \text{ when } x = 2, z = 3, t = 4$$

Solution:

$$\text{Here } y \propto x^3, y \propto \frac{1}{z^2t}$$

In joint variation:

$$y \propto \frac{x^3}{z^2t}$$

$$\Rightarrow y = \frac{kx^3}{z^2t} \rightarrow (i)$$

For value of kPut $y = 16, x = 4, z = 2, t = 3$ in equation (i)

$$16 = \frac{k(4)^3}{(2)^2(3)}$$

$$16 = \frac{k(64)}{12}$$

$$\frac{16 \times 12}{64} = k$$

$$3 = k$$

$$\Rightarrow k = 3$$

$$\therefore y = \frac{3x^3}{z^2t}$$

For value of 'y'Put $x = 2, z = 3, t = 4$ and $k = 3$ in eq (i)

$$y = \frac{3(2)^3}{(3)^2(4)}$$

$$y = \frac{3(8)}{(9)(4)}$$

$$\Rightarrow y = \frac{2}{3}$$

Result

$$y = \frac{2}{3} \text{ when } x = 2, z = 3 \text{ and } t = 4$$

Q.4 Given (K.B + A.E)

$$u \propto x^2 \text{ and } u \propto \frac{1}{yz^3}$$

$$u = 2 \text{ when } x = 8, y = 7, z = 2$$

To find

$$u = ? \text{ when } x = 6, y = 3, z = 2$$

Solution:

Here

$$u \propto x^2 \text{ and } u \propto \frac{1}{yz^3}$$

In joint variation:

$$u \propto \frac{x^2}{yz^3}$$

Or $u = \frac{kx^2}{yz^3} \rightarrow (i)$

For value of k

Put $u = 2, x = 3, y = 7, z = 2$ in equation (i)

$$2 = \frac{k(3)^2}{7(2)^3}$$

$$2 = \frac{k(9)}{7(8)}$$

$$2 \times \frac{7}{8} = k$$

$$\frac{7}{4} = k$$

$$\Rightarrow k = \frac{7}{4}$$

$$\therefore u = \frac{7x^2}{4yz^3}$$

For value of u

Put $k = \frac{7}{4}, x = 6, y = 3$ and $z = 2$

$$u = \frac{7}{4} \cdot \frac{(6)^2}{3(2)^3}$$

$$u = \frac{7}{4} \times \frac{36}{3(8)}$$

$$u = \frac{21}{8}$$

Result

$$u = \frac{21}{8} \text{ when } x = 6, y = 3, z = 2$$

Q.5 Given (K.B + A.B)

$$v \propto xy^2 \text{ and } v \propto \frac{1}{z^2}$$

$$v = 27 \text{ when } x = 7, y = 6, z = 7$$

To find

$$v = ? \text{ when } x = 6, y = 2, z = 3$$

Solution:

Here

$$V \propto xy^2 \text{ and } v \propto \frac{1}{z^2}$$

In joint variation

$$v \propto \frac{xy^2}{z^2}$$

$$v = \frac{kxy^2}{z^2} \rightarrow (i)$$

For value of k

Put $v = 27, x = 7, y = 6$ and $z = 7$

$$27 = \frac{k(7)(6)^2}{(7)^2}$$

$$27 = \frac{k(7)(216)}{49}$$

$$\frac{27 \times 7}{216} = k$$

$$\frac{7}{8} = k$$

Or $k = \frac{7}{8}$

$$\therefore v = \frac{7xy^2}{8z^2}$$

For value of v

Put $x = 6, y = 2, z = 3$ and $k = \frac{7}{8}$ in equation (i)

$$v = \frac{7(6)(2)^2}{8 \cdot 3^2}$$

$$v = \frac{7}{8} \times \frac{6 \times 8}{9}$$

$$v = \frac{14}{3}$$

Result

$$v = \frac{14}{3} \text{ when } x = 6, y = 2, z = 3$$

Q.6 Given (D.G.K 2015) (K.B + A.B)

$$w \propto \frac{1}{u^3}$$

$$w = 5 \text{ when } u = 3$$

To find

$$w = ? \text{ when } u = 6$$

Solution:

Here

$$w \propto \frac{1}{u^3}$$

$$w = \frac{k}{u^3} \rightarrow (i)$$

For value of k

Put $w = 5, u = 3$ in equation (i)

$$5 = \frac{k}{(3)^3}$$

$$5 \times 27 = k$$

$$135 = k$$

$$k = \frac{135}{1}$$

For value of w

Put $k = 135$ and $u = 6$ in equation (i)

$$w = \frac{135}{(6)^3}$$

$$= \frac{135}{216}$$

$$= \frac{45}{72}$$

$$= \frac{15}{24}$$

$$= \frac{5}{8}$$

$$= \frac{5}{8}$$

$$= \frac{5}{8}$$

$$= \frac{5}{8}$$

$$= \frac{5}{8}$$

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$$= \frac{5}{8}$$

$$= \frac{5}{8}$$

Result

$$w = \frac{5}{8} \text{ when } u = 6$$

K-Method

(K.B + A.B)

If $a : b :: c : d$ is a proportion, then

$$\text{Let } \frac{a}{b} = \frac{c}{d} = k$$

$$\text{Or } \frac{a}{b} = k, \quad \frac{c}{d} = k$$

$$a = bk, c = dk$$

These equations are used to evaluate certain expression more easily.

Using these equations is called K-Method.

Example 3: (Page # 66)

(K.B + A.B)

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then show that

$$\frac{a^3 + c^3 + e^3}{b^3 + d^3 + f^3} = \frac{ace}{bdf}$$

Solution:

$$\text{Let } \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$$

$$\text{Then } \frac{a}{b} = k, \frac{c}{d} = k \text{ and } \frac{e}{f} = k$$

$$\text{i.e., } a = bk, c = dk \text{ and } e = fk$$

To prove:

$$\frac{a^3 + c^3 + e^3}{b^3 + d^3 + f^3} = \frac{ace}{bdf}$$

Proof:

$$\begin{aligned} \text{Now L.H.S} &= \frac{a^3 + c^3 + e^3}{b^3 + d^3 + f^3} \\ &= \frac{(bk)^3 + (dk)^3 + (fk)^3}{b^3 + d^3 + f^3} \\ &= \frac{b^3k^3 + d^3k^3 + f^3k^3}{b^3 + d^3 + f^3} \\ &= k^3 \left(\frac{b^3 + d^3 + f^3}{b^3 + d^3 + f^3} \right) = k^3 \end{aligned}$$

$$\begin{aligned} \text{Also R.H.S} &= \frac{ace}{bdf} = \frac{bk + dk + fk}{bdf} \\ &= k^3 \frac{bdf}{bdf} = k^3 \end{aligned}$$

$$\therefore \text{L.H.S} = \text{R.H.S}$$

$$\text{i.e., } \frac{a^3 + c^3 + e^3}{b^3 + d^3 + f^3} = \frac{ace}{bdf}$$

Hence proved

Exercise 3.6

Q.1

(K.B + A.B)

(FSD 2015, SGD 2015)

Given $a : b = c : d$

To prove

$$(i) \quad \frac{4a - 9b}{4a + 9b} = \frac{4c - 9d}{4c + 9d}$$

$$(ii) \quad \frac{6a - 5b}{6a + 5b} = \frac{6c - 5d}{6c + 5d}$$

$$(iii) \quad \frac{a}{b} = \sqrt{\frac{a^2 + c^2}{b^2 + d^2}}$$

$$(iv) \quad a^6 + b^6 : b^6 + d^6 = a^3c^3 : b^3d^3$$

$$(v) \quad p(a + b) + qb : p(c + d) + qd = a : c$$

$$(vi) \quad a^2 + b^2 : \frac{a^3}{a+b} = c^2 + d^2 : \frac{c^3}{c+d}$$

$$(vii) \quad \frac{a}{a-b} : \frac{a+b}{b} = \frac{c}{c-d} : \frac{c+d}{d}$$

Proof:

Let $a : b = c : d = k$

$$\Rightarrow \frac{a}{b} = \frac{c}{d} = k$$

$$\Rightarrow \frac{a}{b} = k, \quad \frac{c}{d} = k$$

$$a = bk, \quad c = dk$$

$$(i) \quad \frac{4a-9b}{4a+9b} = \frac{4c-9d}{4c+9d} \quad \text{(A.B)}$$

$$\text{L.H.S} = \frac{4a-9b}{4a+9b}$$

Putting the values of a

$$= \frac{4bk-9b}{4bk+9b}$$

$$= \frac{b(4k-9)}{b(4k+9)}$$

$$= \frac{b(4k-9)}{b(4k+9)}$$

$$= \frac{4k-9}{4k+9} \rightarrow (i)$$

$$\text{R.H.S} = \frac{4c-9d}{4c+9d}$$

$$\text{Putting the value of 'c'}$$

$$= \frac{4dk-9d}{4dk+9d}$$

$$= \frac{d(4k-9)}{d(4k+9)}$$

$$= \frac{4k-9}{4k+9} \rightarrow (ii)$$

$$\text{From equation (i) and (ii)}$$

$$\text{L.H.S} = \text{R.H.S}$$

$$\frac{4a-9b}{4a+9b} = \frac{4c-9d}{4c+9d}$$

$$\text{Hence Proved}$$

$$(ii) \quad \frac{6a-5b}{6a+5b} = \frac{6c-5d}{6c+5d} \quad \text{(A.B)}$$

$$\text{L.H.S} = \frac{6a-5b}{6a+5b}$$

$$\text{Putting the value of 'a'}$$

$$= \frac{6bk-5b}{6bk+5b}$$

$$= \frac{b(6k-5)}{b(6k+5)} = \frac{6k-5}{6k+5} \rightarrow (i)$$

$$\text{R.H.S} = \frac{6c-5d}{6c+5d}$$

$$= \frac{6dk-5d}{6dk+5d}$$

$$= \frac{d(6k-5)}{d(6k+5)} = \frac{6k-5}{6k+5} \rightarrow (ii)$$

$$\text{From equation (i) and (ii)}$$

$$\text{L.H.S} = \text{R.H.S}$$

$$\text{Hence Proved}$$

$$= \frac{6(bk)-5b}{6(bk)+5b}$$

$$= \frac{b(6k-5)}{b(6k+5)} = \frac{6k-5}{6k+5} \rightarrow (i)$$

$$\text{R.H.S} = \frac{6c-5d}{6c+5d}$$

Putting the value of 'c'

$$= \frac{6dk-5d}{6dk+5d}$$

$$= \frac{d(6k-5)}{d(6k+5)}$$

$$= \frac{6k-5}{6k+5} \rightarrow (ii)$$

$$\text{From equation (i) and (ii)}$$

$$\text{R.H.S} = \text{R.H.S}$$

$$\frac{6a-5b}{6a+5b} = \frac{6c-5d}{6c+5d}$$

$$\text{Hence Proved}$$

$$(iii) \quad \frac{a}{b} = \sqrt{\frac{a^2+c^2}{b^2+d^2}} \quad \text{(A.B)}$$

$$\text{L.H.S} = \frac{a}{b}$$

Putting the value of a

$$= \frac{bk}{b}$$

$$\text{L.H.S} = k \rightarrow (i)$$

$$\text{R.H.S} = \sqrt{\frac{a^2+c^2}{b^2+d^2}}$$

Putting the values

$$= \sqrt{\frac{(bk)^2+(dk)^2}{b^2+d^2}}$$

$$= \sqrt{\frac{b^2k^2+d^2k^2}{b^2+d^2}}$$

$$= \sqrt{\frac{k^2(b^2+d^2)}{b^2+d^2}} = \sqrt{k^2}$$

$$\text{R.H.S} = k \dots\dots\dots (ii)$$

From Equation (i) and (ii)

$$\text{L.H.S} = \text{R.H.S}$$

$$\frac{a}{b} = \sqrt{\frac{a^2 + c^2}{b^2 + d^2}}$$

Hence Proved

(iv) $a^6 + b^6 : b^6 + d^6 = a^3 c^3 : b^3 d^3$ **(A.B)**

L.H.S = $a^6 + c^6 : b^6 + d^6$

Putting the value of 'a' and 'c'

$$= (bk)^6 + (dk)^6 : b^6 + d^6$$

$$= b^6 k^6 + d^6 k^6 : b^6 + d^6$$

$$= k^6 (b^6 + d^6) : (b^6 + d^6)$$

$$= k^6 \rightarrow (i)$$

R.H.S = $a^3 c^3 : b^3 d^3$

Putting the values of a and c

$$= (bk)^3 (dk)^3 : b^3 d^3$$

$$= b^3 k^3 d^3 k^3 : b^3 d^3$$

$$= b^3 d^3 k^6 : b^3 d^3$$

$$= k^6 \rightarrow (ii)$$

From equation (i) and (ii)

L.H.S = R.H.S

$$a^6 + b^6 : b^6 + d^6 = a^3 c^3 : b^3 d^3$$

Hence Proved

(v) $p(a+b) + qb : p(c+d) + qd = a : c$ **(A.B)**

L.H.S = $p(a+b) + qb : p(c+d) + qd$

Putting the value of a and c

$$= p(bk+b) + qb : p(dk+d) + qd$$

$$= pb(k+1) + qb : pd(k+1) + qd$$

$$= b[p(k+1) + q] : d[p(k+1) + q]$$

$$= b : d \rightarrow (i)$$

R.H.S = $a : c$

Putting the value of a and c

$$= bk : dk$$

$$= b : d \rightarrow (ii)$$

From equation (i) and (ii)

L.H.S = R.H.S

$$p(a+b) + qb : p(c+d) + qd = a : c$$

Hence Proved

(vi) $a^2 + b^2 : \frac{a^3}{a+b} = c^2 + d^2 : \frac{c^3}{c+d}$ **(A.B)**

L.H.S = $a^2 + b^2 : \frac{a^3}{a+b}$

Putting the values

$$= (bk)^2 + b^2 : \frac{(bk)^3}{bk+b}$$

$$= b^2 k^2 + b^2 : \frac{b^3 k^3}{b(k+1)}$$

$$= b^2 (k^2 + 1) : \frac{b^2 k^3}{k+1}$$

$$= \frac{b^2 (k^2 + 1)}{b^2 k^3 / k+1}$$

$$= \frac{b^2 (k^2 + 1)(k+1)}{b^2 k^3}$$

L.H.S = $\frac{(k^2 + 1)(k+1)}{k^3} \dots (i)$

R.H.S = $c^2 + d^2 : \frac{c^3}{c+d}$

Putting the values

$$= (dk)^2 + d^2 : \frac{(dk)^3}{dk+d}$$

$$= d^2 k^2 + d^2 : \frac{d^3 k^3}{d(k+1)}$$

$$= d^2 (k^2 + 1) : \frac{d^2 k^3}{k+1}$$

$$= \frac{d^2 (k^2 + 1)}{d^2 k^3 / k+1}$$

$$= \frac{d^2 (k^2 + 1)}{d^2 k^3} \times k+1$$

R.H.S = $\frac{(k^2 + 1)(k+1)}{k^3} \dots (i)$

From Equation (i) and (ii)

L.H.S = R.H.S

$$a^2 + b^2 : \frac{a^3}{a+b} = c^2 + d^2 : \frac{c^3}{c+d}$$

Hence Proved

(vii) $\frac{a}{a-b} : \frac{a+b}{b} = \frac{c}{c-d} : \frac{c+d}{d}$ **(A.B)**

L.H.S = $\frac{a}{a-b} : \frac{a+b}{b}$

Putting the value of 'a'

$$= \frac{bk}{bk-b} : \frac{bk+b}{b}$$

$$= \frac{bk}{b(k-1)} : \frac{b(k+1)}{b}$$

$$= \frac{k}{k-1} : k+1 \rightarrow (i)$$

$$\text{R.H.S} = \frac{c}{c-d} : \frac{c+d}{d}$$

Putting the value of 'c'

$$= \frac{dk}{dk-d} : \frac{dk+d}{d}$$

$$= \frac{dk}{d(k-1)} : \frac{d(k+1)}{d}$$

$$= \frac{k}{k-1} : k+1 \rightarrow (ii)$$

From equation (i) and (ii)

L.H.S = R.H.S

$$\frac{a}{a-b} : \frac{a+b}{b} = \frac{c}{c-d} : \frac{c+d}{d}$$

Hence Proved

Q.2

If

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} \quad (a, b, c, d, e, f \neq 0) \text{ then}$$

show that

(A.B)

$$(i) \quad \frac{a}{b} = \sqrt{\frac{a^2 + c^2 + e^2}{b^2 + d^2 + f^2}}$$

$$(ii) \quad \frac{ac + ce + ea}{bd + df + fb} = \left[\frac{ace}{bdf} \right]^{\frac{1}{2}}$$

$$(iii) \quad \frac{ac}{bd} + \frac{ce}{df} + \frac{ea}{fb} = \frac{a^2}{b^2} + \frac{c^2}{d^2} + \frac{e^2}{f^2}$$

Proof: Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

$$\Rightarrow \frac{a}{b} = k, \frac{c}{d} = k, \frac{e}{f} = k$$

$$\Rightarrow a = bk, c = dk, e = fk$$

$$(i) \quad \frac{a}{b} = \sqrt{\frac{a^2 + c^2 + e^2}{b^2 + d^2 + f^2}}$$

(A.B)

$$\text{L.H.S} = \frac{a}{b}$$

Putting the value of 'a'

$$= \frac{bk}{b}$$

$$= k \rightarrow (i)$$

$$\text{R.H.S} = \sqrt{\frac{a^2 + c^2 + e^2}{b^2 + d^2 + f^2}}$$

Putting the values of a, c and e

$$= \sqrt{\frac{(bk)^2 + (dk)^2 + (fk)^2}{b^2 + d^2 + f^2}}$$

$$= \sqrt{\frac{b^2k^2 + d^2k^2 + f^2k^2}{b^2 + d^2 + f^2}}$$

$$= \sqrt{\frac{k^2(b^2 + d^2 + f^2)}{(b^2 + d^2 + f^2)}}$$

$$= \sqrt{k^2}$$

$$= k \rightarrow (i)$$

From equation (i) and (ii)

L.H.S = R.H.S

$$\frac{a}{b} = \sqrt{\frac{a^2 + c^2 + e^2}{b^2 + d^2 + f^2}}$$

Hence Proved

$$(ii) \quad \frac{ac + ce + ea}{bd + df + fb} = \left[\frac{ace}{bdf} \right]^{\frac{1}{2}} \quad (\text{A.B})$$

(FSD 2016, MTN 2015)

$$\text{L.H.S} = \frac{ac + ce + ea}{bd + df + fb}$$

Putting the values

$$= \frac{(bk)(dk) + (dk)(fk) + (fk)(bk)}{ba + df + fb}$$

$$\text{L.H.S} = \frac{bdk^2 + dfk^2 + fbk^2}{ba + df + fb}$$

$$= \frac{k^2(bd + df + fb)}{(bd + df + fb)}$$

$$\text{L.H.S} = k^2 \dots\dots (i)$$

$$\text{R.H.S} = \left[\frac{ace}{bdf} \right]^{\frac{1}{2}}$$

Putting the values

$$= \left[\frac{(bk)(dk)(fk)}{bdf} \right]^{\frac{2}{3}}$$

$$= \left[\frac{bdfk^3}{bdf} \right]^{\frac{2}{3}}$$

$$= (k^3)^{\frac{2}{3}}$$

$$\text{R.H.S} = k^2 \dots\dots (i)$$

$$\text{From Eq. (i) and (ii)}$$

$$\text{L.H.S} = \text{R.H.S}$$

$$\frac{ac + ce + ea}{bd + df + fb} = \left[\frac{ace}{bdf} \right]^{\frac{1}{2}}$$

Hence Proved

$$(iii) \quad \frac{ac}{bd} + \frac{ce}{df} + \frac{ea}{fb} = \frac{a^2}{b^2} + \frac{c^2}{d^2} + \frac{e^2}{f^2} \quad (\text{A.B})$$

$$\text{L.H.S} = \frac{ac}{bd} + \frac{ce}{df} + \frac{ea}{fb}$$

$$= \frac{bk.dk}{bd} + \frac{dk.fk}{df} + \frac{fk.bk}{fb}$$

$$= \frac{bdk^2}{bd} + \frac{dfk^2}{df} + \frac{fbk^2}{fb}$$

$$= k^2 + k^2 + k^2$$

$$= 3k^2 \longrightarrow (i)$$

$$\text{R.H.S} = \frac{a^2}{b^2} + \frac{c^2}{d^2} + \frac{e^2}{f^2}$$

Putting the value of a, c and e

$$= \frac{(bk)^2}{b^2} + \frac{(dk)^2}{d^2} + \frac{(fk)^2}{f^2}$$

$$= \frac{b^2k^2}{b^2} + \frac{d^2k^2}{d^2} + \frac{f^2k^2}{f^2}$$

$$= k^2 + k^2 + k^2$$

$$= 3k^2 \longrightarrow (i)$$

From equation (i) and (ii)

$$\text{L.H.S} = \text{R.H.S}$$

$$\frac{ac}{bd} + \frac{ce}{df} + \frac{ea}{fb} = \frac{a^2}{b^2} + \frac{c^2}{d^2} + \frac{e^2}{f^2}$$

Hence Proved

Real Life Problems Based on Variation

Example 2 (Page # 68) (A.B + U.B)

The current in a wire varies directly as the electromotive force E and inversely as the resistance R . If $I=32$ amperes, when $E=128$ volts and $R=8$ ohms. Find I , when $E=150$ volts and $R=18$ ohms.

Solution:

In joint variation, we have $I \propto \frac{E}{R}$,

$$I = \frac{kE}{R} \rightarrow (i)$$

For $I = 32$, $E = 128$ and $R = 8$,

$$32 = \frac{k(128)}{8}$$

$$\Rightarrow \frac{32 \times 8}{128} = k$$

$$\Rightarrow k = 2$$

Put in equation (i)

$$I = \frac{2E}{R}$$

Now for $E = 150$ and $R = 18$

$$I = \frac{2(150)}{18} = \frac{50}{3} \text{ amp.}$$

Exercise 3.7

Q.1 The surface area A of a cube varies directly as the square of the length l of an edge and $A = 27$ square units when $l = 3$ units. Find A when $l = 4$ units (ii) l when $A = 12$ sq. units. (A.B + K.B)

Given

$$A \propto l^2$$

$$A = 27 \text{ square unit when } l = 3 \text{ units}$$

To find

$$A = ? \text{ when } l = 4 \text{ units}$$

$$l = ? \text{ when } A = 12 \text{ square unit}$$

Solution:

Here

$$A \propto l^2$$

$$A = kl^2 \longrightarrow (i)$$

$$\text{Put } A = 27, l = 3$$

$$27 = k(3)^2$$

$$27 = 9k$$

$$3 = k$$

$$k = 3$$

For value of A

Put $k = 3$, $l = 4$ in equation (i)

$$A = 3(4)^2$$

$$A = 48$$

For value of l

Put $k = 3$, $A = 12$ square unit in equation (i)

$$12 = 3l^2$$

$$4 = l^2$$

$$\text{Or } l^2 = 4$$

Taking square root on b/s

$$l^2 = \pm 2$$

$$\Rightarrow l = 2 \text{ (Length is always positive)}$$

Result:

$A = 48$ square unit when $l = 4$ units

$l = 2$ unit when $A = 12$ square units

- Q.2 The surface area S of the sphere varies directly as the square of radius r , and $S = 16\pi$ when $r = 2$. Find r when $S = 36\pi$.**

(A.B K.B)

Given

$$S \propto r^2$$

$$S = 16\pi \text{ when } r = 2$$

To Find

$$r = ? \text{ when } s = 36\pi$$

Solution:

$$\text{Here } S \propto r^2$$

$$S = kr^2 \longrightarrow (i)$$

For value of k

Put $S = 16\pi$, $r = 2$ in equation (i)

$$16\pi = k(2)^2$$

$$16\pi = 4k$$

$$4\pi = k$$

$$\text{Or } k = 4\pi$$

$$S = \pi r^2$$

For value of r

Put $k = 4\pi$, $S = 36\pi$ in equation (i)

$$36\pi = 4\pi r^2$$

$$9 = r^2$$

Taking square root on both sides

$$r = \pm 3$$

$$\Rightarrow r = 3 \text{ (Length is positive)}$$

Result:

$$r = 3 \text{ when } S = 36\pi$$

- Q.3 In Hook's law the force F applied to stretch a spring varies directly as the amount of elongation S and $F = 32\text{ lb}$ when $S = 1.6$ in. Find (i) S when $F = 50\text{ lb}$ (ii) F when $S = 0.8$ in**

(K.B +A.B)

Given

$$F \propto S$$

$$F = 32 \text{ lb when } S = 1.6 \text{ m.}$$

To find

$$S = ? \text{ when } F = 50 \text{ lb}$$

$$F = ? \text{ when } S = 0.8 \text{ in}$$

Solution:

$$\text{Here } F \propto S$$

$$F = kS \longrightarrow (i)$$

For value of k

Put $F = 32$ and $S = 1.6$ in equation (i)

$$32 = k(1.6)$$

$$20 = k \text{ Or } k = 20$$

$$F = 20S$$

For value of S

Put $k = 20$ and $F = 50$

$$50 = 20S$$

$$\frac{5}{2} = S$$

$$\text{Or } S = \frac{5}{2}$$

For value of F

Put $k = 20$ and $S = 0.8$ in equation (i)

$$F = 20(0.8)$$

$$F = 16$$

Result:

$$S = \frac{5}{2} \text{ in when } F = 50 \text{ lb}$$

- Q.4 $F = 16$ lb when $S = 0.8$ in. The intensity I of light from a given source varies inversely as the square of the distance d from it. If the intensity is 20 candlepower at a distance of 12ft. From the source, in d the intensity at a point 8ft. from the source.**

(K.B +A.B)

Given

$$I \propto \frac{1}{d^2}$$

$$I = 20 \text{ candle power when } d = 12\text{ft}$$

To find $I = ?$ when $d = 8\text{ft}$ **Solution:**

Here

$$I \propto \frac{1}{d^2}$$

$$I = \frac{k}{d^2} \rightarrow (i)$$

For value of k Put $I = 20$ and $d = 12$ in equation (i)

$$20 = \frac{k}{(12)^2}$$

$$20 \times 144 = k$$

$$\text{Or } 2880 = k$$

$$I = \frac{2880}{d^2}$$

For value of I Put $k = 2880$ and $d = 8$ in equation (i)

$$I = \frac{2880}{(8)^2}$$

$$I = \frac{2880}{64}$$

$$I = 45$$

Result $I = 45$ candle power when $d = 8\text{ft}$.

- Q.5** The pressure P in a body of fluid varies directly as the depth d . If the pressure exerted on the bottom of a tank by a column of fluid 5ft. high is 2.25 lb/sq. in, how deep must the fluid be to exert a pressure of 9 lb/sq. in?

(K.B +A.B)**Given**

$$P \propto d$$

$$P = 2.25 \text{ lb/sq when } d = 5\text{ft}$$

To Find

$$d = ? \text{ when } P = 9 \text{ lb/sq}$$

Solution:

Here

$$P \propto d$$

$$P = kd \rightarrow (i)$$

For value of k

$$\text{Put } P = 2.25 \text{ and } d = 5$$

$$2.25 = k(5)$$

$$\text{Or } k = 0.45$$

$$P = 0.45d$$

For value of d

$$\text{Put } k = 0.45 \text{ and } P = 9$$

$$9 = 0.45d$$

$$\frac{9}{0.45} = d$$

$$20 = d$$

$$\text{Or } d = 20$$

Result $d = 20\text{ft}$ when $P = 9 \text{ lb/sq}$.

- Q.6** Labour costs c varies jointly as the number of workers n and the average number of days d , if the cost of 800 workers for 13 days is Rs. 286000, then find the labour cost of 600 workers for 18 days

(K.B +A.B)**Given**

$$c \propto nd$$

$$c = \text{Rs. } 286000 \text{ when } n = 800 \text{ workers, } d = 13 \text{ days}$$

To find

$$c = ? \text{ when } n = 600 \text{ workers, } d = 18 \text{ days}$$

Solution:

Here

$$c \propto nd$$

$$c = knd \rightarrow (i)$$

For value of k

$$\text{Put } c = 286000, n = 800 \text{ and } d = 13 \text{ in eq (i)}$$

$$286000 = k(800)(13)$$

$$\frac{286000}{800 \times 13} = k$$

$$\Rightarrow \frac{55}{2} = k$$

$$c = \frac{55}{2} nd$$

For value of c

$$\text{Put } k = \frac{55}{2}, n = 600 \text{ and } 18 \text{ in equation (i)}$$

$$c = \frac{55}{2} \times 600 \times 18$$

$$c = 297000$$

Result:

$$c = \text{Rs. } 297000 \text{ when } n = 600 \text{ workers and } d = 18 \text{ days}$$

Q.7 The supporting load c of a pillar varies as the fourth power of its diameter d and inversely as the square of its length l . A pillar of diameter 6 inch and of height 30 feet will support a load of 63 tons. How high a 4 inch pillar must be to support a load of 28 tons?

(K.B + A.B)

Given

$$c \propto d^4 \text{ and } c \propto \frac{1}{l^2}$$

$c = 63$ tons when $d = 6$ inches and $l = 30$ feet

To Find

$l = ?$ when $d = 4$ inches $c = 28$ tons

Solution:

Here

$$c \propto d^4 \text{ and } c \propto \frac{1}{l^2}$$

In joint variation:

$$c \propto \frac{d^4}{l^2}$$

$$\Rightarrow c = k \frac{d^4}{l^2}$$

For value of k

Put $c = 63$, $d = 6$ and $l = 30$ in equation (i)

$$63 = k \frac{(6)^4}{(30)^2}$$

$$63 = k \frac{1296}{900}$$

$$\frac{63 \times 900}{1296} = k$$

$$\frac{175}{4} = k$$

Or $k = \frac{175}{4}$

$$c = \frac{175 d^4}{4 l^2}$$

For value of l

Put $k = \frac{175}{4}$, $d = 4$, $c = 28$ in equation (i)

$$28 = \frac{175 (4)^4}{4 l^2}$$

$$28 \times 4 l^2 = 175 \times 256$$

$$l^2 = 400$$

Taking positive square root on both sides

$$\Rightarrow l = 20$$

Result

$l = 20$ feet when $d = 4$ inches and $c = 28$ tons

Q.8 The time T required for an elevator to lift a weight varies jointly as the weight w and the lifting depth d varies inversely as the power p of the motor. If 25 sec. are required for a 4-hp motor to lift 500 lb through 40 ft, what power is required to lift 800 lb, through 120 ft in 40 sec.?

(K.B + A.B)

Given

$$T \propto wd \text{ and } T \propto \frac{1}{p}$$

$T = 25$ sec when $P = 4$ hp, $w = 500$ lb

To Find

$P = ?$ when $c = 800$ lb, $d = 120$ ft and

$T = 40$ sec.

Solution:

Here

$$T \propto wd \text{ and } T \propto \frac{1}{p}$$

In joint variation:

$$T \propto \frac{wd}{p}$$

$$T = \frac{kwd}{p} \rightarrow (i)$$

For value of k

Put $T = 25$, $P = 4$, $w = 500$, $d = 40$ in

equation (i)

$$25 = k \frac{500 \times 40}{4}$$

$$\frac{25}{5000} = k$$

$$\frac{1}{200} = k$$

Or $k = \frac{1}{200}$

$$T = \frac{wd}{200P}$$

For value of P

$$\text{Put } k = \frac{1}{200}, w = 800, d = 120$$

and $T = 40$ in equation (i)

$$40 = \frac{1}{200} \times \frac{800 \times 120}{P}$$

$$40P = 4 \times 120$$

$$P = 12$$

Result:

$P = 12$ hp when $w = 800$ lb, $d = 120$ ft and

$T = 40$ sec.

Q.9 The kinetic energy ($K.E.$) of a body varies jointly as the mass " m " of the body and the square of its velocity " v ". If the kinetic energy is 4320 ft/lb when the mass is 45 lb and the velocity is 24 ft/sec. Determine the kinetic energy of a 3000 lb automobile travelling 44 ft/sec. **(K.B + A.B)**

Given

$$K.E \propto mv^2$$

$K.E = 4320$ ft/lb when $m = 45$ lb and

$v = 24$ ft/sec

To find

$K.E = ?$ when $m = 3000$ lb, $v = 44$ ft/sec

Solution:

Here

$$K.E \propto mv^2$$

$$K.E = kmv^2 \longrightarrow (i)$$

For value of k

Put $K.E = 4320$, $m = 45$ and $v = 24$

$$4320 = k(45)(24)^2$$

$$\frac{4320}{45 \times 576} = k$$

$$k = \frac{1}{6}$$

$$K.E = \frac{1}{6}mv^2$$

For value of $K.E$

Put $k = \frac{1}{6}$, $m = 3000$, $v = 44$ in

equation (i)

$$K.E = \frac{1}{6} \times 3000 \times (44)^2$$

$$K.E = 968000$$

Result:

$K.E = 968000$ ft/lb when $m = 3000$ lb and $v = 44$ ft/sec

Miscellaneous Exercise 3

Q.1 Multiple Choice Questions

Four possible answers are given for the following question. Tick (✓) the correct answer.

- (1) In a ratio $a:b$, a is called: (SVL 2014, MTN 2015, D.G.K 2014-15) **(K.B + A.B)**
 (a) Relation (b) Antecedent
 (c) Consequent (d) None of these
- (2) In a ratio $x:y$, y is called: (LHR 2014, GRW 2014, RWP 2015) **(K.B + A.B)**
 (a) Relation (b) Antecedent
 (c) Consequent (d) None of these
- (3) In a proportion $a:b::c:d$, a and d are called: (LHR 2015, MTN 2015) **(K.B + A.B)**
 (a) Means (b) Extremes
 (c) Third proportional (d) None of these
- (4) In a proportion $a:b::c:d$, b and c are called: (LHR 2015) **(K.B + A.B)**
 (a) Means (b) Extremes
 (c) Fourth proportional (d) None of these
- (5) In continued proportion $a:b=b:c$, $ac=b^2$, b is said to be _____ proportional between a and c . **(K.B + A.B)**
 (a) Third (b) Fourth
 (c) Mean (d) None of these
- (6) In continued proportion $a:b=b:c$, c is said to be _____ proportional between a and b . **(K.B + A.B)**
 (a) Third (b) Fourth
 (c) Means (d) None of these
- (7) Find x in proportion $4:x::5:15$ **(K.B + U.B)**
 (a) $\frac{75}{4}$ (b) $\frac{4}{3}$
 (c) $\frac{3}{4}$ (d) 12
- (8) If $u \propto v^2$, then (LHR 2014, MTN 2015, D.G.K 2014) **(K.B + U.B)**
 (a) $u = v^2$ (b) $u = kv^2$
 (c) $uv^2 = k$ (d) $uv^2 = 1$
- (9) If $y^2 \propto \frac{1}{x^3}$, then (FSD 2015, SWL 2014, D.G.K 2015) **(K.B + A.B)**
 (a) $y^2 = \frac{k}{x^3}$ (b) $y^2 = \frac{1}{x^3}$
 (c) $y^2 = x^2$ (d) $y^2 = kx^3$
- (10) If $\frac{u}{v} = \frac{v}{w} = k$, then (LHR 2014, D.G.K 2015) **(K.B + U.B)**
 (a) $u = wk^2$ (b) $u = vk^2$
 (c) $u = w^2k$ (d) $u = v^2k$

(11) The third proportional of x^2 and y^2 is;

(K.B +A.B)

(GRW 2014, MTN 2015, D.G.K 2015)

(a) $\frac{y^2}{x^2}$

(b) $x^2 y^2$

(c) $\frac{y^4}{x^2}$

(d) $\frac{y^2}{x^4}$

(12) The fourth proportional of $x : y : z$ is; (FSD 2014, 15, RWP 2014)

(K.B +U.B)

(a) $\frac{xy}{y}$

(b) $\frac{vy}{x}$

(c) xyv

(d) $\frac{x}{vy}$

(13) If $a : b = x : y$, then alternando property is; (SGD 2014)

(K.B +U.B)

(a) $\frac{a}{x} = \frac{b}{y}$

(b) $\frac{a}{b} = \frac{x}{y}$

(c) $\frac{a+b}{x} = \frac{x+y}{y}$

(d) $\frac{a-b}{x} = \frac{x-y}{y}$

(14) If $a : b = x : y$, then invertendo property is;

(K.B +U.B)

(a) $\frac{a}{x} = \frac{b}{y}$

(b) $\frac{a}{a-b} = \frac{x}{x-y}$

(c) $\frac{a+b}{b} = \frac{x+y}{y}$

(d) $\frac{b}{a} = \frac{y}{x}$

(15) If $\frac{a}{b} = \frac{c}{d}$, then componendo property is;

(K.B +U.B)

(FSD 2014, SGD 2014, RWP 2015)

(a) $\frac{a}{a+b} = \frac{c}{c+d}$

(b) $\frac{a}{a-b} = \frac{c}{c-d}$

(c) $\frac{ad}{bc}$

(d) $\frac{a-b}{b} = \frac{c-d}{d}$

ANSWER KEY

1	b	6	a	11	c
2	c	7	d	12	b
3	b	8	b	13	a
4	a	9	a	14	d
5	c	10	a	15	a

Q.2 Write short answers of the following questions.

(i) Define ratio and give one example.

Ans. See definition Page # 87. **(K.B)**

(ii) Define proportion. **(K.B)**

Ans. See definition Page # 87.

(iii) Define direct variation. **(K.B)**

Ans. See definition Page # 92.

(iv) Define inverse variation. **(K.B)**

Ans. See definition Page # 93.

(v) State theorem of componendo-dividendo. **(K.B)**

Ans. See theorem on proportion Pg # 103.

(vi) **Given.** **(A.B)**

$$6 : x :: 3 : 5$$

Required

Value of $x = ?$

Solution:

Here

$$6 : x :: 3 : 5$$

Product of extremes = product of means

$$6 \times 5 = 3 \times x$$

$$\frac{30}{3} = x$$

$$\Rightarrow x = 10$$

(vii) **Given** **(A.B)**

$$x \propto y^2$$

$$x = 27 \text{ when } y = 4$$

To find

$$y = ? \text{ when } x = 3$$

Solution:

Here

$$x \propto y^2$$

$$x = ky^2 \rightarrow (i)$$

For value of k

$$\text{Put } x = 27 \text{ and } y = 4$$

$$27 = k(4)^2$$

$$\frac{27}{16} = k$$

Or $k = \frac{27}{16}$

$$x = \frac{27}{16} y^2$$

For value of y

$$\text{Put } k = \frac{27}{16} \text{ and } x = 3$$

$$3 = \frac{27}{16} y^2$$

$$\frac{48}{27} = y^2$$

$$\frac{16}{9} = y^2$$

$$y^2 = \frac{16}{9}$$

Taking square root

$$\Rightarrow y = \pm \frac{4}{3}$$

(viii) **Given** **(A.B)**

$$u \propto \frac{1}{v}$$

$$u = 8, v = 3$$

To find

$$v = ? \text{ when } u = 12$$

Solution:

Here

$$u \propto \frac{1}{v}$$

$$u = \frac{k}{v} \rightarrow (i)$$

For value of k

$$\text{Put } u = 8 \text{ and } v = 3 \text{ in equation (i)}$$

$$8 = \frac{k}{3}$$

$$\Rightarrow k = 24$$

$$u = \frac{24}{v}$$

For value of v

$$\text{Put } k = 24 \text{ and } u = 12 \text{ in equation (i)}$$

$$12 = \frac{24}{v}$$

$$v = 2$$

Result

$$v = 2 \text{ when } u = 12$$

(ix) Let fourth proportional = x **(A.B)**
(LHR 2014, GRW 2017, BWP 2016)

According to given condition

$$8 : 7 :: 6 : x$$

$\therefore$ Product of extremes = Product of means

$$8x = 7 \times 6$$

$$x = \frac{42}{8}$$

$$x = \frac{21}{4}$$

Result:

$$\text{Fourth proportional} = x = \frac{21}{4}$$

- (x) Let mean proportional = x **(A.B)**
(GRW 2015, 17 FSD 2016, MTN 2015, 17, BWP 2015)

According to given condition

$$15 : x :: x : 49$$

$\therefore$ Product of means = product of extremes

$$x^2 = 15 \times 49$$

$$x^2 = 735$$

Taking square root on both sides

$$x = 28$$

Result:

Mean proportional = 28

- (xi) Let third proportional = x **(A.B)**
(SWL 2014, SGD 2014, D.G.K 2016)

According to given condition

$$28 : 4 :: 4 : x$$

Product of extremes = product of means

$$28x = 4 \times 4$$

$$x = \frac{16}{28}$$

$$\Rightarrow x = \frac{4}{7}$$

Result:

$$\text{Third proportional} = \frac{4}{7}$$

- (xii) **Given** **(A.B)**

$$y \propto \frac{x^2}{z}$$

$$y = 28 \text{ when } x = 7, z = 2$$

Required

Value of $y = ?$

Solution:

$$\text{Given that } y \propto \frac{x^2}{z}$$

$$y = k \cdot \frac{x^2}{z} \rightarrow (i)$$

For value of k

Put $y = 28, x = 7, z = 2$ in equation (i)

$$28 = k \cdot \frac{(7)^2}{2}$$

$$28 \times 2 = 49k$$

$$\frac{56}{49} = k \text{ or } k = \frac{8}{7}$$

For value of y

$$\text{Put } k = \frac{8}{7} \text{ in equation (i)}$$

$$y = \frac{8}{7} \cdot \frac{x^2}{z} = \frac{8x^2}{7z}$$

- (xiii) **Given data:** **(A.B)**

$$z \propto xy$$

$$z = 36 \text{ when } x = 2, y = 3$$

Required

$$z = ?$$

Solution:

$$z \propto xy$$

$$z = kxy \rightarrow (i)$$

For value of k

Put $z = 36, x = 2, y = 3$ in equation (i)

$$36 = k(2)(3)$$

$$\frac{36}{6} = k$$

$$6 = k$$

For value of z

Put $k = 6$ in equation (i)

$$z = 6xy$$

- (xiv) **Given data** **(A.B)**

$$w \propto \frac{1}{v^2}$$

$$w = 2 \text{ when } v = 3$$

Required

Value of $w = ?$

Solution:

$$\text{Here } w \propto \frac{1}{v^2}$$

$$w = k \times \frac{1}{v^2} \rightarrow (i)$$

For value of k

Put $w = 2, v = 3$ in equation (i)

$$2 = k \times \frac{1}{(3)^2}$$

$$2 \times 9 = k$$

$$18 = k$$

For value of w

Put $k = 18$ in equation (i)

$$w = 18 \times \frac{1}{v^2}$$

$$w = \frac{18}{v^2}$$

Q.3 Fill in the blanks

- (i) The simplest form of the ratio $\frac{(x+y)(x^2+xy+y^2)}{x^3-y^3}$ is _____. **(K.B)**
- (ii) In a ratio $x:y$; x is called _____. **(K.B)**
- (iii) In a ratio $a:b$; b is called _____. **(K.B)**
- (iv) In a proportion $a:b::x:y$; a and y are called _____. **(K.B)**
- (v) In a proportion $p:q::m:n$; q and m are called _____. **(K.B)**
- (vi) In proportion $7:4::p:8$, $p =$ _____. **(A.B)**
- (vii) If $6:m::9:12$, then $m =$ _____. **(A.B)**
- (viii) If x and y varies directly, then $x =$ _____. **(A.B)**
- (ix) If v varies directly as u^3 , then $u^3 =$ _____. **(A.B)**
- (x) If w varies inversely as p^2 , then $k =$ _____. **(A.B)**
- (xi) A third proportional of 12 and 4, is _____. **(A.B)**
- (xii) The fourth proportional of 15, 6, 5 is _____. **(A.B)**
- (xiii) The mean proportional of $4m^2n^4$ and p^6 is _____. **(A.B)**
- (xiv) The continued proportion of 4, m and 9 is _____. **(A.B)**

ANSWER KEY

(i) $\frac{x+y}{x-y}$

(ii) Antecedent

(iii) Consequent

(iv) Extremes

(v) Means

(vi) $p = 14$

(vii) $m = 8$

(viii) ky

(ix) $\frac{v}{k}$

(x) p^2w

(xi) $\frac{4}{3}$

(xii) 2

(xiii) $K = \pm 2mn^2p^3$

(xiv) $m = \pm 6$



CUT HERE

SELF TEST

Time: 40 min

Marks: 25

Q.1 Four possible answers (A), (B), (C) & (D) to each question are given, mark the correct answer. (7×1=7)

1 The third proportional of x^2 and y^2 is:

(A) $\frac{y^2}{x^2}$

(B) x^2y^2

(C) $\frac{y^4}{x^2}$

(D) $\frac{y^2}{x^4}$

2 If 16, a and 4 are in continued proportion, then a is equal to:

(A) $\pm\sqrt{20}$

(B) ± 8

(C) ± 64

(D) $\pm\sqrt{12}$

3 Find x in proportion $4:x::5:15$

(A) $\frac{75}{4}$

(B) $\frac{4}{3}$

(C) $\frac{3}{4}$

(D) 12

4 In a ratio $x:y$, “y” called

(A) Relation

(B) Antecedent

(C) Consequent

(D) None

5 If $\frac{u}{v} = \frac{v}{\omega} = k$, then

(A) $u = \omega k^2$

(B) $u = vk^2$

(C) $u = \omega^2 k$

(D) $u = v^2 k$

6 If $a:b = x:y$, then invertendo property is:

(A) $\frac{a+b}{b} = \frac{x+y}{y}$

(B) $\frac{a}{a-b} = \frac{x}{x-y}$

(C) $\frac{a}{c} = \frac{b}{y}$

(D) $\frac{b}{a} = \frac{y}{x}$

7 As ‘x’ and ‘y’ varies inversely and $x = 2$ and $k = 6$ then ‘y’ is:

(A) 3

(B) 12

(C) $3/2$

(D) 8

Q.2 Give Short Answers to following Questions.**(5×2=10)**

- (i) Find the third proportional of: $a^2 - b^2, a - b$
- (ii) If $a:b=c:d, (a, b, c, d \neq 0)$, then show that $\frac{a}{b} = \sqrt{\frac{a^2 + c^2}{b^2 + d^2}}$
- (iii) If y varies inversely as x^2 and $y = 16$, when $x = 5$, so find x , when $y = 100$.
- (iv) Find a , if the ratios $a + 3 : 7 + a$ and $4:5$ are equal.
- (v) If y varies jointly as x^2 and z and $y = 6$ when $x = 4, z = 9$. Write y as a function of x and z and determine the value of y , when $x = -8$ and $z = 12$.

Q.3 Answer the following Questions.**(4+4=8)**

- (a) Solve the following by using componendo-dividendo property. $\frac{\sqrt{x+3} + \sqrt{x-3}}{\sqrt{x+3} - \sqrt{x-3}} = \frac{4}{3}$.
- (b) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} (a, b, c, d, e, f \neq 0)$, then show that $\frac{ac + ce + ea}{bd + df + fb} = \left[\frac{ace}{bdf} \right]^{2/3}$

NOTE: Parents or guardians can conduct this test in their supervision in order to check the skill of students.

$$\frac{1}{x^2 - 1} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

UNIT

4

PARTIAL FRACTIONS

Fraction

(K.B)

(MTN 2017, BWP 2014, 15, 17, RWP 2016, D.G.K 2015, 17)

The quotient of two numbers or algebraic expressions is called a fraction. The quotient is indicated by a bar (—). The dividend is written on the top of the bar and divisor below the bar.

For example: $\frac{2}{3}, \frac{x^2 + 4}{x - 2}$ where $x \neq 2$

Note

(K.B + U.B)

If $x = 2$ in second example then the fraction is not defined because $x = 2$ makes the denominator zero.

Rational Fraction

(K.B)

(LHR 2014, 16, GRW 2016, FSD 2015, SGD 2015, 16, MTN 2015, D.G.K 2016)

An expression of the form $\frac{N(x)}{D(x)}$, where

$N(x)$ and $D(x)$ are polynomials in x with real coefficients is called a rational fraction. The polynomial $D(x) \neq 0$

For example $\frac{x^2 + 4}{x - 2}$ where $x \neq 2$

Types of Fractions

(K.B + U.B)

There are two types of fractions.

- (i) Proper Fraction
- (ii) Improper Fraction

Proper Fraction

(K.B)

A rational fraction $\frac{N(x)}{D(x)}$ where $D(x) \neq 0$ is called proper fraction, if degree of the polynomial $N(x)$ is less than degree of the polynomial $D(x)$

For example: $\frac{2}{x+1}, \frac{5x-3}{x^2+4}$ etc.

Improper Fraction

(U.B + K.B)

(LHR 2014, 15, GRW 2014, 17, FSD 2015, SGD 2017, RWP 2017, MTN 2015)

A rational fraction $\frac{N(x)}{D(x)}$, where $D(x) \neq 0$ is

called an improper fraction, if degree of the polynomial $N(x)$ is greater than or equal to degree of the polynomial $D(x)$.

For example: $\frac{5x}{x+2}, \frac{6x^4}{x^3+1}$ etc.

Identity

(K.B)

(GRWP 2014, 15, 17, RWP 2016, SGD 2016, D.G.K 2015, 17)

An identity is an equation, which is satisfied by all the values of the variables involved

For example: $(x+3)^2 = x^2 + 6x + 9$,

$2(x+1) = 2x + 2$ etc.

Conditional Equation

(K.B)

An equation which is true for some specific value(s) of the variable involved.

For example: $x + 2 = 3$ is true only for $x = 1$.

Partial Fraction

(K.B)

(LHR 2014, 16, 17, GRW 2015, FSD 2015, 17, RWP 2015, 16, BWP 2015,)

Decomposition of resultant fraction into its components or into different fractions is called partial fraction.

Note

(K.B + U.B)

General method applicable to resolve all rational

fractions of the form $\frac{N(x)}{D(x)}$, is as follows:

- The numerator $N(x)$ must be of lower degree than the denominator $D(x)$.
- Make substitutions constant accordingly.
- Multiply both sides by L.C.M.
- Arrange terms on both sides by decreasing order.
- Make the equations and solve to find constants.

Resultant Fraction**(K.B)**

Sum of two or more than two proper fractions in the form of a single fraction is called the resultant fraction.

For example:

$\frac{1}{x-1} - \frac{2}{x+1} = \frac{-x+3}{(x-1)(x+1)}$ is resultant fraction.

Example 2: (Page # 78)**(A.B)**

Resolve $\frac{1}{3+x-2x^2}$ into partial fractions.

Solution:

$\frac{1}{3+x-2x^2}$ can be written as for

convenience $\frac{-1}{2x^2-x-3}$

The denominator

$$D(x) = 2x^2 - x - 3 = 2x^2 - 3x + 2x - 3$$

$$= x(2x-3) + 1(2x-3) = (x+1)(2x-3)$$

Let,

$$\frac{-1}{2x^2-x-3} = \frac{-1}{(x+1)(2x-3)} = \frac{A}{x+1} + \frac{B}{2x-3} \rightarrow (i)$$

multiplying both the sides by $(x+1)(2x-3)$, we get

$$-1 = A(2x-3) + B(x+1) \rightarrow (ii)$$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$$-1 = A[2(-1)-3]$$

$$-1 = -5A$$

$$\Rightarrow A = \frac{1}{5}$$

Put $2x-3=0 \Rightarrow x=\frac{3}{2}$ in equation (ii)

$$-1 = B\left(\frac{3}{2}+1\right)$$

$$-1 = \frac{5}{2}B$$

$$\Rightarrow B = -\frac{2}{5}$$

$$\text{Thus, } \frac{1}{3+x-2x^2} = \frac{1}{5(x+1)} - \frac{2}{5(2x-3)}$$

Note

There are two methods to resolve into partial fraction:

- (i) Zero Method
- (ii) Equating coefficient

Exercise 4.1

Resolve into partial fractions.

Q.1 $\frac{7x-9}{(x+1)(x-3)}$ (FSD 2015) **(A.B)**

Solution:

$$\text{Let } \frac{7x-9}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} \rightarrow (i)$$

Multiplying with $(x+1)(x-3)$

$$\begin{aligned} & \frac{7x-9}{(x+1)(x-3)} \times \cancel{(x+1)} \cancel{(x-3)} \\ &= \frac{A}{\cancel{x+1}} \times \cancel{(x+1)} (x-3) + \frac{B}{\cancel{x-3}} \times (x+1) \cancel{(x-3)} \end{aligned}$$

$$\Rightarrow 7x-9 = A(x-3) + B(x+1) \rightarrow (ii)$$

Put $x+1=0 \Rightarrow x=-1$ in eq.(ii)

$$7(-1)-9 = A(-1-3) + B(0)$$

$$-7-9 = -4(A) + 0$$

$$-16 = -4A$$

$$\text{Or } A = \frac{-16}{-4}$$

$$A = 4$$

Put $x-3=0$ or $x=3$ in eq.(ii)

$$7(3)-9 = A(0) + B(3+1)$$

$$21-9 = 0 + B(4)$$

$$12 = 4B$$

$$4B = 12$$

$$B = \frac{12}{4}$$

$$B = 3$$

Putting the values in equation. (i)

$$\Rightarrow \frac{7x-9}{(x+1)(x-3)} = \frac{4}{x+1} + \frac{3}{x-3}$$

Q.2 $\frac{x-11}{(x-4)(x+3)}$ **(A.B)**

Solution:

Let $\frac{x-11}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3} \rightarrow (i)$

Multiplying equation (i) by $(x-4)(x+3)$

$$\frac{x-11}{(x-4)(x+3)} \times (x-4)(x+3) = \frac{A}{(x-4)} \times (x-4)(x+3) + \frac{B}{(x+3)} \times (x-4)(x+3)$$

$$x-11 = A(x+3) + B(x-4) \rightarrow (ii)$$

Put $x-4=0 \Rightarrow x=4$ in eq.(ii)

$$4-11 = A(4+3) + B(0)$$

$$-7 = A(7) + 0$$

$$-7 = 7A$$

Or $7A = -7$

$$A = \frac{-7}{7}$$

$$\Rightarrow A = -1$$

Put $x+3=0$ or $x=-3$ in eq. (ii)

$$-3-11 = A(-3+3) + B(-3-4)$$

$$-14 = A(0) + B(-7)$$

$$-14 = 0 + (-7B)$$

Or $-7B = -14$

$$B = \frac{-14}{-7}$$

$$\Rightarrow B = 2$$

Putting the values of A and B in equation (i)

$$\therefore \frac{x-11}{(x-4)(x+3)} = \frac{-1}{x-4} + \frac{2}{x+3}$$

Q.3 $\frac{3x-1}{x^2-1}$ **(A.B)**

(GRWP 2017, SWL 2014, BWP 2017, D.G.K 2014)

Solution:

$$\frac{3x-1}{x^2-1} = \frac{3x-1}{(x+1)(x-1)}$$

Let $\frac{3x-1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

Multiplying by $(x+1)(x-1)$

$$\frac{3x-1}{(x+1)(x-1)} \times (x+1)(x-1) = \frac{A}{(x+1)} \times (x+1)(x-1) + \frac{B}{(x-1)} \times (x+1)(x-1)$$

$$3x-1 = A(x-1) + B(x+1) \rightarrow (ii)$$

Put $x-1=0 \Rightarrow x=1$ in eq (ii)

$$3(1)-1 = A(1-1) + B(1+1)$$

$$3-1 = 0 + 2B$$

$$2 = 2B$$

Or $2B = 2$

$$B = \frac{2}{2}$$

$$B = 1$$

Put $x+1=0 \Rightarrow x=-1$ in eq(ii)

$$3(-1)-1 = A(-1-1) + B(0)$$

$$-3-1 = A(-2)$$

$$-4 = -2A$$

$$-2A = -4$$

$$A = \frac{-4}{-2}$$

$$A = 2$$

Now putting the values in eq (i)

$$\Rightarrow \frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1}$$

Q.4 $\frac{x-5}{x^2+2x-3}$ **(A.B)**

(FSD 2015, MTN 2016, SGD 2015)

Solution:

$$\begin{aligned} \frac{x-5}{x^2+2x-3} &= \frac{x-5}{x^2-x+3x-3} \\ &= \frac{x-5}{x(x-1)+3(x-1)} \\ &= \frac{x-5}{(x-1)(x+3)} \end{aligned}$$

Let $\frac{x-5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} \rightarrow (i)$

Multiplying by $(x-1)(x+3)$, we get

$$x-5 = A(x+3) + B(x-1) \rightarrow (ii)$$

Put $x-1=0 \Rightarrow x=1$ in eq (ii)

$$1-5 = A(1+3) + B(1-1)$$

$$-4 = A(4) + B(0)$$

$$-4 = 4A$$

$$4A = -4$$

$$A = \frac{-4}{4}$$

$$A = -1$$

$$\text{Put } x+3=0 \Rightarrow x=-3 \text{ in eq (ii)}$$

$$-3-5 = A(-3+3) + B(-3-1)$$

$$-8 = -4B$$

$$-4B = -8$$

$$B = \frac{-8}{-4}$$

$$B = 2$$

$$B=2$$

Now putting values in eq (i)

$$\frac{x-5}{(x-1)(x+3)} = \frac{-1}{x-1} + \frac{2}{x+3}$$

$$\Rightarrow \frac{x-5}{x^2+2x-3} = \frac{-1}{x-1} + \frac{2}{x+3}$$

Q.5

$$\frac{3x+3}{(x-1)(x+2)}$$

(A.B)

(GRW 2015, 16, FSD 2016, 17, BWP 2015, 16, SGD 2015)

Solution:

$$\text{Let } \frac{(3x+3)}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \rightarrow (i)$$

Multiplying equation (i) by (x-1)(x+2)

$$\frac{3x+3}{(x-1)(x+2)} \times (x-1)(x+2) = \frac{A}{x-1} \times (x-1)(x+2) + \frac{B}{x+2} \times (x-1)(x+2)$$

$$3x+3 = A(x+2) + B(x-1) \rightarrow (ii)$$

$$\text{Put } x-1=0 \text{ or } x=1 \text{ in eq (ii)}$$

$$3(1)+3 = A(1+2) + B(1-1)$$

$$3+3 = 3(A) + 0$$

$$6 = 3A$$

$$3A = 6$$

$$A = \frac{6}{3}$$

$$A = 2$$

$$\text{Put } x+2=0 \Rightarrow x=-2 \text{ in eq (ii)}$$

$$3(-2)+3 = A(-2+2) + B(-2-1)$$

$$-6+3 = A(0) + B(-3)$$

$$-3 = 0 - 3B$$

$$3B = 3$$

$$B = \frac{3}{3}$$

$$B = 1$$

Now putting values in eq (i)

$$\Rightarrow \frac{3x+3}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{1}{x+2}$$

Q.6

$$\frac{(7x-25)}{(x-4)(x-3)}$$

(A.B)

(RWP 2016, SGD 2017, D.G.K 2014, 17)

Solution:

$$\text{Let } \frac{(7x-25)}{(x-4)(x-3)} = \frac{A}{x-4} + \frac{B}{x-3} \rightarrow (i)$$

Multiplying by (x-4)(x-3)

$$7x-25 = A(x-3) + B(x-4) \rightarrow (ii)$$

$$\text{Put } x-4=0 \text{ or } x=4 \text{ in eq (ii)}$$

$$7(4)-25 = A(4-3) + B(4-4)$$

$$28-25 = A(1) + B(0)$$

$$3 = A$$

$$\text{Or } A = 3$$

$$\text{Put } x-3=0 \text{ or } x=3 \text{ in eq (ii)}$$

$$7(3)-25 = A(0) + B(3-4)$$

$$21-25 = B(-1)$$

$$-B = -4$$

$$B = 4$$

Now putting values in equation (i)

$$\Rightarrow \frac{7x-25}{(x-4)(x-3)} = \frac{3}{x-4} + \frac{4}{x-3}$$

Q.7

$$\frac{x^2+2x+1}{(x-2)(x+3)}$$

(A.B + K.B)

Solution:

$$\frac{x^2+2x+1}{(x-2)(x+3)} = \frac{x^2+2x+1}{x^2+x-6} \text{ (improper)}$$

$$\therefore x^2+x-6 = x^2+3x-2x-6$$

$$= x(x+3) - 2(x+3)$$

$$= (x+3)(x-2)$$

$$x^2 + x - 6 \overline{) \frac{1}{x^2 + 2x + 1}}$$

$$\underline{-x^2 \pm x \mp 6}$$

$$x + 7$$

$$\frac{x^2 + 2x + 1}{x^2 + x - 6} = 1 + \frac{x + 7}{x^2 + x - 6}$$

$$= 1 + \frac{x + 7}{(x + 3)(x - 2)} \rightarrow (i)$$

Consider

$$\frac{x + 7}{(x + 3)(x - 2)} = \frac{A}{x + 3} + \frac{B}{x - 2} \rightarrow (ii)$$

Multiplying by $(x + 3)(x - 2)$

$$x + 7 = A(x - 2) + B(x + 3) \rightarrow (iii)$$

Put $x + 3 = 0$ or $x = -3$ in eq (iii)

$$-3 + 7 = A(-3 - 2) + B(-3 + 3)$$

$$4 = A(-5) + B(0)$$

$$4 = -5A$$

$$A = \frac{4}{-5}$$

$$A = -\frac{4}{5}$$

Put $x - 2 = 0$ or $x = 2$ in equation (ii)

$$2 + 7 = A(2 - 2) + B(2 + 3)$$

$$9 = A(0) + B(5)$$

$$9 = 5B$$

$$5B = 9$$

$$B = \frac{9}{5}$$

Putting the values of A and B in equation (ii)

$$\frac{x + 7}{(x + 3)(x - 2)} = \frac{\frac{9}{5}}{x - 2} + \frac{-\frac{4}{5}}{x + 3}$$

$$= \frac{9}{5(x - 2)} - \frac{4}{5(x + 3)}$$

Now putting the values in equation (i)

$$\Rightarrow \frac{x^2 + 2x + 1}{(x - 2)(x + 3)} = 1 + \frac{9}{5(x - 2)} - \frac{4}{5(x + 3)}$$

Q.8 $\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$ **(A.B + K.B)**

Solution:

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} \quad (\text{improper fraction})$$

$$3x^2 - 2x - 1 \overline{) \frac{6x^3 + 5x^2 - 7}{2x + 3}}$$

$$\underline{\pm 6x^3 \mp 4x^2 \mp 2x}$$

$$9x^2 + 2x - 7$$

$$\underline{\pm 9x^2 \mp 6x \mp 3}$$

$$8x - 4$$

$$\Rightarrow \frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = 2x + 3 + \frac{8x - 4}{3x^2 - 2x - 1}$$

$$= 2x + 3 + \frac{8x - 4}{3x^2 - 3x + x - 1}$$

$$= 2x + 3 + \frac{8x - 4}{3x(x - 1) + 1(x - 1)}$$

$$= 2x + 3 + \frac{8x - 4}{(x - 1)(3x + 1)} \rightarrow (i)$$

Consider

$$\frac{8x - 4}{(x - 1)(3x + 1)} = \frac{A}{x - 1} + \frac{B}{3x + 1} \rightarrow (ii)$$

Multiplying by $(x - 1)(3x + 1)$

$$8x - 4 = A(3x + 1) + B(x - 1) \rightarrow (iii)$$

Put $x - 1 = 0$ or $x = 1$ in equation (iii)

$$8 - 4 = A(3 + 1) + B(0)$$

$$4 = A(4) + 0$$

$$4 = 4A$$

$$\rightarrow A = 1$$

Put $3x + 1 = 0$ or $x = -\frac{1}{3}$ in equation (iii)

$$8\left(-\frac{1}{3}\right) - 4 = A\left[3\left(-\frac{1}{3}\right) + 1\right] + B\left(-\frac{1}{3} - 1\right)$$

$$\frac{-8 - 12}{3} = A(0) + B\frac{(-1 - 3)}{3}$$

$$-\frac{20}{3} = 0 + B\left(-\frac{4}{3}\right)$$

$$-\frac{20}{3} = -\frac{4}{3}B$$

$$-\frac{20}{3} \left(-\frac{3}{4} \right) = B$$

$$5 = B$$

Or $B = 5$

Now putting values in equation (ii)

$$\frac{8x-4}{(x-1)(3x+1)} = \frac{1}{x-1} + \frac{5}{3x+1}$$

Putting the values in equation (i)

$$\Rightarrow \frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = 2x + 3 + \frac{1}{x-1} + \frac{5}{3x+1}$$

Example: (Page # 79)

(A.B)

Resolve $\frac{1}{(x-1)^2(x-2)}$ into partial fractions.

Solution:

$$\text{Let, } \frac{1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

Multiplying both sides by $(x-1)^2(x-2)$, we get

$$1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

$$A(x^2 - 3x + 2) + B(x-2) + C(x^2 - 2x + 1) = 1 \rightarrow (i)$$

Since (i) is an identity and is true for all values of x

Put $x-1=0$ or $x=1$ in (i), we get

$$B(1-2)=1 \Rightarrow -B=1 \text{ or } B=-1$$

Put $x-2=0$ or $x=2$ in (i), we get

$$C(2-1)^2=1 \Rightarrow C=1$$

Equating coefficients of x^2 on both sides of (i)

$$A+C=0 \Rightarrow A=-C \text{ so } A=-1$$

Hence required partial fractions are

$$\frac{-1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{(x-2)}$$

Thus,

$$\frac{1}{(x-1)^2(x-2)} = \frac{1}{x-2} - \frac{1}{(x-1)} - \frac{1}{(x-1)^2}$$

Exercise 4.2

Resolve into partial fractions.

Q.1 $\frac{x^2-3x+1}{(x-1)^2(x-2)}$ **(A.B)**

Solution:

$$\text{Let } \frac{x^2-3x+1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} \rightarrow (i)$$

Multiplying by $(x-1)^2(x-2)$

$$x^2-3x+1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2 \rightarrow (ii)$$

$$x^2-3x+1 = A(x^2-3x+2) + B(x-2) + C(x^2-2x+1)$$

$$= Ax^2 - 3Ax + 2A + Bx - 2B + Cx^2 - 2Cx + C$$

$$= Ax^2 + Cx^2 - 3Ax + Bx - 2Cx + 2A + C - 2B$$

$$\Rightarrow x^2-3x+1 = (A+C)x^2 + (-3A+B-2C)x$$

$$+ (2A-2B+C)$$

By comparing coefficients of alike powers of x

$$1 = A+C \rightarrow (iii)$$

$$-3 = -3A+B-2C \rightarrow (iv)$$

$$1 = 2A-2B+C \rightarrow (v)$$

Put $x-1=0 \Rightarrow x=1$ in equation (ii)

$$(1)^2 - 3(1) + 1 = A(1-1)(1-2) + B(1-2) + C(0)^2$$

$$1-3+1=0+B(-1)+0$$

$$-1=-1B$$

$$B=1$$

Put $B=x-2=0 \Rightarrow x=2$ in equation (ii)

$$(2)^2 - 3(2) + 1 = A(2-1)(2-2) + B(2-2) + C(2-1)^2$$

$$4-6+1=A(1)(0)+B(0)+C(1)^2$$

$$-1=C$$

$$C=-1$$

Put $C=x-1=0$ in equation (i)

$$1=A+(-1)$$

$$1+1=A$$

$$\text{Or } A=2$$

Now putting values in equation (i)

$$\frac{x^2-3x+1}{(x-1)^2(x-2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{-1}{(x-2)}$$

$$= \frac{2}{x-1} + \frac{1}{(x-1)^2} - \frac{1}{(x-2)}$$

Q.2 $\frac{x^2+7x+11}{(x+2)^2(x+3)}$ **(A.B)**

(LHR 2016, SGD 2016, RWP 2015)

Solution: $\frac{x^2+7x+11}{(x+2)^2(x+3)}$

Consider

$$\frac{x^2+7x+11}{(x+2)^2(x+3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+3} \rightarrow (i)$$

Multiplying by $(x+2)^2(x+3)$

$$x^2+7x+11 = A(x+2)(x+3) + B(x+3) + C(x+2)^2 \rightarrow (ii)$$

$$= A(x^2+5x+6) + B(x+3) + C(x^2+4x+4)$$

$$= Ax^2 + 5Ax + 6A + Bx + 3B + Cx^2 + 4Cx + 4C$$

$$= Ax^2 + Cx^2 + 5Ax + Bx + 4Cx + 6A + 3B + 4C$$

$$= (A+C)x^2 + (5A+B+4C)x + (6A+3B+4C)$$

By comparing coefficients of alike powers of 'x'

$$1 = A + C \quad \text{_____ (iii)}$$

$$7 = 5A + B + 4C \quad \text{_____ (iv)}$$

$$11 = 6A + 3B + 4C \quad \text{_____ (v)}$$

Put $x+2 \Rightarrow x = -2$ in equation _____ (ii)

$$(-2)^2 + 7(-2) + 11 = A(-2+2)(-2+3)$$

$$+ B(-2+3) + C(-2+2)^2$$

$$4 - 14 + 11 = A(0)(1) + B(1) + C(0)^2 \quad -10 + 11 = B$$

$$\begin{aligned} 1 &= B \\ \text{Or } B &= 1 \end{aligned}$$

Put $x+3=0 \Rightarrow x = -3$ in equation (ii)

$$(-3)^2 + 7(-3) + 11 = A(-3+2)(-3+3)$$

$$+ B(-3+3) + C(-3+2)^2$$

$$9 - 21 + 11 = A(-1)(0) + B(0) + C(-1)^2$$

$$-1 = 0 + 0 + C$$

$$C = -1$$

Put $C = -1$ in equation (iii)

$$1 = A + (-1)$$

$$1 + 1 = A$$

$$\text{Or } A = 2$$

Now putting the values of A, B, C in (i)

$$\frac{x^2+7x+11}{(x+2)^2(x+3)} = \frac{2}{x+2} + \frac{1}{(x+2)^2} + \frac{-1}{x+3}$$

$$= \frac{2}{x+2} + \frac{1}{(x+2)^2} - \frac{1}{x+3}$$

Q.3 $\frac{9}{(x-1)(x+2)^2}$ **(A.B)**

(SWL 2014, RWP 2017, SGD 2015)

Solution:

$$\text{Let } \frac{9}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \rightarrow (i)$$

Multiplying by $(x-1)(x+2)^2$

$$9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1) \rightarrow (ii)$$

$$9 = A(x^2+4x+4) + B(x^2+x-2) + C(x-1)$$

$$0x^2+0x+9 = Ax^2+4Ax+4A+Bx^2+Bx-2B+Cx-C$$

$$0x^2+0x+9 = Ax^2+Bx^2+4Ax+Bx$$

$$+Cx+4A-2B-C$$

$$0x^2+0x+9 = (A+B)x^2 + (4A+B+C)x$$

$$+ (4A-2B-C)$$

By comparing coefficients of alike powers of 'x'

$$0 = A + B \rightarrow \text{_____ (iii)}$$

$$0 = 4A + B + C \rightarrow \text{_____ (iv)}$$

$$9 = 4A - 2B - C \rightarrow \text{_____ (v)}$$

Put $x-1=0 \Rightarrow x = 1$ in equation (ii)

$$9 = A(1+2)^2 + B(1-1)(1+2) + C(1-1)$$

$$9 = A(3)^2 + B(0) + C(0)$$

$$9 = 9A + 0 + 0$$

$$9 = 9A$$

$$\text{Or } A = 1$$

Put $x+2=0 \Rightarrow x = -2$ in equation (ii)

$$9 = A(-2+2) + B(-2-1)(-2+2) + C(-2-1)$$

$$9 = A(0) + B(-3)(0) + C(-3)$$

$$9 = 0 + 0 - 3C$$

$$C = -\frac{9}{3}$$

$$C = -3$$

Put $A = 1$ in equation (iii)

$$0 = 1 + B$$

Now putting the values in equation (i)

$$\Rightarrow \frac{9}{(x-1)(x+2)^2} = \frac{1}{x-1} + \frac{-1}{x+2} + \frac{-3}{(x+2)^2}$$

$$= \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$$

Q.4 $\frac{x^4+1}{x^2(x-1)}$ **(K.B + A.B)**

Solution:

$$\frac{x^4+1}{x^2(x-1)} = \frac{x^4+1}{x^3-x^2} \quad (\text{improper fraction})$$

$$\begin{array}{r} x^3 - x^2 \overline{) x^4 + 1} \\ \underline{+ x^4 + x^3} \\ x^3 + 1 \\ \underline{+ x^3 + x^2} \\ x^2 + 1 \end{array}$$

$$\begin{aligned} \frac{x^4+1}{x^2(x-1)} &= x+1 + \frac{x^2+1}{x^3-x^2} \\ &= x+1 + \frac{x^2+1}{x^2(x-1)} \rightarrow (i) \end{aligned}$$

$$\text{Let } \frac{x^2+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \rightarrow (ii)$$

$$x^2+1 = Ax(x-1) + B(x-1) + C(x^2)$$

$$x^2+1 = Ax^2 - Ax + Bx - B + Cx^2$$

$$\begin{aligned} x^2+0x+1 &= Ax^2 + Cx^2 + Bx - Ax - B \\ &= (A+C)x^2 + (B-A)x - B \end{aligned}$$

By comparing coefficients of alike powers of 'x'

$$1 = A + C \rightarrow (iii)$$

$$0 = B - A \rightarrow (iv)$$

$$1 = -B \rightarrow (v)$$

From equation (v)

$$B = -1$$

Put in equation (iv)

$$0 = -1 - A$$

$$A = -1$$

Put in equation (iii)

$$1 = -1 + C$$

$$1+1 = C$$

$$C = 2$$

Now putting the values in equation (ii)

$$\frac{x^2+1}{x^2(x-1)} = \frac{-1}{x} + \frac{-1}{x^2} + \frac{2}{x-1}$$

$$= -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

Now putting the values in equation (i)

$$\Rightarrow \frac{x^4+1}{x^2(x-1)} = x+1 - \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

Q.5 $\frac{7x+4}{(3x+2)(x+1)^2}$ **(A.B)**

Solution:

$$\text{Let } \frac{7x+4}{(3x+2)(x+1)^2} = \frac{A}{3x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \rightarrow (i)$$

Multiplying by $(3x+2)(x+1)^2$

$$7x+4 = A(x+1)^2 + B(3x+2) + C(3x+2) \rightarrow (ii)$$

$$0x^2 + 7x + 4 = A(x^2 + 2x + 1)$$

$$+ B(3x^2 + 3x + 2) + 3Cx + 2C$$

$$= Ax^2 + 2Ax + A + 3Bx^2 + 3Bx + 2B + 3Cx + 2C$$

$$= Ax^2 + 3Bx^2 + 2Ax + 3Bx + 3Cx + A + 2B + 2C$$

$$\begin{aligned} 0x^2 + 7x + 4 &= (A+3B)x^2 + (2A+3B+3C)x \\ &\quad + (A+2B+2C) \end{aligned}$$

By Comparing coefficients of alike powers of x

$$A = ?$$

$$0 = A + 3B \rightarrow (iii)$$

$$7 = 2A + 3B + 3C \rightarrow (iv)$$

$$4 = A + 2B + 2C \rightarrow (v)$$

Put $x+1=0$, $x=-1$ in equation (ii)

$$7(-1)+4 = A(-1+1)^2 + B(-1+1) + C[3(-1)+2]$$

$$-7+4 = 0+0+C(-3+2)$$

$$-3 = (-1)C$$

$$\text{Or } C = 3$$

Put $A = -6$ in equation (iii)

$$0 = -6 + 3B$$

$$6 = 3B$$

$$\text{Or } B = 2$$

Putting all the values in equation (i)

$$\frac{7x+4}{(3x+2)(x+1)^2} = \frac{-6}{3x+2} + \frac{2}{x+1} + \frac{3}{(x+1)^2}$$

Q.6 $\frac{1}{(x-1)^2(x+1)}$ **(A.B)**

Solution:

Let $\frac{1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \rightarrow (i)$

Multiplication by $(x-1)^2(x+1)$

$1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \rightarrow (ii)$

$0x^2 + 0x + 1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

$0x^2 + 0x + 1 = A(x^2 - 1) + B(x+1) + C(x^2 - 2x + 1)$

$= Ax^2 + Cx^2 + Bx - 2Cx - A + B + C$

$= (A+C)x^2 + (B-2C)x + (-A+B+C)$

Comparing coefficients of powers 'x'

$0 = A + C \quad \text{--- (iii)}$

$0 = B - 2C \quad \text{--- (iv)}$

$1 = -A + B + C \quad \text{--- (v)}$

Put $x-1=0 \Rightarrow x=1$ in equation (ii)

$1 = A(0)(1+1) + B(1+1) + C(0)^2$

$1 = 0 + B(2) + C(0)$

$1 = 2B$

Or $B = \frac{1}{2}$

Put $B = \frac{1}{2}$ in equation (iv)

$0 = \frac{1}{2} - 2C$

$2C = \frac{1}{2}$

$C = \frac{1}{4}$

Put $C = \frac{1}{4}$ in equation (iii)

$0 = A + \frac{1}{4}$

$A = -\frac{1}{4}$

Now putting the values in equation (i)

$\frac{1}{(x-1)^2(x+1)} = \frac{-1/4}{x-1} + \frac{1/2}{(x-1)^2} + \frac{1/4}{x+1}$

$\frac{1}{(x-1)^2(x+1)} = \frac{-1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{1}{4(x+1)}$

Q.7 $\frac{3x^2+15x+16}{(x+2)^2}$ **(A.B)**

Solution:

$\frac{3x^2+15x+16}{(x+2)^2} = \frac{3x^2+15x+16}{x^2+4x+4}$

(improper fraction)

$$\begin{array}{r} 3 \\ x^2+4x+4 \overline{) 3x^2+15x+16} \\ \underline{+3x^2+12x+12} \\ 3x+4 \end{array}$$

$\frac{3x^2+15x+16}{(x+2)^2} = 3 + \frac{3x+4}{x^2+4x+4} \quad \text{--- (i)}$

Let $\frac{3x+4}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} \quad \text{--- (ii)}$

Multiplying by $(x+2)^2$

$3x+4 = A(x+2) + B$

$3x+4 = Ax + 2A + B$

By comparing coefficients of powers of 'x'

$3 = A \quad \text{--- (iii)}$

$4 = 2A + B \quad \text{--- (iv)}$

From equation (iii)

$A = 3$

Put in equation (iv)

$4 = 2(3) + B$

$4 = 6 + B$

$4 - 6 = B$

$B = -2$

Now putting values in equation (ii)

$$\begin{aligned} \frac{3x+4}{(x+2)^2} &= \frac{3}{x+2} - \frac{2}{(x+2)^2} \\ &= \frac{3}{x+2} - \frac{2}{(x+2)^2} \end{aligned}$$

Now putting the values in equation (i)

$\Rightarrow \frac{3x^2+15x+16}{(x+2)^2} = 3 + \frac{3}{x+2} - \frac{2}{(x+2)^2}$

Q.8 $\frac{1}{(x^2-1)(x+1)}$ **(K.B + A.B)**

Solution:

$$\frac{1}{(x^2-1)(x+1)} = \frac{1}{(x+1)(x-1)(x+1)}$$

$$= \frac{1}{(x+1)^2(x-1)}$$

Let $\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \rightarrow (i)$

Multiplication by $(x-1)(x+1)^2$

$$1 = A(x+1) + B(x-1)(x+1) + C(x-1) \rightarrow (ii)$$

Put $x-1=0 \Rightarrow x=1$ in equation (ii)

$$1 = A(1+1) + B(0) + C(0)$$

$$1 = A(2)$$

$$1 = 4A$$

$$\Rightarrow A = \frac{1}{4}$$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$$1 = A(-1+1) + B(-1-1)(-1+1) + C(-1-1)$$

$$1 = A(0) + B(0) + C(-2)$$

$$1 = -2C$$

$$\Rightarrow C = -\frac{1}{2}$$

Equation (ii)

$$0x^2 + 0x + 1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$= A(x^2 + 2x + 1) + B(x^2 - 1) + Cx - C$$

$$= Ax^2 + 2Ax + A + Bx^2 - B + Cx - C$$

$$= Ax^2 + Bx^2 + 2Ax + Cx + A - B - C$$

$$= (A+B)x^2 + (2A+C)x + (A-B-C)$$

By comparing coefficients of alike powers of x ,

$$0 = A + B \quad (iii)$$

$$0 = 2A + C \quad (iv)$$

$$1 = A - B - C \quad (v)$$

Put $A = \frac{1}{4}$ in equation (iii)

$$0 = \frac{1}{4} + B$$

$$B = -\frac{1}{4}$$

Now putting values in equation (i)

$$\frac{1}{(x-1)(x+1)^2} = \frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}}{x+1} + \frac{-\frac{1}{2}}{(x+1)^2}$$

$$\Rightarrow \frac{1}{(x-1)(x+1)^2} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x+1)^2}$$

Example: (Page # 80)

(A.B)

Resolve $\frac{11x+3}{(x-3)(x^2+9)}$ into partial

fractions.

