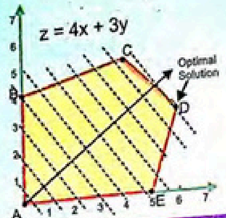
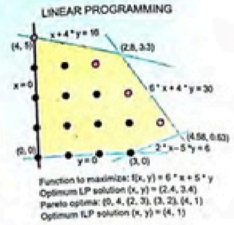
UNIT 9

LINEAR PROGRAMMING





STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Define linear programming (LP) as planning of allocation of limited resources to obtain an optimal result.
- Find algebraic solutions of linear inequalities in one variable and represent them on number line.
- Interpret graphically the linear inequalities in two variables.
- Determine graphically the region bounded by up to 3 simultaneous linear inequalities of non-negative variables and shade the region bounded by them.
- Define
  - linear programming problem,
  - objective function,
  - problem constraints,
  - decision variables.
- Define and show graphically the feasible region (or solution space) of an LP problem.
- Identify the feasible region of simple LP problems.
- Define optimal solution of an LP problem.
- Find optimal solution (graphical) through the following systematic procedure:
  - establish the mathematical formulation of LP problem,
  - construct the graph,
  - identify the feasible region,
  - locate the solution points,
  - evaluate the objective function,
  - select the optimal solution,
  - verify the optimal solution by actually substituting values of variables from the feasible region.
- Solve real life simple LP problems.

9.1 Introduction

In business and industry; the decision makers want to utilize the limited resources in a best possible manner with the view to minimize cost of production and maximize profit. The limited resources may be in the form of capital, labour, money, time manpower, machine capacity, etc. The linear programming is the mathematical method used in decision making in business to maximize the profit or minimize the expenditure subject to certain restrictions which are a result of limitations on resources.

The term programming means planning and refers to a process of determining a particular program. The term linear means that all relationships involved in a particular program which can be solved by this method are linear.

Thus linear programming is a method for solving problems in which a linear function (representing, cost, profit, distance, weight etc.) is to be maximized or minimized. Such problems are usually referred to as **optimization problems** or more commonly known as **linear programming problems.**

The theory of linear programming is a fairly recent advancement in mathematics. It was developed over the past four decades to deal with the increasingly more complicated problems of our technological society.

Linear programming (LP) is planning of allocation of limited resources to obtain an optimal result.

9.2 Linear Inequalities

Recall that an inequality is a statement that one mathematical quantity is less than (or greater than) or less than or equal to (or greater than or equal to) another quantity. Thus, if *a* and b are real numbers, we can compare their positions on the real line by using the relations of less than, greater than, less than or equal to, and greater than or equal to, denoted by inequality symbols $<, >, \leq$ and $\geq$ respectively. The following table describes both algebraic and geometric interpretations of the inequality symbols.

Algebraic Statement	Equivalent Statement	Geometric Statement
$a < b$	*a* is less than b	*a* lies to the left of b.
$a > b$	*a* is greater than b	*a* lies to the right of b.
$a \leq b$	*a* is less than or equal to b	*a* coincides with b or lies to the left of b.
$a \geq b$	*a* is greater than or equal to b	*a* coincides with b or lies to the right of b.

In this section, we shall consider linear inequalities in one variable and two variables. We shall also interpret these inequalities graphically.

9.2.1. Linear inequalities in one variable

Inequalities of the form $ax < b$, $ax \le b$, $ax > b$ or $ax \ge b$ where $a \ne 0$, b are constants are called **linear inequalities in one variable** or **first degree inequalities in one variable.**

For example, $x < -2$, $2x \le 6$, $4 - 3x > -1 - x$, $2x + 5 \ge x - 3$ are linear inequalities in one variable.

The **solutions** of a linear inequality in one variable x are the values of x which satisfy the linear inequality. The set consisting of all solutions of the linear inequality is called the **solution set.**

For example, the solution set of the linear inequality $x > 5$ consists of all values of x that are greater than 5.

We solve a linear inequality in the same way as we solve a linear equation. Following are the steps involved in solving a linear inequality in one variable.

Step I Shift all terms containing x on one side of the inequality.

Step II Shift all other terms on the other side of the inequality.

Step III Simplify the resulting inequality to find the values of x.

Example 1: Solve the linear inequality $x - 5 > 0$.

Solution: Since the only term containing x is on the left side, we need to shift the constant term to the other side. To do this, we add 5 to both sides and then simplify. $x - 5 > 0$

$(x - 5) + 5 > 0 + 5$

$x > 5$

Thus, the solution of the inequality are all values of x that are greater than 5.

The solution set = $\{ x : x \in \mathbb{R} \text{ and } x > 5 \}$

The solution set can also be written alternatively in the form of interval $(5, \infty)$.

Example 2: Solve the inequality $3x - 2 \ge 8 + 5x$

Solution: To solve the given linear inequality, we use step (I)–(III) to obtain the following equivalent inequalities.

$3x - 2 \ge 8 + 5x$

$(3x - 2) - 5x \ge (8 + 5x) - 5x$

$-2x - 2 \ge 8$

$(-2x - 2) + 2 \ge 8 + 2$

Did You Know ?

When both sides of an inequality are multiplied by a negative number, the order (or sense) of the inequality is reversed, that is from <to>, from $\le$ to $\ge$, from $>$ to $<$ or from $\ge$ to $\le$.

$-2x \ge 10$

$\left(-\frac{1}{2}\right)(-2x) \le \left(-\frac{1}{2}\right)(10)$.

$x \le -5$ Thus, the solution set = $\{ x : x \in \mathbb{R} \text{ and } x \le -5 \} = (-\infty, -5]$

Note !

In the above graphical representation of linear inequalities in one variable on the real line, the open (unshaded) circle at the point indicates that the point does not belong to solution set. The filled in (shaded) circle at the point indicates that the point belongs to the solution set.

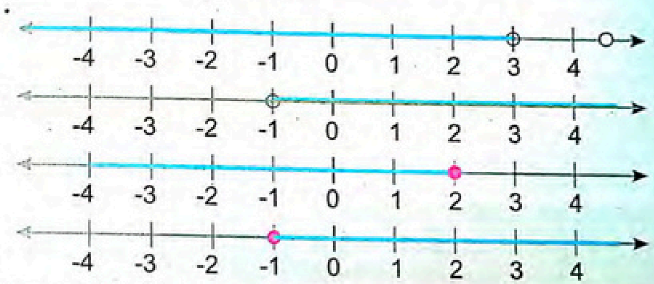
The solutions of linear inequalities in one variable are graphically represented on the real line in the following examples.

