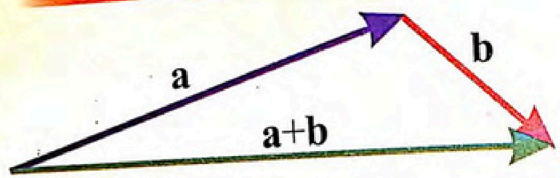
UNIT 3 Vectors



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Define a scalar and a vector.
- Give geometrical representation of a vector.
- Give the following fundamental definitions using geometrical representation.
  - magnitude of a vector,
  - equal vectors,
  - negative of a vector,
  - unit vector,
  - zero/null vector,
  - position vector,
  - parallel vectors,
  - addition and subtraction of vectors,
  - triangle, parallelogram and polygon laws of addition,
  - scalar multiplication.
- Represent a vector in a Cartesian plane by defining fundamental unit vectors i and j.
- Recognize all above definitions using analytical representation.
- Find a unit vector in the direction of another given vector.
- Find the position vector of a point which divides the line segment joining two points in a given ratio.
- Use vectors to prove simple theorems of descriptive geometry.
- Recognize rectangular coordinate system in space.
- Define unit vectors i, j and k.
- Recognize components of a vector.
- Give analytic representation of a vector.
- Find magnitude of a vector.
- Repeat all fundamental definitions for vectors in space which, in the plane, have already been discussed.
- State and prove
  - commutative law for vector addition.
  - associative law for vector addition.

STUDENTS LEARNING OUTCOMES

- Prove that:
  - 0 as the identity for vector addition.
  - $-A$ as the inverse for A.
- State and prove:
  - commutative law for scalar multiplication,
  - associative law for scalar multiplication,
  - distributive laws for scalar multiplication.
- Define dot or scalar product of two vectors and give its geometrical interpretation.
- Prove that.
  - $i \cdot i = j \cdot j = k \cdot k = 1$,
  - $i \cdot j = j \cdot k = k \cdot i = 0$
- Express dot product in terms of components.
- Find the condition for orthogonality of two vectors.
- Prove the commutative and distributive laws for dot product.
- Explain direction cosines and direction ratios of a vector.
- Prove that the sum of the squares of direction cosines is unity.
- Use dot product to find the angle between two vectors.
- Find the projection of a vector along another vector.
- Find the work done by a constant force in moving an object along a given vector.
- Define cross or vector product of two vectors and give its geometrical interpretation.
- Prove that:
  - $i \times i = j \times j = k \times k = 0$,
  - $i \times j = -j \times i = k$,
  - $j \times k = -k \times j = i$,
  - $k \times i = -i \times k = j$.
- Express cross product in terms of components.
- Prove that the magnitude of $A \times B$ represents the area of a parallelogram with adjacent sides A and B.
- Find the condition for parallelism of two non-zero vectors.
- Prove that $A \times B = -B \times A$.
- Prove the distributive laws for cross product.
- Use cross product to find the angle between two vectors.
- Find the vector moment of a given force about a given point.
- Define scalar triple product of vectors.
- Express scalar triple product of vectors in terms of components (determinantal form).
- Prove that:
  - $i \cdot j \times k = j \cdot k \times i = k \cdot i \times j = 1$,
  - $i \cdot k \times j = j \cdot i \times k = k \cdot j \times i = -1$.
- Prove that dot and cross are inter-changeable in scalar triple product.
- Find the volume of
  - a parallelepiped,
  - a tetrahedron, determined by three given vectors.
- Define coplanar vectors and find the condition for coplanarity of three vectors.

3.1 Introduction

Physical quantities such as mass, temperature and work are measured by numbers referred to some chosen unit. These numbers are called **scalars**. Scalars being just numbers, can therefore be added, subtracted, multiplied and divided by using the fundamental laws of elementary algebra.

Other quantities exist such as displacement, velocity, acceleration and force, which require for their complete specification a direction as well as a scalar. These quantities are called **vectors** and may be represented by a straight line with an arrow. Vectors cannot be added, subtracted, multiplied or divided by ordinary mathematical rules but we use methods of vector addition (triangle rule or parallelogram rule) or other analytical methods for their multiplication, for this purpose.

Vectors have many applications in Geometry, Physics and Engineering. We begin with geometrical interpretation of a vector. However, in the sequel we shall apply vector methods to prove some fundamental results of descriptive geometry.

3.1.1 Scalar and Vector

Scalar Quantity: A quantity which has only magnitude and no direction is called a scalar quantity or simply a scalar.
Examples of scalar are mass, temperature, volume, work etc.

Vector Quantity: A quantity which has magnitude as well as direction is called a vector quantity or simply a vector.
Examples of vector are displacement, velocity, acceleration, force etc.

