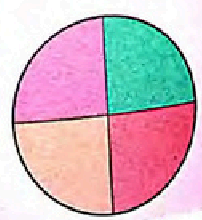
(vii) The number of all possible matrices of order 3x3 with each entry 0 and 1 is:
(a) 18 (b) 27 (c) 512 (d) 81

(viii) How many diagonals can be drawn in a plane figure of 8 sides?
(a) 21 (b) 20 (c) 35 (d) 81

(ix) If $P(A)=\frac{1}{2}, P(B)=0$, then $P(A|B)$ is
(a) 0 (b) $\frac{1}{2}$ (c) not defined (d) 1

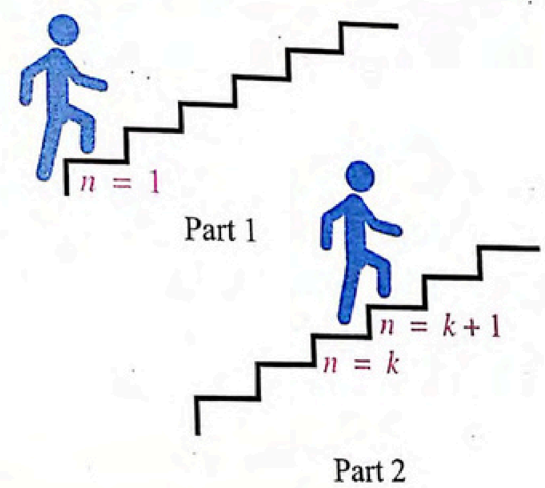
(x) If A and B are events such that $P(A|B)=P(B|A)$ then
(a) $A\subset B$ but $A\neq B$ (b) $A=B$ (c) $A\cap B=\varphi$ (d) $P(A)=P(B)$

2. (i) If ${}^{2n}C_r={}^{2n}C_{r+2}$; find r. (ii) If ${}^{18}C_r={}^{18}C_{r+2}$; find ${}^rC_5$.
3. ${}^{56}P_{r+6}:{}^{54}P_{r+3}=30800:1$, find r.
4. In how many distinct ways can $x^4y^3z^5$ be expressed without exponents?
5. In how many different ways can be six children seated at a round table if a certain two children (i) refuse to sit next to each other? (ii) insist on sitting next to each other?
6. Six people including Faisal and Saima are to be seated around a circular table. Find the probability that Faisal and Saima are seated next to each other.
7. If $P(A)=0.8, P(B)=0.5, P(B|A)=0.4$,
find (i) $P(A\cap B)$ (ii) $P(A|B)$ (iii) $P(A\cup B)$.
8. How many 6-digited telephone numbers can be constructed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, if each number starts with 35 and no digits appears more than once.
9. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?
10. A party of n men is to be seated round a circular table. Find the probability that two particular men sit together.
11. Given the following spinner, determine the probability:
P (Orange)
P (Red or Green)
P (Not Red)
P (Pink)



UNIT 7

MATHEMATICAL INDUCTION AND BINOMIAL THEOREM



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Describe the principle of mathematical induction.
- Apply the principle to prove the statements, identities or formulae.
- Use Pascal's triangle to find the expansion of $(x+y)^n$ where n is a small positive integer.
- State and prove binomial theorem for positive integral index
- Expand $(x+y)^n$ using binomial theorem and find its general term.
- Find the specified term in the expansion of $(x+y)^n$
- Expand $(1+x)^n$ where n is a positive integer and extend this result for all rational values of n.
- Expand $(1+x)^n$ in ascending powers of x and explain its validity/ convergence for $|x|<1$ where n is a rational number.
- Determine the approximate values of the binomial expansions having indices as –ve integers or fractions.

7.1. Introduction

To understand the basic principles of mathematical induction, suppose a set of thin rectangular tiles are placed as shown in the following Figure (7.1).

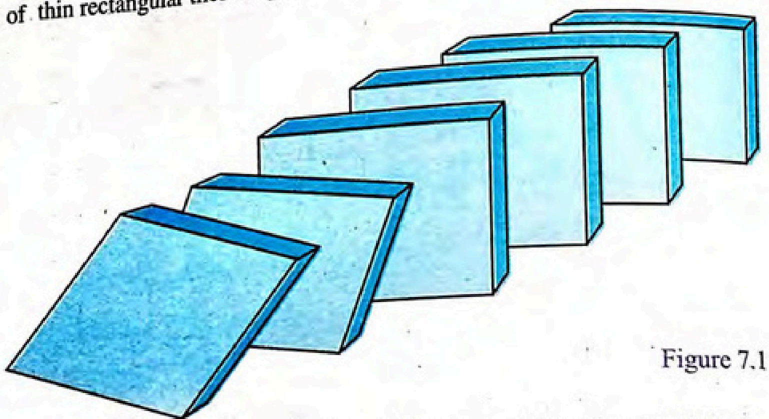


Figure 7.1

When the first tile is pushed in the indicated direction, all the tiles will fall. To be absolutely sure that all the tiles will fall, it is sufficient to know that

(a) The first tile falls, and

(b) In the event that any tile falls its successor necessarily falls.

This is the underlying principle of mathematical induction.

We know, the set of natural numbers **N** is a special ordered subset of the real numbers. In fact, **N** is the smallest subset of **R** with the following property.

A set **S** is said to be an inductive set if $1 \in \mathbf{S}$ and $x+1 \in \mathbf{S}$ whenever $x \in \mathbf{S}$.

Since **N** is the smallest subset of **R** which is an inductive set, it follows that any subset of **R** that is an inductive set must contain **N**.

Mathematical induction is one of the developed techniques of proof in the history of mathematics. It is used to check conjectures about the outcomes of processes that occur repeatedly and according to definite patterns.

For example:

$$1+3+5+\ldots\ldots\ldots\ldots+(2n-1) = n^2 \qquad (1)$$

$$1+2+3+\ldots\ldots\ldots\ldots+n = \frac{n(n+1)}{2} \qquad (2)$$

$$1^2+2^2+3^2+\ldots\ldots\ldots+n^2 = \frac{n(n+1)(2n+1)}{2} \qquad (3)$$

are all propositions, statements which involve the natural number n. Equation (1) above asserts that the sum of first n positive odd integers is equal to the square of n. We see that the L.H.S. of (1) reduces simply to:

$$1 = 1 \quad \text{if} \quad n=1$$

$$1+3=4=2^2 \quad \text{if} \quad n=2$$

$$1+3+5=9=3^2 \quad \text{if} \quad n=3 \text{ and so on.}$$

It is impossible to verify (1) for each $n \in N$, because it involves infinitely many calculations which never end. To avoid such situations, the principle of mathematical induction is applied.

7.1.1. The Principle of Mathematical Induction

The principle of mathematical induction is stated as follows.

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer.

Suppose the following two statements are true.

1. P(a) is true.
2. For all integers $k \geq a$, if P(k) is true then P(k+1) is true.

Then the statement for all integers $n \geq a$; P(n) is true.

The principle of mathematical induction is explained through the following examples.

