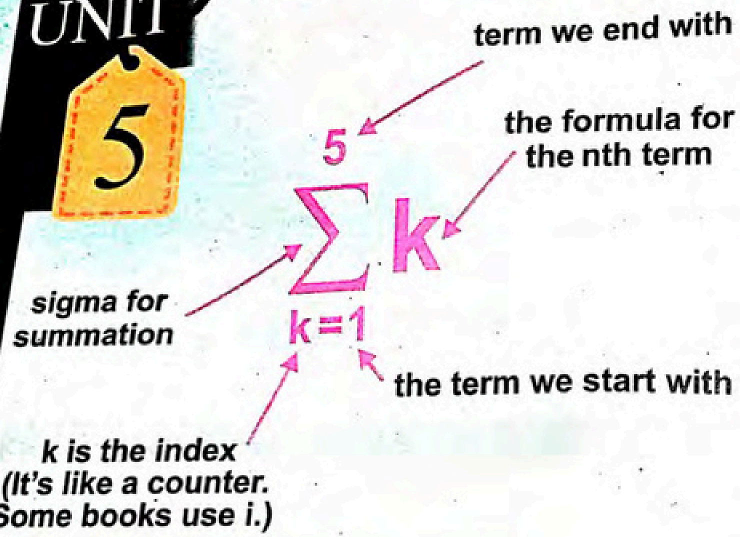
UNIT 5

MISCELLANEOUS SERIES



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Recognize sigma (Σ) notation.
- Find sum of
  - the first n natural numbers (Σn),
  - the squares of the first n natural numbers (Σn^2),
  - the cubes of the first n natural numbers (Σn^3).
- Define arithmetico-geometric series.
- Find sum to n terms of the arithmetico-geometric series.
- Define method of differences. Use this method to find the sum of n terms of the series whose differences of the consecutive terms are either in arithmetic or in geometric sequence.
- Use partial fractions to find the sum to n terms and to infinity the series of the type $\frac{1}{a(a+d)}+\frac{1}{(a+d)(a+2d)}+...$

5.1 Introduction

In the previous chapter, we computed the sums of arithmetic and geometric sequences. In this chapter, we discuss a few more techniques for computing sums of some other sequences. Since we are already familiar with the standard notation, called the sigma notation (Σ) and its rules. However here we properly define it with a few examples of summation notation.

5.1.1 Sigma Notation

The letter "Σ" of the Greek alphabet (pronounced as sigma) is used to denote the sum of a given series. The letter Σ is placed before the rth term, say, a_r. We, thus write Σa_r to denote the sum of terms of the type a_r. If we want to sum up terms a_r for values of r corresponding to $r=1,2,3,....n$, we denote the sum by $\sum_{r=1}^{r=n} a_r$ or by $\sum_{1}^{n} a_r$

Example 1: Find the following sum.

(i) $\sum_{k=1}^{4} k^2(k-3)$ (ii) $\sum_{k=0}^{3} \frac{2^k}{(k+1)}$ (iii) $\sum_{k=2}^{6} (-1)^k\sqrt{k}$

Solution: (i) $\sum_{k=1}^{4} k^2(k-3) = 1^2(1-3)+2^2(2-3)+3^2(3-3)+4^2(4-3)$
$= (-2)+(-4)+0+16 = 10$

(ii) $\sum_{k=0}^{3} \frac{2^k}{(k+1)} = \frac{2^0}{(0+1)} + \frac{2^1}{(1+1)} + \frac{2^2}{(2+1)} + \frac{2^3}{(3+1)}$
$= 1+1+\frac{4}{3}+2 = \frac{16}{3}$

(iii) $\sum_{k=2}^{6} (-1)^k\sqrt{k}$
$= (-1)^2\sqrt{2}+(-1)^3\sqrt{3}+(-1)^4\sqrt{4}+(-1)^5\sqrt{5}+(-1)^6\sqrt{6}$
$= \sqrt{2}-\sqrt{3}+2-\sqrt{5}+\sqrt{6}$

