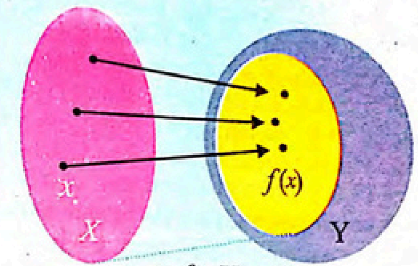
2. *Find the middle term in the expansion of* $(2x^3 + 3y)^8$
3. *What is the coefficient of the fourth term in the expansion of* $(2x - 4y)^7$?
4. 2^7xy^3 *is a term in the expansion of* $(ax + 2y)^4$. Find a.
5. What is the constant term in the expansion of $\left(\frac{2}{x^2} + \frac{x^2}{2}\right)^{10}$
6. Find an approximation of $(0.99)^5$ using the first three terms of its expansion.
7. For every positive integer n, prove that $7^n - 3^n$ is divisible by 4.
8. Prove that $(1+x)^n \geq (1+nx)$, for all natural number n where $x > -1$

1																
1	1															
1	2	1														
1	3	3	1													
1	4	6	4	1												
1	5	10	10	5	1											
1	6	15	20	15	6	1										
1	7	21	35	35	21	7	1									
1	8	28	56	70	56	28	8	1								
1	9	36	84	126	126	84	36	9	1							
1	10	45	120	210	252	210	120	45	10	1						
1	11	55	165	330	462	462	330	165	55	11	1					
1	12	66	220	495	792	924	792	495	220	66	12	1				
1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1			
1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1		
1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1	
1	16	120	560	1820	4368	8008	11440	12870	11440	8008	4368	1820	560	120	16	1

UNIT 8 FUNCTIONS AND GRAPHS



$f : X \rightarrow Y$

STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Recall
  - function as a rule of correspondence,
  - domain, co-domain and range of a function,
  - one to one and onto functions.
- Know linear, quadratic and square root functions.
- Define inverse functions and demonstrate their domain and range with examples.
- Sketch graphs of
  - linear functions (e.g. $y = ax + b$),
  - non-linear functions (e.g. $y = x^2$).
- Sketch the graph of the function $y = x^n$ where n is
  - a + ve integer,
  - a −ve integer (x),
  - a rational number for $x > 0$.
- Sketch graph of quadratic function of the form $y = ax^2 + bx + c$, $a(\neq 0)$, b, c are integers.
- Sketch graph using factors.
- Predict functions from their graphs (use the factor form to predict the equation of a function of the type $f(x) = ax^2 + bx + c$, if two points where the graph crosses x-axis and third point on the curve, are given).
- Find the intersecting point graphically when intersection occurs between
  - a linear function and coordinate axes,
  - two linear functions,
  - a linear and a quadratic function.
- Solve, graphically, appropriate problems from daily life.

8.1. Introduction

In many practical situations the value of one quantity depends on the value of another quantity. Such dependence of one quantity on another is described mathematically as **function.** For example, one of the indicators on the dashboard of a car shows that the amount of petrol in gallons in the tank is decreasing and another indicator shows that the distance travelled in **kilometers** is increasing. In this example, we observe that there are two variable quantities and there is a relation between them. The variable quantities are the number of gallons of petrol in the tank and the number of **kilometers** travelled. Thus, the distance travelled in **kilometers** is the function of numbers of gallon of petrol in the tank.

As another example, the temperature of air throughout the day depends on the instant of time, so we can say that temperature of air is a function of instant of time. In general, if a variable denoted by y (say) is associated in a definite way with a variable x, then y is said to be a function of x.

> **Remember**
>
> A constant is a symbol that always represents the same number, on the other hand, A variable is a symbol that may represents different values in the same problem.

To be more specific,"If the values of y depend on x in such a way that each value of x determines exactly one and only one value of y, then y is a function of x".

Symbolically, we write $\quad y = f(x)$. $\quad$ (1)

Which reads as"y is a function of x or simply y is equal to f of x". In equation (1) the variable x is called the **independent variable** (or **argument**) whereas y is called the **dependent variable.**

8.1.1 Function as a rule or correspondence

In this section, we give formal definition of a function.

A function from a set X to a set Y is a rule or correspondence that assigns to each element x in X a unique element y in Y. Symbolically, we write it as $f: X \rightarrow Y$ and read as "f is a function from X to Y".

The elements of X are called **pre-images** and the corresponding elements of Y are called the **images.** If $y \in Y$ is an image of $x \in X$ under the functions f, we write it as $y = f(x)$. Equivalently, we say that y is the **value** of the function f at x, see **(Figure 8.1).**

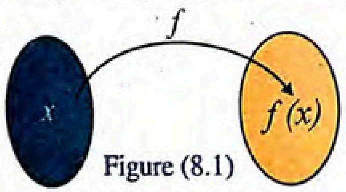


Figure (8.1)

Illustration: The following is a function, which relates the time of day to the temperature.

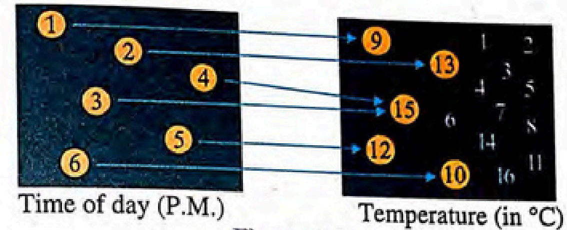


Figure (8.2)

Example 1: Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$. State whether or not the rules indicated by the following figures are functions from X to Y.

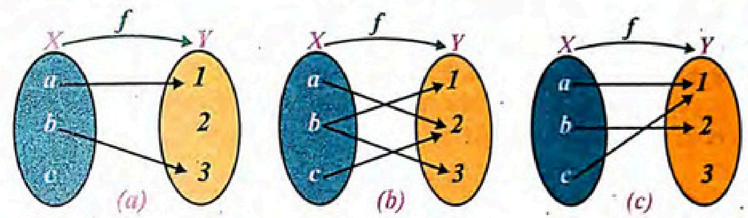


Figure (8.3)

Solution:

(1) The figure *(a)* does not define a function, because the element c of the set X has not been assigned any element of Y.

(2) The figure *(b)* does not define a function, because the element b of X has been assigned two elements of Y.

(3) The figure *(c)* does define a function, because every element of X has been assigned a unique element of Y. It may be noted that definition of function does not require that each element of Y should be an image of some element of X.

Example 2: Evaluating a function

Let $g(x) = -x^2 + 4x + 1$.

Find each function value. a. $g(2)$ b. $g(t)$ c. $g(x+2)$

Solution:

a. Replacing x with 2 in $g(x) = -x^2 + 4x + 1$ yields the following.

$$g(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5$$

b. Replacing x with t yields the following.

$$g(t)=-(t)^2+4(t)+1=-t^2+4t+1$$

c. Replacing x with $x+2$ yields the following.

$$g(x+2)=-(x+2)^2+4(x+2)+1$$
$$=-(x^2+4x+4)+4x+8+1$$
$$=-x^2-4x-4+4x+8+1$$
$$=-x^2+5$$

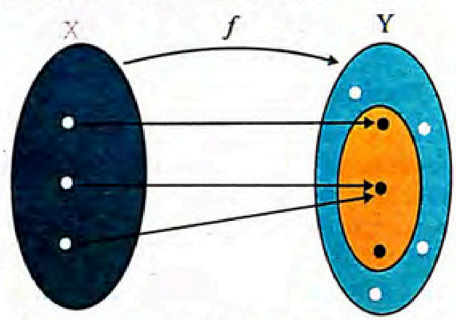
For You Information

Although f is often used as a convenient function name and x is often used as the independent variable, other letters can also be used. For example, $f(x)=x^2-7x+12$, $f(t)=t^2-7t+12$, and $g(s)=s^2-7s+12$ all define the same function.

8.1.2 Domain and Range of a Function

Let $f: X \to Y$ be a function from a set X to a set Y. Then set X is called **domain** and the set Y is called **codomain** of the function f. The set of all those elements of Y which f is assuming is called **range** of the function f.

If the domain is not specified, then it is assumed to be the set of all real numbers. If f is a function of X into Y, the range is a subset of Y but need not be all of Y. This has been shown in (Figure 8.4).