Example 1: Prove that for every $n \in N$, $1+2+3+\ldots\ldots+n = \frac{n(n+1)}{2}$

Solution: Step 1. For n=1, the statement becomes

$$1 = \frac{1(1+1)}{2}. \quad \leftarrow \textit{basis } (p(1))$$

Thus the statement is true for n = 1

Step 2. Let us assume that the statement be true for $n=k \in N$, that is, we assume

$$1+2+\ldots+k = \frac{k(k+1)}{2}. \quad \leftarrow \text{inductive hypothesis } (P(k))$$

Step 3. Let n = k+1 and consider

$$(1+2+\ldots+k)+(k+1) = \frac{k(k+1)}{2}+(k+1)$$

(adding k+1 to both sides of P(k))

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+2)(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Which is just the form taken by the proposition when n = k+1. So the above proposition is true for n=k+1 and thus by the principle of mathematical induction, it is true for all positive integers n.

Example 2: (i) Find 2+4+6+……+500

(ii) Find 5+6+7+8+……+50

(iii) Find an integer $h \geq 2$, find 1+2+3+…..+(h−1)

Solution:

(i) $2+4+6+\cdots+500 = 2\cdot(1+2+3+\cdots+250)$

$= 2\cdot\left(\frac{250\cdot 251}{2}\right)$ (by applying the formula for the sum of the first n with n =250)

$= 62,750$

(ii) $5+6+7+8+\cdots+50 = (1+2+3+\cdots+50) - (1+2+3+4)$ (by applying the formula for the sum of the first n with n =50)

$= \frac{50\cdot 251}{2} - 10$

$= 1265$ (by applying the formula for the sum of the first n with n =h− 1)

(iii) $1+2+3+\cdots+(h-1) = \frac{(h-1).[(h-1)+1]}{2} = \frac{(h-1).h}{2}$

Example 3: Prove that $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution: **Step 1.** For n = 1, the proposition becomes

$1^2 = 1 = \frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = 1$. Thus it is true for n =1

Step 2. Suppose the proposition is true for n = k, then

$$1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \text{(i)}$$

Step 3. let n = k + 1 and consider

$$1^2+2^2+3^2+\cdots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{(Adding } (k+1)^2 \text{ to both sides of (i))}$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)\{k(2k+1)+6(k+1)\}}{6}$$

$$= \frac{(k+1)\{2k^2+k+6k+6)\}}{6} = \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Which is just the form taken by the proposition for n = k + 1. So the above proposition is true for n = k+1 and hence by the principle of mathematical induction, it is true for all positive integer n.

It must be noted that the application of the principle of mathematical induction is not limited only to P(n) stated by means of an equation. The principle can also be applied in cases where no equation is involved as we shall see in the following examples.

Example 4: Show that a–b is a factor of $a^n - b^n$ for all positive integer n.

Solution: To show that $a - b$ is a factor of $a^n - b^n$, we will use induction on n.

Step 1. Let n = 1, then $a^n - b^n = a - b$ and since $a - b$ divides $a - b$, so $a - b$ is a factor of $a - b$. Therefore the above statement is true for n = 1.

Step 2. Let the above statement is true for n = k then $a-b$ is a factor of $a^k - b^k$.

$\Rightarrow$ $a - b$ divides $a^k - b^k$ and as such we can write

$a^k - b^k = (a-b)\,Q$……..(1) where Q is the quotient.

Step 3. Let n = k + 1 and consider $a^{k+1} - b^{k+1}$. We can write

$a^{k+1} - b^{k+1} = a^k a - b^k b$ (Adding and subtracting the term ab^k)

$= a^k a - ab^k + ab^k - b^k b$

$= a(a^k - b^k) + b^k(a-b) = a(a-b)\,Q + b^k(a-b)$ (Using 1)

$= (a-b)\,[aQ + b^k]$

$\Rightarrow$ $a - b$ divides $a^{k+1} - b^{k+1}$ with quotient $aQ + b^k$

$\Rightarrow$ $a - b$ is a factor of $a^{k+1} - b^{k+1}$

Therefore the above statement is true for $n = k + 1$ and hence by the principle of induction it is true for all positive integer n.

Example 5: Prove that if n is a positive odd integer then $x + y$ is a factor of $x^n + y^n$

Solution: Since n is given to be a positive odd integer, so we can write $n=2m-1$ where m is a positive integer. Therefore $x^n+y^n=x^{2m-1}+y^{2m-1}$

To prove the above statement, we will use the method of induction on m.

Step 1. Let $m = 1$, then $x^{2m-1} + y^{2m-1} = x^{2\cdot1-1} + y^{2\cdot1-1} = x + y$ and since $x + y$ divides $x + y$, so $x + y$ is a factor of $x + y$. Therefore the above statement is true for $m = 1$.

Step 2. Let the above statement is true for $m = k$ then $x + y$ is a factor of $x^{2k-1}+y^{2k-1}$

$\Rightarrow$ $x + y$ divides $x^{2k-1}+y^{2k-1}$.

So we can write $x^{2k-1}+y^{2k-1} = (x+y)Q$ (1) where Q is the quotient.

Step 3. Now let $m = k+1$ and consider

$$x^{2(k+1)-1}+y^{2(k+1)-1} = x^{2k+2-1}+y^{2k+2-1} = x^{2k-1+2}+y^{2k-1+2} = x^{2k-1}.x^2+y^{2k-1}.y^2$$

$$= x^{2k-1}x^2 + y^{2k-1}x^2 - y^{2k-1}.x^2 + y^{2k-1}y^2$$

$$= x^2[x^{2k-1}+y^{2k-1}] + y^{2k-1}(y^2-x^2)$$

$$= x^2(x+y)Q + y^{2k-1}(y-x)(y+x) \quad \text{(Using 1)}$$

$$= x^2(x+y)Q + y^{2k-1}(y-x)(x+y) = (x+y)[x^2Q + y^{2k-1}(y-x)]$$

or $x^{2(k+1)-1}+y^{2(k+1)-1} = (x+y)Q_1$ where $Q_1 = x^2Q + y^{2k-1}(y-x)$

$\Rightarrow x+y$ is a factor of $x^{2(k+1)-1} - y^{2(k+1)-1}$

So the above statement is true for $m = k+1$ and hence by induction it is true for all positive integral values of m.

Therefore $x+y$ is factor of x^n+y^n where n is a positive odd integer.

7.1.2 General (extended) form of principle of Mathematical Induction

Sometimes it happens that a given statement and proposition does not hold for first few positive integral values of n but after those values of n it becomes true; For example let us consider the statement $n^2 > n+3$

We see that when n=1 then $1 > 1+3$ or $1 > 4$ which is false.

When $n = 2$ then $2^2 > 2+3$ and $4 > 5$ which is again false.

When $n = 3$ then $3^2 > 3 + 3$ or $9 > 6$ which is true. That is, the above statement is false for $n = 1$ and 2 but is true for all values of n greater than 2.

Similarly if we consider the statement $n^3 > 4n^2 + n + 1$ then this statement is not true for $n = 1,2,3,4$ but it becomes true forn $= 5$ and higher values.

In such situations the principle of mathematical induction is defined as under:

Let P(n) is a given statement or proposition such that.

(i) P(n) is true for $n = m$, where m is the least positive integer.

(ii) If P(n) is true for $n = k$ where $k > m$ then p(n) is also true for $n = k+1$

We then say that P(n) is true for all integral values of $n \geq m$.

This is called general (extended) form of the principle of mathematical induction.