Example 2: Simplify $\sum_{j=2}^{10} \frac{1}{j} - \sum_{j=1}^{8} \frac{1}{j+2}$

Solution: It can be seen that most terms are common to both sums and will cancel. In the second sum, let $k = j+2$ and in the first sum, let $k = j$, then we have

$$\sum_{j=2}^{10} \frac{1}{j} - \sum_{j=1}^{8} \frac{1}{j+2} = \sum_{k=2}^{10} \frac{1}{k} - \sum_{k=3}^{10} \frac{1}{k}$$

$$=\frac{1}{2}+\sum_{k=3}^{10}\frac{1}{k}-\sum_{k=3}^{10}\frac{1}{k}=\frac{1}{2}$$

This example illustrates how changing the index can simplify expressions involving several sums.

5.1.2 Evaluation of sum of the first n

i. Natural numbers
ii. Squares of natural numbers
iii. Cubes of natural numbers

Before evaluation of the above mentioned sums, here we discuss a general principle that will allow us to compute a wide variety of sums.

Suppose $b_1, b_2, \ldots, b_{n+1}$ is a sequence

and $a_j = b_{j+1} - b_j$

then $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n}(b_{j+1}-b_j)$

$$=(b_2-b_1)+(b_3-b_2)+\ldots+(b_{n+1}-b_n)$$

$$=-b_1+(b_2-b_2)+(b_3-b_3)+\ldots+(b_n-b_n)+b_{n+1}$$

$$=b_{n+1}-b_1$$

Thus if $a_j = b_{j+1} - b_j$

then $\sum_{j=1}^{n} a_j = b_{n+1} - b_1$

This statement seems very simple, yet in practice it can be very powerful.

Suppose we want to compute $\sum_{j=1}^{n} a_j$. If we can find a sequence $b_1, b_2, \ldots$ such that $b_{j+1} - b_j = a_j$, then we can write down the answer immediately, that is $b_{n+1} - b_1$.

(i) Let $b_j = j^2$ (1)

then $b_{j+1} - b_j = (j+1)^2 - j^2$

$$=2j+1$$

thus here, we take $a_j = 2j+1$

Now using $\sum_{j=1}^{n} a_j = b_{n+1} - b_1$

$$\sum_{j=1}^{n}(2j+1)=(n+1)^2-1^2 \quad \text{by (1)}$$

$$2\sum_{j=1}^{n} j+\sum_{j=1}^{n} 1=n^2+2n$$

$$2\sum_{j=1}^{n} j+n=n^2+2n$$

$$2\sum_{j=1}^{n} j=n^2+2n-n$$

$$2\sum_{j=1}^{n} j=n^2+n$$

Hence $\sum_{j=1}^{n} j=\frac{n(n+1)}{2}$

(ii) let $b_j = j^3$ (2)

then $b_{j+1}-b_j=(j+1)^3-j^3$

$$=3j^2+3j+1$$

thus here $a_j = 3j^2+3j+1$.

Now, using the following $\sum_{j=1}^{n} a_j = b_{n+1}-b_1$

$$\sum_{j=1}^{n}(3j^2+3j+1)=(n+1)^3-1^3 \quad \text{by (2)}$$

$$3\sum_{j=1}^{n} j^2+3\sum_{j=1}^{n} j+\sum_{j=1}^{n} 1=(n+1)^3-1$$

$$3\sum_{j=1}^{n} j^2+3\left(\frac{n(n+1)}{2}\right)+n=(n+1)^3-1$$

$$3\sum_{j=1}^{n} j^2=(n+1)^3-1-3\left(\frac{n(n+1)}{2}\right)-n$$

$$=(n+1)^3-(n+1)-\frac{3}{2}n(n+1)$$

$$=\frac{n+1}{2}\left[2(n+1)^2-2-3n\right]$$

$$=\frac{n+1}{2}\left[2n^2+2+4n-2-3n\right]=\frac{n+1}{2}\left[2n^2+n\right]$$

$$=\frac{n(n+1)(2n+1)}{2}$$

Hence $\sum_{j=1}^{n} j^2 = \frac{1}{6}[n(n+1)(2n+1)]$ (3)

