UNIT 4

SEQUENCES AND SERIES

$$a_n = a_1 + (n-1)\,d$$

- a_n: n^{th} term in the sequence
- a_1: 1st term in the sequence
- $(n-1)$: number of terms in the sequence
- d: common difference

STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Define a sequence (progression) and its terms.
- Know that a sequence can be constructed from a formula or an inductive definition.
- Recognize triangle, factorial and pascal sequences.
- Define an arithmetic sequence.
- Find the nth or general term of an arithmetic sequence.
- Solve problems involving arithmetic sequence.
- Know arithmetic mean between two numbers.
- Insert n arithmetic means between two numbers.
- Define an arithmetic series.
- Establish the formula to find the sum to n terms of an arithmetic series.
- Show that sum of n arithmetic means between two numbers is equal to n times their arithmetic mean.
- Solve real life problems involving arithmetic series.
- Define a geometric sequence.
- Find the nth or general term of a geometric sequence.
- Solve problems involving geometric sequence.
- Know geometric mean between two numbers.
- Insert n geometric means between two numbers.
- Define a geometric series.
- Find the sum of n terms of a geometric series.
- Find the sum of an infinite geometric series.
- Convert the recurring decimal into an equivalent common fraction.
- Solve real life problems involving geometric series.
- Recognize a harmonic sequence.
- Find nth term of harmonic sequence.
- Define a harmonic mean.
- Insert n harmonic means between two numbers.

4.1 Introduction

In practical life you must have observed many things follow a certain pattern, such as the petals of a sunflower, the holes of a honeycomb, the grains on a maize cob, the spirals on a pineapple on a pipe cone etc.
In our day-to-day life, we see patterns of geometric figures on clothes, pictures, posters etc. They make the learners motivated to form such new patterns.
Number patterns are faced by learners in their study. Number patterns play an important role in the field of mathematics. Let us study the following number patterns

(i) $2, 4, 6, 8, 10,...$ (ii) $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3,...$ (iii) $10, 7, 4, 1, -2,...$ (iv) $2, 4, 8, 16, 32,...$
(v) $4, \frac{1}{2}, \frac{1}{16}, \frac{1}{28},...$ (vi) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4},...$ (vii) $1, 11, 111, 1111, 11111,...$

It is an interesting study to find whether some specific names have been given to some of the above number patterns and the methods of finding some next terms of the given patterns.
Observing various patterns various sequences were defined to solve various summation problems.
Among various sequences A.P. (Arithmetic progression), G.P. (Geometric progression) and H.P (Harmonic progression) are most common.
Idea on A.P. was given by mathematician Carl Friedrich Gauss, who, as a young boy, stunned his teacher by adding up $1 + 2 + 3 + ... + 99 + 100$ within a few minutes. Here's how he did it.
He realized that adding the first and last numbers, 1 and 100, gives, 101 and adding the second, and second last numbers, 2 and 99, gives 101, as well as $3 + 98 = 101$ and so on, Thus he concluded that there are 50 sets of 101. So the sum of the series is $50(1 + 100) = 5050$.

4.1.1. Sequence

A **sequence** is a function whose domain is the set of positive integers. The numbers in the range of a sequence are real numbers, called terms of the sequence.

4.1.2 Construction of a sequence from a formula (inductive definition)

Let f be a function defined by

$$f(n) = 2n,\ n \in \{1, 2, 3, ...\}$$

then $f(1) = 2$, the first term

$f(2) = 4$, the second term

$f(3) = 6$, the third term ....

Thus the required sequence is 2, 4, 6, ...

Remember

A Sequence is also called a progression

In sequences, instead of using a symbol such as $f(n)$ for the nth term (usually called the general term) which denotes the number that corresponds to a given integer n, it is customary to use the symbol a_n for f(n). When the nth term of a sequence is known then we denote the entire sequence by the symbol $\{a_n\}$, where $a_1, a_2, a_3, \ldots$ are the first term, the second term, the third term of the sequence $\{a_n\}$ and so on. Since the order among the positive integers induce the ordering among the corresponding terms of the sequence, this clearly shows that the ordering has a vital role in the definition of a sequence, so we can also define a sequence as follows.

A **sequence** is a collection of numbers arranged in particular order.

The sequence 1,1,2,3,5,8,…can be written as $x_1 = x_2 = 1,\ x_n = x_{n-1} + x_{n-2},\ n > 2,\ n \in N$.

This sequence of numbers is called the Fibonacci sequence. Some sequences may not be described by any rule 2, 3, 5, 7, 11, 13, 17,… the formula for a_n, the nth prime number has not been found yet.

Remember

A sequence may be described by specifying first few terms and a formula (or a set of formulae) giving a relation between successive terms. Such a formula is called RECURSIVE FORMULA (or RECURRENCE RELATION).

Example 1: Write the first four terms a_1, a_2, a_3, and a_4 of each sequence, where $a_n = f(a)$.

(a) $f(n) = 2n - 5$ (b) $f(n) = 4(2)^{n-1}$

(c) $f(n) = (-1)^n \left(\frac{n}{n+1}\right)$

Solution:

a) Since $f(n) = 2n - 5$

$a_1 = f(1) = 2(1) - 5 = -3$

$a_2 = f(2) = 2(2) - 5 = -1$

In a similar manner, $a_3 = f(3) = 1$ and $a_4 = f(4) = 3$.

b) Since $f(n) = 4(2)^{n-1}$

$a_1 = f(1) = 4(2)^{1-1} = 4$

Similarly, $a_2 = 8$, $a_3 = 16$ and $a_4 = 32$.

c) $a_1 = f(1) = (-1)^1 (1/1+1) = -1/2$,

$a_2 = f(2) = (-1)^2 (2/2+1) = 2/3$,

$a_3 = f(3) = (-1)^3 (3/3+1) = -3/4$,

$a_4 = f(4) = (-1)^4 (4/4+1) = 4/5$.

Remember

A sequence is said to be finite if there is a first and last term otherwise it is said to be infinite.

Did You Know

The factor $(-1)^n$ causes the terms of the sequence to alternate signs

Example 2: Find the first four terms of the recursive sequence that is defined by $a_n = 2a_{n-1} + 1;\quad a_1 = 3$.

Solution: The sequence is defined recursively, so we must find the terms in order.

$a_1 = 3$

$a_2 = 2a_1 + 1 = 2(3) + 1 = 7$

$a_3 = 2a_2 + 1 = 2(7) + 1 = 15$

$a_4 = 2a_3 + 1 = 2(15) + 1 = 31$

The first four terms are 3,7,15, and 31.

Did You Know

There is no unique representation of a general term of a sequence. It is not always possible to determine the general term of a sequence.

4.1.3 Some special Sequences

Some well-known sequences are given in the following example.

Example 3: Write down the first five terms of each recursively defined sequence.

(a) $t_1 = 1,\ t_{n+1} = t_n + (n+1),\ n = 1,2,3,\ldots$

(b) $f_0 = 1,\ f_{r+1} = (r+1)f_r,\ r = 0,1,2,3,\ldots$

(c) $p_0 = 1,\ p_{r+1} = \frac{4-r}{r+1} p_r,\ r = 0,1,2,3,\ldots$

Solution: (a)

$t_1 = 1$

$t_2 = t_1 + 2 = 1 + 2 = 3$

$t_3 = t_2 + 3 = 1 + 2 + 3 = 6$

$t_4 = t_3 + 4 = 1 + 2 + 3 + 4 = 10$

$t_5 = t_4 + 5 = 1 + 2 + 3 + 4 + 5 = 15$

(b)

$f_0 = 1$

$f_1 = 1.f_0 = 1 \times 1 = 1$

$f_2 = 2.f_1 = 2 \times 1 = 2$

$f_3 = 3.f_2 = 3 \times 2 \times 1 = 6$

$f_4 = 4.f_3 = 4 \times 3 \times 2 \times 1 = 24$

This sequence is so important that it has its own special notation, r !, read as ‘r factorial’. It is defined as: 0! = 1 and (r + 1)! = (r + 1) × r!, r = 0, 1, 2, . . .

(c) $p_0 = 1$

$p_1 = \frac{4}{1} p_0 = \left(\frac{4}{1}\right)(1) = 4$ $\qquad p_2 = \frac{3}{2} p_1 = \left(\frac{3}{2}\right)(4) = 6$

$p_3 = \frac{2}{3} p_2 = \left(\frac{2}{3}\right)(6) = 4$ $\qquad p_4 = \frac{1}{4} p_3 = \left(\frac{1}{4}\right)(4) = 1$

The sequences given in example 3(a), 3(b) and 3(c) are special type of sequences, called the triangle number sequence, the factorial sequence and Pascal sequence respectively. These sequences play an important role in the expansion of binomial expressions like $(x + y)^n$.
The complete Pascal sequence in 3(c) is

$$1, 4, 6, 4, 1, 0, 0, 0, 0, \ldots$$

This is only one of the families of Pascal sequences. Due to its importance, it has a special notation,

$$\binom{4}{r}, \; r = 0,1,2,3,\ldots$$

For example $\binom{4}{0}=1, \binom{4}{1}=4, \binom{4}{2}=6, \binom{4}{3}=4, \binom{4}{4}=1, \binom{4}{5}=0$ and so on.

