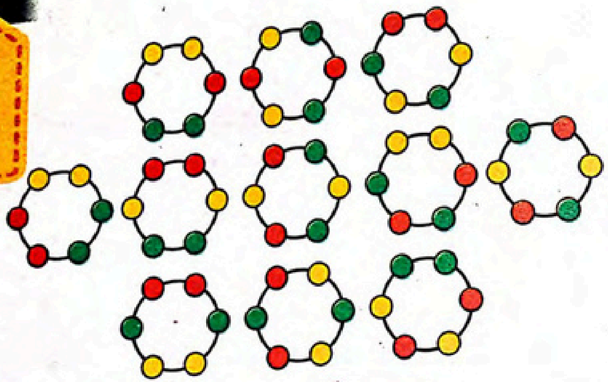
UNIT 6

PERMUTATION, COMBINATION AND PROBABILITY



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Know Kramp's factorial notation to express the product of first n natural numbers by n!.
- Recognize the fundamental principle of counting and illustrate this principle using tree diagram.
- Explain the meaning of permutation of n different objects taken r at a time and know the notation ${}^nP_r$.
- Prove that ${}^nP_r = n(n-1)(n-2)\ldots(n-r+1)$ and hence deduce that

$${}^nP_r = \frac{n!}{(n-r)!},$$

- ${}^nP_n = n!$
- $0! = 1$.
- Apply ${}^nP_r$ to solve relevant problems of finding the number of arrangements of n objects taken r at a time (when all n objects are different and when some of them are alike).
- Find the arrangement of different objects around a circle.
- Define combination of n different objects taken r at a time.

STUDENTS LEARNING OUTCOMES

- Prove the ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ and deduce that
  - $\binom{n}{n} = \binom{n}{0} = 1$
  - $\binom{n}{r} = \binom{n}{n-r} \cdot \binom{n}{1} = \binom{n}{n-1} = n,$
  - $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$
- Solve problems involving combination.
- Define the following:
  - statistical experiment,
  - sample space and an event,
  - mutually exclusive events,
  - equally likely events,
  - dependent and independent events,
  - simple and compound events.
- Recognize the formula for probability of occurrence of an event E, that is $P(E) = \frac{n(E)}{n(S)}$, $0 \le P(E) \le 1$
- Apply the formula for finding probability in simple cases.
- Use Venn diagrams and tree diagrams to find the probability for the occurrence of an event.
- Define the conditional probability
- Recognize the addition theorem (or law) of probability $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where A and B are mutually exclusive events.
- Recognize multiplication theorem (or law) of probability $P(A \cap B) = P(A)\,P(B|A)$ or $P(A \cap B) = P(B)\,P(A|B)$ where $P(B|A)$ and $P(A|B)$ are conditional probabilities.
  Deduce that $P(A \cap B) = P(A)\,P(B)$ where A and B are independent events.
- Use theorem of addition and multiplication of probability to solve related problems

6.1 Introduction

Counting is one of the most fundamental skills. People start to count on their fingers when they are in kindergarten or even earlier. But how to count quickly, correctly, and systematically is a lifelong course.

In order to study probability, it is first necessary to learn about combinatorics, the theory of counting.

In this unit, we will develop techniques and formulae for counting the number of objects in a set. These formulae are used in computer science to analyze algorithms. They are also used to determine probabilities, the likelihood that a certain outcome of a random experiment will occur.

6.1.1 Kramp's Factorial notation to express the product of first n natural numbers by n!

Factorial Notation

If n is a positive integer, the notation n! (read "n factorial") is the product of all positive integers from n down through 1.

$n! = n(n-1)(n-2)\text{-------}(3)(2)(1)$

0! (zero factorial), by definition, $0! = 1$

Remember

The First Ten Factorials

$0! = 1$
$1! = 1$
$2! = 2\cdot 1 = 2$
$3! = 3\cdot 2\cdot 1 = 6$
$4! = 4\cdot 3\cdot 2\cdot 1 = 24$
$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$
$6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$
$7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5{,}040$
$8! = 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 40{,}320$
$9! = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 362{,}880$

Technology

Most calculators have factorial keys. To find 5!, most calculators use one of the following:

Many Scientific Calculators
5 [x!]

Many Graphing Calculators
5 [!] [ENTER]

Note The Difference

$2\cdot 3! = 2(3\cdot 2\cdot 1) = 12$
$(2\cdot 3)! = 6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$

Example 1: Simplify the following expressions:

$a.\ \frac{8!}{7!}\quad b.\ \frac{5!}{2!.3!}\quad c.\ \frac{1}{2!.4!}+\frac{1}{3!.3!}\quad d.\ \frac{(n+1)!}{n!}\quad e.\ \frac{n!}{(n-3)!}$

Solution: a. $\frac{8!}{7!}=\frac{8.7!}{7!}=8$

b. $\frac{5!}{2!\cdot 3!}=\frac{5.4.3!}{2!\cdot 3!}=\frac{5.4}{2\cdot 1}=10$

c. $\frac{1}{2!\cdot 4!}+\frac{1}{3!\cdot 3!}=\frac{1}{2!\cdot 4!}\cdot\frac{3}{3}+\frac{1}{3!\cdot 3!}\cdot\frac{4}{4}=\frac{3}{3\cdot 2!\cdot 4!}+\frac{4}{3!\cdot 4\cdot 3!}$

$=\frac{3}{3!\cdot 4!}+\frac{4}{3!\cdot 4!}=\frac{7}{3!\cdot 4!}=\frac{7}{144}$

d. $\frac{(n+1)!}{n!}=\frac{(n+1).n!}{n!}=n+1$

e. $\frac{n!}{(n-3)!}=\frac{n.(n-1).(n-2).(n-3)!}{(n-3)!}$

$= n.(n-1).(n-2) = n^3-3n^2+2n$

Practice

Evaluate each factorial expression:

a. $\frac{14!}{2!12!}$ b. $\frac{n}{(n-1)!}$

Example 2: Write the following in factorial form:

(i) $\frac{13\cdot 17}{9\cdot 8\cdot 7\cdot 5}$ (ii) $\frac{(n-3)(n-2)(n-1)}{n(n-4)}$

Solution: (i) $\frac{13\cdot 17}{9\cdot 8\cdot 7\cdot 5}=\frac{17\cdot 16\cdot 15\cdot 14\cdot 13\cdot 12!}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4!}\cdot\frac{13\cdot 6\cdot 4!}{16\cdot 15\cdot 14\cdot 13\cdot 12!}$

$=\frac{17!}{9!}\cdot\frac{6\cdot 5!\cdot 4!\cdot 13\cdot 12!}{16!\cdot 5!\cdot 12!}=\frac{17!\cdot 13!\cdot 6!\cdot 4!}{16!\cdot 12!\cdot 9!\cdot 5!}$