(i) $x < 3$

(ii) $x > -1$

(iii) $x \le 2$

(iv) $x \ge -1$



9.2.2. Linear inequalities in two variables

A linear inequality in two variables x and y is an expression of one of the following forms.

(i) $ax + by < c$ (ii) $ax + by > c$

(iii) $ax + by \le c$ (iv) $ax + by \ge c$

where a and b are not both 0 and a, b and c are real numbers.

If $a = 0$ or $b = 0$ in the above inequalities, then the resulting inequalities reduce to the corresponding linear inequalities in one variable.

For example, (i) $3x < 2$ (ii) $4x + 3 \ge 0$ (iii) $x - 2y > 1$ (iv) $5x + 3y \le 1$ are linear inequalities. Inequalities (i) and (ii) are in one variables while (iii) and (iv) are in two variables. With each linear inequality in two variables x and y is associated a linear equation in two variables x and y called the **associated or corresponding equation.**

For example, the associated equation of $ax + by \ge c$ is $ax + by = c$.............. (1)

To find the associated equation of a linear inequality in two variables, simply substitute an "equals" sign for the symbol of inequality. In our later work we will see that the linear equation (1) in two variables represents a straight line.

The **solution set** of an inequality is the set of all numbers, which when substituted for the variable (or variables) in the inequality, make the inequality a true statement. To solve an inequality is to find its solution set.

9.2.2.1 Graphing Inequalities in Two Variables

Since linear inequalities are closely related to linear equations, graphing them is very similar to graphing linear equations. The graph of a linear equation of the form $ax + by = c$ is a line which divides the plane into disjoint regions as stated below.

(1) The set of ordered pairs (x, y) such that $ax + by < c$.

(2) The set of ordered pairs (x, y) such that $ax + by > c$.

The regions (1) and (2) are called **half-planes.** The line $ax + by = c$ that divides the plane is called the **boundary** of both half planes. **(See figure 9.1).** If the boundary line is included in either plane then it is called **closed half plane.** Since a plane has infinite length and breadth, it cannot be completely shown by a figure. Only a segment of the plane has been shown in the figure.

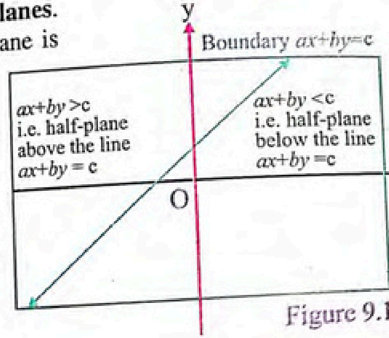


Figure 9.1

Note

A vertical line divides the plane into left and right half–planes while a non– vertical line divides the plane into upper and lower half–planes.

A **Solution** of a linear inequality in two variables x and y is an ordered pair of real numbers (a, b) such that the inequality is satisfied when we substitute $x = a$ and $y = b$.

For example, the ordered pair (–1, 2) is a solution of the inequality $3x + y < 5$, since $3(-1) + 2 = -3 + 2 = -1 < 5$ which is true.

The graph of a linear inequality in two variables x and y is the set of all ordered pairs that satisfy the inequality.

Note

The graph of a single inequality, in more than two variables, is a half–plane.

9.2.2.2 Procedure for Graphing a Linear Inequality in Two variables

To graph a linear inequality, we follow the following procedure.

Step–1: Replace the inequality sign with an equal sign and draw the line. Make the line solid if the inequality involves ≤ or ≥ , make the line dashed if the inequality involves < or >.

Step–2: Choose any point that is not on the line as a test point. If the origin is not on the line, it is the most appropriate choice.

Step–3: Substitute the coordinates of the test point into the original inequality. If the test point satisfies the inequality, shade the half–plane that includes the test point, otherwise, shade the half–plane on the other sides of the line.

Did You Know

If a line intersects x–axis at $(a,0)$, then a is called x–intercept of the line.
If a line intersects y–axis at $(0,b)$, then b is called y–intercept of the line.

Example 3: Graph the inequality $2x - 5y \geq 10$.

Solution: The associated equation of the inequality is

$$2x - 5y \geq 10 \quad \text{(i)}$$

$$2x - 5y = 10 \quad \text{(ii)}$$

Graph the line (ii) by finding x–and y–intercepts.
To find the x– intercept, let y=0.
To find y–intercept, let x=0.

x	y
5	0
0	– 2

We have $2x - 5(0) = 10$

$\Rightarrow x = 5$

and $2(0) - 5y = 10$

$\Rightarrow y = -2$

Therefore, the boundary line passes through (5, 0) and (0,–2).
The line is solid because the inequality involves ≥.
We choose O(0,0) as a test point, because it is not on the line (ii)
Substituting $x = 0, y = 0$ into the original inequality,

$$2x - 5y \geq 10$$

we get $2(0) - 5(0) \geq 10$

$\Rightarrow 0 \geq 10$

which is not true. Therefore the test point does not satisfy the inequality, and so the solution is not the half–plane that includes the origin.
Thus the solution is the half–plane not containing the origin (see figure 9.2).

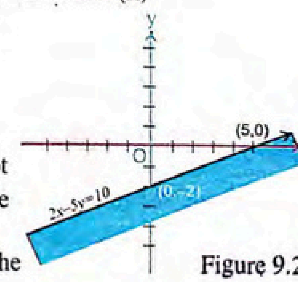


Figure 9.2

Example 4: Graph the inequality $y > x - 4$.

Solution: The associated equation of the inequality is

x	y
4	0
0	− 4

$y > x - 4$ (i)

$y = x - 4$ (ii)

To find the x–intercept put $y=0$ in (ii)

$0 = x - 4 \Rightarrow \quad x = 4$

Similarly to find the y–intercept put $x=0$ in (ii),

$y = 0 - 4$

$\Rightarrow \quad y = -4$

Therefore the boundary line passes through (4, 0) and (0, –4). The line is dashed because the inequality involves>. We choose O (0, 0) as a test point, because it is not on the line (ii).

Substituting $x = 0$, $y=0$ into the original inequality

$y > x - 4$ we get $0 > 0 - 4 \quad \Rightarrow 0 > -4$

which is true. Therefore the solution is the half–plane that includes the origin (see figure 9.3).