Obviously, different Pascal sequences will have different multiplying factor.

The general definition of a Pascal sequence is:

$$\binom{n}{0}=1 \text{ and } \binom{n}{r+1}=\frac{n-r}{r+1}\binom{n}{r}, \quad r=0,1,2,3,\ldots$$

We obtain the following Pascal sequences for $n = 0,1,2,3,4,\ldots$ by using the bove general definition:

= 0: 1, 0, 0, 0, 0, 0, . . .
= 1: 1, 1, 0, 0, 0, 0, . . .
= 2: 1, 2, 1, 0, 0, 0, . . .
= 3: 1, 3, 3, 1, 0, 0, . . .
= 4: 1, 4, 6, 4, 1, 0, . . .

m which we extract the well-known triangle, called

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Did You Know ?

Summation Notation
Summation notation is used to write series effectively. The symbol $\sum$, the uppercase Greek letter sigma, indicates a sum.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots\ldots + a_n$$

The letter k is called the index of summation. The numbers 1 and n represent the sub-scripts of the first and last terms in the series. They are called the lower limit and upper limit of the summation, respectively.

Practice !

Using summation notation
Evaluate each series

(a) $\sum_{k=1}^{n} k^2$

(b) $\sum_{k=1}^{n} 5$

(c) $\sum_{k=1}^{n} (2k-5)$

Remember !

Sequences and Series
A sequence is an ordered list, whereas a series is a sum of the terms of a sequence.
For example.
1, 3, 5, 7, 9, 11, 13, 15 is a sequence, and
1+3+5+7+9+11+13+15 a series.

Example 4: Find the sum $\sum_{k=1}^{4} k^2(k-2)$

Solution:
$$\sum_{k=1}^{4} k^2(k-2) = 1^2(1-2)+2^2(2-2)+3^2(3-2)+4^2(4-2)$$
$$=(-1)+0+(9)+(32) = 40$$

Example 5: Find the sum $\sum_{k=1}^{10} c$, c is constant

Solution:
$$\sum_{k=1}^{10} c = c + c + c + \ldots + c = 10c$$

EXERCISE 4.1

1. Classify the following into finite and infinite sequences.
   (i) 2,4,6,8,...,50 (ii) 1,0,1,0,1,...
   (iii) ...,–4,0,4,8,...,60 (iv) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \dots, -\frac{1}{2187}$
2. Find the first four terms of a sequence with the given general terms:
   (i) $\frac{n(n+1)}{2}$ (ii) $(-1)^{n-1}2^{n+1}$
   (iii) $\left(\frac{1}{3}\right)^n$ (iv) $\frac{n(n-1)(n-2)}{6}$
3. Write down the nth term of each sequence as suggested by the pattern.
   (i) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$ (ii) $2, -4, 6, -8, 10, \dots$ (iii) $1, -1, 1, -1, \dots$
4. Write down the first five terms of each sequence defined recursively.
   (i) $a_1 = 3, a_{n+1} = 5 - a_n$ (ii) $a_1 = 3, a_{n+1} = \frac{a_n}{n}$
5. Write each of the following series in expanded form.
   (i) $\sum_{j=1}^{6}(2j-3)$ (ii) $\sum_{k=1}^{5}(-1)^k 2^{k-1}$
   (iii) $\sum_{j=1}^{\infty}\frac{1}{2^j}$ (iv) $\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k$
6. Find the Pascal sequences for: (i) n=5 (ii) n=6 (iii) n=8 by using its general recursive definition.

4.2 Arithmetic Sequence (A.P)

4.2.1. Numbers are said to be in **Arithmetic Sequence (A.S)** ***or Arithmetic Progression (A.P)*** when its terms increase or decrease by a common difference. Thus each of the following ***sequence*** forms an Arithmetic Progression:

3, 7, 11, 15, ...

8, 2, –4, –10, ...

a, a+d, a+2d, a+3d, ...

The common difference is found by subtracting any term of the series from that which follows *it*. In the first of the above examples the common difference is 4; in the second it is –6; in the third it is d.

4.2.2 The nth term of an Arithmetic Sequence

We find a formula for the nth term of an arithmetic sequence. Let a_1 be its first term and d be its common difference. Then consecutive terms of the sequence are given by

$$a_1 = a_1 + 0.d = a_1 + (1-1)d$$
$$a_2 = a_1 + d = a_1 + 1\cdot d = a_1 + (2-1)d$$
$$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2\cdot d = a_1 + (3-1)d$$
$$a_4 = a_3 + d = (a_1 + 2\cdot d) + d = a_1 + 3\cdot d = a_1 + (4-1)d$$
$$a_5 = a_4 + d = (a_1 + 3\cdot d) + d = a_1 + 4\cdot d = a_1 + (5-1)d$$
$$\vdots$$
$$\Rightarrow a_n = a_1 + (n-1)d$$

Example 7: Find the 15th term of the arithmetic sequence whose first three terms are 20, 16.5 and 13.

Solution: Here $a_1 = 20, d = 16.5 - 20 = -3.5$ and $n = 15$. Substituting these values in the formula: $a_n = a_1 + (n-1)d$

We obtain, $a_{15} = 20 + (15-1)(-3.5) = 20 - 49 = -29$

If any two terms of an Arithmetic *sequence* be given, the series can be completely determined; for the data furnish two simultaneous equations, the solution of which will give the first term and the common difference.

Example 8: The 8th term of an arithmetic sequence is 75 and the 20th term is 39. Find the first term and the common difference. Give a recursive formula for the sequence.

Solution: We know that $a_n = a_1 + (n-1)d$

then $a_8 = a_1 + 7d = 75$ (i)

and $a_{20} = a_1 + 19d = 39$ (ii)

Subtracting (ii) from (i), we obtain

$-12d = 36 \Rightarrow d = -3$

From (i) we get $a_1 + 7(-3) = 75$ or $a_1 = 96$

Since $a_n = a_1 + (n-1)d = 96 + (n-1)(-3) = 99 - 3n$

$a_{n+1} = 99 - 3(n+1) = 99 - 3n - 3 = 96 - 3n = (99 - 3n) - 3 = a_n - 3$

$\therefore$ $a_{n+1} = a_n - 3$ is the required recursive formula for the given arithmetic sequence.

4.3 Arithmetic Mean (A.M)

4.3.1 When three numbers are in Arithmetic Progression, the middle one is said to be the arithmetic mean of the other two.

Thus arithmetic mean of two numbers a and b is $\frac{a+b}{2}$, where a and b are called the extremes. Mathematically, it is derived in the following way:

Let A be the arithmetic mean between two numbers a and b, then a, A, b, form an arithmetic sequence. By definition, we have

$$A-a=b-A$$

$$2A=a+b$$

Hence $$A=\frac{a+b}{2}$$

Thus the arithmetic mean of two numbers is equal to one–half of their sum.

Example 9: Find the arithmetic mean of $\sqrt{2}-3$ and $\sqrt{2}+3$

Solution: Here $a=\sqrt{2}-3$, $b=\sqrt{2}+3$

$$A=\frac{a+b}{2}=\frac{\sqrt{2}-3+\sqrt{2}+3}{2}=\sqrt{2}$$

Between two given numbers it is always possible to insert any number of terms such that the whole series thus formed shall be an A. P.; the terms thus inserted are called the arithmetic means.

4.3.2 Inserting n Arithmetic Means (A.Ms)

Let $A_1, A_2, \ldots, A_n$ be n A.Ms between a and b then $a, A_1, A_2, \ldots A_n, b$ form a finite arithmetic sequence of $n+2$ terms, that is:

$$a_{n+2}=b$$

$$a+(n+2-1)d=b, \text{ where d is the common difference}$$

$$(n+1)d=b-a$$

$$\therefore d=\frac{b-a}{n+1}$$

Thus $$A_1=a+d=a+\frac{b-a}{n+1}$$

$$A_2=a+2d=a+2\left(\frac{b-a}{n+1}\right)$$

$$A_3=a+3\left(\frac{b-a}{n+1}\right)$$

Similarly $$A_4=a+4\left(\frac{b-a}{n+1}\right)$$

$$\vdots$$

$$A_n=a+n\left(\frac{b-a}{n+1}\right)$$

which are the required n A.Ms between a and b. Thus, $A_1, A_2, \ldots, A_n$ are real numbers such that $a, A_1, A_2, \ldots A_n, b$ is an arithmetic sequence, then $A_1, A_2, \ldots, A_n$ are called the n arithmetic means between the numbers a and b. The process of determining these numbers is referred to as inserting n arithmetic means between a and b.

Example 10: Insert three arithmetic means between 2 and 9.