(ii) $\frac{(n-3)(n-2)(n-1)}{n(n-4)}=\frac{(n-1)(n-2)(n-3)}{n(n-4)}=\frac{(n-1)(n-2)(n-3)(n-4)!}{n(n-4)(n-4)!}$

$=\frac{(n-1)!(n-1)(n-2)(n-3)(n-5)!}{n(n-1)(n-2)(n-3)(n-4)(n-5)!(n-4)!}$

$=\frac{(n-1)!(n-5)!}{n!(n-4)!}\cdot\frac{(n-1)(n-2)(n-3)(n-4)!}{(n-4)!}$

$=\frac{(n-1)!(n-5)!}{n!(n-4)!}\cdot\frac{(n-1)!}{(n-4)!}=\frac{((n-1)!)^2(n-5)!}{n!((n-4)!)^2}$

EXERCISE 6.1

1. Evaluate the following.

   (i) $\frac{10!}{3!3!4!}$ (ii) $\frac{3!+4!}{5!-4!}$ (iii) $\frac{(n-1)!}{(n+1)!}$ (iv) $\frac{10!}{(5!)^2}$

2. Write the following in terms of factorials.

   (i) $19\cdot18\cdot17\cdot16\cdot15\cdot14$ (ii) $2\cdot4\cdot6\cdot8\cdot10\cdot12$

   (iii) $n(n^2-1)$ (iv) $\frac{n(n+1)(n+2)}{3}$

3. Prove the following.

   (i) $\frac{1}{6!}+\frac{2}{7!}+\frac{3}{8!}=\frac{75}{8!}$ (ii) $\frac{(n+5)!}{(n+3)!}=n^2+9n+20$

4. Find the value of n, when

   (i) $\frac{n(n!)}{(n-5)!}=\frac{12(n!)}{(n-4)!}$ (ii) $\frac{n!}{(n-4)!}:\frac{(n-1)!}{(n-4)!}=9:1$

5. Show that (i) $\frac{(2n)!}{n!}=2^n(1\cdot3\cdot5\cdots(2n-1))$

   (ii) $\frac{(2n+1)!}{n!}=2^n(1\cdot3\cdot5\cdots(2n-1)(2n+1))$

6.2 Permutation

As we know that counting plays a vital role in many areas, such as probability, statistics and computer science. In this section and in the next, we shall look at special types of counting problems and develop general formulae for solving them.

The following principle of counting will be helpful and basic to all our work.

6.2.1 Fundamental Principle of Counting

Let $E_1, E_2, \ldots, E_k$ be a sequence of k events. If for each i, E_i can occur in m_i different ways, then the total number of ways the events may take place is the product $m_1m_2\ldots m_k$.

This principle is also known as the multiplication principle.

Example 3: How many different 6-place vehicle number plates are possible if the first 3 places are to be occupied by letters and the final 3 by numbers?

Solution: Since the first three places are to be occupied by the letters A,B,C,…,Z and the final 3 places by the numbers 0, 1, 2, . . . , 9.

Hence each event E_i, $i = 1, 2, 3$ occurs in $m_i = 26$, $i = 1, 2, 3$ different ways and each E_i, $i = 4, 5, 6$ occurs in $m_i = 10$, $i = 4, 5, 6$ different ways. Then by the fundamental counting principle the total number of vehicle number plates is

$$m_1\cdot m_2\cdot m_3\cdot m_4\cdot m_5\cdot m_6 = 26\cdot26\cdot26\cdot10\cdot10\cdot10 = 17576000$$

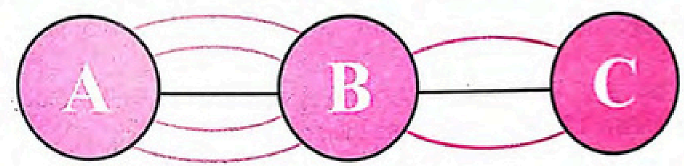
Example 4: How many functions defined on n points is possible if each functional value is either 0 or 1?

Solution: Let the points be 1, 2, 3, . . . , n.

Since $f(i) = 0$ or 1 for each $i = 1, 2, 3, \ldots, n$. Hence each event E_i, $i = 1, 2, 3, \ldots, n$ has $m_i = 2$, $i = 1, 2, 3, \ldots, n$ possibilities. Thus by the fundamental counting principle the total numbers of possible functions is

$$m_1\cdot m_2\cdot m_3\cdots m_n = 2\cdot2\cdot2\cdots2 = 2^n$$

Example 5: There are 5 roads joining A to B and 3 roads joining B to C. Find how many different routes there are from A to C via B.



Solution: There are two operations to be performed in succession.

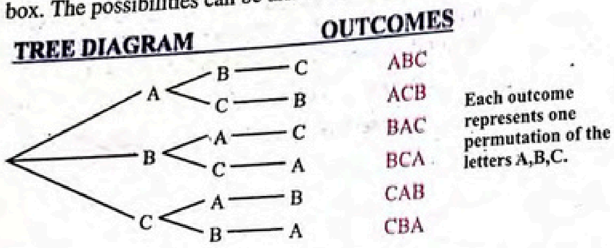
A to B	5 ways
B to C	3 ways

Number of routes from A to C $=5\times3=15$

Example 6: How many 3-letter code symbols can be formed with the letters A, B, C without repetition?

Solution: Consider placing the letters in these boxes.

We can select any of the 3 letters for the first letter in the symbol. Once this letter has been selected, the second must be selected from the 2 remaining letters. After this, the third letter is already determined, since only 1 possibility is left. That is, we can place any of the 3 letters in the first box, either of the remaining 2 letters in the second box, and the only remaining letter in the third box. The possibilities can be arrived at using a tree diagram, as shown below.



We see that there are 6 possibilities. The set of all the possibilities is {ABC, ACB, BAC, BCA, CAB, CBA}.

Example 7: How many 3-letter code symbols can be formed with the letters A, B, C, D, and E with repetition (that is, allowing letters to be repeated)?

Solution: Since repetition is allowed, there are 5 choices for the first letter, 5 choices for the second, and 5 for the third. Thus there are $5\cdot5\cdot5$, or 125 code symbols.

Example 8: How many 5-letter code symbols can be formed with the letters A, B, C, and D if we allow a letter to occur more than once?

Solution: We can select each of the 5 letters in 4 ways. That is, we can select the first letter in 4 ways, the second in 4 ways, and so on. Thus there are 4^5, or 1024 arrangements.

6.2.2 Explaining the meaning of permutation

An ordered arrangement of a finite number of elements taken some or all at a time is called a **permutation** of these elements.

We use the notation ^nP_r or $P(n,r)$ to denote the number of permutations of n elements taken r at a time, where r is a positive integer such that $r \le n$.

Now, we develop general formula for the solution of special types of counting problems.