Domain Figure (8.4) ***Range*** $\subseteq$ codomain

Did You Know

Function Notation

$y=f(x)$, f is the *name* of the function.

y is the dependent variable.

x is the independent variable.

y is the *value of the function at* x

8.1.3 One-to-one and onto Function

(a) A function $f: X \to Y$ is said to be **one-to-one** (or **injective**) if distinct elements of X have distinct images in Y i.e. if x_1 and x_2 are distinct elements of X, then $f(x_1) \neq f(x_2)$ in Y. Equivalently, if $f(x_1)=f(x_2)$, then $x_1=x_2$.

Sometimes we write 1-1 function for one-to-one function.

(b) A function $f: X \to Y$ is said to be **onto** (or **surjective**) if each element of Y is the image of some element in X i.e. the range of f is the whole set Y.

A function f which is both one – to – one and onto is called bijective function. Consider the functions f and g as shown in [Figure (8.5) (i) and (ii)].

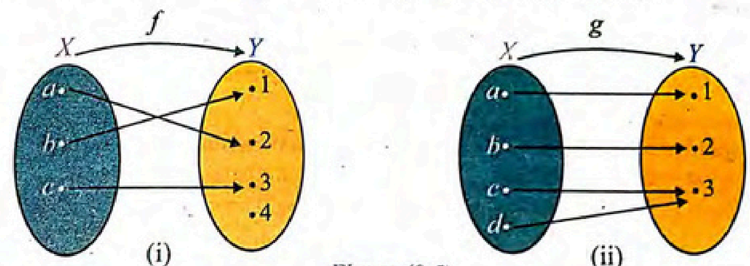


Figure (8.5)

Figure (i) represents a function f which is one –to – one but not onto (why?)

Figure (ii) represents a function g which is onto but not one-to-one (why?)

Example 3: Show that the function $f: I\!R \to I\!R$ defined by $f(x)=3-5x$ is both one-to-one and onto i.e. bijective.

Solution: For any two elements x_1 and x_2 of X, we have

$$f(x_1)=3-5x_1 \text{ and } f(x_2)=3-5x_2$$

If $f(x_1)=f(x_2)$, then $3-5x_1=3-5x_2 \Rightarrow x_1=x_2$.

Thus f is one-to-one.

Now the range of $f(x)=3-5x$ is the whole set $I\!R$ so it is onto.

Hence f is both one-to-one and onto i.e bijective.

Example 4: Show that the function $g: I\!R \to I\!R$ defined by $g(x)=2x^2+1$ is neither one-to-one nor onto.

Solution: The function $g(x)=2x^2+1$ is not one-to-one, because $g(-2)=2(-2)^2+1=9=2(2)^2+1=g(2)$, that is -2 and 2 both have the same image 9.

Now the range of $g(x)=2x^2+1$ is the set of real numbers greater than or equal to 1, that is, Range $g=[1,\infty) \neq I\!R$, so g is not onto function. Thus g is neither one-to-one nor onto.

8.1.4 Linear, Quadratic and Square Root Functions

We begin with the definition of:

(a) Linear Functions

A function f is a **linear function** if it can be written as $f(x)=mx+b$, where m and b are constants.

(If m =0, the function is a **constant function** $f(x) = b$, if m =1 and b= 0, the function is the **identity function** $f(x) = x$)

For example,

$$f(x)=x+1, g(x)=-3x+4, h(x)=3x-8 \text{ are linear functions.}$$

The domain of a linear function is the set of all real numbers.

(b) Quadratic Functions

A **quadratic function** f is a function that can be written in the form $f(x)= ax^2 + bx + c$, $a \neq 0$, where a, b and c are real numbers.

For example, $f(x)=3x^2+4x+1$, and $g(x)=5x^2-x-7$ are quadratic functions.

The domain of quadratic function is the set of all real numbers.

(c) Square Root Function

A function of the form $f(x)=\sqrt{x}$ where $x \geq 0$, is called a square root function.

The domain of square root function is the set of all non-negative real numbers.

8.2 Inverse Function

Let $f: X \to Y$ be a one-to-one and onto function. Then for each element in the domain of f, there is a unique element in the range of f and for each element in the range of f, there is a unique element in the domain of f. In this case the correspondence $f^{-1}: Y \to X$ is also a function, which is called an inverse function of f. Thus the inverse function f^{-1} of f is defined by

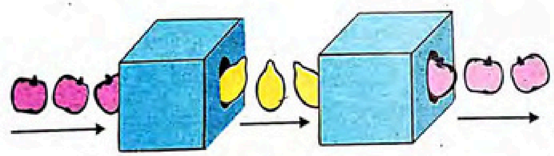


Figure (8.6)

$$x = f^{-1}(y), \forall y \in Y \text{ if and only if } y = f(x), \forall x \in X$$

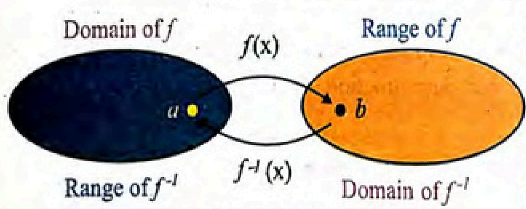


Figure (8.7)

Remember

(i) Not every function has an inverse

(ii) A function has an inverse if and only if it is 1-1 and onto

It is evident that $(f^{-1})^{-1} = f$. Thus f and f^{-1} are inverses of each other. The above figure illustrates the concept of inverse function.

8.2.1 Domain and Range of Inverse Functions

It is clear from the definition of inverse function f^{-1} that *domain* f^{-1} = *range* f and *range* f^{-1} = *domain* f

Example 5: If $f: X \to Y$ is given by

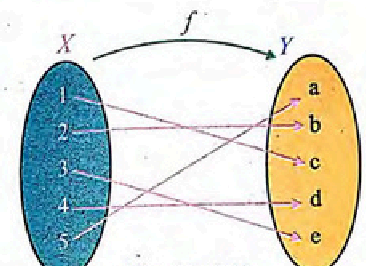


Figure (8.8)

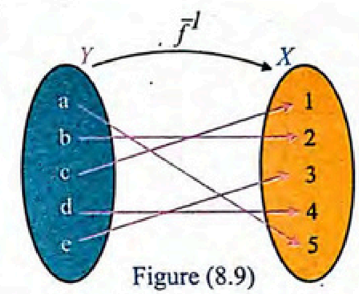


Figure (8.9)

Find f^{-1}.

Solution: Since f is both one-to-one and onto, so it's inverse exists, shown in the (Figure 8.9). We note that f^{-1} is also bijective.

Algebraic method for finding the inverse of a function

If the function f is given by a simple formula, then the inverse function f^{-1} can be found by an algebraic method which involves the following steps.

Step-I Write $y = f(x)$

Step-II Solve the equation in step-I for x in terms of y.

Step-III In the resulting equation in step-II, replace x by $f^{-1}(y)$.

Step-IV Replace each y in the result of step-III by x to get $f^{-1}(x)$

Step-V Check the answer by verifying that $f^{-1}(f(x)) = x$.

Example 6: Let $f = \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x)=2x-1$, find $f^{-1}(x)$.