Solution: Let A_1, A_2 and A_3 be the arithmetic means between 2 and 9 such that 2, A_1, A_2, A_3, 9 forms a finite arithmetic sequence of 5 terms with $a=2$, $b=9$. Let d be the common difference, then $a_5=b$ gives that

$$a+4d=9 \Rightarrow 2+4d=9$$

$$4d=7 \Rightarrow d=\frac{7}{4}$$ Thus the three arithmetic means are

$$A_1=a+d=2+\frac{7}{4}=\frac{15}{4}$$

$$A_2=a+2d=2+2\left(\frac{7}{4}\right)=\frac{11}{2}$$

$$A_3=a+3d=2+3\left(\frac{7}{4}\right)=\frac{29}{4}$$

EXERCISE 4.2

1. Find the 15th term of the arithmetic sequence 2, 5, 8, ...
2. The 1^{st} term of an arithmetic sequence is 8 and the 21^{st} term is 108. Find the 7th term.
3. Find the number of terms in the arithmetic progression 6, 9, 12, ..., 78.
4. The nth term of a sequence is given by $a_n = 2n + 7$. Show that it is an A.P. Also, find its 7^{th} term.
5. Show that the sequence log a, log(ab), log(ab^2), log(ab^3), ... is an A.P. Find its nth term.
6. Find the value of 'k' if 2k+7, 6k–2, 8k –4 are in A.P. Also find the sequence.

7. If $a_6 + a_4 = 6$ and $a_6 - a_4 = \frac{2}{3}$, find the arithmetic sequence.
8. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P. then prove $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.
9. A ball rolling up an incline covered 24m during the first second, 21m during the second, 18m during the third second. Find how many meters it covered in the eighth second?
10. The population of a town is decreasing by 500 inhabitants each year If its population in 1960 was 20135, what was its population in 1970?
11. Ahmad and Akram can climb 1000 feet in the first hour and 100 feet in each succeeding hour. When will they reach the top of a 5400 feet hill?
12. A man earned $3500 the first year he worked. If he received a raise of $750 at the end of each year for 20 years, what was his salary during his twenty first year of work?
13. Find the arithmetic mean between the given numbers:
    (i) 12, 18 (ii) $\frac{1}{3}, \frac{1}{4}$ (iii) $-6, -216$ (iv) $(a+b)^2, (a-b)^2$
14. Insert: (i) Three arithmetic means between 6 and 41.
    (ii) Four arithmetic means between 17 and 32.
15. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the arithmetic mean between a and b?
16. Insert five arithmetic means between 5 and 8 and show that their sum is five times the arithmetic mean between 5 and 8.
17. There are n arithmetic means between 5 and 32 such that the ratio of the 3rd and 7th means is 7:13, find the value of n.

4.4 Arithmetic Series

4.4.1 As we know that associated with every sequence is a series, the indicated sum of the terms of the sequence. If the sequence happens to be the arithmetic sequence, then the associated series is called the arithmetic series.

Let $\{a_n\}$ be the arithmetic sequence then the series

$$a_1 + a_2 + ... + a_n = \sum_{k=1}^{n} a_k \text{ is called the arithmetic series.}$$

The arithmetic series in the general form or standard form is given as:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + ... + [a_1 + (n-1)d] = \sum_{k=1}^{n}[a_1 + (k-1)d] \quad (1)$$

where S_n denotes the sum of the first n terms of the arithmetic series.

4.4.2 Sum of first n terms of an Arithmetic Series

The next result gives a formula for finding the sum of the first n terms of an arithmetic sequence.

Theorem: For an arithmetic sequence $\{a_n\}$, the sum S_n of the first n terms is given by

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$
$$= \frac{n}{2}(a_1 + a_n)$$

Proof: The sum of the first n terms of an arithmetic sequence is denoted by S_n.

Let $S_n = a_1 + a_2 + a_3 + ... + a_n$

Since d is the common difference between terms, S_n can be written forward and backward as follows.

Forward: Start with the first term, a_1. Keep adding d.

Backward: Start with the last term, a_n. Keep subtracting d.

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + --- + a_n$$
$$S_n = a_n + (a_n - d) + (a_n - 2d) + --- + a_1$$

Add the two equations.

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + --- + (a_1 + a_n)$$
$$= n \text{ sums of } (a_1 + a_n)$$
$$2S_n = n(a_1 + a_n)$$
$$S_n = \frac{n}{2}(a_1 + a_n)$$

Solve for S_n, dividing both sides by 2.

$$a_n = a_1 + (n-1)d,$$
$$\therefore S_n = \frac{n}{2}\{a_1 + a_1 + (n-1)d\}.$$
$$\therefore S_n = \frac{n}{2}\{2a_1 + (n-1)d\}.$$

Example 11:

Finding the sum of a finite arithmetic series. Use a formula to find the sum of the arithmetic series $2 + 4 + 6 + 8 + ... + 100$.

Solution: The series $2 + 4 + 6 + 8 + ... + 100$ has n=50 terms with $a_1 = 2$ and $a_{50} = 100$. We can use the formula $S_n = \frac{n}{2}(a_1 + a_n)$ to find its sum.

$$S_{50} = \frac{50}{2}(2 + 100) = 2550$$

We can also use the formula $S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$.

$$S_{50} = 50/2\ (2(2) + (50-1)\ 2) = 2550$$

Example 12: The sum of an arithmetic series with 15 terms is 285. If $a_{15} = 40$, find a_1

Solution: To find a_1, we apply the sum formula

$$S_n = \frac{n}{2}(a_1 + a_n) \quad \text{with } n = 15 \text{ and } a_{15} = 40$$

$$\frac{15}{2}(a_1 + 40) = 285$$

$$15(a_1 + 40) = 570 \quad \text{Multiply by 2.}$$

$$(a_1 + 40) = 38 \quad \text{Divide by 15.}$$

$$a_1 = -2 \quad \text{Subtract 40}$$

Example 13: The first term of a series is 5, the last 45 and the sum 400. Find the numebr of terms, and the common difference.

Solution: If n be the number of terms, then from

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$400 = \frac{n}{2}(5 + 45);$$

Hence $n = 16.$

If d be the common difference

$$45 = \text{the } 16^{th} \text{ term} = 5 + 15d;$$

Hence $d = 2\frac{2}{3}.$

Example 14: Find the sum of the first 200 positive odd integers.

Solution: Since the positive odd integers:

1,3,5,...,2n–1,... form an arithmetic sequence with

$$a_1 = 1, d = 2, n = 200 \text{ then } a_n = a_1 + (n-1)d$$

$$= 1 + (200 - 1)(2) = 399$$

$$\therefore S_n = \frac{n}{2}(a_1 + a_n) = \frac{200}{2}(1 + 399) = \frac{200}{2}(400) = 40000$$

Example 15: Find the 18^{th} term and the sum of the 18 terms of the arithmetic sequence: $-8, -3, 2, 7, \ldots$

Solution: Since we are given that:

$-8, -3, 2, 7, \ldots$ is an arithmetic sequence.

Then $a_1 = -8, d = 5$ and $n = 18$. We have to find a_{18} and S_{18}

Since $a_n = a_1 + (n-1)d$

$a_{18} = -8 + 17(5)$; *putting values of a_1 & d*

$$= 77$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{18} = \frac{18}{2}(-8 + 77) \text{ putting values of } a_1 \text{ \& } a_n$$

$$= 9(69) = 621$$

Example 16: The 10th term of an arithmetic sequence is 32 and the 18^{th} term is 48, what is the sum of the first twelve terms?

Solution: Let a_1 be the first term, d be the common difference and n be the number of terms of the given arithmetic sequence.

Then $a_{10} = 32, \quad a_{18} = 48$

$$a_1 + (10-1)d = 32, \quad a_1 + (18-1)d = 48$$

$$a_1 + 9d = 32 \quad \text{(i)}$$

$$a_1 + 17d = 48 \quad \text{(ii)}$$

Subtracting (i) from (ii) we obtain

$$8d = 16$$

$$d = 2$$

(i) gives that $a_1 + 18 = 32$

$$a_1 = 14$$

Now $S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$

$$S_{12} = \frac{12}{2}\{2(14) + 11(2)\} = 6\{28 + 22\} = 300$$

Example 17: Find the sum of all the integers lying between 100 and 600 that end in 5.

Solution: The integers lying between 100 and 600 that end in 5 are

105,115, 125, ..., 595

which form an arithmetic sequence with

$$a_1 = 105, d = 10, a_n = 595$$

then $a_n = a_1 + (n-1)d$

$$595 = 105 + 10n - 10$$

$$10n = 595 - 105 + 10$$

$$= 500$$

$$n = 50$$

Since $S_n = \frac{n}{2}(a_1 + a_n)$

Which gives $S_{50} = \frac{50}{2}(105+595)$
$= 25(700)$
$= 17500$

4.4.3 Relation of A.M of two numbers with n A.Ms between them

Theorem: The sum of n arithmetic means between two numbers is equal to n times their arithmetic mean.

Proof: Let there be n arithmetic means between a and b such that

$$a, a+d, a+2d, ..., a+nd, b$$

forms an arithmetic sequence with n+2 terms. Then

$$a+(a+d)+(a+2d)+...+(a+nd)+b = \frac{n+2}{2}[2a+(n+2-1)d]$$
$$= \frac{n+2}{2}[a+\{a+(n+1)d\}]$$
$$= \frac{n+2}{2}[a+b] \;, b = a + (n+1)d$$

$$(a+d)+(a+2d)+...+(a+nd) = \frac{n+2}{2}(a+b)-(a+b)$$
$$= (a+b)\left[\frac{n+2}{2}-1\right]$$
$$= n\left(\frac{a+b}{2}\right)$$

Thus the sum of n arithmetic means = n (arithmetic mean)

4.4.4 Real life problems involving arithmetic series

Example 18: Finding the sum of a finite arithmetic series

A person has a starting annual salary of Rs.300,000 and receives a 1500 raise each year.