(iii) Let $b_j = j^4$

then $b_{j+1} - b_j = (j+1)^4 - j^4 = 4j^3 + 6j^2 + 4j + 1$

we take $a_j = 4j^3 + 6j^2 + 4j + 1$

Now, using the following

$$\sum_{j=1}^{n} a_j = b_{n+1} - b_1$$

$$\sum_{j=1}^{n} (4j^3 + 6j^2 4j + 1) = (n+1)^4 - 1^4 \text{ by(1)}$$

$$4\sum_{j=1}^{n} j^3 + 6\sum_{j=1}^{n} j^2 + 4\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 = (n+1)^4 - 1$$

$$4\sum_{j=1}^{n} j^3 + 6\left(\frac{n(n+1)(2n+1)}{6}\right) + 4\left(\frac{n(n+1)}{2}\right) + n = (n+1)^4 - 1$$

$$4\sum_{j=1}^{n} j^3 = (n+1)^4 - 1 - n(n+1)(2n+1) - 2n(n+1) - n$$

$$= (n+1)^4 - (n+1) - n(n+1)(2n+1) - 2n(n+1)$$

$$= (n+1)[(n+1)^3 - 1 - 2n^2 - n - 2n]$$

$$= (n+1)[n^3 + 3n^2 + 3n + 1 - 1 - 2n^2 - 3n] = (n+1)\ [n^3 + n^2]$$

$$= n^2(n+1)^2$$

Hence $\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$

$$\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 3: Find the sum of the n terms of the series.

$$1{\cdot}2 + 2{\cdot}3 + 3{\cdot}4 + \cdots$$

Solution: Let T_j be the general term of the given series, then

$$T_j = j(j+1)$$

and $\sum_{j=1}^{n} T_j = \sum_{j=1}^{n} (j^2 + j)$

$$= \sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} j$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{6}[2n+1+3]$$

$$= \frac{n(n+1)}{6}(2n+4)$$

$$= \frac{n(n+1)(n+2)}{3}$$

Example 4: Find the sum of the n terms of the series

$$1{\cdot}2^2 + 2{\cdot}3^2 + 3{\cdot}4^2 + \ldots$$

Solution: Here $T_j = j(j+1)^2$

then $\sum_{j=1}^{n} T_j = \sum_{j=1}^{n} (j^3 + 2j^2 + j)$

$$= \sum_{j=1}^{n} j^3 + 2\sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} j$$

$$= \frac{n^2(n+1)^2}{4} + 2\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{12}[3n^2 + 3n + 8n + 4 + 6]$$

$$= \frac{n(n+1)}{12}[3n^2 + 11n + 10]$$

$$= \frac{1}{12}\ n(n+1)(n+2)(3n+5)$$

Example 5: Find the sum of n terms of the series whose nth term is $2^{n-1} + 8n^3 - 6n^2$.

Solution: Given that $T_n = 2^{n-1} + 8n^3 - 6n^2$

then $T_j = 2^{j-1} + 8j^3 - 6j^2$

$$\sum_{j=1}^{n} T_j = \sum_{i=1}^{n} (2^{j-1} + 8j^3 - 6j^2)$$

$$= \sum_{j=1}^{n} 2^{j-1} + 8\sum_{j=1}^{n} j^3 - 6\sum_{j=1}^{n} j^2$$

$$=\left[\frac{1(2^n-1)}{2-1}\right]+8\left[\frac{n^2(n+1)^2}{4}\right]-6\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$=2^n-1+n(n+1)[2n^2+2n-2n-1]$$

$$=2^n-1+n(n+1)(2n^2-1)$$

EXERCISE 5.1

1. Sum the following series up to n terms

   (i) $1^2+3^2+5^2+7^2+\cdots$ (ii) $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\cdots$