Solution: We have $f(x)=2x-1$

Step-I Write $f(x)=2x-1=y$

Step-II Then $2x-1=y$

$$\Rightarrow \quad 2x=y+1 \Rightarrow \quad x=\frac{y+1}{2}$$

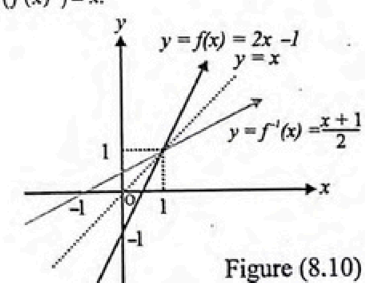


Figure (8.10)

Step-III. Replace x by $f^{-1}(y)$ so that

$$f^{-1}(y)=\frac{y+1}{2}$$

Step-IV. To find $f^{-1}(x)$, replace y by x, we have

$$f^{-1}(x)=\frac{x+1}{2}$$

Step-V. **Verification:** $f^{-1}(f(x))=f^{-1}(2x-1)=\frac{2x-1+1}{2}=\frac{2x}{2}=x.$

EXERCISE 8.1

1. $f(x)=x^2+x-1,$
   (i) Find the images $-2,0,2,5$ (ii) If $f(x)=5$, then find the values of x
   (iii) Find $f(x+1)$ (iv) Find $\frac{f(x+h)-f(x)}{h}$

2. If $f(x)=7x-2$, $g(x)=\frac{2x}{x^2-4}$, $h(x)=4\sqrt{25-x^2}$, $k(x)=x^2+1$, then determine (i) $f(6), g(-1), h(4), k\left(\frac{1}{2}\right)$ (ii) $\frac{f(x)-f(2)}{x-2}$

3. Find all real values of x such that $f(x)=0$.
   (i) $f(x)=15x-3$ (ii) $f(x)=x^2-8x+15$
   (iii) $f(x)=x^3-x$ (iv) $f(x)=x^3-x^2-5x+5$

4. Find the domain and range of the function f(x).
   (i) $f(x)=5x^2+2x-1$ (ii) $f(x)=\sqrt{x^2-16}$

5. Find the inverse function of the following functions
   (i) $f(x)=2x-3$ (ii) $f(x)=\frac{1}{3}x-5$ (iii) $f(x)=\frac{2-x}{5}$ (iv) $f(x)=4+\sqrt{2x}$

6. If $f(x)=x^3-2$, find (i) $f^{-1}(x)$ (ii) $f^{-1}(3)$

7. If $f(x)=\frac{x-4}{x-3}$

   Find (i) Domain and range of f. (ii) Domain and range of f^{-1}

8.3 Graphical Representation of Functions

This section is devoted to the representation of functions by graph. The **graph of a function** is a pictorial representation of function that is obtained by using the xy-plane.

Let f be a function defined by $y=f(x)$. The set of all points (x,y) such that x is in the domain of f is called the **graph** of f and we say that the point (x,y) is on the graph of f. To be more specific, if G **denotes** the graph of f, then $G=\{(x,y): y=f(x)$ where x is in the domain of $f\}$.

Equivalently, the graph of f is the graph of the equation $y=f(x)$.

The graph of a function may be obtained by constructing a table of corresponding values x of f. Each of these points may be plotted by placing a dot at appropriate location in the xy-plane. Then joining them together by means of a smooth curve gives the required graph of the function.

8.3.1(a) Graphs of Linear Functions

We sketch the graph of linear functions of the form $y=ax+b$ where $a,b\in IR$ and $a\neq 0$.

Example 7: Sketch the graph of the function

$$f(x)=2x+1,\ x\in\{0,1,2,3,4\}$$

Solution: For graph of this function, we assign values to x from its domain and find the corresponding values of y in the range of f as shown in the table:

$y=f(x)=2x+1$

x	0	1	2	3	4
y	1	3	5	7	9

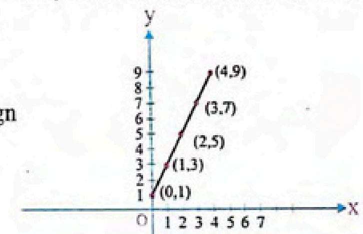


Figure 8.11

Plotting the points (x,y) in Cartesian plane and joining them with curve, we get graph of the given function as shown in the (Figure 8.11).

Note
It is clear from the above figure, that the graph of a linear function is a straight line.

Example 8: Draw the graph of the function $y=f(x)=2x+1, x\in IR$.

Solution: The domain of the function is the set of all real numbers IR. For the graph of $y=f(x)=2x+1$, we assign some values to x from its domain and find corresponding values y in the range of f as shown in the table:

$y = f(x) = 2x+1$

x	−3	−2	−1	0	1	2	3
y	−5	−3	−1	1	3	5	7

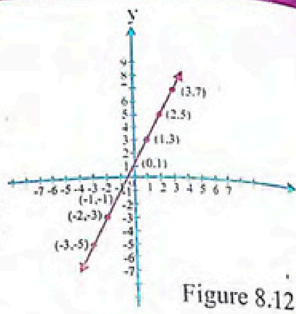


Figure 8.12

The graph of the function is shown in figure (8.12). As x can be any real number, the line is infinite in both the directions representing all the real numbers in the line. The domain and range of linear function are the set of all real numbers.

(b) Graph of Non-linear functions

In this section, we will sketch the graph of non-linear functions, that is functions of the form $f(x) = x^2, f(x) = x^3$ and so on.

Example 9: Sketch the graph of the function $y = f(x) = x^2$

Solution: In the following table some of the corresponding values of x and y are given $y = f(x) = x^2$.

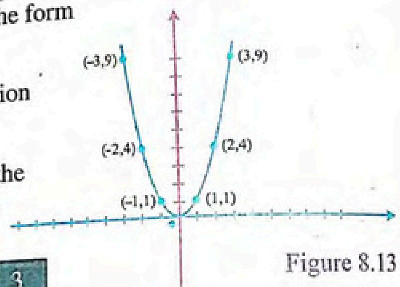
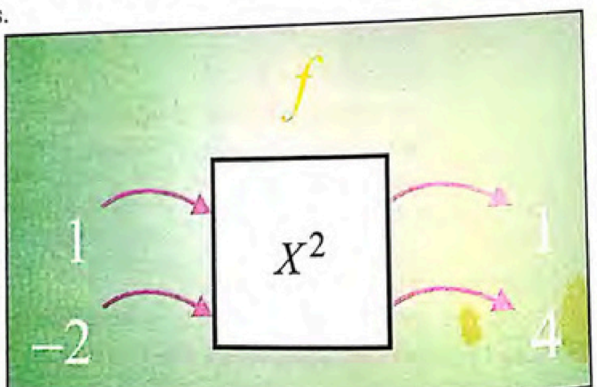


Figure 8.13

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

The graph of the function $f(x) = x^2$ is shown in figure (8.13). The function $f(x) = x^2$ is called a **squaring function.** The graph of squaring function is called a **parabola.** Its domain is the set of all real numbers and its range is the set of non-negative real numbers.



Example 10: Let $f(x) = x^3$. Sketch the graph of f.

Solution: We construct a table of values for $f(x) = x^3$ as follows:

$y = x^3$

x	−3	−2	−1	0	1	2	3
y	−27	−8	−1	0	1	8	27

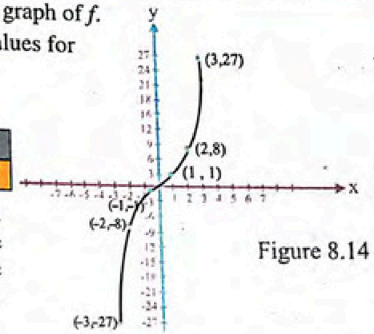


Figure 8.14

Plotting the corresponding points and joining them by a smooth curve, we obtain the graph of the function in figure (8.14). The function $f(x) = x^3$ is called a **cubing function.**

The domain and range of the cubing function are the set of all real numbers.

Example 11: Sketch the graph of the function $f(x) = \sqrt{x}$.

Solution: The given function f is a square root function. The following table gives some values of y corresponding to values of x.

$y = f(x) = \sqrt{x}$

x	0	1	4
y	0	1	2

The graph the function is shown in figure 8.15.

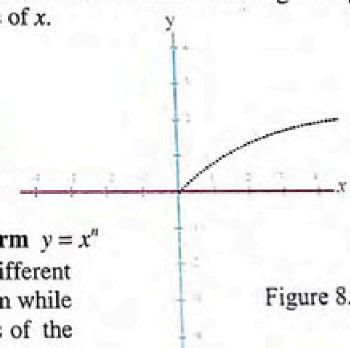


Figure 8.15

8.3.2 Graph of the function of the form $y = x^n$

Sometimes we group together different functions and write them in a single form while observing the definition and properties of the functions. For example, consider the **power function** $y = x^m$ where m is any constant.

Now, if

(a) $m = n$ i.e. a positive integer, we have another function of the form $y = x^n$

(b) $m = -n$ i.e. a negative integer, we have another function of the form

$$y = x^{-n} = \frac{1}{x^n}, x \neq 0$$

(c) $m = \frac{1}{n}$ i.e. a rational number, we have yet another function of the form

$$y = x^{\frac{1}{n}}, x > 0$$

We see that all these functions are represented by a single function of the form

$$y = x^n \qquad (1)$$

where n is any constant. The single function in (1) representing different functions is called a **family of function.**

In this section, we will sketch the graph of the family of functions $y = x^n$. The power function can also have fractional and irrational exponents. However, the discussion of such power functions is beyond the scope of this book.