(a) Calculate the total amount earned over 9 years.

(b) Verify this value using a calculator.

Solution: (a) Using $S_n = \frac{n}{2}\{2a_1+(n-1)d\}$,

$$S_{10} = \frac{10}{2}\{2\times300000+(10-1)1500\} = 3,067,500$$

(b) To verify this result with a calculator, compute the sum $a_1+a_2+a_3+\cdots+a_{10}$

$= 300000+301500+303000+304500+306000+307500+309000+310500$
$+312000+313500 = 3,067,500$

Example 19: A new car costs Rs.1200000. Assume that it depreciates 24% the first year, 20% the second year, 16% the third year and continues in the same manner for six years. If all the depreciations apply to the original cost, what is the value of the car in six years?

Solution: Since the depreciations 24%, 20%, 16%, ... form an arithmetic sequence with

a_1=24, d = –4 and n = 6

Calculating the sum of the depreciations over six years

$$S_n = \frac{n}{2}\{2a_1+(n-1)d\}$$
$$S_6 = \frac{6}{2}\{48+5(-4)\}$$
$$= 3(28) = 84$$

Now the total depreciation in six years is 84% of 1200000

$$= \frac{84}{100}\times1200000 = Rs.1008000$$

Thus the value of the car in six years = 1200000 – 1008000= Rs.192000.

Example 20: A display of cans in a grocery store consists of 24 cans in the bottom row, 21 cans in the next row and so on in an arithmetic sequence. The top row has 3 cans. Find the total number of cans in the display.

Solution: Since the display of cans are in arithmetic sequence with

$a_1 = 24$, $a_n = 3$ and $d = -3$ calculating the number of rows, we have

$$a_n = a_1+(n-1)d$$
$$3 = 24-3n+3$$
$$3n = 24$$
$$n = 8$$

Now the total number of cans is given by $S_n = \frac{n}{2}(a_1+a_n)$

$$S_8 = \frac{8}{2}(24+3)$$
$$= 4(27)$$
$$= 108\ cans$$

EXERCISE 4.3

1. Find the indicated term and the sum of the indicated number of terms in case of each of the following arithmetic sequence:
   (i) $9, 7, 5, 3, \ldots$; 20th term; 20 terms
   (ii) $3, \frac{8}{3}, \frac{7}{3}, 2, \ldots$; 11^{th} term; 11 terms
2. Some of the components a_1, a_n, n, d and S_n are given. Find the ones that are missing:
   (i) $a_1 = 2, n = 17, d = 3$ (ii) $a_1 = -40, S_{21} = 210$
   (iii) $a_1 = -7, d = 8, S_n = 225$ (iv) $a_n = 4,\ S_{15} = 30$
3. Find the sum of all the numbers divisible by 5 from 25 through 350.
4. The sum of three numbers in an arithmetic sequence is 36 and the sum of their cubes is 6336. Find them. [Hint: suppose the numbers are $a - d,\ a,\ a + d$]
5. Find four numbers in arithmetic sequence, whose sum is 20 and the sum of whose squares is 120. [Hint: suppose the numbers are $a - 3d,\ a - d,\ a + d,\ a + 3d$]
6. $x_1, x_2, x_3, \ldots$ are in A.P. If $x_1 + x_7 + x_{10} = -6$ and $x_3 + x_8 + x_{12} = -11$, find $x_3 + x_8 + x_{22}$.
7. Find: 1+3–5+7+9–11+13+15–17+… up to 3n terms.
8. Show that the sum of the first n positive odd integers is n^2.
9. Find the sum of all multiples of 9 between 300 and 700.
10. The sum of Rs.1000 is distributed among four people so that each person after the first receives Rs. 20 less than the preceding person. How much does each person receive?
11. The distance which an object dropped from a cliff will fall 16ft the first second, 48 ft the next second, 80 ft the third second and so on. What is the total distance the object will fall in six seconds?
12. Afzal Khan saves Rs.1 the first day, Rs.2 the second, Rs.3 the third and Rs. N on the nth day for thirty days. How much does he save at the end of the thirtieth day?
13. A theater has 40 rows with 20 seats in the first row, 23 in the second row, 26 in the third row and so forth. How many seats are in the theater?
14. Insert enough arithmetic means between 1 and 50 so that the sum of the resulting series will be 459.

4.5 Geometric Sequence

4.5.1 In nature, certain phenomena can be described by geometric sequences. For example, archaeologists use the half–life of carbon 14 to estimate the age of ancient objects. Carbon 14 is a radioactive element that decays gradually, changing to nitrogen 14. The half–life (i.e. the time it takes for half of a given amount to decay) of carbon 14 is about 5600 years. Thus, one kg of carbon 14 will be reduced to $\frac{1}{2}$ kg in 5600 years, to $\frac{1}{4}$ kg in 11200 years, to $\frac{1}{8}$ kg in 16800 years and so on. Which is obviously a geometric sequence with $r = \frac{1}{2}$.

A geometric sequence (progression) is a sequence for which every term after the first is the product of the preceding term and a fixed number, called the common ratio of the sequence. We use the same notations as we use in A.P. with one exception that is instead of d, the common difference, we use r, the common ratio in geometric sequence.

Thus each of the following is a geometrical *sequence*.

$3, 6, 12, 24, \ldots\ldots$

$1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots\ldots$

$a, ar, ar^2, ar^3, \ldots\ldots, ar^{n-1}, \ldots\ldots$

The common ratio, and it is found by dividing any term by that which immediately precedes it.

In the first of the above examples the common ratio is 2; in the second it is $-\frac{1}{3}$; in the third it is r. A geometric sequence is recursively defined by equations of the form:

$$a_1 = a_1$$

and $a_{n+1} = ra_n$ where a_1 and r are real numbers, $a_1 \neq 0, r \neq 0$, and $n \in N$

4.5.2 The nth term of a Geometric Sequence

The nth term of a geometric sequence is given by: $a_n = a_1 r^{n-1}$

To find a formula for the nth term of a geometric sequence, we write down the first few terms using the recursive definition to observe the pattern:

$$\text{1st term} = a_1 = a_1 r^0 = a_1 r^{1-1}$$

$$\text{2nd term} = a_2 = a_1 r = a_1 r^{2-1}$$

$$\text{3rd term} = a_3 = a_2 r = a_1 r^2 = a_1 r^{3-1}$$

4th term $= a_4 = a_3 r = a_1 r^3 = a_1 r^{4-1}$

.

.

.

nth term $= a_n = a_1 r^{n-1}$

Example 21: Find the first five terms and the tenth term of the geometric sequence having first term 3 and common ratio $-\frac{1}{2}$.

Solution: Here $a_1 = 3$, $r = -\frac{1}{2}$

Then the first five terms are

$$3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \frac{3}{16}$$

Substituting the values in the formula $a_n = a_1 r^{n-1}$

we have $a_{10} = (3)\left(-\frac{1}{2}\right)^{10-1}$ with n = 10

$$= 3\left(-\frac{1}{2}\right)^9 = -\frac{3}{512}$$

Example 22: Show that the sequence $\{a_n\} = 2^{-n}$ is geometric and find its' common ratio.

Solution: Since $a_n = 2^{-n}$

then $a_{n+1} = 2^{-(n+1)}$

and $r = \frac{a_{n+1}}{a_n} = \frac{2^{-(n+1)}}{2^{-n}} = \frac{1}{2}$

The ratio of successive terms is a nonzero number independent of n, thus $\{a_n\}$ is geometric sequence with $r = \frac{1}{2}$

Example 23: If the third term of a geometric sequence is 5 and the sixth term is – 40, find the eighth term.

Solution: Here $a_3 = 5$ and $a_6 = -40$ then, we have $a_1 r^{3-1} = 5$ and $a_1 r^{6-1} = -40$

or $a_1 r^2 = 5$ (i)

and $a_1 r^5 = -40$ (ii)

Dividing the equation (ii) by the (i) we obtain

$$\frac{a_1 r^5}{a_1 r^2} = \frac{-40}{5}$$

or $r^3 = -8 = (-2)^3 \Rightarrow r = -2$ and $a_1 = \frac{5}{4}$ by (i)

Now $a_8 = a_1 r^{8-1} = \left(\frac{5}{4}\right)(-2)^7 = -160$

4.6 Geometric Means (G.Ms)

4.6.1 when three numbers are in Geometrical Progression, the middle one is called the **geometric mean** between the other two.

Mathematically, it is derived in the following way:

Let a and b be the two numbers; G the geometric *mean then*

$\frac{b}{G} = \frac{G}{a}$, since a, G, b are in G. P.,

$\therefore G^2 = ab$;

$G = \pm\sqrt{ab}$.

Example 24: Find the geometric mean of each of the following pairs of numbers.

(a) 9 and 16 (b) $\frac{-3}{10}$ and $-\frac{5}{6}$

Solution: (a) By the above definition

$G = \sqrt{ab} = \sqrt{9\times16} = \sqrt{144} = 12$, Since a,b >0 $\therefore$ G>0

(b) $G = -\sqrt{ab}$, Since a,b <0 $\therefore$ G<0

$$= -\sqrt{\left(\frac{-3}{10}\right)\left(\frac{-5}{6}\right)} = -\sqrt{\frac{1}{4}} = -\frac{1}{2}$$

4.6.2 To insert n Geometric Means between two numbers *a* and b

Since the terms between a and b of a geometric sequence are called the geometric means of a and b. Thus $G_1, G_2, \ldots, G_n$, are the n geometric means between a and b if a, $G_1, G_2, \ldots, G_n$, b form a geometric sequence. Moreover it is a finite geometric sequence of n+2 terms with $a_1 = a$ and $a_{n+2} = b$.