6.2.3 $^nP_r = n(n-1)(n-2)\ldots(n-r+1)$

Theorem: Prove that $^nP_r = n(n-1)(n-2)\ldots(n-r+1)$ and hence deduce the following: (i) $^nP_r = \dfrac{n!}{(n-r)!}$ (ii) $^nP_n = n!$ (iii) $0! = 1$

Proof: To find a formula for ^nP_r, we note that the task of obtaining an ordered arrangement of n elements in which only $r \le n$ of them are used without repetitions, requires making r selections. Therefore, for the first selection, there are n choices; for the second selection, there are (n – 1) choices; for the third, there are (n – 2) choices; and so on. Hence the events:

E_1 occurs in $m_1 = n$ ways

E_2 occurs in $m_2 = (n-1)$ ways

E_3 occurs in $m_3 = (n-2)$ ways

.

.

and E_r occurs in $m_r = \left(n-(r-1)\right) = (n-r+1)$ ways

Thus by the Fundamental Counting Principle

$$^nP_r = m_1\cdot m_2\cdot m_3\cdots m_r = n(n-1)(n-2)\cdots(n-r+1)$$

(i) Since $^nP_r = n(n-1)(n-2)\cdots(n-r+1)$

$$^nP_r = n(n-1)(n-2)\cdots(n-r+1).\frac{(n-r)!}{(n-r)!}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

(ii) Since $^nP_r = n(n-1)(n-2)\cdots(n-r+1)$

Now, putting r = n in the above, we obtain:

$$^nP_n = n(n-1)(n-2)\cdots(n-n+1) = n(n-1)(n-2)\cdots1$$

$$= n(n-1)(n-2)\cdots3\cdot2\cdot1 = n!$$

(iii) Since $^nP_n = n!$ then by using (ii), we obtain: $\dfrac{n!}{(n-n)!} = n!$

or $\dfrac{1}{0!} = 1 \Rightarrow 0! = 1$

Example 9: How many distinct six digit numbers can be formed from the integers 1, 2, 3, 4, .., 9 if each integer is used only once?

Solution: Since the total number of digits is 9 and each number we have to find, consists of six digits. No repetition is allowed. Therefore, this is a problem of permutation.

$\therefore$ The required number of six digit numbers $= {}^9P_6$

$$= \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3!}{3!} = \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3!}{3!}$$

$$= 9\cdot8\cdot7\cdot6\cdot5\cdot4 = 60480$$

Example 10: How many different words can be made out of the letters of the word "triangle"? How many of these will begin with t and end with e?

Solution:

(i) There are 8 different letters in the word "triangle". Therefore, the number of different words $= {}^8P_8 = \frac{8!}{(8-8)!} = \frac{8!}{0!} = 8! = 40320$

(ii) If 't' and 'e' occupy the first and last places, then we are left only with 6 different letters. Thus the number of different words in this case is ${}^6P_6 = 6! = 720$

Example 11: How many different arrangements of 10 objects taken 4 at a time can be made with one particular object (i) never occurs (ii) always?

Solution: There are 10 different objects and we are taking 4 at a time. Then the possible arrangements are ${}^{10}P_4 = \frac{10!}{(10-4)!} = 5040$ (1)

(i) Since one of the objects never occurs, so we are left with 9 objects. Thus the possible arrangements taking 4 at a time $= {}^9P_4 = \frac{9!}{(9-4)!} = 3024$ (2)

(ii) The possible arrangements that the particular object always occurs is obtained by subtracting (2) from (1), i.e. $5040 - 3024 = 2016$

6.2.4. Permutations with Repeated Elements

Consider the example of finding the number of different 9 digit numerals that can be formed from the digits: 6, 6, 6, 6, 5, 5, 5, 4, 3 and consider one such numeral: 665566543 (i)

With this ordering of the 9 digits, there are 4! Permutations of the digits 6 and 3! Permutations of the digits 5 which have no effect on the above numeral. Therefore, there are $4!\cdot3!$ arrangements of digits in the numeral given in (i) which do not result in a distinguishable permutation of the given nine digits. Hence if X is the number of distinguishable permutations of the given 9 digits, then $4!\cdot3!\cdot X = 9!$, where 9! is the number of permutations of 9 distinct elements taken 9 at a time.

$$\therefore X = \frac{9!}{4!\cdot3!} = 2520$$

The above example shows that in case of repeated elements, the number of permutations is reduced. Hence we have the following result.

Theorem: The number of distinguishable permutations of n elements taken all at a time, in which m_1 are alike, m_2 are alike, . . . and m_k are alike is $\frac{n!}{m_1!.m_2!...m_k!}$

Proof: Let X be the required number of distinguishable permutations. Now, if we replace m_1 alike elements by m_1 different elements, then the number of permutations of m_1 distinct elements taken all at a time is $m_1!$. Similarly the replacement of $m_2, \dots, m_k$ alike elements by different elements give rise to $m_2!, \dots m_k!$ permutations respectively.

Thus the simultaneous replacement of alike elements by different elements increases the number of permutations to $X.\ m_1!.\ m_2! \dots m_k!$

Since $n = m_1 + m_2 + \dots + m_k$, then the number of permutations of n distinct elements is n!

$$\therefore X.m_1!.m_2!...m_k! = n! \Rightarrow X = \frac{n!}{m_1!.m_2!...m_k!}.$$

Where X is generally denoted by, $\binom{n}{m_1, m_2, \dots, m_k}$

Thus $\binom{n}{m_1, m_2, \dots, m_k} = \frac{n!}{m_1!.m_2!...m_k!}$

> **Remember**
> We usually omit those digits, which occur once.

Example 12: Find the number of different arrangements that can be made out of the letters of the word "assassination" taken all together.

Solution: The total number of letters is 13, out of which 4 are s, 3 are a, 2 are i and 2 are n, so $n = 13$, $m_1 = 4$, $m_2 = 3$, $m_3 = 2$, $m_4 = 2$

Thus the required number of permutations $= \binom{n}{m_1, m_2, m_3, m_4} = \binom{13}{4, 3, 2, 2} = \frac{13!}{4!\cdot3!\cdot2!\cdot2!}$

$$= 10810800$$

Example 13: How many eight – digit different numbers are possible using all of the digits 1, 1, 1, 1, 2, 2, 3, 4?

Solution: The total number of digit is 8, out of which four are 1s and two are 2s.

So here $n = 8$, $m_1 = 4$, $m_2 = 2$,

thus the total eight digit number = $\begin{pmatrix} n \\ m_1, m_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4,2 \end{pmatrix} = \frac{8!}{4!\cdot 2!} = 840$

6.2.5 Arrangements of Distinct Elements Round a circle

We have been arranging elements in a row and have seen that 4 elements can be arranged in a row in 4! = 24 different ways. Suppose we arrange these same 4 elements in a symmetric circular pattern. For example let us arrange A, B, C, D around a circle. One such arrangement is shown in Figure 6.1 and others in Figure 6.2.