(a) Graph of $y = x^n$ where n is a positive integer

Clearly the domain of $y = x^n$ is the set of real numbers.

(1) When $n = 1$, we have $y = x$. The following table gives the values of the function $y = f(x) = x$

x	−2	−1	0	1	2
y	−2	−1	0	1	2

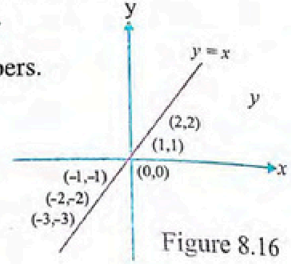


Figure 8.16

The graph is shown in figure (8.16) which is a straight line passing through the origin.

(2) When $n = 2$, we have $y = x^2$. The graph of the squaring function $y = x^2$ was sketched in example 9 which is reproduced in figure (8.17). The graph of $y = x^2$ is a parabola.

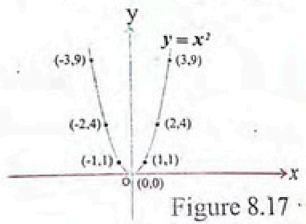


Figure 8.17

(3) When $n = 3$, we have $y = x^3$ which is called **cubing function.** The following table gives some values of the cubing function $y = x^3$.

x	−2	−1	0	1	2
y	−8	−1	0	1	8

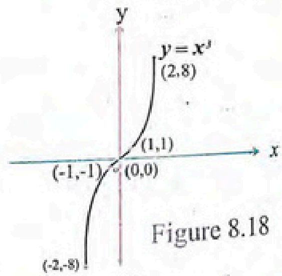


Figure 8.18

The graph of the function is shown in figure (8.18)

(4) When $n = 4$, we have $y = x^4$,

The following table gives some values of the function $y = x^4$

x	−2	−1	0	1	2
y	16	1	0	1	16

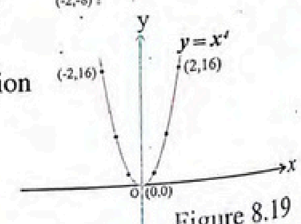


Figure 8.19

The graph is shown in figure (8.19)

(5) When $n = 5$, we have $y = x^5$. The following table gives some values of the function $y = x^5$

x	−2	−1	0	1	2
y	−32	−1	0	1	32

The graph of the function is shown in figure (8.20)

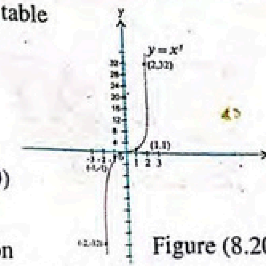


Figure (8.20)

Remember that

(i) When the values of n are even, the function $f(x) = x^n$ are even functions and the graphs of the function $f(x) = x^n$ are symmetric about the y – axis. In this case, all the graphs have the same general shape as the parabola $y = x^2$

(ii) When the values of n are odd, the functions $f(x) = x^n$ are odd functions and the graphs of the function $f(x) = x^n$ are symmetric about the origin. In this case, all the graphs have the same general shape as $y = x^3$ for odd n greater than 1.

(iii) By increasing n the graphs in both cases become flatter over the interval $-1 < x < 1$ and steeper over the interval $x > 1$ and $x < -1$ as shown in figure(8.21) and figure (8.22).

$f(x) = x^n$ when n is even

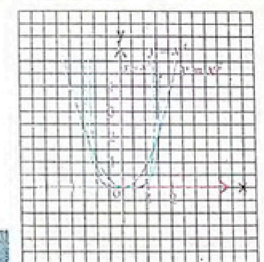


Figure 8.21

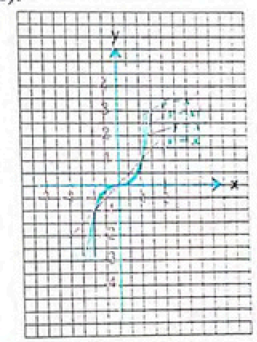


Figure 8.22

$f(x) = x^n$ when n is odd

(b) Graph of $y = x^n$ where n is a negative integer

The domain of the function $y = \frac{1}{x^n}$ is the set of all real numbers except $x \neq 0$.

(1) when $n = -1$ we have $y = \frac{1}{x}$. Some of the values of the function are given in the following table.

$y = \frac{1}{x}$

x	-3	-2	$-\frac{1}{2}$	-1	1	2	$\frac{1}{2}$	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-2	-1	1	$\frac{1}{2}$	2	$\frac{1}{3}$

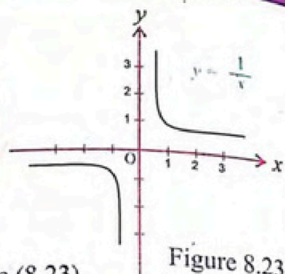


Figure 8.23

The graph of the function is shown in figure (8.23)

(2) When $n=-2$, we have $y=\frac{1}{x^2}$. In the following table some of the values of the function are given.

$y=\frac{1}{x^2}$

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$\frac{1}{4}$	1	4	4	1	$\frac{1}{4}$

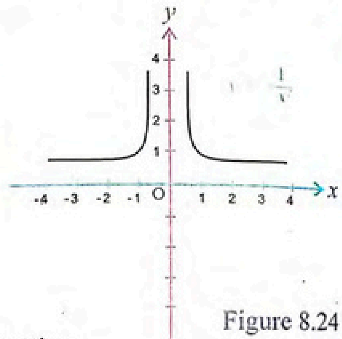


Figure 8.24

Figure (8.24) represents graph of the function.

(3) When $n=-3$, we have $y=\frac{1}{x^3}$. The following tables gives some values of the function.

$y=\frac{1}{x^3}$

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$-\frac{1}{8}$	-1	-8	8	1	$\frac{1}{8}$

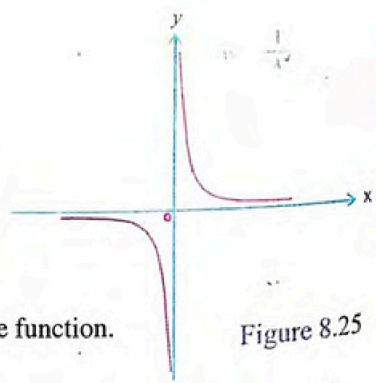


Figure (8.25) shows the graph of the function.

Figure 8.25

(4) When $n=-4$, we have $y=\frac{1}{x^4}$. The following table gives some values of the function.

$y=\frac{1}{x^4}$

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$\frac{1}{32}$	1	32	32	1	$\frac{1}{32}$

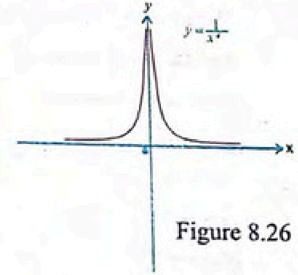


Figure 8.26

The graph of the function is shown in figure (8.26)

Remember that

(i) When the values of n are even, the functions $f(x)=\frac{1}{x^n}$ are even, and their graphs are symmetric about $y-axis$. In this case, all the graphs have the same general shape as $y=\frac{1}{x^2}$.

(i) When the values of n are odd, the functions $f(x)=\frac{1}{x^n}$ are odd, and their graphs are symmetric about the origin. In this case, all the graphs have the same general shape as $y=\frac{1}{x}$.

(ii) By increasing n, the graphs in both cases become steeper over the intervals $-1<x<0$ and $0<x<1$, and flatter over the intervals $x>1$ and $x<-1$ as shown in figure (8.27) and figure (8.28) respectively.