Let r be its common ratio, then $a_{n+2} = b$ gives that $ar^{n+1} = b$, $\because a_n = ar^{n-1}$

$$\Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Hence $G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$

.
.
.

$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

Example 25: Insert two geometric means between 64 and 125.

Solution: Let G_1, G_2 be the two geometric means between 64 and 125 such that

64, G_1, G_2, 125 is a geometric sequence.

Thus $a_1 = 64, n = 4$ and $a_4 = 125$

Let r be the common ratio of the geometric sequence, then

$a_4 = 125$ gives $a_1r^3 = 125$

$64r^3 = 125$ putting value of a_1

$$r^3 = \frac{125}{64} = \left(\frac{5}{4}\right)^3 \quad \Rightarrow r = \frac{5}{4}$$

Hence $G_1 = a_1r = (64)\left(\frac{5}{4}\right) = 80$ and $G_2 = a_1r^2 = (64)\left(\frac{5}{4}\right)^2 = 100$

Example 26: Insert three geometric means between 2 and 32.

Solution: Let G_1, G_2, G_3 be the three geometric means between 2 and 32 such that

2, G_1, G_2, G_3, 32

is a geometric sequence.

We have $a_1 = 2, n = 5$ and $a_5 = 32$

let r be the common ratio, then

$a_5 = 32$

gives. $a_1r^4 = 32$

$2r^4 = 32 \Rightarrow r^4 = 16 = (2)^4 \Rightarrow r = \pm 2$

∴ we have two sets of geometric means given below: If r = 2

then $G_1 = a_1r = (2)(2) = 4$

$G_2 = a_1r^2 = (2)(2)^2 = 8$

$G_3 = a_1r^3 = (2)(2)^3 = 16$

If r = −2 then

$G_1 = a_1r = (2)(-2) = -4$

$G_2 = a_1r^2 = (2)(-2)^2 = 8$

$G_3 = a_1r^3 = (2)(-2)^3 = -16$

Did You Know ?

It can be seen from Example 25 and 26 that if the number of required geometric means is even, a single set of geometric means is obtained, if the number of required geometric means is odd, two sets of geometric means are obtained.

EXERCISE 4.4

1. Write the first five terms of a geometric sequence given that:
   (i) $a_1 = 5; r = 3$ (ii) $a_1 = 8; r = -\frac{1}{2}$
   (iii) $a_1 = -\frac{9}{16}; r = -\frac{2}{3}$ (iv) $a_1 = \frac{x}{y}; r = -\frac{y}{x}$
2. Suppose that the third term of a geometric sequence is 27 and the fifth term is 243. Find the first term and common ratio of the sequence.
3. Find the seventh term of a geometric sequence that has 2 and $-\sqrt{2}$ for its second and third terms respectively.
4. How many terms are there in a geometric sequence in which the first and the last terms are 16 and $\frac{1}{64}$ respectively and $r = \frac{1}{2}$?
5. Find x so that $x+7, x-3, x-8$ form a three term geometric sequence in the given order. Also give the sequence.
6. If $a_{10} = \ell, a_{13} = m, a_{16} = n$; show that $ln = m^2$
7. Show that the reciprocals of the terms of a geometric sequence also form a geometric sequence.
8. Find the geometric mean of the following:
   (i) 3.14 and 2.71 (ii) − 6 and − 216
   (iii) x + y and x − y (iv) $\sqrt{2}+3$ and $\sqrt{2}-3$
9. (i) Insert 5 geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.
   (ii) Insert 6 geometric means between 14 and $-\frac{7}{64}$.

10. Find two numbers if the difference between them is 48 and their A.M. exceeds their G.M. by 18.

11. Prove that the product of n geometric means between a and b is equal to the nth power of the single geometric mean between them.

12. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the geometric mean between a and b?

4.7 Geometric Series

4.7.1 Since with any geometric sequence we have an associated geometric series, which is the indicated sum of the terms of the geometric sequence.

Let $\{a_n\}$ is a geometric sequence, then $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + ... + a_n + ...$ is called a geometric series.

If r is the common ratio, then the above series can be written in the form

$$\sum_{i=1}^{\infty} a_1 r^{i-1} = a_1 + a_1 r + a_1 r^2 + ... a_1 r^{n-1} + ... \qquad (1)$$

known as the general form of the geometric series.

4.7.2 Sum of first n terms of a Geometric Series

Theorem: For a geometric sequence with first term a_1 and common ratio $r \neq 1$, the sum S_n of the first n terms is:

$$S_n = \frac{a_1(1-r^n)}{1-r} \qquad (2)$$

Proof: Let $S_n = a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-1}$

$S_n = a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-2} + a_1 r^{n-1}$ — S_n is the sum of the first n terms of the sequence.

$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^{n-1} + a_1 r^n$ — Multiply both sides of the equation by r.

$S_n - rS_n = a_1 - a_1 r^n$ — Subtract the second equation from the first equation

$S_n(1-r) = a_1(1-r^n)$ — Factor out S_n on the left and a_1 on the right.

$S_n = a_1 \frac{(1-r^n)}{1-r}$ — Solve for S_n by dividing both sides by $1-r$ (assuming that $r \neq 1$).

which is the required sum of the first n terms of a geometric sequence.

Since
$$S_n = \frac{a_1(1-r^n)}{1-r}$$
$$= \frac{a_1 - a_1 r^n}{1-r} = \frac{a_1 - (a_1 r^{n-1})r}{1-r}$$
$$= \frac{a_1 - a_n r}{1-r}, a_n = a_1 r^{n-1} \text{ is the last term}$$
$$= \frac{a_1 - a_n r}{1-r}, \ r \neq 1 \qquad (3)$$

is the alternative form of the result given in (2)

Remember

It is better to use the forms

(i) $S_n = \frac{a_1(1-r^n)}{1-r}$ and

$S_n = \frac{a_1 - a_n r}{1-r}$, whenever $|r| < 1$

(ii) If $|r| > 1$ then the following forms are used $S_n = \frac{a_1(r^n-1)}{r-1}$

and $S_n = \frac{a_n r - a_1}{r-1}$

because the numerators and denominators are positive.

(iii) If r=1, we have the trivial geometric series:

$S_n = a_1 + a_1 + ... + a_1 = na_1$

Example 27: Approximate the sum for the given values of n.

(a) $1 + 1/2 + 1/4 + ... + (1/2)^{n-1}$; n=5, 10, and 20

(b) $3 - 6 + 12 - 24 + 48 - .... + 3(-2)^{n-1}$; n = 3, 8, and 13

Solution: (a) This geometric series has $a_1 = 1$ and $r = ½ = 0.5$.

$$S_5 = \frac{1(1-0.5^5)}{1-0.5} = 1.9375$$

$$S_{10} = \frac{1(1-0.5^{10})}{1-0.5} = 1.998047$$

$$S_{20} = \frac{1(1-0.5^{20})}{1-0.5} = 1.999998$$

(b) This geometric series has $a_1 = 3$ and $r = \frac{-6}{3} = -2$

$$S_3 = \frac{3(1-(-2)^3)}{1-(-2)} = 9$$

$$S_8 = \frac{3(1-(-2)^8)}{1-(-2)} = -255$$

$$S_{13} = \frac{3(1-(-2)^{13})}{1-(-2)} = 8193$$

Example 28: Sum the series $\frac{2}{3}, -1, \frac{3}{2}, \ldots\ldots\ldots to\ 7$ terms.

Solution: The common ratio $= -\frac{3}{2}$; hence by formula $\frac{a_1(1-r^n)}{1-r}$

The sum $= \frac{\frac{2}{3}\left\{1-\left(-\frac{3}{2}\right)^7\right\}}{1+\frac{3}{2}} = \frac{\frac{2}{3}\left\{1-\left(-\frac{3}{2}\right)^7\right\}}{1+\frac{3}{2}}$

$= \frac{\frac{2}{3}\left\{1+\frac{2187}{128}\right\}}{\frac{5}{2}} = \frac{2}{3}\times\frac{2315}{128}\times\frac{2}{5} = \frac{463}{95}$.

Example 29: Compute: 2+6+18+54+162+486

Solution: In this case $a_1 = 2; r = \frac{6}{2} = 3 > 1, n = 6$

Substituting the values in

$$S_n = \frac{a_1(r^n-1)}{r-1}$$

$$S_6 = \frac{2(3^6-1)}{3-1} = 729 - 1 = 728$$

Example 30: Given that $a_1 = \frac{3}{4}, a_n = 48$ and $S_n = 32\frac{1}{4}$, find r and n

Solution: Since $a_1 = \frac{3}{4}, a_n = 48$

Then $a_1 r^{n-1} = 48$

and $\frac{3}{4}r^{n-1} = 48$

$\Rightarrow \quad r^{n-1} = 64 \quad$ (i)

Also, we have $S_n = \frac{a_1 - ra_n}{1-r}$

$$\frac{129}{4} = \frac{\frac{3}{4} - 48r}{1-r}$$

$$129 - 129r = 3 - 192r$$

$$63r = -126$$

$$r = -2$$

From (i) we have

$$(-2)^{n-1} = 64 \Rightarrow (-2)^{n-1} = (-2)^6$$

$$n-1 = 6 \Rightarrow \quad n = 7$$

Example 31: Suppose that the third term of a geometric sequence is 27 and the fifth term is 243. Find a_1, r and S_5.