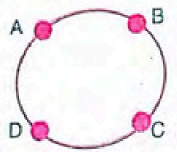


Figure 6.1.

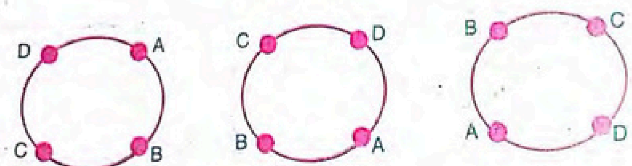


Figure 6.2

Now to check, whether these four arrangements are different or not. Let us ignore the positions of A, B, C, D and consider only their relative order as we go around the circle in a specific direction. We see that these four arrangements are the same. For example if we begin at A and move clockwise around any of the circles we get the same arrangement, ABCD and then back to A again. Thus the four different arrangements ABCD, BCDA, CDAB and DABC are not distinguishable in a circular arrangement.

In general, if there are X distinct circular arrangements of four elements, there would be $4{\cdot}X$ arrangements of these elements along a row. But since the number of arrangements along a row is 4! ,

then we have $4{\cdot}X = 4! \Rightarrow X = \frac{4!}{4} = \frac{4{\cdot}3!}{4} = 3!$

Extending this, we have the following:

The number of distinguishable circular permutations of n elements is (n–1)!.

In arranging keys on a ring or different beads on a necklace, it is agreed that two arrangements are the same if one arrangement can be obtained from the other by turning over the ring (or the necklace) is reflection of one another. Thus in case of the Example 15 of four elements A, B, C, D, the following two arrangements are the same under such conditions (reflection of one another).

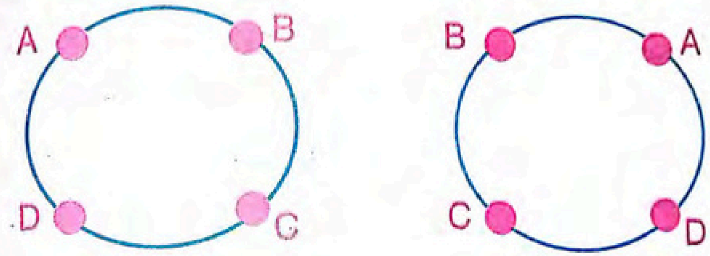


Figure 6.3

Consequently, there are three different arrangements of four different keys on a ring (or four different beads on a necklace), that is, the number of different arrangements is $\frac{(4-1)!}{2} = \frac{3!}{2} = 3$

More generally, the number of different arrangements of keys on a ring (or n different beads on a necklace) is $\frac{(n-1)!}{2}$.

Example 14: In how many ways can six people be seated around a circular table?

Solution: In this case n = 6, so that six people can be seated around a circular table in $(6-1)! = 5! = 120$ ways.

Example 15: How many different necklaces can be formed by stringing eight beads of different colors?

Solution: The number of different necklaces is $\frac{(n-1)!}{2}$

So for n = 8 , we have $\frac{(8-1)!}{2} = \frac{7!}{2} = 2520$ different necklaces.

EXERCISE 6.2

1. Evaluate (i) ${}^{6}P_{6}$ (ii) ${}^{20}P_{2}$ (iii) ${}^{16}P_{3}$
2. Solve for n (i) ${}^{n}P_{5}=56({}^{n}P_{3})$ (ii) ${}^{n}P_{5}=9({}^{n-1}P_{4})$ (iii) ${}^{n^2}P_{2}=600$
3. Prove the following by Fundamental Principle of counting

   (i) ${}^{n}P_{r}=n({}^{n-1}P_{r-1})$ (ii) ${}^{n}P_{r}={}^{n-1}P_{r}+r({}^{n-1}P_{r-1})$
4. In how many ways can a police department arrange eight suspects in a line up?
5. In how many ways can letters of the word 'Fasting' be arranged?
6. How many 4 digit numbers can be formed with the digits 2, 4, 5, 7, 9. (Repetitions not being allowed). How many of these are even?
7. How many three digit numbers can be formed from the digits 1, 2, 3, 4 and 5 if repetitions (i) are allowed (ii) are not allowed.
8. How many different arrangements can be formed of the word "equation" if all the vowels are to be kept together?
9. How many signals can be given by six flags of different colors when any number of them are used at a time?
10. In how many ways can five students be seated in a row of eight seats if a certain two students (i) insist on sitting next to each other?

    (ii) refuse to sit next to each other?
11. How many numbers each lying between 10 and 1000 can be formed with digits 2, 3, 4, 0, 8, 9 using only once?
12. How many different words can be formed from the letters of the following words if the letters are taken all at a time?

    (i) Bookworm (ii) Bookkeeper (iii) Abbottabad (iv) Letter
13. Find the number of permutations of the word 'EXCELLENCE'. How many of these permutations (i) begin with E (ii) begin with E and end with C (iii) begin with E and end with E (iv) do not begin with E. (v) contain two 2L's together (vi) do not contain 2L's together.
14. If five distinct keys are placed on a key ring, how many different orders are possible?
15. In how many ways can 7 people be arranged at a round table so that 2 particular persons always sit together?

6.3 Combination

So for, we have been concerned with permutations, which are ordered arrangements of elements of a set. Now, we focus our discussion on arrangements in which order is not important that is, subsets of a set.

6.3.1 Let S be a set containing n elements and suppose r is a positive integer such that $r \le n$ .Then any subset of S containing r distinct elements is called a combination of n elements taken r at a time.

Notation: The notation, we use for the number of combinations of n elements taken r at a time is ${}^{n}C_{r}$ or $\binom{n}{r}$.

Example 16: Suppose S = {a, b, c, d}. Find the number of combinations by taking 3 letters at a time.

Solution: The subsets of S taken three elements at a time are:

{ a, b, c}, { a, b, d}, { a, c, d}, {b, c, d}

Therefore, ${}^{4}C_{3}=4$

The distinction between permutations and combinations is that changing the order of a set of elements gives a different permutation but the same combination. For example in the above example there are four subsets of {a,b,c,d}, taken three at a time , because ${}^{4}C_{3}=4$. But the elements of each one of the four subsets can be arranged in a definite order in 3! or 6 different ways. Thus the total number of different arrangements in a definite order in all four subsets is

$$6\cdot 4={}^{4}P_{3} \quad \text{or} \quad 3!\cdot{}^{4}C_{3}={}^{4}P_{3}$$

$$\text{or } {}^{4}C_{3}=\frac{{}^{4}P_{3}}{3!}=\frac{4!}{3!(4-3)!}$$ and we have the following important formula.