Solution:

Let, $\frac{11x+3}{(x-3)(x^2+9)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+9}$

Multiplying both the sides by

$(x-3)(x^2+9)$ on both sides

$$\Rightarrow 11x+3 = A(x^2+9) + (Bx+C)(x-3)$$

$$11x+3 = A(x^2+9) + B(x^2-3x) + C(x-3) \rightarrow (i)$$

Since (i) is an identity, we have on substituting $x=3$

$$33+3 = A(9+9)$$

$$\Rightarrow 18A = 36$$

$$\Rightarrow A = 2$$

Comparing the coefficients of x^2 and x on both sides of (i), we get

$$A+B=0$$

$$\Rightarrow B = -2$$

$$-3B+C=11$$

$$\Rightarrow -3(-2)+C=11$$

$$\Rightarrow C=5$$

Therefore, the partial fractions are

$$\frac{2}{x-3} + \frac{-2x+5}{x^2+9}$$

Thus, $\frac{11x+3}{(x-3)(x^2+9)} = \frac{2}{x-3} + \frac{-2x+5}{x^2+9}$

Exercise 4.3**Resolve into partial fractions.**

Q.1 $\frac{3x-11}{(x+3)(x^2+1)}$ **(A.B)**

Solution:

Let $\frac{3x-11}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1} \rightarrow (i)$

Multiplying by $(x+3)(x^2+1)$

$$3x-11 = A(x^2+1) + (Bx+C)(x+3) \rightarrow (ii)$$

$$3x-11 = Ax^2 + A + Bx^2 + 3Bx + 3C + Cx$$

$$3x-11 = Ax^2 + Bx^2 + 3Bx + Cx + A + 3C$$

$$0x^2 + 3x - 11 = (A+B)x^2 + (3B+C)x + (A+3C)$$

By comparing co-efficient of alike powers of x

$$0 = A + B \quad (iii)$$

$$3 = 3B + C \quad (iv)$$

$$-11 = A + 3C \quad (v)$$

Put $x+3=0 \Rightarrow x=-3$ in equation (ii)

$$3(-3)-11 = A[(-3)^2+1] + (B(-3)+C)(-3+3)$$

$$-9-11 = A(9+1) + (Bx+C)(0)$$

$$-20 = A(10) + 0$$

$$\text{Or } A = -2$$

Put in equation (iii)

$$0 = -2 + B$$

$$B = 2$$

Put in equation (iv)

$$3 = 3(2) + C$$

$$3 = 6 + C$$

$$3-6 = C$$

$$-3 = C$$

$$C = -3$$

Now putting in equation (i)

$$\frac{3x-11}{(x+3)(x^2+1)} = -\frac{2}{x+3} + \frac{2x-3}{x^2+1}$$

$$\text{Or } \frac{3x-11}{(x+3)(x^2+1)} = \frac{2x-3}{x^2+1} - \frac{2}{x+3}$$

Q.2 $\frac{3x+7}{(x^2+1)(x+3)}$ **(A.B)**

(SWL 2015 MTN 2015)

Solution:

Let $\frac{3x+7}{(x^2+1)(x+3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+3} \rightarrow (i)$

Multiplying by $(x^2+1)(x+3)$

$$3x+7 = (Ax+B)(x+3) + C(x^2+1) \rightarrow (ii)$$

$$\begin{aligned} 0x^2 + 3x + 7 &= Ax^2 + 3Ax + Bx + 3B + Cx^2 + C \\ &= Ax^2 + Cx^2 + Bx + 3Ax + 3B + C \\ &= (A+C)x^2 + (B+3A)x + (3B+C) \end{aligned}$$

By comparing coefficients of alike powers of x

$$0 = A + C \quad (iii)$$

$$3 = 3A + B \quad (iv)$$

$$7 = 3B + C \quad (v)$$

Put $x+3=0 \Rightarrow x=-3$ in equation (ii)

$$3(-3)+7 = (A(-3)+B)(0) + C[(-3)^2+1]$$

$$-9+7 = C(9+1)$$

$$-2 = 0 + C(10)$$

$$-\frac{2}{10} = C$$

$$\Rightarrow C = -\frac{1}{5}$$

Put in equation (iii)

$$0 = A - \frac{1}{5}$$

$$\text{Or } A = \frac{1}{5}$$

Put in equation (iv)

$$3 = 3\left(\frac{1}{5}\right) + B$$

$$3 = \frac{3}{5} + B$$

$$3 - \frac{3}{5} = B$$

$$\frac{15-3}{5} = B$$

$$\frac{12}{5} = B$$

$$\Rightarrow B = \frac{12}{5}$$

Now putting values in equation (i)

$$\begin{aligned} \frac{3x+7}{(x^2+1)(x+3)} &= \frac{\frac{1}{5}x + \frac{12}{5}}{x^2+1} + \frac{-\frac{1}{5}}{x+3} \\ &= \frac{\frac{1}{5}(x+12)}{x^2+1} - \frac{1}{5(x+3)} \\ \Rightarrow \frac{3x+7}{(x^2+1)(x+3)} &= \frac{x+12}{5(x^2+1)} - \frac{1}{5(x+3)} \end{aligned}$$

Q.3 $\frac{1}{(x+1)(x^2+1)}$ **(A.B)**

Solution:

$$\text{Let } \frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \text{ --- (i)}$$

Multiplying by $(x+1)(x^2+1)$

$$1 = A(x^2+1) + (Bx+C)(x+1) \text{ --- (ii)}$$

$$= Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$= Ax^2 + Bx^2 + Bx + Cx + A + C$$

$$0x^2 + 0x + 1 = (A+B)x^2 + (B+C)x + (A+C)$$

By comparing coefficients of alike powers of x

$$0 = A + B \text{ --- (iii)}$$

$$0 = B + C \text{ --- (iv)}$$

$$1 = A + C \text{ --- (v)}$$

Put $x+1=0 \rightarrow x=-1$ in equation (ii)

$$1 = A[(-1)^2+1] + (B(-1)+C)(0)$$

$$1 = A(1+1) + 0$$

$$1 = A(2)$$

$$\text{Or } A = \frac{1}{2}$$

Put in equation (iii)

$$0 = \frac{1}{2} + B$$

$$B = -\frac{1}{2}$$

Put in equation (iv)

$$0 = -\frac{1}{2} + C$$

$$\Rightarrow C = \frac{1}{2}$$

$$\frac{1}{(x+1)(x^2+1)} = \frac{\frac{1}{2}}{2(x+1)} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1}$$

$$\frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} + \frac{-\frac{1}{2}(x-1)}{x^2+1}$$

$$\Rightarrow \frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} - \frac{x-1}{2(x^2+1)}$$

Q.4 $\frac{9x-7}{(x+3)(x^2+1)}$ **(A.B)**

Solution:

$$\text{Let } \frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1} \text{ --- (i)}$$

Multiplication by $(x+3)(x^2+1)$

$$9x-7 = A(x^2+1) + (Bx+C)(x+3) \text{ --- (ii)}$$

$$= Ax^2 + A + Bx^2 + Cx + 3Bx + 3C$$

$$= Ax^2 + Bx^2 + 3Bx + Cx + A + 3C$$

$$0x^2 + 9x - 7 = (A+B)x^2 + (3B+C)x + (A+3C)$$

By comparing coefficients of alike powers of x

$$0 = A + B \text{ --- (iii)}$$

$$9 = 3B + C \text{ --- (iv)}$$

$$-7 = A + 3C \text{ --- (v)}$$

Put $x+3=0 \Rightarrow x=-3$ in equation (ii)

$$9(-3) - 7 = A[(-3)^2+1] + [B(-3)+C](-3-3)$$

$$-27 - 7 = A(9+1) + 0$$

$$-34 = A(10)$$

$$A = -\frac{34}{10}$$

$$A = -\frac{17}{5}$$

Put in equation (iii)

$$0 = -\frac{17}{5} + B$$

$$B = \frac{17}{5}$$

Put in equation (iv)

$$9 = 3\left[\frac{17}{5}\right] + C$$

$$9 = \frac{51}{5} + C$$

$$9 - \frac{51}{5} = C$$

$$\frac{45 - 51}{5} = C$$

$$-\frac{6}{5} = C$$

$$C = -\frac{6}{5}$$

Now putting the values in equation (i)

$$\frac{9x-7}{(x+3)(x^2+1)} = -\frac{\frac{17}{5}}{(x+3)} + \frac{\frac{17}{5}x + \left(-\frac{6}{5}\right)}{x^2+1}$$

$$= -\frac{17}{5(x+3)} + \frac{\frac{1}{5}(17x-6)}{x^2+1}$$

$$= -\frac{17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$$

$$\Rightarrow \frac{9x-7}{(x+3)(x^2+1)} = \frac{17x-6}{5(x^2+1)} - \frac{17}{5(x+3)}$$

Q.5 $\frac{3x+7}{(x+3)(x^2+4)}$ **(A.B)**

(SWL 2015, MTN 2015)

Solution:

Let $\frac{3x+7}{(x+3)(x^2+4)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4}$ (i)

Multiplying by $(x+3)(x^2+4)$

$$3x+7 = A(x^2+4) + (Bx+C)(x+3)$$
 (ii)

$$= Ax^2 + 4A + Bx^2 + 3Bx + Cx + 3C$$

$$= Ax^2 + Bx^2 + 3Bx + Cx + 4A + 3C$$

$$0x^2 + 3x + 7 = (A+B)x^2 + (3B+C)x + (4A+3C)$$

By comparing coefficients of alike powers of x ,

$$0 = A + B \text{ (iii)}$$

$$3 = 3B + C \text{ (iv)}$$

$$7 = 4A + 3C \text{ (v)}$$

Put $x+3=0 \Rightarrow x=-3$ in equation (iii)

$$3(-3)+7 = A[(-3)^2+4] + (B(-3)+C)(-3+3)$$

$$-9+7 = A(9+4)+0$$

$$-2 = A(13)$$

$$A = -\frac{2}{13}$$

Put in equation (iii)

$$0 = -\frac{2}{13}B$$

$$B = \frac{2}{13}$$

Put in equation (iv)

$$3 = 3\left[\frac{2}{13}\right] + C$$

$$3 = \frac{6}{13} + C$$

$$3 - \frac{6}{13} = C$$

$$\frac{39-6}{13} = C$$

$$\frac{33}{13} = C$$

$$C = \frac{33}{13}$$

Now putting the values in equation (i)

$$\frac{3x+7}{(x+3)(x^2+4)} = \frac{-\frac{2}{13}}{x+3} + \frac{\frac{2}{13}x + \frac{33}{13}}{x^2+4}$$

$$= -\frac{2}{13(x+3)} + \frac{\frac{1}{13}(2x+33)}{x^2+4}$$

$$\Rightarrow \frac{3x+7}{(x+3)(x^2+4)} = \frac{2x+33}{13(x^2+4)} - \frac{2}{13(x+3)}$$

Q.6 $\frac{x^2}{(x+2)(x^2+4)}$ **(A.B)**

Solution:

Let $\frac{x^2}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$ — (i)

Multiplying by $(x+2)(x^2+4)$

$$\begin{aligned} x^2 &= A(x^2+4) + (Bx+C)(x+2) \text{ — (ii)} \\ &= Ax^2 + 4A + Bx^2 + 2Bx + Cx + 2C \\ &= Ax^2 + Bx^2 + 2Bx + Cx + 4A + 2C \\ x^2 + 0x + 0 &= (A+B)x^2 + (2B+C)x + (4A+2C) \end{aligned}$$

By comparing coefficients of alike powers of x

$$1 = A + B \text{ — (iii)}$$

$$0 = 2B + C \text{ — (iv)}$$

$$0 = 4A + 2C \text{ — (v)}$$

Put $x+2=0 \Rightarrow x=-2$ in equation (ii)

$$(-2)^2 = A[(-2)^2 + 4] + [B(-2) + C](0)$$

$$4 = A(4+4) + 0$$

$$4 = A(8)$$

$$4 = 8A$$

$$A = \frac{1}{2}$$

Put in equation (iii)

$$1 = \frac{1}{2} + B$$

$$1 - \frac{1}{2} = B$$

$$\frac{2-1}{2} = B$$

$$B = \frac{1}{2}$$

Put in equation (iv)

$$0 = 2\left(\frac{1}{2}\right) + C$$

$$0 = 1 + C$$

$$C = -1$$

Now putting values of A, B, C in equation (i)

$$\begin{aligned} \frac{x^2}{(x+2)(x^2+4)} &= \frac{\frac{1}{2}}{(x+2)} + \frac{\frac{1}{2}x + (-1)}{x^2+4} \\ &= \frac{1}{2(x+2)} + \frac{\frac{1}{2}(x-2)}{x^2+4} \\ \Rightarrow \frac{x^2}{(x+2)(x^2+4)} &= \frac{1}{2(x+2)} + \frac{x-2}{2(x^2+4)} \end{aligned}$$

Q.7 $\frac{1}{x^3+1}$ **(K.B + A.B)**

Solution:

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$$

Let $\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ — (i)

Multiplying by $(x+1)(x^2-x+1)$

$$1 = A(x^2-x+1) + (Bx+C)(x+1) \text{ — (ii)}$$

$$1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$1 = Ax^2 + Bx^2 + Bx + Cx - Ax + A + C$$

$$0x^2 + 0x + 1 = (A+B)x^2 + (B+C-A)x + (A+C)$$

By comparing coefficients of alike power of x

$$0 = A + B \text{ — (iii)}$$

$$0 = B + C - A \text{ — (iv)}$$

$$1 = A + C \text{ — (v)}$$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$$1 = A[(-1)^2 - (-1) + 1] + [B(-1) + C](-1+1)$$

$$1 = A(1+1+1) + (Bx+C)(0)$$

$$1 = A(3) + 0$$

$$A = \frac{1}{3}$$

Put in equation (iii)

$$0 = \frac{1}{3} + B$$

$$B = -\frac{1}{3}$$

Put $A = \frac{1}{3}$ and $B = -\frac{1}{3}$ in equation (iv)

$$0 = -\frac{1}{3} + C - \frac{1}{3}$$

$$C = \frac{1}{3} + \frac{1}{3}$$

$$C = \frac{2}{3}$$

Now putting values in equation (i)

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{\frac{1}{3}}{(x+1)} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}$$

$$= \frac{1}{3(x+1)} + \frac{-\frac{1}{3}(x-2)}{x^2-x+1}$$

$$\Rightarrow \frac{1}{(x+1)(x^2-x+1)} = \frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)}$$

Q.8 $\frac{x^2+1}{x^3+1}$ **(K.B + A.B)**

Solution:

$$\frac{x^2+1}{x^3+1} = \frac{x^2+1}{(x+1)(x^2-x+1)}$$

Let $\frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \rightarrow (i)$

Multiplying by $(x+1)(x^2-x+1)$

$$x^2+1 = A(x^2-x+1) + (Bx+C)(x+1) \quad (ii)$$

$$= Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$= Ax^2 + Bx^2 - Ax + Bx + Cx + A + C$$

$$x^2 + 0x + 1 = (A+B)x^2 + (B+C-A)x + (A+C)$$

By comparing coefficients of power of x

$$1 = A + B \quad (iii)$$

$$0 = B + C - A \quad (iv)$$

$$1 = A + C \quad (v)$$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$$(-1)^2 + 1 = A[(-1)^2 - (-1) + 1] + [B(-1) + C](0)$$

$$1+1 = A(1+1+1) + 0$$

$$2 = A(3)$$

$$\frac{2}{3} = A$$

$$A = \frac{2}{3}$$

Put in equation (ii)

$$1 = \frac{2}{3} + E$$

$$1 - \frac{2}{3} = B$$

$$\frac{3-2}{3} = B$$

$$\frac{1}{3} = B$$

$$B = \frac{1}{3}$$

Put $P = \frac{2}{3}$ in equation (v)

$$1 = \frac{2}{3} + C$$

$$1 - \frac{2}{3} = C$$

$$C = \frac{1}{3}$$

Putting the values in equation (i)

$$\frac{x^2+1}{x^3+1} = \frac{\frac{2}{3}}{3(x+1)} + \frac{\frac{1x}{3} + \frac{1}{3}}{x^2-x+1}$$

$$= \frac{2}{3(x+1)} + \frac{\frac{1}{3}(x+1)}{x^2-x+1}$$

$$\Rightarrow \frac{x^2+1}{x^3+1} = \frac{2}{3(x+1)} + \frac{x+1}{3(x^2-x+1)}$$

Example: (Page # 31) (K.B + A.B)

Resolve $\frac{x^3-2x^2-2}{(x^2+1)^2}$ into partial fractions

Solution:

$$\frac{x^3-2x^2-2}{(x^2+1)^2} \text{ is a proper fraction as}$$

degree of numerator is less than the degree of denominator.

$$\text{Let, } \frac{x^3 - 2x^2 - 2}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

Multiplying both the sides by $(x^2 + 1)^2$,
we have

$$x^3 - 2x^2 - 2 = (Ax + B)(x^2 + 1) + Cx + D$$

$$x^3 - 2x^2 - 2 = A(x^3 + x) + B(x^2 + 1) + Cx + D \rightarrow (i)$$

Equating the coefficients of x and constant on
the both sides of (i).

$$\text{Coefficients of } x^3: A = 1$$

$$\text{Coefficients of } x^2: B = -2$$

$$\text{Coefficients of } x: A + C = 0 \Rightarrow C = -1$$

$$\text{Constants: } B + D = -2 \rightarrow (ii)$$

Putting the value of B in equation (ii)

$$D = -2 - B$$

$$= -2 - (-2)$$

$$= -2 + 2 = 0$$

$$\Rightarrow D = 0$$

Thus,

$$\frac{x^3 - 2x^2 - 2}{(x^2 + 1)^2} = \frac{x - 2}{x^2 + 1} + \frac{-x + 0}{(x^2 + 1)^2}$$

$$\Rightarrow \frac{x^3 - 2x^2 - 2}{(x^2 + 1)^2} = \frac{x - 2}{x^2 + 1} - \frac{x}{(x^2 + 1)^2}$$

Exercise 4.4

Resolve into partial fractions.

Q.1 $\frac{x^3}{(x^2 + 4)^2}$ **(K.B + A.B)**

Solutions:

$$\text{Let } \frac{x^3}{(x^2 + 4)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2} \quad (i)$$

Multiplying by $(x^2 + 4)^2$

$$x^3 = (Ax + B)(x^2 + 4) + Cx + D \quad (ii)$$

$$x^3 = Ax^3 + 4Ax + Bx^2 + 4B + Cx + D$$

$$x^3 + 0x^2 + 0x$$

$$= Ax^3 + Bx^2 + (4A + C)x + 4B + D$$

By comparing coefficients of alike
powers of 'x'

$$1 = A \quad (iii)$$

$$0 = B \quad (iv)$$

$$0 = 4A + C \quad (v)$$

$$0 = 4B + D \quad (vi)$$

From equation (iii) and (iv)

$$A = 1; B = 0$$

Put A = 1 in equation (v)

$$0 = 4(1) + C$$

$$0 = 4 + C$$

$$C = -4$$

Put B = 0 in equation (vi)

$$0 = 4B + D$$

$$0 = 4(0) + D$$

$$D = 0$$

Putting the values in equation (i)

$$\frac{x^3}{(x^2 + 4)^2} = \frac{x + 0}{x^2 + 4} + \frac{-4x + 0}{(x^2 + 4)^2}$$

$$\Rightarrow \frac{x^3}{(x^2 + 4)^2} = \frac{x}{x^2 + 4} - \frac{4x}{(x^2 + 4)^2}$$

Q.2 $\frac{x^4 + 3x^2 + x + 1}{(x + 1)(x^2 + 1)^2}$ **(K.B + A.B)**

Solution:

$$\text{Let } \frac{x^4 + 3x^2 + x + 1}{(x + 1)(x^2 + 1)^2} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} \rightarrow (i)$$

Multiplying by $(x + 1)(x^2 + 1)^2$

$$x^4 + 3x^2 + x + 1 = A(x^2 + 1)^2$$

$$+ (Bx + C)(x + 1)(x^2 + 1) + (Dx + E)(x + 1) \rightarrow (ii)$$

$$= A(x^4 + 2x^2 + 1) + (Bx^4 + Bx^3 + Cx^3 + Cx^2 + Dx^3 + Dx^2 + Ex + E)$$

$$+ Dx^2 + Dx + Ex + E$$

$$x^4 + 0x^3 + 3x^2 + x + 1$$

$$= Ax^4 + Bx^4 + Bx^3 + Cx^3 + 2Ax^2 + Bx^2 +$$

$$Cx^2 + Dx^2 + Bx + Cx + Dx + Ex + A + C + E$$

$$= (A + B)x^4 + (B + C)x^3 + (2A + B + C + D)x^2$$

$$+ (B + C + D + E)x + (A + C + E)$$

By comparing coefficients of alike powers
'x'

$$1 = A + B \quad \text{_____ (iii)}$$

$$0 = B + C \quad \text{_____ (iv)}$$

$$3 = 2A + B + C + D \quad \text{_____ (v)}$$

$$1 = B + C + D + E \quad \text{_____ (vi)}$$

$$1 = A + C + E \quad \text{_____ (vii)}$$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$$(-1)^4 + 3(-1)^2 + (-1) + 1 = A[(-1)^2 + 1]^2$$

$$+ [B(-1) + C](0) + [D(-1) + E](0)$$

$$1 + 3(1) - 1 + 1 = A(1+1)^2 + 0$$

$$4 = 4A$$

$$A = 1$$

Put in equation (iii)

$$1 + B = 1$$

$$B = 1 - 1$$

$$B = 0$$

Put in equation (iv)

$$0 + C = 0$$

$$C = 0$$

$$C = 0$$

Put $A=1, B=0$ and $C=0$ in equation (v)

$$2(1) + 0 + 0 + D = 3$$

$$2 + D = 3$$

$$D = 3 - 2$$

$$D = 1$$

Put $A=1, C=0$ in equation (vii)

$$1 + 0 + E = 1$$

$$E = 1 - 1$$

$$E = 0$$

Now putting the values in equation (i)

$$\frac{x^4 + 3x^2 + x + 1}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{0(x)+0}{x^2+1} + \frac{1(x)+0}{(x^2+1)^2}$$

$$\Rightarrow \frac{x^4 + 3x^2 + x + 1}{(x+1)(x^2+1)^2} = \frac{1}{x+1} + \frac{x}{(x^2+1)^2}$$

Q.3

$$\frac{x^2}{(x+1)(x^2+1)^2} \quad \text{(K.E. + A.B.)}$$

Solution:

$$\text{Let } \frac{x^2}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2} \rightarrow \text{(i)}$$

Multiplying by $(x+1)(x^2+1)^2$

$$x^2 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1)$$

$$+ (Dx+E)(x+1) \rightarrow \text{(ii)}$$

$$x^2 = A(x^4 + 1 + 2x^2) + (Bx+C)(x+1)(x^2+1)$$

$$+ (Dx+E)(x+1)$$

$$x^2 = Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Bx^3 + Bx$$

$$+ Cx^3 + C + Dx^2 + Dx + Ex + E$$

$$\Rightarrow 0x^4 + 0x^3 + x^2 + 0x + 0$$

$$= (A+B)x^4 + (B+C)x^3 + (2A+B+D)x^2$$

$$+ (B+D+E)x + (A+C+E)$$

By comparing coefficients of alike powers of 'x'

$$A + B = 0 \quad \text{_____ (iii)}$$

$$B + C = 0 \quad \text{_____ (iv)}$$

$$2A + B + D = 1 \quad \text{_____ (v)}$$

$$B + D + E = 0 \quad \text{_____ (vi)}$$

$$A + C + E = 0 \quad \text{_____ (vii)}$$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$$(-1)^2 = A[(-1)^2 + 1]^2 + [B(-1) + C](0)$$

$$D = ?$$

$$+ [D(-1) + E](0)$$

$$1 = A(1+1)^2 + 0 + 0$$

$$1 = A(2)^2$$

$$1 = 4A$$

$$A = \frac{1}{4}$$

Put in equation (iii)

$$\frac{1}{4} + B = 0$$

$$B = -\frac{1}{4}$$

Put in equation (iv)

$$-\frac{1}{4} + C = 0$$

$$C = \frac{1}{4}$$

Put $A = \frac{1}{4}$ and $C = \frac{1}{4}$ in equation (vii)

$$\frac{1}{4} + \frac{1}{4} + E = 0$$

$$E = -\frac{1}{4} - \frac{1}{4}$$

$$= -\frac{2}{4}$$

$$E = -\frac{1}{2}$$

Now putting the values in equation (i)

$$\frac{\frac{1}{x^2}}{(x+1)(x^2+1)} = \frac{\frac{1}{4}}{4(x+1)}$$

$$+ \frac{-\frac{1}{4}x + \frac{1}{4}}{x^2+1} + \frac{\frac{1}{2}x + \left(-\frac{1}{2}\right)}{(x^2+1)^2}$$

$$= \frac{1}{4(x+1)} + \frac{-\frac{1}{4}(x-1)}{x^2+1} + \frac{\frac{1}{2}(x-1)}{(x^2+1)^2}$$

$$\frac{x^2}{(x+1)(x^2+1)} = \frac{1}{4(x+1)} - \frac{x-1}{4(x^2+1)} + \frac{x-1}{2(x^2+1)^2}$$

Q.4 $\frac{x^2}{(x-1)(x^2+1)^2}$ **(K.B + A.B)**

Solution:

$$\frac{x^2}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \rightarrow (i)$$

Multiplying by $(x-1)(x^2+1)^2$

$$x^2 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1)$$

$$+ (Dx+E)(x-1) \rightarrow (ii)$$

$$x^2 = A(x^4 + 2x^2 + 1) + (Bx^2 - Bx + Cx - C)(x^2 + 1)$$

$$+ Dx^2 - Dx + Ex - E$$

$$0x^4 + 0x^3 + x^2 - 0x + 0 = Ax^4 + 2Ax^2 + A + Bx^3$$

$$+ Bx^2 - Bx + Cx^2 + Cx - Cx^2 - C$$

$$+ Dx^2 - Dx + Ex - E$$

$$= (A+B)x^4 + (-B+C)x^3 + (2A+B-C+D)x^2$$

$$+ (-B+C-D+E)x + (A-C-E)$$

By comparing coefficients of alike powers of x

$$0 = A + B \quad \text{--- (iii)}$$

$$0 = -B + C \quad \text{--- (iv)}$$

$$1 = 2A - E - C + D \quad \text{--- (v)}$$

$$0 = -B + C - D + E \quad \text{--- (vi)}$$

$$0 = A - C - E \quad \text{--- (vii)}$$

Put $x-1=0 \Rightarrow x=1$ in equation (ii)

$$(1)^2 = A[(1)^2 + 1]^2$$

$$+ (B(1)+C)(1-1)[(1)^3 + 1] + [D(1)+E](0)$$

$$1 = A(1+1)^2 + 0 + 0$$

$$1 = A(2)^2$$

$$1 = 4A$$

$$A = \frac{1}{4}$$

Put in equation (iii)

$$0 = \frac{1}{4} + B$$

$$B = -\frac{1}{4}$$

Put in equation (iv)

$$0 = -\left[-\frac{1}{4}\right] + C$$

$$0 = \frac{1}{4} + C$$

$$\Rightarrow C = -\frac{1}{4}$$

Put $A = \frac{1}{4}$ and $C = -\frac{1}{4}$ in equation (vii)

$$0 = \frac{1}{4} - \left[-\frac{1}{4}\right] - E$$

$$0 = \frac{1}{4} + \frac{1}{4} - E$$

$$E = \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$$

$$\Rightarrow E = \frac{1}{2}$$

Put in equation (vi)

$$0 = -\left[-\frac{1}{4}\right] + \left[-\frac{1}{4}\right] - D + \frac{1}{2}$$

$$0 = \frac{1}{4} - \frac{1}{4} - D + \frac{1}{2}$$

$$\Rightarrow D = \frac{1}{2}$$

Put in equation (i)

$$\frac{x^2}{(x-1)(x^2+1)} = \frac{\frac{1}{4}}{x-1} + \frac{\left(-\frac{1}{4}\right)x - \frac{1}{4}}{x^2+1} + \frac{\frac{1}{2}x + \frac{1}{2}}{(x^2+1)^2}$$

$$= \frac{1}{4(x-1)} + \frac{-\frac{1}{4}(x+1)}{x^2+1} + \frac{\frac{1}{2}(x+1)}{(x^2+1)^2}$$

$$\Rightarrow \frac{x^2}{(x-1)(x^2+1)^2} = \frac{1}{4(x-1)} - \frac{x+1}{4(x^2+1)} + \frac{x+1}{2(x^2+1)^2}$$

Q.5 $\frac{x^4}{(x^2+2)^2}$ **(K.B + A.B)**

Solution:

$$\frac{x^4}{(x^2+2)^2} = \frac{x^4}{x^4 + 4x^2 + 4}$$

(improper fraction)

$$\begin{array}{r} x^4 + 4x^2 + 4 \overline{) x^4} \\ \underline{\pm x^4 \pm 4x^2 \pm 4} \\ -4x^2 - 4 \\ \underline{-(4x^2 + 4)} \end{array}$$

$$\frac{x^4}{(x^2+2)^2} = 1 - \frac{4x^2+4}{(x^2+2)^2} \rightarrow (i)$$

Consider

$$\frac{4x^2+4}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} \rightarrow (ii)$$

Multiply by $(x^2+2)^2$ on both sides

$$4x^2+4 = (Ax+B)(x^2+2) + Cx+D$$

$$4x^2+4 = Ax^3 + 2Ax + Bx^2 + Cx + D + 2B$$

$$0x^3 + 4x^2 + 0x + 4 = Ax^3 + Bx^2 + 2Ax + Cx + D + 2B$$

By comparing coefficients of alike powers of 'x'

$$0 = A \quad \text{--- (iii)}$$

$$4 = B \quad \text{--- (iv)}$$

$$0 = 2A + C \quad \text{--- (v)}$$

$$4 = D + 2B \quad \text{--- (vi)}$$

$$\text{Equation (iii)} \Rightarrow A = 0$$

$$\text{Equation (iv)} \Rightarrow B = 4$$

$$\text{Put } A = 0 \text{ in equation (v)}$$

$$0 = 2(0) + C$$

$$\Rightarrow C = 0$$

$$\text{Equation (vi)}$$

$$4 = D + 2B$$

$$4 = D + 2(4)$$

$$4 = D + 8$$

$$D = 4 - 8$$

$$D = -4$$

Putting the values in equation (ii)

$$\begin{aligned} \frac{4x^2+4}{(x^2+2)^2} &= \frac{0x+4}{x^2+2} + \frac{0x-4}{(x^2+2)^2} \\ &= \frac{4}{x^2+2} - \frac{4}{(x^2+2)^2} \end{aligned}$$

Now putting the value in equation

$$\frac{x^4}{(x^2+2)^2} = 1 - \left(\frac{4}{x^2+2} - \frac{4}{(x^2+2)^2} \right)$$

$$\Rightarrow \frac{x^4}{(x^2+2)^2} = 1 - \frac{4}{x^2+2} + \frac{4}{(x^2+2)^2}$$

Q.6 $\frac{x^5}{(x^2+1)^2}$ **(K.B + A.B)**

Solution:

$$\frac{x^5}{(x^2+1)^2} = \frac{x^5}{x^4+2x^2+1}$$

(improper fraction)

$$\begin{array}{r} x \\ x^4 + 2x^2 + 1 \overline{) x^5} \\ \underline{\pm x^5 \pm 2x^3 \pm x} \\ -2x^3 - x \end{array}$$

$$\Rightarrow \frac{x^5}{x^4+2x^2+1} = x + \frac{-2x^3-x}{x^4+2x^2+1}$$

$$\frac{x^5}{(x^2+1)^2} = x - \frac{2x^3+x}{x^4+2x^2+1} \quad \text{--- (i)}$$

$$\text{Let } \frac{2x^3+x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \quad \text{--- (ii)}$$

Multiplying by $(x^2+1)^2$

$$2x^3+x = (Ax+B)(x^2+1) + (Cx+D)$$

$$2x^3+x = Ax^3+Ax+Bx^2+B+Cx+D$$

$$2x^3+x = Ax^3+Bx^2+Ax+Cx+B+D$$

$$2x^3+x = Ax^3+Bx^2+(A+C)x+(B+D)$$

By comparing coefficients of like powers of x

$$A = 2 \quad \text{--- (iii)}$$

$$B = 0 \quad \text{--- (iv)}$$

$$A+C = 1 \quad \text{--- (v)}$$

$$B+D = 0 \quad \text{--- (vi)}$$

From equation (iii) and (iv)

$$\Rightarrow A = 2$$

$$B = 0$$

Put $A = 2$ in equation (v)

$$2+C = 1$$

$$C = 1-2$$

$$C = -1$$

Put $B = 0$ in equation (vi)

$$B+D = 0$$

$$0+D = 0$$

$$D = 0$$

Putting the values in equation (ii)

$$\frac{2x^3+x}{(x^2+1)^2} = \frac{2x}{x^2+1} - \frac{x}{(x^2+1)^2}$$

Put in equation (i)

$$\frac{x^5}{(x^2+1)^2} = x - \left[\frac{2x}{x^2+1} - \frac{x}{(x^2+1)^2} \right]$$

$$\Rightarrow \frac{x^5}{(x^2+1)^2} = x - \frac{2x}{x^2+1} + \frac{x}{(x^2+1)^2}$$

Miscellaneous Exercise 4

Q.1 Multiple Choice Questions

Four possible answers are given for the following question. Tick (✓) the correct answer.

- (i) The identity $(5x+4)^2 = 25x^2 + 40x + 16$ is true for; (SWL 2015) **(K.B + A.B)**

(a) One value of x (b) Two values of x
(c) All values of x (d) None of these

- (ii) A function of the form $f(x) = \frac{N(x)}{D(x)}$, with $D(x) \neq 0$, where $N(x)$ and $D(x)$ are polynomials in x is called; **(K.B + A.B)**

(a) An identity (b) An equation
(c) A fraction (d) Algebraic relation

- (iii) A fraction in which the degree of numerator is greater or equal to the degree of denominator is called; (GRW 2017, RWP 2015, 17, FSD 2015) **(K.B + A.B)**

(a) A proper fraction (b) An improper fraction
(c) An equation (d) Algebraic relation

- (iv) A fraction in which the degree of numerator is less than the degree of the denominator is called (GRW 2014, 15, 16, FSD 2017) **(K.B + A.B)**

(a) An equation (b) An improper fraction
(c) An identity (d) A proper fraction

- (v) $\frac{2x+1}{(x+1)(x-1)}$ is; **(K.B + A.B)**

(a) An improper fraction (b) An equation
(c) A proper fraction (d) None of these

- (vi) $(x+3)^2 = x^2 + 6x + 9$ is; (LHR 2014, 15, 16, MTN 2015, SWL 2015, D.G.K 2015) **(K.B + A.B)**

(a) A linear equation (b) An equation
(c) An identity (d) None of these

- (vii) $\frac{x^3+1}{(x-1)(x+2)}$ is; **(K.B + A.B)**

(a) A proper fraction (b) An improper fraction
(c) An identity (d) A constant term

- (viii) Partial fraction of $\frac{x-2}{(x-1)(x+2)}$ are of the form; **(K.B + A.B)**

(a) $\frac{A}{x-1} + \frac{B}{x+2}$ (b) $\frac{Ax}{x-1} + \frac{E}{x+2}$
(c) $\frac{A}{x-1} + \frac{Bx+C}{x+2}$ (d) $\frac{Ax+B}{x-1} + \frac{C}{x+2}$

- (ix) Partial fraction of $\frac{x+2}{(x+1)(x^2+2)}$ are of the form; (GRW 2014, RWP 2017) **(K.B + A.B)**

(a) $\frac{A}{x+1} + \frac{B}{x^2+2}$ (b) $\frac{A}{x+1} + \frac{Bx+C}{x^2+2}$
(c) $\frac{Ax+B}{x+1} + \frac{C}{x^2+2}$ (d) $\frac{A}{x+1} + \frac{Bx}{x^2+2}$

(x) Partial fraction of $\frac{x^2+1}{(x+1)(x-1)}$ are of the form; **(K.B + A.B)**

(a) $\frac{A}{x+1} + \frac{B}{x-1}$

(b) $1 + \frac{A}{x+1} + \frac{Bx+C}{x-1}$

(c) $1 + \frac{A}{x+1} + \frac{B}{x-1}$

(d) $1 + \frac{Ax+B}{(x+1)} + \frac{C}{x-1}$

ANSWER KEY

i	ii	iii	iv	v	vi	vii	viii	ix	x
c	c	b	d	c	c	b	a	b	c

Q.2 Write short answers of the following questions.

(i) Define a rational fraction. **(K.B)**

Ans:

Rational Fraction

An expression of the form $\frac{N(x)}{D(x)}$, where

$N(x)$ and $D(x)$ are polynomials in x with real coefficients is called a rational fraction. The polynomial $D(x) \neq 0$

For example $\frac{x^2+4}{x-2}$ where $x \neq 2$

(ii) What is a proper fraction? **(K.B)**

Ans:

Proper Fraction

A rational fraction $\frac{N(x)}{D(x)}$, where $D(x) \neq 0$ is

called proper fraction, if degree of the polynomial $N(x)$ is less than degree of the polynomial $D(x)$

For example: $\frac{2}{x+1}, \frac{5x-3}{x^2+4}$ etc.

(iii) What is an improper fraction? **(K.B)**

Ans:

Improper Fraction

A rational fraction $\frac{N(x)}{D(x)}$, where $D(x) \neq 0$ is

called an improper fraction, if degree of the polynomial $N(x)$ is greater than or equal to degree of the polynomial $D(x)$.

For example: $\frac{5x}{x+2}, \frac{6x^4}{x^3+1}$ etc.

(iv) What are partial fractions? **(K.B)**

Ans:

Partial Fraction

(LHR 2016, FSD 2016, BWP 2014, RWP 2016, 17, MTN 2016, 17, SWL 2017, SGD 2017)

Decomposition of resultant fraction into its components or into different fractions is called partial fraction.

(v) How can we make partial fractions of $\frac{x-2}{(x+2)(x+3)}$? **(K.B)**

Solution:

$$\text{Let } \frac{x-2}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} \rightarrow (i)$$

Multiply by $(x+2)(x+3)$

$$x-2 = A(x+3) + B(x+2) \rightarrow (ii)$$

Put $x+2=0$ or $x=-2$ in equation (ii)

$$-2-2 = A(-2+3) + B(0)$$

$$-4 = A(1) + 0$$

$$\Rightarrow A = -4$$

Put $x+3=0$ or $x=-3$ in equation (ii)

$$-3-2 = A(0) + B(-3+2)$$

$$-5 = 0 + B(-1)$$

$$-5 = -B$$

$$\Rightarrow B = 5$$

Now putting the values in equation (i)

$$\frac{x-2}{(x+2)(x+3)} = \frac{-4}{x+2} + \frac{5}{x+3}$$

(vi) Resolve $\frac{1}{x^2-1}$ into partial fractions.

(A.B + K.B)

Solution:

(LHR 2016, 17, GRW 2016, 17, SWL 2017, RWP 2017, BWP 2015, 16, 17, MTN 2017, SGD 2014, 15, 16, 17)

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)}$$

$$\text{Let } \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \rightarrow (i)$$

Multiply $(x+1)(x-1)$ on both sides

$$1 = A(x-1) + B(x+1) \rightarrow (ii)$$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$$1 = A(-1-1) + B(0)$$

$$1 = -2A + 0$$

$$1 = -2A$$

$$\Rightarrow A = \frac{-1}{2}$$

Put $x-1=0 \Rightarrow x=1$ in equation (ii)

$$1 = A(0) + B(1+1)$$

$$1 = 0 + 2B$$

$$1 = 2B$$

$$\Rightarrow B = \frac{1}{2}$$

Now putting the values in equation $\rightarrow (i)$

$$\frac{1}{(x+1)(x-1)} = \frac{-1}{2(x+1)} + \frac{1}{2(x-1)}$$

$$\Rightarrow \frac{1}{x^2-1} = -\frac{1}{2(x+1)} + \frac{1}{2(x-1)}$$

(vii) Find partial fractions of $\frac{3}{(x+1)(x-1)}$.

(BWP 2015, SGD 2014)

Solution:

$$\text{Let } \frac{3}{(x+1)(x-1)} = \frac{A}{x-1} + \frac{B}{x+1} \rightarrow (i)$$

Multiply by $(x+1)(x-1)$ on both sides

$$3 = A(x-1) + B(x+1) \rightarrow (ii)$$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$$3 = A(-1-1) + B(0)$$

$$3 = -2A + 0$$

$$3 = -2A$$

$$\frac{3}{-2} = A$$

$$\text{Or } A = -\frac{3}{2}$$

Put $x-1=0$ or $x=1$ in equal (ii)

$$3 = A(0) + B(1+1)$$

$$3 = 0 + 2B$$

$$3 = 2B$$

$$\frac{3}{2} = B$$

$$\text{Or } B = \frac{3}{2}$$

Now put the values in equation (i)

$$\frac{3}{(x+1)(x-1)} = \frac{-\frac{3}{2}}{x+1} + \frac{\frac{3}{2}}{x-1}$$

$$\frac{3}{(x+1)(x-1)} = -\frac{3}{2(x+1)} + \frac{3}{2(x-1)}$$

$$\text{Or } \frac{3}{(x+1)(x-1)} = \frac{3}{2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right)$$

(viii) Resolve $\frac{x}{(x-3)^2}$ into partial fractions.

(A.B + K.B)

(FSD 2015, RWP 2014, D.G.K 2014)

Solution:

$$\text{Let } \frac{x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} \rightarrow (i)$$

Multiply by $(x-3)^2$ on both sides

$$x = A(x-3) + B$$

$$x = Ax - 3A + B$$

$$x + 0 = Ax - 3A + B$$

By comparing coefficient of alike powers of 'x'

$$1 = A \rightarrow (ii)$$

$$0 = -3A + B \rightarrow (iii)$$

$A=1$ Put in equation (iii)

$$0 = -3(1) + B$$

$$0 = -3 + B$$

$$\text{Or } B = 3$$

Now putting the values in equation (i)

$$\frac{x}{(x-3)^2} = \frac{1}{x-3} + \frac{3}{(x-3)^2}$$

(ix) How we can make the partial fractions

$$\text{of } \frac{x}{(x+a)(x-a)}?$$

(A.B + K.B + U.B)

Solution:

$$\text{Let } \frac{x}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a}$$

Multiplying by L.C.M i.e. $(x+a)(x-a)$

$$(x+a)(x-a) \frac{x}{(x+a)(x-a)}$$

$$= (x+a)(x-a) \frac{A}{x+a} + (x+a)(x-a) \frac{B}{x-a}$$

$$x = A(x-a) + B(x+a) \rightarrow (i)$$

Put $x = -a$ in equation (i)

$$-a = A(-a-a) + B(-a+a)$$

$$-a = A(-2a) + B(0)$$

$$-a = -2aA$$

$$A = \frac{-a}{-2a}$$

$$A = \frac{1}{2}$$

Put $x = a$ in equation (i)

$$a = A(a-a) + B(a+a)$$

$$a = 0 + B(2a)$$

$$a = 2aB$$

$$B = \frac{a}{2a}$$

$$B = \frac{1}{2}$$

$$\therefore \frac{x}{(x+a)(x-a)} = \frac{1}{2(x+a)} + \frac{1}{2(x-a)}$$

(x) Whether $(x+3)^2 = x^2 + 6x + 9$ is an identity? (A.B + K.B + U.B)

$$(x+3)^2 = x^2 + 6x + 9$$

Let $x = 1$

$$(1+3)^2 = (1)^2 + 6(1) + 9$$

$$(4)^2 = 1 + 6 + 9$$

$$16 = 16 \quad (\text{True})$$

Let $x = 25$

$$(2+3)^2 = (2)^2 + 6(2) + 9$$

$$(5)^2 = 4 + 12 + 9$$

$$25 = 25 \quad (\text{True})$$

Hence:

Given equation is an identity because it is true for all values of variable 'x'



CUT HERE

SELF TEST**Time: 40 min****Marks: 25**

Q.1 Four possible answers (A), (B), (C) & (D) to each question are given, mark the correct answer. (7×1=7)

1 A function of the form $f(x) = \frac{N(x)}{D(x)}$, with $D(x) \neq 0$, where $N(x)$ and $D(x)$ are polynomials in x is called:

- (A) An identity (B) An equation
(C) A fraction (D) None of these

2 Partial fractions of $\frac{x+2}{(x+1)(x^2+2)}$ are of the form:

- (A) $\frac{A}{x+1} + \frac{B}{x^2+2}$ (B) $\frac{A}{x+1} + \frac{Bx+C}{x^2+2}$
(C) $\frac{Ax+B}{x+1} + \frac{C}{x^2+2}$ (D) $\frac{A}{x+1} + \frac{Bx}{x^2+2}$

3 An improper rational fraction can be reduced by division to a:

- (A) Rational fraction (B) Irrational fraction
(C) Polynomial and rational form (D) None of these

4 $(x+3)^2 = x^2 + 6x + 9$ is:

- (A) An identity (B) An equation
(C) A linear equations (D) None

5 A fraction in which the degree of the numerator is less than the degree of the denominator is called:

- (A) An improper fraction (B) An equation
(C) A proper fraction (D) None

6 Partial fractions of $\frac{1}{(x+1)(x^2-1)}$ will be of the form:

- (A) $\frac{A}{x+1} + \frac{Bx+C}{x^2-1}$ (B) $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$
(C) $\frac{A}{x+1} + \frac{B}{x+1} + \frac{C}{x-1}$ (D) None of these

7 The degree of the expression $4x^3 + 2x + 5$ is:

- (A) 4 (B) 5
(C) 2 (D) 3

Q.2 Give Short Answers to following Questions.**(5×2=10)**

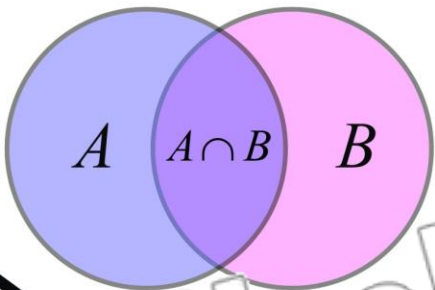
- (i) Define partial fractions.
- (ii) What is difference between conditional equation and identity?
- (iii) Resolve $\frac{x^2+1}{(x-1)(x+2)}$ into partial fractions.
- (iv) Define improper rational fraction.

(v) Convert into proper fraction: $\frac{3x^2-2x-1}{x^2-x+1}$.

Q.3 Answer the following Questions.**(4+4=8)**

- (a) Resolve into partial fractions: $\frac{x^5}{(x^2+1)^2}$.
- (b) Resolve into partial fractions $\frac{6x^3+5x^2-7}{3x^2-2x-1}$

NOTE: Parents or guardians can conduct this test in their supervision in order to check the skill of students.



UNIT 5

SETS AND FUNCTIONS

Set (K.B + U.B)

A well define collection of distinct objects is called Set. A set is represented by capital English alphabets.

For example: A = Set of integers,
 $B = \{1, 2, 3, \dots, 10\}$ etc.

Presentation of Sets (K.B + U.B)

A set is presented by 3 ways

- Tabular form
- Descriptive Form
- Set builder notation

Some Important Sets (K.B + U.B)

N = The set of natural numbers = $\{1, 2, 3, \dots\}$

W = The set of whole numbers = $\{0, 1, 2, 3, \dots\}$

Z = The set of all integers = $\{0, \pm 1, \pm 2, \pm 3, \dots\}$

E = The set of all even integers
 $= \{0, \pm 2, \pm 4, \pm 6, \dots\}$

O = The set of all odd integers
 $= \{\pm 1, \pm 3, \pm 5, \dots\}$

P = The set of prime numbers = $\{2, 3, 5, 7, \dots\}$

Q = The set of rational numbers
 $= \left\{ x/x = \frac{m}{n}, \text{ where } m, n \in \mathbb{Z} \wedge n \neq 0 \right\}$

Q' = The set of irrational numbers
 $= \left\{ x/x \neq \frac{m}{n}, \text{ where } m, n \in \mathbb{Z} \wedge n \neq 0 \right\}$

R = The set of real numbers = $\mathbb{Q} \cup \mathbb{Q}'$

Empty Set or Null Set (K.B + U.B)

A set having no element in it is called an empty set. It is represented by ϕ (phi) or $\{\}$.

Singleton Set (K.B + U.B)

A set having only one element in it is called singleton set.

For example: $\{a\}, \{3\}$ etc.

Subset (K.B + U.B)

(LHR 2016, D.G.K 2016)

“If A and B are two sets, such that every element of set A is present in set B, then set A is called subset of set B”. It is represented as $A \subseteq B$.

For Example

If $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$

Then $A \subseteq B$.

Note (K.B + U.B)

- Number of all possible subsets = 2^n
- Number of all possible proper subsets = $2^n - 1$.
- Number of improper subsets = 1

Proper Subset (K.B + U.B)

“If A and B are two sets, such that every element of set A is present in set B and there is at least one element in set B which is not in set A, then set A is called proper subset of set B”. It is represented as $A \subset B$.

For example:

If $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$

Then set $A \subset B$.

Improper Subset (K.B + U.B)

If A and B are two sets, such that every element of set A is present in set B and there is no more element in set B which is not in set A, then set A is called improper subset of set B.

For example: $A = \{1, 2, 3\}$, $B = \{3, 2, 1\}$

Then A is improper subset of set B.

Equal Sets (K.B + U.B)

If A and B are two sets, such that $A \subseteq B$ and $B \subseteq A$ then $A = B$

For example

If $A = \{1, 2, 3\}$, $B = \{1, 2, 3\}$

Then $A = B$

Power Set (K.B + U.B)

A set consisting of all possible subsets of a set A is called power set of set A represented as $P(A)$.

For example:

If $A = \{1, 2\}$ then $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Note (K.B + U.B)

- Formula to find number of elements in the power set = 2^n

Disjoint Sets (K.B + U.B)

Two sets having no common elements are called disjoint sets.

i.e. If $A \cap B = \phi$ then A and B are called disjoint sets.

For example

If $A = \{1, 2\}$, $B = \{a, b\}$, then A and B are disjoint sets.

Overlapping Sets (K.B + U.B)

Two sets are called overlapping sets if they have:

- At least one element is common
- Neither set is subset of other.

For example

If $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8, 10\}$

Then A and B are overlapping sets.

Operation on Sets**Union of Two Sets****(BWP 2014) (K.B + U.B)**

The union of two sets A and B written as $A \cup B$ (read as A union B) is a set consisting of all the elements which are either in A or in B or in both.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in A \text{ and } B \text{ both}\}$$

For example

If $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8, 10\}$

Then $A \cup B = \{1, 2, 3, 4, 5, 8, 10\}$

Intersection of Two Sets**(LHR 2015) (K.B + U.B)**

If A and B are two sets then a set consisting of common elements of set A and B is called A intersection B , represented by $A \cap B$.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

For example:

If $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8, 10\}$

Then $A \cap B = \{2\}$

Difference of Two Sets (K.B + U.B)

If A and B are two sets then their difference written as $A - B$ or $A \setminus B$ is a set consisting of all the elements of A which are not in B .

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

$$B - A = \{x \mid x \in B \text{ and } x \notin A\}$$

For example:

If $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8, 10\}$

Then $A - B = \{1, 3\}$

Universal Set (K.B + U.B)

A set consists of all the elements of the sets under consideration is called universal set. It is represented by capital English alphabet U or X .

For example:

U = Set of integers, $A = \{1, 2, 3\}$,

$B = \{2, 4, 6, 8, 10\}$, then U is universal set.

Complement of a Set (K.B + U.B)**(GRW 2014, SWL 2015, BWP 2016)**

If U is a universal set and A is its subset then $U - A$ is called complement of A , represented as A' or A^c .

For example:

If $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$

$$\begin{aligned} A' &= U - A \\ &= \{1, 2, 3, 4, 5\} - \{1, 2, 3\} \\ &= \{4, 5\} \end{aligned}$$

Exercise 5.1**Q.1 (A.B)****Given**

$$X = \{1, 4, 7, 9\}$$

$$Y = \{2, 4, 5, 9\}$$

To Find

(i) $X \cup Y$

(ii) $X \cap Y$

(iii) $Y \cup X$

(iv) $Y \cap X$

Solution:

$$\begin{aligned} \text{(i)} \quad X \cup Y &= \{1, 4, 7, 9\} \cup \{2, 4, 5, 9\} \\ &= \{1, 2, 4, 5, 7, 9\} \end{aligned}$$

$$(ii) \quad X \cap Y = \{1, 4, 7, 9\} \cap \{2, 4, 5, 9\} \\ = \{4, 9\}$$

$$(iii) \quad Y \cup X = \{2, 4, 5, 9\} \cup \{1, 4, 7, 9\} \\ = \{1, 2, 4, 5, 7, 9\}$$

$$(iv) \quad Y \cap X = \{2, 4, 5, 9\} \cap \{1, 4, 7, 9\} \\ = \{4, 9\}$$

Q.2 (A.B)**Given:**

X = set of prime numbers less than or equal to 17.

Y = set of first 12 natural number

To Find:

$$(i) \quad X \cup Y$$

$$(ii) \quad Y \cup X$$

$$(iii) \quad X \cap Y$$

$$(iv) \quad Y \cap X$$

Solution:**Here**

X = Set of prime numbers less than or equal to 17.

$$= \{2, 3, 5, 7, 11, 13, 17\}$$

Y = Set of first 12 natural numbers

$$= \{1, 2, 3, \dots, 12\}$$

$$(i) \quad X \cup Y = \{2, 3, 5, 7, 11, 13, 17\} \cup \{1, 2, 3, \dots, 12\} \\ = \{1, 2, 3, \dots, 12, 13, 17\}$$

$$(ii) \quad Y \cup X = \{1, 2, 3, \dots, 12\} \cup \{2, 3, 5, 7, 11, 13, 17\} \\ = \{1, 2, 3, \dots, 12, 13, 17\}$$

$$(iii) \quad X \cap Y = \{2, 3, 5, 7, 11, 13, 17\} \cap \{1, 2, 3, \dots, 12\} \\ = \{2, 3, 5, 7, 11\}$$

$$(iv) \quad Y \cap X = \{1, 2, 3, \dots, 12\} \cap \{2, 3, 5, 7, 11, 13, 17\} \\ = \{2, 3, 5, 7, 11\}$$

Q.3 (A.B)**Given**

$$X = \phi, \quad Z = Z^+, \quad T = O^+$$

To Find

$$(i) \quad X \cup Y$$

$$(ii) \quad X \cup T$$

$$(iii) \quad Y \cup T$$

$$(iv) \quad X \cap Y$$

$$(v) \quad X \cap T$$

$$(vi) \quad Y \cap T$$

Solution:

$$(i) \quad X \cup Y = \phi \cup Z^+ \\ = Z^+$$

$$(ii) \quad X \cup T = \phi \cup O^+ \\ = O^+$$

$$(iii) \quad Y \cup T = Z^+ \cup O^+ \\ = Z^+$$

$$(iv) \quad X \cap Y = \phi \cap Z^+ \\ = \phi$$

$$(v) \quad X \cap T = \phi \cap O^+ \\ = \phi$$

$$(vi) \quad Y \cap T = Z^+ \cap O^+ \\ = O^+$$

Q.4 Given (A.B)

$$U = \{x | x \in N \wedge 3 < x \leq 25\}$$

$$X = \{x | x \text{ is prime} \wedge 8 < x < 25\}$$

$$Y = \{x | x \in W \wedge 4 \leq x \leq 17\}$$

To Find

$$(i) \quad (X \cup Y)' \quad (ii) \quad X' \cap Y'$$

$$(ii) \quad (X \cap Y)' \quad (iv) \quad X' \cup Y'$$

Solution: Here

$$U = \{x / x \in N \wedge 3 < x \leq 25\}$$

$$= \{4, 5, 6, \dots, 25\}$$

$$X = \{x | x \text{ is prime} \wedge 8 < x < 25\}$$

$$= \{11, 13, 17, 19, 23\}$$

$$Y = \{x | x \in W \wedge 4 \leq x \leq 17\}$$

$$= \{4, 5, 6, 7, \dots, 17\}$$

$$(i) \quad X \cup Y = \{11, 13, 17, 19, 23\} \cup \{4, 5, 6, \dots, 17\}$$

$$= \{4, 5, 6, \dots, 17, 19, 23\}$$

$$(X \cup Y)' = U - (X \cup Y)$$

$$= \{4, 5, 6, \dots, 25\} - \{4, 5, 6, \dots, 17, 19, 23\}$$

$$= \{18, 20, 21, 22, 24, 25\}$$

$$(ii) \quad X' = U - X$$

$$= \{4, 5, 6, \dots, 25\} - \{11, 13, 17, 19, 23\}$$

$$= \{4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25\}$$

$$Y' = U - Y$$

$$= \{4, 5, 6, \dots, 25\} - \{4, 5, 6, \dots, 17\}$$

$$= \{18, 19, 20, 21, 22, 23, 24, 25\}$$

$$X' \cap Y' = \{4, 5, 6, \dots, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25\} \\ \cap \{18, 19, 20, \dots, 25\}$$

- $= \{18, 20, 21, 22, 24, 25\}$
- (iii) $X \cap Y = \{11, 13, 17, 19, 23\} \cap \{4, 5, 6, \dots, 17\}$
 $= \{11, 13, 17\}$
- $(X \cap Y)' = U - (X \cap Y)$
 $= \{4, 5, 6, \dots, 25\} - \{11, 13, 17\}$
 $= \{4, 5, 6, \dots, 10, 12, 14, 15, 16, 18, 19, 20, \dots, 25\}$
- (iv) $X' \cup Y'$
 $= \{4, 5, 6, \dots, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25\}$
 $\cup \{18, 19, 20, 21, 22, 23, 24, 25\}$
 $= \{4, 5, 6, \dots, 10, 12, 14, 15, 16, 18, 19, 20, \dots, 25\}$

Q.5 Given (A.B)

$X = \{2, 4, 6, \dots, 20\}$ (LHR 2014)

$Y = \{4, 8, 12, \dots, 24\}$ (FSD 2015)

To Find

(i) $X - Y$

(ii) $Y - X$

Solution:

(i) $X - Y = \{2, 4, 6, \dots, 20\} - \{4, 8, 12, \dots, 24\}$
 $= \{2, 6, 10, 14, 18\}$

(ii) $Y - X = \{4, 8, 12, \dots, 24\} - \{2, 4, 6, \dots, 20\}$
 $= \{24\}$

Q.6 Given (BWP 2014) (A.B)

$A = N, B = W$ (LHR 2015)

To Find (D.G.K 2014)

(i) $A - B$

(ii) $B - A$

Solution:

(i) $A - B = N - W = \{1, 2, 3, \dots\} - \{0, 1, 2, 3, \dots\}$
 $= \phi$

(ii) $B - A = W - N = \{0, 1, 2, 3, \dots\} - \{1, 2, 3, \dots\}$
 $= \{0\}$

Properties of Union and Intersection of Sets (K.B)

- **Commutative property of Union**
 $A \cup B = B \cup A$
- **Commutative property of intersection**
 $A \cap B = B \cap A$
- **Associative property of union**
 $A \cup (B \cap C) = (A \cup B) \cap C$
- **Associative property of intersection**
 $A \cap (B \cup C) = (A \cap B) \cup C$
- **Distributive property of union over intersection**
 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- **Distributive property of intersection over union**

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- **De-Morgan's laws**

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

Commutative Property of Union

(K.B + U.B)

For any two sets A and B, prove that $A \cup B = B \cup A$.

Proof

Let $x \in A \cup B$

$\Rightarrow x \in A$ or $x \in B$ (by definition of union of sets)

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in (B \cup A)$

$\Rightarrow (A \cup B) \subseteq (B \cup A) \rightarrow (i)$

Now let $y \in (B \cup A)$

$\Rightarrow y \in B$ or $y \in A$ (by definition of union of sets)

$\Rightarrow y \in A$ or $y \in B$

$\Rightarrow y \in (A \cup B)$

$\Rightarrow (B \cup A) \subseteq (A \cup B) \rightarrow (ii)$

From (i) and (ii), we have $A \cup B = B \cup A$.

(by definition of equal sets)

Commutative Property of Intersection

(MTN 2014) (K.B + U.B)

For any two sets A and B, prove that $A \cap B = B \cap A$

Proof

Let $x \in (A \cap B)$

$\Rightarrow x \in A$ and $x \in B$

(by definition of intersection of sets)

$\Rightarrow x \in B$ and $x \in A$

$\Rightarrow x \in (B \cap A)$

$\therefore (A \cap B) \subseteq (B \cap A) \rightarrow (i)$

Now let $y \in (B \cap A)$

$\Rightarrow y \in B$ and $y \in A$

(by definition of intersection of sets)

$\Rightarrow y \in A$ and $y \in B$

$\Rightarrow y \in (A \cap B)$

Therefore, $(B \cap A) \subseteq (A \cap B) \rightarrow (ii)$

From (i) and (ii), we have $A \cap B = B \cap A$ (by definition of equal sets)

Associative Property of Union

(K.B + U.B)

For any three sets A, B and C, prove that $(A \cup B) \cup C = A \cup (B \cup C)$

Proof

Let $x \in (A \cup B) \cup C$

$\Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C$

(Associative property)

$\Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C)$

$\Rightarrow x \in A \text{ or } x \in (B \cup C)$

$\Rightarrow x \in A \cup (B \cup C)$

$\Rightarrow (A \cup B) \cup C \subseteq A \cup (B \cup C) \rightarrow (i)$

Now let $y \in A \cup (B \cup C)$

$\Rightarrow y \in A \text{ or } y \in (B \cup C)$

$\Rightarrow y \in A \text{ or } (y \in B \text{ or } y \in C)$

$\Rightarrow y \in (A \text{ or } y \in B) \text{ or } y \in C$

$\Rightarrow y \in (A \cup B) \text{ or } y \in C$

$\Rightarrow y \in (A \cup B) \cup C$

$\therefore A \cup (B \cup C) \subseteq (A \cup B) \cup C \rightarrow (ii)$

From (i) and (ii), we have

$(A \cup B) \cup C = A \cup (B \cup C)$

Associative Property of Intersection

(K.B + U.B)

For any three sets A, B and C, prove that $(A \cap B) \cap C = A \cap (B \cap C)$

Proof

Let $x \in (A \cap B) \cap C$

$\Rightarrow x \in (A \cap B)$ and $x \in C$

$\Rightarrow (x \in A \text{ and } x \in B) \text{ and } x \in C$

$\Rightarrow x \in A \text{ and } (x \in B \text{ and } x \in C)$

$\Rightarrow x \in A \text{ and } x \in (B \cap C)$

$\Rightarrow x \in A \cap (B \cap C)$

$\Rightarrow (A \cap B) \cap C \subseteq A \cap (B \cap C) \rightarrow (i)$

Now let $y \in A \cap (B \cap C)$

$\Rightarrow y \in A \text{ and } y \in (B \cap C)$

$\Rightarrow y \in A \text{ and } (y \in B \text{ and } y \in C)$

$\Rightarrow (y \in A \text{ and } y \in B) \text{ and } y \in C$

$\Rightarrow y \in (A \cap B) \text{ and } y \in C$

$\Rightarrow y \in (A \cap B) \cap C$

$\therefore A \cap (B \cap C) \subseteq (A \cap B) \cap C \rightarrow (ii)$

From (i) and (ii), we have

$(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Property of Union over Intersection

(K.B + U.B)

For any three sets A, B and C, prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof

Let $x \in A \cup (B \cap C)$

$\Rightarrow x \in A \text{ or } x \in (B \cap C)$

$\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C)$

$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$

$\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

Therefore

$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \rightarrow (i)$

Similarly, now let $y \in (A \cup B) \cap (A \cup C)$

$\Rightarrow y \in (A \cup B) \text{ and } y \in (A \cup C)$

$\Rightarrow (y \in A \text{ or } y \in B) \text{ and } (y \in A \text{ or } y \in C)$

$\Rightarrow y \in A \text{ or } (y \in B \text{ and } y \in C)$

$\Rightarrow y \in A \text{ or } y \in (B \cap C)$

$\Rightarrow y \in A \cup (B \cap C)$

$\Rightarrow (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \rightarrow (ii)$

From (i) and (ii), we have

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Distributive Property of Intersection over Union

(K.B + U.B)

For any three sets A, B and C, prove that

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof

Let $x \in A \cap (B \cup C)$

$$\Rightarrow x \in A \text{ and } x \in (B \cup C)$$

$$\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$$

$$\Rightarrow (x \in A \text{ and } x \in B)$$

$$\text{or } (x \in A \text{ and } x \in C)$$

$$\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C)$$

$$\Rightarrow x \in (A \cap B) \cup (A \cap C)$$

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \rightarrow (i)$$

Now let

$$\Rightarrow y \in (A \cap B) \cup (A \cap C)$$

$$\Rightarrow y \in (A \cap B) \text{ or } y \in (A \cap C)$$

$$\text{or } (y \in A \text{ and } y \in C)$$

$$\Rightarrow (y \in A \text{ and } y \in B)$$

$$\Rightarrow y \in A \text{ and } (y \in B \text{ or } y \in C)$$

$$\Rightarrow y \in A \text{ and } y \in (B \cup C)$$

$$y \in A \cap (B \cup C)$$

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \rightarrow (ii)$$

From (i) and (ii), we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

De-Morgan's Laws (K.B + U.B)

(LHR 2016, MTN 2016, SGD 2015, BWP 2016, D.G.K 2016)

For any two sets A and B belonging from universal set U.

$$(A \cup B)' = A' \cap B' \rightarrow (i)$$

$$(A \cap B)' = A' \cup B' \rightarrow (ii)$$

(i) Proof

Let $x \in (A \cup B)'$

$$\Rightarrow x \notin (A \cup B) \quad (\text{by definition of complement of set})$$

$$\Rightarrow x \notin A \text{ and } x \notin B$$

$$\Rightarrow x \in A' \text{ and } x \in B'$$

$$\Rightarrow x \in A' \cap B' \quad (\text{by definition of intersection of sets})$$

$$\Rightarrow (A \cup B)' \subseteq A' \cap B' \rightarrow (1)$$

Let $y \in A' \cap B'$

$$\Rightarrow y \in A' \text{ and } y \in B'$$

$$\Rightarrow y \notin A \text{ and } y \notin B$$

$$\Rightarrow y \notin (A \cup B) \quad (\text{by definition of union of set})$$

$$y \in (A \cup B)'$$

$$A' \cap B' \subseteq (A \cup B)' \rightarrow (ii)$$

Using (i) and (ii), we have

$$(A \cup B)' = A' \cap B'$$

(ii) Proof

Let $x \in (A \cap B)'$

$$\Rightarrow x \notin A \cap B \quad (\text{by definition of complement of set})$$

$$\Rightarrow x \notin A \text{ or } x \notin B$$

$$\Rightarrow x \in A' \text{ or } x \in B'$$

$$\Rightarrow x \in A' \cup B' \quad (\text{by definition of union of set})$$

$$\Rightarrow (A \cap B)' \subseteq A' \cup B' \rightarrow (i)$$

Let $y \in A' \cup B'$

$$\Rightarrow y \in A' \text{ or } y \in B'$$

$$\Rightarrow y \notin A \text{ or } y \notin B$$

$$\Rightarrow y \notin A \cap B \quad (\text{by definition of intersection of sets})$$

$$\Rightarrow y \in (A \cap B)'$$

$$\Rightarrow A' \cup B' \subseteq (A \cap B)' \rightarrow (ii)$$

From (i) and (ii), we have proved that

$$(A \cap B)' = A' \cup B'$$

Exercise 5.2

Q.1 Given $X = \{1, 3, 5, 7, \dots, 19\}$ **(A.B)**

$$Y = \{0, 2, 4, 6, \dots, 20\}$$

$$Z = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$$

To Find

(i) $X \cup (Y \cap Z)$

(ii) $(X \cup Y) \cup Z$

(iii) $X \cap (Y \cap Z)$

(iv) $(X \cap Y) \cap Z$

(v) $X \cup (Y \cap Z)$

(vi) $(X \cup Y) \cap (X \cup Z)$

(vii) $X \cap (Y \cup Z)$

(viii) $(X \cap Y) \cup (X \cap Z)$

Solution:

(i) $X \cup (Y \cap Z)$ **(RWP 2015) (A.B)**

$$= \{1, 3, 5, 7, \dots, 19\} \cup$$

$$(\{0, 2, 4, 6, \dots, 20\} \cap \{2, 3, 5, 7, 11, 13, 17, 19, 23\})$$

$$= \{1, 3, 5, \dots, 19\} \cup$$

$$\{0, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 23\}$$

$$= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 23\}$$

(ii) $(X \cup Y) \cup Z$ (A.B)

$$X \cup Y = \{1, 3, 5, 7, \dots, 19\} \cup \{0, 2, 4, 6, \dots, 20\}$$

$$= \{0, 1, 2, 3, \dots, 20\}$$

$$(X \cup Y) \cup Z$$

$$= \{0, 1, 2, 3, \dots, 20\} \cup \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$$

$$= \{0, 1, 2, 3, \dots, 20, 23\}$$

(iii) $X \cap (Y \cap Z)$ (A.B)

$$Y \cap Z = \{0, 2, 4, 6, \dots, 20\} \cap \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$$

$$= \{2\}$$

$$X \cap (Y \cap Z) = \{1, 3, 5, 7, \dots, 19\} \cap \{2\}$$

$$= \{\}$$

(iv) $(X \cap Y) \cap Z$ (A.B)

$$X \cap Y = \{1, 3, 5, 7, \dots, 19\} \cap \{0, 2, 4, 6, \dots, 20\}$$

$$= \{\}$$

$$(X \cap Y) \cap Z = \{\} \cap \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$$

$$= \{\}$$

(v) $X \cup (Y \cap Z)$ (A.B)

$$Y \cap Z = \{0, 2, 4, 6, \dots, 20\} \cap \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$$

$$= \{2\}$$

$$X \cup (Y \cap Z) = \{1, 3, 5, 7, \dots, 19\} \cup \{2\}$$

$$= \{1, 2, 3, 5, 7, \dots, 19\}$$

(vi) $(X \cup Y) \cap (X \cup Z)$ (A.B)

$$X \cup Y = \{1, 3, 5, \dots, 19\} \cup \{0, 2, 4, 6, \dots, 20\}$$

$$= \{0, 1, 2, 3, \dots, 20\}$$

$$X \cup Z = \{1, 3, 5, \dots, 19\} \cup \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$$

$$= \{1, 2, 3, 5, 7, \dots, 19, 23\}$$

$$(X \cup Y) \cap (X \cup Z) = \{0, 1, 2, 3, \dots, 20\}$$

$$\cap \{1, 2, 3, 5, 7, \dots, 19, 23\}$$

$$= \{1, 2, 3, 5, 7, \dots, 19\}$$

(vii) $X \cap (Y \cup Z)$ (A.B)

$$Y \cup Z = \{0, 2, 4, 6, \dots, 20\}$$

$$\cup \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$$

$$= \{0, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 23\}$$

$$X \cap (Y \cup Z) = \{1, 3, 5, 7, \dots, 19\} \cap$$

$$\{0, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 23\}$$

$$= \{3, 5, 7, 11, 13, 17, 19\}$$

(viii) $(X \cap Y) \cup (X \cap Z)$ (A.B)

$$X \cap Y = \{1, 3, 5, 7, \dots, 19\} \cap \{0, 2, 4, 6, \dots, 20\}$$

$$= \{\}$$

$$X \cap Z = \{1, 3, 5, 7, \dots, 19\}$$

$$\cap \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$$

$$= \{3, 5, 7, 11, 13, 17, 19\}$$

$$(X \cap Y) \cup (X \cap Z) = \{\} \cup \{3, 5, 7, 11, 13, 17, 19\}$$

$$= \{3, 5, 7, 11, 13, 17, 19\}$$

Q.2 Given $A = \{1, 2, 3, 4, 5, 6\}$

(A.B + K.B)

$$B = \{2, 4, 6, 8\}$$

$$C = \{1, 4, 8\}$$

To Prove

(i) $A \cap B = B \cap A$

(ii) $A \cup B = B \cup A$

(iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof

(i) $A \cap B = B \cap A$ (A.B + K.B)

$$\text{L.H.S} = A \cap B$$

$$= \{1, 2, 3, 4, 5, 6\} \cap \{2, 4, 6, 8\}$$

$$= \{2, 4, 6\} \rightarrow (i)$$

$$\text{R.H.S} = B \cap A$$

$$= \{2, 4, 6, 8\} \cap \{1, 2, 3, 4, 5, 6\}$$

$$= \{2, 4, 6\} \rightarrow (ii)$$

From equation (i) and (ii)

$$\text{L.H.S} = \text{R.H.S}$$

$$A \cap B = B \cap A$$

Hence Proved

(ii) $A \cup B = B \cup A$ (A.B + K.B)

Proof

$$\text{L.H.S} = A \cup B$$

$$= \{1, 2, 3, 4, 5, 6\} \cup \{2, 4, 6, 8\}$$

$$= \{1, 2, 3, 4, 5, 6, 8\} \rightarrow (i)$$

$$\text{R.H.S} = B \cup A$$

$$= \{2, 4, 6, 8\} \cup \{1, 2, 3, 4, 5, 6\}$$

$$= \{1, 2, 3, 4, 5, 6, 8\} \rightarrow (ii)$$

From equation (i) and (ii)

$$\text{L.H.S} = \text{R.H.S}$$

$$A \cup B = B \cup A$$

Hence Proved

$$(iii) \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(A.B + K.B)

Proof

$$L.H.S = A \cap (B \cup C)$$

$$B \cup C = \{2, 4, 6, 8\} \cup \{1, 4, 8\}$$

$$= \{1, 2, 4, 6, 8\}$$

$$A \cap (B \cup C)$$

$$= \{1, 2, 3, 4, 5, 6\} \cap \{1, 2, 4, 6, 8\}$$

$$= \{1, 2, 4, 6\} \rightarrow (i)$$

$$R.H.S = (A \cap B) \cup (A \cap C)$$

$$A \cap B = \{1, 2, 3, 4, 5, 6\} \cap \{2, 4, 6, 8\}$$

$$= \{2, 4, 6\}$$

$$A \cap C = \{1, 2, 3, 4, 5, 6\} \cap \{1, 4, 8\}$$

$$= \{1, 4\}$$

$$(A \cap B) \cup (A \cap C) = \{2, 4, 6\} \cup \{1, 4\}$$

$$= \{1, 2, 4, 6\} \rightarrow (ii)$$

From equation (i) and (ii)

$$L.H.S = R.H.S$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Hence Proved

$$(iv) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(A.B + K.B)

Proof

$$L.H.S = A \cup (B \cap C)$$

$$B \cap C = \{2, 4, 6, 8\} \cap \{1, 4, 8\}$$

$$= \{4, 8\}$$

$$L.H.S = A \cup (B \cap C)$$

$$= \{1, 2, 3, 4, 5, 6\} \cup \{4, 8\}$$

$$= \{1, 2, 3, 4, 5, 6, 8\} \rightarrow (i)$$

$$R.H.S = (A \cup B) \cap (A \cup C)$$

$$A \cup B = \{1, 2, 3, 4, 5, 6\} \cup \{2, 4, 6, 8\}$$

$$= \{1, 2, 3, 4, 5, 6, 8\}$$

$$A \cup C = \{1, 2, 3, 4, 5, 6\} \cup \{1, 4, 8\}$$

$$= \{1, 2, 3, 4, 5, 6, 8\}$$

$$R.H.S = (A \cup B) \cap (A \cup C)$$

$$= \{1, 2, 3, 4, 5, 6, 8\} \cap \{1, 2, 3, 4, 5, 6, 8\}$$

$$= \{1, 2, 3, 4, 5, 6, 8\} \rightarrow (ii)$$

From equation (i) and (ii)

$$L.H.S = R.H.S$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Hence Proved

$$Q.3 \quad \text{Given } U = \{1, 2, 3, \dots, 10\}$$

(A.B + K.B)

$$A = \{1, 3, 5, 7, 9\}$$

$$B = \{2, 3, 5, 7\}$$

To Prove

$$(i) \quad (A \cap B)' = A' \cup B'$$

$$(ii) \quad (A \cup B)' = A' \cap B'$$

$$(i) \quad (A \cap B)' = A' \cup B'$$

Proof

$$L.H.S = (A \cap B)'$$

$$A \cap B = \{1, 3, 5, 7, 9\} \cap \{2, 3, 5, 7\}$$

$$= \{3, 5, 7\}$$

$$(A \cap B)' = U - (A \cap B)$$

$$= \{1, 2, 3, \dots, 10\} - \{3, 5, 7\}$$

$$= \{1, 2, 4, 6, 8, 9, 10\} \rightarrow (i)$$

$$R.H.S = A' \cup B'$$

$$A' = U - A$$

$$= \{1, 2, 3, \dots, 10\} - \{1, 3, 5, 7, 9\}$$

$$= \{2, 4, 6, 8, 10\}$$

$$B' = U - B$$

$$= \{1, 2, 3, \dots, 10\} - \{2, 3, 5, 7\}$$

$$= \{1, 4, 6, 8, 9, 10\}$$

$$A' \cup B' = \{2, 4, 6, 8, 10\} \cup \{1, 4, 6, 8, 9, 10\}$$

$$= \{1, 2, 4, 6, 8, 9, 10\} \rightarrow (ii)$$

From equation (i) and (ii)

$$L.H.S = R.H.S$$

$$(A \cap B)' = A' \cup B'$$

Hence Proved

(ii) $(A \cup B)' = A' \cap B'$ **(A.B + K.B)**

Proof

L.H.S = $(A \cup B)'$

$A \cup B = \{1, 3, 5, 7, 9\} \cup \{2, 3, 5, 7\}$

$= \{1, 2, 3, 5, 7, 9\}$

L.H.S = $(A \cup B)' = U - (A \cup B)$

$= \{1, 2, 3, \dots, 10\} - \{1, 2, 3, 5, 7, 9\}$

$= \{4, 6, 8, 10\} \rightarrow (i)$

R.H.S = $A' \cap B'$

$= \{2, 4, 6, 8, 10\} \cap \{1, 4, 6, 8, 9, 10\}$

$= \{4, 6, 8, 10\} \rightarrow (ii)$

From equation (i) and (ii)

L.H.S = R.H.S

$(A \cup B)' = A' \cap B'$

Hence Proved

Q.4 Given (A.B + K.B)

$U = \{1, 2, 3, \dots, 20\}$

$X = \{1, 3, 7, 9, 15, 18, 20\}$

$Y = \{1, 3, 5, \dots, 17\}$

To Prove

(i) $X - Y = X \cap Y'$

(ii) $Y - X = Y \cap X'$

Proof

(i) $X - Y = X \cap Y'$ **(A.B + K.B)**

L.H.S = $X - Y$

$= \{1, 3, 7, 9, 15, 18, 20\} - \{1, 3, 5, \dots, 17\}$

$= \{18, 20\} \rightarrow (i)$

$Y' = U - Y$

$= \{1, 2, 3, \dots, 20\} - \{1, 3, 5, \dots, 17\}$

$= \{2, 4, 6, \dots, 18, 19, 20\}$

R.H.S = $X \cap Y'$

$= \{1, 3, 7, 9, 15, 18, 20\} \cap \{2, 4, 6, \dots, 18, 19, 20\}$

$= \{18, 20\} \rightarrow (ii)$

From equation (i) and (ii)

L.H.S = R.H.S

$X - Y = X \cap Y'$

Hence Proved

(ii) $Y - X = Y \cap X'$ **(A.B + K.B)**

Proof

L.H.S = $Y - X$

$= \{1, 3, 5, \dots, 17\} - \{1, 3, 7, 9, 15, 18, 20\}$

$= \{5, 11, 13, 17\} \rightarrow (i)$

L.H.S = $Y \cap X'$

$X' = U - X$

$= \{1, 2, 3, \dots, 20\} - \{1, 3, 7, 9, 15, 18, 20\}$

$= \{2, 4, 5, 6, 8, 10, 11, 12, 13, 14, 16, 17, 19\}$

$Y \cap X' = \{1, 3, 5, \dots, 17\}$

$\cap \{2, 4, 5, 6, 8, 10, 11, 12, 13, 14, 16, 17, 19\}$

$= \{5, 11, 13, 17\} \rightarrow (ii)$

From equation (i) and (ii)

L.H.S = R.H.S

$Y - X = Y \cap X'$

Hence Proved

Venn Diagram (K.B)

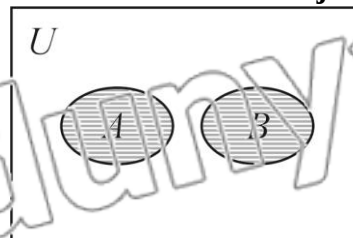
British mathematician John Venn introduced Venn Diagrams.