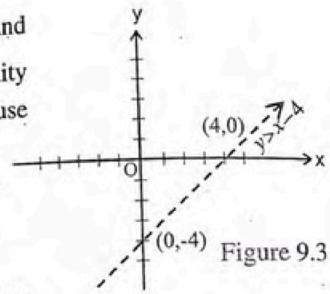


Figure 9.3

9.2.3. Region bounded by 2 or 3 simultaneous inequalities

(i.e. System of Linear Inequalities in Two Variables)

Two or more linear inequalities together form a **system of linear inequalities**. The graph of a system of linear inequalities in two variables x and y is the set of all ordered pairs (x, y) that satisfy simultaneously each of the linear inequalities in the system. Thus, the graph of a system of linear inequalities can be obtained by graphing each inequality individually and then taking intersection of all the graphs. The common region so obtained is called the **solution region** for the system of linear inequalities.

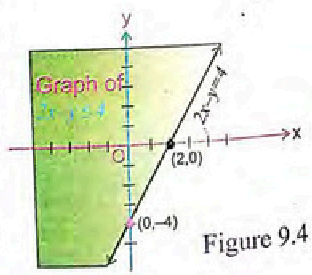


Figure 9.4

Example 5: Graph the system of linear inequalities.

$$\left.\begin{aligned} 2x - y &\le 4 \\ x + y &\ge 2 \end{aligned}\right\}$$

Solution: Following the procedure for graphing linear inequalities, the graph of the line $2x - y = 4$ is drawn by joining the points (2, 0) and (0, –4). The test point (0,0) satisfies the inequality, so the graph of the inequality $2x - y \le 4$ is the upper half–plane including the graph of the line $2x - y = 4$. The closed half–plane is partially shown as a shaded region in Figure 9.4.

The graph of the line $x + y = 2$ is drawn by joining the points (2, 0) and (0, 2). The test pint (0, 0) does not satisfy the original inequality, so the graph of the inequality $x + y \ge 2$ is the closed half–plane not on the origin side of the line $x+y=2$. The closed half–plane is partially shown by shading in the figure 9.5.

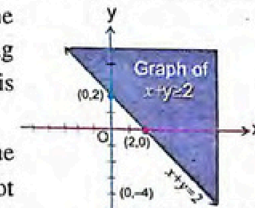


Figure 9.5

The solution region of the given system of linear inequalities is displayed in figure 9.6 by the shaded overlapping region of the graphs shown in figure 9.4 and figure 9.5. The point (2, 0) is the intersection point of the graph of the system of inequalities that can also be found by solving the equations $2x - y = 4$ and $x + y = 2$

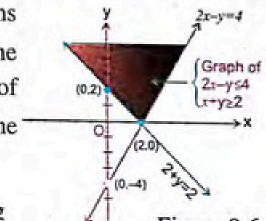


Figure 9.6

Example 6: Graph the solution region of the following system of linear inequalities in each case.

a) $\left.\begin{aligned} 2x - y &\le 4 \\ x + y &\ge 2 \\ -x + 2y &\le 4 \end{aligned}\right\}$ b) $\left.\begin{aligned} x - 2y &\le 6 \\ 2x + y &\ge 2 \\ x + 2y &\le 10 \end{aligned}\right\}$

Solution: (a) The graph of the inequalities $2x - y \le 4$ and $x + y \ge 2$ have already been plotted in figure 9.4 and figure 9.5 respectively and their solution region partially shown in figure 9.6 of example (5).

Following the procedure for graphing of linear inequalities, the graph of the inequality $-x + 2y \le 4$ is partially shown in figure 9.7.

The intersection of the three graphs is the required solution region which is the shaded triangular region ABC (including its sides) shown in figure 9.8.

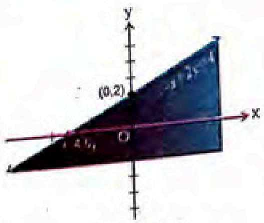


Figure 9.7

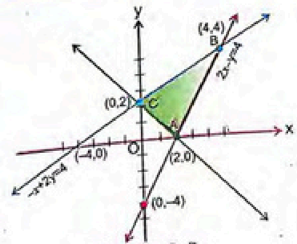


Figure 9.8

(b) The graph of the line $x-2y=6$ is drawn by joining the points (6,0) and (0,–3). Since the test point (0, 0) satisfies the inequality $x-2y \le 6$, thus the graph of $x-2y \le 6$ is the upper half-plane including the graph of the line which is partially shown by a shaded region in **figure 9.9.**

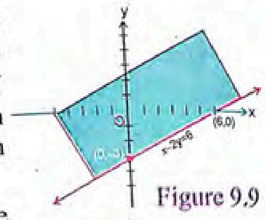


Figure 9.9

The graph of the line $2x + y = 2$ is drawn by joining the points (1, 0) and (0, 2). Since the test Point (0, 0) does not satisfy the inequality $2x + y \ge 2$, thus the graph of $2x+y\ge 2$ is the closed half-plane which is shown partially as shaded region in **figure 9.10.**

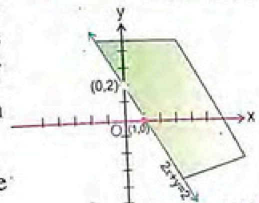


Figure 9.10

The graph of the line $x + 2y = 10$ is drawn by joining the points (10,0) and (0,5). Since the test point (0,0) satisfies the inequality $x+2y \le 10$, thus the graph of $x + 2y \le 10$ is the lower half-plane including the graph of the line which is partially shown by shading in **figure 9.11.**

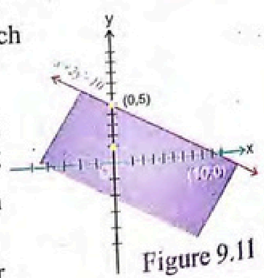


Figure 9.11

The required graph of the solution region of the system is the shaded overlapping triangular region ABC (including its sides) termed by the three graphs as shown in **figure 9.12.**

In **example (6)**, we see that the solution region of either system is the shaded triangular region ABC as solution in **figures 9.8** and **9.12** respectively where A, B and C are the points of the solution regions, obtained by the intersection of its boundary lines. Such points are termed as **corner points** or **vertices** of the solution region. Thus, a point of a solution region where two of its boundary lines intersect, is called a corner point or a vertex of the solution region. The corner points of the solution region can be obtained by solving the associated equations of linear inequalities in pairs.