Example 6: Prove that $n^3 > 4n^2 + n + 1$ for $n \geq 5$

We are to prove that $n^3 > 4n^2 + n + 1$ for $n \geq 5$

Solution: In this case our induction will start from n=5

Step 1. Let $n = 5$, then $n^3 = 5^3 = 125$ and

$4n^2+n+1 = 4(5)^2 + 5 + 1 = 100 + 5 + 1 = 106$

Clearly $125 > 106$ so the above statement is true for $n = 5$

Step 2. Let us assume that the above statement is true for $n = k \geq 5$ then,

$k^3 > 4k^2 + k + 1$ (1)

Step 3. Now let $n = k+1$, then $n^3 = (k+1)^3$

and so $(k+1)^3 = k^3+3k^2+3k+1 > 4k^2+k+1+3k^2+3k+1$

$\Rightarrow (k+1)^3 > 4k^2+3k^2+4k+2$ (using (1))

$\Rightarrow (k+1)^3 > 4k^2+3k\cdot k+4k+2 \Rightarrow (k+1)^3 > 4k^2+3k\cdot5+4k+2$ as $k \geq 5$

$\Rightarrow (k+1)^3 > 4k^2+15k+4k+2$ $\Rightarrow (k+1)^3 > 4k^2+9k+6k+4k+2$

$\Rightarrow (k+1)^3 > 4k^2+9k+10k+2$ $\Rightarrow (k+1)^3 > 4k^2+9k+6k+4k+2$

$\Rightarrow (k+1)^3 > 4k^2+9k+6$ (as $6k+4k+2>6$)

$\Rightarrow (k+1)^3 > 4k^2+8k+k+4+2$ $\Rightarrow (k+1)^3 > 4k^2+8k+4+k+2$

$\Rightarrow (k+1)^3 > 4(k^2+2k+1)+k+1+1$ $\Rightarrow (k+1)^3 > 4(k+1)^2+(k+1)+1$

Which is of the form (1) for $n = k + 1$, so the given proposition is true for $n=k+1$, thus by induction it is true for all $n \geq 5$.

Example 7: Prove that $2^n < \binom{2n}{n}$ for $n > 1$

Solution: We are to prove that $2^n < \binom{2n}{n}$ for $n > 1$ .... (1)

Step 1.
Let $n = 2$, then $2^n = 2^2 = 4$ and $\binom{2n}{n} = \binom{2\cdot 2}{2} = \binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4\cdot 3\cdot 2}{2\cdot 2} = \frac{24}{4} = 6$

Therefore 4<6 and once $2^n < \binom{2n}{n}$ is true for n = 2

Step 2. Let us suppose that the above assertion is true for $n=k$ for $k>1$, then

$$2^k < \binom{2k}{k} \quad \text{or} \quad 2^k < \frac{2k!}{k!(2k-k)!}$$

$$2^k < \frac{2k!}{k!\,k!} \quad \ldots\ldots\ldots\ldots \quad (2)$$

Step 3. Let $n = k+1$ and consider 2^{k+1}, we can write

$$2^{k+1} = 2^k \cdot 2 < \frac{2k!\cdot 2}{k!\,k!} \quad \ldots\ldots\ldots\ldots \quad (3)$$

Now $\frac{(2k+2)(2k+1)}{(k+1)^2} = \frac{2(k+1)(k+k+1)}{(k+1)^2} = \frac{2(k+k+1)}{k+1} = 2\left[\frac{k}{k+1} + \frac{k+1}{k+1}\right]$

$$= 2\left[\frac{k}{k+1} + 1\right] = \frac{2k}{k+1} + 2 > 2 \text{ as } k > 1$$

$\Rightarrow 2 < \frac{(2k+2)(2k+1)}{(k+1)^2}$ From (3), we have

$2^{k+1} < \frac{2k!\cdot 2}{k!\cdot k!} < \frac{2k!}{k!\cdot k!}\cdot\frac{(2k+2)(2k+1)}{(k+1)^2}$ or, $2^{k+1} < \frac{(2k+2)(2k+1)\,2k!}{k!(k+1)\,k!(k+1)}$

$2^{k+1} < \frac{(2k+2)!}{k!(k+1)\,k!(k+1)} \Rightarrow 2^{k+1} < \frac{(2k+2)!}{(k+1)!\,(k+1)!}$

$2^{k+1} < \binom{2k+2}{k+1}$ which is of the form (1) when n is replaced by $k+1$.

So the given statement is true for $n = k+1$ and hence it is true for all $n > 1$.

Thus $2^n < \binom{2n}{n}$ for $n > 1$.

EXERCISE 7.1

Establish the formulas given below by mathematical induction.

1. $2+4+6+\cdots+2n = n(n+1)$
2. $1+5+9+\ldots\ldots+(4n-3) = n(2n-1)$
3. $3+6+9+\ldots\ldots+3n = \frac{3n(n+1)}{2}$
4. $3+7+11+\cdots+(4n-1) = n(2n+1)$
5. $1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$
6. $1(1!)+2(2!)+3(3!)+\cdots+n(n!) = (n+1)!-1$
7. $1\cdot 2+2\cdot 3+3\cdot 4+\ldots\ldots+n(n+1) = \frac{n(n+1)(n+2)}{3}$
8. $1+2+2^2+2^3+\ldots\ldots+2^{n-1} = 2^n-1$
9. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\ldots+\frac{1}{3^n} = \frac{1}{2}\left[1-\frac{1}{3^n}\right]$
10. $\binom{5}{5}+\binom{6}{5}+\binom{7}{5}+\ldots\ldots+\binom{n+4}{5} = \binom{n+5}{6}$
11. $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots\ldots+\binom{n}{2} = \binom{n+1}{3}$ for $n \geq 2$
12. Show by mathematical induction that
    (i) $\frac{5^{2n}-1}{24}$ is an integer. (ii) $\frac{10^{n+1}-9n-10}{81}$ is an integer.
13. (i) $2^n > n \;\forall n \in N$. (ii) $n! > n^2$ for every integer $n \geq 4$
14. (i) Show that 5 is a factor of $3^{2n-1}+2^{2n-1}$ where n is any positive integer.
    (ii) Prove that $2^{2n}-1$ is a multiple of 3 for all positive integers.
15. Show that $a+b$ is a factor of a^n-b^n for all even positive integer n.

7.2 The Binomial Theorem

In algebra a sum of two terms, such as a + b, is called a binomial. The binomial theorem gives an expression for the powers of a binomial $(a+b)^n$, for each positive integer n and all real numbers a and b.

7.2.1 Statement and proof of the binomial theorem

The binomial theorem in its explicit form is stated as under.