   (iii) $2^2+4^2+6^2+\cdots$ (iv) $1^3+3^3+5^3+\cdots$ (v) $1^3+5^3+9^3+\cdots$

2. Find the sum $1\cdot2+2\cdot3+3\cdot4+\cdots+99\cdot100$
3. Find the sum $1^2+3^2+5^2+7^2+\cdots+99^2$
4. Find the sum $2+(2+5)+(2+5+8)+\cdots$ to n terms
5. Sum $2+5+10+17+\cdots$ to n terms
6. Sum to n terms. $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots$
7. Sum to n terms $1\cdot5\cdot9+2\cdot6\cdot10+3\cdot7\cdot11+\cdots$
8. Find the sum to 2n terms of the series whose nth term is $4n^2+5n+1$
9. Find the sum of n terms of the series whose nth term is:

   (i) $n^2(2n+3)$ (ii) $3(4^n+2n^2)-4n^3$

5.2 Arithmetico-Geometric series

Since we are already familiar with the arithmetic and geometric sequences and their related series. Now, we discuss here another important sequence and its related series, which we obtain from arithmetic and geometric sequences.

5.2.1 A series which is obtained by multiplying the corresponding terms of an arithmetic series and a geometric series is called **Arithmetico–Geometric series.**

For example,

$$[a+(a+d)+(a+2d)+\ldots+(a+(n-1)d)][1+r+r^2+\ldots+r^{n-1}]$$

$$=a+(a+d)r+(a+2d)r^2+\ldots+(a+(n-1)d)\ r^{n-1}$$

which is arithmetico–geometric series.

nth term of Arithmetico-Geometric Series

A series which is formed by multiplying the corresponding terms of an A. P. and a G. P. is called an arithmetico-geometric series. Thus nth term of such series has the form $[a+(n-1)d]\times r^{n-1}$

5.2.2 Sum of n terms of Arithmetico-Geometric Series

Let $S_n=a+(a+d)r+(a+2d)r^2+\ldots+[a+(n-1)d]\ r^{n-1}$ (1)

then $rS_n=ar+(a+d)r^2+\ldots+[a+(n-2)d]\ r^{n-1}+[a+(n-1)d]\ r^n$ (2)

subtracting (2) from (1) we obtain

$$(1-r)S_n=a+(dr+dr^2+\ldots+dr^{n-1})-[a+(n-1)d]\ r^n$$

$$\therefore S_n=\frac{a}{1-r}+\frac{1}{1-r}\left[\frac{dr(1-r^{n-1})}{1-r}\right]-\frac{1}{1-r}[a+(n-1)d]\ r^n$$

$$=\frac{a}{1-r}+\frac{dr}{(1-r)^2}-\frac{dr^n}{(1-r)^2}-\frac{[a+(n-1)d]r^n}{(1-r)} \quad (3)$$

which is the required sum of the n terms of arithmetico–geometric series.

Example 6: Sum the series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots..$ to n terms.

Solution: Let $S=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots..+\frac{3n-2}{5^{n-1}}$

$$\therefore \frac{1}{5}S=\frac{1}{5}+\frac{4}{5^2}+\frac{7}{5^3}+\ldots..+\frac{3n-5}{5^{n-1}}+\frac{3n-2}{5^n}$$

$$\therefore \frac{4}{5}S=1+\left(\frac{3}{5}+\frac{3}{5^2}+\frac{3}{5^3}+\ldots\ldots..+\frac{3}{5^{n-1}}\right)-\frac{3n-2}{5^n}$$

$$=1+\frac{3}{5}\left(1+\frac{1}{5}+\frac{1}{5^2}+\ldots\ldots..+\frac{1}{5^{n-2}}\right)-\frac{3n-2}{5^n}$$

$$=1+\frac{3}{5}\left(\frac{1-\frac{1}{5^{n-1}}}{1-\frac{1}{5}}\right)-\frac{3n-2}{5^n}$$