Solution: Since $a_3 = 27$ and $a_5 = 243$

Then we have $a_1 r^2 = 27$ (i) $\quad a_1 r^4 = 243$ (ii),

$$\therefore \quad a_n = a_1 r^{n-1}$$

Dividing (ii) by (i) we obtain

$$\frac{a_1 r^4}{a_1 r^2} = \frac{243}{27}$$

$$r^2 = 9 \Rightarrow r = \pm 3$$

We obtain two different solutions since there are two values of r.

$r = 3$	$r = -3$
$a_1 r^2 = 27$	$a_1 r^2 = 27$
$a_1(3)^2 = 27$	$a_1(-3)^2 = 27$
$a_1.9 = 27$	$a_1.9 = 27$
$a_1 = 3$	$a_1 = 3$
The first sequence is	The second sequence is
3,9,27,81,243,...	3,–9,27,–81,243,...
$S_5 = \frac{ra_5 - a_1}{r-1}$	$S_5 = \frac{ra_5 - a_1}{r-1}$
$= \frac{(3)(243)-3}{3-1}$	$S_5 = \frac{(-3)(243)-3}{-3-1}$
$= \frac{729-3}{2} = 363$	$= \frac{-732}{-4} = 183$

Example 32: Find the sum $\sum_{i=1}^{10} 6\cdot 2^i$

Solution: $\sum_{i=1}^{10} 6\cdot 2^i = 6\cdot 2 + 6\cdot 2^2 + 6\cdot 2^3 + \cdots + 6\cdot 2^{10}$

Do you see that each term after the first is obtained by multiplying the preceding term by 2? To find the sum of the 10 terms (n=10), we need to know the first term, a_1, and the common ratio, r. The first term is 6. 2 or 12: a_1= 12. The common ratio is 2.

$S_n = \frac{a_1(r^n-1)}{r-1}$	Use the formula for the sum of the first n terms of a geometric sequence.
$S_{10} = \frac{12(2^{10}-1)}{2-1}$	a_1 (the first term) = 12, r = 2, and n = 10 because are adding ten terms
= 12,276	Use a calculator

Thus, $\sum_{i=1}^{10} 6\cdot 2^i = 12,276$

4.7.3 Sum of infinite Geometric series

Our discussion of series has so far been restricted to those associated with finite sequences. The series associated with the infinite sequence:

$$a_1, a_1r, a_1r^2, ..., a_1r^{n-1}, ...$$

is denoted by:

$$a_1 + a_1r + a_1r^2 + ..., a_1r^{n-1} + ... = \sum_{i=1}^{\infty} a_1r^{i-1}$$

and is called an infinite series. Important questions arise over here are, what do we mean by the "sum" of an infinite number of terms, and under what circumstances does such a "sum" exist? The answers to these questions depend upon the concept of "limit" which is studied in a course in the calculus. However, for some particular infinite series we can give an intuitive idea of the concept of "sum".

Consider the formula for the sum of the first n terms in a geometric sequence, we have already proved that:

$$S_n = \begin{cases} a_1 + a_1 + a_1 + ... + a_1 = na_1, r = 1 \\ a_1 - a_1 + a_1 - ... + (-1)^{n-1}a_1, r = -1 \\ \frac{a_1(r^n-1)}{r-1}, |r| > 1 \\ \frac{a_1(1-r^n)}{1-r}, |r| < 1 \end{cases} \quad (4)$$

(i) Since $S_n = na_1$, when r=1

As n increases, the sum of the infinite geometric series increases without limit. Symbolically it is written as:

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} na_1 = \infty$

Thus the infinite geometric series in this case does not have a finite sum.

(ii) Here $S_n = a_1 - a_1 + a_1 - ... + (-1)^{n-1}a_1$ when r =–1

The sum of the first n terms is $a_1$or 0 according as n is odd or even; therefore the sum oscillates between the values 0 and a_1.

(iii) $$S_n = \frac{a_1(r^n-1)}{r-1}, |r| > 1 = \frac{a_1r^n}{1-r} - \frac{a}{r-1}, |r| > 1$$

Since $|r| > 1$, then the absolute value of each term is greater than the absolute value of the preceding term. Therefore such an infinite series cannot have a finite "sum".

Mathematically, it is shown that:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a_1r^n}{1-r} - \frac{a_1}{1-r}\right) = \lim_{n\to\infty}\left(\frac{a_1r^n}{1-r}\right) - \frac{a_1}{1-r} = \infty - \frac{a_1}{1-r} = \infty$$

(iv) $$S_n = \frac{a_1(1-r^n)}{1-r}, |r| < 1$$

$$= \frac{a_1}{1-r} - \frac{a_1r^n}{1-r}, |r| < 1$$

This is the case which provides us a quite different situation and we have some useful result.

Since $|r| < 1$, then r^n approaches zero as n increases with out bound, that is, we can make r^n or $\frac{ar^n}{1-r}$ as close as we wish to 0 by taking n sufficiently large. It follows that S_n approaches $\frac{a_1}{1-r}$ as n increases without a bound and we write

$$S_\infty = \frac{a_1}{1-r}$$

Mathematically, it can be shown as $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a_1}{1-r} - \frac{a_1 r^n}{1-r}\right)$

$$= \frac{a_1}{1-r} - \frac{a_1}{1-r}\lim_{n\to\infty} r^n$$

$$= \frac{a_1}{1-r} - \frac{a_1}{1-r}(0)$$

$$S_\infty = \frac{a_1}{1-r}$$

This gives us the following:

Theorem: If $|r|<1$, then the infinite geometric series: $a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} + \cdots$ has the sum: $\frac{a_1}{1-r}$

Example 33: Find the sum of the infinite geometric series:

$$\frac{3}{8} - \frac{3}{16} + \frac{3}{32} - \frac{3}{64} + \cdots$$

Solution: Before finding the sum, we must find the common ratio.

$$r = \frac{a_2}{a_1} = -\frac{3/16}{3/8} = -\frac{3}{16}\cdot\frac{8}{3} = -\frac{1}{2}$$

Because $r = -1/2$, the condition that $|r| < 1$ is met. Thus, the infinite geometric series has a sum

$S = \frac{a_1}{1-r}$ This is the formula for the sum of an infinite geometric series. Here $a_1 = 3/8$ and $r = -1/2$

$$= \frac{3/8}{1-(-1/2)} = \frac{3/8}{3/2} = \frac{3}{8}\cdot\frac{2}{3} = \frac{1}{4}$$

Thus, the sum of $\frac{3}{8} - \frac{3}{16} + \frac{3}{32} + \cdots\cdots$ is $\frac{1}{4}$. Put in an informal way, as we continue to add more and more terms, the sum is approximately $\frac{1}{4}$.

Example 34: The sum of an infinite number of terms in G. P. is 15, and the sum of their squares is 45. Find the series.

Solution: Let a denote the first term, r the common ratio; then the sum of the term is $\frac{a}{1-r}$; and the sum of their square is $\frac{a^2}{1-r^2}$.

Hence $\frac{a}{1-r} = 15$.......................(1)

$\frac{a^2}{1-r^2} = 45$.......................(2)

Dividing (2) by (1) $\frac{a}{1+r} = 3$.......................(3)

And from (1) and (3) $\frac{1+r}{1-r} = 5$; $\Rightarrow r = \frac{2}{3}$, and therefore $a = 5$.

Thus the series is $5, \frac{10}{3}, \frac{20}{9}, \ldots\ldots$

Example 35: Find the sum of the infinite geometric sequence: $1, \frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^n}, \ldots$

Solution: Here $a_1 = 1, r = \frac{1}{3}$ and $|r| = \frac{1}{3} < 1$

Thus the sum exists and is given by the formula:

$$S_\infty = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

4.7.4 Conversion of recurring Decimals into an equivalent fraction

Recurring decimals furnish a good illustration of infinite Geometrical Progressions.

Example 36: Convert $2.3\overline{4}$ to a common fraction.

Solution: Since $2.3\overline{4} = 2.3 + 0.0\overline{4}$

$$= \frac{23}{10} + 0.04444\ldots\ldots$$

$$= \frac{23}{10} + 0.04 + 0.004 + 0.0004 + \ldots\ldots$$

$$= \frac{23}{10} + \left(\frac{a_1}{1-r}\right), a_1 = 0.04, |r| = 0.1 < 1$$

$$= \frac{23}{10} + \left(\frac{0.04}{1-0.1}\right) = \frac{23}{10} + \frac{4}{90} = \frac{211}{90}$$

Example 37: Convert $0.\overline{21}$ to a common fraction

Solution: Since $0.\overline{21} = 0.212121... = 0.21 + 0.0021 + 0.000021 + \ldots$

$$= 0.21 + (0.01)(0.21) + (0.01)^2(0.21) + \ldots$$

Which is an infinite geometric series with $a_1 = 0.21$, $r = 0.01$,

and $|r| = 0.01 < 1$, so the sum exists and is given by $S = \dfrac{a_1}{1-r} = \dfrac{0.21}{1-0.01} = \dfrac{7}{33}$

Thus $0.\overline{21} = \dfrac{7}{33}$

4.7.5 Real life problems involving Geometric series

Example 38: Computing a lifetime salary

A union contract specifies that each worker will receive a 5 % pay increase each year for the next 30 years. One worker is paid Rs. 20,000 the first year. What is this person's total lifetime salary over a 30-years period?