“A graphical method to represent sets in which Universal Set is represented by a rectangle and its subsets by a closed shaped (i.e. circle or oval) in it, is called Venn Diagram”.

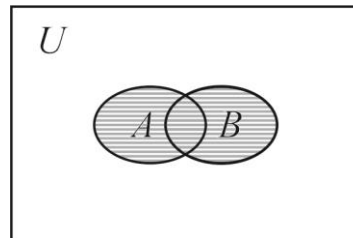
Operations on Sets Through Venn Diagrams (K.B)

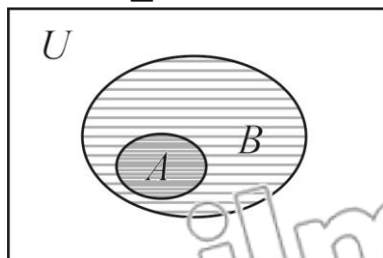
Union of Two Sets

When A and B are Disjoint (K.B)

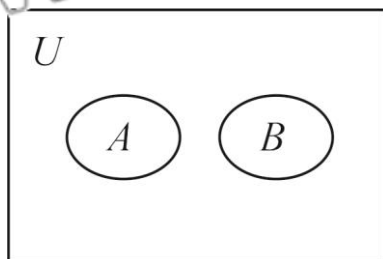


When A and B are overlapping (K.B)

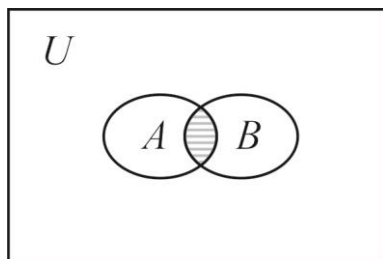
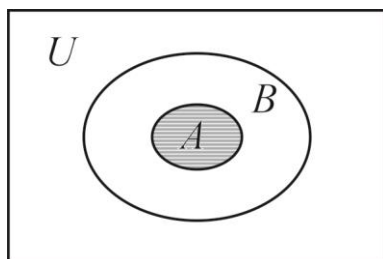
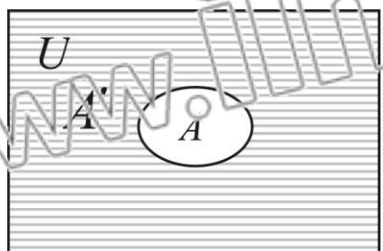


When $A \subseteq B$ **Intersection of Two Sets**

When A and B are Disjoint



When A and B are overlapping

When $A \subseteq B$ **Complement of a Set****(K.B)****(K.B)****(K.B)****(K.B)****Exercise 5.3****Q.1** Given $U = \{1, 2, 3, 4, \dots, 10\}$

$$A = \{1, 3, 5, 7, 9\} \quad \text{(K.B)}$$

$$B = \{1, 4, 7, 10\} \quad \text{(A.B)}$$

(LHR 2017, GRW 2016, FSD 2017, SWL 2017, RWP 2016, BWP 2016, D.G.K 2016)

To Prove

(i) $A - B = A \cap B'$

(ii) $B - A = B \cap A'$

(iii) $(A \cup B)' = A' \cap B'$

(iv) $(A \cap B)' = A' \cup B'$

(v) $(A - B)' = A' \cup B$

(vi) $(B - A)' = B' \cup A$

Proof

(i) $A - B = A \cap B'$

L.H.S = $A - B$

$$= \{1, 3, 5, 7, 9\} - \{1, 4, 7, 10\}$$

$$= \{3, 5, 9\} \rightarrow \text{(i)}$$

R.H.S = $A \cap B'$

$$B' = U - B$$

$$= \{1, 2, 3, \dots, 10\} - \{1, 4, 7, 10\}$$

$$= \{2, 3, 5, 6, 8, 9\}$$

$$A \cap B' = \{1, 3, 5, 7, 9\} \cap \{2, 3, 5, 6, 8, 9\}$$

$$= \{3, 5, 9\} \rightarrow \text{(ii)}$$

From equation (i) and (ii)

L.H.S = R.H.S

$$A - B = A \cap B'$$

Hence Proved

(ii) $B - A = B \cap A' \quad \text{(K.B + A.B)}$

L.H.S = $B - A$

$$= \{1, 4, 7, 10\} - \{1, 3, 5, 7, 9\}$$

$$= \{4, 10\} \rightarrow \text{(i)}$$

R.H.S = $B \cap A'$

$$A' = U - A = \{1, 2, 3, \dots, 10\} - \{1, 3, 5, 7, 9\}$$

$$= \{2, 4, 6, 8, 10\}$$

$$B \cap A' = \{1, 4, 7, 10\} \cap \{2, 4, 6, 8, 10\}$$

$$= \{4, 10\} \rightarrow \text{(ii)}$$

From equation (i) and (ii)

L.H.S = R.H.S

$$B - A = B \cap A'$$

Hence Proved

(iii) $(A \cup B)' = A' \cap B'$ (K.B + A.B)

$$A \cup B = \{1, 3, 5, 7, 9\} \cup \{1, 4, 7, 10\}$$

$$= \{1, 3, 4, 5, 7, 9, 10\}$$

$$\text{L.H.S} = (A \cup B)' = U - (A \cup B)$$

$$= \{1, 2, 3, \dots, 10\} - \{1, 3, 4, 5, 7, 9, 10\}$$

$$= \{2, 6, 8\} \rightarrow (i)$$

$$\text{R.H.S} = A' \cap B'$$

$$A' = U - A = \{1, 2, 3, \dots, 10\} - \{1, 3, 5, 7, 9\}$$

$$= \{2, 4, 6, 8, 10\}$$

$$B' = U - B = \{1, 2, 3, \dots, 10\} - \{1, 4, 7, 10\}$$

$$= \{2, 3, 5, 6, 8, 9\}$$

$$\text{R.H.S} = A' \cap B'$$

$$= \{2, 4, 6, 8, 10\} \cap \{2, 3, 5, 6, 8, 9\}$$

$$= \{2, 6, 8\} \rightarrow (ii)$$

From equation (i) and (ii)

$$\text{L.H.S} = \text{R.H.S}$$

$$(A \cup B)' = A' \cap B'$$

Hence Proved(iv) $(A \cap B)' = A' \cup B'$ (K.B + A.B)**Proof**

$$A \cap B = \{1, 3, 5, 7, 9\} \cap \{1, 4, 7, 10\}$$

$$= \{1, 7\}$$

$$\text{L.H.S} = (A \cap B)' = U - (A \cap B)$$

$$= \{1, 2, 3, \dots, 10\} - \{1, 7\}$$

$$= \{2, 3, 4, 5, 6, 8, 9, 10\} \rightarrow (i)$$

$$\text{R.H.S} = A' \cup B'$$

$$= \{2, 4, 6, 8, 10\} \cup \{2, 3, 5, 6, 8, 9\}$$

$$= \{2, 3, 4, 5, 6, 8, 9, 10\} \rightarrow (ii)$$

From equation (i) and (ii)

$$\text{L.H.S} = \text{R.H.S}$$

$$(A \cap B)' = A' \cup B'$$

Hence Proved(v) $(A - B)' = A' \cup B$ (K.B + A.B)**Proof**

$$\text{L.H.S} = (A - B)'$$

$$A - B = \{1, 3, 5, 7, 9\} - \{1, 4, 7, 10\}$$

$$= \{3, 5, 9\}$$

$$\text{L.H.S} = (A - B)' = U - (A - B)$$

$$= \{1, 2, 3, \dots, 10\} - \{3, 5, 9\}$$

$$= \{1, 2, 4, 6, 7, 8, 10\} \rightarrow (i)$$

$$\text{R.H.S} = A' \cup B$$

$$A' = U - A$$

$$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1, 3, 5, 7, 9\}$$

$$= \{2, 4, 6, 8, 10\}$$

$$A' \cup B = \{2, 4, 6, 8, 10\} \cup \{1, 4, 7, 10\}$$

$$= \{1, 2, 4, 6, 7, 8, 10\} \rightarrow (ii)$$

From (i) and (ii), we get

$$\text{L.H.S} = \text{R.H.S}$$

$$(A - B)' = A' \cup B$$

Hence Proved(vi) $(B - A)' = B' \cup A$ (K.B + A.B)

(FSD 2015)

Proof

$$B - A = \{1, 4, 7, 10\} - \{1, 3, 5, 7, 9\}$$

$$B - A = \{4, 10\}$$

$$\text{L.H.S} = (B - A)' = U - (B - A)$$

$$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{4, 10\}$$

$$= \{1, 2, 3, 5, 6, 7, 8, 9\} \rightarrow (i)$$

$$\text{R.H.S} = B' \cup A$$

$$B' = U - B$$

$$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1, 4, 7, 10\}$$

$$= \{2, 3, 5, 6, 8, 9\}$$

$$B' \cup A = \{2, 3, 5, 6, 8, 9\} \cup \{1, 3, 5, 7, 9\}$$

$$= \{1, 2, 3, 5, 6, 7, 8, 9\} \rightarrow (ii)$$

From (i) and (ii) we get

$$\text{L.H.S} = \text{R.H.S}$$

$$(B - A)' = B' \cup A$$

Hence Proved

Q.2 If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$$A = \{1, 3, 5, 7, 9\} \quad (\mathbf{K.B + A.B})$$

$$B = \{1, 4, 7, 10\}$$

$$C = \{1, 5, 8, 10\}$$

Then Verify

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(K.B + A.B)

Proof

$$A \cup B = \{1, 3, 5, 7, 9\} \cup \{1, 4, 7, 10\}$$

$$= \{1, 3, 4, 5, 7, 9, 10\}$$

$$\text{L.H.S} = (A \cup B) \cup C$$

$$= \{1, 3, 4, 5, 7, 9, 10\} \cup \{1, 5, 8, 10\}$$

$$= \{1, 3, 4, 5, 7, 8, 9, 10\} \rightarrow (i)$$

$$B \cup C = \{1, 4, 7, 10\} \cup \{1, 5, 8, 10\}$$

$$= \{1, 4, 5, 7, 8, 10\}$$

$$\text{R.H.S} = A \cup (B \cup C)$$

$$= \{1, 3, 5, 7, 9\} \cup \{1, 4, 5, 7, 8, 10\}$$

$$= \{1, 3, 4, 5, 7, 8, 9, 10\} \rightarrow (ii)$$

From (i) and (ii), we get

$$\text{L.H.S} = \text{R.H.S}$$

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Hence proved

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

(K.B + A.B)

Proof

$$\text{L.H.S} = (A \cap B) \cap C$$

$$= (\{1, 3, 5, 7, 9\} \cap \{1, 4, 7, 10\}) \cap \{1, 5, 8, 10\}$$

$$= \{1, 7\} \cap \{1, 5, 8, 10\}$$

$$= \{1\} \rightarrow (i)$$

$$\text{R.H.S} = A \cap (B \cap C)$$

$$= \{1, 3, 5, 7, 9\} \cap (\{1, 4, 7, 10\} \cap \{1, 5, 8, 10\})$$

$$= \{1, 3, 5, 7, 9\} \cap \{1, 10\}$$

$$= \{1\} \rightarrow (ii)$$

From (i) and (ii), we get

$$\text{L.H.S} = \text{R.H.S}$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Hence proved

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(K.B + A.B)

Proof

$$\text{L.H.S} = A \cup (B \cap C)$$

$$= \{1, 3, 5, 7, 9\} \cup (\{1, 4, 7, 10\} \cap \{1, 5, 8, 10\})$$

$$= \{1, 3, 5, 7, 9\} \cup \{1, 10\}$$

$$= \{1, 3, 5, 7, 9, 10\} \rightarrow (i)$$

$$A \cup B = \{1, 3, 5, 7, 9\} \cup \{1, 4, 7, 10\}$$

$$= \{1, 3, 4, 5, 7, 9, 10\}$$

$$A \cup C = \{1, 3, 5, 7, 9\} \cup \{1, 5, 8, 10\}$$

$$= \{1, 3, 5, 7, 8, 9, 10\}$$

$$\text{R.H.S} = (A \cup B) \cap (A \cup C)$$

$$= \{1, 3, 4, 5, 7, 9, 10\} \cap \{1, 3, 5, 7, 8, 9, 10\}$$

$$= \{1, 3, 5, 7, 9, 10\} \rightarrow (ii)$$

From (i) and (ii), we get

$$\text{L.H.S} = \text{R.H.S}$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Hence Proved

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(K.B + A.B)

Proof

$$B \cup C = \{1, 4, 5, 7, 8, 10\} \cup \{1, 5, 8, 10\}$$

$$= \{1, 4, 5, 7, 8, 10\}$$

$$\text{L.H.S} = A \cap (B \cup C)$$

$$= \{1, 3, 5, 7, 9\} \cap \{1, 4, 5, 7, 8, 10\}$$

$$= \{1, 5, 7\} \rightarrow (i)$$

$$A \cap B = \{1, 3, 5, 7, 9\} \cap \{1, 4, 7, 10\}$$

$$= \{1, 7\}$$

$$A \cap C = \{1, 3, 5, 7, 9\} \cap \{1, 5, 8, 10\}$$

$$= \{1, 5\}$$

$$\text{R.H.S} = (A \cap B) \cup (A \cap C)$$

$$= \{1, 7\} \cup \{1, 5\}$$

$$= \{1, 5, 7\} \rightarrow (ii)$$

From (i) and (ii), we get

$$L.H.S = R.H.S$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Hence Proved

Q.3 Given $U = N$, $A = \phi$, $B = P$

(K.B + A.B)

(LHR 2016, GRV 2016, RWT 2015, 17, MTN 2016)

To prove:

De-Morgan's Laws

$$(i) \quad (A \cup B)' = A' \cap B'$$

$$(ii) \quad (A \cap B)' = A' \cup B'$$

$$(i) \quad (A \cup B)' = A' \cap B'$$

Proof

$$L.H.S = (A \cup B)'$$

$$A \cup B = \phi \cup P$$

$$= P$$

$$L.H.S = (A \cup B)' = U - (A \cup B)$$

$$= N - P \rightarrow (i)$$

$$R.H.S = A' \cap B'$$

$$A' = U - A = N - \phi$$

$$= N$$

$$B' = U - B = N - P$$

$$R.H.S = A' \cap B' = N \cap (N - P)$$

$$= N - P \rightarrow (ii)$$

From equation (i) and (ii)

$$L.H.S = R.H.S$$

$$(A \cup B)' = A' \cap B'$$

Hence Proved

$$(ii) \quad (A \cap B)' = A' \cup B' \quad \text{(K.E + A.B)}$$

Proof

$$A \cap B = \phi \cap P$$

$$= \phi$$

$$L.H.S = (A \cap B)'$$

$$= U - (A \cap B)$$

$$= N - \phi$$

$$= N$$

$$R.H.S = A' \cup B'$$

$$= N \cup (N - P)$$

$$= N \rightarrow (ii)$$

From equation (i) and (ii)

$$L.H.S = R.H.S$$

$$(A \cap B)' = A' \cup B'$$

Hence Proved

Method-(II)

$$(i) \quad (A \cup B)' = A' \cap B' \quad \text{(K.B + A.B)}$$

Proof

$$A \cup B = \{ \} \cup \{2, 3, 5, 7, \dots\}$$

$$= \{2, 3, 5, 7, \dots\}$$

$$L.H.S = (A \cup B)' = U - (A \cup B)$$

$$= \{1, 2, 3, \dots\} - \{2, 3, 5, 7, \dots\}$$

$$= \{1, 4, 6, 8, 9, 10, 12, \dots\} \rightarrow (i)$$

$$A' = U - A$$

$$= \{1, 2, 3, \dots\} - \{ \}$$

$$= \{1, 2, 3, 4, \dots\}$$

$$B' = U - B$$

$$= \{1, 2, 3, 4, \dots\} - \{2, 3, 5, 7, \dots\}$$

$$= \{1, 4, 6, 8, 9, 10, \dots\}$$

Now

$$R.H.S = A' \cap B' = \{1, 2, 3, \dots\} \cap \{1, 4, 6, 8, 9, 10, \dots\}$$

$$= \{1, 4, 6, 8, 9, 10, \dots\} \rightarrow (ii)$$

From equation (i) and (ii)

$$L.H.S = R.H.S$$

$$(A \cup B)' = A' \cap B'$$

Hence Proved

Q.4 If $U = \{1, 2, 3, \dots, 10\}$ **(K.B + A.B)**

$$A = \{1, 3, 5, 7, 9\}$$

$$B = \{2, 3, 4, 5, 8\}$$

then prove the following question by venn diagram.

$$(i) \quad A - B = A \cap B'$$

$$(ii) \quad B - A = B \cap A'$$

$$(iii) \quad (A \cup B)' = A' \cap B'$$

$$(iv) \quad (A \cap B)' = A' \cup B'$$

(v) $(A - B)' = A' \cup B$

(vi) $(B - A)' = B' \cup A$
(Through Venn diagram)

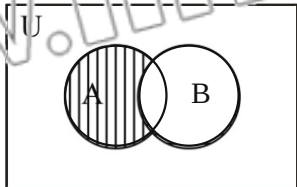
(i) $A - B = A \cap B'$ **(K.B)**

Proof

A and B are overlapping sets because

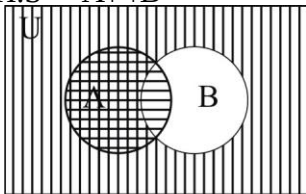
$$A \cap B = \{3, 5\}$$

$$\text{L.H.S} = A - B$$



$$B - A \equiv$$

$$\text{R.H.S} = A \cap B'$$



$$A \equiv$$

$$B' \equiv$$

$$A \cap B' \equiv$$

Same regions represent that

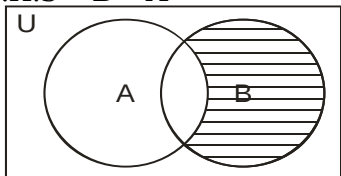
$$\text{L.H.S} = \text{R.H.S}$$

$$A - B = A \cap B'$$

Hence Proved

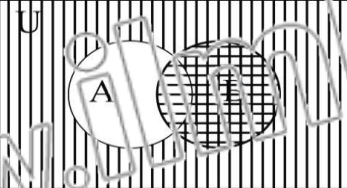
(ii) $B - A = B \cap A'$ **(K.B)**

$$\text{L.H.S} = B - A$$



$$B - A \equiv$$

$$\text{R.H.S} = B \cap A'$$



$$B \equiv$$

$$A' \equiv$$

$$B \cap A' \equiv$$

Same regions represent that

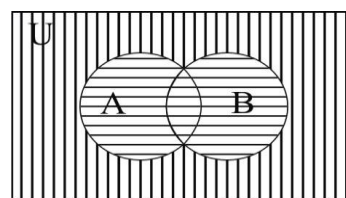
$$\text{L.H.S} = \text{R.H.S}$$

$$B - A = B \cap A'$$

Hence Proved

(iii) $(A \cup B)' = A' \cap B'$ **(K.B)**

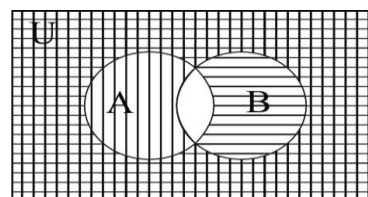
$$\text{L.H.S} = (A \cup B)'$$



$$A \cup B \equiv$$

$$(A \cup B)' \equiv$$

$$\text{R.H.S} = A' \cap B'$$



$$A' \equiv$$

$$B' \equiv$$

$$A' \cap B' \equiv$$

Same regions represent that

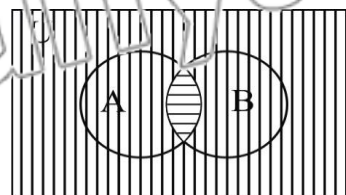
$$\text{L.H.S} = \text{R.H.S}$$

$$(A \cup B)' = A' \cap B'$$

Hence Proved

(iv) $(A \cap B)' = A' \cup B'$ **(K.B)**

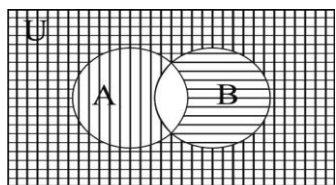
$$\text{L.H.S} = (A \cap B)'$$



$$A \cap B \equiv$$

$$(A \cap B)' \equiv$$

$$\text{R.H.S} = A' \cup B'$$



$$A' \equiv \text{|||||}$$

$$B' \equiv \text{|||||}$$

$$A' \cup B' \equiv \text{|||||}$$

Same regions represent that

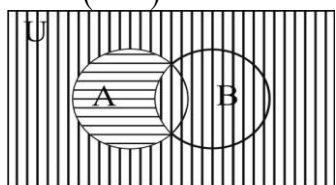
$$\text{L.H.S} = \text{R.H.S}$$

$$(A - B)' = A' \cup B'$$

Hence Proved

(v) $(A - B)' = A' \cup B$ (K.B)

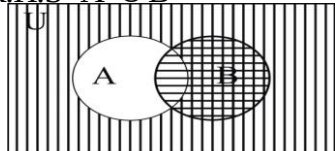
$$\text{L.H.S} = (A - B)'$$



$$A - B \equiv \text{|||||}$$

$$(A - B)' \equiv \text{|||||}$$

$$\text{R.H.S} = A' \cup B$$



$$A' \equiv \text{|||||}$$

$$B \equiv \text{|||||}$$

$$A' \cup B \equiv \text{|||||}$$

Same regions represent that

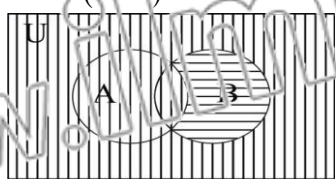
$$\text{L.H.S} = \text{R.H.S}$$

$$(A - B)' = A' \cup B$$

Hence Proved

(vi) $(B - A)' = B' \cup A$ (K.B)

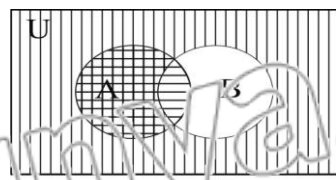
$$\text{L.H.S} = (B - A)'$$



$$B - A \equiv \text{|||||}$$

$$(B - A)' \equiv \text{|||||}$$

$$\text{R.H.S} = B' \cup A$$



$$A \equiv \text{|||||}$$

$$B' \equiv \text{|||||}$$

$$B' \cup A \equiv \text{|||||}$$

Same regions represent that

$$\text{L.H.S} = \text{R.H.S}$$

$$(B - A)' = B' \cup A$$

Hence Proved

Ordered Pairs (K.B)

Any two numbers x and y , written in the form (x, y) is called an ordered pair. In an ordered pair (x, y) , the order of numbers is important.

For example: $(3, 2)$ is different from $(2, 3)$.

Hence $(x, y) \neq (y, x)$ unless $x = y$.

Cartesian Product (K.B)

(FSD 2015, D.G.K 2015, 17)

Cartesian product of two non-empty sets A and B denoted by $A \times B$ consists of all the ordered pairs (x, y) such that $x \in A$ and $y \in B$

$$\text{i.e. } A \times B = \{(x, y) | x \in A \wedge y \in B\}$$

For example:

$$\text{If } A = \{1, 2\}, B = \{3, 4\}$$

$$\text{Then } A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

Exercise 5.4

Q.1 Given $A = \{a, b\}$ (GRW 2014) (A.E)

$$B = \{c, d\} \quad (\text{RWP 2015})$$

To Find

(i) $A \times B$

(ii) $B \times A$

Solution:

$$\begin{aligned} \text{(i)} \quad A \times B &= \{a, b\} \times \{c, d\} \\ &= \{(a, c), (a, d), (b, c), (b, d)\} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad B \times A &= \{c, d\} \times \{a, b\} \\ &= \{(c, a), (c, b), (d, a), (d, b)\} \end{aligned}$$

Q.2 Given $A = \{0, 2, 4\}$ **(A.B)**

$$B = \{-1, 3\}$$

(FSD 2015, SWL 2017, BWP 2015)

To Find

$$A \times B \quad B \times A \quad A \times A \quad B \times B$$

Solution:

$$(i) \quad A \times B = \{0, 2, 4\} \times \{-1, 3\} \\ = \{(0, -1), (0, 3), (2, -1), (2, 3), (4, -1), (4, 3)\}$$

$$(ii) \quad B \times A = \{-1, 3\} \times \{0, 2, 4\} \\ = \{(-1, 0), (-1, 2), (-1, 4), (3, 0), (3, 2), (3, 4)\}$$

$$(iii) \quad A \times A = \{0, 2, 4\} \times \{0, 2, 4\} \\ = \{(0, 0), (0, 2), (0, 4), (2, 0), (2, 2), (2, 4), (4, 0), (4, 2), (4, 4)\}$$

$$(iv) \quad B \times B = \{-1, 3\} \times \{-1, 3\} \\ = \{(-1, -1), (-1, 3), (3, -1), (3, 3)\}$$

Q.3 **(A.B)**

(i) Given $(a - 4, b - 2) = (2, 1)$

(GRW 2016, 17, FSD 2017, SWL 2015, SGD 2017, MTN 2016)

Required

Values of a and b

Solution:

Given that

$$(a - 4, b - 2) = (2, 1)$$

By comparing, we get

$$a - 4 = 2 \quad \text{and} \quad b - 2 = 1$$

$$a = 2 + 4 \quad b = 1 + 2$$

$$\Rightarrow a = 6, \quad b = 3$$

(ii) Given $(2a + 5, 3) = (7, b - 4)$ **(A.B)**

(SWL 2017, MTN 2017, RWP 2016, D.G.K 2015)

Required

Values of a and b

Solution:

Given that

$$(2a + 5, 3) = (7, b - 4)$$

By comparing, we get

$$2a + 5 = 7 \quad \text{and} \quad 3 = b - 4$$

$$2a = 7 - 5 \quad 3 + 4 = b$$

$$2a = 2 \quad 7 = b$$

$$a = 1 \quad b = 7$$

(iii) Given $(3 - 2a, b - 1) = (a - 7, 2b + 5)$

(GRW 2014) **(A.B)**

Required

Values of a and $b = ?$

Solution:

Given that

$$(3 - 2a, b - 1) = (a - 7, 2b + 5)$$

By comparing, we get

$$3 - 2a = a - 7 \quad \text{and} \quad b - 1 = 2b + 5$$

$$-2a - a = -7 - 3 \quad b - 2b = 5 + 1$$

$$-3a = -10 \quad -b = 6$$

$$a = \frac{-10}{-3} \quad b = -6$$

$$\Rightarrow a = \frac{10}{3}$$

Q.4 **Given**

$$X \times Y = \{(a, a), (b, a), (c, a), (d, a)\}$$

(A.B)

Required

Set X and Y

Solution:

Given that

$$X \times Y = \{(a, a), (b, a), (c, a), (d, a)\}$$

$$X = \{a, b, c, d\}$$

$$Y = \{a\}$$

Q.5 **Given** $X = \{a, b, c\}$ **(A.B)**

$$Y = \{d, e\}$$

Required

Number of elements in

$$(i) \quad X \times Y$$

$$(ii) \quad Y \times X$$

$$(iii) \quad X \times X$$

Solution:

$$(i) \quad X \times Y$$

$$n(X) = 3$$

$$n(Y) = 2$$

$$n(X \times Y) = n(X) \times n(Y)$$

$$= 3 \times 2$$

$$= 6$$

Method-2

(K.B)

(i) Number of elements in $X \times Y = m \times n$

$$= 3 \times 2$$

$$= 6$$

(ii) Number of elements in $Y \times X = m \times n$

$$= 2 \times 3$$

- (iii) Number of elements in $X \times X = m \times n$
 $= 3 \times 3$
 $= 9$

Binary Relation**(K.B)**

(GRW 2017, RWP 2016, SWL 2016, MTN 2015)
 If A and B are any two non-empty sets then a subset $R \subseteq A \times B$ is called a binary relation from set A into set B.

i.e. $A = \{1, 2\}$, $B = \{a\}$

Then $A \times B = \{(1, a), (2, a)\}$ and

$R_1 = \{(1, a)\}$, $R_2 = \{(2, a)\}$

$R_3 = \{(1, a), (2, a)\}$, $R_4 = \phi$

Are all possible relations.

Note**(K.B + U.B)**

Formula to find number of binary relations is $2^{m \times n}$ where m = number of elements in set A and n = number of elements in set B.

Domain of Relation**(K.B)**

Domain of a relation denoted by $\text{Dom}(R)$ is a set consisting of all the first elements of each ordered pair in the relation.

For example:

If $R = \{(0, 2), (2, 3), (3, 3), (3, 4)\}$

Then $\text{Dom}(R) = \{0, 2, 3\}$

Range of a Relation**(K.B)**

Range of a relation is a set containing all the second elements of each ordered pair of the relation.

For Example:

If $R = \{(0, 2), (2, 3), (3, 3), (3, 4)\}$

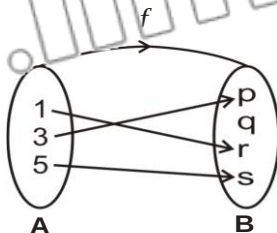
Then $\text{Range}(R) = \{2, 3, 4\}$

Function**(K.B + U.B + A.B)****(LHR 2014)**

Suppose A and B are two non-empty sets, then relation $f : A \rightarrow B$ is called a function if

- (i) $\text{Dom}(f) = A$
- (ii) Every $x \in A$ appears in one and only one ordered pair in f .

Example:

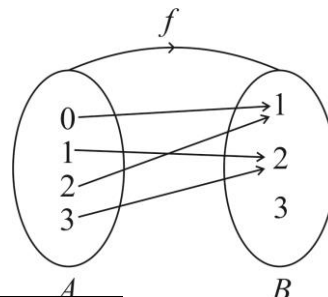
**Types of Function****(K.B + U.B)****Into Function****(A.B)**

A function $f : A \rightarrow B$ is called an into function, if at least one element in set B is not an image of some element of set A i.e., $\text{Range of } f \subset B$.

For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3\}$, then $f : A \rightarrow B$ such that

$f = \{(0, 1), (1, 2), (2, 1), (3, 2)\}$.

$\therefore \text{Range}(f) = \{1, 2\} \subset B$. Thus f is an into function.

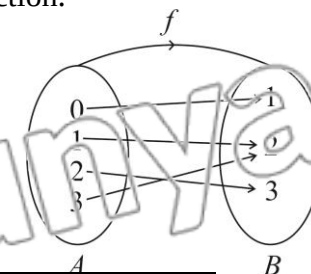
**Onto Function****(K.B + U.B + A.B)****(SWL 2016, BWP 2014, 16, 17)**

A function $f : A \rightarrow B$ is called an onto function, if every element of set B is an image of at least one element of set A i.e., $\text{Range of } f = B$.

For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3\}$, then $f : A \rightarrow B$ such that

$f = \{(0, 1), (1, 2), (2, 3), (3, 2)\}$.

$\therefore \text{Range}(f) = \{1, 2, 3\} = B$. Thus f is an onto function.

**One-to-One Function****(LHR 2015) (K.B + U.B + A.B)**

A function $f : A \rightarrow B$ is called one-one function if all distinct elements of A have distinct images in B, i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \in A$

or $x_1 \neq x_2 \in A \forall x_1 \in A \Rightarrow f(x_1) \neq f(x_2)$

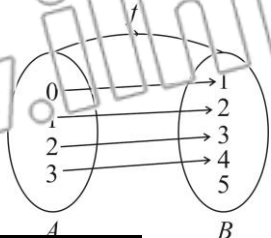
For example, if $A = \{0, 1, 2, 3\}$ and

$B = \{1, 2, 3, 4, 5\}$, then we define a function

$f: A \rightarrow B$ such that

$$f = \{(0, 1), (1, 2), (2, 3), (3, 4)\}$$

f is one-one function because no element in B is repeated.



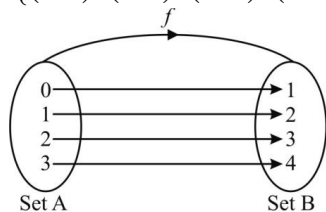
Bijjective Function

(GRW 2014) **(K.B + U.B + A.B)**

A function $f: A \rightarrow B$ is called bijective function iff function f is one-one and onto.

For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$

Then $f = \{(0, 1), (1, 2), (2, 3), (3, 4)\}$



Note

(K.B)

- Every function is ration but converse may not be true.
- Every function may not be one-one.
- Every function may not be onto.

Exercise 5.5

Q.1 Given (LHR 2014) (A.B)

$$L = \{a, b, c\}$$

$$M = \{3, 4\}$$

To Find

Two binary relations of $L \times M$ and $M \times L$

Solution:

$$L \times M = \{a, b, c\} \times \{3, 4\}$$

$$= \{(a, 3), (a, 4), (b, 3), (b, 4), (c, 3), (c, 4)\}$$

Two binary relations of $L \times M$ are:

$$R_1 = \{(a, 3), (a, 4)\}$$

$$R_2 = \{(a, 3), (b, 3), (c, 4)\}$$

$$\text{Now } M \times L = \{3, 4\} \times \{a, b, c\}$$

$$= \{(3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}$$

Two binary relations of $M \times L$ are:

$$R_1 = \{(3, a)\}$$

$$R_2 = \{(3, a), (4, b)\}$$

Q.2 Given

(A.B)

(LHR 2015, GRW 2014)

$$Y = \{-2, 1, 2\}$$

Required:

Two binary relations of $Y \times Y$ and also find their domain and range.

Solution:

$$Y \times Y = \{-2, 1, 2\} \times \{-2, 1, 2\}$$

$$= \{(-2, -2), (-2, 1), (-2, 2), (1, -2),$$

$$(1, 1), (1, 2), (2, -2), (2, 1), (2, 2)\}$$

Two Binary relations for $Y \times Y$ are:

$$R_1 = \{(-2, -2), (-2, 1), (-2, 2)\}$$

$$R_2 = \{(-2, -2), (1, 1)\}$$

Domain and Range

$$\text{Dom}(R_1) = \{-2\}$$

$$\text{Range}(R_1) = \{-2, 1, 2\}$$

$$\text{Dom}(R_2) = \{-2, 1\}$$

$$\text{Range}(R_2) = \{-2, 1\}$$

Q.3 Given

(A.B)

$$L = \{a, b, c\}$$

$$M = \{d, e, f, g\}$$

Required

Two binary relation for

(i) $L \times L$ (ii) $L \times M$ (iii) $M \times M$

Solution:

$$L \times L = \{a, b, c\} \times \{a, b, c\}$$

$$= \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

$$L \times M = \{a, b, c\} \times \{d, e, f, g\}$$

$$= \{(a, d), (a, e), (a, f), (a, g), (b, d), (b, e),$$

$$(b, f), (b, g), (c, d), (c, e), (c, f), (c, g)\}$$

$$M \times M = \{d, e, f, g\} \times \{d, e, f, g\}$$

$$= \{(d,d), (d,e), (d,f), (d,g), (e,d), (e,e), (e,f), (e,g), \\ (f,d), (f,e), (f,f), (f,g), (g,d), (g,e), (g,f), (g,g)\}$$

Two binary relations for $L \times L$ are:

$$R_1 = \{(a,a), (b,b), (c,c)\}$$

$$R_2 = \{(a,b)\}$$

Two Binary Relations for $L \times M$ are:

$$R_1 = \{(a,d), (a,e)\}$$

$$R_2 = \{(a,c), (b,e), (c,f)\}$$

Two Binary Relations for $M \times M$ are:

$$R_1 = \{(d,d)\}$$

$$R_2 = \{(d,e), (f,g)\}$$

Q.4 If set M has 5 elements, then find the number of binary relations in M .

(A.B)

Solution:

$$\begin{aligned} \text{No of binary relations in } M &= 2^{mn} \\ &= 2^{5 \times 5} \\ &= 2^{25} \end{aligned}$$

Q.5 Given $L = \{x \mid x \in \mathbb{N} \wedge x \leq 5\}$

$$M = \{y \mid y \in \mathbb{P} \wedge y < 10\}$$

(A.B + K.B)

To Find

(i) $R_1 = \{(x,y) \mid y < x\}$

(ii) $R_2 = \{(x,y) \mid y = x\}$

(iii) $R_3 = \{(x,y) \mid x + y = 6\}$

(iv) $R_4 = \{(x,y) \mid y - x = 2\}$

Also find the domain and range of each relation.

Solution:

Here

$$L = \{x \mid x \in \mathbb{N} \wedge x \leq 5\} = \{1, 2, 3, 4, 5\}$$

$$\text{And } M = \{y \mid y \in \mathbb{P} \wedge y < 10\} = \{2, 3, 5, 7\}$$

$$L \times M = \{1, 2, 3, 4, 5\} \times \{2, 3, 5, 7\}$$

$$\begin{aligned} &= \{(1,2), (1,3), (1,5), (1,7), (2,2), (2,3), (2,5), \\ &\quad (2,7), (3,2), (3,3), (3,5), (3,7), (4,2), (4,3), \\ &\quad (4,5), (4,7), (5,2), (5,3), (5,5), (5,7)\} \end{aligned}$$

(i) Relation (A.B + K.B)

$$R_1 = \{(x,y) \mid y < x\}$$

$$= \{(3,2), (4,2), (4,3), (5,2), (5,3)\}$$

Domain and Range

$$\text{Dom } (R_1) = \{3, 4, 5\}$$

$$\text{Range } (R_1) = \{2, 3\}$$

(ii) Relation (A.B + K.B)

$$R_2 = \{(x,y) \mid y = x\}$$

$$= \{(2,2), (3,3), (5,5)\}$$

Domain and Range

$$\text{Dom } (R_2) = \{2, 3, 5\}$$

$$\text{Range } (R_2) = \{2, 3, 5\}$$

(iii) Relation (A.B + K.B)

$$R_3 = \{(x,y) \mid x + y = 6\}$$

$$= \{(1,5), (3,3), (4,2)\}$$

Domain and Range of R_3

$$\text{Dom } (R_3) = \{1, 3, 4\}$$

$$\text{Range } (R_3) = \{2, 3, 5\}$$

(iv) Relation (A.B + K.B)

$$R_4 = \{(x,y) \mid y - x = 2\}$$

$$= \{(1,3), (3,5), (5,7)\}$$

Domain and Range of R_4

$$\text{Dom } (R_4) = \{1, 3, 5\}$$

$$\text{Range } (R_4) = \{3, 5, 7\}$$

Q.6 Indicate relations, into function, one-one function, onto function and bijective function from the following. Also find their domain and the range. (In each part (i) $\rightarrow$ (vi) the co-domain set is equal to the range set)

(A.B + K.B + U.B)

(i) $R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$

(ii) $R_2 = \{(1,2), (2,1), (3,4), (3,5)\}$

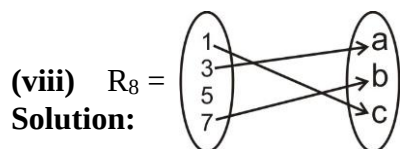
(iii) $R_3 = \{(b,a), (c,a), (d,a)\}$

(iv) $R_4 = \{(1,1), (2,3), (3,4), (4,3), (5,4)\}$

(v) $R_5 = \{(a,b), (b,a), (c,d), (d,e)\}$

(vi) $R_6 = \{(1,2), (2,3), (1,3), (3,4)\}$

(vii) $R_7 =$



Solution:

(i) $R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$

- $\text{Dom}(R_1) = \{1, 2, 3, 4\}$

There is no repetition in 1st element of any two ordered pairs, so R_1 is a function.

- $\text{Range}(R_1) = \{1, 2, 3, 4\}$

Also there is no repetition in 2nd element of any two ordered pairs. So R_1 is bijective function.

(ii) $R_2 = \{(1,2), (2,1), (3,4), (3,5)\}$

- $\text{Dom}(R_2) = \{1, 2, 3\}$

There is a repetition in first element of last two ordered pairs. So R_2 is not a function.

(iii) $R_3 = \{(b,a), (c,a), (d,a)\}$

- $\text{Dom}(R_3) = \{b, c, d\}$

There is no repetition in 1st element of any two pairs, so R_3 is a function.

- $\text{Range}(R_3) = \{a\}$

There is a repetition in second element of all ordered pairs.

So R_3 is on to function.

(iv) $R_4 = \{(1,1), (2,3), (3,4), (4,3), (5,4)\}$

- $\text{Dom}(R_4) = \{1, 2, 3, 4, 5\}$

There is no repetition in 1st element of any two pairs, so R_4 is a function.

- $\text{Range}(R_4) = \{1, 3, 4\}$

There is a repetition in second element of all ordered pairs.

So R_4 is on to function.

(v) $R_5 = \{(a,b), (b,a), (c,d), (d,e)\}$

- $\text{Dom}(R_5) = \{a, b, c, d\}$

There is no repetition in 1st element of any two ordered pairs, so R_5 is a function.

- $\text{Range}(R_5) = \{a, b, d, e\}$

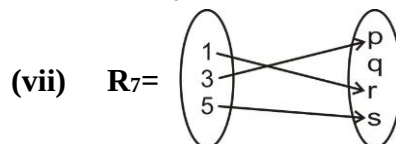
Also there is no repetition in 2nd element of any two ordered pairs. So R_5 is bijective function.

(vi) $R_6 = \{(1,2), (2,3), (1,3), (3,4)\}$

- $\text{Dom}(R_6) = \{1, 2, 3\}$

There is a repetition in first element of two ordered pairs.

So R_6 is not a function.



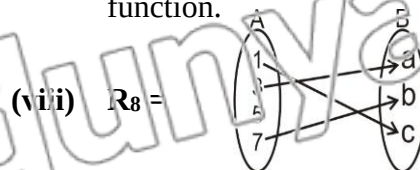
- $\text{Dom}(R_7) = \{1, 3, 5\} = A$

Also there is no repetition in 1st element of any two ordered pairs. So R_7 is a function.

- $\text{Range}(R_7) = \{p, r, s\} \subset B$

Also there is no repetition in 2nd element of any two ordered pairs.

So R_7 is one-to-one (injective) function.



- $\text{Dom}(R_8) = \{1, 3, 7\} \neq A$

Therefore R_8 is not a function.

Miscellaneous Exercise 5

Q.1 Multiple choice questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

- (1) A collection of well-defined distinct objects is called; **(K.B)**
 (a) Subset (b) Power set
 (c) Set (d) None of these
- (2) A set $Q = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \wedge b \neq 0 \right\}$ is called a set of; **(K.B)**
 (a) Whole numbers (b) Natural numbers
 (c) Irrational numbers (d) Rational numbers
- (3) The different number of ways to describe a set are; **(K.B)**
 (a) 1 (b) 2
 (c) 3 (d) 4
- (4) A set with no element is called; **(K.B)**
 (a) Subset (b) Empty set
 (c) Singleton set (d) Super set
- (5) The set $\{x \mid x \in W \wedge x \leq 101\}$ is; **(K.B)**
 (a) Infinite set (b) Subset
 (c) Null set (d) Finite set
- (6) The set having only one element is called; **(K.B)**
 (a) Null set (b) Power set
 (c) Singleton set (d) Subset
- (7) Power set of an empty set is; **(K.B)**
 (a) ϕ (b) $\{a\}$
 (c) $\{\phi, \{a\}\}$ (d) $\{\phi\}$
- (8) The number of elements in power set $\{1, 2, 3\}$ is; **(K.B)**
 (a) 4 (b) 6
 (c) 8 (d) 9
- (9) If $A \subseteq B$, then $A \cup B$ is equal to; **(K.B)**
 (a) A (b) B
 (c) ϕ (d) None of these
- (10) If $A \subseteq B$ and $A \cap B$ is equal to; **(K.B)**
 (a) A (b) B
 (c) ϕ (d) None of these
- (11) If $A \subseteq B$ and $A - B$ is equal to, **(K.B)**
 (a) A (b) B
 (c) ϕ (d) $B - A$
- (12) $(A \cup B) \cup C$ is equal to; **(K.B)**
 (a) $A \cap (B \cup C)$ (b) $(A \cup B) \cap C$
 (c) $A \cup (B \cap C)$ (d) $A \cap (B \cap C)$

- (13) $A \cup (B \cap C)$ is equal to; **(K.B)**
 (a) $(A \cup B) \cap (A \cup C)$ (b) $(A \cap B) \cap C$
 (c) $(A \cap B) \cup (A \cap C)$ (d) $A \cap (B \cap C)$
- (14) If A and B are disjoint sets, then $A \cup B$ is equal to; **(K.B)**
 (a) A (b) B
 (c) ϕ (d) $B \cup A$
- (15) If number of elements in set A is 5 and in set B is 4, then number of elements in $A \times B$ is; **(K.B)**
 (a) 3 (b) 4
 (c) 12 (d) 7
- (16) If number of elements in set A is 3 and in set B is 2, then number of binary relations in $A \times B$ is; **(K.B)**
 (a) 2^3 (b) 2^6
 (c) 2^8 (d) 2^2
- (17) The domain of $R = \{(0, 2), (2, 3), (3, 3), (3, 4)\}$ is; **(K.B)**
 (a) $\{0, 3, 4\}$ (b) $\{0, 2, 3\}$
 (c) $\{0, 2, 4\}$ (d) $\{2, 3, 4\}$
- (18) The range of $R = \{(1, 3), (2, 2), (3, 1), (4, 4)\}$ is; **(K.B)**
 (a) $\{1, 2, 4\}$ (b) $\{3, 2, 4\}$
 (c) $\{1, 2, 3, 4\}$ (d) $\{1, 3, 4\}$
- (19) point $(-1, 4)$ lies in the quadrant; **(K.B)**
 (a) I (b) II
 (c) III (d) IV
- (20) the relation $\{(1, 2), (2, 3), (3, 3), (3, 4)\}$ is; **(K.B)**
 (a) Onto function (b) Into function
 (c) Not a function (d) One-One function

ANSWER KEY

i	c	vi	c	xi	c	xvi	b
ii	d	vii	a	xii	c	xvii	b
iii	c	viii	c	xiii	a	xviii	c
iv	b	ix	b	xiv	d	xix	b
v	d	x	a	xv	c	xx	c

Q.2 Write short answers of the following questions.

- (i) Define a subset and give one example.

(K.B + U.B)**Answer****Subset**

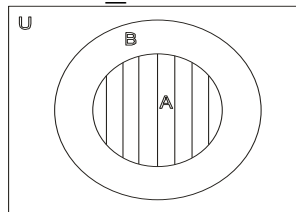
If A and B are two sets such that every element of set A is an element of set B , then set A is called subset of set B .

For example $A = \{1, 3, 5, 7\}$ and $B = \{1, 2, 3, \dots, 10\}$ then A is subset of B and represented by $A \subseteq B$.

- (ii) Write all the subsets of the set
- $\{a, b\}$

AnswerLet $S = \{a, b\}$ All possible subset of set S are: $\phi, \{a\}, \{b\}, \{a, b\}$

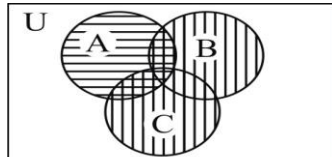
- (iii) Show
- $A \cap B$
- by Venn diagram. When

 $A \subseteq B$. (MTN 2014, SGD 2016) **(K.B)****Answer**If $A \subseteq B$  $A \cap B$ □

- (iv) Show by Venn diagram
- $A \cap (B \cup C)$

(K.B + A.B)**Answer**

Let A , B and C are overlapping (General Case)

 $A \cap (B \cup C)$ ■■ $B \cup C$ ■■ ■■ $A \cap (B \cup C)$ ■■ ■■

- (v) Define intersection of two sets

(K.B)**Answer****Intersection of Two Sets****(K.B)**

The intersection of two sets A and B , written as $A \cap B$ (read as 'A intersection B') is the

set consisting of all the common elements of A and B . Thus

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Clearly $x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$

For example, if $A = \{a, b, c, d\}$ and $B = \{c, d, e, f\}$, then $A \cap B = \{c, d\}$

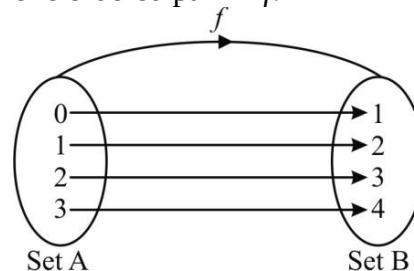
- (vi) Define a function

Answer**Function****(K.B)**

Suppose A and B are two non-empty sets, then relation $f : A \rightarrow B$ is called a function if

- (i)
- $\text{Dom } f = A$

- (ii) Every
- $x \in A$
- appears in one and only one ordered pair in
- f
- .



- (vii) Define one-one function

Answer**One - One Function****(K.B)**

A function $f : A \rightarrow B$ is called one-one function if all distinct elements of A have distinct images in B , i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \in A$ or $x_1 \neq x_2 \in A \Rightarrow f(x_1) \neq f(x_2)$.

For example, if $A = \{0, 1, 2, 3\}$ and

$B = \{1, 2, 3, 4, 5\}$, then we define a function

$f : A \rightarrow B$ such that

$$f = \{(0, 1), (1, 2), (2, 3), (3, 4)\}$$

f is one-one function because no element in B is repeated.

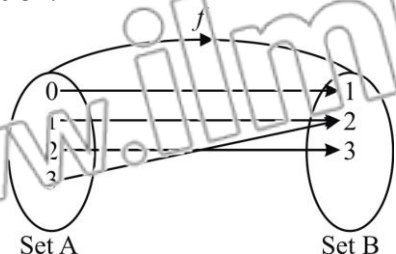
- (viii) Define an Onto function or Surjective function.

Answer**Onto (Surjective) Function****(K.B)**

A function $f : A \rightarrow B$ is called an onto function, if every element of set B is an image of at least one element of set A i.e., Range of $f = B$.

For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3\}$, then $f : A \rightarrow B$ such that

$f = \{(0, 1), (1, 2), (2, 3), (3, 2)\}$. Here Range $f = \{1, 2, 3\} = B$. Thus f so defined is an onto function.



(ix) Define a Bijective function

Answer

Bijective Function

(K.B)

A function $f : A \rightarrow B$ is called bijective function iff function f is one-one and onto.

For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$

Then $f = \{(0, 1), (1, 2), (2, 3), (3, 4)\}$

Evidently this function is one-one because distinct elements of A have distinct images

Q.3 Fill in the blanks

(i) If $A \subseteq B$, then $A \cup B = \underline{\hspace{2cm}}$.

(K.B + A.B)

(ii) If $A \cap B = \phi$ then A and B are ____.

(K.B + A.B)

(iii) If $A \subseteq B$ and $B \subseteq A$ then ____.

(K.B + A.B)

(iv) $A \cap (B \cup C) = \underline{\hspace{2cm}}$.

(K.B + A.B)

(v) $A \cup (B \cap C) = \underline{\hspace{2cm}}$.

(K.B + A.B)

(vi) The complement of U is ____.

(K.B + A.B)

(vii) The complement of ϕ is ____.

(K.B + A.B)

(viii) $A \cap A = \underline{\hspace{2cm}}$.

(K.B + A.B)

(ix) $A \cup A^c = \underline{\hspace{2cm}}$.

(K.B + A.B)

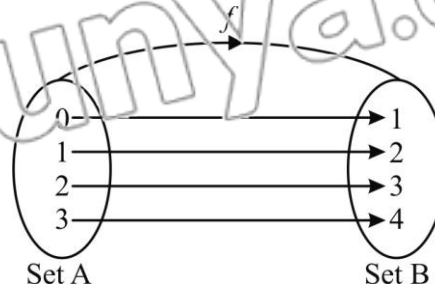
(x) The set $\{x | x \in A \text{ and } x \notin B\} = \underline{\hspace{2cm}}$.

(K.B + A.B)

(xi) The point $(-5, -7)$ lies in ____ quadrant.

(K.B + A.B)

in B . This is an onto function also because every element of B is the image of at least one element of A .



(x) Write De Morgan's Laws. (K.B)

Answer

De Morgan's Laws

For any two sets A and B belonging to universal set U ,

(i) $(A \cap B)' = A' \cup B'$

(ii) $(A \cup B)' = A' \cap B'$ are called De Morgan's laws.

- (xii) The point $(4, -6)$ lies in _____ quadrant. **(K.B + A.B)**
- (xiii) The y co-ordinate of every point is _____ on-x-axis. **(K.B + A.B)**
- (xiv) The x co-ordinate of every point is _____ on-y-axis. **(K.B + A.B)**
- (xv) The domain of $\{(a, b), (b, c), (c, d)\}$ is _____. **(K.B + A.B)**
- (xvi) The range of $\{(a, a), (b, b), (c, c)\}$ is _____. **(K.B + A.B)**
- (xvii) Venn-diagram was first used by _____. **(K.B + A.B)**
- (xviii) A subset of $A \times A$ is called the _____ in A. **(K.B + A.B)**
- (xix) If $f : A \rightarrow B$ and range of $f = B$, then f is an _____ function. **(K.B + A.B)**
- (xx) The relation $\{(a, b), (b, c), (a, d)\}$ is _____ a function. **(K.B + A.B)**

ANSWER KEY

- | | |
|-----------------------------------|-------------------------|
| (i) B | (xiii) Zero |
| (ii) Disjoint sets | (xiv) Zero |
| (iii) $A = B$ | (xv) $\{a, b, c\}$ |
| (iv) $(A \cap B) \cup (A \cap C)$ | (xvi) $\{a, b, c\}$ |
| (v) $(A \cup B) \cap (A \cup C)$ | (xvii) John Venn |
| (vi) ϕ | (xviii) Binary relation |
| (vii) U | (xix) Onto |
| (viii) ϕ | (xx) Not |
| (ix) U | |
| (x) $A - B$ | |
| (xi) III rd quadrant | |
| (xii) IV th quadrant | |



CUT HERE

SELF TEST

Time: 40 min

Marks: 25

Q.1 Four possible answers (A), (B), (C) & (D) to each question are given, mark the correct answer. (7×1=7)

1 Power set of an empty set is:

- (A) ϕ (B) $\{a\}$
 (C) $\{0, \{a\}\}$ (D) $\{\phi\}$

2 $(A \cup B) \cap C$ is equal to:

- (A) $A \cap (B \cup C)$ (B) $(A \cup B) \cap C$
 (C) $A \cup (B \cap C)$ (D) $A \cap (B \cap C)$

3 The number of ways to describe a set are:

- (A) 1 (B) 2
 (C) 3 (D) 4

4 If $A \subseteq B$, then $A \cup B$ is equal to:

- (A) A (B) B
 (C) ϕ (D) U

5 The domain of $R = \{(0, 2), (2, 3), (3, 3), (3, 4)\}$ is:

- (A) $\{0, 3, 4\}$ (B) $\{0, 2, 3\}$
 (C) $\{0, 2, 4\}$ (D) $\{2, 3, 4\}$

6 Point $(-1, 4)$ lies in the quadrant:

- (A) I (B) II
 (C) III (D) IV

7 If the number of elements in set A is 3 and in set B is 4, then the number of elements in $A \times B$ is:

- (A) 3 (B) 4
 (C) 12 (D) 7

Q.2 Give Short Answers to following Questions.

(5×2=10)

(i) Define bijective function.

(ii) Show $A \cap B$ by Venn diagram when $A \subseteq B$

(iii) Find "a" and "b" if $(3 - 2a, b - 1) = (a - 7, 2b + 5)$.

(iv) If $P = \{-2, -1\}$ then make two binary relations for $P \times P$.

(v) If $U = \{x \mid x \in N \wedge 3 < x \leq 25\}$, $X = \{x \mid x \text{ is prime} \wedge 8 < x < 25\}$ and $Y = \{x \mid x \in W \wedge 4 \leq x \leq 17\}$. Find the $(X \cap Y)'$.

Q.3 Answer the following Questions.

(4+4=8)

(a) If $U = \{1, 2, 3, \dots, 10\}$, $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 4, 5, 8\}$, then prove that $(A - B)' = A' \cup B$.

(b) If $L = \{x \mid x \in N \wedge x \leq 5\}$, $M = \{y \mid y \in P \wedge y < 10\}$, then make the following relations from L to M. $R = \{(x, y) \mid x + y = 6\}$ Also write the domain and range of the relation.

NOTE: Parents or guardians can conduct this test in their supervision in order to check the skill of students.



UNIT 6

BASIC STATISTICS

Statistics (K.B)

The branch of mathematics which deals with collection and manipulation of the numeric data.

Information Handling (K.B)

To present the information in a manageable way so that some useful conclusions can be drawn is called information handling.

Data (K.B)

The numerical figure obtained from any field of study is known as data.

Types of Data (K.B)

There are two types of data

- (i) Primary data,
- (ii) Secondary data

Primary Data (K.B)

The data directly collected from its source is called primary data.

For example:

Prices of fruits collected from market.

Secondary Data (K.B)

The data which have been passed through some statistical treatment at least once is called secondary data.

For example:

Office records, printed materials.

Constant (K.B)

Any quantity that has a single value is known as constant.

Example:

All real numbers can be taken as constant.

Variables (K.B)

Any quantity whose value is not fixed is called variable.

Examples:

Age of person, weight of person, height of person.

Types of Variable (K.B)

There are two types of variable.

- (i) Discrete variable
- (ii) Continuous variable.

Discrete Variable (K.B + U.B)

A variable which can take / assume some specific values in the given data is called discrete variable.

Note (K.B + U.B)

- (i) A discrete variable usually takes the values which are integers or whole numbers
i.e. 0,1,2,3.
- (ii) It represents the countable data.

Examples:

- (1) Numbers of heads appeared in tossing 5 coins.
- (2) No of children in a family.
- (3) No of students in a class.

Continuous Variable (K.B + U.B)

A variable which can take / assure every possible value with in the given range or interval i.e. (a to b) is called continuous variable.

Note (K.B + U.B)

- (1) It may be a whole figure or a fraction.
- (2) It represents the measurable data

Examples:

- (i) Age of an employee.
- (ii) Temperature of a place.
- (iii) Height / weight of an individual.

Ungrouped Data (K.B)

The numerical facts obtained on first hand and record as they stand are called ungrouped data.

Grouped Data (K.B)

The data which have gone / passed through some statistical process is called grouped

data. It may be classified into certain groups or into rows and columns.

Classification (K.B)

The process of arranging the data into certain groups or classes having similar characteristics is called classification.

Note about Number of Classes (K.B)

- More than 15 groups or classes are generally not advisable.
- There is no hard and fast rule to find exact number of classes.
- Groups / classes should be between 5 and 15.
- Too small number of groups / classes results into loss of important information.
- Taking too many groups do not pay for the labour involved condensing the information.

Example: (K.B)

Marks obtained in maths out of 100. Form a frequency distribution.
60, 62, 65, 66, 67, 68, 66, 60, 73, 69, 70, 63, 64, 63, 67

Given Data:

Groups	No. of Students	Class Marks (X)	Class Boundaries
60 ——— 62	3	$\frac{60 + 62}{2} = \frac{122}{2} = 61$	59.5 - 62.5
63 ——— 65	4	$\frac{63 + 65}{2} = \frac{128}{2} = 64$	62.5 - 65.5
66 ——— 68	5	$\frac{66 + 68}{2} = \frac{134}{2} = 67$	65.5 - 68.5

Class Mark / Mid Point (K.B)

The average value of lower and upper class limits of any class interval is called class mark, it is the mid point of any class. Class marks are represented by X.

For C.I, 60 – 62, 63 – 65,....

$$\text{Class mark} = (X) = \frac{60 + 62}{2} = \frac{122}{2} = 61$$

Class Boundaries (K.B)

The values which describe true class limits of a class are called class boundaries.

The smaller value is called lower class boundary.

The larger value is called upper class boundary.

Total number of observations = 15

The smallest number = 60,

The largest number = 73

Frequency Distribution

Groups	No of Students
60 ——— 62	3
63 ——— 65	4
66 ——— 68	5
69 ——— 71	2
72 ——— 74	1
	$n = 15$

Class Limits (K.B)

Each class / group is defined by two values one small and other large these values are called class limits. The smaller one is called lower class limit.

The large one is called upper class limit.

Size of Class Interval (K.B)

The difference between lower or upper class limits of any two consecutive classes / groups is called size or length of class interval. It is denoted by h.

Class Frequency (K.B)

The number of occurrence of items corresponding to the class interval is called class frequency. It is denoted by f.

Frequency Distribution (K.B)

The tabular arrangement of data in which various items are arranged into classes and the number of items lying in each class is called frequency distribution.

Formation of Frequency Distribution

(K.B)

By Tally Bar Method

By Direct Method

Cumulative Frequency (K.B)

The total of frequency upto an upper class limit or boundary is called the cumulative frequency.

Histogram (K.B)

A histogram is a graph of adjacent rectangle constructed on XY-plane. It is a graph of frequency distribution, on which class boundaries or class marks are marked on x-axis and frequency on y-axis.

Frequency Polygon (K.B)

Frequency polygon is a many sided closed figure on XY plane, on which class marks are marked on X-axis and frequencies on Y-axis.

Cumulative Frequency Polygon or Ogive (K.B)

A cumulative frequency polygon or ogive is a graph of many sided closed figure of less than cumulative frequency distribution, in which cumulative frequencies are marked on Y-axis and upper class boundaries on X-axis.

Exercise 6.1

Q.1 The following data shows the number of members in various families. Construct frequency distribution. Also find cumulative frequencies. 9, 11, 4, 5, 6, 8, 4, 3, 7, 8, 5, 5, 8, 3, 4, 9, 12, 8, 9, 10, 6, 7, 7, 11, 4, 4, 8, 4, 3, 2, 7, 9, 10, 9, 7, 6, 9, 5, 7. **(A.B)**

Solution:

Min. Value = 2, Max. Value = 12, Total values = 39

Discrete Frequency Distribution

No. of members	Tally bars	No. of families (f)	C.F
2		1	1
3		3	1+3=4
4		6	4+6=10
5		4	10+4=14
6		3	17
7		6	23
8		5	28
9		6	34
10		2	36
11		2	38
12		1	39

Q.2 The following data has been obtained after weighing 40 students of class V. Make a frequency distribution taking class interval size as 5. Also find the class boundaries and midpoints. 34, 26, 33, 32, 24, 21, 37, 40, 41, 28, 28, 31, 33, 34, 37, 23, 27, 31, 31, 36, 29, 35, 36, 37, 38, 22, 27, 28, 29, 31, 35, 35, 40, 21, 32, 33, 27, 29, 30, 23. **(A.B)**
Also make a less than cumulative frequency distribution. (Hint: Make classes 20-24, 25-29...).

Solution:

Min. Value = 21, Max. Value = 41, Size of class interval = 5, Total values = 40

Frequency Distribution

C – I	Tally Bars	Frequency	C – B	Mid points
15-19		-	14.5-19.5	17
20-24		6	19.5-24.5	22
25-29		10	24.5-29.5	27
30-34		12	29.5-34.5	32
35-39		9	34.5-39.5	37
40-44		3	39.5-44.5	42
Total		40		

Less than cumulative frequency distribution

C – B	C.f
Less than 19.5	0
Less than 24.5	6
Less than 29.5	16
Less than 34.5	28
Less than 39.5	37
Less than 44.5	40

- Q.3** From the following data representing the salaries of 30 teachers of a school. Make a frequency distribution taking class interval size of Rs. 100, 450, 500, 550, 580, 670, 1200, 1150, 1120, 950, 1130, 1230, 890, 780, 760, 670, 880, 890, 1050, 980, 970, 1020, 1130, 1220, 760, 690, 710, 750, 1120, 760, 1240. **(A.B + U.B + K.B)**
(Hint: Make classes 450-549, 550-29...).

Solution:

Min. Value = 450, Max. Value = 1240, Size of class interval = 100 , total values = n = 30

Frequency Distribution

C – I	Tally Bars	Frequency (f)
450 – 549		2
550 – 649		2
650 – 749		4
750 – 849		5
850 – 949		3
950 – 1049		4
1050 – 1149		5
1150 - 1249		5
Total		30

- Q.4** The following data shows the daily load shedding duration in hours in 31 localities of a certain city. Make a frequency distribution of the load shedding duration taking 2 hours as class interval size and answer the following questions. 6, 12, 5, 7, 8, 3, 6, 7, 10, 7, 14, 11, 12, 8, 6, 8, 9, 7, 11, 6, 9, 12, 13, 10, 14, 7, 6, 10, 11, 14, 12.

(A.B + U.B + K.B)

- (a) Find the most frequent load shedding hours?
(b) Find the least load shedding intervals?
(Hint: Make classes 2 - 3, 4 - 5, 6 - 7....)

Solution:

Min. Value = 2, Max. Value = 14, Size of class interval = 2, total values = $n = 31$

Frequency Distribution (By Direct Method)

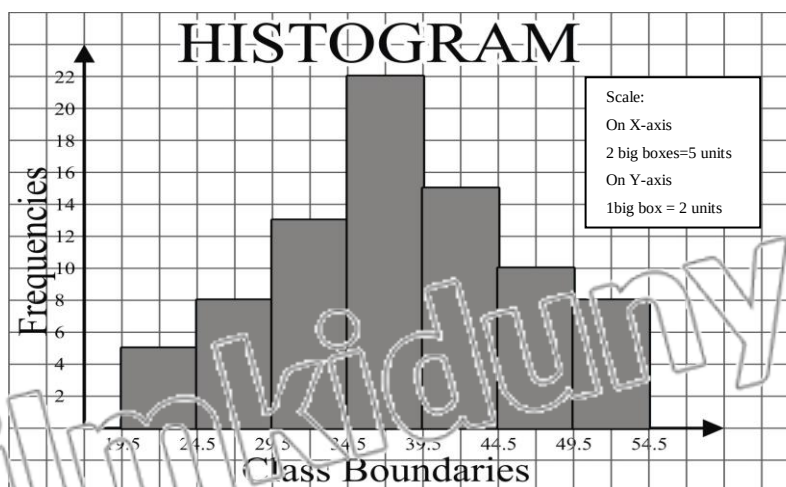
C - I	Values	Frequency (f)
2 - 3	3, 2	2
4 - 5	5	1
6 - 7	6, 7, 6, 7, 6, 7, 6, 7, 6	9
8 - 9	8, 8, 8, 9, 9,	5
10 - 11	10, 11, 11, 10, 10, 11	6
12 - 13	12, 12, 12, 13, 12	5
14 - 15	14, 14, 14	3
Total		31

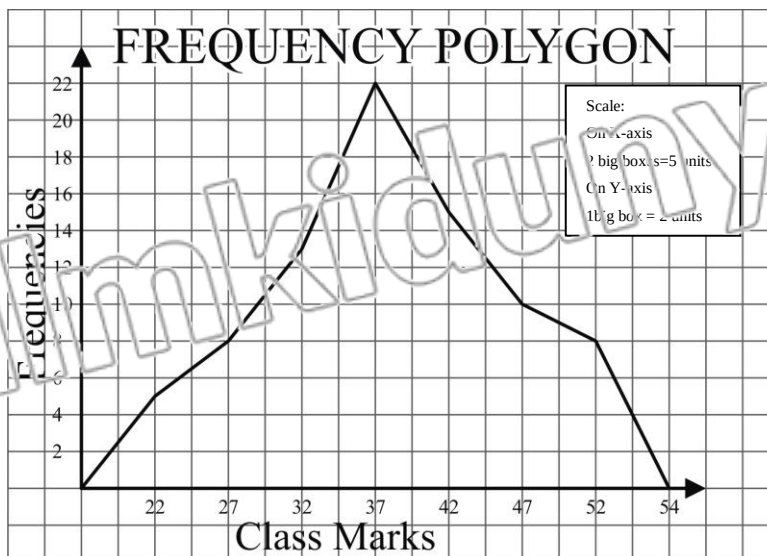
(a) The most frequent load shading hours = 6 - 7

(b) The least load shading intervals = 4 - 5

Q.5 Construct a Histogram and frequency Polygon for the following data showing weights of students in kg. (A.B + U.B + K.B)

Weights	Frequency / No. of Students	Mid Values	Class Boundaries
20-24	5	22	19.5-24.5
25-29	8	27	24.5-29.5
30-34	13	32	29.5-34.5
35-39	22	37	34.5-39.5
40-44	15	42	39.5-44.5
45-49	10	47	44.5-49.5
50-54	8	52	49.5-54.5



**Average****(K.B)**

A single value which represents the data is called average.