Theorem: If a and b are any two real numbers and n is a positive integer, then

$$(a+b)^n=\binom{n}{0}a^nb^o+\binom{n}{1}a^{n-1}b^1+\binom{n}{2}a^{n-2}b^2+\dots+\binom{n}{r}a^{n-r}b^r+\dots+\binom{n}{n}a^0b^n$$

which more compactly can be written in summation form as:

$$(a+b)^n=\sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$$

Proof: Mathematical induction provides us the best way for confirming the validity of the binomial theorem.

$$(a+b)^n=\binom{n}{o}a^nb^0+\binom{n}{1}a^{n-1}b^1+\binom{n}{2}a^{n-2}b^2+\dots+\binom{n}{r}a^{n-r}b^r+\dots+\binom{n}{n}a^0b^n\ \dots\dots\text{(i)}$$

Step 1. If n = 1, then from (i), we obtain

$$(a+b)^1=\binom{1}{0}a^1b^0+\binom{1}{1}a^{1-1}b^1=a+b \qquad \left(\because \binom{1}{0}=\binom{1}{1}=1\right)$$

which is true. Thus the statement is true for n = 1

Step 2. Suppose that the statement is true for $n=k$, then

$$(a+b)^k=\binom{k}{0}a^kb^0+\binom{k}{1}a^{k-1}b^1+\binom{k}{2}a^{k-2}b^2+\dots+\binom{k}{r}a^{k-r}b^r+\dots+\binom{k}{k}a^0b^k\ \dots\dots\text{(ii)}$$

Step 3. We now prove that the theorem is true for $n=k+1$. Multiplying both sides of equation (ii) by $(a+b)$, we have

$$(a+b)(a+b)^k=(a+b)\left[\binom{k}{0}a^kb^0+\binom{k}{1}a^{k-1}b^1+\binom{k}{2}a^{k-2}b^2+\dots+\binom{k}{r}a^{k-r}b^r\dots+\binom{k}{k}a^0b^k\right]$$

$$\Rightarrow (a+b)^{k+1}=\left[\binom{k}{0}a^{k+1}b^0+\binom{k}{1}a^kb^1\binom{k}{2}a^{k-1}b^2+\dots+\binom{k}{r}a^{k-r+1}b^r+\dots\dots+\binom{k}{k}ab^k\right]$$

$$+\left[\binom{k}{0}a^kb+\binom{k}{1}a^{k-1}b^2+\binom{k}{2}a^{k-1}b^3+\dots+\binom{k}{r}a^{k-r}b^{r+1}+\dots+\binom{k}{k}a^0b^{k+1}\right]$$

$$\Rightarrow (a+b)^{k+1}=\binom{k}{0}a^{k+1}+\left[\binom{k}{1}+\binom{k}{0}\right]a^kb+\left[\binom{k}{2}+\binom{k}{1}\right]a^{k-1}b^2+\dots\dots\dots\dots+$$

$$\left[\binom{k}{r}+\binom{k}{r-1}\right]a^{k-r+1}b^r+\dots\dots+\binom{k}{k}a^0b^{k+1}$$

We know that $\binom{k}{0}=\binom{k+1}{0}=1$ and $\binom{k}{k}=\binom{k+1}{k+1},\binom{k}{r-1}+\binom{k}{r}=\binom{k+1}{r}$ for $0\le r\le k$,

therefore,

$$(a+b)^{k+1}=\binom{k+1}{0}a^{k+1}b^0+\binom{k+1}{1}a^kb+\binom{k+1}{2}a^{k-1}b^2+\dots+\binom{k+1}{r}a^{k+1-r}b^r+\dots+\binom{k+1}{k+1}a^0b^{k+1}$$

which is of the form (i) for $n=k+1$

So the given statement is true for $n=k+1$ and thus by the method of induction it is true for all positive integers n.

7.2.2 Properties of the Binomial Expansion

The expansion of $(a+b)^n$ has the following properties.

(i) The number of terms in the expansion of $(a+b)^n$ are $n+1$ i.e. the number of terms are one more than the exponent n.

Thus the expansion of $(a+b)^8$ will contain 8+1 = 9 terms.

(ii) In the expansion of $(a+b)^n$ the first term is a^nb^0, the second term is n $a^{n-1}b^1$and the third term is $\frac{n(n-1)}{2!}a^{n-2}b^2$ and so on . In each term the exponent of a decreases progressively by 1 and the exponent of b increases progressively by 1, but the sum of the exponents of a and b in each terms is always equal to n.

(iii) In the expansion of $(a+b)^n$ the terms $\binom{n}{r}a^{n-r}b^r$ and $\binom{n}{n-r}a^rb^{n-r}$ are equidistant from the beginning and the end. For $\binom{n}{r}a^{n-r}b^r$ is preceeded by r terms and followed by $n-r$ terms while $\binom{n}{n-r}a^rb^{n-r}$ is preceded by $n-r$ terms and followed by r terms. Also since $\binom{n}{n-r}=\frac{n!}{(n-r)!r!}=\frac{n!}{r!(n-r)!}=\binom{n}{r}$

So the coefficients of terms equidistant from the beginning and end are equal.

(iv) In the expansion of $(a+b)^n$, if n is even, the number of terms are odd and there will be only one middle term. If n is odd, the number of terms are even and there will be two middle terms.

(v) For n even in $(a+b)^n$, the $\left(\frac{n+2}{2}\right)$th term is the only one middle term and for n odd the $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms are the two middle terms.

(vi) In $(a+b)^n$ if b is replaced by $-b$ then $(a-b)^n$ has expansion of the form

$(a-b)^n = \binom{n}{0}a^n b^0 - \binom{n}{1}a^{n-1}b + \binom{n}{0}a^{n-2}b^2 + \dots\dots\dots + (-1)^n\binom{n}{n}a^0 b^n$

or $(a-b)^n = a^n - \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots\dots\dots + (-1)^n b^n$.

We note that in the expansion of $(a-b)^n$ the terms are alternately positive and negative.

(vii) In the expansion of $(a+b)^n$ the (r+1)th term which is $\binom{n}{r}a^{n-r}b^r$ is usually called the **general term** and is denoted by T_{r+1}.

Thus $T_{r+1} = \binom{n}{r}a^{n-r}b^r = \frac{n!}{r!(n-r)!}a^{n-r}.b^r$

We note that for using binomial formula for given value of n, in the expansion of $(a+b)^n$, the most important task is to find the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots\dots$etc.

7.2.3 Pascal's Triangle

Consider the following expanded powers of $(a+b)^n$, where $a+b$ is any binomial and n is a whole number. Look for patterns.

$(a+b)^0 = 1$

$(a+b)^1 = a + b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Each expansion is a polynomial. There are some patterns to be noted.

(i) There is one more term than the power of the exponent, n. That is, there are $n+1$ terms in the expansion of $(a+b)^n$.

(ii) In each term, the sum of the exponents is n, the power to which the binomial is raised.

(iii) The exponents of a start with n, the power of the binomial, and decrease to 0. The last term has no factor of a. The first term has no factor of b, so powers of b start with 0 and increase to n,

(iv) The coefficients start at 1 and increase through certain values about "half-way" and then decrease through these same values back to 1.

The above binomial expansions can be written in the following triangular form

$$\begin{array}{c} 1 \\ a + b \\ a^2 + 2ab + b^2 \\ a^3 + 3a^2b + 3ab^2 + b^3 \\ a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \end{array}$$

Did You Know

Pascal's triangle is most convenient to obtain the coefficients of the binomial expansion $(a+b)^n$ when n is a small number.

For each of the above expansions, we write down the binomial coefficients in the following fashion

n	Values of binomial coefficients
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

The above configuration of numbers is called **Pascal's Triangle.**

Example 8: Find the expansion of $(x+y)^6$.

Solution: By the formula,

$$(x+y)^6 = x^6 + {}^6C_1x^5y + {}^6C_2x^4y^2 + {}^6C_3x^3y^3 + {}^6C_4x^2y^5 + {}^6C_5xy^5 + {}^6C_6y^6$$

$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

On calculating the value of ${}^6C_1$, ${}^6C_2$, ${}^6C_3$,........

Example 9: Find the 6th term in the expansion of $(3x+2y)^{12}$.