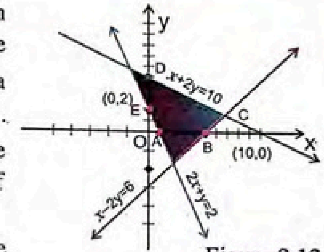


Figure 9.12

For example, in example 6 (a) the following three corner points are obtained by solving the associated equations of the inequalities in pairs.

Associated Equations of Inequalities		Corner Points
$2x-y=4$,	$x+y=2$	A (2, 0)
$2x-y=4$,	$-x+2y=4$	B (4, 4)
$x+y=2$,	$-x+2y=4$	C (0, 2)

The graph of a solution region of the system of linear inequalities may be either bounded or unbounded. The graph of the solution region is **bounded** if it can be enclosed within some circle of sufficiently large radius while the graph of the solution region is **unbounded**, if it cannot be enclosed in any circle how large its radius may be. In example (5), the solution region is unbounded while in example (6), the solution region of both systems (a) and (b) is bounded.

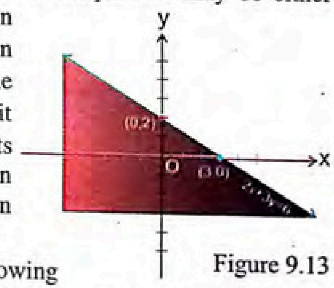


Figure 9.13

Example 7: Graph the solution region of the following system of linear inequalities and find their corner points. Also check whether the graph of the solution region is bounded or not.

$$\left.\begin{aligned} 2x + 3y &\le 6 \\ 2x - 3y &\le 6 \\ x &\ge 0 \end{aligned}\right\}$$

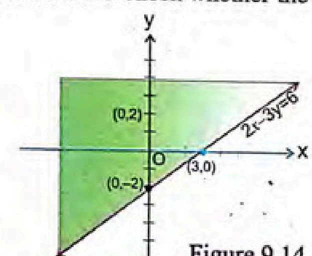


Figure 9.14

Solution: The associated equations of the linear inequalities

$$2x+3y \le 6 \quad \text{and} \quad 2x-3y \le 6$$

are $2x+3y = 6$ (i) and $2x-3y = 6$ (ii)

The graph of line (i) is drawn by joining the points (3, 0) and (0, 2). The test point (0, 0) satisfies the inequality. Thus the graph of the inequality $2x + 3y \le 6$ is the lower half plane including the graph of the line. The closed half plane is partially shown as a shaded region in figure 9.13.

The graph of the line (ii) is drawn by joining the points (3, 0) and (0, –2). The test point (0, 0) satisfies the inequality. Thus the graph of the inequality $2x - 3y \le 6$ is the upper half-plane including the graph of the line. The closed half plane is partially shown by a shaded region in figure 9.14.

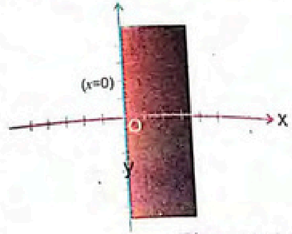


Figure 9.15

The graph of $x \ge 0$ is the right half–plane including the graph of the line $x = 0$ (the y–axis) of the linear inequality $x \ge 0$. The graph of $x \ge 0$ is partially shown in figure 9.15. The solution region of the given system of linear inequalities is the intersection of the graph partially shown in figure 9.13, figure 9.14 and figure 9.15. This region is displayed as the shaded overlapping region in (Figure 9.16).

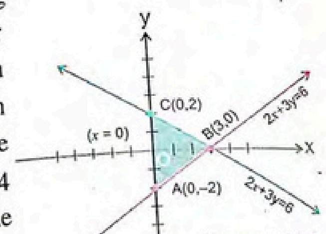


Figure 9.16

The corner points are A(0, –2), B(3, 0) and C(0,2). The graph of the solution region is clearly bounded.

EXERCISE 9.1

1. Solve the following inequalities and graph the solution set in each case

(i) $x + 3 < 7$ (ii) $-3x - 2 \le 4$

(iii) $x + y \le 2$ (iv) $2x - 3y > 6$

2. Graph the following systems of linear inequalities.

(i) $\left.\begin{aligned} 2x - 3y &\le 12 \\ 3x + 2y &\le 6 \end{aligned}\right\}$ (ii) $\begin{aligned} x - y &\le 1 \\ x + y &\ge 4 \end{aligned}$

(iii) $\left.\begin{aligned} 2x + y &\ge 4 \\ x + y &\ge 3 \\ x &\ge 0 \end{aligned}\right\}$ (iv) $\left.\begin{aligned} 2x + y &\le 8 \\ x + y &\le 6 \\ y &\ge 0 \end{aligned}\right\}$

3. Graph the solution region of the following system of linear inequalities and find the corner points in each case. Also tell whether the graph is bounded or unbounded.

(i) $\left.\begin{aligned} 2x + y &\le 6 \\ x + 2y &\le 6 \\ x &\ge 0 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} 2x + 3y &\ge 6 \\ x + y &\ge 4 \\ y &\ge 0 \end{aligned}\right\}$

4. Graph the solution region of the following system of linear inequalities and find the corner points in each case. Also tell whether the graph is bounded or unbounded.

(i) $\left.\begin{aligned} 2x + 3y &\le 12 \\ 3x + y &\le 12 \\ x + y &\ge 2 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} 2x + y &\ge 3 \\ x + y &\le 5 \\ x - y &\ge 2 \end{aligned}\right\}$

9.3 Feasible region

9.3.1 Define linear programming problem, objective function, problem constraints and decision variables

As mentioned earlier, linear programming consists of methods for finding the maximum or minimum value of a linear function in two variables of the form $f(x, y) = ax + by$; $a, b \in \mathbb{R}$,

where the variables x and y are subject to the set of conditions or constraints given in the form of linear inequalities. In order to maximize or minimize the linear function $f(x, y) = ax + by$, called the **objective function,** we need to find points (x, y) that make the function largest (or smallest) possible. Such points occur at the corner of the feasible region as the following theorem asserts.

"The maximum (or minimum) value of the objective function is achieved at one of the corner of the feasible region."

Many practical problems arising in the field of business, economics, the sciences and engineering involve systems of linear inequalities. In such problems the choice of values of the variables is not entirely free but subject to some restrictions or conditions given in the form of linear inequalities. The linear inequalities involved in the problem are called **problem constraints.** The variables used in the system of linear inequalities relating to the problem are non–negative and called **non–negative constraints** or **decision variables**.