Types of Average**(A.B + K.B)**

- (i) Arithmetic mean
- (ii) Median
- (iii) Mode
- (iv) Geometric mean
- (v) Harmonic mean
- (vi) Quartiles

Arithmetic Mean**(A.B + K.B)**

A value obtained by dividing the sum of all the observations by their number of observations is called arithmetic mean it is denoted by $\bar{X}$

Let $X_1, X_2, X_3, \dots, X_n$ be the values of a data thus the A.M is defined as:

$$\bar{X} = \frac{X_1 + X_2 + X_3 + \dots + X_n}{n} = \frac{\sum_{i=1}^n X_i}{n}$$

where $i = 1, 2, 3, 4, \dots, n$

$$\Rightarrow \bar{X} = \frac{\sum X}{n} \quad \text{(For ungrouped data)}$$

$$\bar{X} = \frac{\sum fx}{\sum f} \quad \text{(For grouped data)}$$

Arithmetic Mean**(By short cut method) (A.B + K.B)**

$$\bar{X} = A + \frac{\sum D}{n} \quad \text{(For ungrouped data.)}$$

$$\bar{X} = A + \frac{\sum fD}{\sum f} \quad \text{(For grouped data)}$$

where $D = X - A$

A = Provisional Mean or Assumed Mean
and X = Mid point of a class

Arithmetic Mean (By Coding**Method)****(A.B + U.B)**

$$\bar{X} = A + \frac{\sum u}{n} \times h \quad \text{(For ungrouped data)}$$

where 'h' is common difference / size of C-I

$$\bar{X} = A + \frac{\sum fu}{\sum f} \times h \quad \text{(For grouped data)}$$

where $u = \frac{X - A}{h}$

Median**(U.B + K.B)**

A value which divides an arranged data into two equal parts (i.e., 50% data before the median and 50% after it is called median.

Or

Median is the middle most observation in an arranged data set. It divides the data set into two equal parts.

It is represented on $\tilde{x}$.

Formulas**(K.B)****(For ungrouped data)****(i)** If “n” is odd number

$$\text{Median} = \frac{n+1}{2}^{\text{th}} \text{ term}$$

If “n” is even number

$$\text{Median} = \frac{1}{2} \left[\left(\frac{n}{2} \right)^{\text{th}} \text{ item} + \left(\frac{n+2}{2} \right)^{\text{th}} \text{ item} \right]$$

(For grouped data)

$$\text{Median} = l + \frac{\frac{n}{2} - c}{f} \times h$$

where

l = lower class boundary of median class

h = Size of median class

f = Frequency of median class

c = Preceding commutative frequency of median class

 $n/2$ is used to decide the median class

$$n = \sum f$$

Mode**(K.B)**

The most repeated value or most frequent value in the given data is called mode.

Formula:**For Ungrouped data**

Mode = the most repeated value in given data. (No Formula)

For Grouped data

$$\text{Mode} = l + \frac{(f_m - f_1)}{2f_m - f_1 - f_2} \times h$$

Where

l = lower class boundary of modal class.

 f_m = Frequency of mode class f_1 = Preceding frequency of modal frequency class. f_2 = Following frequency of modal class.

h = size of modal class.

Empirical relation among Mean,**Median and Mode****(K.B)**

Mode = 3 Median – 2 Means

Geometric Mean**(K.B)**

Geometric mean of a variable X is the n^{th} positive root of the product of the $x_1, x_2, x_3, \dots, x_n$ observation. In symbols, we write

$$\text{G.M.} = (x_1, x_2, x_3, \dots, x_n)^{\frac{1}{n}}$$

The above formula can also be written by using logarithm.

For Ungrouped data

$$\text{G.M.} = \text{Anti log} \left(\frac{\sum \log x}{n} \right)$$

For Grouped data

$$\text{G.M.} = \text{Anti log} \left(\frac{\sum f \log X}{\sum f} \right)$$

Harmonic Mean**(K.B)**

Harmonic Mean refers to the value obtained by reciprocating the mean of the reciprocal of $x_1, x_2, x_3, \dots, x_n$ observations. In symbols,

For ungrouped data,

$$\text{H.M.} = \frac{n}{\sum \frac{1}{X}}$$

For Grouped data,

$$\text{H.M.} = \frac{\sum f}{\sum \frac{f}{X}}$$

Properties of Arithmetic Mean**(K.B + U.B)**

- (i)** Mean of a variable with similar observations say constant k is the constant k itself.
- (ii)** Mean is affected by change in origin.
- (iii)** Mean is affected by change in scale.
- (iv)** Sum of the deviations of the variable X from its mean is always zero.

The Weighted Arithmetic Mean**(K.B + U.B)**

The relative importance of a number is called its weight. When numbers $x_1, x_2, \dots, x_n$ are not equally important, we associate them with certain weight, $w_1, w_2, w_3, \dots, w_n$ depending on the importance or significance. Average of such data is called weighted arithmetic mean. It can be calculated by the formula

$$\bar{x}_w = \frac{\sum wx}{\sum w}$$

Moving Averages**(K.B + U.B)**

Moving averages are defined as the successive averages (arithmetic means) which are computed for a sequence of days/months/years at a time. Place the average of each period against the mid of the each period. Dropping first value and adding succeeding value to this group and continue the process until all values from first to last are used.

Quartiles**(K.B)**

It divides the data in four equal parts.

Formula: (for grouped data)

$$Q_i = l + \frac{h}{f} \left(\frac{in}{4} - c \right) ; i = 1, 2, 3$$

Note**(K.B)**

- i) Q_2 = Median
- ii) Q_1 = Lower Quartiles
- iii) Q_3 = Upper Quartiles

Deciles**(K.B)**

It divides the data in ten equal parts.

Formula: (for grouped data)

$$D_i = l + \frac{h}{f} \left(\frac{in}{10} - c \right) ; i = 1, 2, \dots, 9$$

Note**(K.B)**

D_5 = Median

Example 1: (Page # 119)**(A.B)**

The marks of seven students in Mathematics are as follows. Calculate the Arithmetic Mean and interpret the result.

Student No	1	2	3	4	5	6	7
Marks	45	60	74	58	65	63	49

Solution:

Let X = marks of a students

$$\bar{X} = \frac{\sum X}{n} = \frac{x_1 + x_2 + x_3 + \dots + x_7}{7}$$

$$\text{Or } \bar{X} = \frac{45 + 60 + 74 + 58 + 65 + 63 + 49}{7} = \frac{414}{7} = 59.14 \text{ marks}$$

Example 2 (Page # 120)**(GRW 2014, FSD 2017, D.G.K 2017)****(A.B)**

The salaries of five teachers are as follows. Find the mean salary using direct and indirect methods and compare the results. 11500, 12400, 15000, 14500, 14800.

Solution:

We proceed as follows:

(a) Using Direct method

$$\begin{aligned} \bar{X} &= \frac{\sum x_i}{5} = \frac{11500 + 12400 + 15000 + 14500 + 14800}{5} \\ &= \frac{74000}{5} = 13640 \text{ Rupees.} \end{aligned}$$

(b) Indirect methods:

We assume $A = 13000$, $D_i = (x_i - 13000)$, $h = 100$ and $u_i = \frac{(x_i - A)}{100}$, the computations are shown in the following table:

X	$D_i = (x_i - 13000)$	$u_i = \frac{(x_i - A)}{100}$
11500	-1500	-15
12400	-600	-6
15000	2000	20
14500	1500	15
14800	1800	18
$\sum X_i = 74000$	$\sum D_i = 3200$	$\sum u_i = 32$

(i) **Short formula:**

$$\bar{X} = 13000 + \frac{3200}{5} = 13000 + 640 = 13640 \text{ Rupees}$$

(ii) **Coding method:**

$$\bar{X} = 13000 + \frac{32}{5} \times 100 = 13640 \text{ Rupees}$$

Example 5 (Page # 122)**(A.B)**

Find arithmetic mean using short formula taking $X = 34.5$ as the provisional mean in example 4.

Solution:

We use the following formula

$$(i) \quad \bar{X} = A + \frac{\sum fD}{\sum f}$$

$$(ii) \quad \bar{X} = A + \frac{\sum fu}{\sum f} \times h$$

Given $A = 34.5$, we note that the distribution has equal class interval size of 10. So we may take $h = 10$ and make the following calculations:

Classes/groups	f	Midpoints x	$D = X - 34.5$	$u = (X - A) / 10$	fD	fu
0---9	2	4.5	-30	-3	-60	-6
10---19	10	14.5	-20	-2	-200	-20
20---29	5	24.5	-10	-1	-50	-5
30---39	9	34.5	0	0	0	0
40---49	6	44.5	10	1	60	6
50---59	7	54.5	20	2	140	14
60---69	1	64.5	30	3	30	3
Total	40		1300		-80	-8

Substituting the totals in the above formulae we get

$$(i) \quad \bar{X} = 34.5 + \frac{(-80)}{40} = 34.5 - 2 = 32.5 \text{ gm}$$

$$(ii) \quad \bar{X} = 34.5 + \left(\frac{-8}{40} \right) \times 10 = 34.5 - 2 = 32.5 \text{ gm}$$

Example 6 (Page # 123)**(A.B)**

On 5 term test in mathematics, a student has made marks of 82, 93, 86, 92 and 79. Find the median for the marks.

Solution:

By arranging the grades in ascending order, the arranged data is 79, 82, 86, 92, 93

Since number of observation is odd i.e., $n = 5$.

$$\tilde{x} = \text{size of } \left(\frac{5+1}{2}\right)^{\text{th}} \text{ observation}$$

$$\tilde{x} = \text{size of } 3^{\text{rd}} \text{ observation}$$

$$\tilde{x} = 86$$

Example 2 (Page # 123)

(IHR 2017, FSD 2017, 18, SGD 2014, MTN 2016)

(A.B)

The sugar contents for a random sample of 6 packs of juices of a certain brand are found to be 2.3, 2.7, 2.5, 2.9, 3.1 and 1.9 milligram. Find the median.

Solution:

Arranging the values by increasing order of magnitude

1.9, 2.3, 2.5, 2.7, 2.9, 3.1

Since the number of observations are even i.e., $n = 6$.

$$\tilde{x} = \frac{1}{2} \left[\text{size of } \left(\frac{6}{2}\right)^{\text{th}} + \left(\frac{6+2}{2}\right)^{\text{th}} \text{ observations} \right]$$

$$\tilde{x} = \frac{1}{2} [\text{size of } (3^{\text{rd}} + 4^{\text{th}}) \text{ observations}]$$

$$\tilde{x} = \frac{2.5 + 2.7}{2} = 2.6 \text{ milligram.}$$

Example 4 (Page # 125)**(A.B)**

The following data is the time taken by 40 students to solve a problem is recorded. Find the median time taken by the students.

138	164	150	132	144	125	149	157
146	158	140	147	136	148	152	144
168	126	138	176	163	119	154	165
146	173	142	147	135	153	140	135
161	145	135	142	150	156	145	128

Solution:

Class Intervals	Frequency	Class boundaries	Cumulative Frequency
118 — 126	3	117.5 – 126.5	3
127 — 135	5	126.5 – 135.5	8
136 — 144	9	135.5 – 144.5	19
145 — 153	12	144.5 – 153.5	29
154 — 162	5	153.5 – 162.5	34
163 — 171	4	162.5 – 171.5	38
172 — 180	2	171.5 – 180.5	40
Total	$40 = \sum f$	—	—

Median class = class containing $\left(\frac{n}{2}\right)^{\text{th}}$ observation

Median class = class containing $\left(\frac{40}{2}\right)^{\text{th}} = 20^{\text{th}}$ observation

$$\tilde{x} = l + \frac{h}{f} \left(\frac{n}{2} - c \right) = 144.5 + \frac{9}{12} (20 - 17) = 146.8$$

Example 1 (Page # 127)

(SGD 2014)

(A.B)

Find the geometric mean of the observations 2,4,8 using
 (i) basic formula and (ii) using logarithmic formula.

Solution:**Basic formula**

$$G.M = (2 \times 4 \times 8)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = 4$$

Using log method:

X	$\log X$
2	0.3010
4	0.6021
8	0.9031
Total	1.8062

$$G.M = \text{Antilog} \left(\frac{\sum \log X}{n} \right) = \text{Antilog} \left(\frac{1.8062}{3} \right)$$

$$= \text{Antilog} (0.6021) = 4.00003 = 4$$

Example 2: (Page # 130)**(A.B + U.B)**

A variable X take the following values 4,5,8,6,2. Find the mean of X , also find the mean when (a) 5 is added to each observation (b) 10 is multiplied with each observation (c) Prove that sum of the deviation from mean is zero.

Solution:Given the values of X , $X: 4 \ 5 \ 8 \ 6 \ 2.$

We may introduce two new variables Y and Z under (a) and (b) respectively. So we are given that (a) $Y = X + 5$ (b) $Z = 10X$. The following table shows the desired result:

X	$Y = X + 5$	$Z = 10X$	$X - \bar{X}$
4	9	40	-1
5	10	50	0
8	13	80	3
6	11	60	1
2	7	20	-3
25	50	250	$\sum (X - \bar{X}) = 0$

From the above table we get,

$$\bar{X} = \frac{\sum X}{n} = \frac{25}{5} = 5 \quad ; \quad \bar{Y} = \frac{\sum Y}{n} = \frac{50}{5} = 10 \quad ; \quad \bar{Z} = \frac{\sum Z}{n} = \frac{250}{5} = 50$$

Note**(A.B + U.B + K.B)**

$$(a) \quad \bar{Y} = 10 = 5 + 5 = \bar{X} + 5$$

$$(b) \quad \bar{Z} = 50 = 10(5) = 10\bar{X}$$

Which shows that mean is affected by change in origin and scale.

(c) From the last column of the table we note that $\sum (X - \bar{X}) = 0$, the sum of the deviations from mean is zero.

Example 1: (Page # 131)**(A.B)**

The following table gives the monthly earnings and the number of workers in a factory, compute the weighted average.

No. of employees	Monthly earnings. Rs.
4	800
22	45
20	100
30	30
30	35
300	15

Solution:

Number of employees are treated as a weight (w) and monthly earnings as variable (x)

No. of employees (w)	Monthly earning in Rs. (x)	(xw)
4	800	3200
22	45	990
20	100	2000
30	30	900
30	35	2800
300	15	4500
$\sum w = 456$	—	$\sum xw = 14390$

$$\bar{X}_w = \frac{\sum xw}{\sum w} = \frac{14390}{456} = 31.5$$

Example 2: (Page # 131)**(A.B)**

Calculate three days moving average for the following record of attendance:

Week	Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	24	55	28	45	51	54	60

Solution:

Week and days	Attendance	3-days moving	
		Total	Average
Sun.	24	—	—
Mon.	55	107	$107/3 = 35.67$
Tue.	28	128	$128/3 = 42.67$
Wed.	45	124	$124/3 = 41.33$
Thu.	51	150	$150/3 = 50.00$
Fri.	54	168	$168/3 = 56.00$
Sat.	60	—	—

Example 1: (Page # 132)**(A.B)**

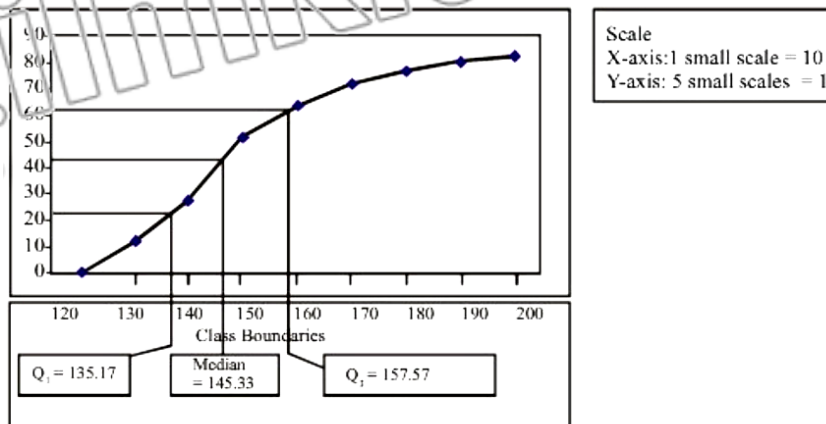
For the following distribution locate Median and Quartiles on graph.

Class boundaries	Cumulative frequency
Less than 120	0
Less than 130	12
Less than 140	27
Less than 150	51

Less than 160	64
Less than 170	71
Less than 180	76
Less than 190	80
Less than 200	82

Solution:

We locate median and Quartiles by using the following cumulative frequency polygon.



Example 2: (Page # 133)

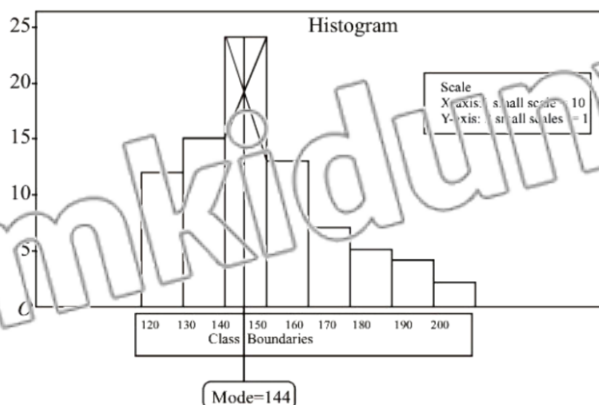
(A.B + U.B + K.B)

For the following distribution locate Mode on graph.

Salaries in Rupees	No. of teachers
120 — 130	12
130 — 140	15
140 — 150	24
150 — 160	13
160 — 170	7
170 — 180	5
180 — 190	4
190 — 200	2

Solution:

On Histogram, the mode is located on X-axis as shown below:



Exercise 6.2

Q.1 What do you understand by measures of central tendency? (A.B)

Ans: A single value which represents the data is called as average. As average tends to lie in the centre of the data, so to locate an average is called measure of central tendency.

Q.2 Define Arithmetic mean, Geometric mean, Harmonic mean, mode and median. (A.B)

Ans: See Definition page # 188, 189

Q.3 Find arithmetic mean by direct method for the following set of data: (A.B)

(i) 12, 14, 17, 20, 24, 29, 35, 45.

(ii) 200, 225, 350, 375, 270, 320, 290.

Ans:

(i) **Given Data:**

12, 14, 17, 20, 24, 29, 35, 45

(LHR 2014, 16, GRW 2017, FSD 2017, RWP 2015, 17, MTN 2016, D.G.K 2014, 16)

Required:

Arithmetic mean by direct method

Solution:

X	12	14	17	20	24	29	35	45	$\sum X = 196$
---	----	----	----	----	----	----	----	----	----------------

Formula:

$$\begin{aligned}\bar{X} &= \frac{\sum X}{n} \\ &= \frac{196}{8} \\ &= 24.5\end{aligned}$$

Result

$$\bar{X} = 24.5$$

(ii) **Given Data:**

200, 225, 350, 375, 270, 320, 290

(GRW 2014, 16, D.G.K 2016)

Required:

Arithmetic mean by direct method

Solution:

X	200	225	350	375	270	320	290	$\sum X = 2030$
---	-----	-----	-----	-----	-----	-----	-----	-----------------

We know that

$$\begin{aligned}\bar{X} &= \frac{\sum X}{n} = \frac{2030}{7} \\ &= 290\end{aligned}$$

Result: $\bar{X} = 290$

Q.4 For each of the data in Q. No 3, compute arithmetic mean using indirect method. (A.B)

(i) **Give Data:**

12, 14, 17, 20, 24, 29, 35, 45

Required:

Arithmetic mean by indirect method

Solution:

X	D = X - 20
12	-8
14	-6
17	-3
20	0
24	4
29	9
35	15
45	25
	$\Sigma D = 36$

We know that

$$\bar{X} = A + \frac{\Sigma D}{n}$$

Putting the values

$$= 20 + \frac{36}{8}$$

$$= 20 + 4.5 = 24.5$$

Result:

$$\bar{X} = 24.5$$

(ii) **Give Data:**

200, 225, 350, 375, 270, 320, 290

(A.B)**Required:**

Arithmetic mean by indirect method

Solution: (By using short-cut and coding method)

X	D = X - 200	U = $\frac{X - 200}{5}$
200	0	0
225	25	5
270	70	14
290	90	18
320	120	24
350	150	30
375	175	35
	$\Sigma D = 630$	$\Sigma U = 126$

(H.C.F of the difference of class marks is 5, so h is 5)

Arithmetic mean by short-cut method**Formula:**

$$\bar{X} = A + \frac{\Sigma D}{n}$$

$$= 200 + \frac{630}{7}$$

$$= 200 + 90$$

$$\bar{X} = 290$$

Arithmetic mean by coding method

Formula:

$$\begin{aligned}\bar{X} &= A + \frac{\sum U}{n} \times h \\ &= 200 + \frac{126}{7} \times 5 \\ &= 200 + \frac{630}{7} \\ &= 200 + 90 = 290\end{aligned}$$

Result: $\bar{X} = 290$

Q.5 The marks obtained by students of class XI in mathematics are given below.

Compute arithmetic mean by direct and indirect methods. (LHR 2015) (A.B)

Given Data:

Classes/Groups	Frequency
0 – 9	2
10 – 19	10
20 – 29	5
30 – 39	9
40 – 49	6
50 – 59	7
60 – 69	1

Required:

Arithmetic mean by direct and indirect method

Solution:

C – I	f	X	fX	D = X - 34.5	fD	$U = \frac{X - 34.5}{10}$	fU
0 – 9	2	4.5	9	-30	-60	-3	-6
10 – 19	10	14.5	145	-20	-200	-2	-20
20 – 29	5	24.5	122.5	-10	-50	-1	-5
30 – 39	9	34.5	310.5	0	0	0	0
40 – 49	6	44.5	267	10	60	1	6
50 – 59	7	54.5	381.5	20	140	2	14
60 – 69	1	64.5	64.5	30	30	3	3
	$\sum f = 40$		$\sum fX = 1300$		$\sum fD = -80$		$\sum fU = -8$

Arithmetic mean by direct method**Formula**

$$\begin{aligned}\bar{X} &= \frac{\sum fx}{\sum f} \\ &= \frac{1300}{40} \\ &= 32.5\end{aligned}$$

Results

$$\bar{X} = 32.5$$

Arithmetic mean by short cut method**Formula**

$$\bar{X} = A + \frac{\sum fD}{\sum f}$$

Putting the values

$$\Rightarrow \bar{X} = 34.5 + \left(\frac{-80}{40} \right)$$

$$= 34.5 - 2$$

$$\bar{X} = 32.5$$

Results

$$\bar{X} = 32.5$$

Arithmetic mean by coding method**Formula**

$$\bar{X} = A + \frac{\sum fD}{\sum f} \times h$$

$$= 34.5 + \frac{-8}{40} \times 10$$

$$= 34.5 - 2$$

$$\bar{X} = 32.5$$

Results

$$\bar{X} = 32.5$$

Q.6 The following data relates to the ages of children in a school. Compute the mean age by direct and short-cut method taking any Provisional mean. (Hint. Take A = 8)

(A.B)**Given Data:**

Class limits	Frequency
4 – 6	10
7 – 9	20
10 – 12	13
13 – 15	7
Total	50

Required:

- (i) Arithmetic mean by direct and short-cut method
- (ii) G.M
- (iii) H.M

Solution:

C – I	f	X	fX	D = X – 8	fD	log X	f log X	f/X
4 – 6	10	5	50	-3	-30	0.6990	6.99	2.0000
7 – 9	20	8	160	0	0	0.9031	18.062	2.5
10 – 12	13	11	143	3	39	1.0414	13.5382	1.1818
13 – 15	7	14	98	6	42	1.1461	8.0227	0.5
	$\sum f = 50$		$\sum fX = 451$		$\sum fD = 51$		$\sum f \log X = 46.6129$	$\sum \frac{f}{X} = 6.1818$

Arithmetic mean by direct method

$$\begin{aligned}\bar{X} &= \frac{\sum fX}{\sum f} \\ &= \frac{451}{50} \\ &= 9.02\end{aligned}$$

Arithmetic mean by short method

$$\begin{aligned}\bar{X} &= A + \frac{\sum fD}{\sum f} \\ &= 8 + \frac{51}{50} \\ &= 8 + 1.02 \\ &= 9.02\end{aligned}$$

Geometric Mean:

Q.7 The following data shows the number of children in various families. Find mode and median. 9, 11, 4, 5, 6, 8, 4, 3, 7, 8, 5, 5, 8, 3, 4, 9, 12, 8, 9, 10, 6, 7, 7, 11, 4, 4, 8, 4, 3, 2, 7, 9, 10, 9, 7, 6, 9, 5. (FSD 2015, SGD 2015) **(A.B)**

Required:

- (i) Mode
(ii) Median

Solution:

Min Value = 2, Max. Value = 12

Frequency Distribution

X	Tally Bars	Frequency	C.f
2		1	1
3		3	4
4		6	10
5		4	14
6		3	17
7		5	22
8		5	27
9		6	33
10		2	35
11		2	37
12		1	38

Mode

Mode = 4 and 9 (Repeating maximum number of times)

Median

$$\text{Median} = \left(\frac{n}{2} \right)^{\text{th}} \text{ term}$$

$$\begin{aligned}G.M &= \text{antilog} \left(\frac{\sum f \log X}{\sum f} \right) \\ &= \text{antilog} \left(\frac{46.5129}{50} \right) \\ &= \text{antilog} (0.9323)\end{aligned}$$

$$G.M = 8.5566$$

Harmonic Mean:

$$\begin{aligned}H.M &= \frac{\sum f}{\sum \left(\frac{f}{X} \right)} \\ &= \frac{50}{6.1818} \\ H.M &= 8.0883\end{aligned}$$

(A.B + U.B)**(A.B + U.B)**

$$= \left(\frac{38}{2}\right)^{th} \text{ term}$$

$$= 19^{th} \text{ term}$$

$$\text{Median} = 7$$

Q.8 Find Modal number of heads for the following distribution showing the number of heads when 5 coins are tossed. Also determine median. (A.B + U.B + K.B)

Given Data:

X (No. of heads)	Freq. (No. of times)
1	3
2	8
3	5
4	3
5	1

Required:

(i) Mode

(ii) Median

Solution:

X	f	C.f
1	3	3
2	8	11
3	5	16
4	3	19
5	1	20

Mode

Mode = 2 (Repeating max. number of times)

Median

$$\text{Median} = \left(\frac{n}{2}\right)^{th} \text{ term} = \left(\frac{20}{2}\right)^{th} \text{ term} = 10^{th} \text{ term} = 2$$

Q.9 The following frequency distribution the weights of boys in kilogram. Compute mean, median, mode. (A.B + U.B + K.B)

Given Data:

Class intervals	Frequency
1 – 3	2
4 – 6	3
7 – 9	5
10 – 12	4
13 – 15	6
16 – 18	2
19 – 21	1

Required:

(i) Mean =? (ii) Median =? (iii) Mode =?

Solution:

C - I	f	X	fX	C - B	C.f
1 - 3	2	2	4	0.5 - 3.5	2
4 - 6	3	5	15	3.5 - 6.5	5
7 - 9	5	8	40	6.5 - 9.5	10
10 - 12	4f ₁	11	44	9.5 - 12.5	14
13 - 15	6f _m	14	84	12.5 - 15.5	20
16 - 18	2f ₂	17	34	15.5 - 18.5	22
19 - 21	1	20	20	18.5 - 21.5	23
	Σf=23		ΣfX = 241		

(i) **Mean:** (A.B)

Formula:

$$\bar{X} = \frac{\sum fX}{\sum f}$$

Putting the values

$$= \frac{241}{23}$$

$$\Rightarrow \bar{X} = 10.478kg$$

(ii) **Median:** (A.B)

Formula:

$$\begin{aligned} \text{Median} &= l + \frac{h}{f} \left(\frac{n}{2} - c \right) \\ &= 9.5 + \frac{3}{4} (11.5 - 10) \\ &= 9.5 + 1.125 \end{aligned}$$

$$= 10.625$$

$$\Rightarrow \text{Median} = 10.625kg$$

(iii) **Mode:** (A.B)

Formula:

$$\text{Mode} = l + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h$$

Putting the values

$$= 12.5 + \frac{6 - 4}{2(6) - 4 - 2} \times 3$$

$$= 12.5 + \frac{2 \times 3}{6}$$

$$= 12.5 + 1$$

$$\Rightarrow \text{Mode} = 13.5kg$$

Q.10 A student obtained the following marks at a certain examination: English 73, Urdu 82, Mathematics 80, History 67 and Science 62. (A.B + U.B)

(i) If the weights accorded these marks are 4, 3, 3, 2 and 2, respectively, what is an appropriate average mark?

(ii) What is the average mark if equal weights are used?

Given Data:

Subject	Marks	Weights
English	73	4
Urdu	82	3
Mathematics	80	3
History	67	2
Science	62	2

Required:

(i) What is an appropriate average mark? (i-e Weighted means)

(ii) What is the average mark if equal weights are used (simple means)

Solution:

Subject	Marks (x)	Weights (w)	Wx
English	73	4	292
Urdu	82	3	246
Mathematics	80	3	240
History	67	2	134
Science	62	2	124
Total	364	14	1036

(i) Weighted Mean:

$$\bar{X}_w = \frac{\sum wx}{\sum w}$$

$$= \frac{1036}{14}$$

$$\Rightarrow \bar{X}_w = 74 \text{ marks}$$

(ii) When equal weights are used

$$\bar{X} = \frac{\sum X}{n}$$

$$= \frac{364}{5}$$

$$= 72.8 \text{ marks}$$

Q.11 On a vacation trip a family bought 21.3 liters of petrol at 39.90 rupees per liter, 18.7 liters at 42.90 rupees per liter, and 23.5 liters at 40.90 rupees per liter. Find the mean price paid per liter. **(A.B)**

Given Data:

No. of liters	Prices (Rs)
21.3	39.90
18.7	42.90
23.5	40.90

Required:

Mean price paid per liters = $\bar{X}_w = ?$

No. of liters (w)	Price (x)	wx
21.3	39.90	849.87
18.7	42.90	802.23
23.5	40.90	961.15
$\sum w = 63.5$		$\sum wx = 2613.25$

Formula

$$\bar{X}_w = \frac{\text{Total payment}}{\text{Total liters}} = \frac{\sum wx}{\sum w}$$

$$= \frac{2613.25}{63.5}$$

$$= 41.15$$

Result:

Mean price paid per liter = Rs 41.15

Q.12 Calculate simple moving average of 3 years from the following data: (A.B)

Given Data:

Years	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Values	102	108	130	140	158	180	196	210	220	230

Required:

3 years moving average:

Solution:

Years	Values	3 years moving total	3 years moving average
2001	102	-	-
2002	108	340	113.33
2003	130	378	126
2004	140	428	142.67
2005	158	478	159.33
2006	180	534	178
2007	196	586	195.33
2008	210	626	208.67
2009	220	660	220
2010	230	-	-

Q.13 Determine graphically for the following data and check your answer by using formulae.

(i) Median and Quartiles using cumulative frequency polygon.

(ii) Mode using Histogram.

(A.B)

Given Data:

Class boundaries	Frequency
10 – 20	2
20 – 30	5
30 – 40	9
40 – 50	6
50 – 60	4
60 – 70	1

Required:

(i) Median and quartiles using cumulative frequency polygon and formula.

(ii) Mode using histogram and using formula.

Solution:

C – B	f	cf
0-10	0	0
10 – 20	2	2
20 – 30	5 f_1	7
30 – 40	9 f_m	16
40 – 50	6 f_2	22
50 – 60	4	26
60 – 70	1	27
	27	

Q₁ class
Median class/Modal class
Q₃ class

Median using Formula**(K.B)**

$$\begin{aligned}\text{Median} &= \left(\frac{n}{2}\right)^{\text{th}} \text{ observation} \\ &= \left(\frac{27}{2}\right)^{\text{th}} \text{ observation} \\ &= 13.5^{\text{th}} \text{ observation}\end{aligned}$$

$$\text{Median} = l + \frac{h}{f} \left(\frac{n}{2} - C \right)$$

Putting the values

$$\begin{aligned}\text{Median} &= 30 + \frac{10}{9} (13.5 - 7) \\ &= 30 + \frac{10}{9} (6.5) \\ &= 30 + 7.22\end{aligned}$$

$$\Rightarrow \text{Median} = 37.22$$

Quartiles using Formula**(K.B)****First Quartile**

$$\begin{aligned}Q_1 &= \frac{n}{4} \text{th observation} \\ &= \frac{27}{4} \text{th observation} \\ &= 6.75^{\text{th}} \text{ observation}\end{aligned}$$

$$Q_1 = l + \frac{h}{f} \left(\frac{n}{4} - C \right)$$

Putting the values

$$\begin{aligned}&= 20 + \frac{10}{5} (6.75 - 2) \\ &= 20 + 2 (4.75) \\ &= 20 + 9.50\end{aligned}$$

$$\Rightarrow Q_1 = 29.50$$

Third Quartile (Q_3)

$$\begin{aligned}Q_3 &= \frac{3n}{4} \text{th observation} \\ &= 3 \times \frac{27}{4} \text{th observation} \\ &= \frac{81}{4} \text{th observation} \\ &= 20.25 \text{th observation}\end{aligned}$$

$$Q_3 = l + \frac{h}{f} \left(\frac{3n}{4} - C \right)$$

Putting the values

$$\begin{aligned}&= 40 + \frac{10}{6} (20.25 - 16) \\ &= 40 + \frac{10}{6} (4.25) \\ &= 40 + 7.80\end{aligned}$$

$$\Rightarrow Q_3 = 47.80$$

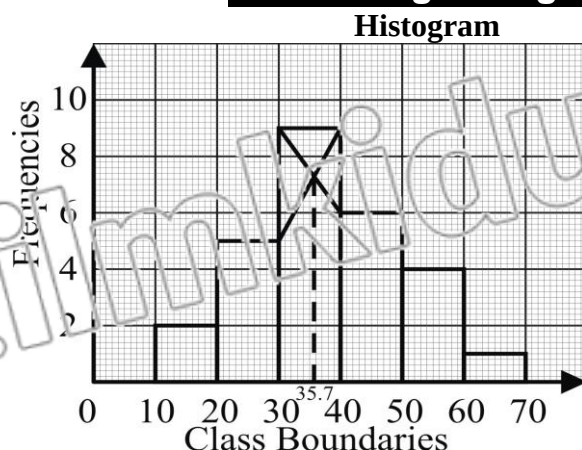
Mode**(K.B)**

$$\text{Mode} = l + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h$$

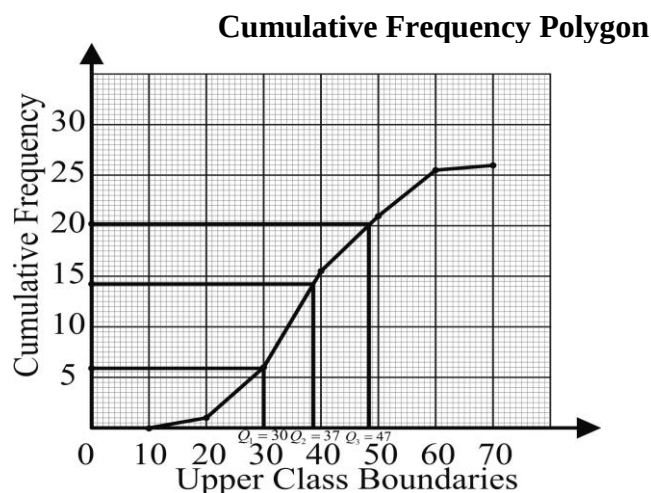
Putting the values

$$\begin{aligned}&= 30 + \frac{9 - 5}{2(9) - 5 - 6} \times 10 \\ &= 30 + \frac{4}{18 - 11} \times 10 \\ &= 30 + \frac{40}{7} \\ &= 30 + 5.71\end{aligned}$$

$$\Rightarrow \text{Mode} = 35.71$$

Mode Using Histogram

Scale:
 On X-axis
 1 big box = 20
 On Y-axis
 1 big box = 5



Scale:
 On X-axis
 1 big box = 20
 On Y-axis
 1 big box = 5

DISPERSION**(K.B)**

Spread or scatter ness of the data is known as dispersion.

Types of Dispersion**(K.B)**

- (i) Range
- (ii) Variance
- (iii) Standard deviation

Measure of Dispersion**(K.B)**

The techniques that are used to determine the degree or extent of variation in a data set is called measure of dispersion.

Range**(K.B)**

The difference between the largest and smallest value in a given data is called range.

Mathematically it is defined as:

$$\text{Range (R)} = X_m - X_1$$

X_m = The largest value

X_1 = The smallest value

Example:

Given

16, 5, 6, 10, 11

$$X_m = 16, \quad X_1 = 5$$

$$\text{Range} = X_m - X_1$$

$$= 16 - 5$$

$$= 11$$

Variance**(K.B)**

A value obtained by dividing the sum of squares of deviations taken from arithmetic mean by the number of observations in the given data is called variance. It is denoted by S^2 .

Formulae

$$(i) \quad S^2 = \frac{\sum (X - \bar{X})^2}{n}$$

(Proper mean formula)

$$(ii) \quad S^2 = \frac{\sum X^2}{n} - \left(\frac{\sum X}{n} \right)^2$$

(Direct Formula)

Standard Deviation (K.B)

The positive square root of variance is called standard deviation. It is abbreviated as S.D and is denoted by S.

Formulae

$$(i) \quad S = \sqrt{\frac{\sum (X - \bar{X})^2}{n}}$$

(Proper mean formula)

$$(ii) \quad S = \sqrt{\frac{\sum X^2}{n} - \left(\frac{\sum X}{n} \right)^2}$$

(Direct formula)

Example 5: (Page # 140) (A.B)

Compare the variation about mean for the two groups of students who obtained the following marks in statistics:

X = Marks (Section A)	Y = Marks (section B)
60	62
70	62
30	65
90	68
80	67
40	48

Solution:

In order to compare variation about mean we compute standard deviation for the two groups as follows:

X	Y	X - $\bar{X}$	(X - $\bar{X}$) ²	Y - $\bar{Y}$	(Y - $\bar{Y}$) ²
60	62	-2	4	0	0
70	62	8	64	0	0
30	65	-32	1024	3	9
90	68	28	784	6	36
80	67	18	324	5	25
40	48	-20	400	-14	196
$\sum x = 372$	$\sum y = 372$	0	$\sum (x - \bar{x})^2 = 2600$	0	$\sum (y - \bar{y})^2 = 266$

$$\text{Mean for group A} = \bar{X} = \frac{\sum X}{n} = \frac{372}{6} = 62$$

$$\text{Mean for group B} = \bar{Y} = \frac{\sum Y}{n} = \frac{372}{6} = 62$$

$$\begin{aligned} S.D(X) &= \sqrt{\frac{\sum (X - \bar{X})^2}{n}} \\ &= \sqrt{\frac{2600}{6}} = \sqrt{433.333} \\ &= 20.82 \text{ gm} \end{aligned}$$

$$\begin{aligned} S.D(Y) &= \sqrt{\frac{\sum (Y - \bar{Y})^2}{n}} \\ &= \sqrt{\frac{266}{6}} = \sqrt{4.333} \\ &= 6.66 \text{ gm} \end{aligned}$$

Comment: we note that the variation in Group B is smaller than that of Group A. This implies the marks of students in Group B are closer to their Mean than that of group A.

Note (K.B + U.2)
Smaller variation is more consistent in performance.

Exercise 6.3

Q.1 What do you understand by dispersion? (K.B)

Ans: See Definition page # 206

Q.2 How do you define measures of dispersion? (K.B)

Ans: See Definition page # 206

Q.3 Define Range, Standard deviation and Variance. (K.B + A.B)

Ans: See Definition page # 206, 207

Q.4 The salaries of five teachers in Rupees are as follows. 11500, 12400, 15000, 14500, 14800. Find the range and standard deviation. (A.B)

Given Data:

11500, 12400, 15000, 14500, 14800

Required

(i) Range (ii) Standard Derivation

Solution:

X	X ²
11500	132250000
12400	153760000
15000	225000000
14500	210250000
14800	219040000
$\Sigma X = 68200$	$\Sigma X^2 = 940300000$

Range

(A.B)

Max. value = $X_m = 15,000$ Min. value = $X_o = 11,500$ Range = $X_m - X_o = 15000 - 11500$
= 3,500 Rs.

Standard Deviation

(A.B)

$$S = \sqrt{\frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2}$$

$$= \sqrt{\frac{940300000}{5} - \left(\frac{68200}{5}\right)^2}$$

$$= \sqrt{188060000 - 186049600}$$

$$= \sqrt{2010400}$$

$$\Rightarrow S = 1417.886$$

Q.5 a- Find the standard deviation "S" of each set of numbers: (A.B)

(i) 12, 6, 7, 3, 15, 10, 18, 5

(ii) 9, 3, 8, 8, 9, 8, 9, 18.

b- Calculate variance for the data:
10, 8, 9, 7, 5, 12, 8, 6, 8, 2.

Given Data:

(i) 12, 6, 7, 3, 15, 10, 18, 5 (FSD 2017)

Required:

Standard Deviation = S = ?

(a) Solution:

X	X ²
12	144
6	36
7	49
3	9
15	225
10	100
18	324
5	25
$\Sigma X = 76$	$\Sigma X^2 = 912$

Standard Deviation

(A.B)

$$S = \sqrt{\frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2}$$

$$= \sqrt{\frac{912}{8} - \left(\frac{76}{8}\right)^2}$$

$$\begin{aligned}
 &= \sqrt{114 - (9.5)^2} \\
 &= \sqrt{114 - 90.25} \\
 &= \sqrt{23.75} \\
 \Rightarrow S &= 4.87
 \end{aligned}$$

(ii) 9, 3, 8, 8, 9, 8, 9, 18 (FSD 2015) (A.B)

X	$X - \bar{X} = X - 9$	$(X - \bar{X})^2$
9	0	0
3	-6	36
8	-1	1
8	-1	1
9	0	0
8	-1	1
9	0	0
18	9	81
$\Sigma X = 72$		120

Mean

$$\begin{aligned}
 \bar{X} &= \frac{\Sigma X}{n} \\
 &= \frac{72}{8} \\
 &= 9
 \end{aligned}$$

Standard Deviation

(A.B)

$$\begin{aligned}
 S &= \sqrt{\frac{\Sigma (X - \bar{X})^2}{n}} \\
 &= \sqrt{\frac{120}{8}} = \sqrt{15} \\
 \Rightarrow S &= 3.87
 \end{aligned}$$

The length of 32 items are given below. Find the mean length and standard deviation of the distribution.

Length	20-22	23-25	26-28	29-31	32-34
Frequency	3	6	12	9	2

Given Data:

Length	Frequency
20 - 22	3
23 - 25	6
26 - 28	12
29 - 31	9
32 - 34	2

Required

- (i) Mean length (ii) Standard Deviation

(b) **Given Data:** (D.G.K 2015) (A.B)

10, 8, 9, 7, 5, 12, 8, 6, 8, 2

RequiredVariance = S^2 **Solution**

X	X^2
10	100
8	64
9	81
7	49
5	25
12	144
8	64
6	36
8	64
2	4
$\Sigma X = 75$	$\Sigma X^2 = 631$

Variance

$$\begin{aligned}
 S^2 &= \frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n} \right)^2 \\
 &= \frac{631}{10} - \left(\frac{75}{10} \right)^2 \\
 &= 63.1 - 56.25
 \end{aligned}$$

$$S^2 = 6.85$$

Result

$$\text{Variance} = S^2 = 6.85$$

Solution:

C - I	f	X	fX	fX ²
20-22	3	21	63	1,323
23-25	6	24	144	3,456
26-28	12	27	324	8,748
29-31	9	30	270	8,100
32-34	2	33	66	2,178
	$\Sigma f = 32$		$\Sigma fX = 867$	$\Sigma fX^2 = 23,805$

Mean

$$\bar{X} = \frac{\Sigma fX}{\Sigma f}$$

$$= \frac{867}{32}$$

$$\Rightarrow \bar{x} = 27.09$$

Standard Deviation

$$S = \sqrt{\frac{\Sigma fX^2}{n} - \left(\frac{\Sigma fX}{n}\right)^2}$$

$$= \sqrt{\frac{23805}{32} - \left(\frac{867}{32}\right)^2}$$

$$= \sqrt{743.906 - 733.87}$$

$$= \sqrt{9.8347} = 3.136$$

Q.6 Find the range of the given data:

(A.B)

Class	33-40	41-50	51-60	61-70	71-75
Frequency	28	31	12	9	5

Solution:

Class	Frequency	C-B
33-40	28	32.5-40.5
41-50	31	40.5-50.5
51-60	12	50.5-60.5
61-70	9	60.5-70.5
71-75	5	70.5-75.5

Here

Range = upper class boundary of last class – lower class boundary of 1st class

$$\text{Range} = 75.5 - 32.5 = 43$$

FORMULAE

(Exercise # 6.2,6.3)

(A.B + K.B + U.B)

For Ungrouped Data		For Grouped Data	
Arithmetic means			
Ungroup Data		Grouped Data	
(Direct Method)		(Short Cut/ Deviation Method)	
$\bar{X} = \frac{\sum x}{n}$		$\bar{X} = \frac{\sum fx}{\sum f}$	
Indirect Method		(Coding Method)	
$\bar{X} = A + \frac{\sum D}{n} \quad ; (D = x - A)$		$\bar{X} = A + \frac{\sum fD}{\sum f}$	
(Coding Method)			
$A.M = A + \frac{\sum u}{n} \times h \quad ; \left(u = \frac{x - A}{h}\right)$		$A.M = A + \frac{\sum fu}{\sum f} \times h$	
(Median)			
Median = $\left(\frac{n+1}{2}\right)^{\text{th}}$ item (n is odd)		Median = $l + \frac{h}{f} \left(\frac{n}{2} - C\right)$	
Median $= \frac{1}{2} \left\{ \left(\frac{n}{2}\right)^{\text{th}}$ item + $\left(\frac{n+2}{2}\right)^{\text{th}}$ item $\right\}$ (n is even)			
(Mode)			
Mode = most repeated value of the data		Mode = $l + \frac{(f_m - f_1)}{2f_m - f_1 - f_2} \times h$	
(Geometric Mean)			
G.M = $(x_1, x_2, x_3, \dots, x_n)^{\frac{1}{n}}$ By definition method		G.M = $\text{Anti log} \left(\frac{\sum f \log X}{\sum f} \right)$	
G.M = $\text{Anti log} \left(\frac{\sum \log X}{n} \right)$ By log method			
Harmonic Mean			
H.M = $\frac{n}{\sum \frac{1}{X}}$		H.M = $\frac{\sum f}{\sum \frac{f}{X}}$	
Weighted Arithmetic Mean = $\bar{X}_w = \frac{\sum wx}{\sum w}$			
Range			
Range = $X_m - X_1$		Range = upper class boundary of last class – lower class boundary of first class	

Variance	
(i) $S^2 = \frac{\sum (X - \bar{X})^2}{n}$ Proper mean formula	(i) $S^2 = \frac{\sum f(X - \bar{X})^2}{\sum f}$ Proper mean formula
(ii) $S^2 = \frac{\sum X^2}{n} - \left(\frac{\sum X}{n} \right)^2$ Direct Formula	(ii) $S^2 = \frac{\sum fX^2}{\sum f} - \left(\frac{\sum fX}{\sum f} \right)^2$ Direct Formula
Standard Deviation	
(i) $S = \sqrt{\frac{\sum (X - \bar{X})^2}{n}}$ Proper mean formula	(i) $S = \sqrt{\frac{\sum f(X - \bar{X})^2}{\sum f}}$ Proper mean formula
(ii) $S = \sqrt{\frac{\sum X^2}{n} - \left(\frac{\sum X}{n} \right)^2}$ Direct Formula	(ii) $S = \sqrt{\frac{\sum fX^2}{\sum f} - \left(\frac{\sum fX}{\sum f} \right)^2}$ Direct Formula

Miscellaneous Exercise 6

Q.1 Multiple choice questions

Three possible answers are given for the following question. Tick (✓) the correct answer.

- (i) A grouped frequency table is also called (SWL 2014, MTN 2015) **(K.B)**
 (a) Data (b) Frequency Distribution (c) Frequency Polygon
- (ii) A histogram is a set of adjacent (LHR 2014, 15, MTN 2015) **(K.B)**
 (a) Squares (b) Rectangles (c) Circles
- (iii) A frequency polygon is a many sided (LHR 2014) **(K.B)**
 (a) Closed figure (b) Rectangle (c) Square
- (iv) A cumulative frequency table is also called **(K.B)**
 (a) Frequency distribution (b) Data (c) Less than cumulative frequency distribution
- (v) In a cumulative frequency polygon frequencies are plotted against **(K.B)**
 (a) Midpoints (b) Upper class boundaries (c) Class limits
- (vi) Arithmetic mean is a measure that determines a value of the variable under study by dividing the sum of all values of the variable by their **(K.B)**
 (a) Number (b) Group (c) Denominator
- (vii) A deviation is defined as a difference of any value of the variable from a **(K.B)**
 (a) Constant (b) Histogram (c) Sum
- (viii) A data in the form of frequency distribution is called **(K.B)**
 (a) Grouped data (b) Ungrouped data (c) Histogram
- (ix) Mean of a variable with similar observations say constant k is **(K.B)**
 (a) Negative (b) k itself (c) Zero
- (x) Mean is affected by change in (LHR 2015, FSD 2014) **(K.B)**
 (a) Value (b) Ratio (c) Origin
- (xi) Mean is affected by change in **(K.B)**
 (a) Place (b) Scale (c) Rate
- (xii) Sum of the deviations of the variable X from its mean is always **(K.B)**
 (a) Zero (b) One (c) Same (FSD 2016, 18, D.G.K 2015)
- (xiii) The n^{th} positive root of the product of the $x_1, x_2, x_3, \dots, x_n$ observations is called **(K.B)**
 (a) Mode (b) Mean (c) Geometric mean
- (xiv) The value obtained by reciprocating the mean of the reciprocal of $x_1, x_2, x_3, \dots, x_n$ observations is called **(K.B)**
 (a) Geometric mean (b) Median (c) Harmonic mean
- (xv) The most frequent occurring observation in a data set is called **(K.B)**
 (a) Mode (b) Median (c) Harmonic mean (GF.W 2014, FSD 2014, 15, SWL 2014)
- (xvi) The measure which determines the middlemost observation in a data set is called **(K.B)**
 (a) Median (b) Mode (c) Mean
- (xvii) The observations that divide a data set into four equal parts are called (SWL 2015, 16) **(K.B)**
 (a) Deciles (b) Quartiles (c) Percentiles
- (xviii) The spread or scatterness of observations in a data set is called **(K.B)**
 (a) Average (b) Dispersion (c) Central tendency (FSD 2014, SGD 2014, D.G.K 2014)
- (xix) The measures that are used to determine the degree or extent of variation in a data set are called measures of **(K.B)**
 (a) Dispersion (b) Central tendency (c) Average

- (xx) The extent of variation between two extreme observations of a data set is measured by (RWP 2015) **(K.B)**
 (a) Average (b) Range (c) Quartiles
- (xxi) The mean of the squared deviations of $x_i (i=1, 2, \dots, n)$ observations from their arithmetic mean is called **(K.B)**
 (a) Variance (b) Standard deviation (c) Range
- (xxii) The positive square root of mean of the squared deviations of $X_i (i=1, 2, \dots, n)$ observations from their arithmetic mean is called **(K.B)**
 (a) Harmonic mean (b) Range (c) Standard deviation

ANSWER KEY

i	b	v	b	ix	b	xiii	c	xvii	b	xxi	a
ii	b	vi	a	x	c	xiv	c	xviii	b	xxii	c
iii	a	vii	a	xi	b	xv	a	xix	a		
iv	c	viii	a	xii	a	xvi	a	xx	b		

Q.2 Write short answers of the following questions.

- (i) Define class limits. (RWP 2015, BWP 2016, D.G.K 2014) **(K.B)**
 Ans: See definition page # 184
- (ii) Define class mark. **(K.B)**
 Ans: See definition page # 184
- (iii) What is cumulative frequency? (GRW 2014, BWP 2011, 14, RWP 2016, SGD 2014, 16) **(K.B)**
 Ans: See definition page # 185
- (iv) Define a frequency distribution. (BWP 2015, SWL 2016, MTN 2016) **(K.B)**
 Ans: See definition page # 184
- (v) What is Histogram? (GRW 2016, FSD 2014, SWL 2017, SGD 2015, RWP 2014) **(K.B)**
 Ans: See definition page # 185
- (vi) Name two measures of central tendency. **(K.B)** (LHR 2014, GRW 2016, SWL 2015, D.G.K 2015)
 Ans: See definition page # 196
- (vii) Define Arithmetic mean. (LHR 2016, GRW 2016, FSD 2015, SGD 2015, D.G.K 2014) **(K.B)**
 Ans: See definition page # 188
- (viii) Write three properties of Arithmetic mean. **(K.B)**
 (LHR 2016, 17, GRW 2014, FSD 2014, 15, 17, SWL 2016, RWP 2014, BWP 2014, SGD 2015, MTN 2017)
 Ans: See definition page # 189
- (ix) Define Median. (LHR 2014, BWP 2015, SGD 2016, RWP 2017, MTN 2017) **(K.B)**
 Ans: See definition page # 188
- (x) Define Mode? **(K.B)**
 (LHR 2014, 17, GRW 2017, SWL 2015, 16, BWP 2014, 17, MTN 2015, 16, 17, D.G.K 2017)
 Ans: See definition page # 189
- (xi) What do you mean by Harmonic mean? **(K.B)** (GRW 2015, SWL 2015, MTN 2015, SGD 2015)
 Ans: See definition page # 189
- (xii) Define Geometric mean. (LHR 2015, FSD 2015, SWL 2016, D.G.K 2016) **(K.B)**
 Ans: See definition page # 189
- (xiii) What is Range? (SWL 2014) **(K.B)**
 Ans: See definition page # 206
- (xiv) Define Standard deviation. **(K.B)**
 (LHR 2016, FSD 2015, SWL 2014, 15, BWP 2017, MTN 2015, 17, SGD 2014, 17, D.G.K 2014)
 Ans: See definition page # 207

**SELF TEST**

Time: 40 min

Marks: 25

Q.1 Four possible answers (A), (B), (C) & (D) to each question are given, mark the correct answer. (7×1=7)

1 The value obtained by reciprocating the mean of the reciprocal of $x_1, x_2, x_3, \dots, x_n$ observations is called:

- (A) Geometric mean (B) Median
(C) Harmonic mean (D) Mode

2 If $D = x - 10$, $\sum D = 50$ and $n = 5$ then arithmetic mean is:

- (A) 10 (B) 20
(C) 50 (D) 15

3 In 4,3,2,1,5,6,7 the mode is:

- (A) 0 (B) 4
(C) 7 (D) No mode

4 If $\sum (x - \bar{x})^2 = 40$, $n = 5$ then standard deviation is:

- (A) 8 (B) 2.83
(C) 200 (D) 14.1

5 In a cumulative frequency polygon frequencies are plotted against:

- (A) Midpoints (B) Upper class boundaries
(C) Class limits (D) Lower class boundaries

6 A histogram is a set of adjacent:

- (A) Squares (B) Rectangles
(C) Circles (D) Closed figure

7 The measure which determines the middle most observation in a data set is called:

- (A) Mean (B) Mode
(C) Median (D) None

Q.2 Give Short Answers to following Questions.

(5×2=10)

(i) Find the standard deviation for the data: 12,6,7,3,15,10,18,5.

(ii) Find arithmetic mean by indirect method for the set of data: 200,225,350,375,270,320,290.

(iii) Find the geometric mean of 2, 4, 8 using logarithmic formula.

(iv) Find median.

Class Mark (X)	5	10	15	20	25	30
Frequency (f)	2	12	25	32	14	5

(v) On a vacation trip a family bought 21.3 liters of petrol at 39.90 rupees per liter, 18.7 liters at 42.90 rupees per liter, and 23.5 liters at 40.90 rupees per liter. Find the mean price paid per liter.

Q.3 Answer the following Questions.

(4+4=8)

(a) The length of 32 items are given below, find the mean length and standard deviation of the distribution.

Length	20-22	23-25	26-28	29-31	32-34
Frequency	3	6	12	9	2

(b) On a prize distribution day, 50 students brought pocket money as under. Find mode.

Rupees	5-10	10-15	15-20	20-25	25-30
Frequency (f)	12	9	18	7	4

NOTE: Parents or guardians can conduct this test in their supervision in order to check the skill of students.

UNIT 7

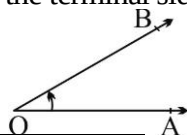
INTRODUCTION TO TRIGONOMETRY

Angle

(K.B)

(LHR 2015, SWL 2015, MTN 2016)

Two non-collinear rays with a common end point form an angle. The rays are called arms of the angle and the common point is called vertex of the angle. The original position of the ray is called initial side and the final position of the ray is called the terminal side of the angle.



Sexagesimal System

(D.G.K 2015) (K.B)

System of measurement of an angle in degrees, minutes and seconds is called sexagesimal system.

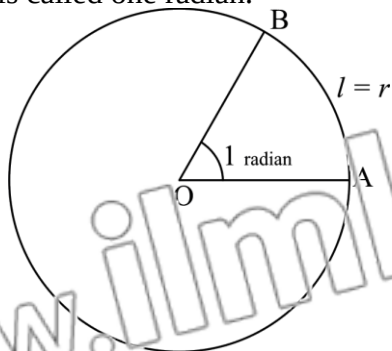
Degree

(RWP 2015, BWP 2015, SGD 2015) (K.B)

If we divide a circle/ circumference into 360 equal parts/arcs then central angle of one arc is called degree it is denoted by 1° .

Radians (GRW 2014, FSD 2015) (K.B)

The angle subtended at the centre of the circle by an arc, whose length is equal to its radius is called one radian.



$m\angle AOB = 1$ radians if $\overline{mOA} = \overline{mAB}$

Circular System

(K.B)

System of measurement of an angle in radians is called circular system.

UNIT

7

Relationship between Radians and Degrees

(K.B)

$$180^\circ = \pi \text{ radians}$$

$$\Rightarrow 1^\circ = \frac{\pi}{180} \text{ radians}$$

$$\Rightarrow x^\circ = \frac{x\pi}{180} \text{ radians}$$

And

$$\Rightarrow 1 \text{ radian} = \frac{180^\circ}{\pi}$$

$$\Rightarrow x \text{ radians} = \frac{x(180^\circ)}{\pi}$$

Note

(K.B)

- $1 \text{ radians} = \frac{180^\circ}{\pi} \approx 57.295795^\circ \approx 57^\circ 17' 45''$
- $1^\circ = \frac{\pi}{180} \text{ radians} \approx 0.0175 \text{ radians}$

Important Formulae

(K.B)

$$x^\circ = \frac{x\pi}{180} \text{ radians}$$

$$x \text{ radians} = \frac{x(180^\circ)}{\pi}$$

$$l = r\theta$$

$$A = \frac{1}{2} r^2 \theta$$

Example 2 (Page # 147)

(K.B)

Convert $12^\circ 23' 35''$ to decimal degrees correct to three decimal places.

Solution:

$$\begin{aligned} 12^\circ 23' 35'' &= \left(12 + \frac{23}{60} + \frac{35}{3600} \right)^\circ \\ &\approx (12 + 0.3833 + 0.00972)^\circ \\ &\approx 12.393^\circ \end{aligned}$$

Example 3 (Page # 148)

(A.B)

Convert 45.36° to $D^\circ M' S''$ form.

Solution:

$$\begin{aligned} 45.36^\circ &= 45^\circ + .36 \times 60' \\ &= 45^\circ + 21.6' \\ &= 45^\circ + 21' + .6 \times 60'' \\ &= 45^\circ + 21' + 36'' = 45^\circ 21' 36'' \end{aligned}$$

Example 4 (Page # 149)

(A.B)

Convert $124^\circ 22'$ into radian measure.

Solution:

$$124^\circ 22' = \left(124 + \frac{22}{60} \right)^\circ = 124.3666^\circ$$

$$= 124.3666 \times \frac{\pi}{180} \text{ radians}$$

$$\approx 2.171 \text{ radians}$$

Example 5 (Page # 149)

(A.B)

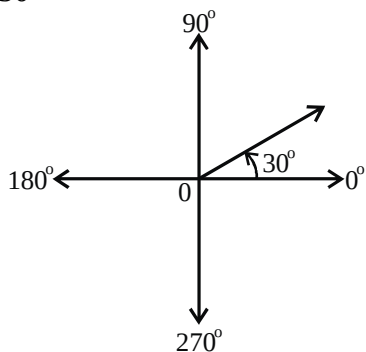
Express $\frac{2\pi}{3}$ radians into degrees.

$$\begin{aligned} \frac{2\pi}{3} \text{ radians} &= \frac{2\pi}{3} \times \frac{180^\circ}{\pi} \because 1 \text{ rad} = \left(\frac{180}{\pi} \right)^\circ \\ &= 120^\circ \end{aligned}$$

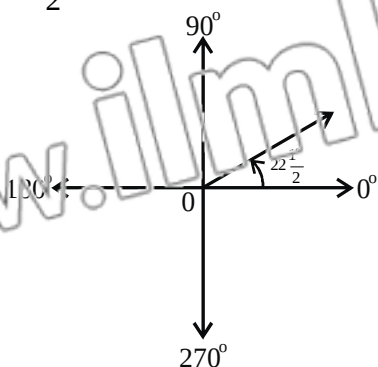
Exercise 7.1

Q.1 Locate the following angles:

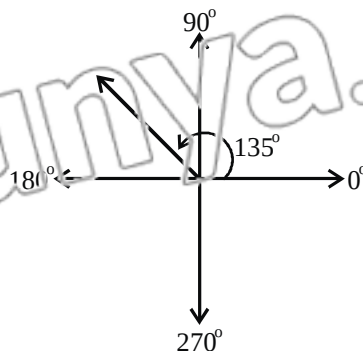
(i) 30° (A.B)



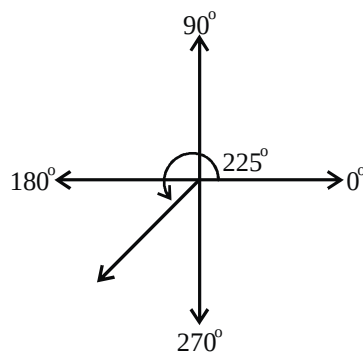
(ii) $22\frac{1}{2}^\circ$ (A.B)



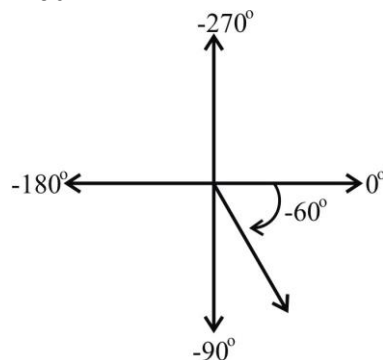
(iii) 135° (A.B)



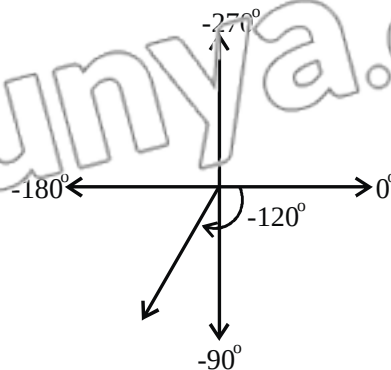
(iv) 225° (A.B)



(v) -60° (A.B)

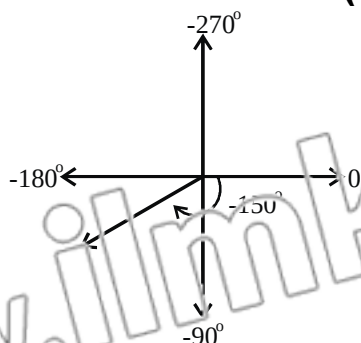


(vi) -120° (A.B)

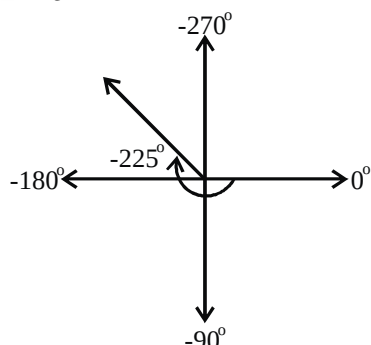


(vii) -150°

(A.B)

(viii) -225°

(A.B)



Q.2 Express the following sexagesimal measures of angles in decimal form:

(i) $45^\circ 30'$

(A.B)

$$\begin{aligned}
 &= 45^\circ + \left(\frac{30}{60}\right)^\circ \\
 &= 45^\circ + 0.5^\circ \\
 &= 45.5^\circ
 \end{aligned}$$

(ii) $60^\circ 30' 30''$

(A.B)

$$\begin{aligned}
 &= 60^\circ + \left(\frac{30}{60}\right)^\circ + \left(\frac{30}{3600}\right)^\circ \\
 &= 60^\circ + 0.5^\circ + 0.0083^\circ \\
 &= 60.5083^\circ
 \end{aligned}$$

(iii) $125^\circ 22' 50''$

(A.B)

$$\begin{aligned}
 &= 125^\circ + \left(\frac{22}{60}\right)^\circ + \left(\frac{50}{3600}\right)^\circ \\
 &= 125^\circ + (0.367)^\circ + (0.0139)^\circ \\
 &= 125.3809^\circ
 \end{aligned}$$

Q.3 Express the following into D°M'S' form. (SWL 2015, BWP 2016)

(i) 47.36°

(A.B)

$$\begin{aligned}
 &= 47^\circ + (0.36 \times 60)' \because 1^\circ = 60' \\
 &= 47^\circ + 21.6'
 \end{aligned}$$

$$\begin{aligned}
 &= 47^\circ + 21' + (0.6 \times 60)'' \\
 &= 47^\circ + 21' + 36'' \\
 &= 47^\circ 21' 36''
 \end{aligned}$$

(ii)

 125.45°

(A.B)

$$\begin{aligned}
 &= 125^\circ + (0.45 \times 60)' \\
 &= 125^\circ + 27' \\
 &= 125^\circ 27'
 \end{aligned}$$

(iii)

 225.75°

(A.B)

$$\begin{aligned}
 &= 225^\circ + (0.75 \times 60)' \\
 &= 225^\circ + 45' \\
 &= 225^\circ 45'
 \end{aligned}$$

(iv)

 -22.5°

(A.B)

$$\begin{aligned}
 &= -(22^\circ + (0.5 \times 60)') \\
 &= -(22^\circ + 30') \\
 &= -22^\circ 30'
 \end{aligned}$$

(v)

 -67.58°

(A.B)

$$\begin{aligned}
 &= -(67^\circ + (0.58 \times 60)') \\
 &= -(67^\circ + 34.8') \\
 &= -(67^\circ + 34' + (0.8 \times 60)') \\
 &= -(67^\circ + 34' + 48'') \\
 &= -(67^\circ 34' 48'')
 \end{aligned}$$

(vi)

 315.18°

(A.B)

$$\begin{aligned}
 &= 315^\circ + (0.18 \times 60)' \\
 &= 315^\circ + 10.8' \\
 &= 315^\circ + 10' + (0.8 \times 60)'' \\
 &= 315^\circ + 10' + 48'' \\
 &= 315^\circ 10' 48''
 \end{aligned}$$

Q.4 Express the following angles into radians.

(i)

 30°

(A.B)

$$\begin{aligned}
 &= 30 \times \frac{\pi}{180} \text{ radians} \\
 &= \frac{\pi}{6} \text{ radians}
 \end{aligned}$$

(ii) 60° (A.B)

$$= 60 \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{\pi}{3} \text{ radians}$$

(iii) 135° (BWP 2014, D.G.K 2016) (A.B)

$$= 135 \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{3\pi}{4} \text{ radians}$$

(iv) 225° (A.B)

$$= 225 \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{5\pi}{4} \text{ radians}$$

(v) -150° (BWP 2014, D.G.K 2016) (A.B)

$$= -150 \times \frac{\pi}{180} \text{ radians}$$

$$= -\frac{5\pi}{6} \text{ radians}$$

(vi) -225° (A.B)

$$= -225 \times \frac{\pi}{180} \text{ radians}$$

$$= -\frac{5\pi}{4} \text{ radians}$$

(vii) 300° (SGD 2015) (A.B)

$$= 300 \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{5\pi}{3} \text{ radians}$$

(viii) 315° (A.B)

$$= 315 \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{7\pi}{4} \text{ radians}$$

Q.5 Convert each of the following to degrees

(i) $\frac{3\pi}{4}$ (LHR 2015, MTN 2016, GRW 2016) (A.B)

$$= \left(\frac{3\pi}{4} \times \frac{180}{\pi} \right)^\circ \because 1 \text{ radians} = \frac{180}{\pi} \text{ degree}$$

$$= 3 \times 45^\circ$$

$$= 135^\circ$$

(ii) $\frac{5\pi}{6}$ (A.B)

(SWL 2014, SGD 2016, MTN 2015, 16)

$$= \left(\frac{5\pi}{6} \times \frac{180}{\pi} \right)^\circ$$

$$= 150^\circ$$

(iii) $\frac{7\pi}{8}$ (A.B)

$$= \left(\frac{7\pi}{8} \times \frac{180}{\pi} \right)^\circ$$

$$= \frac{315^\circ}{2}$$

$$= 157.5^\circ$$

(iv) $\frac{13\pi}{16}$ (A.B)

$$= \left(\frac{13\pi}{16} \times \frac{180}{\pi} \right)^\circ$$

$$= \left(\frac{585}{4} \right)^\circ$$

$$= 146.25^\circ$$

(v) 3 (A.B)

$$= \left(3 \times \frac{180}{\pi} \right)^\circ$$

$$= 171.89^\circ$$

(vi) 4.5 (A.B)

$$= \left(4.5 \times \frac{180}{\pi} \right)^\circ = 257.83^\circ$$

(vii) $-\frac{7\pi}{8}$ (A.B)

(GRW 2017, RWP 2015, D.G.K 2017)

$$= -\left(\frac{7\pi}{8} \times \frac{180}{\pi} \right)^\circ = -157.5^\circ$$

(viii) $-\frac{13\pi}{16}$ (A.B)

$$= -\left(\frac{13\pi}{16} \times \frac{180}{\pi} \right)^\circ = -146.25^\circ$$

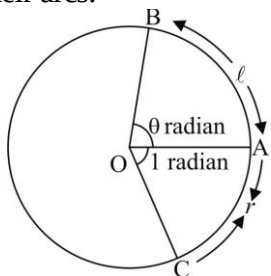
Relation between Length of Arc and Radius of a Circle (K.B)

Statement:

Establish the rule $l = r\theta$, where l is the length of an arc, r is radius of a circle and θ is the central angle measured in radians.

Proof:

Let an arc AB denoted by l , subtends a central angle θ radians. Consider another arc AC whose length is equal to its radius then its central angle will be equal to 1 radian. In plane geometry, measure of central angles of the arcs of a circle are proportional to the lengths of their arcs.



$$\therefore \frac{m\angle AOB}{m\angle AOC} = \frac{mAB}{mAC}$$

$$\frac{\theta \text{ radians}}{1 \text{ radian}} = \frac{l}{r}$$

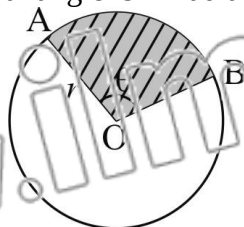
$$\text{Or } r\theta = l$$

$$\text{Or } l = r\theta$$

Area of Sector (K.B)

$$A = \frac{1}{2} r^2 \theta$$

Find area of sector of a circle whose radius is r and central angle is θ radians.



Consider a sector AOB , whose central angle is θ radians. In plane geometry

$$\frac{\text{Area of Sector}}{\text{Area of Circle}} = \frac{\text{Angle of the Sector}}{\text{Angle of the Circle}}$$

$$\therefore \frac{\text{Area of Sector}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\text{Area of sector } AOB = \frac{\theta}{2\pi} \times \pi r^2$$

$$\text{Area of sector } AOB = \frac{1}{2} r^2 \theta$$

Example 2: (Page # 151)

(A.B)

Find the distance travelled by a cyclist moving on a circle of radius $15m$, if he makes 3.5 revolutions.

Solution:

$$1 \text{ revolution} = 2\pi \text{ radians}$$

$$3.5 \text{ revolutions} = 3.5 \times 2\pi \text{ radians} \\ = 7\pi \text{ radians}$$

$$\text{Radius of circle} = 15m$$

$$\text{Distance travelled} = l = ?$$

Formula

$$l = r\theta$$

Putting the values

$$l = 7\pi \times 15m \\ = 105\pi m = 329.87m$$

$$\therefore \text{Distance travelled by cyclist} = 329.87m$$

Example 3: (Page # 152)

(A.B)

(LHR 2014, D.G.K 2014)

Find the area of sector of a circle of radius $16cm$ if the angle at the centre is 60° .