Note

Sum to infinity of an Arithmetico–Geometric Series

Let $|r|<1$

Then $r^n\to0$ as $n\to\infty$

$\therefore$ Equation (3) reduces to

$$S_\infty=\frac{a}{1-r}+\frac{dr}{(1-r)^2}$$

which is the required sum to infinity of arithmetico–geometric series

$$=1+\frac{3}{4}\left(1-\frac{1}{5^{n-1}}\right)-\frac{3n-2}{5^n}=1+\frac{3}{4}-\frac{3}{4\cdot 5^{n-1}}-\frac{3n-2}{5^n}=\frac{7}{4}-\frac{1}{5^{n-1}}\left(\frac{3}{4}+\frac{3n-2}{5}\right)$$

$$=\frac{7}{4}-\frac{1}{5^{n-1}}\left(\frac{15+12n-8}{4\cdot 5}\right)=\frac{7}{4}-\frac{12n+7}{4.5^n}$$

$$\therefore S=\frac{35}{16}-\frac{12n+7}{16.5^{n-1}}$$

Example 7: Sum the series.

$2\cdot1+4\cdot3+6\cdot9+8\cdot27+10\cdot81+\ldots$ to n terms.

$$2\cdot1+4\cdot3+6\cdot9+\ldots+(2n-2)\cdot3^{n-2}+2n\cdot3^{n-1}. \qquad \text{(i)}$$

Solution: Let $S=2\cdot1+4\cdot3+6\cdot9+\ldots+(2n-2)\cdot3^{n-2}+2n\cdot3^{n-1}$.

Multiplying by 3, the common ratio of the geometric series, we get

$$3\cdot S=2\cdot3+4\cdot9+6\cdot27+\ldots+(2n-2)\cdot3^{n-1}+2n\cdot3^n. \qquad \text{(ii)}$$

Subtracting (ii) from (i), we get

$$(1-3)S=2\cdot1+\{3(4-2)+9(6-4)+27(8-6)+\ldots+3^{n-1}(2n-2n-2)\}-2n\cdot3^n.$$

$$\therefore(-2)\cdot S=2\cdot1+\{2(3+9+27+\ldots\text{to (n-1) terms})\}-2n\cdot3^n$$

$$=2+2\left[3\cdot\frac{(3^{n-1}-1)}{3-1}\right]-2n\cdot3^n$$

$$=2+3^n-3-2n\cdot3^n=-1-3^n(2n-1).$$

$$\therefore\ S=\frac{1}{2}\left[1+3^n(2n-1)\right]$$

Example 8: If $x<1$, sum the series

$1+2x+3x^2+4x^3+\ldots$ to infinity

Solution: Let $S=1+2x+3x^2+4x^3+\ldots$ (i)

$\therefore\quad xS=x+2x^2+3x^3+\ldots$ (ii)

Subtracting (ii) from (i), we get

$$\therefore S(1-x)=1+x+x^2+x^3+\ldots$$

The R.H.S is an infinite geometric series with $a_1=1$ and $r=x<1$

$$S(1-x)=\frac{1}{1-x}$$

$$\therefore S=\frac{1}{(1-x)^2}$$

Example 9: Show that $2^{\frac{1}{4}}\times4^{\frac{1}{8}}\times8^{\frac{1}{16}}\times16^{\frac{1}{32}}\times\ldots\infty=2$

Solution: Let $x=2^{\frac{1}{4}}\times4^{\frac{1}{8}}\times8^{\frac{1}{16}}\times16^{\frac{1}{32}}\times\ldots\infty$

$$\log x=\log2^{\frac{1}{4}}+\log4^{\frac{1}{8}}+\log8^{\frac{1}{16}}+\log16^{\frac{1}{32}}+\ldots\infty$$

$$\log x=\frac{1}{4}\log2+\frac{1}{8}\log4+\frac{1}{16}\log8+\frac{1}{32}\log16+\ldots\infty$$

$$\log x=\frac{1}{4}\log2+\frac{1}{8}\log2^2+\frac{1}{16}\log2^3+\frac{1}{32}\log2^4+\ldots\infty$$

$$\log x=\frac{1}{4}\log2+\frac{2}{8}\log2+\frac{3}{16}\log2+\frac{4}{32}\log2+\ldots\infty$$

$$\log x=\left(\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+\ldots\infty\right)\log2 \qquad \text{(i)}$$