6.3.2 Theorem: Prove that ${}^{n}C_{r}=\dfrac{n!}{r!(n-r)!}$ And hence deduce that

(i) $\binom{n}{n}=1$, (ii) $\binom{n}{0}=1$, (iii) $\binom{n}{1}=n$, (iv) $\binom{n}{n-1}=n$, (v) $\binom{n}{r}=\binom{n}{n-r}$

Proof: To find ${}^{n}C_{r}$, we must find the total number of subsets of r elements each of that can be obtained from a set of n elements. Since each of these combinations (subsets) contains r elements, which can be permuted among themselves in r! ways. Thus ${}^{n}C_{r}$ such combinations will give ${}^{n}C_{r}.r!$ permutations. But we know that the number of permutations of n elements taken r at a time is ${}^nP_r$ $\quad \therefore \quad {}^nC_r . r! = {}^nP_r = \dfrac{n!}{(n-r)!} \Rightarrow {}^nC_r = \dfrac{n!}{r!(n-r)!}$

(i) If $r=n$, then ${}^nC_n = \dfrac{n!}{n!(n-n)!} = \dfrac{1}{0!} = 1,$ $\quad \because 0! = 1$

(ii) If $r=0$, then ${}^nC_0 = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{n!} = 1$

(iii) If $r=1$, then ${}^nC_1 = \dfrac{n!}{1!(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n$

(iv) If $r=n-1$, then ${}^nC_{n-1} = \dfrac{n!}{(n-1)!(n-n+1)!} = \dfrac{n(n-1)!}{(n-1)!.1!} = n$

(v) Putting (n – r) for r, we have

$$ {}^nC_{n-r} = \frac{n!}{(n-r)!\,(n-n+r)!} = \frac{n!}{(n-r)!\,r!} = {}^nC_r $$

Example 17: Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Solution: Taking L.H.S $= {}^nC_r + {}^nC_{r-1}$

$$ = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} $$

$$ = \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!} $$

$$ = \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right] $$

$$ = \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right] $$

$$ = \frac{(n+1)n!}{r(r-1)!(n-r+1)(n-r)!} $$

$$ = \frac{(n+1)!}{r!.(n-r+1)!} = \frac{(n+1)!}{r![(n+1)-r]!} = {}^{n+1}C_r = \text{R.H.S} $$

Did You Know ?

The number of combinations of n things r at a time is equal to the number of combinations of n things n – r at a time i.e. ${}^nC_r = {}^nC_{n-r}$

Such combinations are called complementary.

Put $r = n$, then ${}^nC_0 = {}^nC_n = 1$

Example 18: From 12 books in how many ways can a selection of 5 be made, (i) when one specified book is always included, (ii) when one specified book is always excluded?

Solution: (i) Since the specified book is to be included in every selection, we have only to choose 4 out of the remaining 11.

Hence the number of ways $= {}^{11}C_4$

$$ = \frac{11\times10\times9\times8}{1\times2\times3\times4} = 330. $$

(ii) Since the specified book is always to be excluded, we have to choose the 5 books out of the remaining 11.

Hence the number of ways $= {}^{11}C_5 = \dfrac{11\times10\times9\times8\times7}{1\times2\times3\times4\times5} = 462.$

Example 19: Out of 14 men in how many ways can an eleven be chosen?

Solution: The required number $= {}^{14}C_{11} = {}^{14}C_3 = \dfrac{14\times13\times12}{1\times2\times3} = 364.$

EXERCISE 6.3

1. Solve the following for n.
   (i) ${}^nC_2 = 36$ (ii) ${}^{n+1}C_4 = 6.{}^{n-1}C_2$ (iii) ${}^{n^2}C_2 = 30.{}^nC_3$
2. Find n and r if ${}^nP_r = 840$ and ${}^nC_r = 35$
3. Find n when ${}^{2n}C_3 : {}^nC_2 = 36 : 3$
4. Prove that (i) ${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^nC_r$ (ii) $r.{}^nC_r = n.{}^{n-1}C_{r-1}$
5. How many (i) straight lines (ii) triangles are determined by 12 points, no three of which lie on the same straight line.
6. Find the total number of diagonals of a hexagon.
7. Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?
8. A student is to answer 7 out of 10 questions in an examination. How many choices has he, if he must answer the first 3 questions?
9. An 8-person committee is to be formed from a group of 6 women and 7 men. In how many ways can the committee be chosen if (i) the committee must contain four men and four women? (ii) there must be at least two men? (iii) there must be at least two women? (iv) there must be more women than men?

6.4 Probability

Unconscious application of probability theory is very wide and indeed, practically every one is applying it without realizing. The phrases like "He is reliable", "He is a liar", "He is not likely to come" and so on are all probabilistic and we use them by "applying" probability theory. Basically, probability originated in problems related to games of chance and was developed mathematically by Pascal (1623 – 1662) and Fermat (1601 – 1665). Today, probability has grown far beyond the area of games of chance and has applications in genetics, insurance, physics, social sciences, engineering and medicine.

Before defining probability, we define and explain certain terms which are used in its definition.

6.4.1 (i) Statistical Experiment

Intuitively by an experiment one pictures a procedure being carried out under a certain set of conditions. The procedure can be repeated any numbers of times under the same set of conditions and upon completion of the procedure certain results are observed. The experiments are of two types

(a) Deterministic experiment An experiment is deterministic if, given the conditions under which the experiment is carried out, the outcome is completely determined. For example if pure water is brought to a temperature of 100^0 C and 760 mm Hg of atmospheric pressure the outcome is that the water will boil.

(b) Random experiment An experiment for which the outcome cannot be predicted except that it is known to be one of a set of possible outcomes, is called a random experiment.

For example (i) Tossing a coin (ii) Rolling a die.

Since our interest lies in the random experiment, so in this text by experiment we mean random experiment.

(ii) Sample space and an event

The set of all possible outcomes of a random experiment is called a sample space and is donated by S. The elements of S are called sample points or outcomes.

For example (a) Tossing a coin once, then

S= {H, T} where H and T are the possible outcomes.

(b) Tossing a coin twice, then the possible outcomes in the sample space are

HH, HT, TH, TT.

(c) Rolling a pair of dice, then we have the following sample space

$S = \{ (i, j) : i, j = 1, 2, 3, 4, 5, 6 \}$

$$= \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix},$$

Event: Let S be the sample space of an experiment. Any subset E of S is called an event associated with the experiment. For example E = {HH, TT} is an event associated with the experiment of tossing a coin twice.

(iii) Mutually Exclusive events

Two events are said to be mutually exclusive if they cannot both occur at the same time. Mathematically, it is expressed as:

If $A \cap B = \phi$, then A and B are mutually exclusive events.

For example rolling a die, let A be the event that even number has shown up while B be the event that odd number has shown up and C be the event that a number less than 4 has occurred.

Here S = {1, 2, 3, 4, 5, 6}

Let A = {even number has shown up } = {2, 4, 6}

B = {odd number has shown up } = {1, 3, 5}

and C = {a number less than 4 has occurred}= {1, 2, 3}

Now $A \cap B = \phi \Rightarrow$ A and B are mutually exclusive while $A \cap C = \{2\}$ and $B \cap C = \{1,3\}$ showing that A, C and B, C are not mutually exclusive.