Solution:

$$\text{Radius of circle} = r = 16cm$$

$$\text{Angle of sector} = \theta = 60^\circ$$

$$= 60 \times \frac{\pi}{180} = \frac{\pi}{3} \text{ radians}$$

Formula

$$\text{Area of sector} = A = \frac{1}{2} r^2 \theta$$

Putting the values

$$A = \frac{1}{2} (16)^2 \frac{\pi}{3} cm^2$$

$$\Rightarrow A = 143.1 cm^2$$

Exercise 7.2

Q.1

- (i) Find θ , when $l = 2\text{cm}$, $r = 3.5\text{cm}$
(LHR 2015, GRW 2016, BWP 2014, MTN 2015, 16, 17, SGD 2015) **(A.B)**

Solution:

We know that

$$l = r\theta$$

or

$$\theta = \frac{l}{r}$$

Putting the values

$$= \frac{2\text{cm}}{3.5\text{cm}}$$

$$\theta = 0.57\text{radians}$$

- (ii) Given: **(A.B)**

$$l = 4.5\text{m}, r = 2.5\text{m}$$

(FSD 2014, SGD 2014, 16, MTN 2016, D.G.K 2015, 17)

Required:

$$\theta = ?$$

Solution:

We know that

$$\theta = \frac{l}{r}$$

Putting the values

$$\theta = \frac{4.5}{2.5}$$

$$\Rightarrow \theta = 1.8\text{ radians}$$

Q.2

- (i) Given $\theta = 180^\circ$, $r = 4.9\text{cm}$ **(A.B)**
(LHR 2014, GRW 2014, FSD 2015, SWL 2016, D.G.K 2016)

Required:

$$l = ?$$

Solution:

Here

$$\theta = 180^\circ$$

$$= 180 \times \frac{\pi}{180} \text{ radians } \therefore 1^\circ = \frac{\pi}{180} \text{ rad}$$

$$= 3.14 \text{ radians}$$

We know that

$$l = r\theta$$

Putting the values

$$= 4.9\text{ cm} \times 3.14$$

$$\Rightarrow l = 15.39\text{ cm}$$

- (ii) Find l , when $\theta = 60^\circ 30'$, $r = 15\text{mm}$
(LHR 2014, SWL 2016, BWP 2016, MTN 2015)

Solution:

Here

$$\theta = 60^\circ 30'$$

$$= 60^\circ + \left(\frac{30}{60}\right)^\circ$$

$$= 60.5^\circ$$

$$= 60.5 \times \frac{\pi}{180} \text{ radians}$$

$$= 1.0559 \text{ radians}$$

We know that

$$l = r\theta$$

putting the values

$$= (15\text{mm})(1.0559)$$

$$\Rightarrow l = 15.84\text{mm}$$

Q.3 **(A.B)**

- (i) Find r , when $l = 4\text{cm}$, $\theta = \frac{1}{4} \text{radian}$

Solution:

We know that

$$r = \frac{l}{\theta}$$

Putting the values

$$r = \frac{4\text{cm}}{\frac{1}{4}}$$

$$\Rightarrow r = 16\text{cm}$$

- (ii) Given: $l = 52\text{cm}$, $\theta = 45^\circ$ **(A.B)**

(LHR 2015, 17, GRW 2017, SWL 2014, 17, RWP 2010, 15, D.G.K 2017)

Required:

$$r = ?$$

Solution:

Here $\theta = 45^\circ$

$$= 45 \times \frac{\pi}{180} \text{ radians } \therefore 1^\circ = \frac{\pi}{180} \text{ rad}$$

$$= 0.785 \text{ radians}$$

We know that

$$r = \frac{l}{\theta}$$

$$= \frac{52}{0.785} \text{ cm}$$

$$\Rightarrow r = 66.21\text{cm}$$

- Q.4 Given: $r = 12\text{m}$, $\theta = 1.5 \text{radian}$
(SGD 2014) **(A.B)**

Required:

$$l = ?$$

Solution:

We know that

$$l = r\theta$$

Putting the values

$$l = (12\text{m})(1.5)$$

Unit-7

Introduction to Trigonometry

$$\Rightarrow l = 18m$$

Q.5 Given: $r = 10m$, $\theta =$ angle formed by 3.5 revolutions **(A.B)**

Required:

$$l = ?$$

Solution:

Angle in one revolution $= 2\pi$ radians

Angle in 3.5 revolution

$$= (2\pi \times 3.5) \text{ radians}$$

$$= 7\pi \text{ radians}$$

We know that

$$l = r\theta$$

Putting the values

$$= (10m)(7\pi)$$

$$= 70\pi m$$

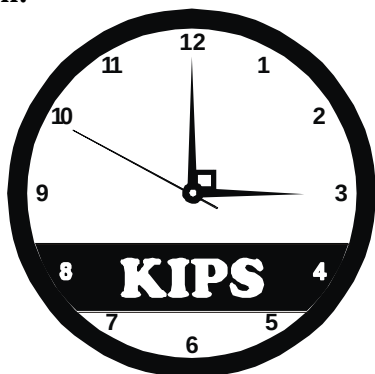
$$= 220m$$

Result:

Distance travelled by point $= 220m$

Q.6 What is the circular measure of the angle between the hands of the watch at 3 O' clock? **(A.B)**

Solution:



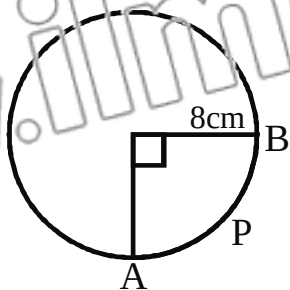
Circular measure of angle between

$$\text{hands of clock} = 90 \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{\pi}{2} \text{ radian}$$

Q.7 What is the length of arc APB? **(A.B)**

Solution:



Here

$$\theta = 90^\circ$$

$$= 90 \times \frac{\pi}{180} \text{ radians} \because 1^\circ = \frac{\pi}{180} \text{ rad}$$

$$= \frac{\pi}{2} \text{ radian}$$

$$r = 8cm$$

We know that

$$l = r\theta$$

Putting the values

$$l = (8cm) \left(\frac{\pi}{2} \text{ radian} \right)$$

$$= 4\pi \text{ radians}$$

$$l = 12.57cm$$

Result:

Length of arc APB $= 12.57cm$

Q.8 In a circle of radius 12cm, how long an arc subtends a central angle of 84° ? **(A.B)**

Given:

$$r = 12cm, \theta = 84^\circ$$

Required:

$$l = ?$$

Solution:

Here

$$\theta = 84^\circ$$

$$= 84 \times \frac{\pi}{180} \text{ radian} \because 1^\circ = \frac{\pi}{180} \text{ rad}$$

$$= 1.466 \text{ radian}$$

We know that

$$l = r\theta$$

Putting the values

$$= (12cm)(1.466)$$

$$= 17.592cm$$

Result:

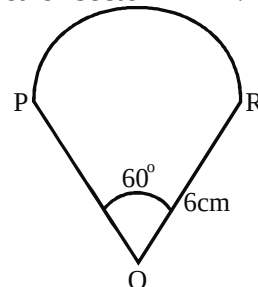
$\therefore$ Length of arc $= 17.592cm$

Q.9 Find the area of the sector OPR. **(A.B)**

$$(a) \text{ Given: } \theta = 60^\circ, r = 6cm$$

Required:

$$\text{Area of sector} = A = ?$$



Solution:

$$\theta = 60^\circ$$

$$= 60 \times \frac{\pi}{180} \text{ radians } \because 1^\circ = \frac{\pi}{180} \text{ rad}$$

$$= \frac{\pi}{3} \text{ radians}$$

We know that

$$A = \frac{1}{2} r^2 \theta$$

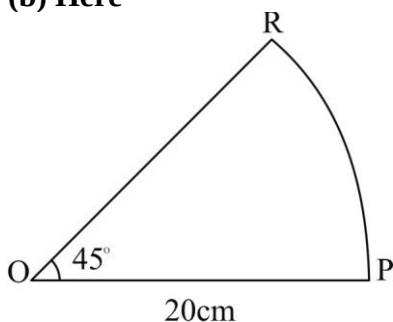
Putting the values

$$= \frac{1}{2} (6\text{cm})^2 \left(\frac{\pi}{3} \right)$$

$$= \frac{1}{2} \times 36 \times \frac{\pi}{3} = 6\pi$$

$$\Rightarrow A = 18.85\text{cm}^2$$

(b) Here



$$\theta = 45^\circ$$

$$= 45 \times \frac{\pi}{180} \text{ radians } \because 1^\circ = \frac{\pi}{180} \text{ rad}$$

$$= \frac{\pi}{4} \text{ radians}$$

We know that

$$A = \frac{1}{2} r^2 \theta$$

Putting the values

$$A = \frac{1}{2} (20\text{cm})^2 \left(\frac{\pi}{4} \right)$$

$$= \frac{1}{2} \times 400 \times \frac{\pi}{4} = 50\pi$$

$$\Rightarrow A = 157.08\text{cm}^2$$

Q.10 Find the area of sector inside a central angle of 20° in a circle of radius 7m. **(A.B)**

Given:

$$\theta = 20^\circ, r = 7\text{m}$$

Area of sector = $A = ?$

Here

$$\theta = 20^\circ$$

$$= 20 \times \frac{\pi}{180} \text{ radians } \because 1^\circ = \frac{\pi}{180} \text{ rad}$$

$$= \frac{\pi}{9} \text{ radians}$$

We know that

$$A = \frac{1}{2} r^2 \theta$$

Putting the values

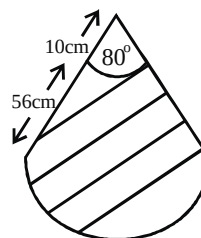
$$A = \frac{1}{2} (7\text{m})^2 \left(\frac{\pi}{9} \right)$$

$$= \frac{1}{2} \times 49 \times \frac{\pi}{9} \text{ m}^2$$

$$= \frac{49}{18} \times 3.14 \text{ m}^2$$

$$\Rightarrow A = 8.552\text{m}^2$$

Q.11 Sehar is making a skirt. Each panel of this skirt is of the shape shown shaded in the diagram. How much material (cloth) is required for each panel? **(A.B)**



Given:

$$\theta = 80^\circ, r_1 = 10\text{cm}$$

$$r_2 = 56\text{cm} + 10\text{cm} = 66\text{cm}$$

Required:

Area of shaded region = $A = ?$

Solution:

Here

$$\theta = 80^\circ$$

$$= 80 \times \frac{\pi}{180} \text{ radians } \because 1^\circ = \frac{\pi}{180} \text{ rad}$$

$$= \frac{20\pi}{45} \text{ radians}$$

Formula:

$$\text{Area of small sector} = \frac{1}{2} r_1^2 \theta$$

$$= \frac{1}{2} \times (10)^2 \times \frac{20\pi}{45}$$

$$= \frac{1}{2} \times 100 \times \frac{20\pi}{45}$$

$$= 50 \times \frac{20\pi}{45}$$

$$= 69.841\text{cm}^2$$

$$\text{Area of big sector} = \frac{1}{2} r_2^2 \times \theta$$

$$= \frac{1}{2} (66\text{cm})^2 \times \frac{20\pi}{45}$$

$$\Rightarrow A = 3042.285\text{cm}^2$$

Area of shaded region =

Area of big sector – Area of small sector

$$= 3042.285 - 69.841$$

$$= 2972.4\text{cm}^2$$

$\therefore 2972.4\text{cm}^2$ material (cloth) is required for each panel.

Q.12 Find the area of sector with

central angle of $\frac{\pi}{5}$ radian in a circle

of radius 10cm.

(A.B)

Given:

$$\theta = \frac{\pi}{5} \text{ radians, } r = 10\text{cm}$$

Required:

$$A = ?$$

Solution:

We know that

$$A = \frac{1}{2} r^2 \theta$$

Putting the values

$$= \frac{1}{2} (10\text{cm})^2 \left(\frac{\pi}{5} \right)$$

$$= \frac{1}{2} \times 100 \times \frac{\pi}{5} = 10\pi$$

$$\Rightarrow A = 31.416\text{cm}^2$$

Q.13 The area of the sector with a central angle θ in a circle of radius 2m is 10 square metre. Find θ in radians.

(A.B)

Given:

Area of Sector = $A = 10$ square m

Radius of circle $r = 2\text{cm}$

Required:

Central Angle = $\theta = ?$

Solution:

We know that

$$A = \frac{1}{2} r^2 \theta$$

$$\text{Or } \theta = \frac{2A}{r^2}$$

$$= \frac{2 \times 10}{(2)^2}$$

$$= \frac{20}{4}$$

$$= 5 \text{ radians}$$

Coterminal Angles

(K.B)

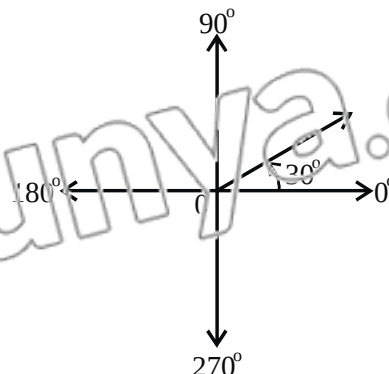
Two or more angles with the same initial and terminal sides are called coterminal angles. For example 30° , 390° , 750° , are coterminal angles $(\theta + 360k), k \in \mathbb{Z}$

Angle in Standard Position

(K.B)

A general angle is said to be in standard position if its vertex is at the origin and its initial side is directed along the positive direction of the x -axis of a rectangular coordinate system.

For example, angle below is in standard position.

**Quadrantal Angles**

(K.B)

If the terminal side of an angle in standard position falls on x -axis or y -axis then it is called a quadrantal angle i.e. 0° , 90° , 180° , 270° , 360° are quadrantal angles

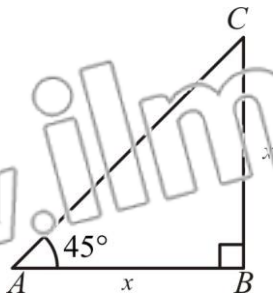
Unit-7

Introduction to Trigonometry

Trigonometric Ratios of 45°

(K.B + U.B + A.B)

Consider a right angled isosceles $\triangle ABC$, in which $m\overline{AB} = m\overline{BC} = x$



By Pythagoras theorem,

$$m\overline{AC} = x^2 + x^2 \\ = 2x^2$$

$$m\overline{AC} = \sqrt{2x^2} \\ = \sqrt{2}x$$

Now trigonometric Ratios are

$$\sin 45^\circ = \frac{m\overline{BC}}{m\overline{AC}} = \frac{x}{\sqrt{2}x} \\ = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{m\overline{AB}}{m\overline{AC}} \\ = \frac{x}{\sqrt{2}x} \\ = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{m\overline{BC}}{m\overline{AB}} = \frac{x}{x} \\ = 1$$

$$\operatorname{cosec} 45^\circ = \frac{1}{\sin 45^\circ} \\ = \sqrt{2}$$

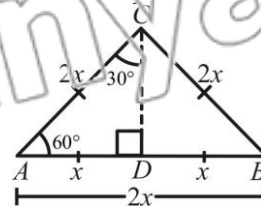
$$\sec 45^\circ = \frac{1}{\cos 45^\circ} \\ = \sqrt{2}$$

$$\cot 45^\circ = \frac{1}{\tan 45^\circ} \\ = 1$$

Trigonometric Ratios of 60° & 30°

(K.B + U.B + A.B)

Consider an equilateral $\triangle ABC$,



In which $m\overline{AB} = m\overline{BC} = m\overline{AC} = 2x$

Draw $\overline{CD} \perp \overline{AB}$, then

$$\overline{AD} = \overline{BD} = x$$

From $\triangle CAD$, using Pythagoras theorem

$$(m\overline{AD})^2 + (m\overline{CD})^2 = (m\overline{AC})^2$$

$$x^2 + (m\overline{CD})^2 = (2x)^2$$

$$(m\overline{CD})^2 = 4x^2 - x^2$$

$$(m\overline{CD})^2 = 3x^2$$

$$\Rightarrow m\overline{CD} = \sqrt{3x^2} \\ = \sqrt{3}x$$

Trigonometric ratios of 60° are:

$$\sin 60^\circ = \frac{m\overline{CD}}{m\overline{AC}} = \frac{\sqrt{3}x}{2x} \\ = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{m\overline{AD}}{m\overline{AC}} = \frac{x}{2x} \\ = \frac{1}{2}$$

$$\tan 60^\circ = \frac{m\overline{CD}}{m\overline{AD}} \\ = \frac{\sqrt{3}x}{x} \\ = \sqrt{3}$$

$$\operatorname{cosec} 60^\circ = \frac{1}{\sin 60^\circ} \\ = \frac{2}{\sqrt{3}}$$

$$\sec 60^\circ = \frac{1}{\cos 60^\circ} \\ = 2$$

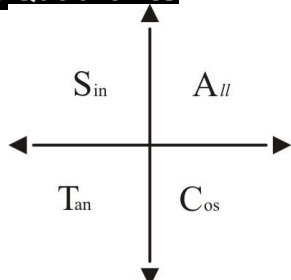
$$\cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}$$

Now Trigonometric Ratios of 30°

In $\triangle ACD$

$$\begin{aligned}\sin 30^\circ &= \frac{m\overline{AD}}{m\overline{AC}} = \frac{x}{2x} = \frac{1}{2} \\ \cos 30^\circ &= \frac{m\overline{CD}}{m\overline{AC}} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2} \\ \tan 30^\circ &= \frac{m\overline{AD}}{m\overline{CD}} = \frac{x}{\sqrt{3}x} = \frac{1}{\sqrt{3}} \\ \operatorname{cosec} 30^\circ &= \frac{1}{\sin 30^\circ} = \frac{2}{1} \\ \sec 30^\circ &= \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}} \\ \cot 30^\circ &= \frac{1}{\tan 30^\circ} = \sqrt{3}\end{aligned}$$

Signs of Trigonometric Ratios in Different Quadrants (K.B)



(ASTC = After school to college)

NOTE:

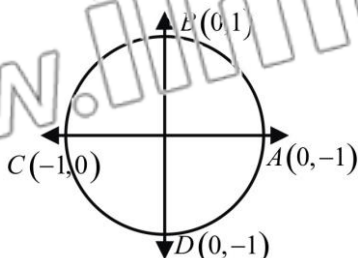
Where a ratio is +ve, its reciprocal is also +ve. And remaining ratios are -ve.

Trigonometric Ratios of Quadrantal Angles (0° , 90° , 180° & 270° & 360°)

Trigonometric Ratios of 0°

(K.B + U.B + A.E)

Consider a unit circle.



Then $m\overline{OA} = m\overline{OB} = m\overline{OC} = m\overline{OD} = 1$

$$\begin{aligned}r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(1)^2 + (0)^2} \\ &= \sqrt{1+0} \\ &= \sqrt{1} \\ &= 1\end{aligned}$$

$$\sin 0^\circ = \frac{y}{r} = \frac{0}{1} = 0$$

$$\cos 0^\circ = \frac{x}{r} = \frac{1}{1} = 1$$

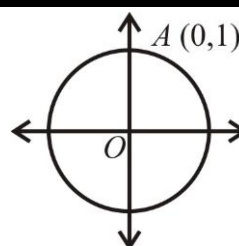
$$\tan 0^\circ = \frac{y}{x} = \frac{0}{1} = 0$$

$$\operatorname{cosec} 0^\circ = \frac{1}{0} = \text{Undefined}$$

$$\sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1$$

$$\cot 0^\circ = \frac{1}{0} = \text{Undefined}$$

Trigonometric Ratios of 90° (K.B)



$$\sin 90^\circ = \frac{y}{r} = \frac{1}{1} = 1$$

$$\cos 90^\circ = \frac{x}{r} = \frac{0}{1} = 0$$

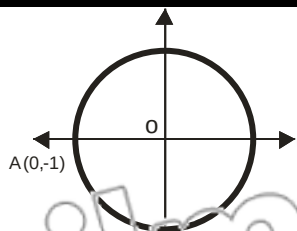
$$\tan 90^\circ = \frac{y}{x} = \frac{1}{0} = \text{Undefined}$$

$$\operatorname{cosec} 90^\circ = \frac{1}{\sin 90^\circ} = 1$$

$$\sec 90^\circ = \frac{1}{0} = \text{Undefined}$$

$$\cot 90^\circ = \frac{1}{\tan 90^\circ} = 0$$

Trigonometric Ratios of 180° (K.B)



$$\sin 180^\circ = \frac{y}{r} = \frac{-1}{1} = -1$$

$$\cos 180^\circ = \frac{x}{r} = \frac{-1}{1} = -1$$

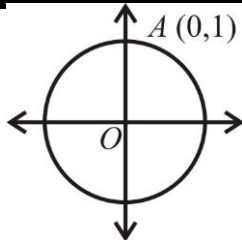
$$\tan 180^\circ = \frac{y}{x} = \frac{-1}{-1} = 1$$

$$\operatorname{cosec} 180^\circ = \frac{1}{\sin 180^\circ} = \frac{1}{-1} = -1$$

$$\sec 180^\circ = \frac{1}{\cos 180^\circ} = \frac{1}{-1} = -1$$

$$\cot 180^\circ = \frac{1}{\tan 180^\circ} = \frac{1}{1} = 1$$

Trigonometric Ratios of 270° (K.B)



$$\sin 270^\circ = \frac{y}{r} = \frac{1}{1} = 1$$

$$\cos 270^\circ = \frac{x}{r} = \frac{0}{1} = 0$$

$$\tan 270^\circ = \frac{y}{x} = \frac{1}{0} = \text{Undefined}$$

$$\operatorname{cosec} 270^\circ = \frac{1}{\sin 270^\circ} = \frac{1}{1} = 1$$

$$\sec 270^\circ = \frac{1}{\cos 270^\circ} = \frac{1}{0} = \text{Undefined}$$

$$\cot 270^\circ = \frac{1}{\tan 270^\circ} = \frac{1}{\text{Undefined}} = 0$$

Note

(K.B + U.B)

$$\sin(-\theta) = -\sin \theta$$

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\sec(-\theta) = \sec \theta$$

$$\tan(-\theta) = -\tan \theta$$

$$\cot(-\theta) = -\cot \theta$$

Example 1: (Page # 159) (K.B + A.B)

If $\sin \theta = \frac{-3}{4}$ and $\cos \theta = \frac{\sqrt{7}}{4}$, then

find the values of $\tan \theta, \cot \theta, \sec \theta$ and $\operatorname{cosec} \theta$

Solution:

Applying the identities that express the remaining trigonometric functions in terms of sine and cosine, we have,

$$\sin \theta = \frac{-3}{4}$$

$$\Rightarrow \operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{-4}{3}$$

$$\Rightarrow \operatorname{cosec} \theta = \frac{-4}{3}$$

$$\cos \theta = \frac{\sqrt{7}}{4}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\Rightarrow \sec \theta = \frac{4}{\sqrt{7}}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{-3}{4}}{\frac{\sqrt{7}}{4}}$$

$$\Rightarrow \tan \theta = \frac{-3}{\sqrt{7}}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{-\sqrt{7}}{3}$$

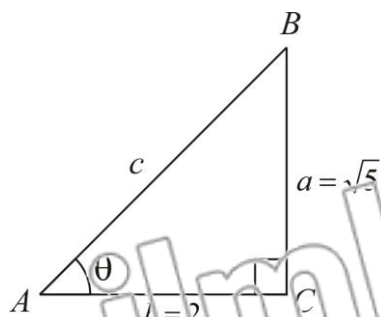
Example 2: (Page # 159) (K.B + A.B) (GRW 2014)

If $\tan \theta = \frac{\sqrt{5}}{2}$, then find the values of other trigonometric ratios at θ .

Solution:

In any right triangle ABC ,

$$\tan \theta = \frac{\sqrt{5}}{2} = \frac{a}{b} \Rightarrow a = \sqrt{5}, b = 2$$



Now by Pythagorean theorem

$$a^2 + b^2 = c^2 \Rightarrow (\sqrt{5})^2 + (2)^2 = c^2$$

$$c^2 = 5 + 4 = 9 \Rightarrow c = \pm 3 \text{ or } c = 3$$

$$\therefore \cot \theta = \frac{1}{\tan \theta}$$

$$\therefore \cot \theta = \frac{1}{\frac{\sqrt{5}}{2}} \Rightarrow \cot \theta = \frac{2}{\sqrt{5}}$$

$$\sin \theta = \frac{a}{c} = \frac{\sqrt{5}}{3}, \quad \operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$\Rightarrow \operatorname{cosec} \theta = \frac{1}{\frac{\sqrt{5}}{3}} \therefore \operatorname{cosec} \theta = \frac{3}{\sqrt{5}}$$

$$\cos \theta = \frac{b}{c} = \frac{2}{3}, \quad \sec \theta = \frac{1}{\cos \theta}$$

Also $\Rightarrow \sec \theta = \frac{1}{\frac{2}{3}} \therefore \sec \theta = \frac{3}{2}$

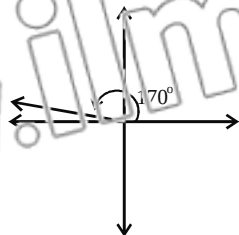
Exercise 7.3

Q.1 Locate each of the following angles in standard positions using protractor or fair free hand guess. Also find a positive and negative coterminal with each given angle.

Solution:

(i) 170°

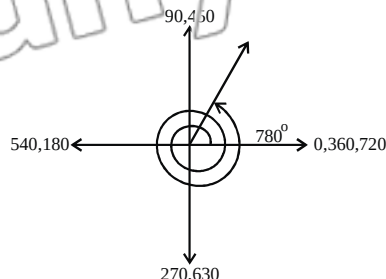
(A.B)



Positive coterminal angle of 170°
 $= 360^\circ + 170^\circ$

$$\begin{aligned} &= 530^\circ \\ \text{Negative coterminal angle} &= -360^\circ + 170^\circ \\ &= -190^\circ \end{aligned}$$

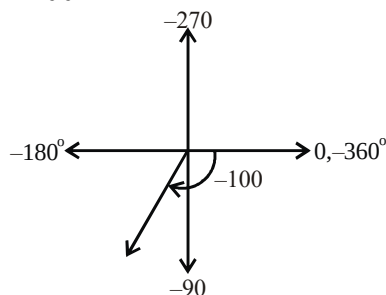
(ii) 780° (A.B)



$$\begin{aligned} \text{Positive coterminal angle} &= 780^\circ - 2(360^\circ) \\ &= 60^\circ \end{aligned}$$

$$\begin{aligned} \text{Negative coterminal angle} &= -360^\circ + 60^\circ \\ &= -300^\circ \end{aligned}$$

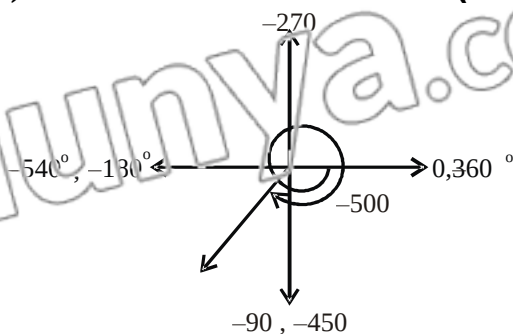
(iii) -100° (A.B)



$$\begin{aligned} \text{Positive coterminal angle} &= 360^\circ + (-100^\circ) \\ &= 260^\circ \end{aligned}$$

$$\begin{aligned} \text{Negative coterminal angle} &= -360^\circ + (-100^\circ) \\ &= -460^\circ \end{aligned}$$

(iv) -500° (A.B)



$$\begin{aligned} \text{Negative coterminal angle} &= -500^\circ + 360^\circ \\ &= -140^\circ \end{aligned}$$

$$\begin{aligned} \text{Positive coterminal angle} &= -140^\circ + 360^\circ \\ &= 220^\circ \end{aligned}$$

Unit-7

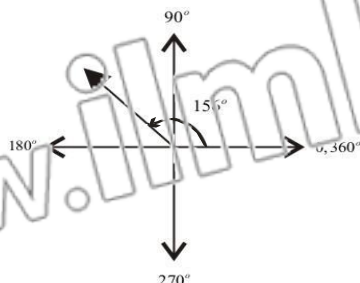
Introduction to Trigonometry

Q.2 Identify the closest quadrantal angles between which the following angles lie.

Solution:

(i) 156°

(A.B)



Closest quadrantal angles are 90° and 180°

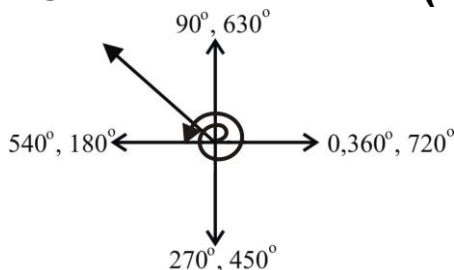
(ii) 318°

(A.B)

Closest quadrantal angles are 270° and 360°

(iii) 572°

(A.B)



Closest quadrantal angles are 540° and 630°

(iv) -330°

(A.B)

Closest quadrantal angles are -270° and -360°

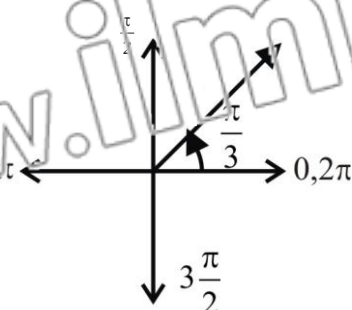
Q.3 Write the closest quadrantal angles between which the angle lies. Write your answer in radian measure.

Solution:

(i) $\frac{\pi}{3}$

(A.B)

Closest quadrantal angles are 0 and $\frac{\pi}{2}$



(ii) $\frac{3\pi}{4}$

(A.B)

Closest quadrantal angles are $\frac{\pi}{2}$ and π

(iii) $-\frac{\pi}{4}$

(A.B)

Closest quadrantal angles are 0 and $-\frac{\pi}{2}$

(iv) $-\frac{3\pi}{4}$

(A.B)

Closest quadrantal angles are $-\frac{\pi}{2}$ and $-\pi$

Q.4 In which quadrant θ lies, when

(i) $\sin \theta > 0, \tan \theta < 0$

(ii) $\cos \theta < 0, \sin \theta < 0$

(iii) $\sec \theta > 0, \sin \theta < 0$

(iv) $\cos \theta > 0, \tan \theta < 0$

(v) $\operatorname{cosec} \theta > 0, \cos \theta < 0$

(vi) $\sin \theta > 0, \sec \theta < 0$

Solution:

(i) $\sin \theta > 0, \tan \theta < 0$

(K.B)

$\sin \theta > 0$, then θ lies I, II quadrant.

$\tan \theta < 0$, then θ lies in II, IV quadrant.

II quadrant is common

$\therefore \theta$ lies in second quadrant

(ii) $\cos \theta < 0, \sin \theta < 0$

(K.B)

$\cos \theta < 0$, then θ lies in II, III quadrant.

$\sin \theta < 0$, then θ lies in III, IV quadrant.

III quadrant is common.

$\therefore \theta$ lies in III quadrant.

(iii) $\sec \theta > 0, \sin \theta < 0$

(K.B)

$\sec \theta > 0$, then θ lies I, IV quadrant.

$\sin \theta < 0$, then θ lies in III, IV quadrant.

IV quadrant is common

$\therefore \theta$ lies in fourth quadrant

(iv) $\cos \theta > 0, \tan \theta < 0$

(K.B)

$\cos \theta > 0$, then θ lies I, IV quadrant.

$\tan \theta < 0$, then θ lies in II, IV quadrant.

IV quadrant is common
 $\therefore \theta$ lies in fourth quadrant

(v) $\operatorname{cosec} \theta > 0, \cos \theta < 0$ **(K.B)**

$\operatorname{cosec} \theta > 0$, then θ lies I, II quadrant.

$\cos \theta < 0$, then θ lies in II, III quadrant.

II quadrant is common

$\therefore \theta$ lies in second quadrant

(vi) $\sin \theta > 0, \sec \theta < 0$ **(K.B)**

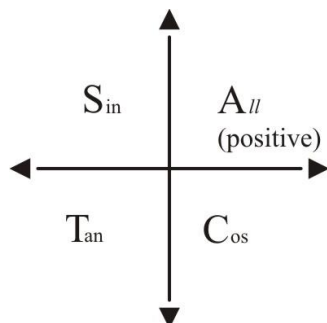
$\sin \theta > 0$, then θ lies I, II quadrant.

$\sec \theta < 0$, then θ lies in II, III quadrant.

II quadrant is common

$\therefore \theta$ lies in second quadrant

Note



(After school to college, (where a ratio is positive its reciprocal is also positive))

Q.5 Fill in the blanks.

(i) $\cos(-150^\circ) = \cos 150^\circ$ **(K.B)**

$$\cos(-\theta) = \cos \theta$$

(ii) $\sin(-310^\circ) = -\sin 310^\circ$ **(K.B)**

$$\therefore \sin(-\theta) = -\sin \theta$$

(iii) $\tan(-210^\circ) = -\tan 210^\circ$ **(K.B)**

$$\therefore \tan(-\theta) = -\tan \theta$$

(iv) $\cot(-45^\circ) = -\cot 45^\circ$ **(K.B)**

$$\therefore \cot(-\theta) = -\cot \theta$$

(v) $\sec(-60^\circ) = \sec 60^\circ$ **(K.B)**

$$\therefore \sec(-\theta) = \sec \theta$$

(vi) $\operatorname{cosec}(-137^\circ) = -\operatorname{cosec} 137^\circ$ **(K.B)**

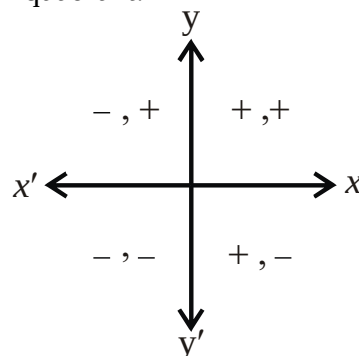
$$\therefore \operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta$$

Q.6 The given point P lies on the terminal side of θ . Find quadrant of θ and all six trigonometric ratios.

Solution:

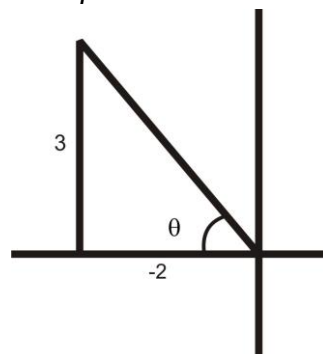
(i) $P(-2, 3)$ **(K.B + A.B)**

Since x is $-ve$ and y is $+ve$, θ lies in II quadrant.



Trigonometric Ratios:

$$\sin \theta = \frac{y}{r}$$



$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{(-2)^2 + (3)^2}$$

$$= \sqrt{4 + 9}$$

$$= \sqrt{13}$$

$$\sin \theta = \frac{3}{\sqrt{13}}$$

$$\cos \theta = \frac{x}{r}$$

$$= \frac{-2}{\sqrt{13}}$$

$$\tan \theta = \frac{y}{x}$$

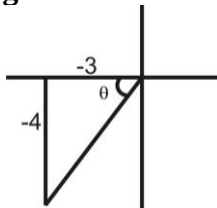
$$\begin{aligned}
 &= \frac{3}{-2} \\
 &= -\frac{3}{2} \\
 \operatorname{cosec} \theta &= \frac{1}{\sin \theta} \\
 &= \frac{1}{-\frac{3}{2}} \\
 \sec \theta &= \frac{1}{\cos \theta} \\
 &= -\frac{\sqrt{13}}{3}
 \end{aligned}$$

$$\begin{aligned}
 \cot \theta &= \frac{1}{\tan \theta} \\
 &= -\frac{2}{3}
 \end{aligned}$$

(ii) P(-3, -4) **(K.B + A.B)**

Since both x and y are $-ve$, θ lies in III quadrant.

Trigonometric Ratios:



$$\begin{aligned}
 &= \sqrt{(-3)^2 + (-4)^2} \\
 &= \sqrt{9+16} \\
 &= \sqrt{25} \\
 &= 5
 \end{aligned}$$

Now

$$\sin \theta = \frac{y}{r} = \frac{-4}{5} = -\frac{4}{5}$$

$$\cos \theta = \frac{x}{r} = \frac{-3}{5} = -\frac{3}{5}$$

$$\tan \theta = \frac{y}{x} = \frac{-4}{-3} = \frac{4}{3}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} = -\frac{5}{4}$$

$$\sec \theta = \frac{1}{\cos \theta} = -\frac{5}{3}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{3}{4}$$

(iii) $(\sqrt{2}, 1)$ **(K.B + A.B)**

Since both x and y are $+ve$, θ lies in I quadrant.

$$\sin \theta = \frac{y}{r} = \frac{1}{\sqrt{3}}$$

$$\cos \theta = \frac{x}{r} = \frac{\sqrt{2}}{\sqrt{3}}$$

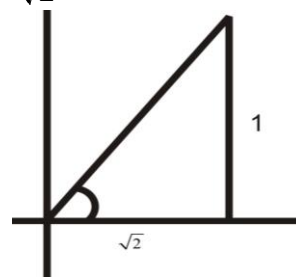
$$\tan \theta = \frac{y}{x} = \frac{1}{\sqrt{2}}$$

$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{(\sqrt{2})^2 + (1)^2}$$

$$= \sqrt{2+1}$$

$$= \sqrt{3}$$



$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \sqrt{3}$$

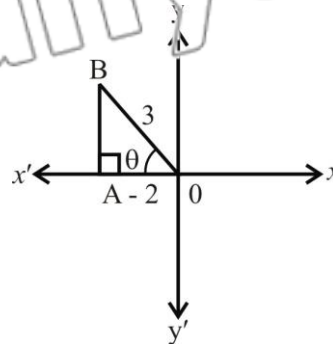
$$\sec \theta = \frac{1}{\cos \theta} = \frac{\sqrt{3}}{\sqrt{2}}$$

$$\cot \theta = \frac{1}{\tan \theta} = \sqrt{2}$$

Q.7 If $\cos \theta = -\frac{2}{3}$ and terminal arm of the angle θ is in quad. II, find the values of remaining trigonometric functions. **(K.B + A.B)**

(SGD 2014, D.G.K 2014)

Solution:



Form $\triangle ABC$

$$\begin{aligned}
 (\overline{AB})^2 &= (\overline{OB})^2 - (\overline{OA})^2 \\
 &= 3^2 - (2)^2 \\
 &= 9 - 4 \\
 &= 5
 \end{aligned}$$

$$\Rightarrow \overline{AB} = \sqrt{5} \text{ (}\theta \text{ lies in II quad)}$$

$$\sin \theta = \frac{\text{Perp}}{\text{Hyp}} = \frac{m\overline{AB}}{m\overline{OB}} = \frac{\sqrt{5}}{3}$$

$$\cos \theta = \frac{m\overline{OA}}{m\overline{OB}} = \frac{-2}{3} = -\frac{2}{3}$$

$$\tan \theta = \frac{m\overline{AB}}{m\overline{OB}} = \frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{3}{\sqrt{5}}$$

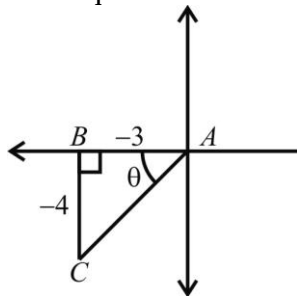
$$\sec \theta = \frac{1}{\cos \theta} = -\frac{3}{2}$$

$$\cot \theta = \frac{1}{\tan \theta} = -\frac{2}{\sqrt{5}}$$

Q.8 If $\tan \theta = \frac{4}{3}$ and $\sin \theta < 0$, find the values of other trigonometric functions at θ **(K.B + A.B)**

Solution:

Since $\tan \theta$ is +ve and $\sin \theta$ is -ve, θ lies in III quadrant.



Form ΔABC

$$\begin{aligned}
 (\overline{AC})^2 &= (\overline{AB})^2 + (\overline{BC})^2 \\
 &= (-3)^2 + (-4)^2 \\
 &= 9 + 16 \\
 &= 25
 \end{aligned}$$

$$m\overline{AC} = 5$$

$$\sin \theta = \frac{m\overline{BC}}{m\overline{AC}} = \frac{-4}{5}$$

$$= -\frac{4}{5}$$

$$\cos \theta = \frac{m\overline{AB}}{m\overline{AC}} = \frac{-3}{5}$$

$$= -\frac{3}{5}$$

$$\tan \theta = \frac{m\overline{BC}}{m\overline{AB}} = \frac{-4}{-3}$$

$$= \frac{4}{3}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} = -\frac{5}{4}$$

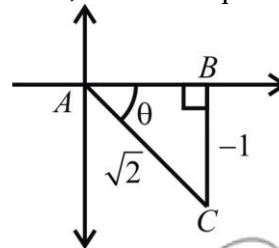
$$\sec \theta = \frac{1}{\cos \theta} = -\frac{5}{3}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{3}{4}$$

Q.9 If $\sin \theta = -\frac{1}{2}$ and terminal side of the angle is not in quadrant III, find the values of $\tan \theta$, $\sec \theta$ and $\operatorname{cosec} \theta$. **(K.B + A.B)**

Solution:

Since $\sin \theta$ is -ve and θ is not in III quadrant, lies in IV quadrant.



Form ΔABC

$$\begin{aligned}
 (\overline{AC})^2 &= (\overline{AB})^2 + (\overline{BC})^2 \\
 &= (1)^2 + (-1)^2 \\
 &= 1 + 1 \\
 &= 2
 \end{aligned}$$

$$\Rightarrow m\overline{AC} = \sqrt{2}$$

$$\sin \theta = \frac{m\overline{BC}}{m\overline{AC}} = \frac{-1}{\sqrt{2}}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} = -\sqrt{2}$$

$$\sec \theta = \frac{m\overline{AC}}{m\overline{AB}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\tan \theta = \frac{m\overline{BC}}{m\overline{AB}} = \frac{-1}{1} = -1$$

Result:

$$\tan \theta = -1$$

$$\sec \theta = \sqrt{2}$$

$$\operatorname{cosec} \theta = -\sqrt{2}$$

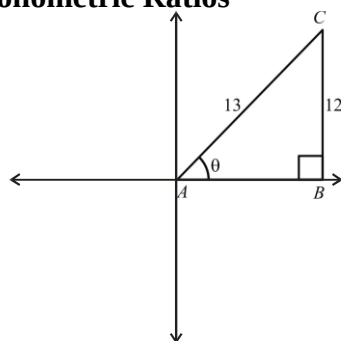
Q.10 If $\operatorname{cosec} \theta = \frac{13}{12}$ and $\sec \theta > 0$, find the remaining trigonometric functions. **(K.B + A.B)**

Solution:

Since both $\operatorname{Cosec} \theta$ and $\sec \theta$ are +ve, θ lies in I quadrant.

Now

Trigonometric Ratios



Form $\triangle ABC$

$$\begin{aligned} |\overline{AB}|^2 &= |\overline{AC}|^2 - |\overline{BC}|^2 \\ &= (13)^2 - (12)^2 \\ &= 169 - 144 \\ &= 25 \end{aligned}$$

$$\Rightarrow m\overline{AB} = 5 (\theta \text{ is in I quad})$$

$$\sin \theta = \frac{1}{\operatorname{cosec} \theta} = \frac{12}{13}$$

$$\cos \theta = \frac{m\overline{AB}}{m\overline{AC}} = \frac{5}{13}$$

$$\tan \theta = \frac{m\overline{BC}}{m\overline{AB}} = \frac{12}{5}$$

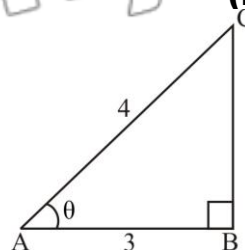
$$\operatorname{cosec} \theta = \frac{13}{12}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{13}{5}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{5}{12}$$

Q.11 Find the values of trigonometric functions at the indicated angle θ in the right angle triangle.

(i) **(K.B + A.B)**



Here

$$\begin{aligned} |m\overline{BC}|^2 &= |m\overline{AC}|^2 - |m\overline{AB}|^2 \\ &= (4)^2 - (3)^2 \\ &= 16 - 9 \\ &= 7 \end{aligned}$$

$$\Rightarrow m\overline{BC} = \sqrt{7}$$

Now Trigonometric Ratios are:

$$\sin \theta = \frac{m\overline{BC}}{m\overline{AC}} = \frac{\sqrt{7}}{4}$$

$$\cos \theta = \frac{m\overline{AB}}{m\overline{AC}} = \frac{3}{4}$$

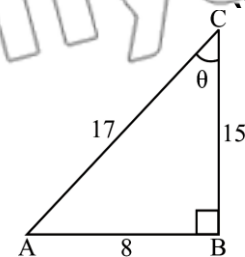
$$\tan \theta = \frac{m\overline{BC}}{m\overline{AB}} = \frac{\sqrt{7}}{3}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{4}{\sqrt{7}}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{4}{3}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{3}{\sqrt{7}}$$

(ii) **Trigonometric Ratios are: (K.B + A.B)**



$$\sin \theta = \frac{m\overline{AB}}{m\overline{AC}} = \frac{8}{17}$$

$$\cos \theta = \frac{m\overline{BC}}{m\overline{AC}} = \frac{15}{17}$$

$$\tan \theta = \frac{m\overline{AB}}{m\overline{BC}} = \frac{8}{15}$$

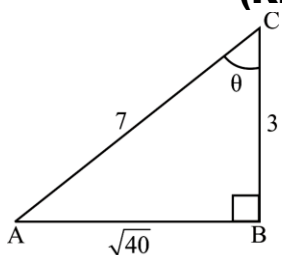
$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{17}{8}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{17}{15}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{15}{8}$$

(iii)

(K.B + A.B)



Here

$$\begin{aligned} |m\overline{AB}|^2 &= |m\overline{AC}|^2 - |m\overline{BC}|^2 \\ &= (7)^2 - (3)^2 \\ &= 49 - 9 \\ &= 40 \end{aligned}$$

$$\Rightarrow m\overline{AB} = \sqrt{40}$$

Now Trigonometric Ratios are:

$$\sin \theta = \frac{m\overline{AB}}{m\overline{AC}} = \frac{\sqrt{40}}{7}$$

$$\cos \theta = \frac{m\overline{BC}}{m\overline{AC}} = \frac{3}{7}$$

$$\tan \theta = \frac{m\overline{AB}}{m\overline{BC}} = \frac{\sqrt{40}}{3}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{7}{\sqrt{40}}$$

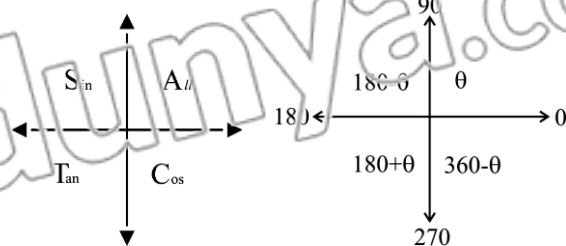
$$\sec \theta = \frac{1}{\cos \theta} = \frac{7}{3}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{3}{\sqrt{40}}$$

Q.12 Find the values of the trigonometric functions. Do not use trigonometric tables or calculator.

Solution:

(i) $\tan 30^\circ = \frac{1}{\sqrt{3}}$ (K.B + A.B)



(ii) $\tan 330^\circ = -\tan 30^\circ$ (K.B + A.B)

$$= -\frac{1}{\sqrt{3}}$$

$$\begin{aligned} 360^\circ - \theta &= 330^\circ \\ \theta &= 360^\circ - 330^\circ \\ &= 30^\circ \end{aligned}$$

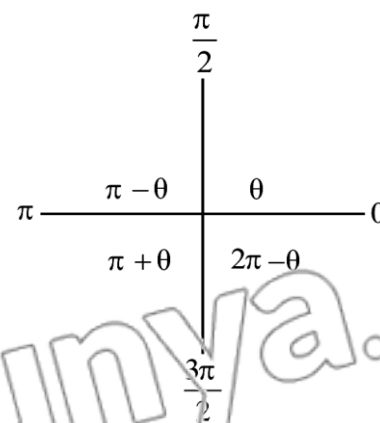
(iii) $\sec 330^\circ = \sec 30^\circ$ (K.B + A.B)

$$= \frac{1}{\cos 30^\circ}$$

$$= \frac{1}{\sqrt{3}/2}$$

$$= \frac{2}{\sqrt{3}}$$

(iv) $\cot \frac{\pi}{4} = 1$ (K.B + A.B)



(v) $\cos \frac{2\pi}{3} = -\cos \frac{\pi}{3}$ (K.B + A.B)

$$= -\frac{1}{2}$$

$$\frac{2\pi}{3} = \pi - \theta$$

$$\theta = \pi - \frac{2\pi}{3}$$

$$= \frac{\pi}{3}$$

$$(vi) \quad \operatorname{Cosec} \frac{2\pi}{3} = \operatorname{Cosec} \frac{\pi}{3} \quad (\text{K.B + A.B})$$

$$= \frac{2}{\sqrt{3}}$$

$$(vii) \quad \cos(-450^\circ) = \cos 450^\circ$$

$$= \cos(360^\circ + 90^\circ) \\ = \cos 90^\circ \\ = 0$$

$$(viii) \quad \tan(-9\pi) = -\tan 9\pi$$

$$= -\tan(4 \times 2\pi + \pi) \\ = -\tan \pi \\ = -0 \\ = 0$$

$$(ix) \quad \cos\left(-\frac{5\pi}{6}\right) \quad \because \cos(-\theta) = \cos \theta$$

$$= -\cos \frac{\pi}{6}$$

$$= -\frac{\sqrt{3}}{2}$$

$$\frac{5\pi}{6} = \pi - \theta$$

$$\theta = \pi - \frac{5\pi}{6}$$

$$= \frac{\pi}{6}$$

$$(x) \quad \sin \frac{7\pi}{6} = -\sin \frac{\pi}{6}$$

$$= -\frac{1}{2}$$

$$\pi + \theta = \frac{7\pi}{6}$$

$$\theta = \frac{7\pi}{6} - \pi$$

$$\theta = \frac{\pi}{6}$$

$$(xi) \quad \cot \frac{7\pi}{6} = \cot \frac{\pi}{6}$$

$$= \sqrt{3}$$

(K.B + A.B)

$$(xii) \quad \cos 225^\circ \\ = -\cos 45^\circ$$

$$= -\frac{1}{\sqrt{2}}$$

$$225^\circ = 180^\circ + \theta \\ \theta = 45^\circ$$

(5) Trigonometric identities:

(K.B + U.B)

$$(i) \quad \sin^2 \theta + \cos^2 \theta = 1$$

$$(ii) \quad 1 + \tan^2 \theta = \sec^2 \theta$$

$$(iii) \quad 1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

Proof:

Consider a $\triangle CAB$ in which
 $m\angle CAB = \theta$ radians



(i) By Pythagorean Theorem:

$$b^2 = a^2 + c^2$$

Div by ' b^2 '

$$\frac{b^2}{b^2} = \frac{a^2}{b^2} + \frac{c^2}{b^2}$$

$$1 = \left(\frac{a}{b}\right)^2 + \left(\frac{c}{b}\right)^2$$

$$1 = \sin^2 \theta + \cos^2 \theta$$

Or $\sin^2 \theta + \cos^2 \theta = 1$

Proved

(ii) By Pythagorean theorem

$$b^2 = a^2 + c^2$$

Div by ' c^2 '

$$\frac{b^2}{c^2} = \frac{a^2}{c^2} + \frac{c^2}{c^2}$$

$$\left(\frac{b}{c}\right)^2 = \left(\frac{a}{c}\right)^2 + 1$$

$$\sec^2 \theta = \tan^2 \theta + 1$$

Or $1 + \tan^2 \theta = \sec^2 \theta$

Proved

(iii) By Pythagorean theorem

$$b^2 = a^2 + c^2$$

Div by ' a^2 '

$$\frac{b^2}{a^2} = \frac{a^2}{a^2} + \frac{c^2}{a^2}$$

$$\left(\frac{b}{a}\right)^2 = 1 + \left(\frac{c}{a}\right)^2$$

$$\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$$

Proved

Example 2: (Page # 164)

(A.B)

Verify that $\tan^4 \theta + \tan^2 \theta = \tan^2 \theta \sec^2 \theta$

Solution:

$$L.H.S = \tan^4 \theta + \tan^2 \theta$$

$$= \tan^2 \theta (\tan^2 \theta + 1)$$

$$\begin{aligned}\therefore \tan^2 \theta + 1 &= \sec^2 \theta \\ &= \tan^2 \theta \sec^2 \theta \\ &= \text{R.H.S}\end{aligned}$$

Example 3: (Page # 164)**(A.B)**

Show that $\frac{\cot^2 \alpha}{\operatorname{cosec} \alpha - 1} = \operatorname{cosec} \alpha + 1$

Solution:

$$\begin{aligned}\text{L.H.S} &= \frac{\cot^2 \alpha}{\operatorname{cosec} \alpha - 1} \\ &\left(\because \operatorname{cosec}^2 \theta - \cot^2 \theta = 1 \right) \\ &\quad \left(\cot^2 \theta = \operatorname{cosec}^2 \theta - 1 \right) \\ &= \frac{(\operatorname{cosec}^2 \alpha - 1)}{\operatorname{cosec} \alpha - 1} \\ &= \frac{(\operatorname{cosec} \alpha - 1)(\operatorname{cosec} \alpha + 1)}{(\operatorname{cosec} \alpha - 1)} \\ &= \operatorname{cosec} \alpha + 1 = \text{R.H.S}\end{aligned}$$

Example 4: (Page # 164)**(A.B)**

Express the trigonometric functions in terms of $\tan \theta$.

Solution:

By using reciprocal identity, we can express $\cot \theta$ in terms of $\tan \theta$.

$$\text{i.e.} \quad \cot \theta = \frac{1}{\tan \theta}$$

$$\text{Since } 1 + \tan^2 \theta = \sec^2 \theta$$

$$\Rightarrow \sec \theta = \pm \sqrt{\tan^2 \theta + 1}$$

we have expressed $\sec \theta$ in terms of $\tan \theta$.

$$\begin{aligned}\therefore \cos \theta &= \frac{1}{\sec \theta} \\ \Rightarrow \cos \theta &= \frac{1}{\pm \sqrt{\tan^2 \theta + 1}}\end{aligned}$$

$$\text{Since } \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\therefore \sin \theta = \tan \theta \left(\frac{1}{\pm \sqrt{\tan^2 \theta + 1}} \right)$$

$$= \frac{\tan \theta}{\pm \sqrt{\tan^2 \theta + 1}}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$= \frac{\pm \sqrt{\tan^2 \theta + 1}}{\tan \theta}$$

Note**(K.B)**

We can express all the trigonometric functions in terms of one trigonometric function.

Exercise 7.4

In problems 1-6, simplify each expression to a single trigonometric function.

Q.1 $\frac{\sin^2 x}{\cos^2 x}$ **(K.B + A.B)**

Solution:

$$\begin{aligned}\frac{\sin^2 x}{\cos^2 x} &= \frac{(\sin x)^2}{(\cos x)^2} \\ &= \left(\frac{\sin x}{\cos x} \right)^2 \left(\because \tan \theta = \frac{\sin \theta}{\cos \theta} \right) \\ &= (\tan x)^2 \\ &= \tan^2 x\end{aligned}$$

Q.2 $\tan x \sin x \sec x$ **(K.B + A.B)**

Solution:

$$\begin{aligned}\tan x \sin x \sec x &= \frac{\sin x}{\cos x} \times \sin x \times \frac{1}{\cos x} \\ &= \frac{\sin^2 x}{\cos^2 x} \left(\because \frac{\sin \theta}{\cos \theta} = \tan \theta \right) \\ &= \left(\frac{\sin x}{\cos x} \right)^2 \left(\because \sec \theta = \frac{1}{\cos \theta} \right) \\ &= \tan^2 x\end{aligned}$$

Q.3 $\frac{\tan x}{\sec x}$ **(K.B + A.B)**

Solution:

$$\frac{\tan x}{\sec x} = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} \quad \therefore \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\therefore \sec \theta = \frac{1}{\cos \theta}$$

$$\begin{aligned}\frac{\tan x}{\sec x} &= \frac{\sin x}{\cancel{\cos x}} \times \frac{\cancel{\cos x}}{1} \\ &= \sin x\end{aligned}$$

Q.4 $1 - \cos^2 x$ (K.B + A.B)

Solution:

$$\begin{aligned}
 &1 - \cos^2 x \\
 &= 1 - (1 - \sin^2 x) \quad (\because \sin^2 \theta + \cos^2 \theta = 1) \\
 &= \cancel{1} - \cancel{1} + \sin^2 x \\
 &= \sin^2 x
 \end{aligned}$$

Q.5 $\sec^2 x - 1$ (LHR 2014) (K.B + A.B)Solution: $\sec^2 x - 1$

$$\begin{aligned}
 &= \left(\frac{1}{\cos x} \right)^2 - 1 \\
 &= \frac{1}{\cos^2 x} - 1 \\
 &= \frac{1 - \cos^2 x}{\cos^2 x} \quad (\because \sin^2 \theta + \cos^2 \theta = 1) \\
 &= \frac{\sin^2 x}{\cos^2 x} \\
 &= \left(\frac{\sin x}{\cos x} \right)^2 \quad \left(\because \frac{\sin \theta}{\cos \theta} = \tan \theta \right) \\
 &= \tan^2 x
 \end{aligned}$$

Q.6 $\sin^2 x \cdot \cot^2 x$ (K.B + A.B)Solution: $\sin^2 x \cdot \cot^2 x$

$$\begin{aligned}
 &= \sin^2 x \cdot \left(\frac{\cos x}{\sin x} \right)^2 \\
 &= \cancel{\sin^2 x} \cdot \frac{\cos^2 x}{\cancel{\sin^2 x}} \\
 &= \cos^2 x
 \end{aligned}$$

In problems 7-12, verify the identities.

Q.7 $(1 - \sin \theta)(1 + \sin \theta) = \cos^2 \theta$

(K.B + A.B)

(FSD 2015, SWL 2014, SGD 2014)

Proof:

$$\begin{aligned}
 \text{L.H.S} &= (1 - \sin \theta)(1 + \sin \theta) \\
 &= (1)^2 - (\sin \theta)^2 \quad [\because (a+b)(a-b) = a^2 - b^2] \\
 &= 1 - \sin^2 \theta \\
 &= \cos^2 \theta \quad (\because \sin^2 \theta + \cos^2 \theta = 1) \\
 &= \text{R.H.S}
 \end{aligned}$$

Q.8 $\frac{\sin \theta + \cos \theta}{\cos \theta} = 1 + \tan \theta$ (K.B + A.B)

(LHR 2016, GRW 2014)

Proof:

$$\begin{aligned}
 \text{L.H.S} &= \frac{\sin \theta + \cos \theta}{\cos \theta} \\
 &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\cos \theta} \\
 &= \tan \theta + 1 \quad \left(\because \tan \theta = \frac{\sin \theta}{\cos \theta} \right) \\
 &= 1 + \tan \theta \\
 &= \text{R.H.S}
 \end{aligned}$$

Hence Proved

Q.9 $(\tan \theta + \cot \theta) \tan \theta = \sec^2 \theta$

(K.B + A.B)

Proof:

$$\begin{aligned}
 \text{L.H.S} &= (\tan \theta + \cot \theta) \tan \theta \\
 &= \tan^2 \theta + \cot \theta \cdot \tan \theta \\
 &= \tan^2 \theta + \frac{1}{\tan \theta} \times \tan \theta \quad \left(\because \cot \theta = \frac{1}{\tan \theta} \right) \\
 &= \tan^2 \theta + 1 \\
 &= \sec^2 \theta \\
 &= \text{R.H.S}
 \end{aligned}$$

Hence Proved

M-II

L.H.S = $(\tan \theta + \cot \theta) \tan \theta$

$$\begin{aligned}
 &= \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right) \times \frac{\sin \theta}{\cos \theta} \\
 &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cdot \cos \theta} \times \frac{\sin \theta}{\cos \theta} \\
 &= \frac{1}{\cos \theta} \\
 &= \sec \theta \\
 &= \text{R.H.S}
 \end{aligned}$$

Proved

Q.10

 $(\cot \theta + \operatorname{cosec} \theta)(\tan \theta - \sin \theta) = \sec \theta - \cos \theta$ (FSD 2017) (K.B + A.B)

Proof:

$$\begin{aligned}
 \text{L.H.S} &= (\cot \theta + \operatorname{cosec} \theta)(\tan \theta - \sin \theta) \\
 &= \left(\frac{\cos \theta}{\sin \theta} + \frac{1}{\sin \theta} \right) \left(\frac{\sin \theta}{\cos \theta} - \sin \theta \right) \\
 &\quad \left(\because \operatorname{cosec} \theta = \frac{1}{\sin \theta}, \cot \theta = \frac{\cos \theta}{\sin \theta}, \tan \theta = \frac{\sin \theta}{\cos \theta} \right) \\
 &= \frac{\cos \theta + 1}{\sin \theta} \cdot \frac{\sin \theta - \sin \theta \cos \theta}{\cos \theta}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1+\cos\theta}{\sin\theta} \cdot \frac{\cancel{\sin\theta}(1-\cos\theta)}{\cos\theta} \\
 &= \frac{(1+\cos\theta)(1-\cos\theta)}{\cos\theta} \\
 &= \frac{1-\cos^2\theta}{\cos\theta} \quad (\because (a+b)(a-b)=a^2-b^2) \\
 &= \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta} \\
 &= \sec\theta - \cos\theta \quad \left(\because \sec\theta = \frac{1}{\cos\theta} \right) \\
 &= \text{R.H.S}
 \end{aligned}$$

Hence Proved

Q.11 $\frac{\sin\theta + \cos\theta}{\tan^2\theta - 1} = \frac{\cos^2\theta}{\sin\theta - \cos\theta}$ **(K.B + A.B)**
(FSD 2018, MTN 2017, SGD 2015)

Proof:

$$\begin{aligned}
 \text{L.H.S} &= \frac{\sin\theta + \cos\theta}{\tan^2\theta - 1} \\
 &= \frac{\sin\theta + \cos\theta}{\frac{\sin^2\theta}{\cos^2\theta} - 1} \quad \left(\because \tan\theta = \frac{\sin\theta}{\cos\theta} \right) \\
 &= \frac{\sin\theta + \cos\theta}{\frac{\sin^2\theta - \cos^2\theta}{\cos^2\theta}} \\
 &= (\sin\theta + \cos\theta) \times \frac{\cos^2\theta}{\sin^2\theta - \cos^2\theta} \\
 &= (\sin\theta + \cos\theta) \times \frac{\cos^2\theta}{(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)} \\
 &= \frac{\cos^2\theta}{\sin\theta - \cos\theta} \\
 &= \text{R.H.S}
 \end{aligned}$$

Hence Proved

Q.12 $\frac{\cos^2\theta}{\sin\theta} + \sin\theta = \text{cosec}\theta$ **(K.B + A.B)**

Proof:

$$\begin{aligned}
 \text{L.H.S} &= \frac{\cos^2\theta}{\sin\theta} + \sin\theta \\
 &= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta} \quad (\because \sin^2\theta + \cos^2\theta = 1) \\
 &= \frac{1}{\sin\theta} \quad \left(\because \text{cosec}\theta = \frac{1}{\sin\theta} \right)
 \end{aligned}$$

$$= \text{cosec}\theta$$

$$= \text{R.H.S}$$

Hence Proved

Q.13 $(\sec\theta - \cos\theta) = \tan\theta \times \sin\theta$
(BWP 2013, D.G.K 2015, 16, 17) **(K.B + A.B)**

Proof:

$$\begin{aligned}
 \text{L.H.S} &= \sec\theta - \cos\theta \\
 &= \frac{1}{\cos\theta} - \cos\theta \quad \left(\because \sec\theta = \frac{1}{\cos\theta} \right) \\
 &= \frac{1 - \cos^2\theta}{\cos\theta} \\
 &= \frac{\sin^2\theta}{\cos\theta} \quad (\because \sin^2\theta + \cos^2\theta = 1) \\
 &= \frac{\sin\theta}{\cos\theta} \times \sin\theta \\
 &= \tan\theta \times \sin\theta \quad \left(\because \tan\theta = \frac{\sin\theta}{\cos\theta} \right) \\
 &= \text{R.H.S}
 \end{aligned}$$

Hence Proved

Q.14 $\frac{\sin^2\theta}{\cos\theta} + \cos\theta = \sec\theta$ **(K.B + A.B)**
(SWL 2017, SGD 2014, D.G.K 2016, 17)

Proof:

$$\begin{aligned}
 \text{L.H.S} &= \frac{\sin^2\theta}{\cos\theta} + \cos\theta \\
 &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} \\
 &= \frac{1}{\cos\theta} \quad (\because \sin^2\theta + \cos^2\theta = 1) \\
 &= \sec\theta \quad \left(\because \sec\theta = \frac{1}{\cos\theta} \right) \\
 &= \text{R.H.S}
 \end{aligned}$$

Hence Proved

Q.15 $\tan\theta + \cot\theta = \sec\theta \text{ cosec}\theta$ **(K.B + A.B)**

Proof:

$$\begin{aligned}
 \text{L.H.S} &= \tan\theta + \cot\theta \\
 &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\
 &\quad \left(\because \tan\theta = \frac{\sin\theta}{\cos\theta}, \cot\theta = \frac{\cos\theta}{\sin\theta} \right) \\
 &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta \sin\theta}
 \end{aligned}$$

$$= \frac{1}{\cos \theta \times \sin \theta} \quad (\because \sin^2 \theta + \cos^2 \theta = 1)$$

$$= \frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta}$$

$$= \sec \theta \times \operatorname{cosec} \theta$$

$$\left(\because \sec \theta = \frac{1}{\cos \theta}, \operatorname{cosec} \theta = \frac{1}{\sin \theta} \right)$$

$$= \text{R.H.S}$$

Hence Proved

Q.16 $(\tan \theta + \cot \theta)(\cos \theta + \sin \theta)$

$$= \sec \theta + \operatorname{cosec} \theta \quad \textbf{(K.B + A.B)}$$

Proof:

$$\text{L.H.S} = (\tan \theta + \cot \theta)(\cos \theta + \sin \theta)$$

$$= \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right) (\cos \theta + \sin \theta)$$

$$\left(\because \tan \theta = \frac{\sin \theta}{\cos \theta} \right)$$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \times (\cos \theta + \sin \theta)$$

$$= \frac{1}{\sin \theta \cos \theta} (\cos \theta + \sin \theta)$$

$$(\because \sin^2 \theta + \cos^2 \theta = 1)$$

$$= \frac{\cos \theta + \sin \theta}{\sin \theta \cos \theta}$$

$$= \frac{\cancel{\cos \theta}}{\sin \theta \cancel{\cos \theta}} + \frac{\cancel{\sin \theta}}{\cancel{\sin \theta} \cos \theta}$$

$$= \frac{1}{\sin \theta} + \frac{1}{\cos \theta}$$

$$\left(\because \operatorname{cosec} \theta = \frac{1}{\sin \theta}, \sec \theta = \frac{1}{\cos \theta} \right)$$

$$= \operatorname{cosec} \theta + \sec \theta$$

$$= \text{R.H.S}$$

Hence Proved

Q.17 $\sin \theta (\tan \theta + \cot \theta) = \sec \theta$

$$\textbf{(K.B + A.B)}$$

Proof:

$$\text{L.H.S} = \sin \theta (\tan \theta + \cot \theta)$$

$$= \sin \theta \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right)$$

$$= \cancel{\sin \theta} \left(\frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \cancel{\sin \theta}} \right)$$

$$= \frac{1}{\cos \theta}$$

$$= \sec \theta$$

$$= \text{R.H.S}$$

Hence Proved

Q.18 $\frac{1 + \cos \theta}{\sin \theta} + \frac{\sin \theta}{1 + \cos \theta} = 2 \operatorname{cosec} \theta$

$$\textbf{(K.B + A.B)}$$

Proof:

$$\text{L.H.S} = \frac{1 + \cos \theta}{\sin \theta} + \frac{\sin \theta}{1 + \cos \theta}$$

$$= \frac{(1 + \cos \theta)^2 + \sin^2 \theta}{\sin \theta (1 + \cos \theta)}$$

$$= \frac{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}{\sin \theta (1 + \cos \theta)}$$

$$= \frac{1 + 2 \cos \theta + 1}{\sin \theta (1 + \cos \theta)}$$

$$= \frac{2 + 2 \cos \theta}{\sin \theta (1 + \cos \theta)}$$

$$= \frac{2(1 + \cos \theta)}{\sin \theta (1 + \cos \theta)}$$

$$= \frac{2}{\sin \theta}$$

$$= 2 \operatorname{cosec} \theta$$

$$= \text{R.H.S}$$

Hence Proved

Q.19 $\frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta} = 2 \operatorname{cosec}^2 \theta$

$$\textbf{(K.B + A.B)}$$

(FSD 2015, BWP 2017, RWP 2016, 17, SGD 2016, MTN 2016, D.C.K 2017)

Proof:

$$\text{L.H.S} = \frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta}$$

$$= \frac{1 + \cancel{\cos \theta} + 1 - \cancel{\cos \theta}}{(1 - \cos \theta)(1 + \cos \theta)}$$

$$= \frac{2}{1 - \cos^2 \theta}$$

$$= \frac{2}{\sin^2 \theta}$$

$$= 2 \operatorname{cosec}^2 \theta$$

$$= \text{R.H.S}$$

Hence Proved

Q.20 $\frac{1+\sin\theta}{1-\sin\theta} - \frac{1-\sin\theta}{1+\sin\theta} = 4\tan\theta\sec\theta$
(D.G.K 2015) **(K.B + A.B)**

Proof:

$$\begin{aligned}\text{L.H.S} &= \frac{1+\sin\theta}{1-\sin\theta} - \frac{1-\sin\theta}{1+\sin\theta} \\ &= \frac{(1+\sin\theta)^2 - (1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)} \\ &= \frac{4(\sin\theta)}{1-\sin^2\theta} \\ &= \frac{4\sin\theta}{\cos^2\theta} \\ &= \frac{4\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta} \\ &= 4\tan\theta\sec\theta \\ &= \text{R.H.S}\end{aligned}$$

Hence Proved

Q.21 $\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$
(LHR 2014, SWL 2015) **(K.B + A.B)**

Proof:

$$\begin{aligned}\text{R.H.S} &= \sin\theta - \sin\theta\cos^2\theta \\ &= \sin\theta(1-\cos^2\theta) \\ &= \sin\theta \times \sin^2\theta \\ &= \sin^3\theta \\ &= \text{L.H.S}\end{aligned}$$

Hence Proved

Q.22 $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$
(K.B + A.B)

Proof:

$$\begin{aligned}\text{L.H.S} &= \cos^4\theta - \sin^4\theta \\ &= (\cos^2\theta)^2 - (\sin^2\theta)^2 \\ &= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) \\ &= 1(\cos^2\theta - \sin^2\theta) \\ &= \cos^2\theta - \sin^2\theta \\ &= \text{R.H.S}\end{aligned}$$

Hence Proved

Q.23 $\frac{1+\cos\theta}{1-\cos\theta} = \frac{\sin\theta}{1-\cos\theta}$
(LHR 2016, FSD 2015, 16, SWL 2016)
(K.B + A.B)

Proof:

$$\begin{aligned}\text{L.H.S} &= \frac{1+\cos\theta}{1-\cos\theta} \\ &\text{Multiply and divide by } 1-\cos\theta \\ &= \frac{1+\cos\theta}{1-\cos\theta} \times \frac{1-\cos\theta}{1-\cos\theta} \\ &= \frac{1-\cos^2\theta}{(1-\cos\theta)^2} \\ &= \frac{\sin^2\theta}{(1-\cos\theta)^2} \\ &= \sqrt{\left(\frac{\sin\theta}{1-\cos\theta}\right)^2} \\ &= \frac{\sin\theta}{1-\cos\theta} \\ &= \text{R.H.S}\end{aligned}$$

Hence Proved

Q.24 $\sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = \frac{\sec\theta+1}{\tan\theta}$ **(K.B + A.B)**

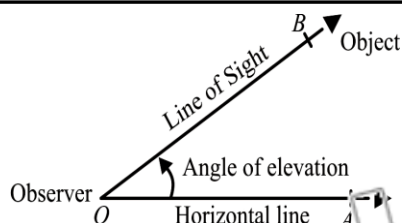
Proof:

$$\begin{aligned}\text{L.H.S} &= \sqrt{\frac{\sec\theta+1}{\sec\theta-1}} \\ &\text{Multiply \& divide by } \sec\theta+1 \\ &= \sqrt{\frac{\sec\theta+1}{\sec\theta-1} \times \frac{\sec\theta+1}{\sec\theta+1}} \\ &= \sqrt{\frac{(\sec\theta+1)^2}{\sec^2\theta-1}} \\ &= \sqrt{\frac{(\sec\theta+1)^2}{\tan^2\theta}} \\ &= \frac{(\sec\theta+1)}{\tan\theta} \\ &= \frac{\sec\theta+1}{\tan\theta} \\ &= \text{R.H.S}\end{aligned}$$

Hence Proved

Angle of Elevation **(K.B + A.B)**
(FSD 2015, MTN 2015)

Angle of elevation is the angle between the horizontal line and the line of sight to an object above the horizontal line.

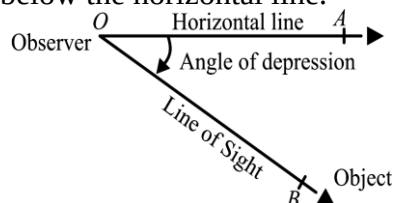


Angle between horizontal line $\overline{OA}$ and a line segment $\overline{OB}$ from eye O to object B above point A is called angle of elevation.

Angle of Depression (K.B + A.B)

(LFR 2015, BWP 2014, RWP 2015)

Angle of depression is the angle between the horizontal line and the line of sight to an object below the horizontal line.

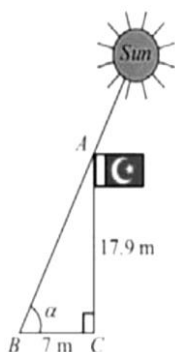


Angle between horizontal line $\overline{OA}$ and a line segment $\overline{OB}$ from eye O to object B below point A is called angle of depression.

Example 1: (Page # 166) (K.B + A.B)

A flagpole 17.9 meter high casts a 7 meter shadow. Find the angle of elevation of the sun.

Solution:



From the figure, we observe that α is the angle of elevation.

Using the fact that

$$\tan \alpha = \frac{AC}{BC} = \frac{17.9}{7}$$

Solving for α gives us

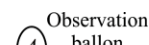
$$\alpha = \tan^{-1}\left(\frac{17.9}{7}\right) = (68.6666)^\circ$$

$$\Rightarrow \alpha = 68^\circ 40'$$

Example 2: (Page # 167) (K.B + A.B)

An observation balloon is 4280 meter above the ground and 9613 meter away from a farmhouse. Find the angle of depression of the farmhouse as observed from the observation balloon.

Solution:



For problems of this type the angle of elevation of A from B is considered equal to the angle of depression of B from A , as shown in the diagram.

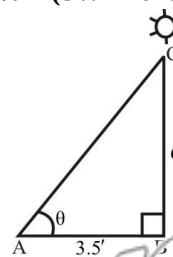
$$\tan \alpha = \frac{AC}{BC} = \frac{4280}{9613} = 0.44523$$

$$\alpha = \tan^{-1}(0.44523) = 24^\circ$$

So, angle of depression 24°

Exercise 7.5

Q.1 Find the angle of elevation of the sun if a 6 feet man casts a 3.5 feet shadow. (SWL 2016, 17) (A.B)



Given:

Height of man = $m\overline{EC} = 6'$

Length of shadow = $m\overline{AB} = 3.5'$

Required:

Angle of elevation = $\theta = ?$

Solution:

From $\triangle ABC$

$$\begin{aligned} \tan \theta &= \frac{m\overline{BC}}{m\overline{AB}} \\ &= \frac{6}{3.5} \end{aligned}$$

$$\tan \theta = 1.714$$

$$\Rightarrow \theta = \tan^{-1} 1.714$$

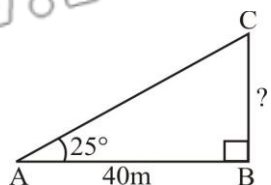
$$\theta = 59.74^\circ$$

or $\theta = 59^\circ 44' 37''$

Result:

$\therefore$ Angle of elevation of the sun is $59^\circ 44' 37''$

Q.2 A tree casts a 40 meter shadow when the angle of elevation of the sun is 25° . Find the height of the tree. **(A.B)**



Given:

Length of shadow = $\overline{mAB} = 40\text{m}$

Angle of elevation = $\theta = 25^\circ$

Required:

Height of tree = $\overline{mBC} = ?$

Solution:

From $\triangle ABC$,

$$\tan \theta = \frac{\overline{mBC}}{\overline{mAB}}$$

$$\tan 25^\circ = \frac{\overline{mBC}}{40\text{m}}$$

$$\overline{mBC} = \tan 25^\circ \times 40\text{m}$$

$$= 18.65\text{m}$$

Result

$\therefore$ Height of tree 18.65m

Q.3 A 20 feet long ladder is leaning against a wall. The bottom of the ladder is 5 feet from the base of the wall. Find the acute angle (angle of elevation) the ladder makes with the ground. **(A.B)**

Given:



Length of ladder

$$= \overline{mAC} = 20'$$

Distance between ladder

and wall = $\overline{mAB} = 5'$

Required:

Angle of elevation = $\theta = ?$

From $\triangle ABC$,

$$\cos \theta = \frac{\overline{mAB}}{\overline{mAC}}$$

$$\cos \theta = \frac{5}{20}$$

$$\theta = \cos^{-1} \frac{1}{4}$$

$$= 75.52$$

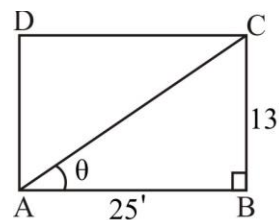
$$= 75^\circ 31' 21''$$

Result:

$\therefore$ Angle of elevation = $75^\circ 31' 21''$

Q.4 The base of a rectangle is 25 feet and the height of the rectangle is 13 feet. Find the angle that the diagonal of the rectangle makes with base. (MTN 2015) **(A.B)**

Given:



Length of rectangle = $\overline{mAB} = 25'$

Width of rectangle = $\overline{mBC} = 13'$

Required:

Angle of elevation = $\theta = ?$

Solution:

From $\triangle ABC$,

$$\tan \theta = \frac{\overline{mBC}}{\overline{mAB}}$$

$$\tan \theta = \frac{13}{25}$$

$$\theta = \tan^{-1} \frac{13}{25}$$

$$= 27.47^\circ$$

$$\theta = 27^\circ 28' 28''$$

Result:

$\therefore$ Angle of elevation = $27^\circ 28' 28''$

- Q.5** A rocket is launched and climbs at a constant angle of 80° . Find the altitude of the rocket after it travels 5000 meters. (SGD 2015) **(A.B)**

Given:



Distance traveled by rocket

$$= \overline{AC} = 5000\text{m}$$

Angle of elevation = 80°

Required:

Height (altitude) of rocket = $\overline{BC} = ?$

Solution:

From $\triangle ABC$,

$$\sin \theta = \frac{\overline{BC}}{\overline{AC}}$$

$$\sin 80^\circ = \frac{\overline{BC}}{5000\text{m}}$$

$$\overline{BC} = \sin 80^\circ \times 5000\text{m}$$

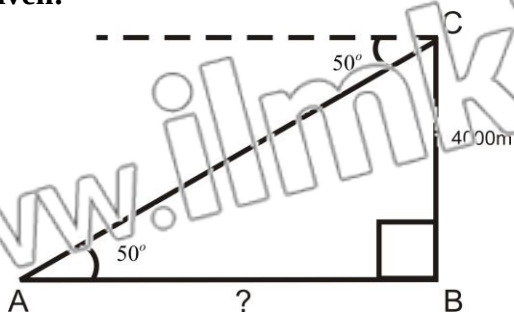
$$= 4924.04\text{m}$$

Result:

$\therefore$ Height of rocket = 4924.04m

- Q.6** An aero plane pilot flying at an altitude of 4000m wishes to make an approach to an airport at an angle of 50° with the horizontal. How far from the airport will the plane be when the pilot begins to descend? **(A.B)**

Given:



Height of plane = $\overline{BC} = 4000\text{m}$

Angle of depression = 50°

Required:

Distance between plane and airport = $\overline{AB} = ?$

Solution:

From $\triangle ABC$,

$$\tan \theta = \frac{\overline{BC}}{\overline{AB}}$$

$$\tan 50^\circ = \frac{4000\text{m}}{\overline{AB}}$$

$$\overline{AB} = \frac{4000\text{m}}{\tan 50^\circ}$$

$$= 3356.40\text{m}$$

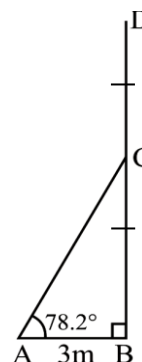
Result:

$\therefore$ Plane is 3356.40m away from airport

- Q.7** **Required:** **(A.B)**

Height of pole = $\overline{BD} = ?$

Given:



Distance between wire and pole = $\overline{AB} = 3\text{m}$

Angle of elevation = 78.2°

Solution:

From $\triangle ABC$,

$$\tan \theta = \frac{\overline{BC}}{\overline{AB}}$$

$$\tan 78.2^\circ = \frac{\overline{BC}}{3\text{m}}$$

$$\overline{BC} = \tan 78.2^\circ \times 3\text{m}$$

$$\overline{BC} = 14.36\text{m}$$

$$\overline{BD} = \overline{BC} + \overline{CD}$$

$$= 2\overline{BC}$$

$$= 2(14.36\text{m})$$

$$= 28.72\text{m}$$

Result:

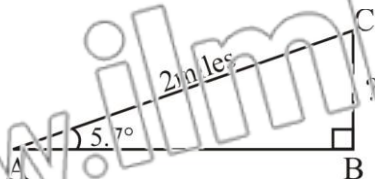
$\therefore$ Height of pole = 28.78m

Unit-7

Introduction to Trigonometry

- Q.8** A road is inclined at an angle 5.7° . Suppose that we drive 2 miles up this road starting from sea level. How high above sea level are we? **(A.3)**

Given:



Distance travelled by car = $m\overline{AC}$ = 2 miles

Angle of elevation = 5.7°

Height of car from sea level = $m\overline{BC}$ = ?