Now, $\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+\ldots\infty$ is an arithmetico-geometric series

Let $$S=\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+\ldots\infty \qquad \text{(ii)}$$

$$\frac{1}{2}S=\frac{1}{8}+\frac{2}{16}+\frac{3}{32}+\frac{4}{64}+\ldots\infty \qquad \text{(iii)}$$

On subtracting Eq(iii) from Eq(ii), we get

$$\frac{1}{2}S=\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots\infty$$

$$\Rightarrow\frac{1}{2}S=\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\ldots\infty$$

$$\frac{1}{2}S=\frac{\frac{1}{2^2}}{1-\frac{1}{2}}$$

$$\frac{1}{2}S=\frac{1}{2}$$

$$S=1$$

$$\therefore(i)\Rightarrow\log x=1\times\log2$$

$$\log x=\log2$$

$$\therefore x=2.$$

EXERCISE 5.2

1. Sum to n terms the following series

   (i) $1\cdot 2+2\cdot 2^2+3\cdot 2^3+4\cdot 2^4+\ldots$ (ii) $1+4x+7x^2+10x^3+\ldots$

   (iii) $1+2x+3x^2+4x^3+\ldots$ (iv) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots$

   (v) $1-7x+13x^2-19x^3+\ldots$

2. Find the sum to infinity of the following series

   (i) $1^2+3^2x+5^2x^2+7^2x^3+\ldots, x<1$ (ii) $1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\ldots$

3. Find the nth term of the following arithmetico–geometric series

   $\frac{0}{1}+\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\ldots$

4. Find the sum of the following Arithmetico–geometric series

   $5+\frac{7}{3}+1+\frac{11}{27}+\ldots$

5. If the sum to infinity of the series $3+5r+7r^2+\ldots\ldots\infty$ is $\frac{44}{9}$, find the value of r.

5.3 The Method of Differences

In the case of some series in which the difference of successive terms form an A.P, or G.P., the following method can be employed to find the nth term. The sum of such a series to n terms may then be obtained.

Example 10: Find the nth term and the sum to n terms of the series

$1+7+17+31+49+\ldots\ldots\ldots$

Solution:

We have

$a_2-a_1=6$

$a_3-a_2=10$

$a_4-a_3=14$

....

....

$a_n-a_{n-1}=$ (n–1)th term of the sequence $6,10,14,\ldots$

Adding column-wise, we get

$a_n-a_1=6+10+14+18+\ldots\ldots\text{to }(n-1)\text{ terms},$

$a_n-a_1=\frac{n-1}{2}[2\cdot 6+(n-2)\cdot 4]$

$a_n-a_1=\frac{n-1}{2}[12+4n-8]=\frac{n-1}{2}[4n+4]=(n-1)(2n+2)\Rightarrow a_n=a_1+(n-1)(2n-2)$

$\therefore a_n=1+2(n-1)(n+1)$ $[\because a_1=1]$

$\therefore a_n=2n^2-1$

$\therefore a_r=2r^2-1$

$\therefore \sum_1^n a_r=2\sum_1^n r^2-\sum_1^n 1=2\cdot\frac{n(n+1)(2n+1)}{6}-n=\frac{n(n+1)(2n+1)}{3}-n$

$=\frac{n(n+1)(2n+1)-3n}{3}=\frac{n(n+2)(2n-1)}{3}$

$\therefore$ the required sum $=\frac{n(n+2)(2n-1)}{3}$

Example 11: Find the nth term and the sum to n terms of the series

$3+5+9+17+31+\ldots\ldots\ldots$

Solution:

We have

$a_2-a_1=2$

$a_3-a_2=4$

$a_4-a_3=8$

....