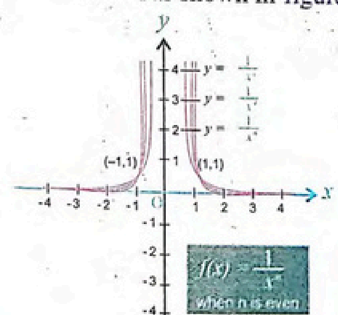


Figure 8.27

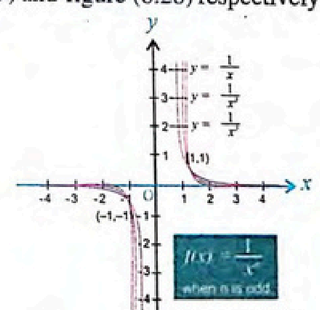


Figure 8.28

(c) Graph of $y=x^n\ (x>0)$ when n is a Rational Number

Generally the domain of the function $y=x^{\frac{1}{n}}$ is the set of all real numbers.

However, at present we will consider $y = x^{\frac{1}{n}}$ with $x > 0$.

(1) When $n = 1$, we have $y = f(x) = x$ which is the **identity function.** It is a special linear function. Its domain and range are the set of all real numbers in general. Some of the values of the function are given in the following table.

$y = x$

x	1	2	3	4	5	6
y	1	2	3	4	5	6

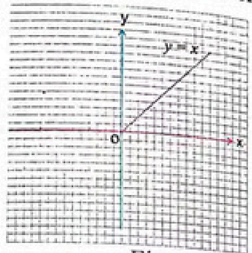


Figure 8.29

The graph of the function is shown in figure (8.29) which is a straight line.

(2) When $n = 2$, we obtain $y = x^{\frac{1}{2}} = \sqrt{x}$ that is, the square root function. The following table gives the values of the function $y = \sqrt{x}$

X	$\frac{1}{4}$	1	4	9	16
Y	$\frac{1}{2}$	1	2	3	4

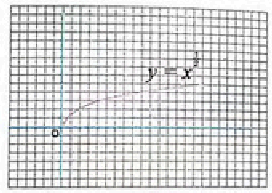


Figure 8.30

The graph of the function is shown in figure(8.30)

(3) When $n = 3$, we have $y = x^{\frac{1}{3}} = \sqrt[3]{x}$. Some of the values of the function are given in the following table.

$y = \sqrt[3]{x}$

x	$\frac{1}{8}$	1	8
y	$\frac{1}{2}$	1	2

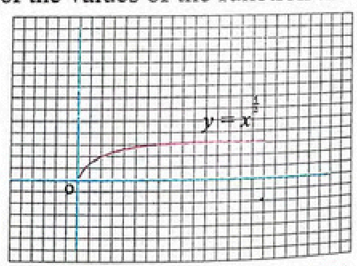


Figure 8.31

The graph of the function is shown in figure(8.31).

(4) When $x = 4$, we have $y = x^{\frac{1}{4}} = \sqrt[4]{x}$. The values ofthe function are given in the following table

$y = \sqrt[4]{x}$

x	$\frac{1}{16}$	1	16
y	$\frac{1}{2}$	1	2

The graph is given in figure(8.32).

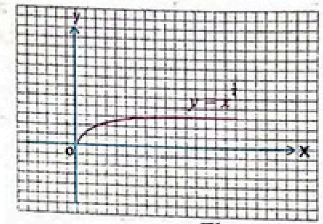


Figure 8.32

Remember that

(i) When the values of n are even, the graphs of the function $y = x^{\frac{1}{n}}$ have the same general shape as the square root function $y = \sqrt{x}$

(ii) When the values of n are odd, the graphs of the functions $y = x^{\frac{1}{n}}$ have the same general shapes as $y = x^{\frac{1}{3}} = \sqrt[3]{x}$.

(iii) The graph of $y = x^{\frac{1}{3}}$ extends over the entire x-axis, because $f(x) = x^{\frac{1}{3}}$ is defined for all real values of x. The reason is that every real number has a cube root.

(iv) The graph of $y = x^{\frac{1}{2}}$ only extends over the non-negative x-axis. The reason is that negative numbers have imaginary roots.

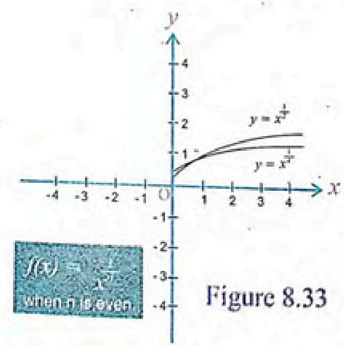


Figure 8.33

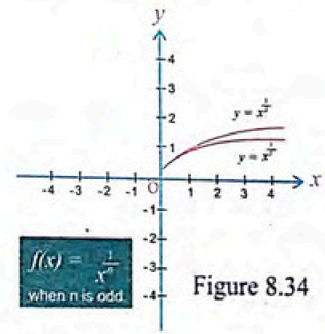


Figure 8.34

8.3.3 The Graph of Quadratic Functions

In this section we want to look at the graph of a quadratic function. The most general form of a quadratic function is, $f(x) = ax^2 + bx + c$

The graphs of quadratic functions are called **parabolas**.

Here are some examples of parabolas

Not For Sale

The lowest or highest point of a parabola is called its **vertex**. The vertical line passing through the vertex of a parabola is called the **axis of symmetry** or more briefly **axis** of the parabola.
In figure (8.35), the dashed line passing through the lowest or highest point i.e. vertex of the parabola is the axis of symmetry.

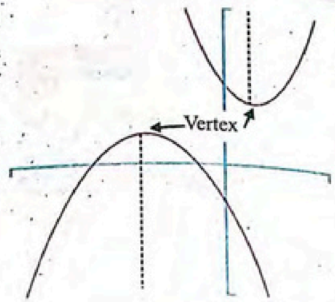


Figure 8.35

The Graph of a General Quadratic Function

Let $f(x)=ax^2+bx+c, a\neq 0$ be an arbitrary quadratic function. In order to sketch graph, we complete the square in $f(x)=ax^2+bx+c$ as follows:

$$f(x)=ax^2+bx+c$$

$$=(ax^2+bx)+c \qquad \text{(Separating c)}$$

$$=a(x^2+\frac{b}{a}x)+c \qquad \text{(Taking a as common)}$$

$$=a(x^2+\frac{b}{a}x+\frac{b^2}{4a^2})+c-a(\frac{b^2}{4a^2}) \qquad \text{(Adding and subtracting the square of half of the co - efficient of } x\text{).}$$

$$=a(x+\frac{b}{2a})^2+(c-\frac{b^2}{4a})$$

$$f(x)=a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right), a\neq 0 \qquad (1)$$

To simplify (1), we let

$$h=\frac{-b}{2a} \text{ and } k=c-\frac{b^2}{4a} \qquad (2)$$

Then (1) becomes

$$f(x)=a(x-h)^2+k \qquad (3)$$

The graph of f is a Parabola with vertex at the point (h,k)
The parabola opens upward if $a>0$ and downwards if $a<0$.
The axis is the vertical line $x=h$. With the help of formula (3), we can draw a reasonably accurate graph of the quadratic function in x by plotting the vertex and at least two points in each side of it.

Example 12: Sketch the graphs of the quadratic functions f and g defined by
(a) $f(x)=x^2$ (b) $g(x)=-x^2$

Solution:(a) The graph of the quadratic function $f(x)=x^2$ with $a=1$, $b=0$, $c=0$ was sketched in Example 9 and is reproduced in figure (8.36).
The vertex of the graph is the lowest point (0,0).

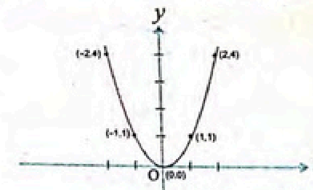


Figure 8.36

(b) In the following table some of the values of x and corresponding values of y of the quadratic equation $y=g(x)=-x^2$ with $a=1$, $b=0$, $c=0$ are given:

$y=g(x)=-x^2$

x	-2	-1	0	1	2
y	-4	-1	0	-1	-4

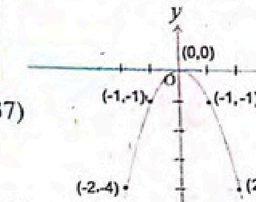


Figure 8.37

The graph of the function is shown in figure (8.37)
The graph of $f(x)=x^2$ **opens upward** and the graph of $y=g(x)=-x^2$ **opens downward.**
In general if, $f(x)=ax^2, a\neq 0$, then the graph of $f(x)$ opens upward if $a>0$ and opens downward if $a<0$

Example 13: Sketch the graph of the function

$$f(x)=x^2-2x+1$$

Solution: We construct a table of values of the function as follows: $y=x^2-2x+1$

x	-3	-2	-1	0	1	2	3	4	5
y	16	9	4	1	0	1	4	9	16

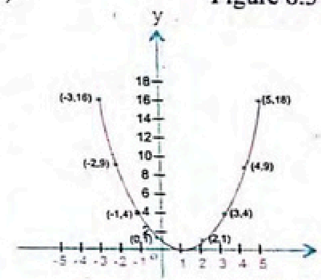


Figure 8.38

The graph of the function is shown in figure (8.38) with vertex at (1,0)

Example 14: Without graphing, find the vertex and axis of the graph of the function $f(x)=-x^2+4x-5$. Also determine whether the graph opens upward or downward.