Solution: Let T_{r+1}th term be the sixth term of the expansion $(3x+2y)^{12}$. We remember that the T_{r+1}th term for the expansion of $(a+b)^n$ is $T_{r+1} = \binom{n}{r}a^{n-r}b^r$

So, for the given expansion $(3x+2y)^{12}$

$T_{r+1} = \binom{12}{r}(3x)^{12-r}(2y)^r$. Here we have $n = 12$, $a = 3x$ and $b = 2y$

Since we are interested in finding the 6th term i.e. T_6, so choosing r = 5 and putting in the last result, we have,

$$T_{5+1} = T_6 = \binom{12}{5}(3x)^{12-5} \cdot (2y)^5 \qquad \Rightarrow T_6 = \frac{12!}{5!\,7!}3^7 \cdot x^7 \cdot 2^5 \cdot y^5$$

$$\Rightarrow T_6 = \frac{12.11.10.9.8.7!}{1.2.3.4.5.7!} \times 2187 \times 32x^7y^5 \qquad \Rightarrow T_6 = 11 \cdot 9 \cdot 8 \cdot 2187 \cdot 32x^7y^5$$

$$\Rightarrow T_6 = 55427328\,x^7y^5$$

Example 10: Find the coefficient of x^5 in the expansion of $(2x^2 - \frac{3}{x})^{10}$

Solution: Let T_{r+1} of $(2x^2 - \frac{3}{x})^{10}$ be the particular terms containing x^5.

Now for the given expansion $(2x^2 - \frac{3}{x})^{10}$

$$T_{r+1} = \binom{10}{r}(2x^2)^{10-r}\left(-\frac{3}{x}\right)^r = \binom{10}{r}2^{10-r}(x^2)^{10-r}(-1)^r \cdot \frac{3^r}{x^r}$$

$$= (-1)^r\binom{10}{r}2^{10-r} \cdot 3^r x^{20-2r} x^{-r} = (-1)^r\binom{10}{r}2^{10-r} \cdot 3^r \cdot x^{20-2r-r}$$

$$T_{r+1} = (-1)^r\binom{10}{r}2^{10-r} \cdot 3^r \cdot x^{20-3r} \qquad (1)$$

But this term contains x^5 and this is only possible if $x^{20-3r} = x^5$ and thus $20-3r = 5$
$\Rightarrow 3r = 20-5$ or $3r = 15 \Rightarrow r = 5$ Putting this value of r = 5 in (1) we get.

$$T_{5+1} = T_6 = (-1)^5\binom{10}{5}2^{10-r} \cdot 3^5x^{20-15} \qquad \Rightarrow T_6 = (-1)^5\binom{10}{5}2^5 \cdot 3^5x^5$$

So the required coefficient is $(-1)^5\binom{10}{5}2^5 \cdot 3^5 = -\frac{10!}{5!\,5!}32 \cdot 243$

$\therefore$ Required coefficient $= -\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 5!} \cdot 32 \cdot 243$, that is the required coefficient of $x^5 = -1959552$

Example 11: Find the term independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

Solution: Let T_{r+1}th term of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ be the particular term which is independent of x. The T_{r+1}th term for the above expansion is

$$T_{r+1} = \binom{9}{r}\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r = \binom{9}{r}\left(\frac{3}{2}\right)^{9-r}x^{2(9-r)}(-1)^r \cdot \frac{1}{3^r} \cdot \frac{1}{x^r}$$

$$= (-1)^r\binom{9}{r}\left(\frac{3}{2}\right)^{9-r}\frac{1}{3^r} \cdot x^{18-2r} \cdot x^{-r} = (-1)^r\binom{9}{r}\left(\frac{3}{2}\right)^{9-r}\frac{1}{3^r}x^{18-2r-r}$$

$$= (-1)^r\binom{9}{r}\left(\frac{3}{2}\right)^{9-r}\frac{1}{3^r}x^{18-3r} \qquad (1)$$

But T_{r+1}th term is free of x and this is possible if $x^{18-3r} = x^0$ giving $18 - 3r = 0$
$\Rightarrow 3r = 18$ and so $r = 6$

Thus $T_{r+1} = T_{6+1} = T_7$ i.e. 7th term of the given expansion is independent of x.

$$T_7 = (-1)^6\binom{9}{6}\left(\frac{3}{2}\right)^{9-6}\frac{1}{3^6}x^{18-18} = \frac{9!}{6!\,3!}\frac{3^3}{2^3} \cdot \frac{1}{3^6} \cdot 1$$

$$= \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6!\,2 \cdot 3} \cdot \frac{1}{2^3} \cdot \frac{1}{3^3} = \frac{9 \cdot 8 \cdot 7}{2 \cdot 3} \cdot \frac{1}{8} \cdot \frac{1}{27} = \frac{7}{18}$$

Thus the 7th term of the expansion $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is independent of x and its value is $\frac{7}{18}$.

Example 12: Find the middle term in the expansion of $\left(\frac{a}{x} + \frac{x}{a}\right)^{10}$.

Solution: Since in $\left(\frac{a}{x} + \frac{x}{a}\right)^{10}$, n = 10 which is even, so that total number of terms in the above expansion = 10+1 = 11. Thus it has only one middle term which is $\left(\frac{n+2}{2}\right)$th term $= \left(\frac{10+2}{2}\right)$th term = 6th term.i.e. 6th term is the middle term

Now T_{r+1} for $(\frac{a}{x} + \frac{x}{a})^{10}$ is given by

$$T_{r+1} = \binom{10}{r}\left(\frac{a}{x}\right)^{10-r}\left(\frac{x}{a}\right)^r. \text{ Putting } r = 5$$

We get $T_6 = \binom{10}{5}\left(\frac{a}{x}\right)^5\left(\frac{x}{a}\right)^5 = \frac{10!}{5!\,5!} \cdot \left(\frac{a^5}{x^5}\right)\left(\frac{x^5}{a^5}\right) = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 252.$

So the 6th term of $(\frac{a}{x} + \frac{x}{a})^{10}$ is the middle term and it is 252.

EXERCISE 7.2

1. Expand by using Binomial theorem.
   (i) $\left(x^2-\frac{1}{y}\right)^4$ (ii) $(1+xy)^7$ (iii) $\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right)^5$
2. Find the indicated term in the expansions.
   (i) 4^{th} term in $(2+a)^7$ (ii) 8^{th} term in $\left(\frac{x}{2}-\frac{3}{y}\right)^{10}$ (iii) 3^{rd} term in $\left(x^2+\frac{1}{\sqrt{x}}\right)^4$
3. Find the term independent of x in the following expansions.
   (i) $\left(\frac{4a^2}{3}-\frac{3}{2a}\right)^9$ (ii) $\left(x-\frac{3}{x^4}\right)^{10}$ (iii) $\left(x-\frac{1}{x^2}\right)^{21}$
4. Find the coefficient of
   (i) x^{23} in $(x^2-x)^{20}$ (ii) $\frac{1}{x^4}$ in $\left(2-\frac{1}{x}\right)^8$ (iii) a^6b^3 in $\left(2a-\frac{b}{3}\right)^9$
5. Find the middle term in the expansion of:
   (i) $\left(\frac{a}{x}+bx\right)^8$ (ii) $\left(3x-\frac{x^2}{2}\right)^9$ (iii) $\left(3x^2-\frac{y}{3}\right)^{10}$
6. Find the constant term in the expansion of $\left(2\sqrt{x}-\frac{3}{x\sqrt{x}}\right)^{23}$.
7. Find
   (i) $(2+\sqrt{3})^5+(2-\sqrt{3})^5$ (ii) $(1+\sqrt{2})^4-(1-\sqrt{2})^4$ (iii) $(a+b)^5+(a-b)^5$
8. Find the numerically greatest term in $(3-2x)^{10}$, when $x=\frac{3}{4}$.
9. Find the numerically greatest term in the expansion of $(x-y)^{20}$ when $x=12$ and $y=4$.
10. Prove that sum of Binomial coefficients of order $n = 2^n$. Also prove the sum of odd binomial coefficients = sum of even Binomial coefficients $=2^{n-1}$.
11. Consider $(1+x)^n$ and take $\binom{n}{r}=C_r$.
    Show that $C_1+2C_2x+3C_3x^2+\dots\dots+nC_nx^{n-1}=n(1+x)^{n-1}$