The graph of the solution region of the system of linear inequalities

$x-2y\leq 6$
$2x+y\geq 2$
$x+2y\leq 10$

is given in (Figure 9.17). We observe that the solution region of the system of linear inequalities is not always within the first quadrant. However, the solution region can be restricted to the first quadrant if the non-negative constraints $x\geq 0, y\geq 0$ are included in the system of linear inequalities. In example 6 (b), if $x\geq 0$ and $y\geq 0$ are included within the system of linear inequalities, then the solution region can be restricted to the first quadrant. It is the polygonal region ABCDE (including its sides) as shown in Figure 9.18

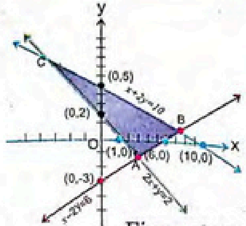


Figure 9.17

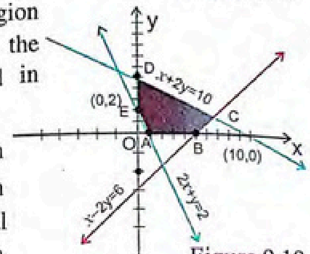


Figure 9.18

9.3.2 A region (which is restricted to the first quadrant) is referred to as a **feasible region.** Each point of the feasible region is called **feasible solution** of the system of linear inequalities (or for the set of given constraints). In other words any ordered pair (x, y) that satisfies all the constraints is called a **feasible solution** of the system of linear inequalities and the set of all feasible solutions is called a **feasible solution set.**

Example 8: Graph the feasible region of the following system of linear inequalities.

$$\left.\begin{aligned} 3x+5y &\leq 15 \\ -x+3y &\leq 3 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}\right\}$$

Solution: The associated equations for the inequalities

$3x+5y\leq 15$ and $-x+3y\leq 3$

are $3x+5y=15$ (i) and $-x+3y=3$ (ii)

The graph of line (i) is drawn by joining the points (5, 0) and (0, 3) by a solid line.

Similarly, the graph of line (ii) is drawn by joining the points (–3, 0) and (0,1) by a solid line.

Since the test point (0, 0) satisfies both the inequalities $3x+5y\leq 15$ and $-x+3y\leq 3$, so both the closed half–planes are on the origin sides of line (i) and (ii).

The intersection of these closed half– planes is the shaded overlapping region as shown in Figure 9.19

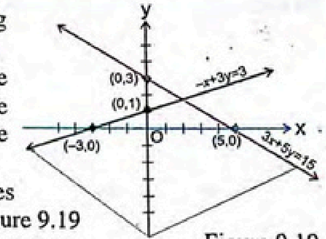


Figure 9.19

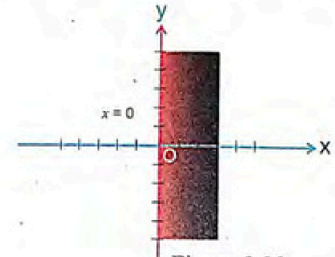


Figure 9.20

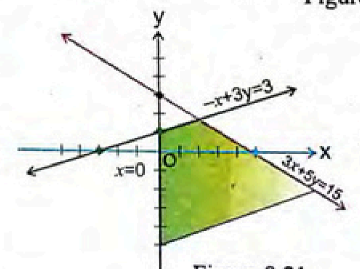


Figure 9.21

The graph of $x\geq 0$ is partially shown in Figure 9.20. The intersection of graphs shown in Figure 9.19 and Figure 9.20 is partially displayed as a shaded region in Figure 9.21.

The graph of $y\geq 0$ is partially plotted in Figure 9.22.

The intersection of graphs shown in Figure 9.19 and Figure 9.22 is partially displayed as shaded region in Figure 9.23.

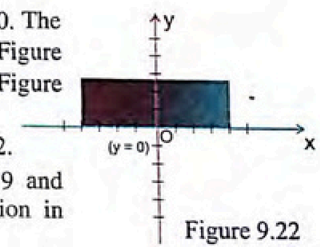


Figure 9.22

The graph of the given system of linear inequalities is the intersection of the graphs shown in Figure 9.21 and Figure 9.23 which is indicated as shaded region in Figure 9.24. This shaded region is the required feasible region of the given system of linear inequalities.

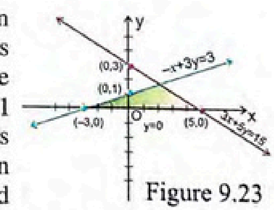


Figure 9.23

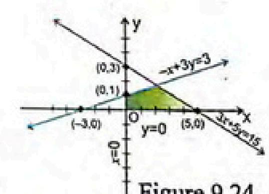


Figure 9.24

Example 9: Graph the feasible region subject to the following constraints.

(a) $\left.\begin{aligned} 3x - 4y &\le 12 \\ 3x + 2y &\ge 6 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}\right\}$ (b) $\left.\begin{aligned} 3x - 4y &\le 12 \\ 3x + 2y &\ge 6 \\ x + 2y &\le 10 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}\right\}$

Did You Know ?

The feasible solution region in example 9(a) is unbounded while the feasible solution region in example 9(b) is bounded.

Solution: (a) The associated equations for the inequalities $3x - 4y \le 12$ and $3x + 2y \ge 6$ are $3x - 4y = 12$ (i) and $3x + 2y = 6$ (ii)

The graph of line (i) is drawn by joining the points (4, 0) and (0, –3) by a solid line. Since the test point (0, 0) satisfies the inequality $3x - 4y \le 12$, so the graph of $3x - 4y \le 12$ is the closed half–plane on the origin side of line $3x - 4y = 12$. The graph of system

$3x - 4y \le 12$
$x \ge 0$
$y \ge 0$

is partially shown as shaded region in Figure 9.25.

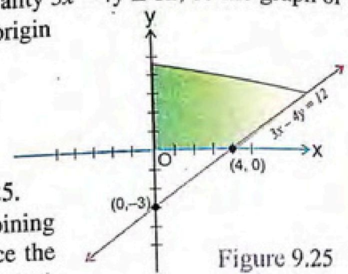


Figure 9.25

Similarly, the graph of line (ii) is drawn by joining the points (2, 0) and (0, 3) by a solid line. Since the test point does not satisfy the inequality $3x + 2y \ge 6$, so the graph of $3x + 2y \ge 6$ is the closed half–plane not on the origin side of the line $3x + 2y = 6$. The graph of system

$3x + 2y \ge 6$
$x \ge 0$
$y \ge 0$ is partially drawn as shaded region in Figure 9.26.