Solution: Here $a=-1$, $b=4$, $c=-5$.

$\therefore$ vertex of the graph of $f=(h,k)=(-\frac{b}{2a}, c-\frac{b^2}{4a})$

$$=(\frac{-4}{2(-1)}, -5-\frac{(4)^2}{4(-1)})=(2,-1).$$

$$\text{Axis} = x=-\frac{b}{2a}=2$$

Since $a=-1<0$, so the graph opens downward.

Example 15: Sketch the graph of the function $f(x)=x^2-2x-2$.

Solution: Here $a=1,\ b=-2,\ c=-2$.

$\therefore$ vertex of the graph of $f=(-\frac{b}{2a},\ c-\frac{b^2}{4a})=(-\frac{-2}{2(1)},-2-\frac{(-2)^2}{4(1)})=(1,-3)$

Axis $= x=-\frac{b}{2a}=1$

Since $a=1>0$, so the graph opens upward.

The two additional values on each side of the vertex are given in following table.

$y=x^2-2x-2$

x	−1	0	1	2	3
y	1	−2	−3	−2	1

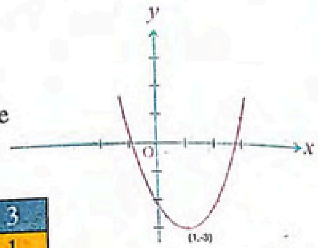
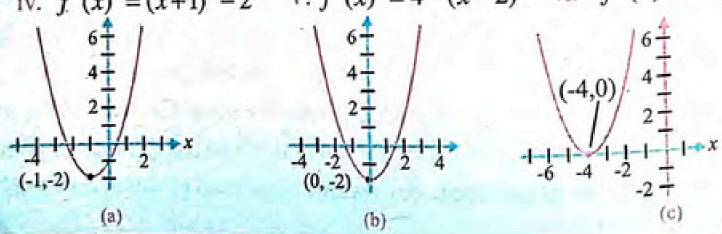


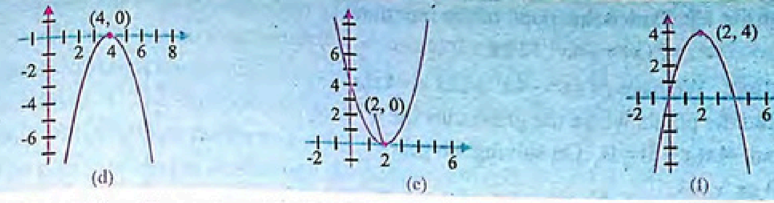
Figure 8.39

The graph of the function is given in figure (8.39).

EXERCISE 8.2

1. Sketch the graph of the given function
   (i) $f(x)=2x+3$ (ii) $f(x)=4x-5$
2. Sketch the graphs of the following functions
   (i) $f(x)=x^2+1$ (ii) $f(x)=-x^2+1$ (iii) $f(x)=x^2+2x+1$
3. Without graphing, find the vertex, all intercepts if any and axis of the graph of the following function. Also determine whether the graphs open upward or downward.
   (i) $f(x)=\frac{3}{4}x^2$ (ii) $f(x)=-2x^2+8$
   (iii) $f(x)=-x^2+6x-5$ (iv) $f(x)=x^2+2x-3$
4. Match the quadratic function with its graph. [The graphs are labeled (a), (b),(c), (d), (e), and (f).]
   i. $f(x)=(x-2)^2$ ii. $f(x)=(x+4)^2$ iii. $f(x)=x^2-2$
   iv. $f(x)=(x+1)^2-2$ v. $f(x)=4-(x-2)^2$ vi. $f(x)=-(x-4)^2$





8.3.4 Using Factors to Sketch Graphs

In the above section we sketched the graphs of quadratic functions by plotting many points. In this section too, we will sketch the graphs of quadratic functions but using their factors.

We know from our previous class knowledge that a quadratic expression can be written as a product of factors. For example, we can write

$$x^2+3x+2=(x+1)(x+2)$$

where $(x+1)$ and $(x+2)$ are the factors of the quadratic expression x^2+3x+2. Similarly, some quadratic functions of the form $f(x)=ax^2+bx+c\ (a\neq 0)$ can be factored and their graphs can be drawn by using the factors. This method of using factors to sketch the graph of quadratic function is explained through the following examples.

Example 16: Sketch the graph of the function $f(x)=x^2+2x-3$.

Solution: We have $f(x)=2x^2+2x-3=(x+3)(x-1)$.

To find the points which lie on the graph of the function $f(x)$, we put $(x+3)(x-1)=0$. The equation is satisfied if $x=-3$ or $x=1$.

Now $f(-3)=0$ and $f(1)=0$. Thus the points lying on the graph of $f(x)$ are $(-3,0)$ and $(1,0)$ that is, the graph cuts the x–axis at $(-3,0)$ and $(1,0)$.

To find the point where the graph cuts the $y-axis$ we put $x=0$ in the function so that $f(0)=-3$. Therefore the required points is $(0,-3)$. All that remains to be done is to obtain few additional points on the graphs in order to sketch it. Some of these are given in the table below.

$y=(x+3)\ (x-1)$

x	−4	−2	−1	0	2
y	5	−3	−4	3	5

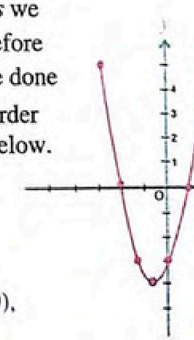


Figure 8.40

The graph of the function is shown in figure (8.40), which opens upward, since $a=1>0$.

Example 17: Sketch the graph of the function

$$f(x) = -4x^2 + 12x.$$

Solution: We have $f(x) = -4x^2 + 12x = -4x(x-3)$

To find the points where the graph cuts the $x-axis$, we put $-4x(x-3) = 0$. On solving we get

$x = 0$ or $x = 3$.

$\therefore f(0) = 0$ and $f(3) = 0$

Thus the graph cuts $x-axis$ at the points (0,0) and (3,0). Also $f(0) = 0$, so the point where the graph cuts $y-axis$ is (0,0).

To draw the graph, we need some additional points, which are given in the table below:

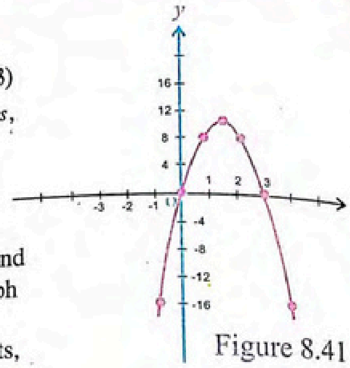


Figure 8.41

$$y = -4x\ (x-3)$$

x	-1	1	$\frac{3}{2}$	2	4
y	-16	8	9	8	-16

The graph of the function is given in figure (8.41) which opens downward, since $a = -4 < 0$.

Remember

We may draw the graph of any quadratic function $f(x)$ which can be factorized as $y = f(x) = a(x-p)(x-q)$ by keeping the following points in mind.

(i) Note the points $(p,0)$ and $(q,0)$ where the graph of the function cuts the x-*axis*.

(ii) By taking $x = 0$ in the function $f(x)$, note the point $(0, y)$ where the graph cuts the $y-axis$

(iii) The sign of the constant a tells whether the graph opens upwards or downwards.

(iv) To draw the graph, obtain some additional points on the graph.

(v) The shape of graphs of all quadratic functions is a parabola.