(iv) Equally likely events

Two events are said to be equally likely if they have equal chances of happening. In other words, each event is as likely to occur as the other. For example rolling a die we have S = { 1, 2, 3, 4, 5, 6 } and each simple event $A_j = \{ j : j = 1, 2, 3, 4, 5, 6 \}$ is as likely to appear as the other. Hence they are equally likely events.

(v) Simple and compound events

Events of the form {s} are called simple events, while an event containing at least two sample points is called a compound event. For example $E_1 = \{HH\}$ is a simple event and $E_2 = \{HH, TT\}$ is a compound event associated with the experiment of tossing a coin twice.

If the random experiment results in s and $s \in A$, we say that the event A occurs or happens. The $\bigcup_j A_j$ occurs if at least one of the A_j occurs. The $\bigcap_j A_j$ occurs if all A_j occur.

If the event A occurs, then $\overline{A}$ (complement of A relative to S) fails to occur.

6.4.2 Let S be the sample space of a random experiment, and E be an event. The probability that an event E will occur, denoted by P (E) is given by

> **Did You Know ?**
>
> **Favorable or Successful Outcomes**
>
> The outcomes which entail the happening of an event are said to be favorable (successful) to the event.
>
> For example rolling a die, the number of outcomes favorable (successful) to the happening of event of even integers are three, i.e 2,4 and 6.

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{\text{the number of favorable (successful) outcome}}{\text{the total number of outcomes}}$$

$$= \frac{\text{no. of elements in the event E}}{\textit{no. of elements in the sample space S}}$$

Since E is a subset of S, then obviously

$0 \le n(E) \le n(S)$ Dividing by n (S), we obtain

$$\frac{0}{n(S)} \le \frac{n(E)}{n(S)} \le \frac{n(E)}{n(S)} \text{ Or } 0 \le P(E) \le 1$$

Hence the probability of an event is always a number between 0 and 1 inclusive.

By the above definition, it is quite clear that $P(\phi) = 0$ and $P(S) = 1$ that is why ϕ is called an impossible event while S is called sure or certain event. If E and F are two events such that $P(E) < P(F)$, then we say that F is more likely to occur than E and if P (E) = P (F), the events E and F are equally likely.

Example 20: (a) If a coin is flipped, find the probability that a head will turn up.

(b) If a fair die is tossed, find the probability that an even number has shown up.

Solution: (a) Here S = { H , T }

Let A = { head has shown up } = { H }

Since, the outcomes are equally likely, then using the formula:

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{2}$$

(b) Here S = { 1, 2, 3, 4, 5, 6 }

Let B = { even number has shown up }= { 2, 4, 6 }

Since, the outcomes are equally likely, then we have $P(B) = \frac{n(B)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

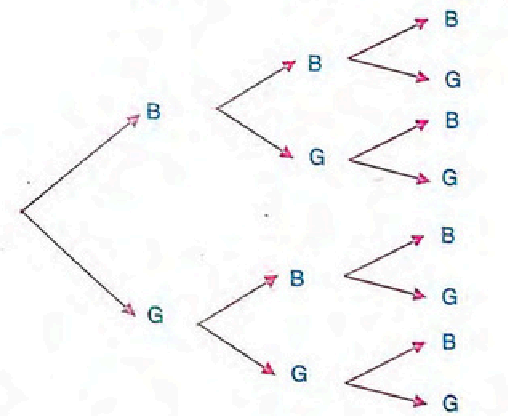
Example 21: In a three child family what is the probability of having

(i) three boys? (ii) at most one boy ?

(iii) at least one boy (iv) exactly one boy ?

Solution: Sometimes a tree diagram is very helpful in constructing a sample space S.

First child | Second child | Third child | Outcomes



BBB
BBG
BGB
BGG
GBB
GBG
GGB
GGG

Hence S = { BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG } and the outcomes are equally likely.

(i) Let A = { having three boys } = { BBB } then $P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$

(ii) Let C = { having at most one boy } = { BGG, GBG, GGB, GGG }

then $P(C) = \frac{n(C)}{n(S)} = \frac{4}{8} = \frac{1}{2}$

(iii) Let D = { having at least one boy }
= { BBB, BBG, BGB, BGG, GBB, GBG, GGB }

then $P(D) = \frac{n(D)}{n(S)} = \frac{7}{8}$

(iv) Let E = { having exactly one boy } = { BGG, GBG, GGB }

then $P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$

EXERCISE 6.4

1. Let S = {1, 2, 3, 4, 5, 6} be the sample space of rolling a die. What is the probability of (i) Rolling a 5? (ii) Rolling a number less than one? (iii) Rolling a number greater than 0? (iv) Rolling a multiple of 3? (v) Rolling a number greater than or equal to 4?

2. A bag contains 4 white, 5 red and 6 green balls. 3 balls are drawn at random. What is the probability that (i) All are green (ii) All are white.

3. A true or false test contains eight questions. If a student guesses the answer for each question, find the probability:
   (i) 8 answers are correct. (ii) 7 answers are correct and 1 is incorrect.
   (iii) 6 answers are correct and 2 are incorrect.
   (iv) at least 6 answers are correct.

4. Three unbiased coins are tossed. What is the probability of obtaining
   (i) all heads (ii) two heads (iii) one head
   (iv) at least one head (v) at least two heads (vi) All tails.

5. A committee of 5 person is to be selected at random from 6 men and 4 women. Find the probability that the committee will consist of
   (i) 3 men and 2 women (ii) 2 men and 3 women.

6. If one card is drawn at random from a well shuffled pack of 52 cards. Then find the probability of each of the following.
   (i) Drawing an ace card, (ii) Drawing either spade or hearts,
   (iii) Drawing a diamond card, (iv) Drawing a face card,
   (v) Not drawing an ace of hearts.

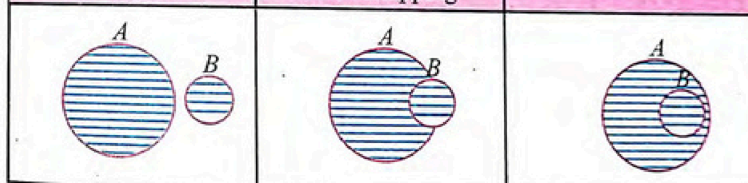
7. Two dice are thrown simultaneously. Find the probability of getting:
   (i) doublet of even numbers (ii) a sum less than 6 (iii) a sum more than 7
   (iv) a sum greater than 10 (v) a sum at least 10 (vi) six as the product
   (vii) an even number as the sum (viii) an odd number as the sum
   (ix) a multiple of 3 as the sum (x) sum as a prime number

6.4.3 Laws of Probability

It is easier to compute the probability of an event from known probabilities of other events. This is true if the event can be expressed as the union or intersection of two other events or as the complement of an event. Some basic elementary laws of probability are given below in the form of theorems.