7.3 Binomial Series

7.3.1 Expansion of $(1+x)^n$ where n is a positive integer

By Binomial theorem, for any two real numbers a and b and for a positive integer n

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}b^3+\dots\dots+b^n \qquad \text{(i)}$$

and this expansion contains (n + 1) terms. Now in particular if $a=1$ and $b=x$ then the above expansion becomes

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\dots\dots+x^n \qquad \text{(ii)}$$

Thus we observe that when n is a positive integer then the binomial expansion $(a+b)^n$ or $(1+x)^n$ terminates after (n + 1)th term.

7.3.2 Expansion of $(1+x)^n$ where n, the exponent, is a negative integer or a fraction

If n is a negative integer or a fraction, then the expansion (ii) never ends and thus in such a case the expansion becomes

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\dots\dots \qquad \text{(iii)}$$

When n is a negative integer or a fraction then the series as given in (iii) is convergent if $-1<x<1$ or $|x|<1$ and it is divergent if $|x|>1$.

Since at this level we will be interested only in those series which are convergent so we will say that if n is a negative integer or a fraction then the series

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\dots \text{ is valid only if } |x|<1.$$

The series of the type $1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\dots$ is called the **binomial series.**

The general term of the binomial series is

$$T_{r+1}=\frac{n(n-1)(n-2)\dots\dots(n-r+1)}{r!}x^r$$

Example 13: Find the first four terms in the expansion of $(1+x)^{-\frac{1}{2}}$

Solution: $(1+x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^2 + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}x^3 + \dots\dots$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}x^2 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}x^3 + \dots\dots$$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}x^2 - \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{6}x^3 + \dots\dots$$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots\dots$$

Example 14: Find the first four terms in the expansion of $\left(9+\frac{4}{x}\right)^{\frac{1}{2}}$ for $|x| > \frac{4}{9}$.

Solution:

$$\left(9+\frac{4}{x}\right)^{\frac{1}{2}} = \left(9(1+\frac{4}{9x})\right)^{\frac{1}{2}} = 9^{\frac{1}{2}}\left(1+\frac{4}{9x}\right)^{\frac{1}{2}} = 3\left(1+\frac{4}{9x}\right)^{\frac{1}{2}}$$

$$= 3\left[1+\frac{1}{2}\cdot\frac{4}{9x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{4}{9x}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{4}{9x}\right)^3+\dots\dots\right]$$

$$= 3\left[1+\frac{2}{9x}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{16}{81x^2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}-\left(\frac{64}{729x^3}\right)+\dots\dots\right]$$

$$= 3\left[1+\frac{2}{9x}-\frac{2}{81x^2}+\frac{3}{8\times 6}\cdot\frac{64}{729x^3}+\dots\dots\right]$$

$$= 3\left[1+\frac{2}{9x}-\frac{2}{81x^2}+\frac{4}{729x^3}+\dots\dots\right]$$

Example 15: Compute $\sqrt[3]{\frac{5}{4}}$ to an accuracy of at least four decimal places using binomial expansion

Solution:

Given $\sqrt[3]{\frac{5}{4}} = \left(\frac{5}{4}\right)^{\frac{1}{3}} = \left(\frac{4+1}{4}\right)^{\frac{1}{3}}$ or

$$\left(\frac{5}{4}\right)^{\frac{1}{3}} = \left(1+\frac{1}{4}\right)^{\frac{1}{3}} = 1+\frac{1}{3}\cdot\frac{1}{4}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{4}\right)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\left(\frac{1}{4}\right)^3+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{4!}\left(\frac{1}{4}\right)^4+\dots\dots$$

or $$\left(\frac{5}{4}\right)^{\frac{1}{3}} = 1+\frac{1}{12}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}\cdot\frac{1}{16}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}\cdot\frac{1}{64}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{24}\cdot\frac{1}{256}+\dots\dots$$

$$= 1+\frac{1}{12}-\frac{1}{9}\cdot\frac{1}{16}+\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{1}{6}\cdot\frac{1}{64}-\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}\cdot\frac{1}{24}\cdot\frac{1}{256}+\dots\dots$$

$$= 1+\frac{1}{12}-\frac{1}{144}+\frac{5}{81\times 64}-\frac{5}{.\,2.3^5.4^3}+\dots\dots$$

$$= 1 + 0.08333 - 0.00694 + 0.00096 - 0.000016 + \dots\dots$$

Taking only these five terms and neglecting the other we can write

$$\sqrt[3]{\frac{5}{4}} \approx 1.00000 + 0.08333 - 0.00694 + 0.00096 - 0.000016.$$

Where $\approx$ stands for 'approximately equal to'. We have used here the symbol $\approx$ because we have omitted all the terms after the first five terms. So we cannot expect even think for exactness.

$$\sqrt[3]{\frac{5}{4}} \approx 1.07719 \approx 1.0772$$

Example 16: Evaluate $\sqrt{35}$ by Binomial theorem

Solution:

$$\sqrt{35} = (36-1)^{\frac{1}{2}} = \left[36\left(1-\frac{1}{36}\right)\right]^{\frac{1}{2}} = (36)^{\frac{1}{2}}\left[1-\frac{1}{36}\right]^{\frac{1}{2}}$$

$$= 6\left[1-\frac{1}{2}\cdot\frac{1}{36}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{36}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{1}{36}\right)^3+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}\left(-\frac{1}{36}\right)^4+\dots\dots\right]$$

$$\sqrt{35}=6\left[1-\frac{1}{72}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{1296}-\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{6}\cdot\frac{1}{46656}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{24}\cdot\frac{1}{1679616}\cdots\cdots\right]$$

$$=6\left[1-\frac{1}{72}-\frac{1}{10368}-\frac{1}{746496}-\frac{5}{214990848}-\cdots\cdots\right]$$

$$\approx 6[1-0.013888-0.00009645-0.000001339]$$

$$\sqrt{35}\approx 6[0.9860142]$$

$$\sqrt{35}\approx 5.9160852$$

$$\sqrt{35}\approx 5.9161$$

Example 17: If x be so small that its square and higher powers may be neglected then evaluate

(i) $\dfrac{\sqrt{4+x}}{4-\frac{x}{3}}$ (ii) $\dfrac{(1+x)^{\frac{1}{2}}(16-5x)^{\frac{1}{2}}}{(9+2x)^{\frac{1}{2}}}$