8.3.5 Predicting Functions from their Graphs

In this section, we are concerned with the use of factor form to predict the equation of a function of the type $f(x) = ax^2 + bx + c, (a \neq 0)$ if two points where the graph cuts the $x-axis$ and third point on the curve are given.

The method employed in doing so is explained through the following example.

Example 18: Find the equation of the graph of the function of the type $y = ax^2 + bx + c$, $(a \neq 0)$ which cuts the $x-axis$ at the point $(-2,0)$ and $(2,0)$ and also passes through the point $(1,-6)$.

Solution: The equation of the curve which passes through $x-axis$ at the points $(p,0)$ and $(q,0)$ has the form $y = a(x-p)(x-q)$ (1)

The curve which passes through the points $(-2,0)$ and $(2,0)$ is shown in figure (8.42).

Here $p = -2$, $q = 2$, so by (1), we have

$$y = a(x+2)(x-2) \quad (2)$$

The point $(1,-6)$ lies on the curve, so it must satisfy equation (2) and so $-6 = a(1+2)\ (1-2)$

$$\Rightarrow -6 = -3a \Rightarrow a = 2$$

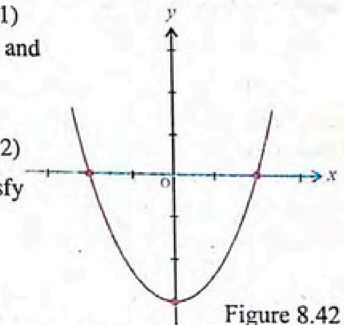


Figure 8.42

Therefore equation (2) of the curve becomes

$$y = 2(x+2)(x-2) \text{ or } y = 2x^2 - 8,$$ which is the required equation.

Example 19: Find the equation in the form $x^2 + bx + c = 0$ of the parabola which crosses the $x-axis$ at the point $(-5,0)$ and $(3,0)$

Solution: The form of the parabola is given by

$$x^2 + bx + c = 0 \quad (1)$$

The general form of the parabola is given by

$$ax^2 + bx + c = 0 \quad (2)$$

Comparing (1) and (2), we have

$$a = 1 > 0$$

so the parabola opens upward. The equation of the curve which cuts the $x-axis$ at the points (p,o) and (q,o) has the form

$$y = a(x-p)(x-q) \quad (3)$$

but $a = 1$, so (3) becomes

$$y = (x-p)(x-q) \quad (4)$$

The curve which cuts the $x-axis$ at the points $(-5,0)$ and $(3,0)$ is shown in figure (8.43).

We have $p = -5$ and $q = 3$

Using (4), we obtain

$y = (x+5)(x-3)$

or $y = x^2 + 2x - 15$

which is the required equation.

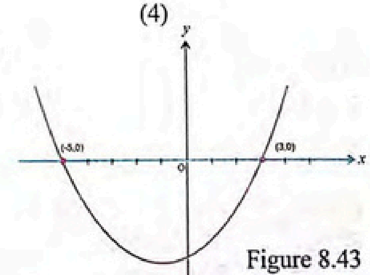


Figure 8.43

8.4 Intersecting Graphs

In this section we aim at to find the intersecting points graphically, when the intersection occurs between a linear function and coordinate axis, two linear functions and a linear and quadratic function. we will also solve graphically appropriate problems from daily life.

(a) Point of intersection of a linear function and coordinate axes

If a line l intersects the *x-axis* at a point $(a, 0)$, the number a is called the ***x- intercept*** of the line l. if a line l intersects the *y-axis* at a point) $(0, b)$ the number b is called the ***y-intercept*** of the line l. see (figure 8.44). Since the graph of a linear function $f(x) = ax + b$, $a, b \in I\!R$ is a straight line, so it will intersect the *x-axis* at the point $(a, 0)$, and *y-axis* at $(0, b)$ thus, the points where a graph of a linear function intersects the coordinate axes are the *x- intercept* and *y-intercept* of the graph.

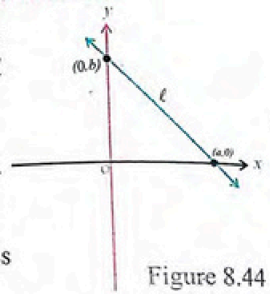


Figure 8.44

Example 20: Find the points of intersection of the linear function $f(x) = x - 4$ with coordinate axes

Solution: By giving some values to x, we find the corresponding values of y in the following table.

$f(x) = y = x - 4$

x	-2	-1	0	1	2	3	4	5	6
y	-6	-5	-4	-3	-2	-1	0	1	2

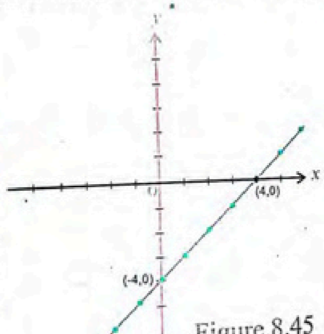


Figure 8.45

The graph of the function is shown in (figure 8.45). The graph intersects *x-axis* at $x=4$ and *y-axis* at $y=-4$. The answer may be easily verified by finding the *x- intercept* and *y-intercept* of the graph.

(b) Point of Intersection of two linear functions

We draw the graph of two linear functions on the same graph paper and then determine where the two graphs of these two linear functions intersect by looking at the graph.

Example 21: Find the point of intersection of the functions $f(x) = x + 3$ and $g(x) = -2x + 9$.

Solution: For $f(x) = x + 3$, we have the following table of values: $y = x + 3$

x	-5	-4	-3	-2	-1	0	1	2
y	-2	-1	0	1	2	3	4	5

For $g(x) = -2x + 9$, we have the following table of values: $y = -2x + 9$

x	-1	0	1	2	3	4	5
y	11	9	7	5	3	1	-1

The graphs of both functions are shown in (figure 8.46). Looking at the graph, we find that the point of intersection is $(2,5)$

Although this seems to be a very simple method of finding the coordinates of the point of intersection of two linear functions, it may not always be very accurate in cases when the coordinates of the point are fractional numbers. In that case, to find where exactly the graphs cross, we must use algebraic rather than graphic method. We can find a value of x and value of y that satisfy both the equations of linear functions simultaneously. For this purpose several methods are available. For example, we may use the method of elimination or method of substitution with whom we are already familiar.

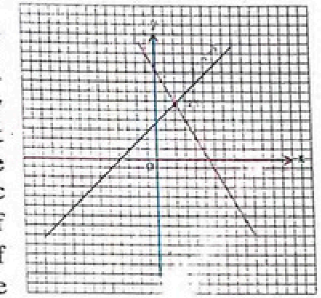


Figure 8.46

(c) Point of intersection of a linear function and a quadratic function

The method for finding the point of intersection of graphs of a linear function and a quadratic function is the same as that for finding the point of intersection of two graphs of linear functions. The method will be clarified by the following example.

Example 22: Find the point of intersection of the functions $f(x) = x^2 - 4x + 6$ and $g(x) = 2x + 1$.

Solution: The following table gives the values of the function $y = f(x) = x^2 - 4x + 6$

$y = x^2 - 4x + 6$

x	-2	-1	0	1	2	3	4	5
y	18	11	6	3	2	3	6	11

The table for values of the function $g(x) = 2x + 1$ is given below: $y = g(x) = 2x + 1$

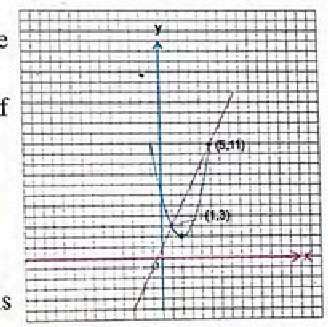


Figure 8.47

x	-3	-2	-1	0	1	2	3	4	5	6
y	-5	-3	-1	1	3	5	7	9	11	13

The graphs of these two functions are shown in figure (8.47).
The points of intersection of the two graphs are $(1,3)$ and $(5,11)$.

8.4.2 Graphical Solutions of Problems from Daily Life

Many problems from daily life can be solved by means of graphs. Here are some examples.

Example 23: It takes a swimmer 2 min to swim 24m downstream in a river and 4 min to swim back. Find the speed of flow of water and the speed at which he can swim in still water.