6.4.4 Use Venn diagrams to find the probability for the occurrence of an event

If A and B are disjoint	If A and B are overlapping	If $B \subset A$



We know that if A and B are two sets, then the shaded parts in the following diagram represent $A \cup B$.
The above diagrams help us in understanding the formulae about the sum of two probabilities.
We know that:
$P(E)$ is the probability of the occurrence of an event E.
If A and B are two events, then
$P(A)$ = the probability of the occurrence of event A;

$P(B)$ = the probability of the occurrence of event B;

$P(A\cup B)$ = the probability of the occurrence of $A\cup B$;

$P(A\cap B)$ = the probability of the occurrence of $A\cap B$;

The formulae for the addition of probabilities are:

i) $P(A\cup B)=P(A)+P(B)$, when A and B are disjoint.

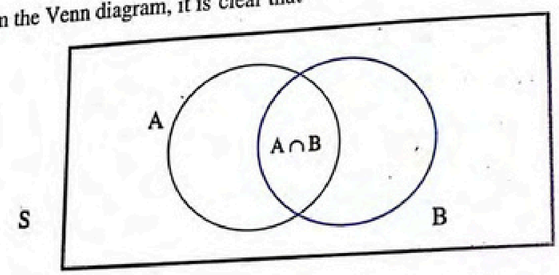
ii) $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

when A and B are overlapping or $B\subseteq A$.

Theorem: If A and B are any two events in a sample space S, then

$$P(A\cup B)=P(A)+P(B)-P(A\cap B).$$

Proof: From the Venn diagram, it is clear that



$$n(A\cup B)=n(A)+n(B)-n(A\cap B)$$

and $n(A\cap B)$ has been subtracted simply because it has been considered twice.

Now, by definition we have

$$P(A\cup B)=\frac{n(A\cup B)}{n(S)}=\frac{n(A)+n(B)-n(A\cap B)}{n(S)}$$

$$=\frac{n(A)}{n(S)}+\frac{n(B)}{n(S)}-\frac{n(A\cap B)}{n(S)}$$

$$=P(A)+P(B)-P(A\cap B)$$

This law is generally called, addition law of probability.

Corollary 1: If A and B are mutually exclusive events, then $P(A\cup B)=P(A)+P(B)$

Proof: Since A and B are mutually exclusive events, then

$A\cap B=\varphi$ and $P(A\cap B)=P(\varphi)=0$

Hence $P(A\cup B)=P(A)+P(B)-P(A\cap B)$ reduces to

$$P(A\cup B)=P(A)+P(B)$$

Now, generalizing the above, we have the following:

Corollary 2: If $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$$P\left(A_1\cup A_2\cup\cdots\cup A_n\right)=P(A_1)+P(A_2)+\cdots+P(A_n)$$

Example 22: One integer is chosen at random from the numbers 1, 2, 3, . . . , 50. What is the probability that the chosen number is divisible by 6 or 8 ?

Solution: Here $S=\{1,2,3,\cdots,50\}$ and $n(S)=50$

Let A = { number is divisible by 6 }= { 6, 12, 18, 24, 30, 36, 42, 48 }

and B = { number is divisible by 8 }

= { 8, 16, 24, 32, 40 , 48 } then $A\cap B=\{24,48\}$

Now, substituting $P(A)=\frac{8}{50}$, $P(B)=\frac{6}{50}$ and $P(A\cap B)=\frac{2}{50}$

in the following, we obtain

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{8}{50}+\frac{6}{50}-\frac{2}{50}=\frac{12}{50}=\frac{6}{25}$$

Example 23: If two dice are rolled, find the probability of obtaining a total of 7 or 11.

Solution: Here $S=\{(i,j): i, j = 1,2,3,4,5,6\}$ and $n(S)=36$

Let A = { a total of 7 occurs }

= { (6,1), (5,2), (4,3), (3, 4), (2,5), (1,6) }

and B = { a total of 11 occurs }

$= \{ (6,5), (5,6) \}$ then $A \cap B = \varphi$

Now $P(A) = \frac{6}{36}$ and $P(B) = \frac{2}{36}$

Since A and B are mutually exclusive, so we have

$$P(A \cup B) = P(A) + P(B) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$$

Complementary events

Suppose we divide a sample space S into two subsets (events) E and E' such that

(i) $E \cap E' = \phi$ and (ii) $E \cup E' = S$

Then E' is called the complement of E relative to S and E and E' are called complementary events.

Theorem: If E and E' are complementary events, then $P(E') = 1 - P(E)$

Proof: Since $E \cup E' = S$ Then $P(E \cup E') = P(S)$

or $P(E) + P(E') = 1$, $\because E \cap E' = \varphi$

or $P(E') = 1 - P(E)$

Example 24: A coin is tossed 6 times in succession. What is the probability that at least one head occurs?

Solution: Tossing a coin 6 times in succession, we have $n(S) = 2^6 = 64$

Let $E = \{$ at least 1 H occurs $\}$ then $E' = \{$ no H occurs $\}$

and $P(E') = \frac{1}{64}$, $\because$ there is only one outcome event, where all tails occur.

$$\therefore \quad P(E) = 1 - P(E') = 1 - \frac{1}{64} = \frac{63}{64}$$

6.4.5 Conditional Probability

The probability of an event may change if the information of the occurrence of another event is given. For example, if an adult is selected at random from certain population, the probability of that person having lung cancer would not be too high. However, if information that the person is also a heavy smoker is provided, then one would certainly want to revise the probability upward.

Let A = { An adult has lung cancer }

and B = { An adult is a heavy smoker }

Then the probability of an event A, given the occurrence of another event B, is called a conditional probability and is denoted by $P(A|B)$.

For events A and B in an arbitrary sample space S, we define the conditional probability of A given B by $P(A|B) = \frac{P(A \cap B)}{P(B)}$, $P(B) > 0$.

Similarly, $P(B|A) = \frac{P(A \cap B)}{P(A)}$, $P(A) > 0$.

Example 25: What is the probability of rolling a prime number in tossing a die, given that an odd number has turned up?

Solution: Here S = { 1, 2, 3, 4, 5, 6 }

Let A= { a prime number has rolled }= { 2, 3, 5 }

and B= { an odd number has turned up } = { 1, 3, 5 }

then $A \cap B = \{3, 5\}$

We have $P(B) = \frac{3}{6}$ and $P(A \cap B) = \frac{2}{6}$

Now, using the formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$

$$P(A|B) = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3} \quad \text{Since} \quad P(B|A) = \frac{P(A \cap B)}{P(A)}$$

or $\quad P(A\cap B)=P(A)P(B|A)$

This shows that the conditional probability can be used in expressing the probability of the intersection of a finite number of events and we have the following theorem known as **the multiplicative theorem**.