Solution: The salary for the first year is 20,000. With a 5% raise, the second-year salary is computed as follows:

Salary for year 2 = 20,000 + 20,000(0.05) = 20,000(1 + 0.05) = 20,000(1.05).

Each year, the salary is 1.05 times what it was in the previous year. Thus, the salary for year 3 is 1.05 times 20,000(1.05), or $20,000(1.05)^2$. Thus

Yearly Salaries					
Year 1	Year 2	Year 3	Year 4	Year 5	...
20,000	20,000(1.05)	$20,000(1.05)^2$	$20,000(1.05)^3$	$20,000(1.05)^4$	...

The numbers in the bottom row form a geometric sequence with $a_1 = 20,000$ and r = 1+5 %=1+.05=1.05. To find the total salary over 30 years, we use the formula for the sum of the first n terms of a geometric sequence, with n = 30.

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{20,000[1-(1.05)^{30}]}{1-1.05} = \frac{20,000[1-(1.05)^{30}]}{-0.05} \approx 1,328,777$$

(Use a calculator)

The total salary over the 30-years period is approximately Rs. 1,328,777.

Example 39: The tip of a pendulum moves back and forth so that it sweeps out an arc 12 inches in length and on each succeeding pass, the length of the arc traveled, is $\frac{7}{8}$ of the length of the preceding pass. What is the total distance traveled by the tip of the pendulum?

Solution: Since the pendulum eventually comes to rest due to friction. We have the following geometric infinite sequence.

$$12, \left(\frac{7}{8}\right)(12), \left(\frac{7}{8}\right)^2(12), \left(\frac{7}{8}\right)^3(12), \ldots$$

and the total distance traveled $S = 12 + \left(\frac{7}{8}\right)(12) + \left(\frac{7}{8}\right)^2(12) + \left(\frac{7}{8}\right)^3(12) + \ldots$

which is an infinite geometric series with

$a_1 = 12, r = \frac{7}{8}$ and $|r| < 1$, so the sum exists.

Thus the total distance traveled $= \dfrac{a_1}{1-r}$

$$= \frac{12}{1-\frac{7}{8}}$$

= 96 inches

Did You Know

The symbol ∞ (infinity) is merely a notational device and does not represent a real number. Loosely, it is the concept of a value beyond any finite value.

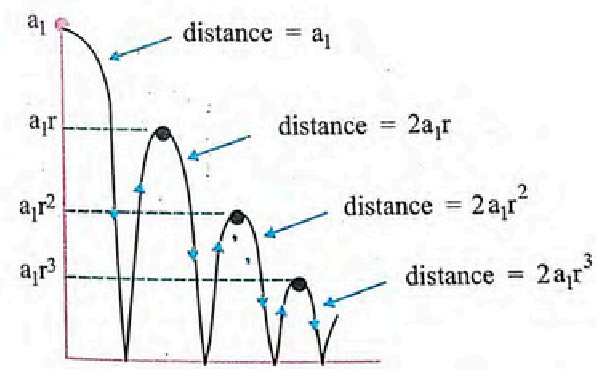
Example 40:

A ball is dropped from x feet above a flat surface. Each time the ball hits the ground after falling a distance h, it rebounds a distance rh where r < 1. Compute the total distance the ball travels.

Solution: The path and the distance the ball travels is shown on the sketch of figure. The total distance s is computed by the geometric series

$$s = a_1 + 2a_1r + 2a_1r^2 + 2a_1r^3 + \ldots \qquad \text{(I)}$$

$$\text{The common ratio is } \frac{2a_1r}{1-r} \qquad \text{(II)}$$



Adding the first term of (I) with (II) we form the total distance as

$$s = a_1 + \frac{2a_1 r}{1-r} = a_1\left(\frac{1+r}{1-r}\right)$$

For example, if $a_1 = 6$ ft and $r = 2/3$, the total distance the ball travels is

$$s = 6\times\frac{1+2/3}{1+2/3} = 30 \text{ ft}$$

EXERCISE 4.5

1. Compute the sum:
   (i) $3+6+12+\ldots+3.2^9$ (ii) $8+4+2+1+\ldots+\frac{1}{16}$
   (iii) $2^4+2^5+2^6+\ldots+2^{10}$ (iv) $\frac{8}{5},-1,\frac{5}{8},\ldots$
   (v) $2,\frac{2}{\sqrt{2}},1,\frac{1}{\sqrt{2}},\frac{1}{2},\ldots$ (vi) $-\frac{1}{3},\frac{1}{2},-\frac{3}{4},\ldots\ldots\ldots$ to 7 terms.
2. Some of the components a_1, a_n, n, r and S_n of a geometric sequence are given. Find the ones that are missing.
   (i) $a_1 = 1, r=-2, a_n = 64$ (ii) $r=\frac{1}{2}, a_9 = 1$
   (iii) $r=-2, S_n=-63, a_n=-96$
3. Find the first five terms and the sum of an infinite geometric sequence having $a_2 = 2$ and $a_3 = 1$
4. Find the value of: (i) $0.\overline{8}$ (ii) $1.\overline{63}$ (iii) $2.1\overline{5}$ (iv) $0.\overline{123}$
5. Find r such that: $S_{10} = 244S_5$.
6. Prove that: $S_n(S_{3n}-S_{2n}) = (S_n - S_{2n})^2$
7. Find the sum S_n of the first n terms of the sequence $\left\{\left(\frac{1}{2}\right)^n\right\}$.
8. The sum of three numbers in G. P. is 38, and their product is 1728; find them.
9. The sum of first 6 terms of a geometric series is 9 times the sum of its first three terms. Find the common ratio.
10. How many terms of the series: $1+\sqrt{3}+3+\ldots$ be added to get the sum $40+13\sqrt{3}$.

11. If p^{th}, q^{th}, r^{th} terms of a G. P. be a,b,c respectively, prove that $a^{q-r}b^{r-p}c^{p-q} = 1$.
12. Find an infinite geometric series whose sum is 6 and such that each term is four times the sum of all the terms that follow it.
13. If $y = \frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\ldots$, where $0<x<3$, then show that $x = \frac{3y}{1+y}$
14. A ball rebounds to half the height from which it is dropped. If it is dropped from 10 ft, how far does it travel from the moment it is dropped until the moment of its eighth bounce?
15. A man wishes to save money by setting aside Rs.1 the first day, Rs.2 the second day, Rs.4 the third day and so on, doubling the amount each day. If this continued, how much must be set aside on the 15^{th} day? What is the total amount saved at the end of 30 days?
16. The number of bacteria in a culture increased geometrically from 64000 to 729000 in 6 days. Find the daily rate of increase if the rate is assumed to be constant.

4.8 Harmonic Sequence

4.8.1 A harmonic sequence is a sequence whose reciprocals form an arithmetic sequence.

The sequence: $\frac{1}{2},\frac{1}{4},\frac{1}{6},\frac{1}{8},\frac{1}{10}$ (1)

is not an arithmetic sequence. However the reciprocals of these numbers, namely: 2, 4, 6, 8,10 do form an arithmetic sequence. Thus the sequence (1) is an example of a harmonic sequence. A harmonic sequence is also called a harmonic progression (H.P).

Example 41: Three numbers a,b,c are in H.P. when $\frac{a}{c} = \frac{a-b}{b-c}$

Solution: Given $\frac{a}{c} = \frac{a-b}{b-c}$ then $a(b-c) = c(a-b)$

or $ab - ac = ca - bc$ Dividing by (abc), we obtain:

$$\frac{1}{c}-\frac{1}{b} = \frac{1}{b}-\frac{1}{a}$$

Thus $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in A.P. and hence a, b, c are in H.P.

4.8.2 Finding nth Term of a Harmonic Sequence

The typical form of a harmonic sequence is

$$\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \ldots, \frac{1}{a_1+(n-1)d}, \ldots$$

The general term or the nth term of this H.P. is

$$\frac{1}{a_1+(n-1)d}$$

whose reciprocal $a_1+(n-1)d$ is the nth term of the A.P.

Remember

Many properties of harmonic progression can be obtained from the corresponding arithmetic progression. However, there is no elementary formula for the sum of a harmonic sequence.

Example 42: Find the twelveth term of the harmonic progression: 6, 4, 3,…

Solution: The 12th term of the corresponding A.P.

$$\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \ldots \quad \text{twelvth}$$

with $a_1 = \frac{1}{6}$, $d = \frac{1}{12}$, $n = 12$

is $a_{12} = \frac{1}{6} + (12-1)\left(\frac{1}{12}\right)$ $\qquad \therefore \; a_n = a_1 + (n-1)d$

$$= \frac{13}{12}$$

Thus the 12th term of the given H.P is $\frac{12}{13}$.