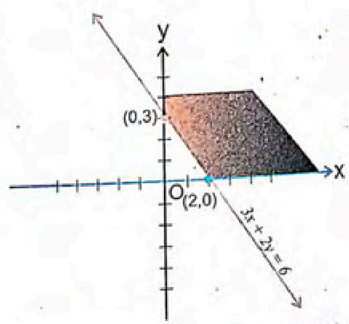


Figure 9.26

The graph of the system

$3x - 4y \le 12$
$3x + 2y \ge 6$
$x \ge 0$
$y \ge 0$

is the intersection of the graphs shown in (Figure 9.25) and figure 9.26 and it is partially displayed in (Figure 9.27) as shaded region. This shaded region in the graph of the feasible region subject to the given constraints.

Corner Points: (2, 0), (4, 0), (0, 3)

(b) The graph of the system

$3x - 4y \le 12$
$3x + 2y \ge 6$
$x \ge 0$
$y \ge 0$ is partially shown in Figure 9.27

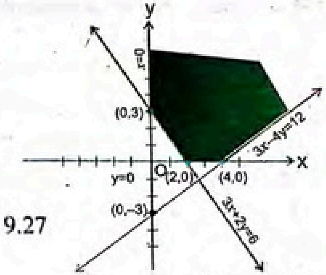


Figure 9.27

The graph of the system

$x + 2y \le 10$
$x \ge 0$
$y \ge 0$ is shown by shaded region in figure 9.28

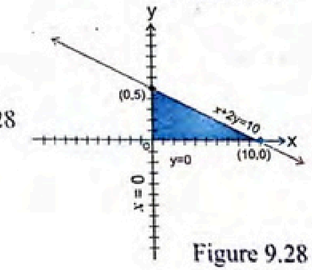


Figure 9.28

The graph of the system

$3x - 4y \le 12$
$3x + 2y \le 6$
$x + 2y \le 10$
$x \ge 0$
$y \ge 0$ is the intersection of the graphs shown in figure 9.27 and figure 9.28 and it is indicated in figure 9.29 as shaded region.

Corner Points:

(2, 0), (4, 0), $\left(\frac{4}{5}, \frac{9}{5}\right)$, (0, 5) (0, 3)

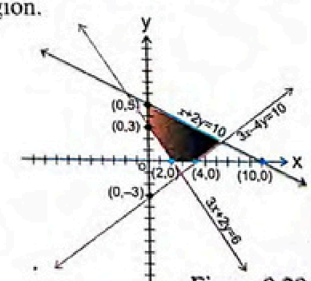


Figure 9.29

9.4. Optimal solution

9.4.1 There are infinitely many feasible solutions in the feasible region. The feasible solution which maximizes or minimizes the objective function is called the **Optimal Solution.**

The procedure for finding the optimal solution (maximum or minimum value) of the objective function $f(x, y) = ax + by$, subject to a set of linear constraints (inequalities) in variables x and y is as following:

9.4.2 Procedure for determining optimal solution

Step–1: Determine the feasible region by graphing the linear inequalities that form the constraints.

Step–2: Find the corner points of feasible region by solving two equations at a time of the boundary lines of the feasible region.

Step–3: Compute the value of the objective function $f(x,y) = ax+by$ at each of the corner points.

Step–4: To find the optimal solution, select the largest value computed in step-3 if $f(x,y) = a\,x+by$ has to be maximized, and select the smallest value if $f(x, y) = ax + by$ has to be minimized.

Example 10: Find the maximum and minimum values of the function $f(x, y) = 2x + 3y$ subject to the constraints

$$3x - y \geq -1$$
$$x + y \leq 5$$
$$x \geq 0$$
$$y \geq 0$$

Solution: The graph of the inequality $3x - y \geq -1$ is the closed half–plane on the origin side of the line $3x - y = -1$ and the graph of the inequality $x + y \leq 5$ is the closed half–plane also on the origin side of the line $x + y = 5$.

The graph of the system

$$3x - y \geq -1$$
$$x + y \leq 5$$
$$x \geq 0$$
$$y \geq 0$$

is shown as a shaded region in Figure 9.30. This shaded region is the feasible region. We see that the feasible region is bounded and its corner points are O(0,0), A (5, 0), B(1, 4) and C (0,1). Evaluating the given function $f(x, y)$ at the corner points, we get

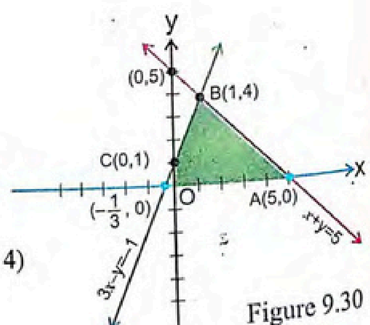


Figure 9.30

$$f(0, 0) = 2(0) + 3(0) = 0$$
$$f(5, 0) = 2(5) + 3(0) = 10$$
$$f(1, 4) = 2(1) + 3(4) = 14$$
$$f(0, 1) = 2(0) + 3(1) = 3$$

Thus the minimum value of $f(x, y)$ is 0 at the corner point O(0, 0) and the maximum value of $f(x, y)$ is 14 at the corner point (1, 4).

Example 11: Find the maximum and minimum value of the function $f(x,y) = 4x + 2y$ subject to the constraints

$$x + 2y \leq 8$$
$$x + y \leq 5$$
$$2x + y \leq 8$$
$$x \geq 0$$
$$y \geq 0$$

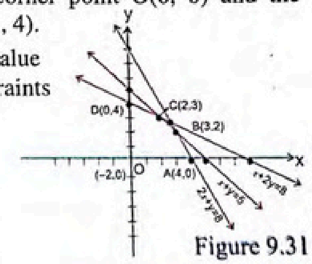


Figure 9.31

Solution: The solution region of the system

$$x + 2y \leq 8$$
$$x + y \leq 5$$
$$2x + y \leq 8$$
$$x \geq 0$$
$$y \geq 0$$

> **Note**
>
> In example 11, the function f(x, y) has maximum value at two corner points (4,0) and (3,2). It follows that f(x, y) has maximum value at all the points of the line segment between the points (3,2) and (2,3).

is the shaded region OABCD shown in figure 9.31. We see that the feasible region is bounded and its corner points are O(0, 0), A (4, 0), B (3, 2), C(2,3) and D (0, 4). We compute the values of the function f(x, y) at the corner points to find its maximum and minimum values. The value of f(x, y) at the corner points are given in the following table.