If A and B are any two events in a sample space S then

$$P(A\cap B)=P(A)P(B|A),\quad P(A)\neq 0$$

$$=P(B)P(A|B),\quad P(B)\neq 0$$

The above theorem can be easily extended to a finite number of events.

For example in case of three events A, B and C it becomes:

$$P(A\cap B\cap C)=P(A)P(B|A)P(C|A\cap B)$$

Example 26: An urn contains three red and seven green balls. A ball is drawn, not replaced and another is drawn. Find the following.

(i) P (red and red) $\qquad$ (ii) P (red and green).

Solution: Total number of balls = 10

(i) Let $\quad A=\{$ the 1^{st} ball drawn is red $\}$

and $\quad B=\{$ the 2^{nd} ball drawn is red $\}$

So using the multiplicative theorem,

$P(\textit{red and red})=P(A\cap B)=P(A)P(B|A)$

Substituting $\quad P(A)=\dfrac{3}{10}$ and $P(B|A)=\dfrac{2}{9}$

We obtain, $\quad P(\textit{red and red})=\dfrac{3}{10}\cdot\dfrac{2}{9}=\dfrac{1}{15}$

(ii) Let $\quad C=\{$ the 1^{st} ball drawn is red $\}$

and $\quad D=\{$ the 2^{nd} ball drawn is green $\}$

So using the multiplicative theorem again.

$P(\textit{red and green})=P(C\cap D)=P(C)P(D|C)$

Substituting $\quad P(C)=\dfrac{3}{10}$ and $P(D|C)=\dfrac{7}{9}$

We obtain, $\quad P(\textit{red and green})=\dfrac{3}{10}\cdot\dfrac{7}{9}=\dfrac{7}{30}$

6.4.6 Dependent and Independent Events

In general $P(A|B)$ and $P(A)$ are not equal. However, there is an important class of events for which they are. If $P(A|B)=P(A)$, then the knowledge of B occurring does not change the probability of A and we say that A is independent of B. Similarly, if $P(B|A)=P(B)$, we say that B is independent of A. Thus two events A and B are said to be independent if the occurrence (or non occurrence) of one does not affect the probability of the occurrence (and hence non occurrence) of the other, otherwise they are called dependent.

Illustration1: In the simultaneous throw of two coins, 'getting a head' on first coin and 'getting a tail on the second coin are independent events.

Illustration2: When a card is drawn from a pack of well shuffled cards and replaced before the second card is drawn, the result of second draw is independent of first draw.

The following theorem gives the probabilities of simultaneous occurrence of two independent events.

Theorem: If A and B are independent events, then $P(A\cap B)=P(A)P(B)$.

Proof: Since multiplicative theorem gives that:

$$P(A\cap B)=P(A)P(B|A) \qquad \text{(i)}$$

$$=P(B)P(A|B) \qquad \text{(ii)}$$

Further, A and B are independent, then we have $P(B|A)=P(B)$ and $P(A|B)=P(A)$ substituting in (i) and (ii) we get the required result:

$$P(A\cap B)=P(A)P(B)$$

The above theorem can be extended to any finite number of mutually independent events. If $A_1, A_2, A_3, \ldots, A_n$ are mutually independent events, then

$$P(A_1\cap A_2\cap A_3\cap\cdots\cap A_n)=P(A_1)P(A_2)P(A_3)\cdots P(A_n)$$

Example 27: A space shuttle has four independent computer control systems. If the probability of failure of any one system is 0.001, what is the probability of failure of all four systems?

Solution: Let $E_i = \{$ failure of system i, $i = 1, 2, 3, 4 \}$

Since the events E_i, $i = 1, 2, 3, 4$ are given to be independent, so using the following.

$$P(E_1\cap E_2\cap E_3\cap E_4)=P(E_1)P(E_2)P(E_3)P(E_4), \quad i=1,2,3,4$$

$$=(0.001)^4=0.000000000001$$

EXERCISE 6.5

1. Suppose events A and B are such that $P(A)=\frac{2}{5}$, $P(B)=\frac{2}{5}$ and $P(A\cup B)=\frac{1}{2}$. Find $P(A\cap B)$.
2. If A and B are 2 events in a sample space S such that $P(A)=\frac{1}{2}, P(\overline{B})=\frac{5}{8}, P(A\cup B)=\frac{3}{4}$. Find (i) $P(A\cap B)$ (ii) $P(\overline{A}\cap\overline{B})$
3. Given $P(A)=0.5$ and $P(A\cup B)=0.6$, find $P(B)$ if A and B are mutually exclusive.
4. A bag contains 30 tickets numbered from 1 to 30. One ticket is selected at random. Find the probability that its number is either odd or the square of an integer.
5. A student finds that the probability of passing an algebra test is $\frac{8}{9}$. What is the probability of failing the test?
6. In the two dice experiment, given that the first die shows 4, what is the probability that the second die shows a number greater than 4?
7. One card is drawn from a pack of 52 cards, what is the probability that the card drawn is neither red nor king.

8. If a pair of dice is thrown, find the probability that the sum of digits is neither 7 nor 11.
9. Ajmal and Bushra appear in an interview for 2 vacancies. The probability of their selection being $\frac{1}{7}$ and $\frac{1}{5}$ respectively. Find the probability that
   (i) both will be selected (ii) only one is selected
   (iii) none will be selected (iv) at least one of them will be selected.
10. A basket contains 20 apples and 10 oranges out of which 5 apples and 3 oranges are defective. If a person takes out 2 at random what is the probability that either both are apples or both are good?

REVIEW EXERCISE 6

1. **Choose the correct option**

(i) In how many ways can we name the vertices of a pentagon using any five of the letters O, P, Q, R, S, T, U in any order?
(a) 2520 (b) 9040 (c) 5140 (d) 4880

(ii) How many two-digit odd numbers can be formed from the digits {1, 2, 3, 4, 5, 6, 7} if repeated digits are allowed?
(a) 14 (b) 42 (c) 28 (d) 21

(iii) How many six-digit numbers can be formed from the digits {2, 3, 4, 6, 7, 8} without repetition if the digits 3 and 7 must be together?
(a) 120 (b) 180 (c) 144 (d) 96

(iv) Evaluate $\frac{(n+2)!(n-2)!}{(n+1)!(n-1)!}$
(a) $(n-3)$ (b) $(n-1)$ (c) $\frac{(n+1)}{(n+2)}$ (d) $\frac{(n+2)}{(n-1)}$

(v) In how many different ways can 5 couples be seated around a circular table if the couples must not be separated?
(a) 768 (b) 724 (c) 844 (d) 696

(vi) A committee of 4 people will be selected from 8 girls and 12 boys in a class. How many different selections are possible if at least one boy must be selected?
(a) 2865 (b) 3755 (c) 4225 (d) 4775