Solution:

(i) $$\frac{\sqrt{4+x}}{4-\frac{x}{3}}=\frac{\sqrt{4\left(1+\frac{x}{4}\right)}}{4\left(1-\frac{x}{12}\right)}=\frac{2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}}{4\left(1-\frac{x}{12}\right)}=\frac{1}{2}\left(1+\frac{x}{4}\right)^{\frac{1}{2}}\left(1-\frac{x}{12}\right)^{-1}$$

$$=\frac{1}{2}\left(1+\frac{1}{2}\cdot\frac{x}{4}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{4}\right)^2+\cdots\cdots\right)\times\left(1+\frac{x}{12}+\frac{x^2}{144}+\cdots\cdots\right)$$

$$=\frac{1}{2}\left(1+\frac{x}{8}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{x^2}{16}+\cdots\cdots\right)\left(1+\frac{x}{12}+\frac{x^2}{144}+\cdots\cdots\right)$$

$$=\frac{1}{2}\left[1+\frac{x}{12}+\frac{x}{8}+\right.$$ Ignoring terms containing $x^2, x^3,\ldots\ldots]$

$$=\frac{1}{2}\left[1+\frac{2x+3x}{24}\right]=\frac{1}{2}\left[1+\frac{5x}{24}\right]$$

$$\therefore\ \frac{\sqrt{4+x}}{4-\frac{x}{3}}=\frac{1}{2}\left[1+\frac{5x}{24}\right]$$

where x is so small such that x^2 and higher powers are neglected.

(ii) Now taking $$\frac{(1+x)^{\frac{1}{2}}(16-5x)^{\frac{1}{2}}}{(9+2x)^{\frac{1}{2}}}=\frac{(1+x)^{\frac{1}{2}}(16)^{\frac{1}{2}}\left(1-\frac{5}{16}x\right)^{\frac{1}{2}}}{9^{\frac{1}{2}}\left(1+\frac{2x}{9}\right)^{\frac{1}{2}}}$$

$$=\frac{4(1+x)^{\frac{1}{2}}\left(1-\frac{5}{16}x\right)^{\frac{1}{2}}}{3\left(1+\frac{2x}{9}\right)^{\frac{1}{2}}}=\frac{4}{3}(1+x)^{\frac{1}{2}}\left(1-\frac{5}{16}x\right)^{\frac{1}{2}}\left(1+\frac{2x}{9}\right)^{-\frac{1}{2}}$$

$$=\frac{4}{3}\left[1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\cdots\cdots\right]\left[1-\frac{1}{2}\cdot\frac{5}{16}x+\cdots\cdots\right]\times\left[1-\frac{1}{2}\cdot\frac{2x}{9}+\cdots\cdots\right]$$

$$=\frac{4}{3}\left[1+\frac{x}{2}+\cdots\cdots\right]\left[1-\frac{5}{32}x+\cdots\cdots\right]\left[1-\frac{x}{9}+\cdots\cdots\right]$$

$$=\frac{4}{3}\left[1+\frac{x}{2}+\cdots\cdots\right]\left[1-\frac{x}{9}-\frac{5}{32}x+\frac{5}{32}\cdot\frac{1}{9}x^2\cdots\cdots\right]$$

$$=\frac{4}{3}\left[1+\frac{x}{2}\right]\left(1-\frac{32x+45x}{9\times 32}\right)$$ Ignoring terms containing $x^2, x^3, x^4,\ldots$

$$=\frac{4}{3}\left[1+\frac{x}{2}\right]\left(1-\frac{77x}{9\times 32}\right)=\frac{4}{3}\left(1-\frac{77x}{9\times 32}+\frac{x}{2}\right)$$ Again ignoring term containing x^2

or $$\frac{(1+x)^{\frac{1}{2}}(16-5x)^{\frac{1}{2}}}{(9+2x)^{\frac{1}{2}}}=\frac{4}{3}\left(1+\frac{x}{2}-\frac{77x}{9\times 32}\right)=\frac{4}{3}\left(1+\frac{9\times 16x-77x}{9\times 32}\right)=\frac{4}{3}\left(1+\frac{144x-77x}{288}\right)$$

$$\Rightarrow\ \frac{(1+x)^{\frac{1}{2}}(16-5x)^{\frac{1}{2}}}{(9+2x)^{\frac{1}{2}}}=\frac{4}{3}\left(1+\frac{67x}{288}\right)$$

7.4 Application of the Binomial Theorem

Approximations: We have seen in the particular cases of the expansion of $(1+x)^n$ that the power of x go on increasing in each expansion. Since $|x|<1$, so $|x|^r<|x|$ *for* 2, 3, 4...

This fact shows that terms in each expansion go on decreasing numerically if $|x|<1$.

Thus some initial terms of the binomial series are enough for determining the approximate values of binomial expansions having indices as negative integers or fractions.

Summation of infinite series: The binomial series are conveniently used for summation of infinite series. The series (*whose sum is required*) is compared with

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots$$

to find out the values of n and x. Then the sum is calculated by putting the values of n and x in $(1+x)^n$.

Example 18: Find the sum of the series $1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2\cdot5}{3\cdot6}\cdot\frac{1}{2^2}+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}\cdot\frac{1}{2^3}$............

Solution: Suppose that the given series is identical with the expansion $(1+x)^n$.

We have $(1+x)^n=1+nx+\frac{n(n-1)}{2!}\cdot x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\dots$ (i)

$$S=1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2\cdot5}{3\cdot6}\cdot\frac{1}{2^2}+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}\cdot\frac{1}{2^3}+\dots\dots \quad \text{(ii)}$$

Comparing (i) and (ii), we get

$$nx=\frac{2}{3}\cdot\frac{1}{2} \text{ and } \frac{n(n-1)}{2!}x^2=\frac{2\cdot5}{3\cdot6}\cdot\frac{1}{2^2}$$

Squaring $n^2x^2=\frac{1}{9}$ and $\frac{n(n-1)}{2!}x^2=\frac{5}{36}$ so that

$$\frac{\frac{n(n-1)}{2!}x^2}{n^2x^2}=\frac{\frac{5}{36}}{\frac{1}{9}} \Rightarrow \frac{n(n-1)}{2}\cdot\frac{1}{n^2}=\frac{5}{36}\times\frac{9}{1} \Rightarrow \frac{n-1}{2n}=\frac{5}{4} \Rightarrow \frac{n-1}{n}=\frac{5}{2}$$

$$\Rightarrow 5n=2n-2 \Rightarrow 5n-2n=-2 \Rightarrow 3n=-2 \Rightarrow n=-\frac{2}{3}$$

Putting this value of n in $nx=\frac{2}{3}\cdot\frac{1}{2}$

We get $-\frac{2}{3}x=\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{3} \Rightarrow -2x=1 \Rightarrow x=-\frac{1}{2}$

So, $S=(1+x)^n=(1-\frac{1}{2})^{\frac{-2}{3}}=\left(\frac{1}{2}\right)^{\frac{-2}{3}}=\frac{1}{\left(\frac{1}{2}\right)^{\frac{2}{3}}}=\frac{1}{\frac{1}{2^{\frac{2}{3}}}}=2^{\frac{2}{3}}=4^{\frac{1}{3}}$

i e. $S=4^{\frac{1}{3}}$ and so from (ii)

$$1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2\cdot5}{3\cdot6}\cdot\frac{1}{2^2}+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}\cdot\frac{1}{2^3}+\dots\dots=4^{\frac{1}{3}}$$