Corner Points	$f(x, y) = 4x + 2y$
(0, 0)	$f(0, 0) = 4(0) + 2(0) = 0$
(4, 0)	$f(4, 0) = 4(4) + 2(0) = 16$
(3, 2)	$f(3, 2) = 4(3) + 2(2) = 16$
(2, 3)	$f(2, 3) = 4(2) + 2(3) = 14$
(0, 4)	$f(0, 4) = 5(0) + 2(4) = 8$

From the above table, we see that the minimum value of the function f(x,y) is 0 at the corner point (0, 0) and the maximum value of f(x, y) is 16 at the corner points (4, 0) and (3,2).

9.4.3. Real life LP Problems

To solve a linear programming problem, first formulate a mathematical model of the problem and then use the procedure given in section 9.4 to solve it.

Mathematical formulation of a linear programming problem

The mathematical formulation of a linear programming problem involves the following basic steps:

Step 1 Identify the decision variable and assign symbol x and y to them. These decision variables are those quantities whose values we wish to determine.

Step 2 Identify the set of constraints and express them as linear equations / inequations in terms of the decision variables. These constraints are the given conditions.

Step 3 Identify the objective function and express it as a linear function of decision variables. It might take the form of maximizing profit or production or minimizing cost.

Step 4 Add the non-negativity restrictions on the decision variables, as in the physical problems, negative values of decision variables have no valid interpretation.

Example 12: A furniture dealer deals in only two items, viz., tables and chairs. He has Rs. 10,000 to invest and a space to store at most 60 pieces. A table costs him Rs. 500 and chair Rs. 200. He can sell a table at a profit of Rs. 50 and a chair at a profit of Rs. 15. Assume that he can sell all the items that he buys. Formulate this problem as on LPP, so that he can maximize the profit.

Solution: Let x and y denote the number of tables and chairs respectively (x and y are decision variables).

The cost of x tables = Rs. $500x$, The cost of y tables = Rs. $200y$

Therefore, the total cost of x tables and y chairs = Rs. $(500x + 200\ \ )$, which cannot be more than 10000. Thus $500x + 200y \leq 10000$ (Constraint)

Also, $x + y \leq 60$ (constraint) as the dealer has the space to store at the most 60 items. It is obvious that $x \geq 0$, $y \geq 0$ (non-negative restrictions) as the number of tables and chairs cannot be negative.

Profit on x tables = $50x$, Profit on y chairs = $15y$

Hence, the profit on x tables and y chairs = Rs. $50x + 15y$ (objective function).

Obviously, the dealer wishes to maximize the profit $Z = 50x + 15y$

Thus, the mathematical formulation of the LPP is

Maximize $Z = 50x + 15y$ subject to the constraints

$$5x + 2y \leq 100$$
$$x + y \leq 60$$
$$x \geq 0,\ y \geq 0$$

Example 13: A factory produces two types of food containers A and B by using two machines M_1 and M_2. To produce container A, M_1 works 2 minutes and M_2, 4 minutes. Similarly, to produce container B, M_1, works 8 minutes and M_2, 4 minutes. The profit for container A is Rs. 29 and for B is Rs. 45. How many container of each type should be produced so that a maximum profit can be achieved?

Solution: Let x = the number of container A per minute and

y = the number of container B per minute.

If per hour production of M_1 and M_2 is x container A and y container B, then the profit per hour is given by the profit function $P(x, y) = 29x + 45y$.

The constraints are

$2x + 8y \leq 60$ (Resulting from machine M_1)

$4x + 4y \leq 60$ (Resulting from machine M_2)

$$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}\right\} \text{(since container cannot be negative)}$$

The above system of linear inequalities/ constraints can be written in the following simplified form

$$x + 4y \leq 30$$
$$x + y \leq 15$$
$$x \geq 0$$
$$y \geq 0$$

We maximize the profit function P under the given constraints.

As before, graphing the linear inequalities, we obtain the feasible region OABC which is shaded in Figure 9.32. Solving the equations $x + 4y = 30$ and $x + y = 15$, we get $x = 10$, $y = 5$, that is, their point of intersection is (10, 5). Thus, the corner points of the feasible region are O(0,0), A(15,0), B(10,5) and C $\left(0, \frac{30}{4}\right)$.

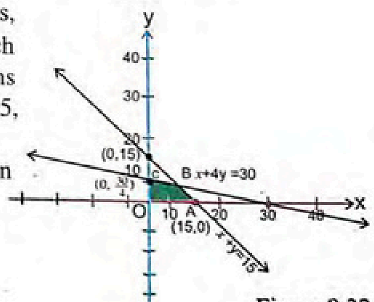


Figure 9.32

We find the value of the function P at the corner points.

Corner Points	P (x, y) = 29x + 45y
(0, 0)	P (0, 0) = 29 (0) + 45 (0) = 0
(15, 0)	P (15, 0) = 29 (15) + 45 (0) = 435
(10, 5)	P (10, 5) = 29 (10) + 45 (5) = 515
$(0, \frac{30}{4})$	$P(0, \frac{30}{4}) = 29(0) + 45(\frac{30}{4}) = 337.50$

From the above table, we see that the maximum profit is Rs. 515 per hour at the corner point B (10, 5). Thus, the optimal production plan that maximizes the profit is achieved by producing 10 containers of A and 5 containers of B.

Example 14: A farmer possesses 80 acres of land and wish to grow two types of crops A and B. Cultivation of crop A requires 3 hours per acres and cultivation of crop B requires 2 hours per acres. Working hours cannot exceed 180. If he gets a profit of Rs. 50 per acres for crop A and Rs.40 per acre for crop B, then how many acres of each crop should be cultivated to maximize his profit.

Solution: Let x = Acres required for cultivation of crop A

and y = Acres required for cultivation of crops B.