From $\triangle ABC$,

$$\sin \theta = \frac{m\overline{BC}}{m\overline{AC}}$$

$$\sin 5.7^\circ = \frac{m\overline{BC}}{2 \text{ miles}}$$

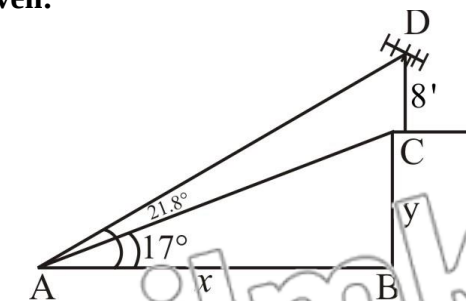
$$m\overline{BC} = \sin 5.7^\circ \times 2 \text{ miles} \\ = 0.1986 \text{ miles}$$

Result

$\therefore$ Car is 0.1986 miles up from sea level.

- Q.9** A television antenna of 8 feet height is located on the top of the house. From a point on the ground the angle of elevation to the top of the house is 17° and the angle of elevation to the top of the antenna is 21.8° . Find the height of the house. **(A.B)**

Given:



Height of antenna = $m\overline{CD}$ = 8 feet

$m\angle BAC = 17^\circ$

$m\angle BAD = 21.8^\circ$

Required:

Height of house = $m\overline{BC}$ = y = ?

Solution:

Let $m\overline{AB} = x$

From $\triangle BAC$,

$$\tan 17^\circ = \frac{m\overline{BC}}{m\overline{AB}}$$

$$0.3057 = \frac{y}{x}$$

$$0.3057x = y \rightarrow (i)$$

From $\triangle BAD$,

$$\tan 21.8^\circ = \frac{m\overline{BD}}{m\overline{AB}}$$

$$0.4 = \frac{y+8}{x}$$

$$m\overline{BD} = m\overline{BC} + m\overline{CD}$$

$$0.4x = y + 8 \rightarrow (ii)$$

Putting the value of 'y'

$$0.4x = 0.3057x + 8$$

$$0.4x - 0.3057x = 8$$

$$0.0943x = 8$$

$$x = \frac{8}{0.0943}$$

$$\Rightarrow x = 84.86$$

Put in equation (i)

$$y = 84.86 (0.3057)$$

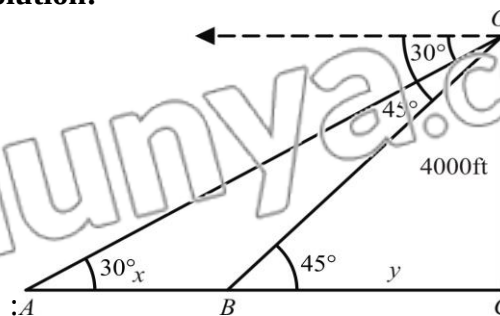
$$y = 25.94$$

Result:

$\therefore$ Therefore, height of house = 25.94 feet

- Q.10** From an observation point, the angles of depression of two boats in line with this point are found to 30° and 45° . Find the distance between the two boats if the point of observation is 4000 feet high. **(A.B)**

Solution:



Let A and B are two

Boat and O is the point of observation.

$$m\angle OAC = 30^\circ$$

$$m\angle OBC = 45^\circ$$

$$\overline{OC} = 4000 \text{ ft}$$

Let distance between two boat = $AB = x$

And $\overline{BC} = y$

In right angled $\triangle OBC$

$$\tan m\angle OBC = \frac{\overline{OC}}{\overline{BC}}$$

$$\tan 45^\circ = \frac{4000}{y}$$

$$1 = \frac{4000}{y}$$

$$y = 4000$$

In $\triangle OAC$

$$\tan m\angle OAC = \frac{\overline{OC}}{\overline{AC}}$$

$$\tan 30^\circ = \frac{4000}{x+y}$$

$$x+y = \frac{4000}{\tan 30^\circ} = \frac{4000}{\frac{1}{\sqrt{3}}}$$

$$= 4000 \times \sqrt{3}$$

$$x + 4000 = 4000\sqrt{3}$$

$$x = 4000\sqrt{3} - 4000$$

$$= (\sqrt{3} - 1)4000$$

$$= (1.732 - 1)4000$$

$$= 0.732 \times 4000$$

$$x = 2928 \text{ ft}$$

Result:

$\therefore$ Distance between the two boats

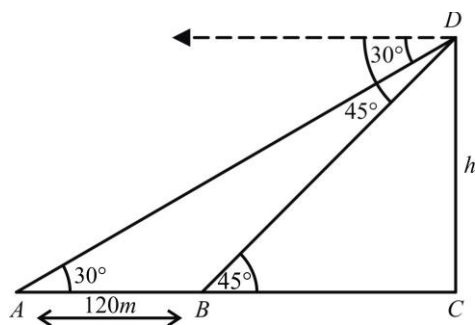
$$= m\overline{AB} = 2928 \text{ ft}$$

Q.11 Two ships, which are in line with the base of a vertical cliff, are 120 meters apart. The angles of depression from the top of the cliff to the ships are 30° and 45° , as show in the diagram. **(A.B)**

(a) Calculate the distance BC

(b) Calculate the height CD , of the cliff.

Solution:



In the figure A and B are two boats and $\overline{CD}$ is height of cliff. height of cliff = $\overline{CD} = h$

Distance between two boats

$$= \overline{AB} = 120 \text{ m}$$

Let $\overline{BC} = x$

$m\angle DAC = 30^\circ$ and $m\angle DBC = 45^\circ$

In $\triangle DBC$

$$\tan m\angle DBC = \frac{\overline{DC}}{\overline{BC}}$$

$$\tan 45^\circ = \frac{h}{x}$$

$$1 = \frac{h}{x}$$

$$x = h \rightarrow (i)$$

In $\triangle DAC$

$$\tan m\angle DAC = \frac{\overline{DC}}{\overline{AC}}$$

$$\tan 30^\circ = \frac{h}{120 + x}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{120 + x}$$

$$120 + x = \sqrt{3}h$$

$$120 + h = \sqrt{3}h$$

$$120 = \sqrt{3}h - h$$

$$= (\sqrt{3} - 1)h$$

$$= (1.732 - 1)h$$

$$120 = 0.732h$$

$$h = \frac{120}{0.732}$$

$$h = 163.93$$

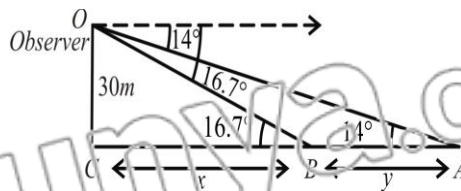
$$\therefore \overline{CD} = h = 163.93m$$

Result:

Hence height of cliff is 163.93 m, and $\overline{BC}$ is also 163.93m.

Q.12 Suppose that we are standing on a bridge 30 feet above a river watching a log (piece of wood) floating toward us. If the angle with the horizontal to the front of the log is 16.7° and angle with the horizontal to the back of the log is 14° how long is the log? **(A.B)**

Solution:



Here O is the observer

Length of log = $m\overline{AB} = y$

$$m\overline{CB} = x$$

$$m\overline{OC} = 30m$$

$$m\angle OBC = 16.7^\circ$$

$$\text{and } m\angle OAC = 14^\circ$$

In $\triangle OBC$

$$\tan m\angle OBC = \frac{m\overline{OC}}{m\overline{BC}}$$

$$\tan 16.7^\circ = \frac{30}{x}$$

$$x \times \tan 16.7^\circ = 30$$

$$x = \frac{30}{\tan 16.7^\circ} = \frac{30}{0.30001}$$

$$x = 99.996$$

In $\triangle OAC$

$$\tan m\angle OAC = \frac{m\overline{OC}}{m\overline{CA}}$$

$$\tan 14^\circ = \frac{30}{x + y}$$

$$x + y = \frac{30}{\tan 14^\circ}$$

$$x + y = \frac{30}{0.24933} = 120.3224$$

$$x + y = 120.3224$$

$$99.996 + y = 120.3224$$

$$y = 120.3224 - 99.996$$

$$y = 20.33m$$

Result:

$\therefore$ Length of log is 20.33m

Miscellaneous Exercise 7

Q.1 Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

- (1) The union of two non-collinear rays, which have common end point is called **(K.B)**
 (a) An angle (b) A degree
 (c) A minute (d) A radian
- (2) The system of measurement in which the angle is measured in radians is called **(K.B)**
 (a) CGS system (b) Sexagesimal
 (c) MKS system (d) Circular system
- (3) $20^\circ =$ **(K.B)**
 (a) $360'$ (b) $630'$
 (c) $1200'$ (d) $3600'$
- (4) $\frac{3\pi}{4}$ radians = **(GRW 2014, RWP 2015, MTN 2015, FSD 2018) (K.B)**
 (a) 115° (b) 135°
 (c) 150° (d) 30°
- (5) If $\tan \theta = \sqrt{3}$, then θ is equal to **(LHR 2014, D.G.K 2015) (K.B)**
 (a) 90° (b) 45°
 (c) 60° (d) 30°
- (6) $\sec^2 \theta =$ **(K.B)**
 (a) $1 - \sin^2 \theta$ (b) $1 + \tan^2 \theta$
 (c) $1 + \cos^2 \theta$ (d) $1 - \tan^2 \theta$
- (7) $\frac{1}{1 + \sin \theta} + \frac{1}{1 - \sin \theta}$ **(SGD 2014) (K.B)**
 (a) $2 \sec^2 \theta$ (b) $2 \cos^2 \theta$
 (c) $\sec^2 \theta$ (d) $\cos \theta$
- (8) $\frac{1}{2} \operatorname{cosec} 45^\circ$ **(GRW 2014, FSD 2015, D.G.K 2014) (K.B)**
 (a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$
 (c) $\sqrt{2}$ (d) $\frac{\sqrt{3}}{2}$
- (9) $\sec \theta \cot \theta =$ **(K.B)**
 (a) $\sin \theta$ (b) $\frac{1}{\cos \theta}$
 (c) $\frac{1}{\sin \theta}$ (d) $\frac{\sin \theta}{\cos \theta}$
- (10) $\operatorname{cosec}^2 \theta - \cot^2 \theta =$ **(LHR 2014, FSD 2014, SWL 2014, RWP 2014) (K.B)**
 (a) -1 (b) 1
 (c) 0 (d) $\tan \theta$

ANSWER KEY

1	2	3	4	5	6	7	8	9	10
a	d	c	b	c	b	a	b	c	b

Q.2 Write short answers. of the following questions.

(i) Define an angle. (K.B)

Ans: See definition Page # 216

(ii) What is the sexagesimal system of measurement of angles? (K.B)

Ans: See definition Page # 216

(iii) How many minutes are in two right angles? (K.B)

Ans: Angle = $\theta = 2 \times 90^\circ$
 $= 180^\circ$
 $= (180 \times 60)'$
 $= 10800'$

(iv) Define radian measure of an angle. (K.B)

(v) Convert $\frac{\pi}{4}$ radian to degree measure. (K.B)

Ans: Convert $\frac{\pi}{4}$ radian to degree measure.

$$\begin{aligned}\frac{\pi}{4} \text{ radian} &= \frac{\pi}{4} \times 1 \text{ radian} \\ &= \frac{\cancel{\pi}}{4} \times \frac{180}{\cancel{\pi}} \\ &= 45^\circ\end{aligned}$$

(vi) Convert 15° to radians. (K.B)

Ans: (D.G.K 2015)

Convert 15° to radian

$$15^\circ = 15 \times 1^\circ = 15 \times \frac{180}{\pi} = \frac{\pi}{12} \text{ radian}$$

(vii) What is radian measure of the central angle of an arc 50m long on the circle of radius 25m? (K.B)

Ans: Arc length = $l = 50\text{m}$
 Radius of circle = $r = 25\text{m}$
 Radian measure = $\theta = ?$

$$l = r\theta$$

$$\theta = \frac{l}{r} = \frac{50}{25}$$

$$\theta = 2 \text{ radian}$$

(viii) Find r when $l = 56\text{cm}$ and $\theta = 45^\circ$ (K.B)

Ans: $r = ?$
 $l = 56\text{cm}$
 $\theta = 45^\circ$

$$= 45 \times 1^\circ$$

$$= 45 \times \frac{\pi}{180} \text{ radian}$$

$$= \frac{\pi}{4}$$

Putting the values

$$l = r\theta$$

$$r = \frac{l}{\theta} = \frac{56}{\pi/4}$$

$$= 56 \times \frac{4}{\pi} = \frac{56 \times 4}{\pi}$$

$$= 56 \times \frac{4^2}{22/7}$$

$$= \frac{784}{11}$$

$$r = 71.27\text{cm}$$

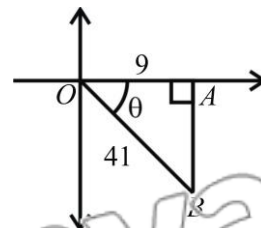
(ix) Find $\tan \theta$ when $\cos \theta = \frac{9}{41}$ and terminal side of the angle θ is in fourth quadrant. (K.B + A.B)

Ans: Solution:

$$\cos \theta = \frac{9}{41} \text{ and terminal side of the}$$

angle θ is in fourth quadrant.

In $\triangle BOA$



$$\text{Base} = \overline{OA} = 9$$

$$\text{Hyp} = \overline{OB} = 41$$

$$\text{Perp} = \overline{AB} = ?$$

$$(\text{Perp})^2 = (\text{Hyp})^2 - (\text{Base})^2$$

$$(\overline{AB})^2 = (\overline{OB})^2 - (\overline{OA})^2$$

$$= (41)^2 - (9)^2$$

$$= 1681 - 81$$

$$(\overline{AB})^2 = 1600$$

$$m\overline{AB} = \sqrt{1600}$$

$$m\overline{AB} = 40$$

$$\tan \theta = \frac{m\overline{AB}}{m\overline{OA}}$$

$$\tan \theta = \frac{40}{9}$$

(x) Prove that $(1 - \sin^2 \theta)(1 + \tan^2 \theta) = 1$

(K.B + A.B)

Ans: L.H.S = $(1 - \sin^2 \theta)(1 + \tan^2 \theta)$

$$= \cos^2 \theta \times \sec^2 \theta$$

$$= \cancel{\cos^2 \theta} \times \frac{1}{\cancel{\cos^2 \theta}}$$

$$= 1$$

$$= \text{R.H.S}$$

Q.3 Fill in the blanks

(i) π radians = _____ degree.

(K.B)

(ii) The terminal side of angle 235° lies in _____ quadrant.

(K.B)

(iii) Terminal side of the angle -30° lies in _____ quadrant.

(K.B)

(iv) Area of a circular sector is _____.

(K.B)

(v) If $r = 2\text{cm}$ and $\alpha = 3\text{radian}$, then area of the circular sector is _____.

(K.B)

(vi) The general form of the angle 480° is _____.

(K.B)

(vii) If $\sin \theta = \frac{1}{2}$, then $\theta =$ _____.

(K.B)

(viii) If $\theta = 300^\circ$ then

$$\sec(-300)^\circ = \text{_____}. \quad (\text{K.B})$$

$$(\text{ix}) \quad 1 + \cot^2 \theta = \text{_____}. \quad (\text{K.B})$$

$$(\text{x}) \quad \sec \theta - \tan \theta = \text{_____}. \quad (\text{K.B})$$

ANSWER KEY

(i) 180°

(ii) III

(iii) IV

$$(\text{iv}) \quad \frac{1}{2} r^2 \theta$$

$$(\text{v}) \quad 6\text{cm}^2$$

$$(\text{vi}) \quad 2k\pi + 120^\circ \text{ where } k = 1$$

$$(\text{vii}) \quad \theta = 30^\circ \text{ or } \frac{\pi}{6} \text{ and}$$

$$(\text{viii}) \quad 2$$

$$(\text{ix}) \quad \operatorname{cosec}^2 \theta$$

$$(\text{x}) \quad \frac{1 - \sin \theta}{\cos \theta}$$



Unit-7

Introduction to Trigonometry

CUT HERE

SELF TEST

Time: 40 min

Marks: 25

Q.1 Four possible answers (A), (B), (C) & (D) to each question are given, mark the correct answer. (7×1=7)

1 The system of measurement in which the angle is measured in radians is called:

- (A) CGS system (B) Sexagesimal system
(C) MKS system (D) Circular system

2 $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta}$

- (A) $2\sec^2\theta$ (B) $2\cos^2\theta$
(C) $\sec^2\theta$ (D) $\cos\theta$

3 Two or more than two angles with the same initial and terminal sides are called _____ angles.

- (A) Adjacent (B) Acute
(C) Conterminal (D) All

4 $\operatorname{cosec}30^\circ =$ _____

- (A) $\sqrt{3}$ (B) 2
(C) $\frac{2}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{2}}$

5 $2\cos^2\theta - 1 =$ _____

- (A) $1 - \sin^2\theta$ (B) $1 - 2\sin^2\theta$
(C) $1 + \sin^2\theta$ (D) $1 + 2\sin^2\theta$

6 A part of the circle bounded by an arc and a chord is called _____ of a circle.

- (A) Arc (B) Segment
(C) Sector (D) Radius

7 $\frac{3\pi}{4}$ radians = _____

- (A) 115° (B) 135°
(C) 150° (D) 30°

Q.2 Give Short Answers to following Questions.

(5×2=10)

- (i) How many minutes are in two right angles?
(ii) Find area of the sector inside a central angle of 20° in a circle of radius 7m.
(iii) Find $\sec(-300^\circ)$ without using table or calculator.

(iv) Verify the identity $\sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = \frac{\sec\theta+1}{\tan\theta}$

(v) A tree casts a 40 meter shadow when the angle of elevation of the sun is 25° . Find the height of the tree.

Q.3 Answer the following Questions.

(4+4=8)

(a) If $\tan\theta = \frac{4}{3}$ and $\sin\theta < 0$, find the values of other trigonometric functions at θ .

(b) From an observation point, the angles of depression of two boats in line with this point are found to 30° and 45° . Find the distance between the two boats if the point of observation is 4000 feet high.

NOTE: Parents or guardians can conduct this test in their supervision in order to check the skill of students.

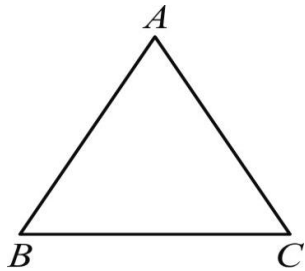
UNIT 8

PROJECTION OF A SIDE OF A TRIANGLE

Acute Angled Triangle (K.B)

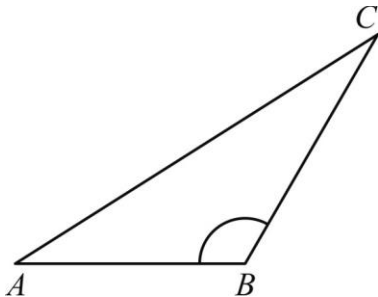
A triangle having all angles less than 90° is called acute angled triangle.

i.e In the fig, $\triangle ABC$ is an acute angle triangle.



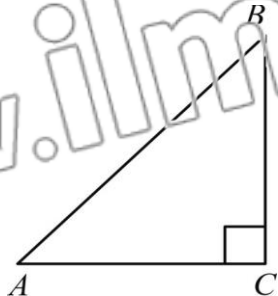
Obtuse Angled Triangle (K.B)

A triangle having one of its angles greater than 90° is called obtuse angle triangle.



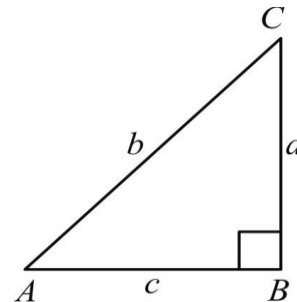
Right Angled Triangle (K.B)

A triangle having one of its angles equal to 90° is called right angle triangle.



Pythagoras's Theorem (K.B)

It states that "In a right angled triangle, square of hypotenuse is equal to, sum of squares of other two sides". From the $\triangle ABC$, $b^2 = a^2 + c^2$



Apollonius Theorem (K.B)

It states that "In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side".

Projection (K.B)

Image on Footing is called projection $\overline{CD}$, is projection of $\overline{AB}$ on $\overline{XY}$



Unit-8

Projection of a Side of a Triangle

Theorem 1

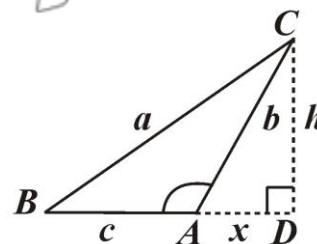
(A.B)

8.1(i)

In an obtuse angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.

Given:

ABC is a triangle having an obtuse angle $\angle BAC$ at A. Draw $\overline{CD}$ perpendicular on $\overline{BA}$ produced. So that $\overline{AD}$ is the projection of $\overline{AC}$ on $\overline{BA}$ produced. Take $m\overline{BC} = a, m\overline{CA} = b, m\overline{AB} = c, m\overline{AD} = x$ and $m\overline{CD} = h$.



To Prove:

$$(\overline{BC})^2 = (\overline{AC})^2 + (\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD}) \text{ i.e., } a^2 = b^2 + c^2 + 2cx$$

Proof:

Statements	Reasons
In $\triangle CDA$, $m\angle CDA = 90^\circ$ $\therefore (\overline{AC})^2 = (\overline{AD})^2 + (\overline{CD})^2$ or $b^2 = x^2 + h^2$ (i)	Given Pythagoras Theorem
In $\triangle CDB$, $m\angle CDB = 90^\circ$ $\therefore (\overline{BC})^2 = (\overline{BD})^2 + (\overline{CD})^2$ or $a^2 = (c + x)^2 + h^2$ $= c^2 + 2cx + x^2 + h^2$ (ii)	Given Pythagoras Theorem $\overline{BD} = \overline{BA} + \overline{AD}$
Hence $a^2 = c^2 + 2cx + b^2$ i.e., $a^2 = b^2 + c^2 + 2cx$ or $(\overline{BC})^2 = (\overline{AC})^2 + (\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD})$	Using (i) and (ii)

Theorem 2

(A.B)

8.1(ii)

In any triangle, the square on the side opposite to acute angle is equal to sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.

Given:

$\triangle ABC$ with an acute angle $\angle CAB$ at A.

Take $m\overline{BC} = a, m\overline{CA} = b$ and $m\overline{AB} = c$

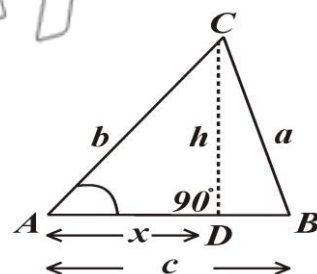
Draw $\overline{CD} \perp \overline{AB}$ so that $\overline{AD}$ is projection of $\overline{AC}$ on $\overline{AB}$

Also, $m\overline{AD} = x$ and $m\overline{CD} = h$

To Prove:

$$(\overline{BC})^2 = (\overline{AC})^2 + (\overline{AB})^2 - 2(m\overline{AB})(m\overline{AD})$$

$$\text{i.e. } a^2 = b^2 + c^2 - 2cx$$



Proof:

Statements	Reasons
In $\triangle CDA$ $m\angle CDA = 90^\circ$ $(\overline{AC})^2 = (\overline{AD})^2 + (\overline{CD})^2$ i.e., $b^2 = x^2 + h^2$ (i)	Given Pythagoras Theorem
In $\triangle CDB$, $m\angle CDB = 90^\circ$ $(\overline{BC})^2 = (\overline{BD})^2 + (\overline{CD})^2$ $a^2 = (c-x)^2 + h^2$ or $a^2 = c^2 - 2cx + x^2 + h^2$ (ii) $a^2 = c^2 - 2cx + b^2$ Hence, $a^2 = b^2 - c^2 - 2cx$ i.e., $(\overline{BC})^2 = (\overline{AC})^2 + (\overline{AB})^2 - 2(\overline{AB})(\overline{AD})$	Given Pythagoras Theorem From the figure Using (i) and (ii)

Theorem 3

(A.B)

(Apollonius' Theorem)

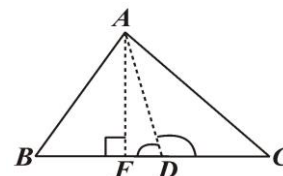
8.1(iii)

In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.

Given:

In a $\triangle ABC$, the median $\overline{AD}$ bisects $\overline{BC}$.

i.e., $m\overline{BD} = m\overline{CD}$



To Prove:

$$(\overline{AB})^2 + (\overline{AC})^2 = 2(\overline{BD})^2 + 2(\overline{AD})^2$$

Construction:

Draw $\overline{AF} \perp \overline{BC}$

Proof:

Statements	Reasons
In $\triangle ADB$ Since $\angle ADB$ is acute at D $\therefore (\overline{AB})^2 = (\overline{BD})^2 + (\overline{AD})^2 - 2m\overline{BD}.m\overline{FD} \rightarrow (i)$	Using Theorem 2
Now in $\triangle ADC$ Since $\angle ADC$ is acute at D $\therefore (\overline{AC})^2 = (\overline{CD})^2 + (\overline{AD})^2 + 2m\overline{CD}.m\overline{FD}$ $= (\overline{BD})^2 + (\overline{AD})^2 + 2m\overline{BD}.m\overline{FD} \rightarrow (ii)$	Using Theorem 1
Thus $(\overline{AB})^2 + (\overline{AC})^2 = 2(\overline{BD})^2 + 2(\overline{AD})^2$	Adding (i) and (ii)

Exercise 8.1

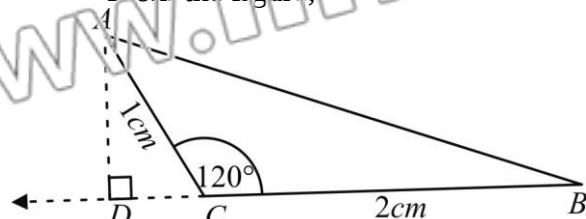
Q.1 (A.B)

Given:

$m\overline{AC} = 1\text{cm}$, $m\overline{BC} = 2\text{cm}$, $m\angle C = 120^\circ$
compute the length AB and the area of $\triangle ABC$

Solution:

For area of $\triangle$
From the figure,



$$m\angle ACD = 180^\circ - m\angle ACB$$

(Supplementary Angles)

$$= 180^\circ - 120^\circ$$

$$= 60^\circ$$

From $\triangle ACD$,

$$\sin 60^\circ = \frac{m\overline{AD}}{m\overline{AC}}$$

$$\frac{\sqrt{3}}{2} = \frac{m\overline{AD}}{1\text{cm}}$$

$$\Rightarrow m\overline{AD} = \frac{\sqrt{3}}{2} \times 1\text{cm} \quad m\overline{AD} = 0.877\text{cm}$$

$$\text{Area of } \triangle ABC = \frac{1}{2} m\overline{BC} \times m\overline{AD}$$

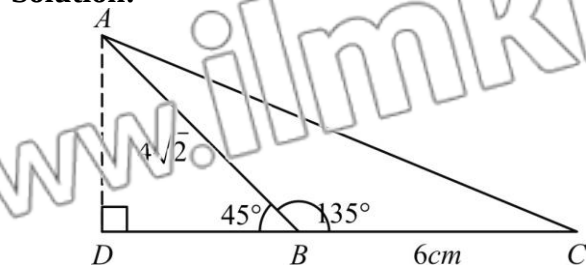
$$\therefore \text{Area of } \triangle = \frac{1}{2} \text{base} \times \text{height}$$

$$\Rightarrow \text{Area of } \triangle ABC = 0.877\text{cm}^2$$

Q.2 Find $m\overline{AC}$ if in $\triangle ABC$ $m\overline{BC} = 6\text{cm}$,
 $m\overline{AB} = 4\sqrt{2}$ and $m\angle ABC = 135^\circ$.

(A.B)

Solution:



$$(m\overline{AC})^2 = (m\overline{AB})^2 + (m\overline{BC})^2 - 2m\overline{BC}m\overline{BD}$$

From $\triangle ABD$

$$\cos 45^\circ = \frac{m\overline{BD}}{m\overline{AB}}$$

$$\frac{1}{\sqrt{2}} = \frac{m\overline{BD}}{4\sqrt{2}}$$

$$4 = m\overline{BD}$$

Now putting the values in formula

$$(m\overline{AC})^2 = (4\sqrt{2})^2 + (6)^2 - 2(6)(4)$$

$$(m\overline{AC})^2 = 16(2) + 36 - 48$$

$$(m\overline{AC})^2 = 32 + 36 - 48$$

$$(m\overline{AC})^2 = 20$$

$$\text{Taking square root } m\overline{AC} = \sqrt{20} = 4.47$$

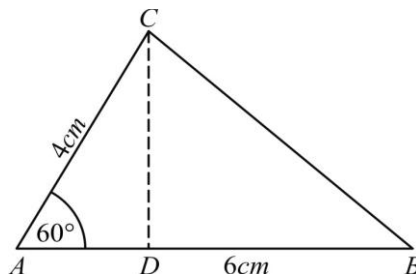
$$\therefore m\overline{AC} = 4.47\text{cm}$$

Exercise 8.2

Q.1 In a $\triangle ABC$ calculate $m\overline{BC}$ when

$m\overline{AB} = 6\text{cm}$, $m\overline{AC} = 4\text{cm}$ and $m\angle A = 60^\circ$

(A.B)



Solution:

By Theorem 8.2, we get

$$(m\overline{BC})^2 = (m\overline{AB})^2 + (m\overline{AC})^2 - 2m\overline{AB}m\overline{AD}$$

For value of $m\overline{AD}$

From $\triangle ADC$,

$$\cos 60^\circ = \frac{m\overline{AD}}{m\overline{AC}}$$

$$\frac{1}{2} = \frac{m\overline{AD}}{m\overline{AC}}$$

$$\frac{1}{2} m\overline{AC} = m\overline{AD}$$

Or

$$(m\overline{BC})^2 = (m\overline{AB})^2 + (m\overline{AC})^2 - 2m\overline{AB} \cdot \frac{1}{2} m\overline{AC}$$

Putting the values

$$(\overline{BC})^2 = (6)^2 + (4)^2 - 2.6.\frac{1}{2}.4$$

$$(\overline{BC})^2 = 36 + 16 - 24$$

$$(\overline{BC})^2 = 28$$

Taking square root on both sides

$$m\overline{BC} = \sqrt{28}$$

Result

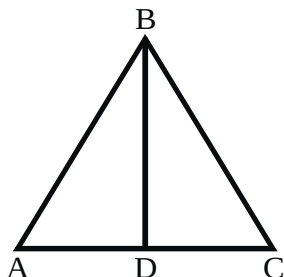
$$m\overline{BC} = \sqrt{28} \text{ cm}$$

Q.2 In a $\triangle ABC$, $m\overline{AB} = 6\text{cm}$, $m\overline{BC} = 8\text{cm}$, $m\overline{AC} = 9\text{cm}$ and D is the mid point of side $\overline{AC}$. Find length of the median $\overline{BD}$. **(A.B)**

Ans:

By Theorem 8.3, we have

$$(\overline{AB})^2 + (\overline{BC})^2 = 2[(\overline{AD})^2 + (\overline{BD})^2]$$



Putting the values

$$(6)^2 + (8)^2 = 2\left[\left(\frac{9}{2}\right)^2 + (\overline{BD})^2\right]$$

$$36 + 64 = 2\left[\frac{49}{4} + (\overline{BD})^2\right]$$

$$100 = 2\left[\frac{49}{4} + (\overline{BD})^2\right]$$

$$50 = \frac{49}{4} + (\overline{BD})^2$$

$$50 - \frac{49}{4} = (\overline{BD})^2$$

$$\frac{200 - 49}{4} = (\overline{BD})^2$$

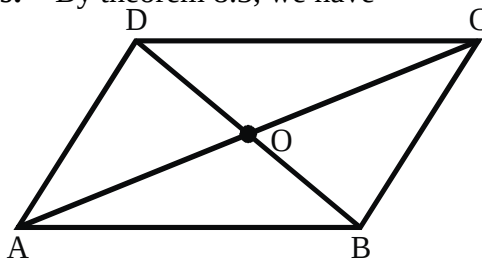
$$\frac{151}{4} = (\overline{BD})^2$$

Taking square root on both sides

$$\frac{\sqrt{151}}{2} = \overline{BD}$$

Q.3 In a parallelogram $AECL$ prove that $(\overline{AC})^2 + (\overline{BD})^2 = 2[(\overline{AB})^2 + (\overline{BC})^2]$ **(A.B)**

Ans: By theorem 8.3, we have



$$(\overline{AB})^2 + (\overline{BC})^2 = 2\left(\frac{1}{2}\overline{AC}\right)^2 + 2\left(\frac{1}{2}\overline{BD}\right)^2$$

$$(\overline{AB})^2 + (\overline{BC})^2 = \frac{1}{2}(\overline{BD})^2 + \frac{1}{2}(\overline{AC})^2$$

$$(\overline{AB})^2 + (\overline{BC})^2 = \frac{1}{2}[(\overline{BD})^2 + (\overline{AC})^2]$$

$\therefore$ O is mid point of $\overline{BD}$ and $\overline{AC}$

$$\text{Or } 2[(\overline{AB})^2 + (\overline{BC})^2] = (\overline{AC})^2 + (\overline{BD})^2$$

Hence Proved

Miscellaneous Exercise 8

Q.1 In a $\triangle ABC$, $m\angle A = 60^\circ$, prove that

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - m\overline{AB}.m\overline{AC}.$$

(A.B)

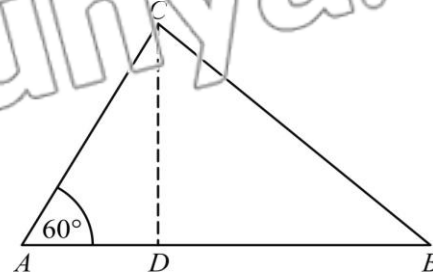
Given:

In a $\triangle ABC$, $m\angle A = 60^\circ$

To prove:

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - m\overline{AB}.m\overline{AC}$$

Proof:



By Theorem 8.2, we get

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - 2m\overline{AB}.m\overline{AC} \rightarrow (i)$$

Unit-8

Projection of a Side of a Triangle

For value of $\overline{mAD}$

From $\triangle ADC$,

$$\cos 60^\circ = \frac{\overline{mAD}}{\overline{mAC}}$$

$$\frac{1}{2} = \frac{\overline{mAD}}{\overline{mAC}}$$

$$\frac{1}{2} \overline{mAC} = \overline{mAD}$$

Put in equation (i)

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - 2\overline{mAB} \cdot \frac{1}{2} \overline{mAC}$$

Or

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - \overline{mAB} \cdot \overline{mAC}$$

Hence Proved

Q.2 In a $\triangle ABC$, $m\angle A = 45^\circ$, prove that

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - \sqrt{2}\overline{mAB} \cdot \overline{mAC}.$$

(A.B)

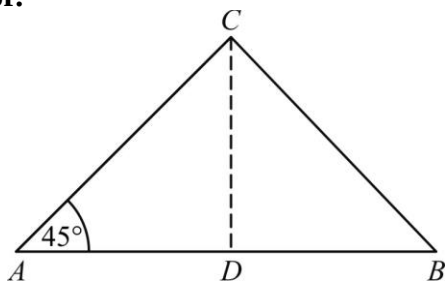
Given:

In a $\triangle ABC$, $m\angle A = 45^\circ$

To prove

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - \sqrt{2}\overline{mAB} \cdot \overline{mAC}$$

Proof:



By Theorem 8.2, we get

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - 2\overline{mAB} \cdot \overline{mAD} \rightarrow (i)$$

For value of $\overline{mAD}$

From $\triangle ADC$,

$$\cos 45^\circ = \frac{\overline{mAD}}{\overline{mAC}}$$

$$\frac{1}{\sqrt{2}} = \frac{\overline{mAD}}{\overline{mAC}}$$

$$\frac{1}{\sqrt{2}} \overline{mAC} = \overline{mAD}$$

Put in equation (i)

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - 2\overline{mAB} \cdot \frac{1}{\sqrt{2}} \overline{mAC}$$

$$\text{Or } (\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - \sqrt{2}\overline{mAB} \cdot \overline{mAC}$$

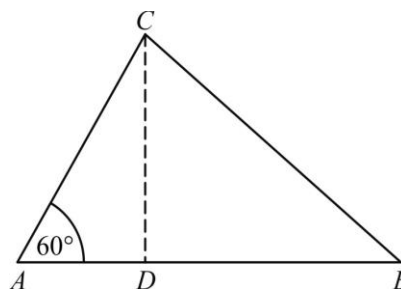
Hence Proved

Q.3 In a $\triangle ABC$, calculate $\overline{mBC}$ when

$$\overline{mAB} = 5\text{cm}, \overline{mAC} = 4\text{cm}, m\angle A = 60^\circ.$$

(A.B)

Solution:



By Theorem 8.2, we get

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - 2\overline{mAB} \cdot \overline{mAD}$$

For value of $\overline{mAD}$

From $\triangle ADC$,

$$\cos 60^\circ = \frac{\overline{mAD}}{\overline{mAC}}$$

$$\frac{1}{2} = \frac{\overline{mAD}}{\overline{mAC}}$$

$$\frac{1}{2} \overline{mAC} = \overline{mAD}$$

Or

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - 2\overline{mAB} \cdot \frac{1}{2} \overline{mAC}$$

Putting the values

$$(\overline{BC})^2 = (5)^2 + (4)^2 - 2 \cdot 5 \cdot \frac{1}{2} \cdot 4$$

$$(\overline{BC})^2 = 25 + 16 - 20$$

$$(\overline{BC})^2 = 21$$

Taking square root on both sides

$$\overline{mBC} = \sqrt{21}$$

Result:

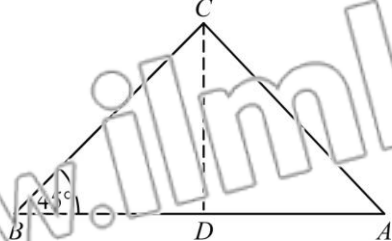
$$\overline{mBC} = \sqrt{21} \text{ cm}$$

Unit-8

Projection of a Side of a Triangle

- Q.4** In a $\triangle ABC$, calculate $m\overline{AC}$ when $m\overline{AB} = 5\text{cm}$, $m\overline{BC} = 4\text{cm}$, $m\angle B = 45^\circ$. **(A.B)**

By theorem 8.2, we have



$$(m\overline{AC})^2 = (m\overline{AB})^2 + (m\overline{BC})^2 - 2m\overline{AB} \times m\overline{BD}$$

For value of $\overline{BD}$

$$\cos 45^\circ = \frac{m\overline{BD}}{m\overline{BC}}$$

$$\frac{1}{\sqrt{2}} = \frac{m\overline{BD}}{4\text{cm}}$$

$$\frac{4\text{cm}}{\sqrt{2}} = m\overline{BD}$$

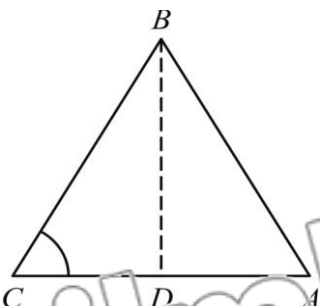
Now putting the values in equation (i)

$$(m\overline{AC})^2 = (5)^2 + (4)^2 - 2(5) \times \frac{4}{\sqrt{2}}$$

$$\Rightarrow m\overline{AC} = 12.71\text{cm}$$

- Q.5** In $\triangle ABC$, $m\overline{BC} = 21\text{cm}$, $m\overline{AC} = 17\text{cm}$, $m\overline{AB} = 10\text{cm}$. Measure the length of projection of $\overline{AC}$ upon $\overline{BC}$ **(A.B)**

Solution:



From the figure, we get

$$(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{AC})^2 - 2m\overline{BC} \cdot m\overline{CD}$$

Putting the values

$$(10)^2 = (21)^2 + (17)^2 - 2(21) \cdot m\overline{CD}$$

$$100 = 441 + 289 - 42m\overline{CD}$$

$$42m\overline{CD} = 730 - 100$$

$$42m\overline{CD} = 630$$

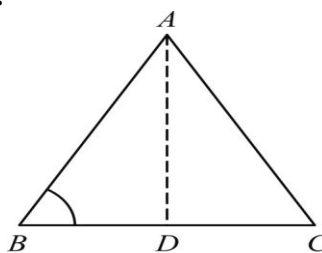
$$m\overline{CD} = \frac{630}{42}$$

$$m\overline{CD} = 15$$

$\therefore$ Length of projection of $\overline{AC}$ upon $\overline{BC} = 15\text{cm}$

- Q.6** In a triangle ABC , $m\overline{BC} = 21\text{cm}$, $m\overline{AC} = 17\text{cm}$ and $m\overline{AB} = 10\text{cm}$. Calculate the projection of $\overline{AB}$ upon $\overline{BC}$. **(A.B)**

Solution:



From the figure, we get

$$(m\overline{AC})^2 = (m\overline{AB})^2 + (m\overline{BC})^2 - 2(m\overline{BD})(m\overline{BC})$$

$$(17)^2 = (10)^2 + (21)^2 - 2m\overline{BD}(21)$$

$$289 = 100 + 441 - 42m\overline{BD}$$

$$42m\overline{BD} = 541 - 289$$

$$42m\overline{BD} = 252$$

$$m\overline{BD} = \frac{252}{42}$$

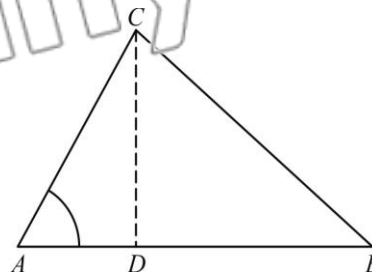
$$m\overline{BD} = 6$$

Result

$\therefore$ Project of $\overline{AB}$ upon $\overline{BC} = 6\text{cm}$

- Q.7** In a $\triangle ABC$, $a = 17\text{cm}$, $b = 15\text{cm}$, $c = 8\text{cm}$. Find $m\angle A$ **(A.B)**

Solution:



From fig, we get

$$(m\overline{BC})^2 = (m\overline{AB})^2 + (m\overline{AC})^2 - 2m\overline{AB} \times m\overline{AD}$$

Putting the values

$$(17)^2 = (15)^2 + (8)^2 - 2 \times 8 \times m\overline{AD}$$

$$289 = 225 + 64 - 16 \times m\overline{AD}$$

$$16 \times m\overline{AD} = 289 - 289$$

$$16 \times m\overline{AD} = 0$$

$$m\overline{AD} = 0$$

$$\text{From } \triangle CAD, \cos m\angle A = \frac{m\overline{AD}}{m\overline{AC}}$$

$$\cos m\angle A = \frac{0}{8}$$

$$\cos m\angle A = 0$$

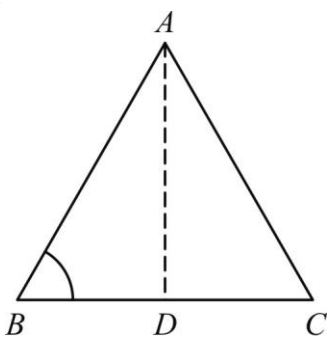
$$m\angle A = \cos^{-1} 0$$

$$m\angle A = 90^\circ$$

Q.8 In a $\triangle ABC$, $a = 17\text{cm}$, $b = 15\text{cm}$ and $c = 8\text{cm}$ find $m\angle B$. **(A.B)**

In $\triangle ABC$

Solution:



$$a = 17\text{cm}, b = 15\text{cm} \text{ and } c = 8\text{cm}$$

By theorem 8.2, we have

$$(m\overline{AC})^2 = (m\overline{AB})^2 + (m\overline{BC})^2 - 2m\overline{BC} \times m\overline{BD}$$

Putting the values

$$(15)^2 = (8)^2 + (17)^2 - 2(17)m\overline{BD}$$

$$225 = 64 + 289 - 34m\overline{BD}$$

$$34m\overline{BD} = 353 - 225$$

$$= 128$$

$$\therefore m\overline{BD} = 3.76\text{cm}$$

Now from $\triangle ABD$

$$\begin{aligned} \cos m\angle B &= \frac{m\overline{BD}}{m\overline{AB}} \\ &= \frac{3.76\text{cm}}{8\text{cm}} \end{aligned}$$

$$\cos m\angle B = 0.48\text{cm}$$

$$m\angle B = \cos^{-1} 0.48$$

$$m\angle B = 61.9^\circ$$

Q.9 Whether the triangle with sides 5cm, 7cm, 8cm is acute, obtuse or right. **(A.B)**

Solution:

$$\text{Let } a = 5\text{cm}, b = 7\text{cm} \text{ and } c = 8\text{cm}$$

$$a^2 + b^2 = (5)^2 + (7)^2 \quad c^2 = (8)^2$$

$$= 25 + 49 \quad = 64$$

$$= 74$$

Since $a^2 + b^2 > c^2$ Δ is acute angled.

Or

$$a^2 = 5^2 = 25$$

$$b^2 = 7^2 = 49$$

$$c^2 = 8^2 = 64$$

Since $a^2 + b^2 > c^2$ Δ is acute angled.

Q.10 Whether the triangle with sides 8cm, 15cm, 17cm is acute, obtuse or right angled. **(A.B)**

Solution:

$$\text{Let } a = 8\text{cm}, b = 15\text{cm} \text{ and } c = 17\text{cm}$$

$$a^2 = (8\text{cm})^2 = 64\text{cm}^2$$

$$b^2 = (15\text{cm})^2 = 225\text{cm}^2$$

$$c^2 = (17\text{cm})^2 = 289\text{cm}^2$$

$$\text{Since } a^2 + b^2 = c^2$$

$\therefore \Delta$ with given sides form a right angled triangle.

Note

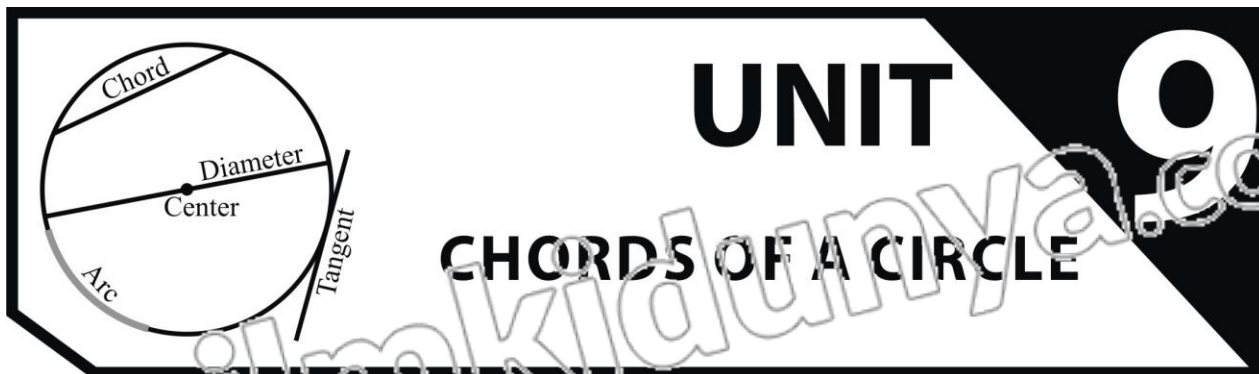
(A.B + U.B + K.B)

If $a^2 + b^2 = c^2$, Δ is a right angled Δ

If $a^2 + b^2 < c^2$, Δ is a obtuse angled Δ

If $a^2 + b^2 > c^2$, Δ is a acute angled Δ

Where c is longest side



Circle

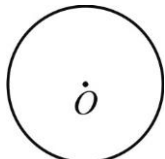
(K.B)

(LHR 2014, GRW 2017, D.G.K 2014, 16)

"It is a set of points on a plane which are equidistant from a fixed point from a circle".

Or

Locus of a point in a plane equidistant from a fixed point is called a circle.



Centre of the Circle

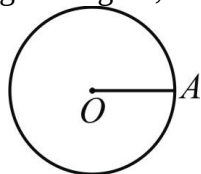
(K.B)

"The fixed point is called center of the circle". In the given figure, O is centre.

Radius of the Circle

(K.B)

"The distance between the centre to any point of the circle is called radius of the circle". In the given figure, $\overline{OA}$ is radius.



Radial Segment

(LHR 2015)

(K.B)

"A line segment which joins centre to any point of the circle is called radial segment".

In the given figure, $\overline{OA}$ is radial segment.

Circumference

(K.B)

(LHR 2014, 15, 16, GRW 2016, FSD 2015, SWL 2016, D.G.K 2014)

"Boundary of the circle is called circumference".

OR

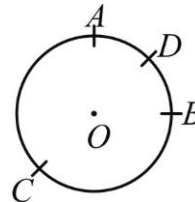
"It is the length of the segment joining all the points of circle is called circumference". Circumference is measured by the formula $2\pi r$, where r is the radius.

Arc of a Circle

(K.B)

(BWP 2014, RWP 2014, SGD 2015, D.G.K 2015)

"It is the part of circumference".



Major Arc

(K.B)

(LHR 2016, FSD 2015, SWL 2016, RWP 2014)

"An arc which is greater than semi-circle is called major arc". In the given figure, $\overline{ACB}$ is major arc.

Minor Arc

(K.B)

(LHR 2016, FSD 2015, SWL 2016, RWP 2014)

"An arc which is less than semi-circle is called minor arc". In the given figure, $\overline{ADB}$ is minor arc.

Chord of the Circle

(K.B)

(LHR 2015, GRW 2014, MTN 2014)

"A line segment whose two end points are any two points of the circle".

OR

"A line segment which joins two points of the circle". In the given figure, $\overline{AB}$ and $\overline{PQ}$ are chords.



Diameter

(K.B)

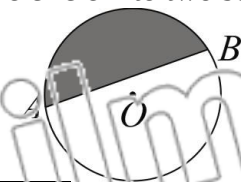
"A chord passing through centre of the circle is called diameter".

OR

"The largest chord is called diameter". In the given figure, $\overline{PQ}$ is diameter.

Segment of a Circle**(K.B)**

“A chord divides a circular region in two parts called segment of a circle”. In the given figure, $\overline{AB}$ divides the circle into two segments.

**Minor Segment****(K.B)**

(LHR 2016, FSD 2015, SWL 2016, RWP 2014)

“Circular region bounded by a chord and minor arc is called minor segment”. In the given figure, shaded part is minor segment.

Major Segment**(K.B)**

“Circular region bounded by a chord and major arc is called major segment”. In the given figure, non shaded part is major segment.

Sector of a Circle**(K.B)**

(LHR 2014, GRW 2014, 17, FSD 2014, SWL 2015, 16, D.G.K 2015)

“Circular region bounded by an arc and its two corresponding radial segments is called sector

of a circle”. In the given figure, AOB is a sector of the circle.

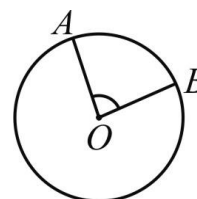
**Central Angle of an Arc****(K.B)**

“An angle subtended by an arc at the center of the circle is called central angle”.

Or

“An angle formed by two radial segments at the centre of the circle is called central angle”.

In the given figure, $\angle AOB$ is central angle of AB .

**Theorem 1**

(FSD 2015)

(A.B)**Statement:**

One and only one circle can pass through three non-collinear points.

Given:

A, B and C are three non collinear points in a plane.

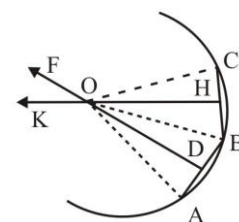
To Prove:

One and only one circle can pass through three non-collinear points A, B and C .

Construction:

Join A with B and B with C .

Draw $\overline{DF} \perp$ bisector to $\overline{AB}$ and $\overline{HK} \perp$ bisector to $\overline{BC}$. So $\overline{DF}$ and $\overline{HK}$ are not parallel and they intersect each other at point O . Also join A, B and C with point O .

**Proof:**

Statements	Reasons
Every point on $\overline{DF}$ is equidistant from A and B .	$\overline{DF} \perp$ bisector to $\overline{AB}$ (construction)
In particular $OA = OB \rightarrow (i)$	
Similarly every point on $\overline{HK}$ is equidistant from B and C .	$\overline{HK}$ is $\perp$ bisector to $\overline{BC}$ (construction)
In particular $OB = OC \rightarrow (ii)$	

Now O is the only point common to $\overline{DF}$ and $\overline{HK}$ which is equidistant from A, B and C .

i.e., $m\overline{OA} = m\overline{OB} = m\overline{OC}$

However there is no such other point except O .

Hence a circle with centre O and radius OA will pass through A, B and C .

Ultimately there is only one circle which passes through three given points A, B and C .

Using (i) and (ii)

Theorem 2:

(RWP 2015)

(A.B)

Statement:

A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.

Given:

M is the mid point of any chord $\overline{AB}$ of a circle with centre at O .

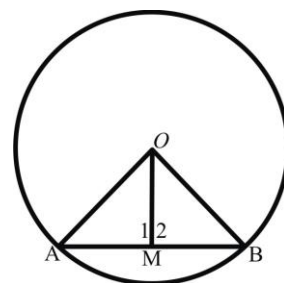
Where chord $\overline{AB}$ is not the diameter of the circle.

To Prove:

$\overline{OM} \perp$ the chord $\overline{AB}$.

Construction:

Join A and B with centre O . Write $\angle 1$ and $\angle 2$ as shown in the figure.



Proof:

Statements	Reasons
In $\triangle OAM \leftrightarrow \triangle OBM$	
$m\overline{OA} = m\overline{OB}$	Radii of the same circle
$m\overline{AM} = m\overline{BM}$	Given
$m\overline{OM} = m\overline{OM}$	Common
$\therefore \triangle OAM \cong \triangle OBM$	S.S.S $\cong$ S.S.S
$\Rightarrow m\angle 1 = m\angle 2 \rightarrow (i)$	Corresponding angles of congruent triangles
i.e., $m\angle 1 + m\angle 2 = m\angle AMB = 180^\circ \rightarrow (ii)$	Adjacent supplementary angles
$\therefore m\angle 1 = m\angle 2 = 90^\circ$	From (i) and (ii)
i.e., $\overline{OM} \perp \overline{AB}$	

Theorem 3

(SWL 2015, D.G.K 2015)

(A.B)

Statement:

Perpendicular from the centre of a circle on a chord bisects it.

Given:

$\overline{AB}$ is the chord of a circle with centre at O .

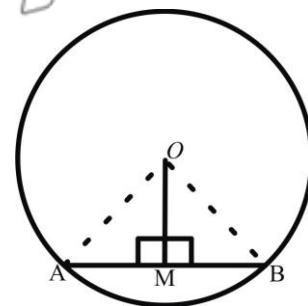
So that $\overline{OM} \perp$ chord $\overline{AB}$.

To Prove:

M is the mid point of chord $\overline{AB}$ i.e., $m\overline{AM} = m\overline{BM}$

Construction:

Join A and B with centre O .



Proof:

Statements	Reasons
In $\triangle OAM \leftrightarrow \triangle OBM$ $m\angle OMA = m\angle OMB = 90^\circ$ $hyp. \overline{OA} = hyp. \overline{OB}$ $m\overline{OM} = m\overline{OM}$ $\therefore \triangle OAM \cong \triangle OBM$ Hence, $m\overline{AM} = m\overline{BM}$ $\Rightarrow \overline{OM}$ bisects the chord $\overline{AB}$.	Given Radii of the same circle Common In $\triangle OAM \cong \triangle OBM$ H.S $\cong$ H.S Corresponding sides of congruent triangles

Corollary 1**(K.B + U.B)**

$\perp$ Bisector of the chord of a circle passes through the centre of a circle.

Corollary 2**(K.B + U.B)**

The diameter of a circle passes through the mid points of two parallel chords of a circle.

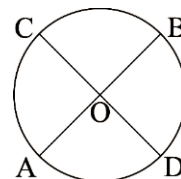
Exercise 9.1**Q. 1 Prove that, the diameters of a circle bisect each other.****(A.B)****Given**

In a circle with centre 'O', $\overline{AB}$ and $\overline{CD}$ are two diameters.

To prove

$\overline{AB}$ and $\overline{CD}$ bisect each other.

i.e O is midpoint of $\overline{AB}$ and $\overline{CD}$.

**Proof**

Statements	Reasons
$\overline{OA} \cong \overline{OB} \rightarrow (i)$ AOB is a straight line $\therefore O$ is midpoint of $\overline{AB} \rightarrow (ii)$ Similarly $\overline{OC} \cong \overline{OD}$ Or O is midpoint of $\overline{CD}$ Hence $\overline{AB}$ and $\overline{CD}$ bisect each other.	Radii of same circle As in (i) From (ii) and (iii) each other

Q. 2 Two chords of a circle do not pass through the centre. Prove that they cannot bisect each other.**(A.B)****Given**

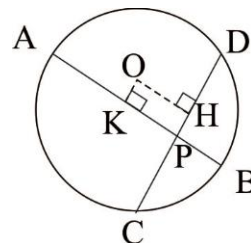
In a circle with centre 'O' chord $\overline{AB}$ and chord $\overline{CD}$ intersect each other at point P.

To prove

P is neither midpoint of $\overline{AB}$ nor $\overline{CD}$.

Construction

Draw $\overline{OK} \perp \overline{AB}$ and $\overline{OH} \perp \overline{CD}$.



Proof

Statements	Reasons
$\overline{AK} \cong \overline{BK}$	The 9.2
$m\overline{AK} = m\overline{BK} = m\overline{KP} + m\overline{PB}$	$m\overline{AK} = m\overline{KP} + m\overline{PB}$
$\therefore P$ is not midpoint of $\overline{AB} \rightarrow$ (i)	
Similarly	
$\overline{CH} \cong \overline{HD}$	
Or $m\overline{CP} + m\overline{PH} = m\overline{HD}$	
$\therefore P$ is not midpoint of $\overline{CD} \rightarrow$ (ii)	
So $\overline{AB}$ and $\overline{CD}$ cannot bisect each other.	From (i) and (ii)

Q. 3 If length of the chord $\overline{AB} = 8\text{cm}$. Its distance from the centre is 3cm , then find the diameter of such circle. (GRW 2014, D.G.K 2014) **(A.B)**

Solution:

From the fig, $m\angle OCA = 90^\circ$

($\overline{OC} \perp \overline{AB}$, $\perp$ is shortest distance)

$$m\overline{AC} = m\overline{BC} = 4\text{cm},$$

From $\triangle OCA$,

$$(\overline{OA})^2 = (\overline{AC})^2 + (\overline{OC})^2$$

$$= (4)^2 + (3)^2$$

$$= 16 + 9$$

$$= 25$$

Taking square root on both side

$$\sqrt{(\overline{OA})^2} = \sqrt{25}$$

$$m\overline{OA} = \sqrt{25}$$

$$m\overline{OA} = 5\text{cm}$$

$$\text{Since } \overline{OA} \text{ is the radius then diameter} = 2m\overline{OA} \\ = 2(5\text{cm})$$

Result:

Diameter = 10cm

Q. 4 Calculate the length of a chord which stands at a distance 5cm from the centre of a circle whose radius is 9cm . **(A.B)**

Solution:

Here $m\overline{OA} = 9\text{cm}$, $m\overline{OC} = 5\text{cm}$

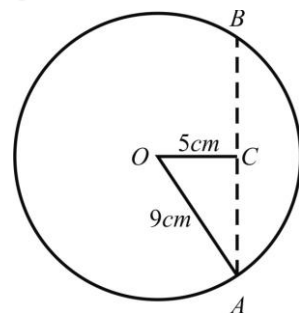
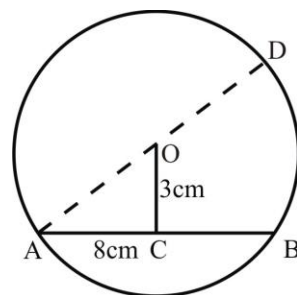
From $\triangle AOC$

$$(\overline{AC})^2 = (\overline{OA})^2 - (\overline{OC})^2$$

$$= (9)^2 - (5)^2 = 81 - 25$$

$$= 56$$

Taking square root on both sides



$$\begin{aligned}
 m\overline{AC} &= \sqrt{56} \\
 &= 2\sqrt{14}\text{cm} \\
 m\overline{AB} &= 2m\overline{AC} \quad \because C \text{ is midpoint of } \overline{AB}, \text{ then by theorem 9.2} \\
 &= 2(2\sqrt{14}\text{cm}) \\
 &= 4\sqrt{14}\text{cm}
 \end{aligned}$$

Theorem 9.1

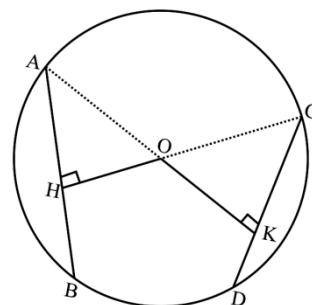
(MTN 2015)

(A.B)

9.1(i)

Statement:**If two chords of a circle are congruent then they will be equidistant from the centre.****Given:** $\overline{AB}$ and $\overline{CD}$ are two equal chords of a circle with centre at O .So that $\overline{OH} \perp \overline{AB}$ and $\overline{OK} \perp \overline{CD}$.**To Prove:**

$$m\overline{OH} = m\overline{OK}$$

Construction:Join O with A and O with C . So thatWe have $\angle rt\Delta^s OAH$ and OCK .**Proof:**

Statements	Reasons
$\overline{OH}$ bisects chord $\overline{AB}$	$\overline{OH} \perp \overline{AB}$ Perpendicular from the centre of a circle on a chord bisects it.
i.e., $m\overline{AH} = \frac{1}{2} m\overline{AB}$ (i)	
Similarly $\overline{OK}$ bisects chord $\overline{CD}$	$\overline{OK} \perp \overline{CD}$ Perpendicular from the centre of a circle on a chord bisects it.
i.e., $m\overline{CK} = \frac{1}{2} m\overline{CD}$ (ii)	
But $m\overline{AB} = m\overline{CD}$ (iii)	Given
Hence $m\overline{AH} = m\overline{CK}$ (iv)	Using (i), (ii) & (iii)
Now in $\angle rt\Delta^s OAH \leftrightarrow OCK$	Given $\overline{OH} \perp \overline{AB}$ and $\overline{OK} \perp \overline{CD}$
$hyp\overline{OA} = hyp\overline{OC}$	Radii of the same circle
$m\overline{AH} = m\overline{CK}$	Already Proved in (iv)
$\therefore \Delta OAH \cong \Delta OCK$	H.S postulate
$\Rightarrow m\overline{OH} = m\overline{OK}$	Corresponding side of congruent triangles

Theorem 5

9.1(v)

Two chords of a circle which are equidistant from the centre, are congruent.

Given:

$\overline{AB}$ and $\overline{CD}$ are two chords of a circle with centre at O

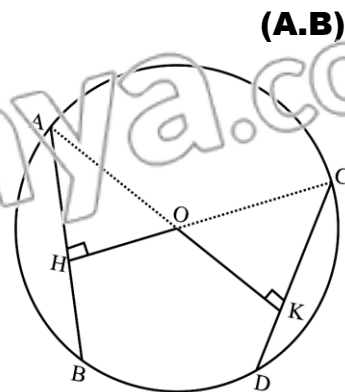
$\overline{OH} \perp \overline{AB}$ and $\overline{OK} \perp \overline{CD}$, so that $m\overline{OH} = m\overline{OK}$

To Prove:

$$m\overline{AB} = m\overline{CD}$$

Construction:

Join A and C with O . So that we can form $\angle rt\Delta^s OAH$ and OCK .



Proof:

Statements	Reasons
In $\angle rt\Delta^s OAH \Leftrightarrow OCK$.	
$\therefore \text{hyp}\overline{OA} = \text{hyp}\overline{OC}$	Radii of the same circle.
$m\overline{OH} = m\overline{OK}$	Given
$\therefore \Delta OAH \cong \Delta OCK$	H.S Postulate
So $m\overline{AH} = m\overline{CK}$ (i)	Corresponding sides of congruent triangles
But $m\overline{AH} = \frac{1}{2} m\overline{AB}$ (ii)	$\overline{OH} \perp \text{chord } \overline{AB}$ (Given)
Similarly $m\overline{CK} = \frac{1}{2} m\overline{CD}$ (iii)	$\overline{OK} \perp \text{chord } \overline{CD}$ (Given)
Since $m\overline{AH} = m\overline{CK}$	Already proved in (i)
$\therefore \frac{1}{2} m\overline{AB} = \frac{1}{2} m\overline{CD}$	Using (ii) & (iii)
or $m\overline{AB} = m\overline{CD}$	Multiplying both sides by 2

Exercise 9.2

Q.1 Two equal chords of a circle intersect, show that the segments of the one are equal corresponding to the segments of the other. (A.3)

Given

In a circle with centre ' O ' $\overline{AB}$ and $\overline{CD}$ are two chords of the circle intersecting at P such that $m\overline{AB} = m\overline{CD}$

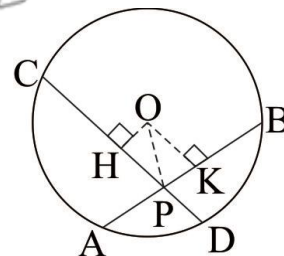
To prove

$$m\overline{AP} = m\overline{CP} \text{ and } m\overline{BP} = m\overline{DP}$$

Construction

Draw $\overline{OH} \perp \overline{CD}$ and $\overline{OK} \perp \overline{AB}$

Join O to P .



Proof

Statements	Reasons
$m\overline{OH} = m\overline{OK}$	Th-9. ($\overline{AB} \cong \overline{CD}$)
Also H is midpoint of $\overline{CD}$ and K is midpoint of $\overline{AB}$	
In rt $\triangle OPH \leftrightarrow \triangle OPK$	
$\overline{OH} \cong \overline{OK}$	Proved
$\overline{OP} \cong \overline{OP}$	Common
$\therefore \triangle OPH \cong \triangle OPK$	S.A.S postulate
$\overline{HP} \cong \overline{KP} \rightarrow (i)$	Corresponding sides of $\cong \Delta^s$
$m\overline{CP} = m\overline{CH} + m\overline{HP} \rightarrow (ii)$	
$\overline{CH} \cong \overline{BK} \rightarrow (ii)$	Both are half of congruent segments
$m\overline{CP} = m\overline{BK} + m\overline{KP}$	
$= m\overline{BP}$	From (i),(ii) and (iii)
Similarly	
$m\overline{AP} = m\overline{DP}$	

Q. 2 $\overline{AB}$ is the chord of a circle and the diameter $\overline{CD}$ is perpendicular bisector of AB .

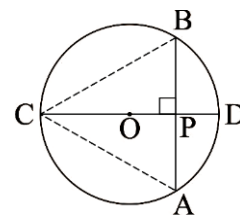
Prove that $m\overline{AC} = m\overline{BC}$.

(A.B)

Given

In a circle with centre 'O', chord $\overline{AB}$ and diameter $\overline{CD}$ intersect each other at point 'P',

Such that $\overline{CD} \perp \overline{AB}$ and $\overline{AP} \cong \overline{BP}$



To prove

$\overline{AC} \cong \overline{BC}$

Construction

Join C to A and B

Proof

Statements	Reasons
In $\triangle ACP \leftrightarrow \triangle BCP$	
$m\overline{AP} = m\overline{BP}$	$\overline{OD} \perp \overline{AB}$ A straight line draw from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.
$\angle APC \cong \angle BPC$	Both = 90° ($\overline{CD} \perp \overline{AB}$)
$\overline{CP} \cong \overline{CP}$	Common
$\therefore \triangle ACP \cong \triangle BCP$	S.A.S postulate
$\overline{AC} \cong \overline{BC}$	Corresponding sides of $\cong \Delta^s$

Q. 3 As shown in the figure, find the distance between two parallel chords $\overline{AB}$ and $\overline{CD}$.

(A.13)

Solution:

Since $\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$ then $\overline{AE} = 3\text{cm}$ and $\overline{CF} = 4\text{cm}$.

From $\triangle OCF$,

$$(\overline{OF})^2 = (\overline{OC})^2 - (\overline{CF})^2$$

$$(\overline{OF})^2 = (5)^2 - (4)^2$$

$$(\overline{OF})^2 = 25 - 16$$

$$(\overline{OF})^2 = 9$$

$$\sqrt{(\overline{OF})^2} = \sqrt{9}$$

$$\overline{OF} = 3$$

Similarly from $\triangle OAE$,

$$(\overline{OE})^2 = (\overline{OA})^2 - (\overline{AE})^2$$

$$(\overline{OE})^2 = (5)^2 - (3)^2$$

$$(\overline{OE})^2 = 25 - 9$$

$$(\overline{OE})^2 = 16$$

Taking sq. root

$$\sqrt{(\overline{OE})^2} = \sqrt{16}$$

$$\overline{OE} = 4$$

Distance between parallel chords $\overline{AB}$ and $\overline{CD}$

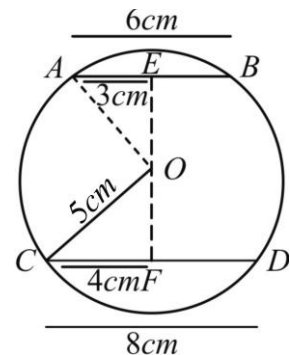
$$\overline{EF} = \overline{OE} + \overline{OF}$$

$$= 4\text{cm} + 3\text{cm}$$

$$= 7\text{ cm}$$

Result:

Distance between parallel chords $\overline{AB}$ and $\overline{CD} = 7\text{cm}$

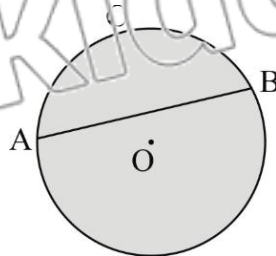


Miscellaneous Exercise 9

Q.1 Multiple choice questions

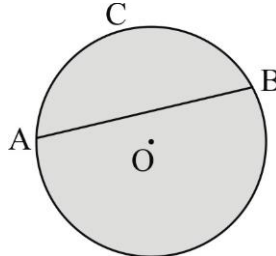
Four possible answers are given for the following question. Tick (✓) the correct answer.

- (i) In the circular figure, AE is called **(K.B)**



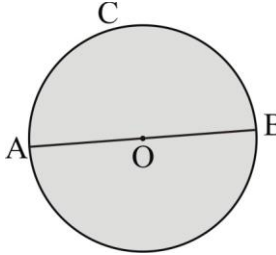
- (a) An arc
(b) A secant
(c) A chord
(d) A diameter

- (ii) In the circular figure, ACB is called **(K.B)**



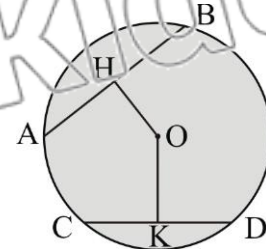
- (a) An arc
(b) A secant
(c) A chord
(d) A diameter

- (iii) In the circular figure, AOB is called **(K.B)**



- (a) An arc
(b) A secant
(c) A chord
(d) A diameter

- (iv) In a circular figure, two chords $\overline{AB}$ and $\overline{CD}$ are equidistant from the centre. They will be **(K.B)**



- (a) Parallel
(b) Non congruent
(c) Congruent
(d) Perpendicular

- (v) Radii of a circle are (LHR 2014, GRW 2014, SWL 2015, 16) **(K.B)**
 (a) All equal (b) Double of diameter
 (c) All unequal (d) Half of any chord
- (vi) A chord passing through the centre of a circle is called **(K.B)**
 (LHR 2015, GRW 2014, FSD 2018)
 (a) Radius (b) Diameter
 (c) Circumference (d) Secant
- (vii) Right bisector of the chord of a circle always passes through the **(K.B)**
 (SWL 2014)
 (a) Radius (b) Circumference
 (c) Centre (d) Diameter
- (viii) The circular region bounded by two radii and the corresponding arc is called **(K.B)**
 (a) Circumference of a circle (b) Sector of a circle
 (c) Diameter of a circle (d) Segment of a circle
- (ix) The distance of any point of the circle to its centre is called **(K.B)**
 (SGD 2014, D.G.K 2014)
 (a) Radius (b) Diameter
 (c) A chord (d) An arc
- (x) Line segment joining any point of the circle to the centre is called **(K.B)**
 (SGD 2014, MTN 1015, RWP 2015)
 (a) Circumference (b) Diameter
 (c) Radial segment (d) Perimeter
- (xi) Locus of a point in a plane equidistant from a fixed point is called **(K.B)**
 (D.G.K 2014)
 (a) Radius (b) Circle
 (c) Circumference (d) Diameter
- (xii) The symbol for a triangle is denoted by **(K.B)**
 (a) $\angle$ (b) Δ
 (c) $\perp$ (d) $\square$
- (xiii) A complete circle is divided into **(K.B)**
 (a) 90 degrees (b) 180 degrees
 (c) 270 degrees (d) 360 degrees
- (xiv) Through how many non collinear points, a circle can pass? **(K.E)**
 (LHR 2015)
 (a) One (b) Two
 (c) Three (d) None

ANSWER KEY

i	c	iv	c	vii	c	x	c	xiii	d
ii	a	v	a	viii	b	xi	b	xiv	c
iii	d	vi	b	ix	a	xii	b		

Q.2 Differentiate between the following terms and illustrate them by diagrams.