$a_n-a_{n-1}=$ (n-1)th term of the sequence $2,4,8,\ldots$

Which is a G.P. Adding column-wise, we get

$a_n-a_1=2+4+8+\ldots\ldots\text{to }(n-1)\text{ terms},$

$a_n-a_1=\frac{2(2^{n-1}-1)}{2-1}=2^n-2$

$\therefore a_n=2^n-2+3$ $[\because a_1=3]$

$\therefore a_n=2^n+1$

$\therefore a_r=2^r+1$

$\therefore \sum_1^n a_r=[2+2^2+2^3+2^4+\ldots\ldots+2^n]+\sum 1$

$=\frac{2(2^n-1)}{2-1}+n=2^{n+1}+n-2$

$\therefore$ the required sum $=2^{n+1}+n-2.$

EXERCISE 5.3

Find the nth term and the sum to n terms of each of the following series:

1. 4+13+28+49+76+.......
2. 4+14+30+52+80+114+.......
3. 4+10+18+28+40+.......
4. 3+5+11+29+83+245+.......
5. 3+9+21+45+93+189+.......
6. 28+32+52+152+652+.......

5.4 Summation by the method of Partial Fractions

If the general term of a series consists of the products of the reciprocals of two or more consecutive terms of an A.P., then the term can be split up into partial fractions and the series can be summed. The method is illustrated in the following examples.

Example 12: Sum the series $\frac{1}{3\cdot 7}+\frac{1}{7\cdot 11}+\frac{1}{11\cdot 15}+\frac{1}{15\cdot 19}+\cdots$ *to n terms*.

Solution: Here, the factors in the denominators are the products of two successive terms of an A. P 3, 7, 11, 15, 19,

$\therefore$ rth term of the given series, $a_r=\dfrac{1}{(4r-1)(4r+3)}$

Expressing a_r as the difference of its partial fractions, we have

$$a_r=\frac{1}{4}\left[\frac{1}{4r-1}-\frac{1}{4r+3}\right]$$

By putting $r=1,2,3,.....(n-1),n$ in succession, we get

$$a_1=\frac{1}{4}\left[\frac{1}{3}-\frac{1}{7}\right]$$

$$a_2=\frac{1}{4}\left[\frac{1}{7}-\frac{1}{11}\right]$$

$$a_3=\frac{1}{4}\left[\frac{1}{11}-\frac{1}{15}\right]$$

... ...

... ...

... ...

$$a_{n-1}=\frac{1}{4}\left[\frac{1}{4n-5}-\frac{1}{4n-1}\right]$$

$$a_n=\frac{1}{4}\left[\frac{1}{4n-1}-\frac{1}{4n+3}\right]$$

Adding column-wise, we get

$$\sum_1^n a_r=\frac{1}{4}\left[\frac{1}{3}-\frac{1}{4n+3}\right]=\frac{n}{3(4n+3)}$$

$\therefore$ The required sum $=\dfrac{n}{3(4n+3)}$.

Example 13: Find the sum of the series:

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots \text{ to infinity.}$$

Solution: Here $T_n=\dfrac{1}{(3n-2)(3n+1)}$

Breaking it into Partial Fractions, we have

$$\frac{1}{(3n-2)(3n+1)}=\frac{A}{3n-2}+\frac{B}{3n+1}$$

Multiplying both sides by $(3n-2)(3n+1)$, we have

$$1=A(3n+1)+B(3n-2)$$

Comparing the coefficient of n and the constants both sides, we get

$$0=3A+3B \qquad \text{(i)}$$

$$1=A-2B \qquad \text{(ii)}$$

Solving (i) and (ii) we get $A=\frac{1}{3}$, $B=-\frac{1}{3}$

$$\therefore \quad T_n=\frac{1}{3(3n-2)}-\frac{1}{3(3n+1)}=\frac{1}{3}\left[\frac{1}{3n-2}-\frac{1}{3n+1}\right]$$

and $$\sum_{k=1}^{\infty}T_k=\frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{1}{3k-2}-\frac{1}{3k+1}\right)$$

$$=\frac{1}{3}\left[\left(1-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\ldots\right]=\frac{1}{3}(1)=\frac{1}{3}$$

EXERCISE 5.4

1. **Find the sum of the following:**

 (i) $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\ldots$ to n terms. (ii) $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\ldots$ to n terms.