If P(x, y) is the profit function, then

$$P(x, y) = 50x + 40y$$

The constraints are

$x + y \le 80$ (Restriction of land)

$3x + 2y \le 180$ (Restriction due to time)

$\left.\begin{array}{l} x \ge 0 \\ y \ge 0 \end{array}\right\}$ (Non–negative constraints, since acres cannot be negative)

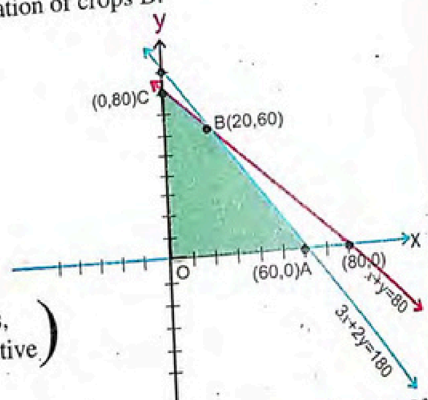


Figure 9.33

Graphing the inequalities, we obtain the feasible region OABC which is displayed by shading in **figure 9.33**. Solving $x + y = 80$ and $3x + 2y = 180$, we get x=20 and y= 60, that is their point of intersection is (20, 60). Thus the corner points of the feasible region are O(0, 0), A (60, 0), B (20, 60) and C (0, 80).

We find the values of the function P at the corner points.

Corner Points	P (x, y) = 50x + 40y
(0, 0)	P (0, 0) = 50 (0) + 40 (0) = 0
(60, 0)	P (60, 0) = 50 (60) + 40(0) = 3000
(20, 60)	P (20, 60) = 50 (20) + 40 (60) = 3400
(0, 80)	P (0, 80) = 50 (0) + 40 (80) = 3200

From the above table, we see that the maximum profit is Rs. 3400 at the corner point (20, 60). Thus, the farmer will get the maximum profit if he cultivates 20 acres of crop A and 60 acres of crop B.

EXERCISE 9.2

1. Graph the feasible region of the following system of linear inequalities and also find the corner points.

(i) $\left.\begin{array}{r} 2x + y \le 6 \\ 4x + y \le 8 \\ x \ge 0 \\ y \ge 0 \end{array}\right\}$ (ii) $\left.\begin{array}{r} 3x - y \ge -4 \\ x + y \le 5 \\ x \ge 0 \\ y \ge 0 \end{array}\right\}$

(iii) $\left.\begin{array}{r} 2x + y \ge 6 \\ 2x + 3y \le 12 \\ -x + y \le 2 \\ x \ge 0 \\ y \ge 0 \end{array}\right\}$ (iv) $\left.\begin{array}{r} x + y \ge 3 \\ 2x + 3y \le 12 \\ x - y \le 2 \\ x \ge 0 \\ y \ge 0 \end{array}\right\}$

2. (i) Maximize f(x, y) = 2x + y subject to the constraints

$$x + y \le 6$$
$$x + y \ge 1$$
$$x, y \ge 0$$

(ii) Maximize f(x, y) = 3x + 5y subject to the constraints

$$2x + 3y \le 12$$
$$3x + 2y \le 12$$
$$x + y \ge 2$$
$$x \ge 0$$
$$y \ge 0$$

3. (i) Find the maximum and minimum values of the function $f(x,y)=5x+2y$ subject to the constraints

$$\begin{aligned} 2x+y &\geq 2 \\ x+2y &\leq 10 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

(ii) Find the maximum and minimum values of the function $f(x,y)=7x+21y$ subject to the constraints

$$\begin{aligned} 2x+y &\geq 2 \\ 2x+3y &\leq 6 \\ x+2y &\leq 8 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

4. A company manufactures two models of bicycles, model A and model B by using two machines M_1 and M_2. Machine M_1 has at most 120 hours available and machine M_2 has a maximum of 144 hours available. Manufacturing a model A bicycle requires 5 hours in machine M_1 and 4 hours in machine M_2 and manufacturing of a model B bicycle requires 4 hours in machine M_1 and 8 hours in machine M_2. If the company gets profit of Rs. 40 per model A bicycle and profit of Rs. 50 per model B bicycle, how many of each model should be manufactured for maximum profit?

5. A machine can produce product A by using 2 units of chemical and 1 unit of a compound or can produce product B by using 1 unit of chemical and 2 units of the compound. Only 800 units of chemical and 1000 units of the compound are available. The profit per unit of A and B are Rs. 30 and Rs. 20 respectively. Determine how many units of each product should be produced to achieve the maximum profit.

6. A company manufactures and sells two models of lamps, L_1, L_2, Use the following table to determine how many of each type of lamps should be produced to achieve a maximum profit?

	Model L_1	Model L_2	Maximum Time Available
Manufacturing time per lamp	2 hours	1 hour	40 hours
Finishing time per lamp	1 hour	1 hour	32 hours
Profit per lamp.	Rs. 70	Rs. 50	—

REVIEW EXERCISE 9

1. **Choose the correct option**

(i) The solution of the system of inequalities $x \geq 0$, $x-5 \leq 0$ and $x \geq y$ is a polygonal region with the vertices as

(a) (0,0), (5,0), (5,5) (b) (0,0), (0,5), (5,5)
(c) (5,5), (0,5), (5,0) (d) (0,0), (0,5), (5,0)

(ii) Find the profit function p if it yields the value 11 and 7 at (3,7) and (1,3) respectively

(a) $p=-8x+5y$ (b) $p=8x-5y$
(c) $p=8x+5y$ (d) $p=-(8x+5y)$

(iii) The vertices of closed convex polygon representing the feasible region of the objective function are (6, 2), (4, 6), (5, 4) and (3, 6). Find the maximum value of the function $f=7x+11y$

(a) 64 (b) 79 (c) 94 (d) 87

(iv) Which of the following is a point in the feasible region determined by the linear inequations $2x+3y \leq 6$ and $3x-2y \leq 16$?

(a) (4, −3) (b) (−2, 4) (c) (3, −2) (d) (3, −4)

(v) The maximum value of the function $f=5x+3y$ subjected to the constraints $x \geq 3$ and $y \geq 3$ is ____________

(a) 15 (b) 9 (c) 24 (d) does not exist

(vi) Maximize $5x+7y$, subject to the constraints $2x+3y \leq 12$, $x+y \leq 5$, $x \geq 0$ and $y \geq 0$

(a) 29 (b) 30 (c) 28 (d) 31

2. Maximize $Z=4x+3y$ subject to the constraints

$$\left.\begin{aligned} 3x+4y &\leq 24 \\ 8x+6y &\leq 48 \\ x &\leq 5 \\ y &\leq 6 \\ x,y &\geq 0 \end{aligned}\right\}$$