(i) A circle and a circumference.

(K.E)

Ans:

Differentiation

Circle	Circumference
- Set of points which are equidistant from a fixed point is called circle - It is graphical figure	- Length of all points on the circle(boundary) which are equidistant from a fixed point is called circumference - It is a constant value

(ii) A chord and the diameter of a circle.

(K.B)

Ans:

Differentiation

Chord	Diameter
- A line segment whose two end points are any two points of the circle - It may/may not pass through centre of the circle - It may/may not be the longest line segment	- A chord passing through centre of the circle is called diameter - It always passes through centre of circle - It is longest line segment inside the circle.

(iii) A chord and an arc of a circle.

(K.B)

Ans:

Differentiation

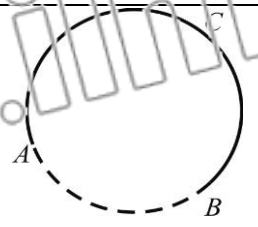
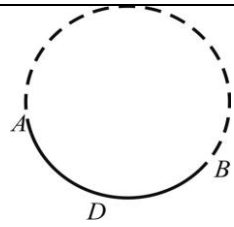
Chord	Arc
- A line segment whose two end points are any two points of the circle - It lies inside the circle - It is straight line	- Any portion of circumference is called an arc - It is part of circumference - It is a curved shape

(iv) Minor arc and major arc of a circle.

(K.B)

Ans:

Differentiation

Major Arc	Minor Arc
- It is bigger than minor arc - Major arc is always greater than semicircle	- It is smaller than major arc - Major arc is always less than semi circle
 ACB is a major arc	 ADB is a minor arc

(v) Interior and exterior of a circle.

(K.B)

Ans:

Differentiation

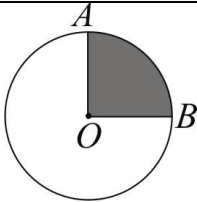
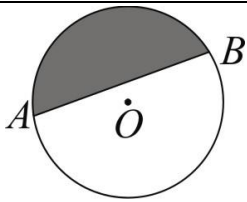
Interior of the Circle	Exterior of the Circle
All the points which lie inside of the circle form interior of the circle	All the point which lie outside of the circle form exterior of the circle
It includes the centre of the circle	It does not include centre of the circle
It is inner part of circle	It is outer part of circle

(vi) A sector and a segment of a circle.

(K.B)

Ans:

Differentiation

Sector of Circle	Segment of Circle
“Circular region bounded by an arc and its two corresponding radial segments is called sector of a circle”. In the given figure, AOB is a sector of the circle.	“A chord divides a circular region in two parts called segment of a circle”. In the given figure, $\overline{AB}$ divides the circle into two segments.
It does not include any chord	It includes one chord
It always included centre of the circle	It may/may not include centre of the circle
	



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SELF TEST

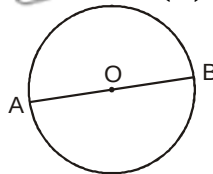
Time: 40 min

Marks: 25

Q.1 Four possible answers (A), (B), (C) & (D) to each question are given, mark the correct answer. (7×1=7)

1 In the circular figure, ACB is called:

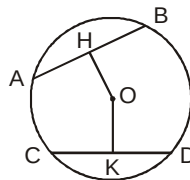
- (A) An arc (B) A secant
(C) A chord (D) A diameter



2 Right bisector of the chord of a circle always passes through the:

- (A) Radius (B) Circumference
(C) Centre (D) Diameter

3 In a circular figure, two chords AB and CD are equidistant from the centre. They will be:

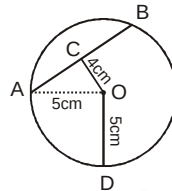


- (A) Parallel (B) Non congruent
(C) Congruent (D) Perpendicular

4 The union of a circle and its interior is called:

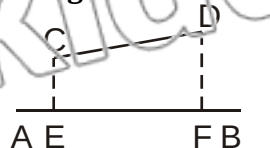
- (A) Diameter (B) Chord
(C) Circular region (D) Circumference

5 In the adjacent circular figure with centre O and radius 5cm. The length of the chord intercepted at 4cm away from the centre of this circle is:



- (A) 4cm (B) 6cm
(C) 7cm (D) 9cm

6 In figure the projection of a line segment $\overline{CD}$ on a line segment $\overline{AB}$ is the portion.



- (A) $\overline{CE}$ (B) $\overline{DF}$
(C) $\overline{EF}$ (D) $\overline{AF}$

7 The triangle with sides 8cm, 15cm, 17cm is:

- (A) Acute (B) Obtuse
(C) Right (D) Can't determine



Unit-9

Chords of a Circle

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Q.2 Give Short Answers to following Questions.

(5×2=10)

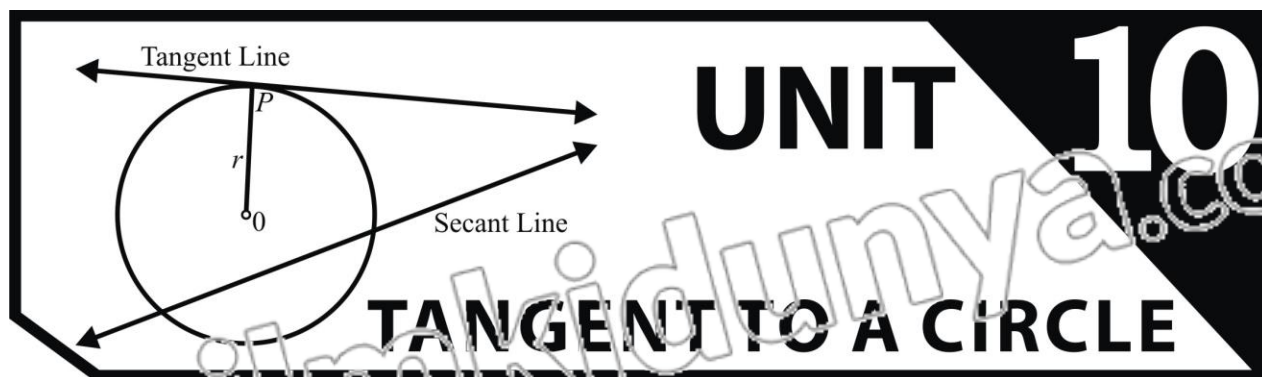
- (i) Define circle.
- (ii) What is the difference between minor segment and major segment?
- (iii) Differentiate between interior and exterior of a circle and illustrate them by diagrams.
- (iv) In a triangle ABC calculate $m\widehat{BC}$ when $m\widehat{AB} = 6cm$, $m\widehat{AC} = 4cm$, $m\angle A = 60^\circ$.
- (v) In a triangle ABC, $m\widehat{BC} = 21cm$, $m\widehat{AC} = 17cm$, $m\widehat{AB} = 10cm$. Measure the length of projection of $\widehat{AC}$ upon $\widehat{BC}$.

Q.3 Prove that,

(8)

A straight line, drawn from the centre of a circle to bisect a chord
(Which is not a diameter) is perpendicular to the chord.

NOTE: Parents or guardians can conduct this test in their supervision in order to check the skill of students.



Tangent Line

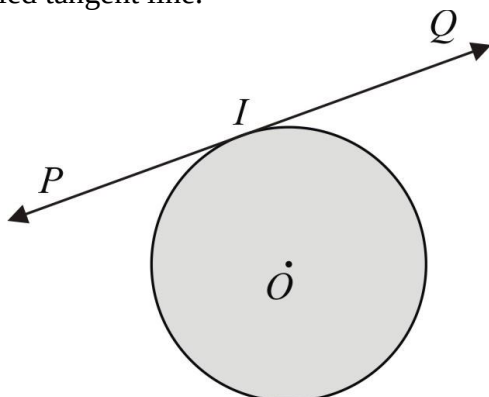
(K.B)

(LHR 2014, 15, GRW 2014, FSD 2014, BWP 2015, SWL 2014, SGD 2014)

A line which has one point common with a circle is called tangent line.

Or

A straight line which touches the circumference of circle at one point only is called tangent line.



In the figure $\overline{PIQ}$ is tangent line.

Secant Line

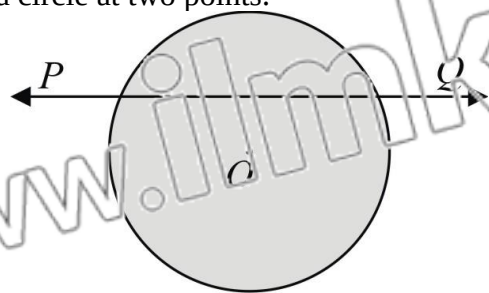
(K.B)

(FSD 2015, SGD 2014, 15, RWP 2014, D.G.K 2014)

A line which has two points common with a circle is called secant line.

Or

A straight line which cuts the circumference of a circle at two points.



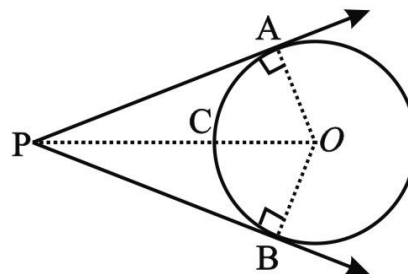
In the figure, $\overline{PQ}$ is secant line.

Length of Tangent

(K.B)

(SWL 2014, RWP 2015, D.G.K 2014, 15)

If a tangent is drawn from a point out side a circle, then distance between that point and the point of contact of the circle is called length of tangent.



In the figure $m\overline{PA} = m\overline{PB}$ is length of tangent.

Concentric Circles

(K.B)

Circles having same centres and different radii are called concentric circles.



In the above figure, circles having centre O are concentric circles.

Theorem 1

(A.B)

10.1(i)

Statement:

If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.

Given:

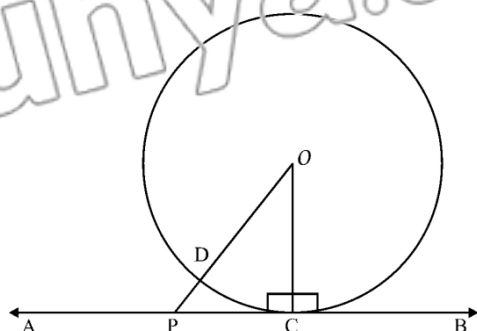
A circle with centre O and $\overline{OC}$ is the radial segment. $\overline{AB}$ is perpendicular to $\overline{OC}$ at its outer end C .

To Prove:

$\overline{AB}$ is a tangent to the circle at C

Construction:

Take any point P other than C on $\overline{AB}$. Join O with P .

**Proof:**

Statements	Reasons
In $\triangle OCP$, $m\angle OCP = 90^\circ$ and $m\angle OPC < 90^\circ$ $m\overline{OP} > m\overline{OC}$ $\therefore P$ is a point outside the circle. Similarly, every point on $\overline{AB}$ except C lies outside the circle. Hence $\overline{AB}$ intersects the circle at one point C only. i.e., $\overline{AB}$ is a tangent to the circle at one point only.	$\overline{AB} \perp \overline{OC}$ (given) Acute angle of right angled Triangle. Greater angle has greater side opposite to it. $\overline{OC}$ is the radial segment.

Theorem 2

(A.B)

10.1(ii)

The tangent to a circle and the radial segment, joining the point of contact and the centre are perpendicular to each other.

Given:

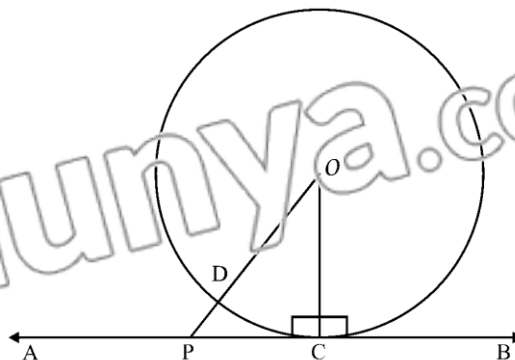
In a circle with centre O and radius $\overline{OC}$.
Also $\overline{AB}$ is the tangent to the circle at point C .

To Prove:

$\overline{AB}$ and radial segment $\overline{OC}$ are perpendicular to each other.

Construction:

Take any point P other than C on the tangent line $\overline{AB}$. Join O with P so that $\overline{OP}$ meets the circle at D .



Proof:

Statements	Reasons
$\overline{AB}$ is the tangent to the circle at point C . Whereas $\overline{OP}$ cuts the circle at D .	Given
$\therefore m\overline{OC} = m\overline{OD} \rightarrow (i)$	Construction
But $m\overline{OD} < m\overline{OP}$ (i)	Radii of the same circle
$\therefore m\overline{OC} < m\overline{OP}$	Point P is outside the circle.
So radius $\overline{OC}$ is shortest of all lines that can be drawn from O to the tangent line $\overline{AB}$	Using (i) and (ii)
Also $\overline{OC} \perp \overline{AB}$	
Hence, radial segment $\overline{OC}$ is perpendicular to the tangent $\overline{AB}$.	

Corollary**(A.B + U.B)**

There can only be one perpendicular draw to the radial segment $\overline{OC}$ at the point C of the circle. It follows that one and only one tangent can be drawn to the circle at the given point C on its circumference.

Theorem 3**(A.B)****10.1(iii)**

Two tangents drawn to a circle from a point outside it, are equal in length.

Given:

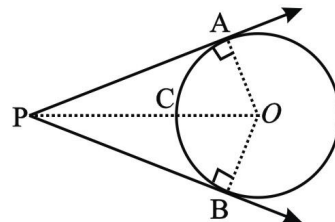
Two tangents $\overline{PA}$ and $\overline{PB}$ are drawn from an external point P to the circle with centre O .

To Prove:

$$m\overline{PA} = m\overline{PB}$$

Construction:

Join O with A , B and P , so that we form $\angle rt\Delta^s OAP$ and OBP .

**Proof:**

Statements	Reasons
In $\angle rt\Delta^s OAP \leftrightarrow OBP$	
$m\angle OAP = m\angle OBP = 90^\circ$	Radii $\perp$ to the tangents $\overline{PA}$ and $\overline{PB}$
hyp. $\overline{OP} = \text{hyp. } \overline{OP}$	Common
$m\overline{OA} = m\overline{OB}$	Radii of the same circle.
$\therefore \Delta OAP \cong \Delta OBP$	$\angle rt\Delta^s$ H.S $\cong$ H.S
Hence $m\overline{PA} = m\overline{PB}$	Corresponding sides of congruent triangles

Note

The length of a tangent to a circle is measured from the given point to the point of contact.

Corollary**(A.B + U.B)**

If O is the centre of a circle and two tangents $\overline{PA}$ and $\overline{PB}$ are drawn from an external point P then $\overline{OP}$ is the right bisector of the chord of contact $\overline{AB}$.

Exercise 10.1

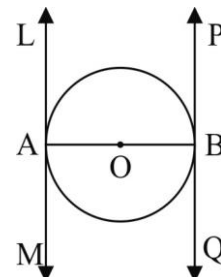
Q.1 Prove that the tangents drawn at the ends of a diameter in a given circle must be parallel. **(A.B)**

Given

In a circle with centre ' O ', $\overline{AB}$ is a diameter, $\overline{LM}$ and $\overline{PQ}$ are two tangents passing through point A and B .

To prove

$$\overline{LM} \parallel \overline{PQ}$$

**Proof**

Statements	Reasons
$\overline{OA}$ and $\overline{OB}$ are radial segments	Given
$\therefore m\angle OAL = 90^\circ \rightarrow (i)$	Tangent is $\perp$ to a radial segment.
Similarly	
$\therefore m\angle OBQ = 90^\circ \rightarrow (ii)$	As in (i)
$\therefore m\angle OAL = m\angle OBQ$	From (i) and (ii)
Or $\overline{LM} \parallel \overline{PQ}$	Alternate angles are congruent.

Q.2 The diameters of two concentric circles are 10 cm and 5 cm respectively. Look for the length of any chord of the outer circle which touches the inner one. **(A.B)**

Solution:

From $\triangle OCB$,

$$\begin{aligned}
 (m\overline{BC})^2 &= (m\overline{OB})^2 - (m\overline{OC})^2 \because (hyp)^2 = (perp)^2 + (base)^2 \\
 &= (5)^2 - (2.5)^2 \\
 &= 25 - 6.25 \\
 &= 18.75
 \end{aligned}$$

Taking square root on both sides

$$m\overline{BC} = \sqrt{18.75}$$

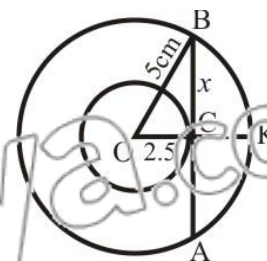
$$x = 4.33$$

$$m\overline{AB} = 2x$$

$$= 2(4.33)$$

$$= 8.66 \text{ cm}$$

$$\therefore \text{Length of chord} = m\overline{AB} = 8.66 \text{ cm}$$



Q.3 $\overline{AB}$ and $\overline{CD}$ are the common tangents drawn to the pair of circles. If A and C are the points of tangency of 1st circle where B and D are the points of tangency of 2nd circle, then prove that $\overline{AC} \parallel \overline{BD}$. (A.B)

Given

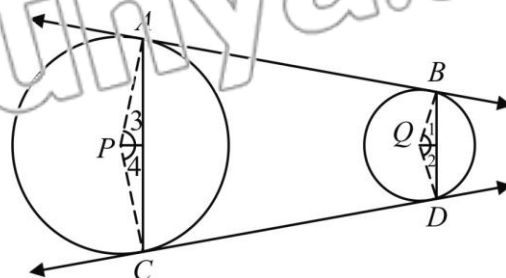
Two circles with centre P and Q . $\overline{AB}$ and $\overline{CD}$ are common tangents of A is joined with C and B with D .

To prove

$\overline{AC} \parallel \overline{BD}$

Construction

Join P with A and C and Q with B and D . Name the angles $\angle 1, \angle 2, \angle 3, \angle 4$ as shown in the figure.



Proof

Statement	Reasons
$\overline{AP} \perp \overline{AB} \rightarrow$ (i)	Theorem 10.2
$\overline{BQ} \perp \overline{AB} \rightarrow$ (ii)	
$\overline{AP} \parallel \overline{BQ}$	From (i),(ii)
$\angle 3 \cong \angle 1 \rightarrow$ (iii)	Corresponding angles
Similarly	
$\angle 4 \cong \angle 2 \rightarrow$ (iv)	
$m\angle 3 + m\angle 4 = m\angle 1 + m\angle 2$	Adding (iii) and (iv)
$m\angle APC = m\angle BQD$	Sum of angles postulate
$\frac{m\overline{AP}}{m\overline{BQ}} = \frac{m\overline{PC}}{m\overline{QD}}$	
$\therefore m\angle PCA = m\angle QDB$	
Hence $\overline{AC} \parallel \overline{BD}$	

Theorem 4 (a)

(A.B)

10.1(iv)

If two circles touch externally then the distance between their centers is equal to the sum of their radii.

Given:

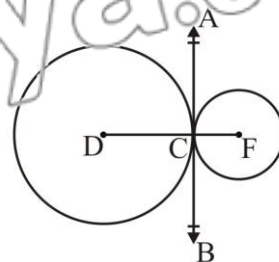
Two circles with centres D and F respectively touch each other externally point C . So that $\overline{CD}$ and $\overline{CF}$ are respectively the radii of the two circles.

To Prove:

Point C lies on the join of centres D and F and $m\overline{DF} = m\overline{DC} + m\overline{CF}$

Construction:

Draw $\overline{ACB}$ as a common tangent to the pair of circles at C .



Proof:

Statements	Reasons
Both circles touch externally at C whereas $\overline{CD}$ is radial segment and $\overline{ACB}$ is the common tangent. $\therefore m\angle ACD = 90^\circ$ (i) Similarly $\overline{CF}$ is radial segment and $\overline{ACB}$ is the common tangent $\therefore m\angle ACF = 90^\circ$ (ii) $m\angle ACD + m\angle ACF = 90^\circ + 90^\circ$ $m\angle DCF = 180^\circ$ (iii) Hence DCF is a line segment with point C between D and F and $m\overline{DF} = m\overline{DC} + m\overline{CF}$	Radial segment $\overline{CD} \perp$ the Tangent line $\overline{AB}$ Radial segment $\overline{CF} \perp$ the tangent line $\overline{AB}$ Adding (i) and (ii) Sum of supplementary adjacent angles.

Exercise 10.2

Q.1 $\overline{AB}$ and $\overline{CD}$ are two equal chords in a circle with centre O . H and K are respectively the midpoints of the chords. Prove that $\overline{HK}$ makes equal angles with $\overline{AB}$ and $\overline{CD}$.

(A.B)**Given**

In a circle with centre ' O ', H and K are midpoints of chord $\overline{AB}$ and $\overline{CD}$ respectively.

Solution

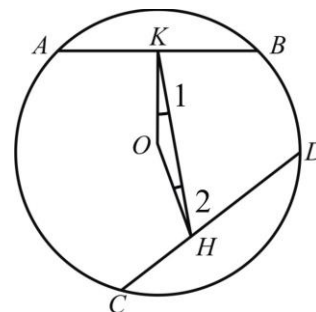
$$m\overline{AB} = m\overline{CD}$$

To prove

$$m\angle BKH = m\angle DHK$$

Construction

Join O to H and K and name the angles as shown in the figure.

**Proof**

Statements	Reasons
$m\overline{OH} = m\overline{OK}$	(Given) ($\overline{AB} \cong \overline{CD}$)
Also $m\angle OHD = m\angle OKB = 90^\circ \rightarrow$ (i)	
$\angle 1 \cong \angle 2 \rightarrow$ (ii)	Angles opposite to congruent sides
$m\angle BKH = m\angle BKO - m\angle 1$ $= 90^\circ - m\angle 1 \rightarrow$ (iii)	
$m\angle DHK = m\angle DHO - m\angle 2$ $= 90^\circ - m\angle 2$ $= 90^\circ - m\angle 1 \rightarrow$ (iv)	From (ii)
$\therefore m\angle BKH = m\angle DHK$	From (iii) and (iv)

Q.2 The radius of a circle is 2.5m. $\overline{AB}$ and $\overline{CD}$ are two chords 3.9 cm apart. If $m\overline{AB} = 1.4\text{cm}$, then find the measurement of other chord. (A.B)

Solution:

In a circle with centre 'O', $\overline{AB}$ and $\overline{CD}$ are two chords and distance between them $\overline{PQ}$ is 3.9cm.

From the fig, $\overline{AP} = \frac{1.4}{2} = 0.7 \quad \therefore \overline{OP} \perp \overline{AB}$

From $\triangle OAP$,

$$\begin{aligned} (m\overline{OP})^2 &= (m\overline{OA})^2 - (m\overline{AP})^2 \\ &= (2.5)^2 - (0.7)^2 \\ &= 6.25 - 0.49 \\ &= 5.76 \end{aligned}$$

Taking square root

$$m\overline{OP} = \sqrt{5.76}$$

$$\Rightarrow m\overline{OP} = 2.4$$

$$\begin{aligned} m\overline{OQ} &= m\overline{PQ} - m\overline{OP} \\ &= 3.9 - 2.4 \\ &= 1.5 \text{ cm} \end{aligned}$$

From $\triangle OCQ$,

$$\begin{aligned} (m\overline{CQ})^2 &= (m\overline{OC})^2 - (m\overline{OQ})^2 \\ (m\overline{CQ})^2 &= (2.5)^2 - (1.5)^2 \\ &= 6.25 - 2.25 \\ &= 4 \end{aligned}$$

Taking square root

$$m\overline{CQ} = 2$$

$$\begin{aligned} \text{Since chord } \overline{CD} &= 2m\overline{CQ} \\ &= 2(2) \\ &= 4 \end{aligned}$$

Result:

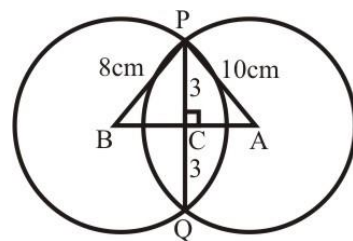
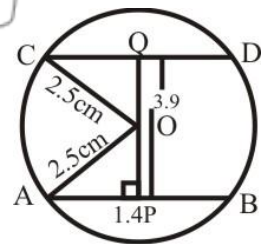
Length of other chord $m\overline{CD} = 4\text{cm}$

Q.3 The radii of two intersecting circles are 10 cm and 8 cm. If the length of their common chord is 6cm then find the distance between the centers. (A.B)

Solution:

In $\triangle ACP$

$$\begin{aligned} (m\overline{AC})^2 &= (m\overline{AP})^2 - (m\overline{PC})^2 \\ (m\overline{AC})^2 &= (10)^2 - (3)^2 \\ (m\overline{AC})^2 &= 100 - 9 \end{aligned}$$



$$(m\overline{AC})^2 = 91$$

$$(m\overline{AC})^2 = \sqrt{91}$$

In $\triangle BCP$

$$(m\overline{BC})^2 = (m\overline{BP})^2 - (m\overline{PC})^2 \quad \because (\text{hypotenuse})^2 = (\text{perpendicular})^2 + (\text{base})^2$$

$$(m\overline{BC})^2 = 8^2 - 3^2$$

$$(m\overline{BC})^2 = 64 - 9$$

$$(m\overline{BC})^2 = 55$$

$$(m\overline{BC})^2 = \sqrt{55}$$

$$\begin{aligned} \text{Distance between centers} = m\overline{AB} &= m\overline{AC} + m\overline{BC} \\ &= \sqrt{91} + \sqrt{55} \\ &= 16.96 \text{ cm} \end{aligned}$$

Q.4 Show that greatest chord in a circle is its diameter. (A.B)

In a circle with centre, 'O'

Given

$\overline{AB}$ is a diameter and $\overline{CD}$ is a chord.

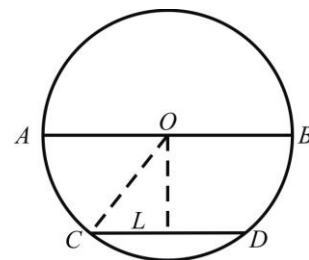
To prove

$$m\overline{AB} > m\overline{CD}$$

Construction

Draw $\overline{OL} \perp \overline{CD}$ and join O to C.

Proof



Statements	Reasons
$\overline{OL} \perp \overline{CD}$	Construction
$\therefore$ L is midpoint of $\overline{CD}$.	Perpendicular drawn from centre bisect the chord.
In $\triangle OLC$	
$m\angle OLC = 90$	$\overline{OL} \perp \overline{CD}$
$m\angle COL < 90$	Acute angle of right triangle
$\therefore m\angle OLC > m\angle COL$	From (i) and (ii)
$\therefore m\overline{OC} > m\overline{CL}$	Side opposite to greater angle
$2m\overline{OC} > 2m\overline{CL}$	Multiply by '2'
Diameter $> m\overline{CD}$	L is midpoint of $\overline{CD}$
$\therefore m\overline{AB} > m\overline{CD}$	(Proved)

Theorem 4 (b)

10.1(v)

If two circles touch each other internally, then the point of contact lies on the line segment through their centres and distance between their centres is equal to the difference of their radii.

Given:

Two circles with centres D and F touch each other

Internally at point C . So that $\overline{CD}$ and $\overline{CF}$ are the radii of two circles.

To Prove:

Point C lies on the join of centres D and F extended and $m\overline{DF} = m\overline{DC} - m\overline{CF}$

Construction:

Draw $\overline{ACB}$ as the common tangent to the pair of circles at C .

Proof:

Statements	Reasons
Both circles touch internally at C whereas $\overline{ACB}$ is the common tangent and $\overline{CD}$ is the radial segment of the first circle.	
$\therefore m\angle ACD = 90^\circ$ (i)	Radial segment $\overline{CD} \perp$ the tangent line $\overline{AB}$
Similarly $\overline{ACB}$ is the common tangent and $\overline{CF}$ is the radial segment of the second circle.	
$\therefore m\angle ACF = 90^\circ$ (ii)	Radial segment $\overline{CF} \perp$ the tangent line $\overline{AB}$
$\Rightarrow m\angle ACD = m\angle ACF = 90^\circ$	Using (i) and (ii)
Where $\angle ACD$ and $\angle ACF$ coincide each other with point F between D and C .	
Hence $m\overline{DC} = m\overline{DF} + m\overline{FC}$	
i.e., $m\overline{DC} - m\overline{FC} = m\overline{DF}$	
Or $m\overline{DF} = m\overline{DC} - m\overline{FC}$	

Exercise 10.3

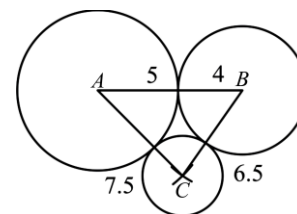
Q.1 Two circles with radii 5cm and 4cm touch each other externally. Draw another circle with radius 2.5cm touching the first pair, externally. **(A.B)**

To construct

A circle of radius 2.5cm touching given two circles externally.

Construction**Steps of construction**

- With centre A , draw an arc of radius 7.5cm ($5 + 2.5 = 7.5$)
- With centre B , draw an arc of radius
- Both arcs cut each other at point C .
- With centre C , draw a circle of radius 2.5cm .



Q.2 If the distance between the centres 6.5cm ($4 + 2.5 = 6.5$) of two circles is the sum or the difference of their radii they will touch each other. (A.E)

Solution:

(i) **Given**

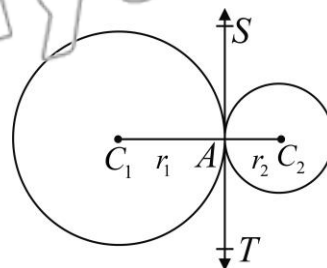
Two circles with centres C_1 and C_2 , radii of measure r_1 and r_2 such that $m\overline{C_1C_2} = r_1 + r_2$

To prove

Circles touch each other externally

Construction

Draw TS tangent to the circle with centre C_1 at A



Proof

Statements	Reasons
TA is tangent to circle with centre C_1 $\therefore m\angle C_1AT = 90^\circ$ $m\angle C_2AT = 180^\circ - 90^\circ$ $m\angle C_2AT = 90^\circ$ TA is perpendicular to radial segment AC_2 So TA is tangent to the circle with centre C_2 $\therefore TA$ is common tangent at A Hence circles touch each other externally	$\because C_1AC_2$ is a straight line

(ii) **Given**

(A.B)

Two circles with centres C_1 and C_2 , radii r_1 and r_2 such that $m\overline{C_1C_2} = r_1 - r_2$

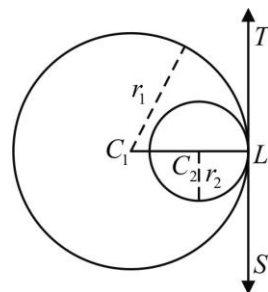
To prove

Circles touch each other internally

Construction

Produce $\overline{C_1C_2}$ to meet the circle with centre C_1 at L

Draw TS tangent to the circle with centre C_1 at L



Proof

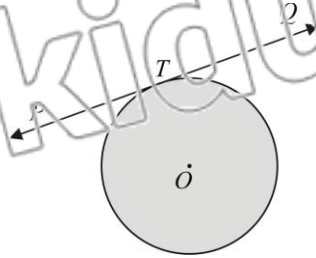
Statements	Reasons
As TL is tangent to the circle with centre C_1 $C_1L \perp TL$ $\therefore m\angle C_1LT = 90^\circ$ But C_1C_2L is a straight line $\therefore m\angle C_2LT = 90^\circ$ i.e. $C_2L \perp TL$ $\therefore TS$ is tangent to the circle with centre C_2 $\therefore TL$ is common tangent Hence circles touch each other internally	Construction Tangent $\perp$ radial segment

Miscellaneous Exerciser 10

Q.1 Multiple choice questions

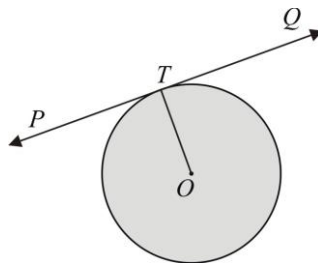
Four possible answers are given for the following question. Tick (✓) the correct answer.

- (i) In the adjacent figure of the circle, the line $\overleftrightarrow{PTQ}$ is named as **(K.B)**



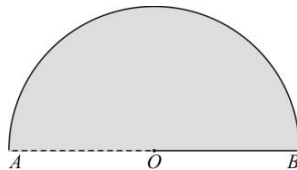
- (a) An arc
(b) A chord
(c) A tangent
(d) A secant

- (ii) In a circle with centre O , if $\overline{OT}$ is the radial segment and $\overleftrightarrow{PTQ}$ is the segment line, then **(K.B)**



- (a) $\overline{OT} \perp \overleftrightarrow{PQ}$
(b) $\overline{OT} \not\perp \overleftrightarrow{PQ}$
(c) $\overline{OT} \parallel \overleftrightarrow{PQ}$
(d) $\overline{OT}$ is right bisector of $\overleftrightarrow{PQ}$

- (iii) In the adjacent figure, find semicircular area if $\pi \approx 3.1416$ and $m\overline{OA} = 20\text{cm}$. **(K.B)**
(GRW 2014)



- (a) 62.83sq cm
(b) 314.16sq cm
(c) 436.20sq cm
(d) 628.32sq cm

- (iv) In the adjacent figure, find half the perimeter of circle with centre O if $\pi \approx 3.1416$ and $m\overline{OA} = 20\text{cm}$. **(K.B)**
(RWT 2015)

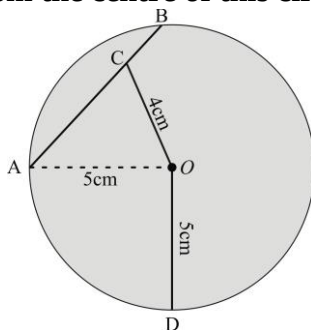


- (a) 31.42cm
(b) 62.832cm
(c) 125.65cm
(d) 188.50cm

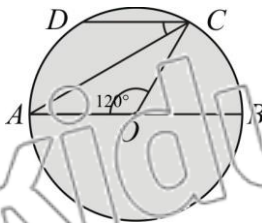
- (v) A line which has two points in common with a circle is called: **(K.B)**

- (a) Sine of a circle
(b) Cosine of a circle
(c) Tangent of a circle
(d) Secant of a circle

- (vi) A line which has only one point in common with a circle is called: **(K.B)** (D.G.K 2014)
 (a) Sine of a circle (b) Cosine of circle
 (c) Tangent of a circle (d) Secant of a circle
- (vii) Two tangents drawn to a circle from a point outside it are of ... in length. **(K.B)**
 (a) Half (b) Equal
 (c) Double (d) Triple
- (viii) A circle has only one: **(K.B)**
 (a) Secant (b) Chord
 (c) Diameter (d) Centre
- (ix) A tangent line intersects the circle at: **(K.B)**
 (a) Three points (b) Two points
 (c) Single point (d) No point at all
- (x) Tangents drawn at the ends of diameter of a circle are ... to each other. **(K.B)**
 (LHR 2015)
 (a) Parallel (b) Non-parallel
 (c) Collinear (d) Perpendicular
- (xi) The distance between the centres of two congruent touching circles externally is: **(K.B)**
 (a) Of zero length (b) The Radius of each circle
 (c) The diameter of each circle (d) Twice the diameter of each circle
- (xii) In the adjacent circular figure with centre O and radius 5cm , the length of the chord intercepted at 4cm away from the centre of this circle is: **(K.B)**



- (a) 4cm (b) 6cm
 (c) 7cm (d) 9cm
- (xiii) In the adjoining figure, there is a circle with centre O . If $\overline{DC} \parallel \text{diameter } \overline{AB}$ and $m\angle AOC = 120^\circ$, then $m\angle ACD$ is: **(K.B)**



- (a) 40° (b) 30°
 (c) 50° (d) 60°

ANSWER KEY

i	c	iv	b	vii	b	x	a	xiii	b
ii	a	v	d	viii	d	xi	c		
iii	d	vi	c	ix	c	xii	b		

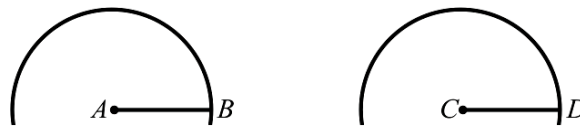
UNIT 11

CHORDS AND ARCS

Congruent Circles

(K.B)

Circles having different centres but same radii are called congruent circles.



In the given figure, circles with centre A and C are congruent, if $m\overline{AB} = m\overline{CD}$.

Congruent Arcs

(K.B)

Two arcs of same circle or of different circles are congruent, if their central angles are congruent.



In the given figure, arcs BC and QR are congruent, if $\angle CAB \cong \angle RPQ$.

Theorem 1

(A.B)

11.1(i)

Statement:

If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.

Given:

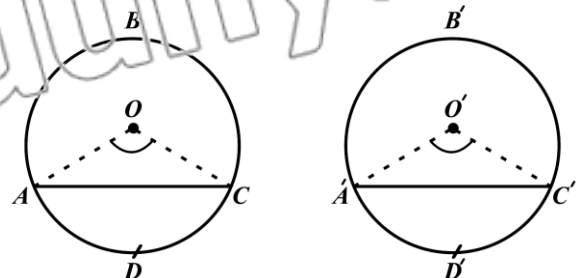
$ABCD$ and $A'B'C'D'$ are two congruent circles. With centres O and O' respectively. So that $m\overline{ADC} = m\overline{A'D'C'}$

To Prove:

$$m\overline{AC} = m\overline{A'C'}$$

Construction:

Join O with A , O with C , O' with A' and O' with C' . So that we can form $\triangle OAC$ and $\triangle O'A'C'$



Proof:

Statements	Reasons
In two equal circles $ABCD$ and $A'B'C'D'$ with centres O and O' respectively.	Given
$mADC = mA'D'C'$	Given
$\therefore m\angle AOC = m\angle A'O'C'$	Central angles subtended by equal arcs of the equal circles.
Now in $\triangle AOC \leftrightarrow \triangle A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\angle AOC = m\angle A'O'C'$	Already Proved
$m\overline{OC} = m\overline{O'C'}$	Radii of equal circles
$\therefore \triangle AOC \cong \triangle A'O'C'$	$S.A.S \cong S.A.S$
and in particular $m\overline{AC} = m\overline{A'C'}$	
Similarly we can prove the theorem in the same circle.	

Theorem 2**(A.B)****11.1(ii)****Statement:**

If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent. In equal circles or in the same circle, if two chords are equal, they cut off equal arcs.

Given:

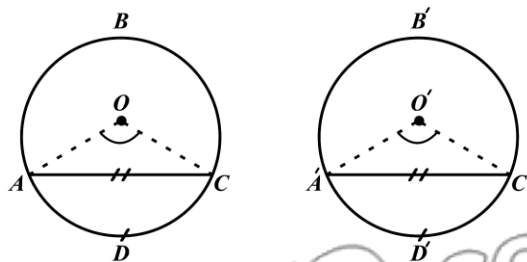
$ABCD$ and $A'B'C'D'$ are two congruent circles with centres O and O' respectively. So that $m\overline{AC} = m\overline{A'C'}$.

To Prove:

$$mADC = mA'D'C'$$

Construction:

Join O with A , O with C , O' with A' and O' with C' .

**Proof:**

Statements	Reasons
In $\triangle AOC \leftrightarrow \triangle A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\overline{OC} = m\overline{O'C'}$	Radii of equal circles
$m\overline{AC} = m\overline{A'C'}$	Given
$\triangle AOC \cong \triangle A'O'C'$	$S.S.S \cong S.S.S$
$\Rightarrow m\angle AOC = m\angle A'O'C'$	Corresponding angles of congruent triangles
Hence $mADC = mA'D'C'$	Arcs corresponding to equal chords in a circle.

Theorem 3**(A.B)****11.1(iii)****Statement:**

Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).

Given:

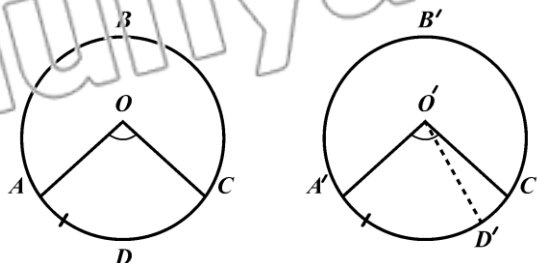
ABC and $A'B'C'$ are two congruent circles with centres O and O' respectively. So that $\overline{AC} = \overline{A'C'}$.

To Prove:

$$\angle AOC \cong \angle A'O'C'$$

Construction:

Let if possible $m\angle AOC \neq m\angle A'O'C'$ then consider $\angle AOC \cong \angle A'O'D'$

**Proof:**

Statements	Reasons
$\angle AOC \cong \angle A'O'D'$	Construction
$\therefore AC \cong A'D' \rightarrow (i)$	Arcs subtended by equal Central angles in congruent circles
$\overline{AC} \cong \overline{A'D'} \rightarrow (ii)$	Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
But $\overline{AC} \cong \overline{A'C'} \rightarrow (iii)$	Given
$\therefore \overline{A'C'} = \overline{A'D'}$	Using (i) and (ii)
Which is only possible, if C' coincides with D' .	
Hence $m\angle A'O'C' = m\angle A'O'D' \rightarrow (iv)$	
But $m\angle AOC = m\angle A'O'D' \rightarrow (v)$	Construction
$\Rightarrow m\angle AOC = m\angle A'O'C'$	Using (iv) and (v)

Corollary 1**(K.B)**

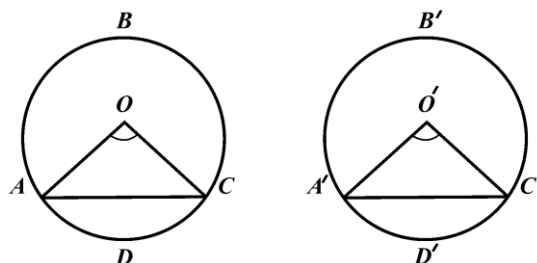
In congruent circles or in the same circle, if central angles are equal then corresponding sectors are equal.

Corollary 2**(K.B)**

In congruent circles or in the same circle, unequal arcs will subtend unequal central angles.

Theorem 4**(A.B)****11.1(iv)****Statement:**

If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.



Given:

$ABCD$ and $A'B'C'D'$ are two congruent circles with centre O and O' respectively. $\overline{AC}$ and $\overline{A'C'}$ are chords of circles $ABCD$ and $A'B'C'D'$ respectively and $m\angle AOC = m\angle A'O'C'$

To Prove:

$$m\overline{AC} = m\overline{A'C'}$$

Proof:

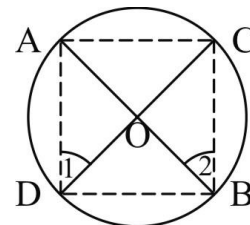
Statements	Reasons
In $\triangle OAC \leftrightarrow \triangle O'A'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of congruent circles
$m\angle AOC = m\angle A'O'C'$	Given
$m\overline{OC} = m\overline{O'C'}$	Radii of congruent circles
$\therefore \triangle OAC \leftrightarrow \triangle O'A'C'$	$SAS \cong SAS$
Hence $m\overline{AC} = m\overline{A'C'}$	Corresponding sides of congruent triangles

Exercise 11.1

Q.1 In a circle two equal diameters $\overline{AB}$ and $\overline{CD}$ intersect each other. Prove that $m\overline{AD} = m\overline{BC}$. **(A.B)**

Given

In a circle with centre ' O ', two chords $\overline{AB}$ and $\overline{CD}$ intersect each other, such that $\overline{AB} \cong \overline{CD}$.

**To prove**

$$m\overline{AD} = m\overline{BC}$$

Proof

Statement	Reasons
In $\triangle CBD$ and BCA	
$\overline{BC} \cong \overline{BC} \rightarrow (i)$	Common
$\overline{AB} \cong \overline{CD} \rightarrow (ii)$	Given
$\angle 2 \cong \angle 1 \rightarrow (iii)$	Circum angles of same segment
$\angle CBD \cong \angle BCA \rightarrow (iv)$	Opposite angles in the same segment
$m\angle CBD - m\angle 2 = m\angle BCA - m\angle 1$	From (iii) and (iv)
$\therefore m\angle CBA = m\angle BCD \rightarrow (v)$	
From (i), (ii) and (v)	
$\triangle CBD \cong \triangle BCA$	S.A.S postulate
$\overline{AC} \cong \overline{BD}$	Corresponding sides of $\cong$ triangles

Q.2 In a circle prove that the arcs between two parallel and equal chords are equal.

(A.B)

Given

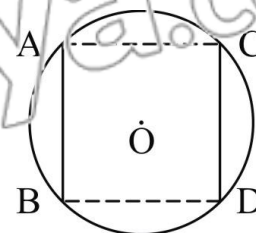
In a circle with centre 'O', $\overline{AB}$ and $\overline{CD}$ are two chords, such that $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$

To prove

$$m\overline{AC} = m\overline{BD}$$

Construction

Join A to C and B to D.



Proof

Statements	Reasons
$m\overline{AB} = m\overline{CD}$ Also $\overline{AB} \parallel \overline{CD}$	Given
$\therefore ABCD$ is a $\parallel^{gm}$	{ A quadrilateral having two sides parallel and congruent is a parallelogram.
$m\overline{AC} = m\overline{BD}$	Opposite sides of a parallelogram
$\therefore m\overline{AC} = m\overline{BD}$	Th-11.2 (chords area equal)

Q.3 Give a geometric proof that a pair of bisecting chords are the diameters of a circle.

(A.B)

Given

In a circle with centre 'O', two chords $\overline{AB}$ and $\overline{CD}$ bisect each other at point 'P'.

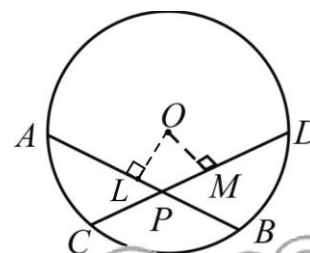
$$\text{i.e } m\overline{PC} = m\overline{PD} \text{ and } m\overline{PB} = m\overline{PA}.$$

To prove

Chords $\overline{AB}$ and $\overline{CD}$ pass through point 'O'.

Construction

Draw $\overline{OM} \perp \overline{CD}$ and $\overline{OL} \perp \overline{AB}$



Proof

Statements	Reasons
L is mid point of $\overline{AB}$	$\overline{OL} \perp \overline{AB}$
P is mid point of $\overline{AB}$	Given
$\therefore P$ and L coincide	
M is the mid point $\overline{CD}$	$\overline{OM} \perp \overline{CD}$
P is mid point of $\overline{CD}$	Given
$\therefore M$ and P coincide	
So L, M and P coincide which is possible only if L, M and P coincide with O	
$\therefore \overline{AB}$ and $\overline{CD}$ pass through O	
Hence $\overline{AB}$ and $\overline{CD}$ are diameters	

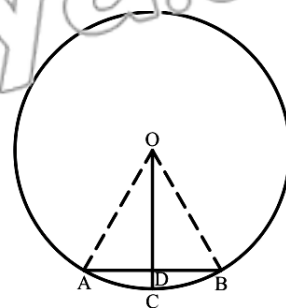
Q.4 If C is the mid point of an arc ACB in a circle with centre O . Show that line segment $\overline{OC}$ bisects the chord AB . **(A.E)**

Given

In a circle with centre ' O ', C is mid point of ACB , $\overline{OC}$ intersect $\overline{AB}$ at point D

To prove

$$\overline{AD} \cong \overline{BD}$$



Construction

Join O to A and B

Proof

Statements	Reasons
C is mid point of ACB	Given
Then $AC \cong BC$	
$\therefore \angle AOC \cong \angle BOC$	{ Central angles of congruent arcs.
Or $\angle AOD \cong \angle BOD$	
In $\triangle OAD \leftrightarrow \triangle OBD$	
$\overline{OA} \cong \overline{OB}$	Radii of same circle
$\angle AOD \cong \angle BOD$	Proved
$\overline{OD} \cong \overline{OD}$	Common
$\therefore \triangle OAD \cong \triangle OBD$	S.A.S postulate
$\overline{AD} \cong \overline{BD}$	Corresponding sides of congruent triangles

Miscellaneous Exercise 11

Q.1 Multiple choice questions

Four possible answers are given for the following question. Tick (✓) the correct answer.

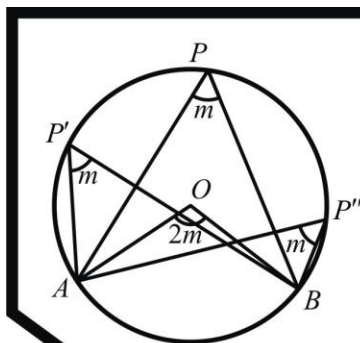
- (1) A 4cm long chord subtends a central angle of 60° . The radial segment of this circle is: (SWL 2015) **(K.B)**
 (a) 1 (b) 2
 (c) 3 (d) 4
- (2) The length of a chord and the radial segment of a circle are congruent, the central angle made by the chord will be: (FSD 2014, RWP 2015) **(K.B)**
 (a) 30° (b) 45°
 (c) 60° (d) 75°
- (3) Out of two congruent arcs of a circle, if one arc makes a central angle of 30° then the other arc will subtend the central angle of: (LHR 2015, SWL 2014) **(K.B)**
 (a) 15° (b) 30°
 (c) 45° (d) 60°
- (4) An arc subtends a central angle of 40° then the corresponding chord will subtend a central angle of: (LHR 2015, GRW 2014, FSD 2014, D.G.K 2014, 15) **(K.B)**
 (a) 20° (b) 40°
 (c) 60° (d) 80°
- (5) A pair of chords of a circle subtending two congruent central angles is: **(K.B)**
 (a) Congruent (b) Incongruent
 (c) Overlapping (d) Parallel
- (6) If an arc of circle subtend a central angle of 60° , then the corresponding chord of the arc will make the central angle of: **(K.B)**
 (a) 20° (b) 40°
 (c) 60° (d) 80°
- (7) The semi circumference and the diameter of a circle both subtend a central angle of: **(K.B)**
 (a) 90° (b) 180°
 (c) 270° (d) 360°
- (8) The chord length of a circle subtending a central angle of 180° is always: **(K.B)**
 (a) Less than radial segment (b) Equal to the radial segment (D.G.K 2014)
 (c) Double of the radial segment (d) None of these
- (9) If a chord of a circle subtend a central angle of 60° , then the length of the chord and the radial segment are: (MTN 2015) **(K.B)**
 (a) Congruent (b) Incongruent
 (c) Parallel (d) Perpendicular
- (10) The arcs opposite to incongruent central angles of a circle arc always: **(K.B)**
 (a) Congruent (b) Incongruent (D.G.K 2014)
 (c) Parallel (d) Perpendicular

ANSWER KEY

1	2	3	4	5	6	7	8	9	10
d	c	b	b	a	c	b	c	a	b

UNIT 12

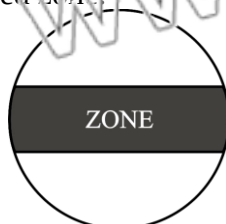
ANGLE IN A SEGMENT OF A CIRCLE



Zone

(K.B)

Circular region bounded by two parallel chords is called zone.



Cyclic Quadrilateral

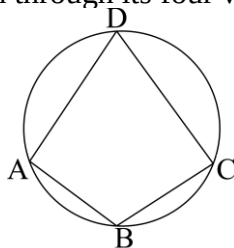
(K.B)

(LHR 2014, 15, GRW 2014, 17, BWP 2016, 17, RWP 2016, MTN 2016, 17, D.G.K 2017)

A quadrilateral whose vertices lie on the circumference of a circle is called cyclic quadrilateral.

Or

A quadrilateral is called cyclic when a circle can be drawn through its four vertices.



Circum Angle

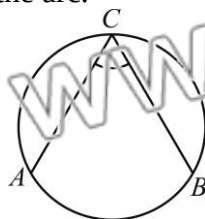
(K.B)

(LHR 2014, 15, GRW 2017, BWP 2016, 17, RWP 2016, MTN 2016, 17, D.G.K 2017)

The angle subtended by an arc of a circle at its circumference is called circum angle.

Or

An angle whose vertex lies on the circumference and whose arms pass through end points of the arc.



In the figure, $\angle ACB$ is the circum angle.

Central Angle

(K.B)

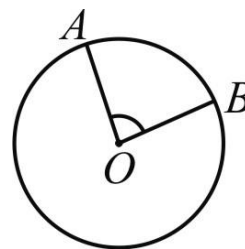
(LHR 2016, GRW 2016, FSD 2015, 16, 17, RWP 2016, 17, SWL 2017, SGD 2016, 17, MTN 2017, D.G.K 2016)

The angle subtended by an arc of a circle at its centre is called circum angle.

Or

“An angle formed by two radial segments at the centre of the circle is called central angle”.

In the given figure, $\angle AOB$ is central angle of AB .



Note

(K.B + U.B)

- (i) The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.
- (ii) Any two angles in the same segment of a circle are equal.
- (iii) The angle
 - in a semi-circle is a right angle,
 - in a segment greater than a semi circle is less than a right angle,
 - in a segment less than a semi-circle is greater than a right angle.
- (iv) The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Theorem 1**(A.B)**

12.1(i)

Statement:

The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.

Given:

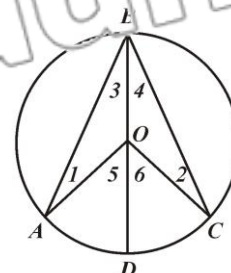
AC is an arc of a circle with centre O . Where as $\angle AOC$ is the central angle and $\angle AEC$ is circum angle.

To Prove:

$$m\angle AOC = 2m\angle ABC$$

Construction:

Join B with O and produce it to meet the circle at D . Write angles $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5$ and $\angle 6$ as shown in the figure.

**Proof:**

Statements	Reasons
As $m\angle 1 = m\angle 3 \rightarrow (i)$	Angles opposite to equal sides in $\triangle OAB$
And $m\angle 2 = m\angle 4 \rightarrow (ii)$	Angles opposite to equal sides in $\triangle OBC$
Now $m\angle 5 = m\angle 1 + m\angle 3 \rightarrow (iii)$	External angle is the sum of internal opposite angles.
Similarly $m\angle 6 = m\angle 2 + m\angle 4 \rightarrow (iv)$	
Again $m\angle 5 = m\angle 3 + m\angle 3 = 2m\angle 3 \rightarrow (v)$	Using (i) and (iii)
And $m\angle 6 = m\angle 4 + m\angle 4 = 2m\angle 4 \rightarrow (vi)$	Using (ii) and (iv)
Then from figure $\Rightarrow m\angle 5 + m\angle 6 = 2m\angle 3 + 2m\angle 4$ $\Rightarrow m\angle AOC = 2(m\angle 3 + m\angle 4) = 2m\angle ABC$	Adding (v) and (vi)

Theorem 2

(GRW 2016, SWL 2016, RWP 2016, MTN 2017, SGD 2017, BWP 2015)

(A.B)

12.1(ii)

Statement:

Any two angles in the same segment of a circle are equal.

Given:

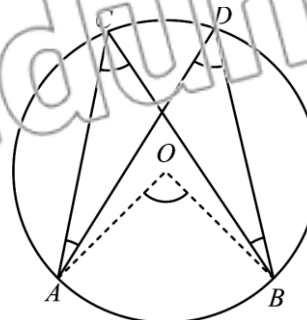
$\angle ACB$ and $\angle ADB$ are the circum angles in the same segment of a circle with centre O .

To Prove:

$$m\angle ACB = m\angle ADB$$

Construction:

Join O with A and O with B . So that $m\angle AOB$ is the central angle.



Proof:

Statements	Reasons
Standing on the same arc AB of a circle. $\angle AOB$ is the central angle whereas $\angle ACB$ and $\angle ADB$ are circum angles $\therefore m\angle AOB = 2m\angle ACB \rightarrow (i)$ and $m\angle AOB = 2m\angle ADB \rightarrow (ii)$ $\Rightarrow 2m\angle ACB = 2m\angle ADB$ Hence $m\angle ACB = m\angle ADB$	Construction Given The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc. Same as above Using (i) and (ii)

Theorem 3

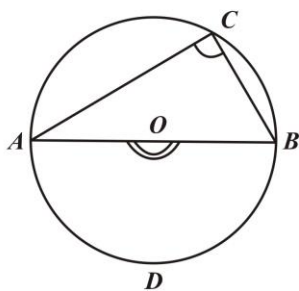
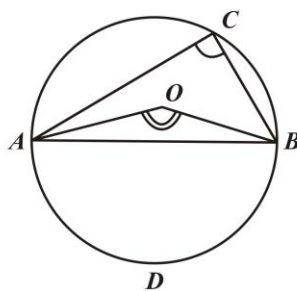
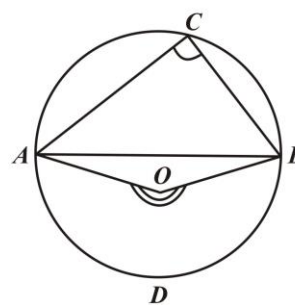
(LHR 2016, MTN 2015)

(A.B)

12.1(iii)

Statement:**The angle**

- In a semi-circle is a right angle,
- In a segment greater than a semi circle is less than a right angle,
- In a segment less than a semi-circle is greater than a right angle.

**Fig I****Fig II****Fig III****Given:**

$\overline{AB}$ is the chord corresponding to an arc ADB . Whereas $\angle AOB$ is a central angle and $\angle ACB$ is a circum angle of a circle with centre O .

To Prove:

- In fig (I) if sector ACB is a semi circle then $m\angle ACB = 1\angle rt$
In fig (II) if sector ACB is greater than a semi circle then $m\angle ACB < 1\angle rt$
In fig (III) if sector ACB is less than a semi circle then $m\angle ACB > 1\angle rt$

Proof:

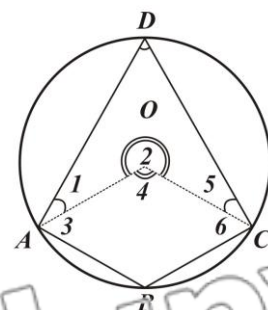
Statements	Reasons
In each figure, $\overline{AB}$ is the chord of a circle with centre O . $\angle AOB$ is the central angle standing on an arc ADB . Whereas $\angle ACB$ is the circum angle	Given
Such that $m\angle AOB = 2m\angle ACB \rightarrow (i)$	The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.
Now in fig (I) $m\angle AOB = 180^\circ$ $\therefore m\angle AOB = 2\angle rt \rightarrow (ii)$ $\Rightarrow m\angle ACB = 1\angle rt$	A straight angle
In fig (II) $m\angle AOB < 180^\circ$ $\therefore m\angle AOB = 2\angle rt \rightarrow (iii)$ $\Rightarrow m\angle ACB < 1\angle rt$	Using (i) and (ii)
In fig (III) $m\angle AOB > 180^\circ$ $\therefore m\angle AOB > 2\angle rt \rightarrow (iv)$ $\Rightarrow 2m\angle ACB > 2\angle rt$ $\Rightarrow m\angle ACB > 1\angle rt$	Using (i) and (iii)
	Using (i) and (iv)

Theorem 4**(A.B)****12.1(iv)****Statement:**

The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Given: $ABCD$ is a quadrilateral inscribed in a circle with centre O .**To Prove:**

$$\begin{cases} m\angle A + m\angle C = 2\angle rts \\ m\angle B + m\angle D = 2\angle rts \end{cases}$$

Construction:Draw $\overline{OA}$ and $\overline{OC}$. Write $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5$ and $\angle 6$ as shown in the figure.**Proof:**

Statements	Reasons
Standing on the same arc ADC , $\angle 2$ is a central angle Whereas $\angle B$ is the circum angle	Arc ADC of the circle with centre O .
$\therefore m\angle B = \frac{1}{2}(m\angle 2) \rightarrow (i)$	The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.
Standing on the same arc ABC , $\angle 4$ is a central angle whereas $\angle D$ is the circum angle	Arc ABC of the circle with centre O .

$\therefore m\angle D = \frac{1}{2}(m\angle 4) \rightarrow (ii)$ $\Rightarrow m\angle B + m\angle D = \frac{1}{2}m\angle 2 + \frac{1}{2}m\angle 4$ $= \frac{1}{2}(m\angle 2 + m\angle 4) = \frac{1}{2}(\text{Total central angle})$ $\text{i.e., } m\angle B + m\angle D = \frac{1}{2}(4\angle rt) = 2\angle rt$ Similarly $m\angle A + m\angle C = 2\angle rt$	The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc. Adding (i) and (ii)
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Exercise 12.1

Q.1 Prove that in a given cyclic quadrilateral, sum of opposite angles is two right angles and conversely. **(A.B)**

Given:

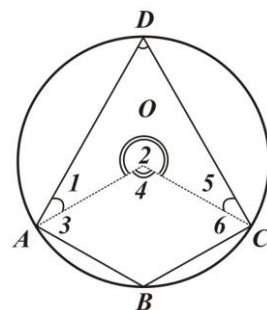
$ABCD$ is a quadrilateral inscribed in a circle with centre O .

To Prove:

$$\begin{cases} m\angle A + m\angle C = 2\angle rt \\ m\angle B + m\angle D = 2\angle rt \end{cases}$$

Construction:

Draw $\overline{OA}$ and $\overline{OC}$. Write $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5$ and $\angle 6$ as shown in the figure.



Proof:

Statements	Reasons
Standing on the same arc ADC , $\angle 2$ is a central angle Whereas $\angle B$ is the circum angle	Arc ADC of the circle with centre O .
$\therefore m\angle B = \frac{1}{2}(m\angle 2) \rightarrow (i)$	The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.
Standing on the same arc ABC , $\angle 4$ is a central angle whereas $\angle D$ is the circum angle	Arc ABC of the circle with centre O .
$\therefore m\angle D = \frac{1}{2}(m\angle 4) \rightarrow (ii)$	The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.
$\Rightarrow m\angle B + m\angle D = \frac{1}{2}m\angle 2 + \frac{1}{2}m\angle 4$	Adding (i) and (ii)
$= \frac{1}{2}(m\angle 2 + m\angle 4) = \frac{1}{2}(\text{Total central angle})$	
i.e., $m\angle B + m\angle D = \frac{1}{2}(4\angle rt) = 2\angle rt$	
Similarly $m\angle A + m\angle C = 2\angle rt$	

Q.2 Show that parallelogram inscribed in a circle will be a rectangle. **(A.B)**

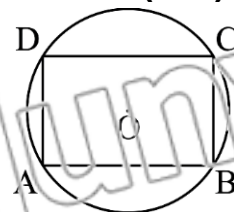
Given

$ABCD$ is a parallelogram inscribed in a circle with centre 'O'.

To prove

$ABCD$ is a rectangle.

Proof



Statements	Reasons
$\angle A \cong \angle C \rightarrow (i)$	Opposite angles of a parallelogram
$m\angle A + m\angle C = 180^\circ \rightarrow (ii)$	Th-12.4 (opposite angles are supplementary)
$2m\angle A = 180^\circ$	From (i) and (ii)
$m\angle A = 90^\circ \rightarrow (iii)$	Div both sides by 2
$m\angle C = 90^\circ \rightarrow (iv)$	
Similarly	
$m\angle B = m\angle D = 90^\circ \rightarrow (v)$	
Hence $ABCD$ is a rectangle.	From (iii), (iv) and (v)

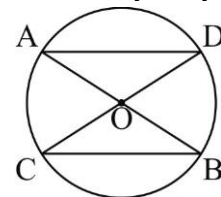
Q.3 AOB and COD are two intersecting chords of a circle. Show that $\triangle AOD$ and BOC are equiangular. **(A.B)**

Given

In a circle with centre 'O', AOB and COD are two intersecting chords.
 AD and BC are joined

To prove

Triangles AOD and BOC are equiangular



Proof

Statement	Reasons
In $\triangle AOD \leftrightarrow \triangle BOC$	
$\angle D \cong \angle B$	Th-12.2 Circum angles
Similarly	Subtended by arc AC .
$\angle A \cong \angle C$	Circum angles subtended by arc BD .
$\angle AOD \cong \angle BOC$	Vertical angle
$\therefore \triangle AOD \cong \triangle BOC$	A.A.A postulate
$\therefore \triangle AOD$ and BOC are equiangular.	

Q.4 $\overline{AD}$ and $\overline{BC}$ are two parallel chords of a circle. Prove that arc $\overline{AB} \cong$ arc $\overline{CD}$ and arc $\overline{AC} \cong$ arc $\overline{BD}$ **(A.B)**

Given:

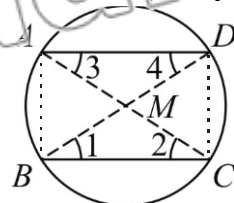
In a circle with centre 'M' chord $\overline{AD} \parallel$ chord $\overline{BC}$

To Prove:

$\overline{AC} \cong \overline{BD}$

Construction:

Join A to C and B to D . $\overline{AC}$ and $\overline{BD}$ intersect at point M . Name of the angles as shown in the figure.



Proof:

Statement	Reasons
$\angle 1 \cong \angle 4 \rightarrow (i)$	Alternate angles
$\angle 2 \cong \angle 3 \rightarrow (ii)$	Alternate angles
$\angle 1 \cong \angle 3 \rightarrow (iii)$	Angles in the same segment
$\angle 3 = \angle 4$	From (i) and (iii)
In $\triangle AMD$, $\overline{AM} \cong \overline{DM} \rightarrow (iv)$	Opposite sides of $\cong$ angles
Similarly $\overline{MC} \cong \overline{BM} \rightarrow (v)$	
$m\overline{AM} + m\overline{CM} = m\overline{DM} + m\overline{BM}$	Adding (iv) & (v)
$\overline{AC} \cong \overline{BD}$	
In $\triangle BCA \leftrightarrow \triangle BCD$	
$\overline{BC} \cong \overline{BC}$	Common
$\angle 1 \cong \angle 2$	From (i) and (ii)
$\overline{AC} \cong \overline{BD}$	Already proved
$\therefore \triangle BCD \cong \triangle BCA$	SAS postulate
$\overline{AB} \cong \overline{CD}$	Corresponding sides of $\cong \Delta^s$.
Hence $AB \cong CD$	

Miscellaneous Exercise 12

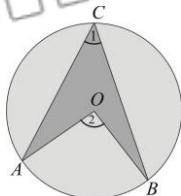
Q.1 Multiple choice questions

Four possible answers are given for the following question. Tick (✓) the correct answer.

- (i) A circle passes through the vertices of a right angled $\triangle ABC$ with $m\overline{AC} = 3\text{cm}$ and $m\overline{BC} = 4\text{cm}$, $m\angle C = 90^\circ$. Radius of the circle is: **(K.B + U.B)**

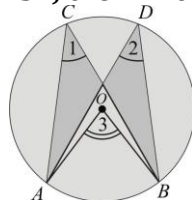
- (a) 1.5cm (b) 2.0cm
(c) 2.5cm (d) 3.5cm

- (ii) In the adjacent circular figure, central and inscribed angles stand on the same arc AB. Then **(FSD 2017) (K.B + U.B)**



- (a) $m\angle 1 = m\angle 2$ (b) $m\angle 1 = 2m\angle 2$
(c) $m\angle 2 = 3m\angle 1$ (d) $m\angle 2 = 2m\angle 1$

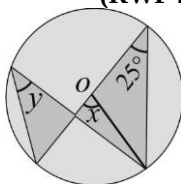
- (iii) In the adjacent figure if $m\angle 3 = 75^\circ$, then find $m\angle 1$ and $m\angle 2$. **(GWR 2014) (K.B + U.B)**



- (a) $37\frac{1}{2}^\circ, 37\frac{1}{2}^\circ$ (b) $37\frac{1}{2}^\circ, 75^\circ$
(c) $75^\circ, 37\frac{1}{2}^\circ$ (d) $75^\circ, 75^\circ$

- (iv) Given that O is the centre of the circle. Then angle marked x will be:

(RWP 2014, SGD 2014, D.G.K 2015) (K.B + U.B)



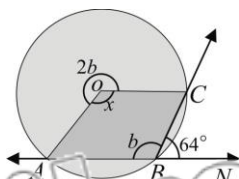
- (a) $12\frac{1}{2}^\circ$ (b) 25°
(c) 50° (d) 75°

- (v) Given that O is the centre of the circle. Then angle marked y will be: **(K.B + U.B)**

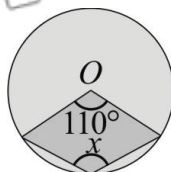


- (a) $12\frac{1}{2}^\circ$ (b) 25°
(c) 50° (d) 75°

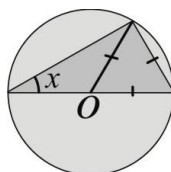
- (vi) In the figure, O is the centre of the circle and $\overline{ABN}$ is a straight line. The obtuse angle $AOC = x$ is: **(K.B + U.B)**



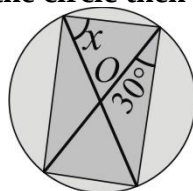
- (a) 32° (b) 64°
(c) 96° (d) 128°
- (vii) In the figure, O is the centre of the circle, then the angle x is: **(K.B + U.B)**
(SWL 2014, SGD 2014, D.G.K 2015)



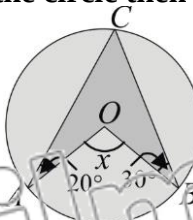
- (a) 55° (b) 110°
(c) 220° (d) 125°
- (viii) In the figure, O is the centre of the circle then angle x is: **(K.B + U.B)**
(GWR 2014, D.G.K 2014)



- (a) 15° (b) 30°
(c) 45° (d) 60°
- (ix) In the figure, O is the centre of the circle then angle x is: **(K.B + U.B)**



- (a) 15° (b) 30°
(c) 45° (d) 60°
- (x) In the figure, O is the centre of the circle then angle x is: (RWP 2015) **(K.B + U.B)**



- (a) 50° (b) 75°
(c) 100° (d) 125°

ANSWER KEY

i	ii	iii	iv	v	vi	vii	viii	ix	x
c	d	a	c	b	d	d	b	d	c



CUT HERE

SELF TEST

Time: 40 min

Marks: 25

Q.1 Four possible answers (A), (B), (C) & (D) to each question are given, mark the correct answer. (7×1=7)

1 A circle passes through the vertices of a right angled triangle ABC with $m\overline{AC} = 3\text{cm}$, $m\overline{BC} = 4\text{cm}$, $m\angle C = 90^\circ$. Radius of the circle is:

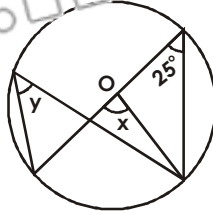
(A) 1.5cm

(A) 2.0cm

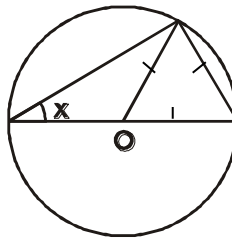
(C) 2.5cm

(D) 3.5cm

2 Given that O is the centre of the circle. The angle marked x will be:

(A) $12\frac{1}{2}^\circ$ (A) 25° (C) 50° (D) 75°

3 In the figure, O is the centre of the circle then angle x is:

(A) 15° (A) 30° (C) 45° (D) 60°

4 Inscribed angle of minor segment is:

(A) Acute

(A) Obtuse

(C) Right

(D) Complete

5 A 4 cm long chord subtends a central angle of 60° . The radial segment of this circle is:

(A) 1

(A) 2

(C) 3

(D) 4

6 Opposite angles in cyclic quadrilateral are:

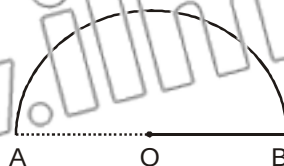
(A) Complementary

(A) Supplementary

(C) Equal

(D) Alternate

7 In the adjacent figure find sectoricular area if $\pi = 3.1416$ and $m\overline{OA} = 20\text{cm}$:



(A) 62.83 sq cm

(A) 314.16 sq cm

(C) 436.20 sq cm

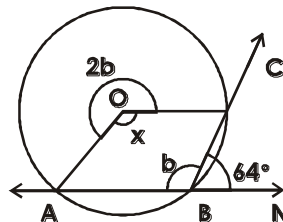
(D) 628.32 sq cm



CUT HERE

Give Short Answers to following Questions.**(5×2=10)**

- (i) Define cyclic quadrilateral.
- (ii) Define circum angle or inscribed angle.
- (iii) Define congruent arcs.
- (iv) Define length of tangent.
- (v) In the figure, O is the centre of the circle and $\overline{ABN}$ is a straight line. Then find obtuse angle $AOB = x$.

**Q.2 Prove that,****(8)**

Prove that the measure of a central angle of a minor arc of a circle, is doubled that of the angle subtended by the corresponding major arc.

NOTE: Parents or guardians can conduct this test in their supervision in order to check the skill of students.

UNIT 13

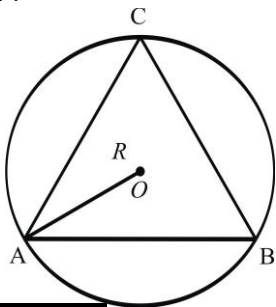
PRACTICAL GEOMETRY - CIRCLE

Geometry (JHR 2014) (K.B)

The word geometry is derived from two Greek words Geo (earth) and Matron (measurement). Therefore the word geometry means measurement of earth. It deals with the shape, size and position of geometrical figures.

Circumcircle (K.B)

A circle passing through the vertices of a triangle is called circumcircle. Its radius is circumradius and centre is circumcentre. Circumcentre is denoted by 'O' and circumradius by 'R'.



Inscribed Circle (K.B)

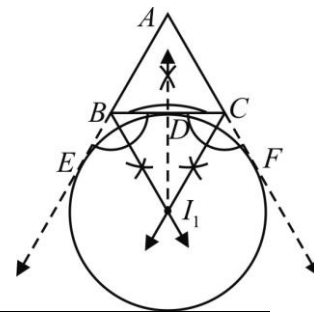
(FSD 2015, D.G.K 2015)

A circle which touches the three sides of a triangle internally is known as incircle. Its radius is in-radius and its centre is in-centre. In centre is denoted by I and in-radius by r.



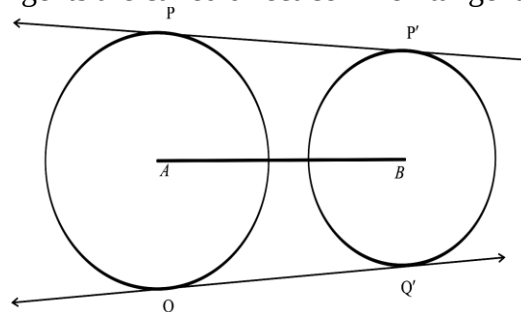
Escribed Circle (JHR 2015) (K.B)

A circle touching one side of a triangle externally and two produced sides internally is called escribed circle or e-circle. Its centre is represented by I_1, I_2, I_3 and radius by r_1, r_2 and r_3 and depending upon the vertex opposite to it.



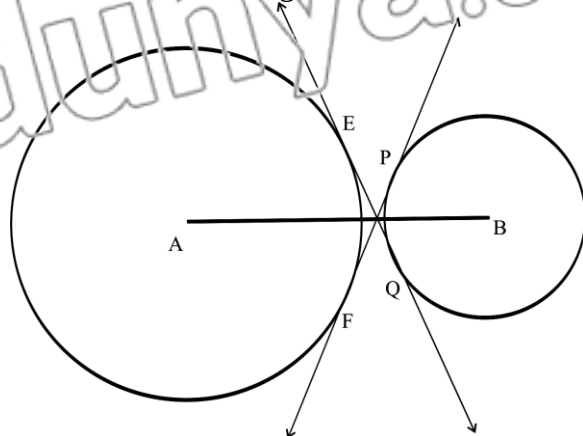
Direct Common Tangent (K.B)

If points of contact of common tangents to the two circles lie on the same side of line segment joining their centres then such tangents are called direct common tangents.



Transverse common tangents (K.B)

If the points of contact of common tangents to the two circles lie on opposite sides of line segment joining their centres, then such tangents are called transverse common tangents.



Note

Greeks Mathematicians contributed a lot, in particular “Euclid’s Elements” have been taught.