 (iii) $\frac{1}{2\cdot5}+\frac{1}{5\cdot8}+\frac{1}{8\cdot11}+\ldots$ to infinity. (iv) $\frac{1}{4\cdot13}+\frac{1}{13\cdot22}+\frac{1}{22\cdot31}+\ldots$ to infinity.

2. Find sum of the series: $\sum_{k=1}^{n}\frac{1}{9k^2+3k-2}$

3. Find sum of the series: $\sum_{k=2}^{n}\frac{1}{(k^2-k)}$

4. Find sum of the series: $\sum_{k=1}^{n}\frac{1}{k^2+7k+12}$

REVIEW EXERCISE 5

1. **Choose the correct option**

(i) If $t_n = 6n+5$, then $t_{n+1} =$
 (a) $6n-1$ (b) $6n+11$ (c) $6n+6$ (d) $6n-5$

(ii) The sum to infinity of the series $1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\cdots$
 (a) 6 (b) 2 (c) 3 (d) 4

(iii) Sum the series: $1+2\cdot2+3\cdot2^2+\ldots\ldots+100\cdot2^{99}$
 (a) $99\cdot2^{100}$ (b) $100\cdot2^{100}$ (c) $99\cdot2^{100}+1$ (d) $1000\cdot2^{100}$

(iv) The nth term of the series $1\cdot2+2\cdot3+3\cdot4+\ldots$ is
 (a) (n^2-n) (b) (n^2+n) (c) n^2 (d) None of these

(v) The sum of n terms of the series whose nth term is $1+2^n$
 (a) $n+2^{n-1}$ (b) $(n+1)+2^{n+1}$ (c) $n+2(2^n-1)$ (d) None of these

(vi) Evaluate $\sum(3+2^r)$, where $r = 1, 2, 3, \ldots. 10$
 (a) 2051 (b) 2049 (c) 2076 (d) 1052

(vii) What is the nth term of the series $1+\frac{(1+2)}{2}+\frac{(1+2+3)}{3}+\ldots?$
 (a) $\frac{n+1}{2}$ (b) $\frac{n(n+1)}{2}$ (c) $n^2-(n+1)$ (d) $\frac{(n+1)(2n+3)}{2}$

(viii) Sum of n terms of the series $1^3+3^3+5^3+7^3+\ldots..$ is
 (a) $n^2(2n^2-1)$ (b) $2n^3+3n^2$ (c) $n^3(n-1)$ (d) n^3+8n+4

2. Sum the series to n terms $1\cdot2+2\cdot3+3\cdot4+\cdots$
3. Sum the series $1\cdot3\cdot5+2\cdot4\cdot6+3\cdot5\cdot7+$ to n terms.
4. Sum the series $\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\frac{1}{10\cdot13}+\ldots$
5. Sum the series $5+12x+19x^2+26x^3+\cdots$ to n terms.
6. Sum the series: $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\ldots$ to n terms.
7. Find the sum of n terms of the series
 (i) Sum the series: $1\cdot2^2+3\cdot3^2+5\cdot4^2+\cdots$ to n terms.
 (ii) Sum the series: $3\cdot1^2+5\cdot2^2+7\cdot3^2+\cdots$ to n terms.
8. Find the sum of n terms of the series whose nth term is
 (i) n^3+3^n (ii) $2n^2+3n$ (iii) $n(n+1)(n+4)$ (iv) $(2n-1)^2$
9. Find the sum of the first n terms of the series
 (i) $3+7+13+21+31+\cdots$ (ii) $2+5+14+41+\cdots$
10. Find the n^{th} term and the sum to n terms of the series
 $1+(1+\frac{1}{2})+(1+\frac{1}{2}+\frac{1}{4})+(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8})+\cdots$