3.1.2 Geometrical representation of a vector

A vector is geometrically represented by an arrow or directed line segment say $\overrightarrow{OP}$, where the arrow indicates the direction of the vector and the length of the arrow specifies, on appropriate scale, the magnitude of the vector. The tail end O of the arrow is called its **origin** or **initial point** and the head (tip) P is called the **terminal point** or **terminus** (Figure 3.1)

In printed work, it is usual to denote all vectors by bold faced letters ***a***, ***b***, ***v*** etc. In hand written work, the vectors are denoted by $\vec{a}$, $\vec{b}$, $\vec{v}$ etc. The other notation used for vector is $\underline{a}, \underline{b}, \underline{v}$ etc.

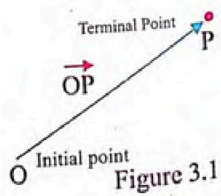


Figure 3.1

3.1.3 Fundamentals of a vector

(i) Magnitude of a vector

The magnitude or modulus of a vector $\overrightarrow{OA}$ or ***a*** is the length of the line segment representing the vector to the scale used. The magnitude of the vector $\overrightarrow{OA}$ is denoted by $|\overrightarrow{OA}|$, $|a|$, $|\underline{a}|$ or a.

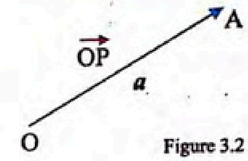


Figure 3.2

(ii) Equal vector

Two vectors ***a*** and ***b*** are said to be equal if they have the same magnitude and direction regardless of the position of their initial point. Symbolically, we write $a = b$ (Figure 3.3)

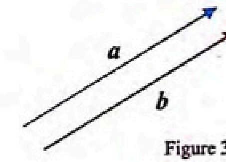


Figure 3.3

(iii) Negative of a vector

A vector having the same magnitude as another vector ***a*** but opposite in direction is called negative of a vector and is denoted by $-a$ as shown in (Figure 3.4)

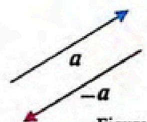


Figure 3.4

(iv) Zero vector or null vector

A vector which has zero magnitude and arbitrary direction is called the zero vector or null vector. Zero vector is denoted by O, $\vec{O}$ or $\underline{O}$.

(v) Unit vector

A vector whose magnitude is one is called unit vector. It is used to represent the direction of a vector. A unit vector is denoted by a letter with a hat over it, such as $\hat{a}$, $\hat{b}$, $\hat{v}$ etc. Any vector ***a*** can be written in terms of unit vector as $a = |a|\hat{a}$

Hence unit vector in the direction of ***a*** is obtained as $\hat{a} = \dfrac{a}{|a|}$

i.e. unit vector in a direction $= \dfrac{\text{Vector in that direction}}{\text{Modulus of the vector}}$

(vi) Parallel vectors

Two vectors ***a*** and ***b*** are parallel if and only if $a = \alpha b$, where α is scalar. See for example (Figure 3.5)

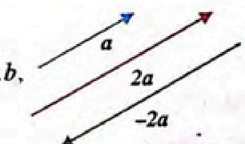


Figure 3.5

For Your Information

The magnitude $|a|$ of a vector ***a*** is a positive scalar quantity and therefore can be added, subtracted, multiplied and divided like all other scalar quantities.

(vii) Position Vector

A vector which joins a given point P in the plane or space with the origin is called position vector of the point P and is denoted by $\overrightarrow{OP}$ (Figure 3.6).

The magnitude of the position vector is equal to the distance between the given point and the origin and whose direction is the direction of the point from the origin.

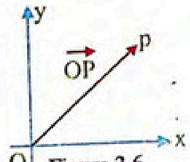


Figure 3.6

Example 1: Using graph paper, draw the vectors.

(a) $2a$ (b) $-a$ (c) $\frac{3}{4}a$

where a is given in (Figure 3.7)

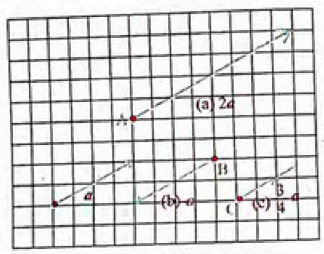


Figure 3.7

Solution: (a) The head of the vector a from its end point is 4 squares to the right and 2 squares up. Hence $2a$ is 8 squares to the right and 4 squares up.

(b) $-a$ is the negative of a, so its direction is opposite to a. Hence $-a$ is 4 squares to the left and 2 squares down from its end point.

(c) $\frac{3}{4}a$ is 3 squares to the right and 1 and a half squares up as shown in (Figure 3.7).

Example 2: In Figure 3.8, vectors a, p, q, r, s are shown. State each of the vectors p, q, r and s in the form ka.

Solution: The direction of a is 2 squares to the right and 4 squares up.