4.9 Harmonic Means (H.Ms)

4.9.1 (i) A number H is said to be the Harmonic Mean (H.M) between two number a and b $(a \neq 0, b \neq 0)$ if a, H, b are in H.P.

Then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P. and $\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$, i.e. $\frac{1}{H}$ is the A.M between $\frac{1}{a}$ and $\frac{1}{b}$.

$$\frac{1}{H} = \frac{a+b}{2ab}$$

$$\therefore \quad H = \frac{2ab}{a+b} \quad \text{is the H.M. between } a \text{ and b}$$

(ii) The numbers $H_1, H_2, \ldots, H_n$ are said to be the n Harmonic Means (H.Ms) between two number a and b $(a \neq 0, b \neq 0)$ if

$$a, H_1, H_2, H_3, \ldots, H_n, b \quad \text{are in H.P.}$$

Then obviously: $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots \frac{1}{H_n}, \frac{1}{b}$ are in A.P with n+2 terms.

$$\therefore \quad \frac{1}{a} + (n+2-1)d = \frac{1}{b}, \text{ utilizing } a_n = a_1 + (n-1)\,d$$

$$\Rightarrow \quad d = \frac{a-b}{ab(n+1)}$$

Thus $\frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{ab(n+1)}$ or $H_1 = \frac{ab(n+1)}{nb+a}$

$\frac{1}{H_2} = \frac{1}{a} + 2\frac{a-b}{ab(n+1)}$ or $H_2 = \frac{ab(n+1)}{(n-1)b+2a}$

$\frac{1}{H_3} = \frac{1}{a} + 3\frac{a-b}{ab(n+1)}$ or $H_3 = \frac{ab(n+1)}{(n-2)b+3a}$

⋮

$\frac{1}{H_n} = \frac{1}{a} + n\frac{a-b}{ab(n+1)}$ or $H_n = \frac{ab(n+1)}{b+na}$

by using $\frac{1}{H_i} = a_1 + id$, $i = 1, 2, 3, \ldots, n$. Hence $H_1, H_2, H_3, \ldots, H_n$, are the n H.Ms between a and b.

Example 43: Find the harmonic mean of 24 and 16

Solution: $H = \frac{2ab}{a+b}$, where $a = 24$, $b = 16$

Then $H = \frac{2(24)(16)}{24+16} = \frac{2 \times 24 \times 16}{40} = \frac{96}{5}$

Example 44: Insert four harmonic means between $-\frac{1}{2}$ and $\frac{1}{13}$.

Solution: Let H_1, H_2, H_3 and H_4 be the required H.Ms, then

$$-\frac{1}{2}, H_1, H_2, H_3, H_4, \frac{1}{13} \quad \text{are in H.P}$$

$$\therefore \quad -2, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, 13 \quad \text{are in A.P}$$

with $a_1 = -2$

$a_6 = 13$

$a_1 + 5d = 13$

$-2 + 5d = 13$

$d = 3$

Now $\frac{1}{H_1} = -2 + 3 = 1 \Rightarrow H_1 = 1$

$\frac{1}{H_2} = 1 + 3 = 4 \Rightarrow H_2 = \frac{1}{4}$

$\frac{1}{H_3} = 4 + 3 = 7 \Rightarrow H_3 = \frac{1}{7}$

$\frac{1}{H_4} = 7 + 3 = 10 \Rightarrow H_4 = \frac{1}{10}$

Hence $1, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}$ are the required 4 H.Ms. between $-\frac{1}{2}$ and $\frac{1}{13}$

Did You Know

Harmonic sequences are called harmonic because of their use in the study of musical chords and their relationship, that is, harmony.

Example 45: Find a relation among Arithmetic, Geometric And Harmonic Means.

Solution: Let $a \neq 0, b \neq 0$ be any two positive numbers,

then $A = \frac{a+b}{2}$

$H = \frac{2ab}{a+b}$ and $G = \sqrt{ab}$

(i) $A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = \left(\sqrt{ab}\right)^2 = G^2 \Rightarrow A, G, H$ are in G.P

(ii) $A > G$ if $\frac{a+b}{2} > \sqrt{ab}$

$a + b > 2\sqrt{ab}$

$a + b - 2\sqrt{ab} > 0,$

$(\sqrt{a} - \sqrt{b})^2 > 0$, which is always true

$\therefore\ A > G$ (1)

$G > H$ if $\sqrt{ab} > \frac{2ab}{a+b}$

$a + b > 2\sqrt{ab}$

$(\sqrt{a} - \sqrt{b})^2 > 0$, which is always true

$\therefore\ G > H$ (2)

(1) and (2) $\Rightarrow A > G > H$

EXERCISE 4.6

1. Find the indicated term of each of the following harmonic progressions:
   (i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$; 9^{th} term (ii) $6, 2, \frac{6}{5}, \ldots$; 20^{th} term
   (iii) $5\frac{2}{3}, 3\frac{2}{5}, 2\frac{3}{7}, \ldots$; 8^{th} term
2. Find five more terms of the H.P. $\frac{1}{3}, 1, -1, \ldots$
3. The second term of an H.P is $\frac{1}{2}$ and the fifth term is $-\frac{1}{4}$. Find the 12^{th} term.
4. Find the arithmetic, harmonic and geometric means of each of the following. Also verify that $A \times H = G^2$.
   (i) 3.14 and 2.71 (ii) −6 and −216 (iii) $x + y$ and $x - y$
5. For what value of n will $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ be the harmonic mean between a and b?
6. Insert two harmonic means between 12 and 48.
7. Insert four harmonic means between $\frac{7}{3}$ and $\frac{7}{11}$.
8. Prove that the square of the geometric mean of two numbers equals the product of the arithmetic mean and the harmonic mean of the two numbers.
9. The arithmetic mean of two numbers is 8, and the harmonic mean is 6. What are the numbers?
10. The harmonic mean of two numbers is $4\frac{4}{5}$ and the geometric mean is 6. What are the numbers?

Review Exercise 4

1. Choose the correct option.
   (i) The sum to 200 terms of the series 1+ 4+ 6+ 5+ 11+ 6+ ....is
   (a) 20,300 (b) 29,800
   (c) 30,200 (d) None of these
   (ii) If the sum of the series 2+ 5+ 8+ 11......is 60100, then the number of the terms is
   (a) 100 (b) 200 (c) 150 (d) 250
   (iii) If a, b, c are in G.P., then
   (a) a^2, b^2, c^2 are in G.P. (b) $a^2(b+c), c^2(a+b), b^2(a+c)$ are in G.P.
   (c) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in G.P. (d) None of these
   (iv) If the nth term of an A.P is 4n+ 1, then the common difference is :
   (a) 3 (b) 4 (c) 5 (d) 6
   (v) Which of the following is not a G.P.?
   (a) 2,4,8,16,........ (b) 5, 25, 125, 625, ...
   (c) 1.5 , 3.0, 6.0, 12.0 ..... (d) 8, 16, 24, 32, ...
   (vi) There are four arithmetic means between 2 and −18. The means are
   (a) −4, −7, −10, −13 (b) 1, −4, −7, −10
   (c) −2, −5, −9, −13 (d) −2, −6, −10, −14
   (vii) If A, G and H are A.M, G.M and H.M. of any two positive numbers, then find the relation between A, G and H.
   (a) $A^2 = GH$ (b) $G^2 = AH$ (c) $H^2 = AG$ (d) $G^2 = A^2H$
   (viii) Find the number of terms to be added in the series 27, 9, 3, ... so that the sum is 1093/27
   (a) 6 (b) 7 (c) 8 (d) 9
   (ix) Find the value of p (p>0) if $\frac{15}{4}+p, \frac{5}{2}+2p$ and 2+p are the three consecutive terms of a geometric progression
   (a) $\frac{3}{4}$ (b) $\frac{1}{4}$ (c) $\frac{5}{3}$ (d) $\frac{1}{2}$
   (x) The 10^{th} term of harmonic progression 1/5, 4/19, 2/9, 4/17,...is
   (a) 11/4 (b) 13/4 (c) 4/13 (d) 4/11
   (xi) Find the sum of 3 geometric means between 1/3 and 1/48 (r>0).
   (a) 1/4 (b) 5/24 (c) 7/24 (d) 1/3

2. If the first term and common difference in an A.P are 8 and −1 respectively, then find:
   (i) General term (ii) The Progression (iii) The 10th term and
   (iv) The expression for sum to n terms and hence sum to 10 terms.
3. If the sum of the n terms of the series 54, 51, 48, ... is 513, then find the value of n.
4. If the sum of n terms of an A.P. is $2n+3n^2$, generate the progression and find the nth term
5. Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3.
6. Find the sum of the series 1, 2/5, 4/25, 8/125, ..., ∞
7. If a, b, c are in A.P. and x, y, z are in G.P, show that $x^b y^c z^a = x^c y^a z^b$
8. Find the arithmetic mean between $10\frac{1}{2}$ and $25\frac{1}{2}$.
9. Find three numbers of a G.P. whose sum is 26 and product is 216.
10. How many odd integers beginning with 15 must be taken for their sum to be equal to 975?
11. A gas-filled balloon has risen 100 feet. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will it rise in 5 minutes?