Solution: Let x = speed of swimmer in still water and y = speed of flow of water

Therefore speed downstream $= x+y$ and speed upstream $= x-y$

We know that time x speed = distance

$\therefore \quad 2(x+y)=24$

$4(x-y)=24$

or $\quad x+y=12$

$x-y=6$

or $\quad y=-x+12 \qquad (1)$

$y=x+6 \qquad (2)$

we see that equations (1) and (2) are the equations of linear functions and they are represented graphically by straight lines. We find the point of intersection of their graphs.

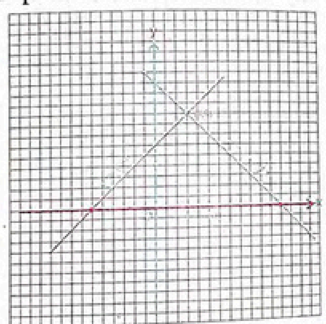


Figure 8.48

The values of functions (1) and (2) are given in the following tables: $y = x + 12$

x	−2	−1	0	1	2	6	12	13	14
y	14	13	12	11	10	6	0	−1	−2

and $y = x + 16$

x	−8	−6	−4	−2	0	2	4
y	−2	0	2	4	6	8	12

The graphs of both functions are shown is (figure 8.48).

We find that their point of intersection is (3,9), that is $x=3$ and $y=9$

Thus the speed of swimmer in still water $= x = 3m/\text{min}$ and the speed of flow of water $= y = 9m/\text{min}$. Use algebraic methods to verify the answer.

Example 24: A group of 45 school children visited a zoo and paid Rs.60.00 altogether as entry ticket. The entry ticket of class 1 was Rs.2.00 per child where as that of class KG Rs.1.00 per child. Find how many children were in the group from each class.

Solution: Let x = the number of children from class 1

and y = the number of children from Class KG.

According to the condition of the question, we have

$x+y=45$

$2x+y=60$

or $\quad y=45-x \qquad (1)$

$y=60-2x \qquad (2).$

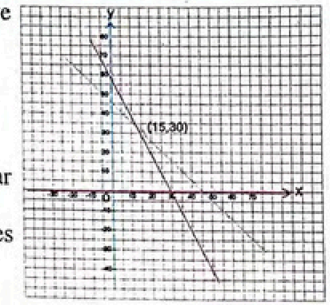


Figure 8.49

Equation (1) and (2) represent the equation of linear functions whose graph are straight line. We find that point of intersection of their graphs. The values of the functions (1) and (2) are given in the following tables.

$y=45-x$

x	−20	−10	0	10	20	30	40	50	60
y	65	55	45	35	25	15	5	−5	−15

and $y=60-2x$

x	−10	0	10	20	30	40	50
y	80	60	40	20	0	−20	−40

The graphs of both functions are shown in figure(8.49). The point of intersection of the graphs is (15,30), that is $x=15$ and $y=30$.

Thus the number of children from class $1 = x = 15$

and the number of children from class KG. $= y = 30$.

EXERCISE 8.3

1. Sketch graphs of the following functions
   (i) $f(x)=(x-1)(x-3)$ (ii) $f(x)=-2(x+1)(x-1)$
2. Using factors to sketch the graphs of the following functions
   (i) $f(x)=x^2-2x-3$ (ii) $f(x)=-(x^2-x-2)$
3. Find the equation of the graph of the function of the type $y=x^2+bx+c$ which crosses the $x-axis$ at the point (3,0) and (4,0).
4. Find the equation of the graph of the function of the type $y=ax^2+bx+c$ which (i) cross the $x-axis$ at the point (–5,0) and (3,0) and also passes through (–1,8)
   (ii) cross the $x-axis$ at the point (–7,0) and (10,0) and also passes through (4,11).
5. Find the point of intersection graphically of the following linear functions with the coordinate axes. (i) $f(x)=x-3$ (ii) $f(x)=2x+1$

6. Find the point of intersection graphically of the following functions.
   (i) $f(x)=-x+2,\quad g(x)=2x+1$
   (ii) $f(x)=3x-2,\quad g(x)=-x+6$
7. Find the point of intersection graphically of the following functions.
   (i) $f(x)=-x^2+4,\quad g(x)=x+2$
   (ii) $f(x)=x^2+x-3,\quad g(x)=-2x-5$
8. The paths of two airplanes A and B in the plane are determined by the straight lines $2x-y=6$ and $3x+y=4$ respectively. Find the point where the two paths cross each other.
9. A pilot makes a check flight in an air. Going directly into the wind, he covers a distance of 24 km in 6 minutes. Going with the wind, he covers the distance in 4 minutes. Find his air speed and velocity of the wind in km/min.

REVIEW EXERCISE 8

1. Choose the correct option.

i. What is the domain of f(x) = $\sqrt{\dfrac{2-x}{x+2}}$?

(a) $[-4,-2)$ (b) $[0, 2]-\{1\}$ (c) $(-2, 2)$ (d) $(-2, 2]$

ii. $A=\{-1, 0, 1, 2\}$, $B=\{0, 1, 4\}$ and $f: A \to B$ defined by $f(x)=x^2$, then f is

(a) Only one-one function (b) Only onto function
(c) bijective (d) not a function

iii. If $f: R \to R$ defined by $f(x)=3x-5$, then $f^{-1}(\{-1, -2, 1, 2\})=$

(a) $\left\{1, \frac{4}{3}, \frac{7}{3}\right\}$ (b) $\left\{-1, 2, \frac{4}{3}\right\}$ (c) $\left\{1, 2, \frac{4}{3}, \frac{7}{3}\right\}$ (d) $\{1, 2, -1, -2\}$

iv. If $f(2x+3)=4x^2+12x+15$, then find the value of $f(3x+2)$ is

(a) $9x^2-12x+36$ (b) $9x^2+12x+10$
(c) $9x^2-12x+24$ (d) $9x^2-12x-5$

v. If $f(x)=x^3-\frac{1}{x^3}$, then $f(x)+f\left(\frac{1}{x}\right)=$

(a) 0 (b) -1 (c) x^3 (d) None of these

vi. If $f(x)=x^2-3x+4$, then find the values of x satisfying the equation $f(x)=f(2x+1)$

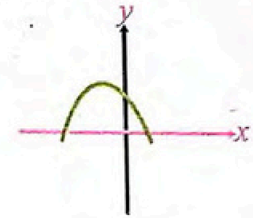
(a) 5/3 (b) 2/3 (c) 1 (d) None of these

vii. The domain of $y=\dfrac{x}{\sqrt{x^2-3x+2}}$ is

(a) $(\infty, 1)$ (b) $(2, \infty)$ (c) $(\infty, 1) \cup (2, \infty)$ (d) $(-\infty, 1) \cup (2, \infty)$

(viii) Guess the quadratic function for the curve given in the figure.

(a) $g(x)=x^2-2x-5$
(b) $g(x)=x^2+2x+5$
(c) $g(x)=-x^2-2x+5$
(d) $g(x)=-x^2+2x+5$



2. Find domain of $f(x)=\sqrt{3-\sqrt{12-x^2}}$
3. Find a polynomial function $f(x)$ of the second degree when $f(0)=5$, $f(-1)=10, f(1)=6$.
4. Find the range of each of the following functions:
   i) $f(x)=x^2+2,\ x \in R$
   ii) $f(x)=x,\ x \in R$
5. The function 't' which maps temperature in Celsius into temperature in degree Fahrenheit is defined by $t(c)=\frac{9c}{5}+32$
   Find (i) t(0) (ii) t(28) (iii) t(−10) (iv) the value of c, when t(c) = 212
6. If $f(x)=8x-7$, find (i) $f^{-1}(9)$ (ii) $f^{-1}\left(\frac{11}{3}\right)$
7. Given that $f(x)=x^3-ax^2+bx+1$. If $f(2)=-3$ and $f(-1)=0$, find the value of a and b.
8. Graph the following. (i) $y=-\frac{1}{2}x+3$ (ii) $y=-3x^2$ (iii) $y=2x^2-7x+3$
9. Sketch the graph of the following.
   (i) $y=x^2+2x-3$ (ii) $y=3(x+1)(x-1)$
10. Find the point of intersection graphically of the following functions.
   (i) $f(x)=x+4,\quad g(x)=-2x+3$
   (ii) $f(x)=x^2-x-2,\quad g(x)=-3x-3$