Example 19: If $y=\frac{3}{4}+\frac{3\cdot5}{4\cdot5}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+\dots\dots$, Show that $y^2+2y-7=0$

Solution: Given that $y=\frac{3}{4}+\frac{3\cdot5}{4\cdot8}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+\dots\dots$

$$\Rightarrow y+1=1+\frac{3}{4}+\frac{3\cdot5}{4\cdot8}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+\dots\dots(1)$$

Let the series on the R.H.S. of (1) be identical with the expansion $(1+x)^n$. We have,

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\dots(2)$$

Comparing right hand sides of (1) and (2), we have,

$nx=\frac{3}{4}$ ....(3) and $\frac{n(n-1)}{2}x^2=\frac{3\cdot5}{4\cdot8}$ ....(4) Squaring equation (3)

$n^2x^2=\frac{9}{16}$ ....(5) Dividing equation (4) by equation (5)

$$\frac{\frac{n(n-1)x^2}{2}}{n^2x^2}=\frac{\frac{3\cdot5}{4\cdot8}}{\frac{9}{16}} \text{ or } \frac{n(n-1)x^2}{2}\times\frac{1}{n^2x^2}=\frac{3\cdot5}{4\cdot8}\times\frac{16}{9}$$

$$\Rightarrow \frac{n(n-1)}{2n^2}=\frac{5}{6} \Rightarrow \frac{n-1}{n}=\frac{5}{3}$$

or $3(n-1)=5n \Rightarrow 3n-3=5n \Rightarrow -3=5n-3n \Rightarrow 2n=-3$

$\Rightarrow n=-\frac{3}{2}$ Putting $n=-\frac{3}{2}$ in $nx=\frac{3}{4}$, we get $\left(-\frac{3}{2}\right)x=\frac{3}{4} \Rightarrow -\frac{x}{2}=\frac{1}{4}$

$x=\frac{-2}{4}$ or $x=-\frac{1}{2}$

So $y+1=(1+x)^n=1+\frac{3}{4}+\frac{3\cdot5}{4\cdot8}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+\dots\dots$ becomes

$y+1=(1-\frac{1}{2})^{\frac{-3}{2}}$ or $y+1=(\frac{1}{2})^{\frac{-3}{2}} \Rightarrow (y+1)^2=(\frac{1}{2})^{-3}$

or $(y+1)^2=\frac{1}{\left(\frac{1}{2}\right)^3}=\frac{1}{\frac{1}{8}}$

i.e. $(y+1)^2=8 \Rightarrow y^2+2y+1-8=0 \Rightarrow y^2+2y-7=0$

EXERCISE 7.3

1. Find the first four terms in the expansions of

 (i) $(1-x)^{\frac{-1}{2}}$ (ii) $(1-x)^{\frac{3}{2}}$ (iii) $(8+12x)^{\frac{2}{3}}$

2. (i) Find $\sqrt{26}$ correct to 3 decimal places.

 (ii) Evaluate $\frac{1}{\sqrt{(.998)}}$ to four significant figures.

 (iii) Find the cube root of 126 correct to five decimal places.

3. Expand: $\sqrt{\frac{1-x}{1+x}}$ up to x^3.

4. If x is such that x^2 and higher powers may be neglected, then show that $\sqrt{\frac{1-3x}{1+4x}} = 1-\frac{7x}{2}$

5. If x is so small that its square and higher powers can be neglected, then show that

 $$\frac{(8+3x)^{\frac{2}{3}}}{(2+3x)\sqrt{4-5x}} = 1-\frac{5}{8}x$$

6. If x is large and if $\frac{1}{x^3}$ may be neglected, then find the approximate value of: $\frac{x\sqrt{x^2-2x}}{(x+1)^2}$

7. If x^4 and higher powers are neglected, such that

 $(1+x)^{\frac{1}{4}}+(1-x)^{\frac{1}{4}} = a-bx^2$. Find a and b.

8. If x is of such a size that its values are considered up to x^3.

 Show that: $\frac{(1+\frac{1}{2}x)^3-(1+3x)^{\frac{1}{2}}}{1-\frac{5}{6}x} = \frac{15x^2}{8}$

9. Find the co-efficients of x^n in $\left(\frac{1+x}{1-x}\right)^2$

10. Find the sum of the following:

 (i) $1-\frac{1}{2^2}+\frac{1\cdot3}{2!}\cdot\frac{1}{2^4}+$.................. (ii) $1+\frac{5}{8}+\frac{5\cdot8}{8\cdot12}+\frac{5\cdot8\cdot11}{8\cdot12\cdot16}+$..................

11. If $y = \frac{1}{2^2}+\frac{1\cdot3}{2!}\cdot\frac{1}{2^4}+\frac{1\cdot3\cdot5}{3!}\cdot\frac{1}{2^6}+$................ then $y^2+2y-1=0$

12. If $2y = \frac{1}{2^2}+\frac{1\cdot3}{2!}\cdot\frac{1}{2^4}+\frac{1\cdot3\cdot5}{3!}\cdot\frac{1}{2^6}+$................then $4y^2+4y-1=0$

13. If x is so small that x^3 and higher powers of x can be ignored. Show that the nth root of $1+x$ is equal to $\frac{2n+(n+1)x}{2n+(n-1)x}$

14. If x is nearly equal to unity then show that $px^p-qx^q=(p-q)x^{p+q}$

REVIEW EXERCISE 7

1. (i) What is the middle term in the expansion of $(2x+5y)^4$?

 (a) $600x^2y^2$ (b) $120xy^2$ (c) $5000xy^3$ (d) $6x^2y^2$

 (ii) What is the coefficient of the term containing $x^{12}y^6$ in the expansion of $(x^3-2y^2)^7$?

 (a) 84 (b) −280 (c) 560 (d) 448

 (iii) The expansion of $\left(x+\sqrt{x^2-1}\right)^5+\left(x-\sqrt{x^2-1}\right)^5$ is a polynomial of degree

 (a) 5 (b) 6 (c) 7 (d) 8

 (iv) Number of terms in expansion of $\left(\sqrt{x}+\sqrt{y}\right)^{10}+\left(\sqrt{x}-\sqrt{y}\right)^{10}$ is

 (a) 6 (b) 11 (c) 20 (d) 5

 (v) $\left(\sqrt{2}+1\right)^5+\left(\sqrt{2}-1\right)^5 =$ ----------------

 (a) 58 (b) $58\sqrt{2}$ (c) −58 (d) $-58\sqrt{2}$

 (vi) $\binom{n-1}{1}+\binom{n-1}{2}+\ldots\ldots\ldots+\binom{n-1}{n-1} = \ldots\ldots$, $n>1$

 (a) 2^n-1 (b) 2^{n-2} (c) $2^{n-1}-1$ (d) 2^{n-1}

 (vii) Sum of coefficients of last 15 terms in expansion of $(1+x)^{29}$ is

 (a) 2^{15} (b) 2^{30} (c) 2^{29} (d) 2^{28}

 (viii) ${}^{10}C_1+{}^{10}C_3+{}^{10}C_5+\ldots\ldots\ldots+{}^{10}C_9 =$ --------------

 (a) 512 (b) 1024 (c) 2048 (d) 1023