Construction of a Circle (K.B)

13.1(i) To locate the centre of a given circle

Given:

A circle

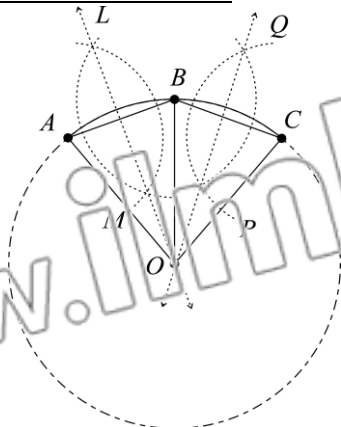
Steps of Construction

1. Draw two chords $\overline{AB}$ and $\overline{CD}$.
 2. Draw $\overline{EFG}$ as perpendicular bisector of chord $\overline{AB}$
 3. Draw $\overline{PQR}$ as perpendicular bisector of chord $\overline{CD}$.
- Perpendicular bisectors $\overline{EFG}$ and $\overline{PQR}$ intersect each other at O . O is the centre of circle.

13.1(ii) Draw a circle passing through three given non-collinear points:

Given:

Three non-collinear points A , B and C .

Steps of Construction

1. Join A with B and B with C .

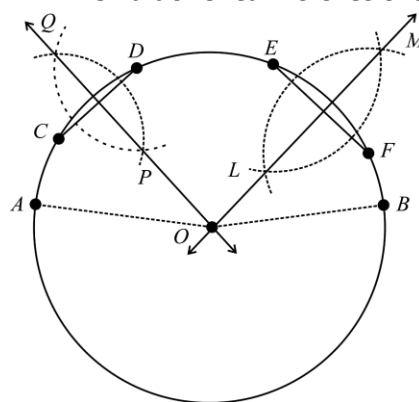
2. Draw $\overline{LM}$ and $\overline{PQ}$ right bisectors of $\overline{AB}$ and $\overline{BC}$ respectively. $\overline{LM}$ and $\overline{PQ}$ intersect at point O .
3. Draw a circle with radius $\overline{OA} = \overline{OB} = \overline{OC}$ having centre at O , which is the required circle.

Page # 223**(K.B)**

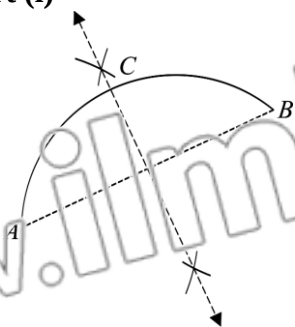
13.1(iii-a) To complete the circle by finding the centre when a part of a circumference is given

Given:

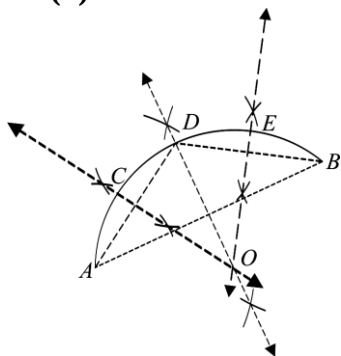
AB is Part of circumference of a circle

**Steps of Construction**

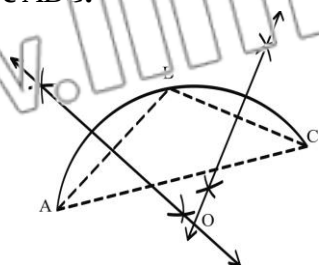
1. Let C , D , E and F be the four points on the given arc AB .
2. Draw chord $\overline{CD}$ and $\overline{EF}$.
3. Draw $\overline{PQ}$ as perpendicular bisector of $\overline{CD}$ and $\overline{LM}$ as perpendicular bisector of $\overline{EF}$.
4. $\overline{LM}$ and $\overline{PQ}$ intersect at O .
 $\therefore O$ is equidistant from points A , B , C , D , E and F .
 Complete the circle with centre O and radius $(\overline{OA} = \overline{OB} = \overline{OC} = \overline{OD} = \overline{OE} = \overline{OF})$.
 This will pass through all the points A , B , C , D , E and F on the given part of the circumference.

Exercise # 13.1**Q.1** Divide an arc of any length**(i) Part (i) (A.B)****Steps of Construction**

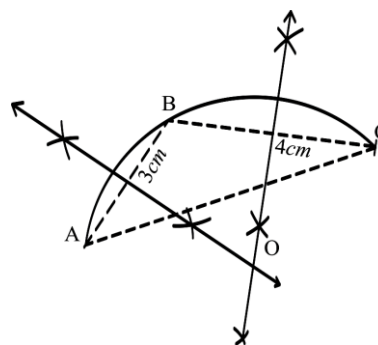
- (i) Join A to B
- (ii) Draw right bisector of $\overline{AB}$, which cuts AB at point C.

Thus $AC \cong BC$ **(ii) Part (ii) (A.B)****Steps of Construction**

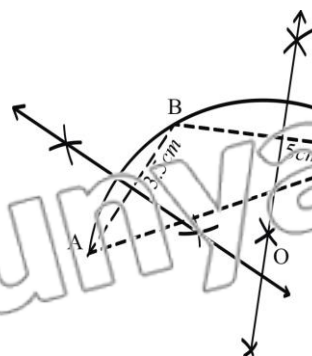
1. Join A to B
2. Draw right bisector of $\overline{AB}$, we get point D on AB.
3. Draw right bisectors of $\overline{AD}$ and $\overline{BD}$, passing through points C and E on AD and BD.

Thus $AC \cong CD \cong DE \cong EB$ **Q.2** Practically find the centre of an arc ABC. **(A.B)****Steps of Construction**

1. Mark two chords $\overline{AB}$ and $\overline{BC}$
2. Draw right bisectors of $\overline{AB}$ and $\overline{BC}$, they intersect each other at point O. Thus O is Centre of the arc.

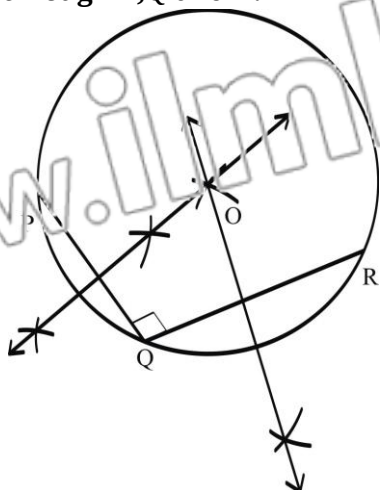
Q.3**(i) If $|\overline{AB}| = 3\text{cm}$ and $|\overline{BC}| = 4\text{cm}$ are the lengths of two chords of an arc, then locate the centre of the arc****(A.B)****Steps of Construction**

1. Draw two chords $\overline{AB} = 3\text{cm}$ and $\overline{BC} = 4\text{cm}$.
2. Draw right bisectors of $\overline{AB}$ and $\overline{BC}$.
3. Both cut each other at point 'O'. Thus O is centre of the arc.

(ii) If $|\overline{AB}| = 3.5\text{cm}$ and $|\overline{BC}| = 5\text{cm}$ are the lengths of two chords of an arc, then locate the centre of the arc.**(A.B)****Steps of Construction**

1. Draw two chords $\overline{AB} = 3.5\text{cm}$ and $\overline{BC} = 5\text{cm}$.
2. Draw right bisectors of $\overline{AB}$ and $\overline{BC}$.
3. Both cut each other at point 'O'. Thus O is centre of the arc.

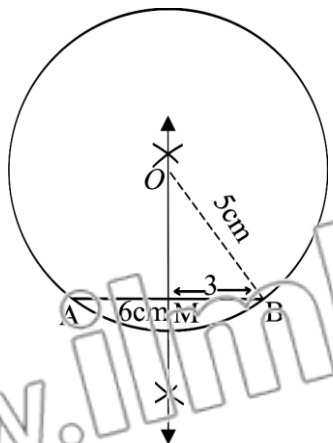
- Q.4** For an arc draw two perpendicular bisectors of the chords $\overline{PQ}$ and $\overline{QR}$ of this arc, construct a circle through P, Q and R . **(A.B)**



Steps of Construction

1. Draw two chords $\overline{PQ}$ and $\overline{QR}$, s.t. $\overline{PQ} \perp \overline{QR}$.
2. Draw right bisectors of $\overline{PQ}$ and $\overline{QR}$. They intersect each other at point O .
3. With centre O , draw a circle of radius $\overline{OP}$.

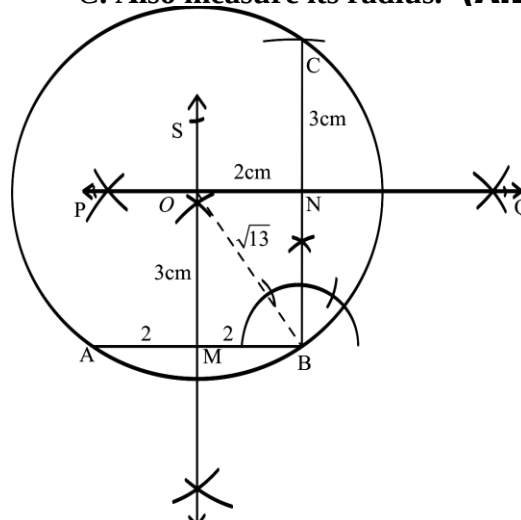
- Q.5** Describe a circle of radius 5cm passing through points A and B , 6cm apart. Also find distance from the centre to the line segment AB . **(A.B)**



Steps of Construction

1. Draw a circle of radius 5cm .
2. Take a point A on circle and draw an arc of radius 6cm to get point B .

3. Draw right bisector of $\overline{AB}$, passing through point ' M '.
 4. Measure O to M . It is 4cm .
- Q.6** If $\overline{AB} = 4\text{cm}$ and $\overline{BC} = 6\text{cm}$, such that $\overline{AB}$ is perpendicular to $\overline{BC}$ construct a circle through A, B and C . Also measure its radius. **(A.B)**

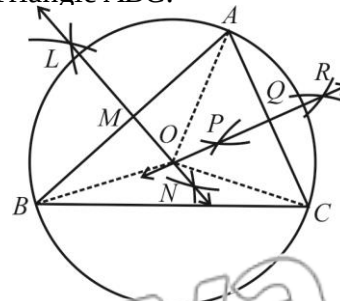


CIRCLES ATTACHED TO POLYGONS

- 13.2(i)** Circumscribe a circle about a given triangle. **(A.B)**

Given:

Triangle ABC .



Steps of Construction

1. Draw $\overline{LMN}$ as perpendicular bisector of side $\overline{AB}$.
2. Draw $\overline{PQR}$ as perpendicular bisector of side $\overline{AC}$.
3. $\overline{LN}$ and $\overline{PR}$ intersect at point O .
4. With centre O and radius $m\overline{OA} = m\overline{OB} = m\overline{OC}$, draw a circle.

This circle will pass through A , B and C whereas O is the circumcentre of the circumscribed circle.

Remember:

The circle passing through the vertices of triangle ABC is known as circumcircle, its radius as circumradius and centre as circumcentre.

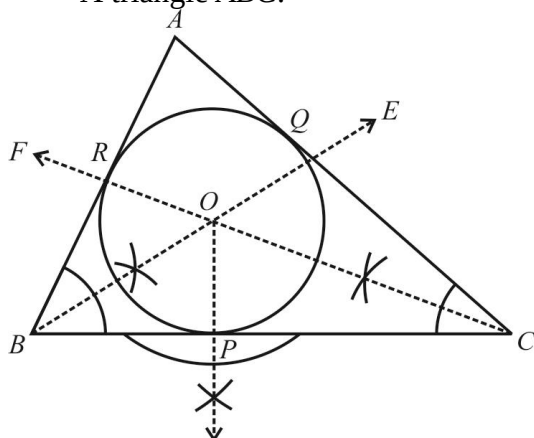
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(A.B)

13.2(ii) Inscribe a circle in a given triangle:

Given:

A triangle ABC .



Steps of Construction

1. Draw $\overline{BE}$ and $\overline{CF}$ to bisect the angles ABC and ACB respectively. Rays $\overline{BE}$ and $\overline{CF}$ intersect each other at point O .
2. O is the centre of the inscribed circle.
3. From O draw $\overline{OP}$ perpendicular to $\overline{BC}$. With centre O and radius $\overline{OP}$ draw a circle. This circle is the inscribed circle of triangle ABC .

Remember:

A circle which touches the three sides of a triangle internally is known as incircle. Its radius as in-radius and centre as in-centre.

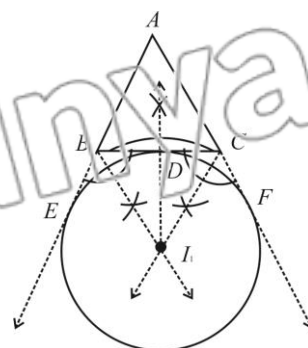
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(A.B)

13.2(iii) Escribe a circle to given triangle:

Given:

A triangle ABC .



Steps of Construction

1. Produce the sides $\overline{AB}$ and $\overline{AC}$ of $\triangle ABC$.
2. Draw bisectors of exterior angles ABC and ACB . Their bisectors of exterior angles meet at I_1 .
3. From I_1 draw perpendicular on side $\overline{BC}$ of $\triangle ABC$. Which $\overline{I_1D}$ intersect $\overline{BC}$ at D . I_1D is the radius of the escribed circle with centre at I_1 .
4. Draw the circle with radius $\overline{I_1D}$ and centre at I_1 that will touch the side BC of the $\triangle ABC$ externally and the produced sides AB and AC .

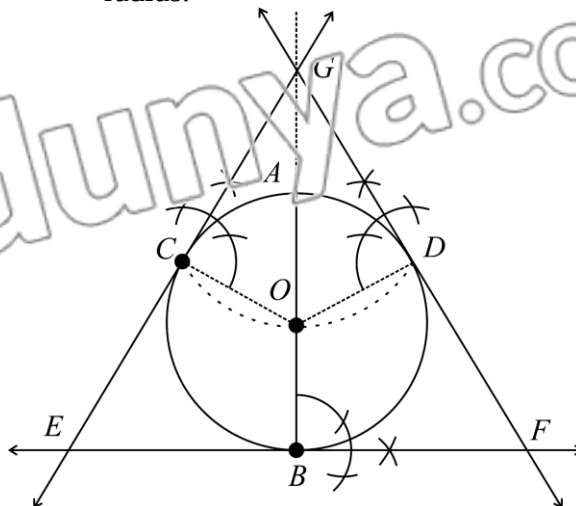
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(A.B)

13.2(iv) Circumscribe an equilateral triangle about a given circle

Given:

A circle with centre O of reasonable radius.



Steps of Construction

1. Draw $\overline{AB}$, the diameter of the circle for locating.
2. Draw an arc of radius $m\overline{OA}$ with centre at A for locating points C and D on the circle.
3. Join O to the points C and D.
4. Draw tangents to the circle at points B, C and D.
5. These tangents intersect at points E, F and G.

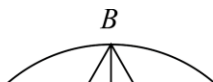
Page 226

(A.B)

13.2(v) Inscribe an equilateral triangle in a given circle.

Given:

A circle with centre O.



Steps of Construction

1. Draw any diameter $\overline{AB}$ of the circle.
2. Draw an arc of radius $\overline{OA}$ from point A. The arc cuts the circle at points C and D.
3. Join the points B, C and D to form straight line segments $\overline{BC}$, $\overline{CD}$ and $\overline{BD}$. Triangle BCD is the required inscribed equilateral triangle.

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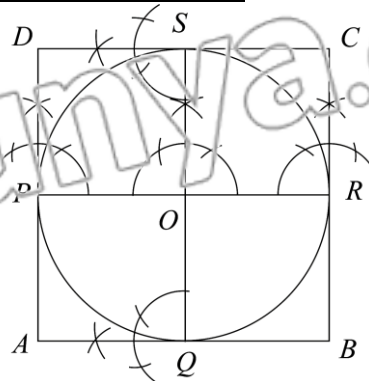
(A.B)

13.2(vi) Circumscribe a square about a given circle.

Given:

A circle with centre O.

Steps of Construction

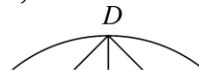


1. Draw two diameter $\overline{PR}$ and $\overline{QS}$ of the circle which bisect each other at right angle.
2. At points P, Q, R and S draw tangents to the circle.
3. Produce the tangents to meet each other at A, B, C and D. ABCD is the required circumscribed square.

13.2(vii) Inscribe a square in a given circle

Given:

A circle, with centre at O.



Steps of Construction

1. Through O draw two diameters $\overline{AC}$ and $\overline{ED}$ which bisect each other at right angle.
2. Join A with B, with C, C with D, and D with A. ABCD is the required square inscribed in the circle.

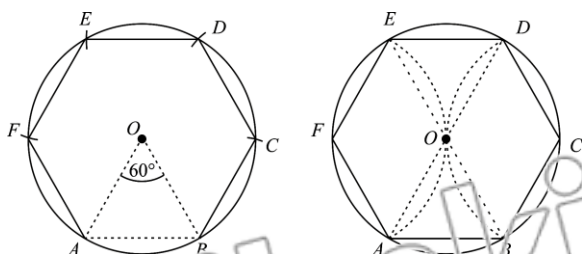
Page # 228

(A.B)

13.2(ix) Inscribe a regular hexagon in a given circle:

Given:

A circle, with centre at O.

**Steps of Construction**

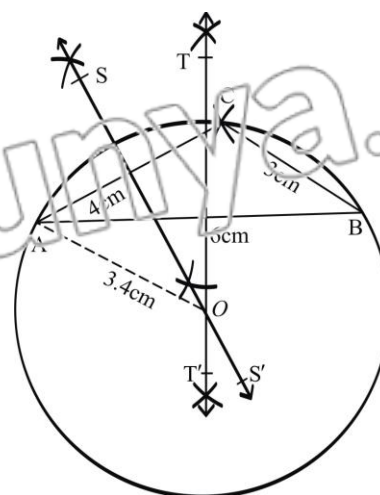
1. Take any point A on the circle and point with O.
2. From point A, draw an arc of radius $\overline{OA}$ which intersects the circle at point B and F.
3. Join O and A with points B and F.
4. $\triangle OAB$ and $\triangle OAF$ are equilateral triangles therefore $\angle AOB$ and $\angle AOF$ are of measure 60° i.e., $m\overline{OA} = m\overline{AB} = m\overline{AF}$
5. Produce $\overline{FO}$ to meet the circle at C. Join B to C. Since in $\angle BOC = 60^\circ$ therefore $m\overline{BC} = m\overline{OA}$.
6. From C and F, draw arcs of radius OA, which intersect the circle at points O and E.
7. Join C to D, D to E and E to F ultimately.

We have

$m\overline{OA} = m\overline{OB} = m\overline{OC} = m\overline{OD} = m\overline{OE} = m\overline{OF}$
Thus the figure ABCDEF is a regular hexagon inscribed in the circle.

Exercise 13.2

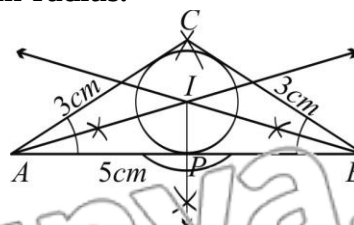
- Q.1** Circumscribe a circle about a triangle ABC with sides $|\overline{AB}| = 5\text{cm}$, $|\overline{BC}| = 3\text{cm}$ and $|\overline{CA}| = 4\text{cm}$. Also measure its circum radius. (A.B)

**Steps of Construction**

1. Construct $\triangle ABC$ with given information.
2. Draw right bisectors of $\overline{AC}$ and $\overline{AB}$. Both cut each other at point O.
3. With centre 'O', draw a circle of radius $\overline{OA}$. Thus required circumscribed circle is formed.

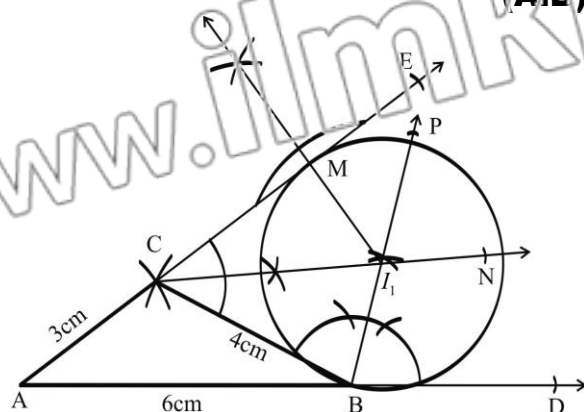
Radius of the circle is 3.4cm.

- Q.2** Inscribe a circle in a triangle ABC with sides $|\overline{AB}| = 5\text{cm}$, $|\overline{BC}| = 3\text{cm}$ and $|\overline{CA}| = 4\text{cm}$. Also measure its in-radius. (A.B)

**Steps of Construction**

1. Construct $\triangle ABC$ with given information.
2. Draw bisectors of $\angle A$ and $\angle B$, both cut each other at point I.
3. From point I, draw $\overline{IP} \perp \overline{AB}$.
4. With centre I, draw a circle of radius $\overline{IP}$. Thus required inscribed circle is formed.

- Q.3** Escribe a circle opposite to vertex A to a triangle ABC with sides $\overline{AB} = 6\text{cm}$, $\overline{BC} = 4\text{cm}$ and $\overline{CA} = 3\text{cm}$. Find its radius also. **(A.E)**

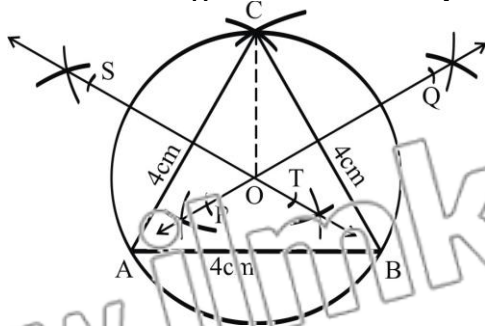


Steps of Construction

1. Construct $\triangle ABC$ with given information.
2. Draw external bisectors of $\angle B$ and $\angle C$, both cut each other at point I_1 .
3. From point I_1 , draw $\overline{I_1M} \perp \overline{AC}$ produced.
4. With centre I_1 , draw a circle of radius I_1M .

Thus required escribed circle is formed.

- Q.4** Circumscribe a circle about an equilateral triangle ABC with each side of length 4cm . **(A.B)**

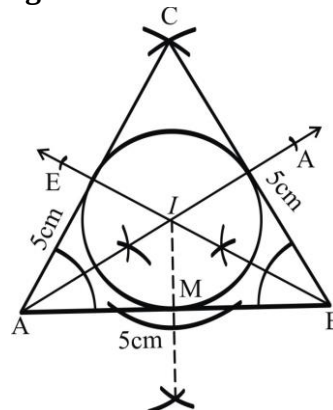


Steps of Construction

1. Construct $\triangle ABC$ with given information.
2. Draw right bisectors of $\overline{AC}$ and $\overline{BC}$. Both cut each other at point O .

3. With centre ' O ', draw a circle of radius $\overline{OC}$. Thus required circumscribed circle is formed.

- Q.5** Inscribe a circle in an equilateral triangle ABC with each side of length 5cm . **(A.B)**



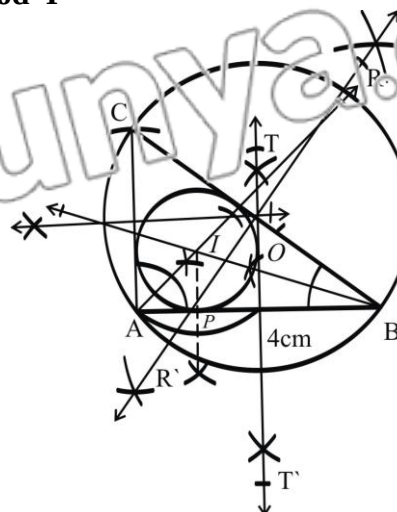
Steps of Construction

1. Construct $\triangle ABC$ with given information.
2. Draw bisectors of $\angle A$ and $\angle B$, both cut each other at point I .
3. From point I , draw $\overline{IM} \perp \overline{AB}$.
4. With centre I , draw a circle of radius $\overline{IM}$.

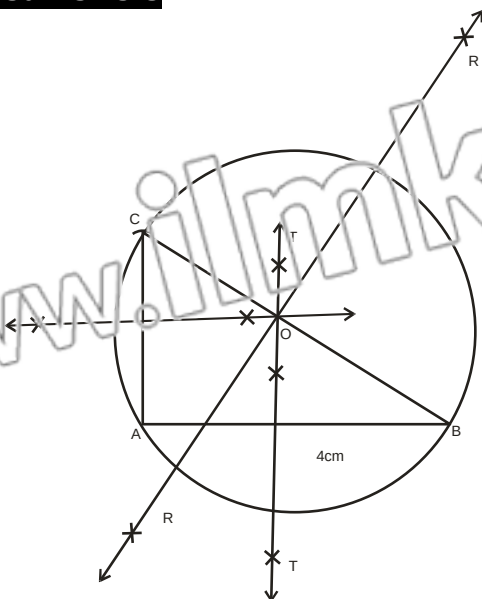
Thus required inscribed circle is formed.

- Q.6** Circumscribe and inscribe circles with regard to a right angle triangle with sides, 3cm , 4cm and 5cm . **(A.B)**

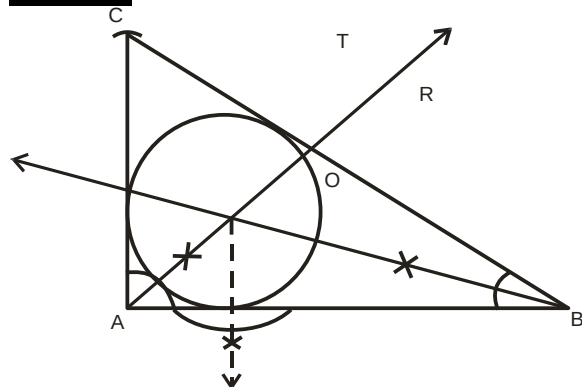
Method-I



Method-II Circumcircle



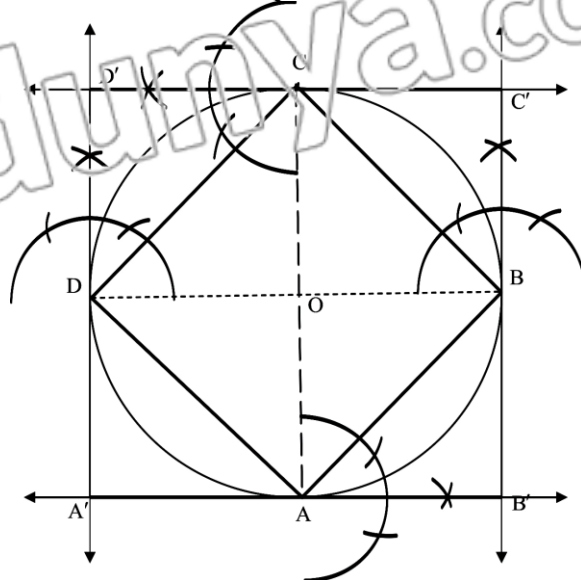
Incircle



Steps of Construction

1. Construct $\triangle ABC$ with given information.
2. Draw right bisectors of $\overline{AC}$ and $\overline{AB}$. Both cut each other at point O .
3. With centre ' O ', draw a circle of radius $\overline{OA}$. Thus required circumscribed circle is formed.
4. Draw bisectors of $\angle A$ and $\angle B$, both cut each other at point I .
5. From point I , draw $\overline{IP} \perp \overline{AB}$.
6. With centre I , draw a circle of radius $\overline{IP}$. Thus required inscribed circle is formed.

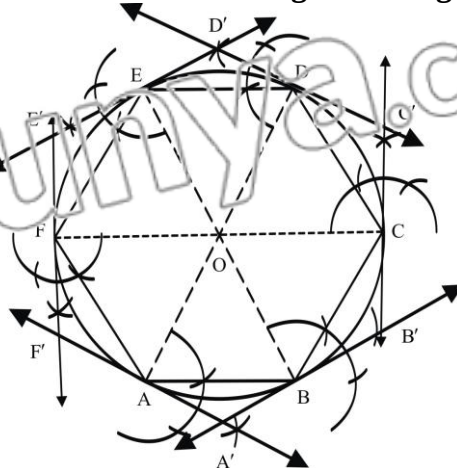
Q.7 In and about a circle of radius 4cm describe a square.



Steps of Construction

1. Draw a circle of radius 4cm .
2. Draw two diameters $\overline{AC}$ and $\overline{BD}$ which bisect each other at right angle.
3. Join B to C , C to D , D to A and A to B . Thus required inscribed square is formed.
4. At point A, B, C and D draw tangents, which intersect each other at points B', C', D' and A' . Thus circumscribed square is formed.

Q.8 In and about a circle of radius 3.5cm describe a regular hexagon.

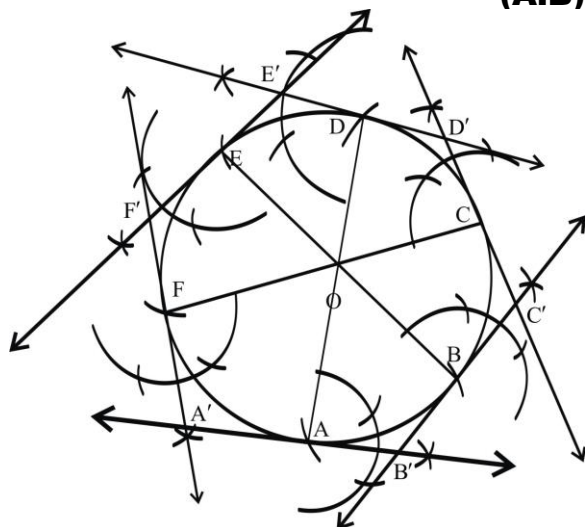


Steps of Construction

1. Draw any diameter $\overline{AD}$.
 2. From point A draw an arc of radius AO (the radius of the circle), which cuts the circle at points B and F .
 3. Join B with O and extend it to meet the circle at E .
 4. Join F with O and extend it to meet the circle at C .
 5. Draw tangents to the circle at points A, B, C, D, E and F intersecting one another at points A', B', C', D', E' and F' respectively.
- Thus $A'B'C'D'E'F'$ is the circumscribed regular hexagon.

Q.9 Circumscribe a regular hexagon about a circle of radius 3cm.

(A.B)

**Steps of Construction**

1. Draw any diameter $\overline{AD}$.
 2. From point A draw an arc of radius AO (the radius of the circle), which cuts the circle at points B and F .
 3. Join B with O and extend it to meet the circle at E .
 4. Join F with O and extend it to meet the circle at C .
 5. Draw tangents to the circle at points A, B, C, D, E and F intersecting one another at points A', B', C', D', E' and F' respectively.
- Thus $A'B'C'D'E'F'$ is the circumscribed regular hexagon.

13.3 TANGENT TO THE CIRCLE

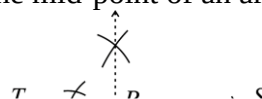
13.3(i) To draw a tangent to a given arc without using the centre through a given point P : (A.B)

Case (i)

When P is the middle point of the arc AB

Given:

P is the mid-point of an arc AB .

**Steps of Construction**

1. Join A and B , to form the chord $\overline{AB}$.
2. Draw the perpendicular bisector of chord $\overline{AB}$ which passes through mid-point P of AB and mid-point R of $\overline{AB}$.
3. At points P construct a right angle TPR .
4. Produce $\overline{PT}$ in the direction of P beyond point S . Thus $\overline{TP}$ is the required tangent to the arc AB at point P .

Page # 229

(A.B)

Case (ii)

When P is at end point of the arc

Given:

P is the end point of arc PQR .



Steps of Construction

1. Take a point A on the arc PQR .
2. Join the points A and P .
3. Draw perpendicular g at A which intersects the arc PQR at B .
4. Join the points B and P .
5. Draw $\angle APD$ of measure equal to that of $\angle AEP$.
6. Now

$$\begin{aligned} m\angle EPD &= m\angle BPA + m\angle APD \\ &= m\angle BPA + m\angle ABP \\ &= 90^\circ \end{aligned}$$

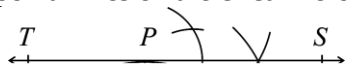
Thus $\overrightarrow{PD}$ is the required tangent.

Page # 230**(A.B)**

13.3(ii-a) To draw a tangent to a circle from a given point P at a given point on the circumference:

Given:

A circle with the centre O and some point P lies on the circumference.

**Steps of Construction**

1. Join point P to the centre O , so that $\overline{OP}$ is the radius of the circle.
 2. Draw a line TPS which is perpendicular to the radius $\overline{OP}$.
- $\therefore \overline{TPS}$ is the required tangent to the circle at given point P .

Page # 231**(A.B)**

13.3(ii-b) To draw a tangent to a circle from a given point P which lies outside the circle:

Given:

A circle with centre O and some point P outside the circle.

**Steps of Construction**

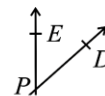
1. Join point P to the centre O .
2. Find M , the mid-point of $\overline{OP}$.
3. Construct a semi-circle on diameter $\overline{OP}$, with M as its centre. This semi-circle cuts the given circle at T .
4. Join P with T and produce $\overline{PT}$ on both sides, then $\overline{PT}$ is the required tangent.

Page # 231**(A.B)**

13.3(iii) To draw two tangents to a circle meeting each other at a given angle:

Given:

A circle with centre O , $\angle MNS$ is a given angle.

**Steps of Construction**

1. Take a point A on the circumference of circle having centre O .
2. Join the points O and A .
3. Draw $\angle COA$ of measure equal to that of $\angle MNS$.

4. Produce $\overline{CO}$ to meet the circle at B.
5. $m\angle AOB = 180^\circ - m\angle COA$
6. Draw $\overline{AD}$ perpendicular to $\overline{OA}$.
7. Draw $\overline{BE}$ perpendicular to $\overline{OB}$.
9. $m\angle AOB + m\angle APB = 180^\circ$, that is
 $m\angle ACB = 180^\circ - m\angle APB$
10. From step 5 and step 9, We have
 180°

$$m\angle COA = 180^\circ - m\angle ABP \Rightarrow m\angle COA = m\angle APB$$

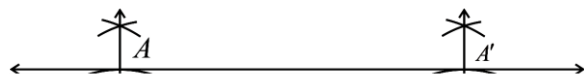
$$\Rightarrow \angle ALP = m\angle MNS \quad (\because m\angle COA = m\angle MNS)$$

$\overline{AP}$ and $\overline{BP}$ are the required tangents meeting at the given $\angle MNS$.

Page # 232

(A.B)

13.3(iv-a) To draw direct or (external) common tangents to equal circles:



Given:

Two circles of equal radii with centres O and O' respectively.

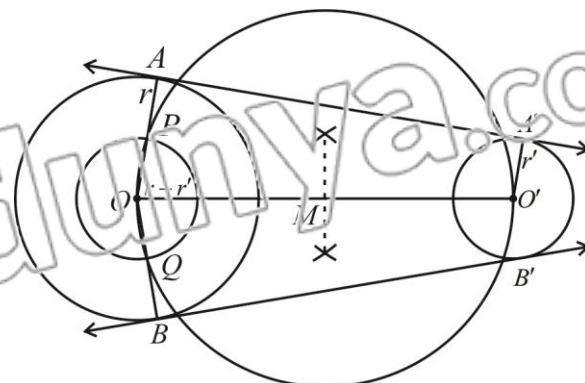
Steps of Construction

1. Join the centres O and O'.
2. Draw diameter AOB of the first circle so that $\overline{AOB} \perp \overline{OO'}$.
3. Draw diameter A'O'B' of the second circle so that $\overline{A'O'B'} \perp \overline{OO'}$.
4. Draw $\overline{AA'}$ and $\overline{BB'}$ which are the required common tangents.

Page # 233

(A.B)

13.3(v-a) To Draw direct or (external) common tangents to (two) unequal circles:



Given: Two unequal circles with centres O, O' and radii r, r' ($r > r'$) respectively.

Steps of Construction

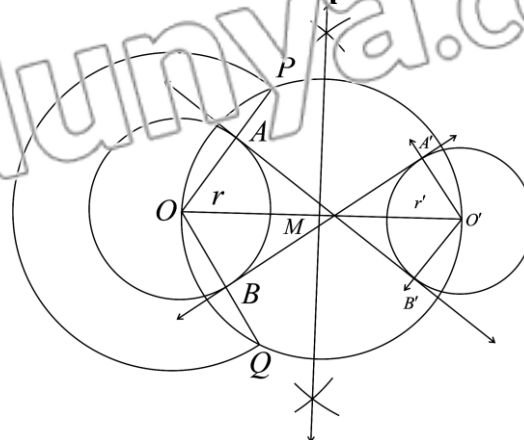
1. Join the centres O and O'.
2. On diameter $\overline{OO'}$, construct a new circle with centre M, the midpoint of $\overline{OO'}$.
3. Draw another circle with centre at O and radius $= r - r'$ cutting the circle with diameter $\overline{OO'}$ at P and Q.
4. Produce $\overline{OP}$ and $\overline{OQ}$ to meet the first circle at A and B respectively.
5. Draw $\overline{O'A'} \parallel \overline{OA}$ and $\overline{O'B'} \parallel \overline{OB}$
6. Join $\overline{AA'}$ and $\overline{BB'}$ which are the required direct common tangents.

Thus $\overline{AA'}$ and $\overline{BB'}$ are the required common tangents.

Page # 233

(A.B)

13.3(v-b) To draw to transverse or internal common tangents to two unequal circles:



Given:

Two unequal circles with centres O, O' and radii r, r' respectively.

Steps of Construction

1. Join the centres O and O' of the given circles.
2. Find mid-point M of $\overline{OO'}$.
3. On diameter $\overline{OO'}$, construct a new circle with centre M .
4. Draw another circle with centre at O and radius $= r + r'$ intersecting the circle of diameter $\overline{OO'}$ at P and Q .
5. Join O with P and Q . $\overline{OP}$ and $\overline{OQ}$ meet the circle with radius r at A and B respectively.
6. Draw $\overline{O'B'} \perp \overline{OA}$ and $\overline{O'A'} \perp \overline{OB}$.
7. Join A with B' and A' with B . Thus $\overline{AB'}$ and $\overline{A'B}$ are the required transverse common tangents.

Page # 234**(A.B)****13.3(vi-a) To draw a tangent to two unequal touching circles:****Case-I****Given:**

Two unequal touching circles with centres O and O' .

Steps of Construction

1. Join O with O' and produce $\overline{OO'}$ to meet the circles at the point A where these circles touch each other.
2. Tangent is perpendicular to the line segment OA .

Draw perpendicular to $\overline{OA}$ at the point A which is the required tangent.

Page # 233**(A.B)****Case-II****Given:**

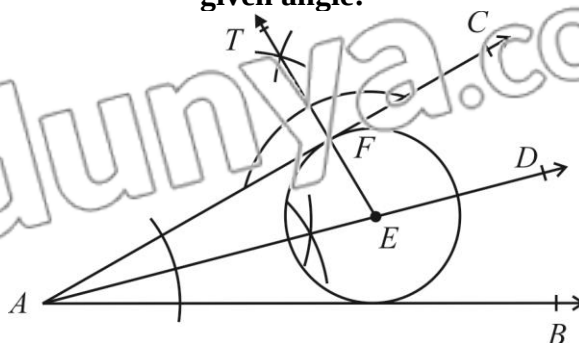
Two unequal touching circles with centres O and O'

Steps of Construction

1. Join O with O' . $\overline{OO'}$ intersects the circles at the point B where these circles touch each other.

See Fig. 2

2. Tangent is perpendicular to line segment containing the centres of the circles.
3. Draw perpendicular to $\overline{OO'}$ at the point B which is the required tangent.

Page # 235**(A.B)****13.3 (vii-a) To draw a circle which touches both the arms of a given angle:****Given:**

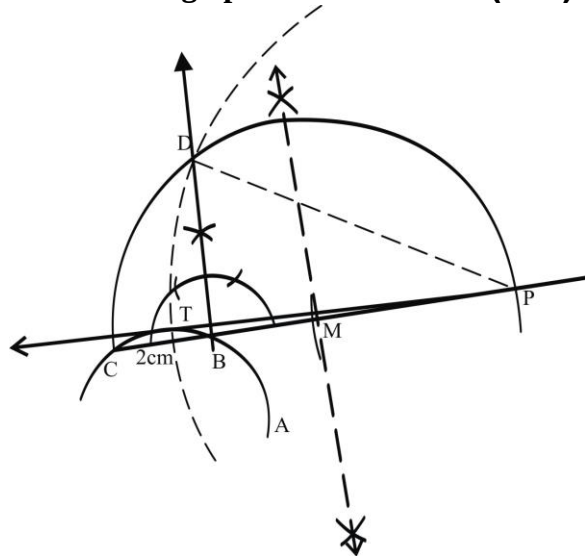
An angle $\angle BAC$.

Steps of Construction

1. Draw $\overline{AD}$ bisecting $\angle BAC$.
2. Take any point E on $\overline{AD}$
3. Draw $\overline{ET}$ perpendicular to $\overline{AC}$ intersecting $\overline{AC}$ at the point F .
4. Draw a circle with centre E and radius $\overline{ET}$.
This circle touches both the arms of $\angle BAC$.

Exercise 13.3

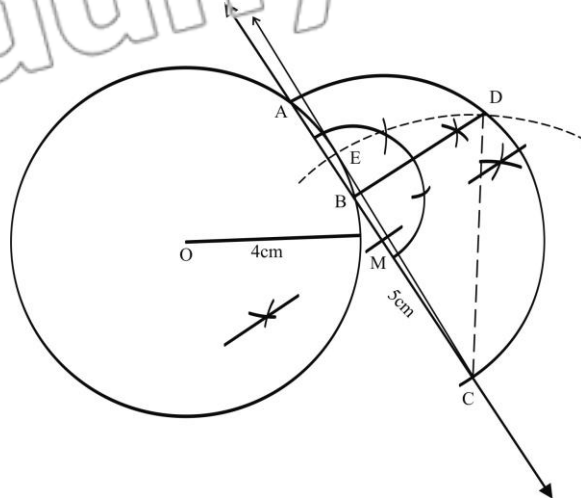
- Q.1** In an arc ABC the length of the chord $\overline{BC} = 2\text{cm}$. Draw a secant $\overline{PBC} = 8\text{cm}$, where P is the point outside the arc. Draw a tangent through point P to the arc. (A.B)



Steps of Construction

1. Draw an arc.
2. Draw a secant line PBC , such that $mBC = 2\text{cm}$.
3. Find the mid-point M of $\overline{PC}$.
4. With centre M , draw a semi circle of radius PM .
5. Draw BD perpendicular to PC .
6. With centre B , draw an arc radius BD . It cut the circle at point T .
7. Join P to T produce it.
Thus required tangent line is formed.

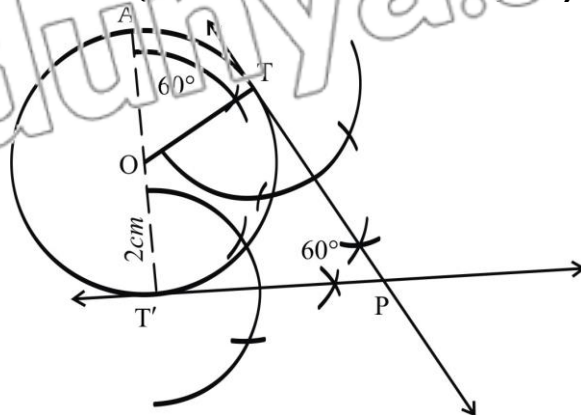
- Q.2** Construct a circle with diameter 8cm . Indicate a point C , 5cm away from its circumference. Draw a tangent from point C to the circle without using its centre. (A.B)



Steps of Construction

1. Draw a circle of radius 4cm .
2. Take a point C , 5cm away from the circumference of circle.
3. Take a secant line CA .
4. Find the mid-point M of $\overline{CA}$.
5. With centre M , draw a semi circle of radius AM .
6. Draw BD perpendicular to AC .
7. With centre C , draw an arc of radius CD . It cut the circle at point E .
8. Join C to E produce it.
Thus required tangent line is formed.

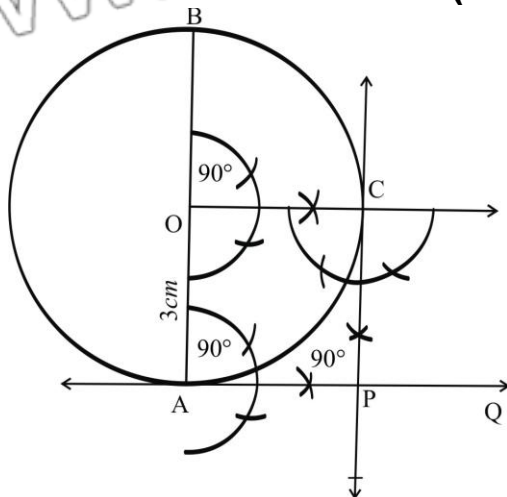
- Q.3** Construct a circle of radius 2cm . Draw two tangents making an angle of 60° with each other. (A.B)



Steps of Construction

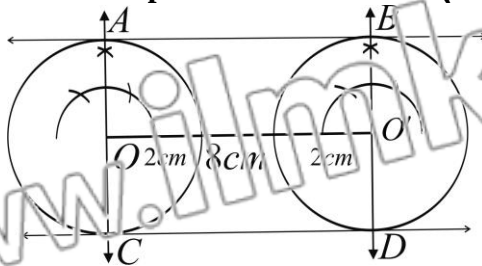
1. Draw a circle of radius 2cm .
2. Draw diameter $\overline{AT'}$.
3. At point T' , draw $\overline{PT'} \perp \overline{OT'}$.
4. At centre ' O ', draw $\angle AOT = 60^\circ$ to get radial segment $\overline{OT}$.
5. At point T , draw $\overline{PT} \perp \overline{OT}$. Thus required tangents are formed.

Q.4 Draw two perpendicular tangents to a circle of radius 3cm . (A.B)

**Steps of Construction**

1. Draw a circle of radius 3cm .
2. Draw diameter $\overline{AB}$.
3. At point A , draw $\overline{AP} \perp \overline{AB}$.
4. At centre ' O ', draw $\angle AOC = 90^\circ$ to get radial segment $\overline{OC}$.
5. At point C , draw $\overline{PC} \perp \overline{OC}$. Thus required tangents are formed.

Q.5 Two equal circles are at 8cm apart. Draw two direct common tangents of this pair of circles. (A.B)

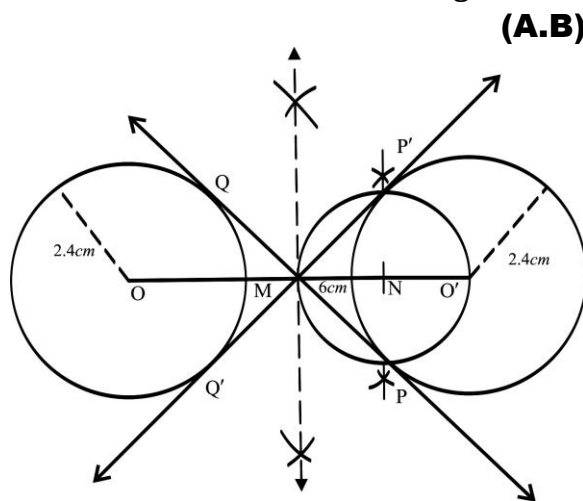
**Steps of Construction**

1. Draw a line segment $\overline{OO'}$ of measure 8cm .

2. With centres O and O' draw circles of radii 2cm .
3. At points C and O' , draw perpendiculars $\overline{AC}$ and $\overline{BD}$, meeting the circles at points A, B, C and D .
4. Join A to B and C to D .

Thus required tangents $\overline{AB}$ and $\overline{CD}$ are formed.

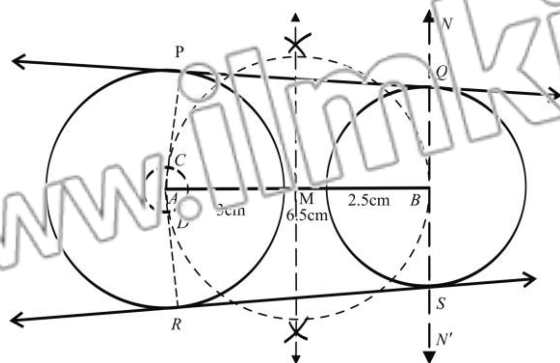
Q.6 Draw two equal circles of each radius 2.4cm . If the distance between their centres is 6cm , then draw their transverse tangents. (A.B)

**Steps of Construction**

1. Draw a line segment $\overline{OO'} = 6\text{cm}$
2. Draw circles with centres as O and O' having radii 2.4cm .
3. Find mid-point M of $\overline{OO'}$.
4. Find mid-point N of $\overline{MO'}$.
5. Taking point N as centre and radius equal to nMN draw a circle intersecting the circle with centre O' at points P and P' .
6. Draw a line through the points M and P touching the second circle at the point Q .
7. Draw a line through the points M and P' touching the second circle at the point Q' .

Thus $\overline{PQ}$ are $\overline{P'Q'}$ the required transverse common tangents to the given circles.

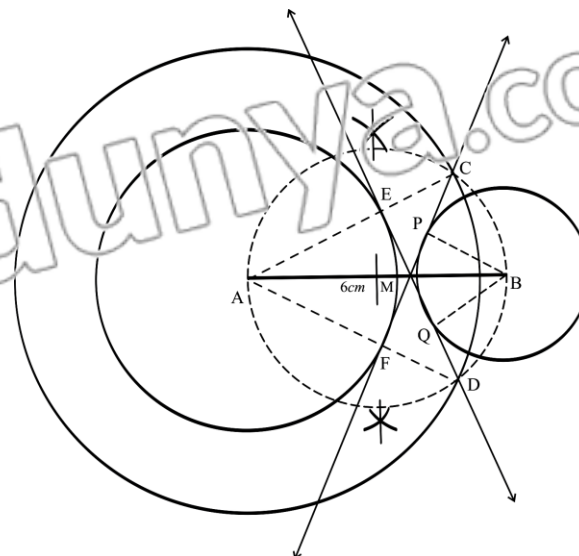
- Q.7** Draw two circles with radii 2.5cm and 3cm . If their centres are 6.5cm apart, then draw two direct common tangents. **(A.B)**



Steps of Construction

1. Take a line segment $\overline{AB}$ of measure 6.5cm .
2. Draw two circles of radii 3 and 2.5 with centres at A and B respectively.
3. Taking centre at A , draw a circle of radius $3 - 2.5 = 0.5\text{cm}$.
4. Bisect the line segment AB at point M .
5. Taking centre at M and radius MA , draw a circle intersecting the circle of radius 0.5cm at C and D .
6. Join the point A with C and produce it to meet the circle with centre A at P . Also join A with D and produce it to meet the circle with centre A at R .
7. Draw $\overline{BQ}$ parallel to $\overline{AP}$.
8. Draw a line joining the P to Q and R to S .
Thus required direct common tangents are formed.

- Q.8** Draw two circles with radii 3.5cm and 2cm . If their centres are 6cm apart, then draw two transverse common tangents. **(A.B)**



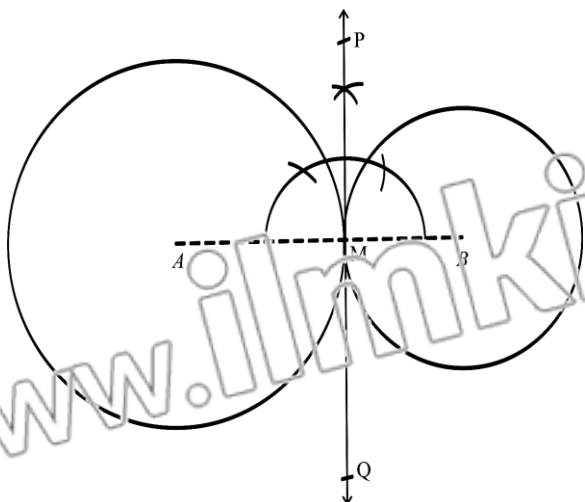
Steps of Construction

1. Take a line segment $\overline{AB}$ of measure 6cm .
2. Draw two circles of radii 3.5 and 2 with centres at A and B respectively.
3. Taking centre at A , draw a circle of radius $3.5 + 2 = 5.5\text{cm}$.
4. Bisect the line segment AB at point M .
5. Taking centre at M and radius MA , draw a circle intersecting the circle of radius 5.5cm at C and D .
6. Join the point A with C . Also join A with D .
7. Draw $\overline{BQ}$ parallel to $\overline{AC}$.
8. Draw a line joining the P to R and Q to S .
Thus required transverse common tangents are formed.

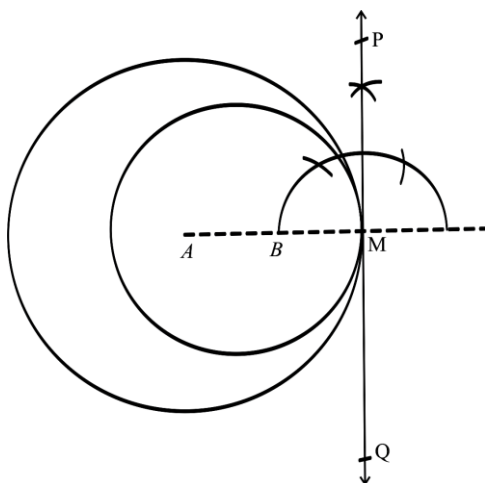
- Q.9** Draw two common tangents to two touching circles of radii 2.5cm and 3.5cm . **(A.B)**

Case-I

Circles touch each other externally

**Case-II**

Circles touch each other internally

**Steps of Construction****Case-I**

(Circles touch each other externally)

1. Draw a line segment $\overline{AB}$ of measure 6cm (sum of radii).
2. With centre A and B, draw two circles of radius 3.5cm and 2.5cm . They touch each other at point M.
3. At point M, draw $\overline{PQ} \perp \overline{AB}$.

Thus required common tangent is formed.

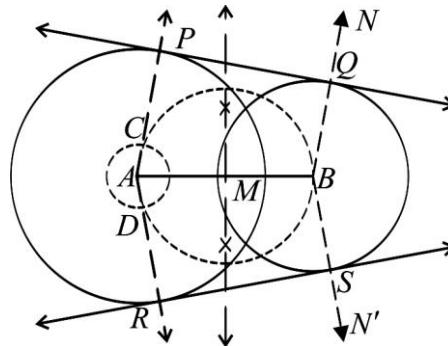
Case-II

(Circles touch each other internally)

4. Draw a line segment $\overline{AB}$ of measure 1cm (difference of radii).
5. With centre A and B, draw two circles of radius 3.5cm and 2.5cm . They touch each other at point M.
6. At point M, draw $\overline{PQ} \perp \overline{AM}$.

Thus required common tangent is formed.

- Q.10 Draw two common tangents to two intersecting circle of radii 3cm and 4cm . (A.B)**

**Steps of Construction**

1. Take a line segment $\overline{AB}$ of measure 6cm .
2. Draw two circles of radii 3 and 2.5 with centres at A and B respectively.
3. Taking centre at A, draw a circle of radius $3 - 2.5 = 0.5\text{cm}$.
4. Bisect the line segment AB at point M.

5. Taking centre at M and radius mAM , draw a circle intersecting the circle of radius 0.5cm at C and D .

6. Join the point A with C and produce it to meet the circle with centre A at P . Also join A with D and produce it to meet the circle with centre A at R .

7. Draw $\overline{BQ}$ parallel to AP .

8. Draw a line joining the P to Q and R to S .

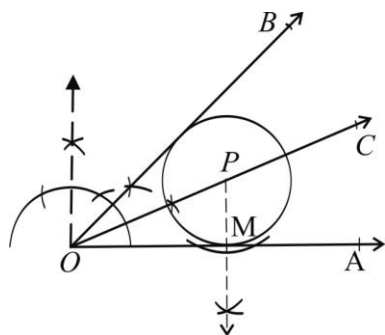
Thus required direct common tangents are formed.

Q.11 Draw circles which touches both the arms of angles (A.B)

(i) 45°

(ii) 60°

(i) 45°

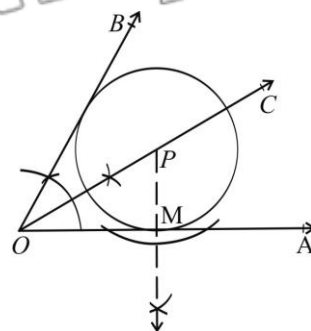


Steps of Construction

1. Draw an angle AOB of measure 45° .
2. Draw $\overline{OC}$ the bisector of $\angle AOB$.
3. Take any point P on $\overline{OC}$. From point P draw $\overline{PM} \perp \overline{OA}$.

4. With centre P , draw a circle of radius $\overline{PM}$ which touches both arms of angle 45° .

Thus required circle is formed. 60°



Steps of Construction

5. Draw an angle AOB of measure 60° .
 6. Draw $\overline{OC}$ the bisector of $\angle AOB$.
 7. Take any point P on $\overline{OC}$. From point P draw $\overline{PM} \perp \overline{OA}$.
 8. With centre P , draw a circle of radius $\overline{PM}$ which touches both arms of angle 60° .
- Thus required circle is formed.

Miscellaneous Exercise 13

Q.1 Multiple choice questions

Four possible answers are given for the following question. Tick (✓) the correct answer.

- (i) The circumference of a circle is called (K.B)
(a) Chord (b) Segment (c) Boundary
- (ii) A line intersecting a circle is called (K.B)
(a) Tangent (b) Secant (c) Chord
- (iii) The portion of a circle between two radii and an arc is called (K.B)
(a) Sector (b) Segment (c) Chord
- (iv) Angle inscribed in a semi-circle is (K.B)
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$
- (v) The length of the diameter of a circle is how many times the radius of the circle (K.B)
(a) 1 (b) 2 (c) 3
- (vi) The tangent and radius of a circle at the point of contact are (K.B)
(a) Parallel (b) Not perpendicular (c) Perpendicular
- (vii) Circles having three points in common (K.B)
(a) Overlapping (b) Not coincide (c) Coincide
- (viii) If two circles touch each other, their centres and point of contact are (K.B)
(a) Coincident (b) Non-collinear (c) Collinear
- (ix) The measure of the external angle of a regular hexagon is (GRW 2014) (K.B)
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$
- (x) If the incentre and circumcentre of a triangle coincide, the triangle is (K.B)
(a) An isosceles (b) A right triangle (c) An equilateral
- (xi) The measure of the external angle of a regular octagon is (K.B)
(LHR 2015, FSD 2018)
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{8}$
- (xii) Tangents drawn at the end points of the diameter of a circle are (K.B)
(a) Parallel (b) Perpendicular (c) Intersecting
- (xiii) The length of two transverse tangents to a pair of circles are
(a) Unequal (b) Equal (c) Overlapping
- (xiv) How many tangents can be drawn from a point outside the circle? (K.B)
(a) 1 (b) 2 (c) 3
- (xv) If the distance between the centres of two circles is equal to the sum of their radii, then the circles will (K.B)
(a) Intersect (b) Do not intersect (c) Touch each other externally
- (xvi) If the two circles touches externally, then the distance between their centers is equal to the (K.B)
(a) Difference of their radii (b) Sum of their radii
(c) Product of their radii
- (xvii) How many common tangents can be drawn for two touching circles? (K.B)
(a) 2 (b) 3 (c) 4
- (xviii) How many common tangents can be drawn for two disjoint circles (K.B)
(a) 2 (b) 3 (c) 4

ANSWER KEY

i	c	Iv	a	vii	d	x	c	xiii	b	xvi	b
ii	b	V	b	viii	c	xi	a	xiv	b	xvii	b
iii	a	Vi	c	ix	a	xii	a	xv	c	xviii	c

Q.2 Define and draw the following geometric figures:

(i) Define and draw the following geometric figures: (A.B)

(a) The segment of a circle.

Ans:

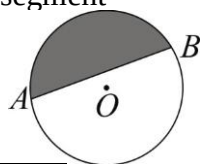
Segment of a Circle (K.B)

“A chord divides a circular region in two parts called segment of a circle”.

In the given figure, $\overline{AB}$ divides the circle into two segments.

(i) Major segment

(ii) Minor segment



Minor Segment (K.B)

“Circular region bounded by a chord and minor arc is called minor segment”. In the given figure, shaded part is minor segment.

Major Segment (K.B)

“Circular region bounded by a chord and major arc is called major segment”. In the given figure, non shaded part is major segment.

(b) The tangent to a circle. (K.B)

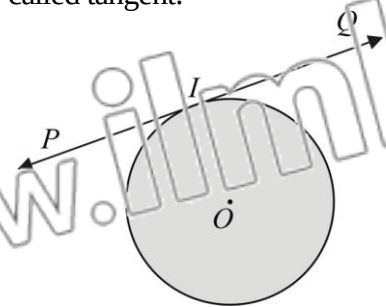
Ans:

Tangent Line

A line which has one point common with a circle is called tangent line.

Or

A straight line which touches the circumference at one point only is called tangent.



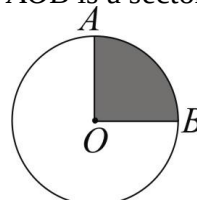
In the figure $\overline{PIQ}$ is tangent line.

(c) The sector of a circle. (K.B)

Ans:

Sector of a Circle

“Circular region bounded by an arc and its two corresponding radial segments is called sector in the given figure, AOB is a sector of the circle.



(d) The inscribed circle. (K.B)

Ans: See definition page # 304

(e) The circumscribed circle. (K.B)

Ans: See definition page # 304

(f) The escribed circle. (K.B)

Ans: See definition page # 304

(ii) The length of each side of a regular octagon is 3cm. Measure its perimeter. (K.B)

Ans: Length of each side of octagon = 3cm

Sides of a octagon = 8

Perimeter of Octagon = $8 \times 3\text{cm}$

$\therefore$ Perimeter = $n \times l$,
= 24cm

(iii) Write down the formula for finding the angle subtended by the side of a n-sided polygon at the centre of the circle. (K.B)

Ans: Formula for finding the angle subtended by the side of a n-sided polygon at the centre of the circle

$= \frac{360^\circ}{n}$

(iv) The length of the side of a regular pentagon is 5cm what is its perimeter? (K.B)

Ans: Length of each side of octagon = 5cm

Sides of a pentagon = 5

Perimeter of pentagon = $5 \times 5\text{cm}$

$\therefore$ Perimeter = $n \times l$,
= 25cm

Q.3 Fill in the blanks

- (i) The boundary of a circle is called _____. (K.B)
- (ii) The circumference of a circle is called _____ of the circle. (K.B)
- (iii) The line joining the two points of circle is called _____. (K.B)
- (iv) The point of intersection of perpendicular bisectors of two non-parallel chords of a circle is called the _____. (K.B)
- (v) Circles having three points in common will _____. (K.B)
- (vi) The distance of a point inside the circle from its centre is _____ than the radius. (K.B)
- (vii) The distance of a point outside the circle from its centre is _____ than the radius. (K.B)
- (viii) A circle has only _____ centre. (K.B)
- (ix) One and only one circle can be drawn through three _____ points. (K.B)
- (x) Angle inscribed in a semi-circle is a _____ angle. (K.B)
- (xi) If two circles touch each other, the point of _____ and their _____ are collinear. (K.B)
- (xii) If two circles touch each other, their point of contact and centres are _____. (K.B)
- (xiii) From a point outside the circle _____ tangents can be drawn. (K.B)
- (xiv) A tangent is _____ to the radius of a circle at its point of contact. (K.B)
- (xv) The straight line drawn $\perp$ to the radius of a circle is called the _____ to the circle. (K.B)
- (xvi) Two circles can not cut each other at more than _____ points. (K.B)
- (xvii) The $\perp$ -bisector of a chord of circle passes through the _____. (K.B)
- (xviii) The length of two direct common tangents to two circles are _____ to each other. (K.B)
- (xix) The length of two transverse common tangents to two circles are _____ to each other. (K.B)
- (xx) If the in-centre and circum-centre of a triangle coincide the triangle is _____. (K.B)
- (xxi) Two intersecting circles are not _____. (K.B)
- (xxii) The centre of an inscribed circle is called _____. (K.B)
- (xxiii) The centre of a circumscribed circle is called _____. (K.B)
- (xxiv) The radius of an inscribed circle is called _____. (K.B)
- (xxv) The radius of a circumscribed circle is called _____. (K.B)

ANSWER KEY

- | | |
|-----------------------|----------------------|
| (i) Circumference | (xiv) Perpendicular |
| (ii) Boundary | (xv) Tangent |
| (iii) Chord | (xvi) Two |
| (iv) Centre | (xvii) Centre |
| (v) Coincide | (xviii) Equal |
| (vi) Less | (xix) Equal |
| (vii) Greater | (xx) Equilateral |
| (viii) One | (xxi) Concentric |
| (ix) Non-collinear | (xxii) Incentre |
| (x) Right | (xxiii) Circumcentre |
| (xi) Contact, centres | (xxiv) In-radius |
| (xii) Collinear | (xxv) Circum-radius |
| (xiii) Two | |



CUT HERE

SELF TEST

Time: 40 min

Marks: 25

Q.1 Four possible answers (A), (B), (C) & (D) to each question are given, mark the correct answer. (7×1=7)

1 A line intersecting a circle is called:

- (A) Tangent (B) Secant
(C) Chord (D) Radius

2 Angle inscribed in a semi-circle is:

- (A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$
(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

3 If two circles touch each other, their centres and point of contact are:

- (A) Coincident (B) Non-collinear
(C) Collinear (D) None of these

4 The measure of the external angle of a regular hexagon is:

- (A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$
(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{2}$

5 If the incentre and circumcentre of a triangle coincide, the triangle is:

- (A) An isosceles (B) A right triangle
(C) An equilateral (D) Scalene

6 If the distance between the centers of two circles is equal to the sum of their radii, then the circles will:

- (A) Intersect (B) Do not intersect
(C) Touch each other externally (D) None of these

7 How many common tangents can be drawn for two touching circles?

- (A) 2 (B) 3
(C) 4 (D) 1

Q.2 Give Short Answers to following Questions.

(5×2=10)

(1) The length of the side of a regular pentagon is 5cm what is its perimeter?

(2) If $|AB| = 3\text{cm}$ and $|BC| = 4\text{cm}$ are the lengths of two chords of an arc, then locate the centre of the arc.

(3) Draw circles which touches both the arms of angles 60° .

(4) Define and draw the following geometric figure of the inscribed circle.

(i) Draw two perpendicular tangents to a circle of radius 3cm.

Q.3 Answer the following Questions in detail.

(4+4=8)

(a) Escribe a circle opposite to vertex A to a triangle ABC with sides $|AB| = 5\text{cm}$, $|BC| = 4\text{cm}$, $|CA| = 3\text{cm}$. Find its radius also.

(b) Draw two circles with radii 3.5 cm and 2 cm of their centers are 6 cm apart then draw two transverse common tangents.

NOTE: Parents or guardians can conduct this test in their supervision in order to check the skill of students.

Formulas of Different Units

Unit # 1

Quadratic Equations

- $(a+b)^2 = a^2 + b^2 + 2ab$
- $(a-b)^2 = a^2 + b^2 - 2ab$
- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Unit # 2

Theory of Quadratic Equations

- Discriminant $= b^2 - 4ac$
- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $1 + \omega + \omega^2 = 0$
- $\omega^3 = 1$
- $\omega = \frac{1}{\omega^2}$ or $\omega^2 = \frac{1}{\omega}$
- $(x+y+z)(x^2+y^2+z^2-xy-yz-xz) = x^3+y^3+z^3-3xyz$
- $S = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$
- $P = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$
- $x^2 - Sx + P = 0$ Or $x^2 - (\text{Sum of roots})x + \text{Product of roots} = 0$
- Function is Symmetric if $f(\alpha, \beta) = f(\beta, \alpha)$
- $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
- $\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

Unit # 3

Variations

- Fourth proportional: $d = \frac{bc}{a}$
- Third proportional: $c = \frac{b^2}{a}$
- Mean proportional: $b^2 = ac$
- Continued proportional: $b^2 = ac$
- Invertedo Theorem. If $a : b = c : d$ then $b : a = d : c$

Formulas of Different Units

- **Alternendo Theorem.**
If $a:b = c:d$ then $a:c = b:d$
- **Componendo Theorem**
If $a:b = c:d$ then
(i) $a+b:b = c+d:d$
(ii) $a:a+b = c:c+d$
- **Dividendo Theorem**
If $a:b = c:d$ then
(i) $a-b:b = c-d:d$
(ii) $a:a-b = c:c-d$
- **Componendo–Dividendo Theorem**
If $a:b = c:d$ then
(i) $a+b:a-b = c+d:c-d$
(ii) $a-b:a+b = c-d:c+d$
- **K Method**
Let $\frac{a}{b} = \frac{c}{d} = k$
Then,
 $a = bk, c = dk$

Unit # 5

Sets and Functions

- Number of all possible subsets = 2^n
- Number of all possible proper subsets = $2^n - 1$.
- Number of improper subsets = 1
- Formula to find number of elements in the power set = 2^n
- $A - B = \{x | x \in A \text{ and } x \notin B\}$
- If $A \cap B = \phi$, then A and B are disjoint sets.
- $A' = A^c = U - A$
- **Commutative property of Union**
 $A \cup B = B \cup A$
- **Commutative property of intersection**
 $A \cap B = B \cap A$
- **Associative property of union**
 $A \cup (B \cap C) = (A \cup B) \cap C$
- **Associative property of intersection**
 $A \cap (B \cup C) = (A \cap B) \cup C$
- **Distributive property of union over intersection**
 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **Distributive property of intersection over union**
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Formulas of Different Units

- **De-Morgan's laws**
 $(A \cup B)' = A' \cap B'$
 $(A \cap B)' = A' \cup B'$
- $A - B = A \cap B'$
- $(A - B)' = A' \cup B$
- Number of elements in Cartesian product $X \times Y = m \times n$
- Number of binary relations = $2^{m \times n}$
- If $A \subseteq B$, then $A \cup B = B$
- If $A \subseteq B$ and $B \subseteq A$ then $A = B$
- $A \cap A^c = \phi$
- $A \cup A^c = U$

Unit # 6

Basic Statistics

For Ungrouped Data	For Grouped Data
Arithmetic means	
Ungroup Data	(Direct Method)
$\bar{X} = \frac{\sum x}{n}$	$\bar{X} = \frac{\sum fx}{\sum f}$
Indirect Method	(Short Method)
$\bar{X} = A + \frac{\sum D}{n} \quad ; (D = x - A)$	$\bar{X} = A + \frac{\sum fD}{\sum f}$
(Coding Method)	
$A.M = A + \frac{\sum u}{n} \times h \quad ; \left(u = \frac{x - A}{h} \right)$	$A.M = A + \frac{\sum fu}{\sum f} \times h$
(Median)	
Median = $\left(\frac{n+1}{2} \right)^{th}$ item (n is odd)	
Median = $\frac{1}{2} \left\{ \left(\frac{n}{2} \right)^{th} \text{ item} + \left(\frac{n+2}{2} \right)^{th} \text{ item} \right\}$ (n is even)	Median = $l + \frac{h}{f} \left(\frac{n}{2} - C \right)$

Formulas of Different Units

(Mode)	
Mode = most repeated value of the data	$\text{Mode} = l + \frac{(f_m - f_1)}{2f_m - f_1 - f_2} \times h$
(Geometric Mean)	
$\text{G.M} = (x_1, x_2, x_3, \dots, x_n)^{\frac{1}{n}}$ By definition method	$\text{G.M} = \text{Anti log} \left(\frac{\sum f \log X}{\sum f} \right)$
$\text{G.M} = \text{Anti log} \left(\frac{\sum \log X}{n} \right)$ By log method	
Harmonic Mean	
$\text{H.M} = \frac{n}{\sum \frac{1}{X}}$	$\text{H.M} = \frac{n}{\sum \frac{f}{X}}$
Weighted Arithmetic Mean = $\bar{X}_w = \frac{\sum wx}{\sum w}$	
Range	
Range = $X_m - X_1$	Range = upper class boundary of last class – lower class boundary of first class
Variance	
(i) $S^2 = \frac{\sum (X - \bar{X})^2}{n}$ Proper mean formula (ii) $S^2 = \frac{\sum X^2}{n} - \left(\frac{\sum X}{n} \right)^2$ Direct Formula	(i) $S^2 = \frac{\sum f (X - \bar{X})^2}{\sum f}$ Proper mean formula (ii) $S^2 = \frac{\sum fX^2}{\sum f} - \left(\frac{\sum fX}{\sum f} \right)^2$ Direct Formula
Standard Deviation	
(i) $S = \sqrt{\frac{\sum (X - \bar{X})^2}{n}}$ Proper mean formula (ii) $S = \sqrt{\frac{\sum X^2}{n} - \left(\frac{\sum X}{n} \right)^2}$ Direct Formula	(i) $S = \sqrt{\frac{\sum f (X - \bar{X})^2}{\sum f}}$ Proper mean formula (ii) $S = \sqrt{\frac{\sum fX^2}{\sum f} - \left(\frac{\sum fX}{\sum f} \right)^2}$ Direct Formula

Formulas of Different Units

Unit # 7

Introduction to Trigonometry

- $1^\circ = \frac{\pi}{180}$ radians
- $x^\circ = \frac{x\pi}{180}$ radians
- $1 \text{ radian} = \frac{180^\circ}{\pi}$
- $x \text{ radians} = \frac{x(180^\circ)}{\pi}$
- $l = r\theta$
- $A = \frac{1}{2}r^2\theta$
- $\theta \pm 360k = \theta, k \in \mathbb{Z}$
- $\sin(-\theta) = -\sin \theta$
- $\operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta$
- $\cos(-\theta) = \cos \theta$
- $\sec(-\theta) = \sec \theta$
- $\tan(-\theta) = -\tan \theta$
- $\cot(-\theta) = -\cot \theta$
- Trigonometric table

	0°	30°	45°	60°	90°	180°	270°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined	0	Undefined

- $\sin \theta = \frac{y}{r}$
- $\cos \theta = \frac{x}{r}$
- $\tan \theta = \frac{y}{x}$
- $\sin^2 \theta + \cos^2 \theta = 1$
- $1 + \tan^2 \theta = \sec^2 \theta$
- $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$

Formulas of Different Units

Unit # 8

Projection of a Side of a Triangle

- $(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2 - 2m\overline{AB}.m\overline{AC}$, when angle opposite to BC is acute.
- $(\overline{BC})^2 = (\overline{AC})^2 + (\overline{AB})^2 + 2(m\overline{AB}).(m\overline{AD})$, when angle opposite to BC is obtuse.
- If $a^2 + b^2 = c^2$, Δ is a right angled Δ
- If $a^2 + b^2 < c^2$, Δ is a obtuse angled Δ
- If $a^2 + b^2 > c^2$, Δ is a acute angled Δ
where, c is longest side

Unit # 10

Tangent to a Circle

- If two circles touch each other externally then distance between their centers is equal to sum of radii.
- Two circles with centres C_1 and C_2 , radii of measure r_1 and r_2 such that $m\overline{C_1C_2} = r_1 + r_2$
- If two circles touch each other internally then distance between their centers is equal to difference of radii.
- Two circles with centres C_1 and C_2 , radii r_1 and r_2 such that $m\overline{C_1C_2} = r_1 - r_2$
- Area of Circle = πr^2
- Area of semi circle = $\frac{1}{2} \pi r^2$
- Perimeter or Circumference of Circle = $2\pi r = \pi d$
- Semi Perimeter or Half Circumference of Circle = πr

Unit # 12

Chords and Arcs

- $m\angle AOC = 2m\angle ABC$, where $m\angle AOC$ is central angle and $m\angle ABC$ is circum angle.
The angle
- In a semi-circle is a right angle,
- In a segment greater than a semi circle is less than a right angle,
- In a segment less than a semi-circle is greater than a right angle.
- $m\angle A + m\angle C = 180^\circ$ and $m\angle B + m\angle D = 180^\circ$, where ABCD is cyclic quadrilateral
- $m\angle ACB = m\angle ADB$, where $m\angle ACB$ and $m\angle ADB$ are angles of same segment.

Unit # 13

Angle in a Segment of a Circle

- Perimeter of a regular polygon = $n \times l$, where n is number of sides and l is length of a side.
 - The measure of the external angle of a regular hexagon is $\frac{\pi}{3}$.
 - The measure of the external angle of a regular octagon is $\frac{\pi}{4}$.
- Formula for finding the angle subtended by the side of a n-sided polygon at the centre of the circle = $\frac{360^\circ}{n}$