Hence $p = -a$, $q = \frac{1}{2}a$

$r = 2a$, $s = \frac{3}{2}a$

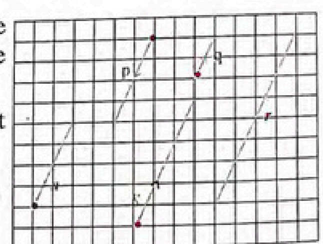


Figure 3.8

Example 3: What type of quadrilateral is ABCD, if (i) $\overrightarrow{AB} = \overrightarrow{CD}$ ii. $\overrightarrow{AB} = 3\overrightarrow{CD}$

Solution: (i) $\overrightarrow{AB} = \overrightarrow{CD}$ means that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal in length i.e. $|\overrightarrow{AB}| = |\overrightarrow{CD}|$ and $\overrightarrow{AB} \parallel \overrightarrow{CD}$. Hence ABCD is a parallelogram as shown in (Figure 3.9.)

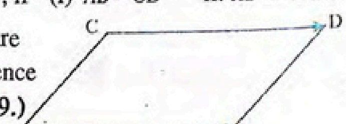


Figure 3.9

(ii) $\overrightarrow{AB} = 3\overrightarrow{CD}$ means $|\overrightarrow{AB}| = 3|\overrightarrow{CD}|$ and $\overrightarrow{AB} \parallel \overrightarrow{CD}$.

Hence ABCD is a trapezium as shown in (Figure 3.10.)

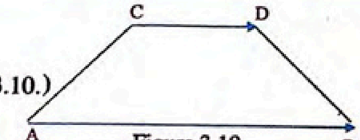


Figure 3.10

(viii) Addition and subtraction of vectors

(a) Addition of vectors

Any two vectors can be added by the following two laws.

- **Head – to – tail or Triangle law of addition**

To add two vectors a and b that is, to combine them into one vector, we draw them in such a way that the head of the first vector coincides with the tail of the second vector. The **sum** or **resultant vector** $a+b$ is obtained by joining the tail of the first vector with the head of the second vector as shown in (Figure 3.11).

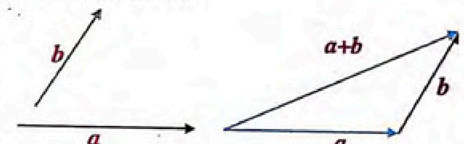


Figure 3.11

We call this way of adding the vectors as **Head–to–Tail** or **Triangle law of addition.**

- **Parallelogram Law of Addition**

If the two adjacent sides AB and AC of a parallelogram represent the vectors a and b as shown in (Figure 3.12), then the diagonal AD represents the vector sum or resultant $a + b$ of vectors a and b. Thus $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC} = a + b$

We call this way of adding the vectors as the **parallelogram law of addition.**

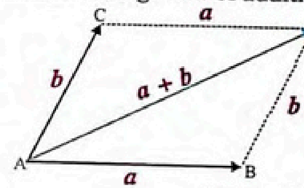


Figure 3.12

- **Polygon Law of Addition of Vectors**

The method of vector addition of two vectors can be extended to more than two vectors. Let **a**, **b**, **c**, **d** be four given vectors. Let O be any point and let us draw the vectors $\overrightarrow{OA}$ = a. From the terminal point A of the vector **a**, draw $\overrightarrow{AB}$ to represent vector **b**. From the terminal point B, draw $\overrightarrow{BC}$ to represent vector **c**. From the terminal point C, draw $\overrightarrow{CD}$ to represent vector **d**. Join OD. Then, from (Figure 3.13),

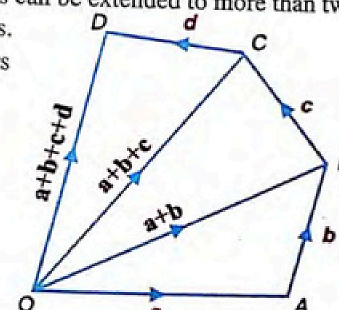


Figure 3.13

we have $a+b+c+d = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$

$= \overrightarrow{OB} + \overrightarrow{BC} + \overrightarrow{CD}$

$= \overrightarrow{OC} + \overrightarrow{CD}$ $\quad [\because \overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}]$

$= \overrightarrow{OD}$ $\quad [\because \overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OC}]$

Thus the vector $\overrightarrow{OD}$ joining the initial point of the first vector **a** and the terminal point of the last vector **d** represents sum of the given vectors. This method of addition is called the polygon law of addition.

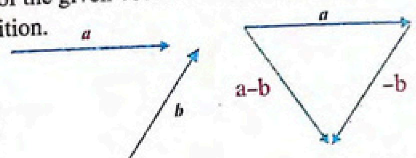


Figure 3.14

(b) Subtraction of two vectors

The difference of two vectors a and b, denoted by $a-b$, is the vector c obtained by adding vector a and the negative of b, that is $\quad c = a - b = a + (-b)$

Thus, the difference $a - b$ of vectors a and b is equal to a vector c which when added to b yields the vector a. The difference $a - b$ is shown in (Figure 3.14.)

(ix) Scalar multiplication

In dealing with vectors, we refer to real numbers as scalars. If k is a scalar and a is a vector, then the multiplication of a by k, denoted as ka, is a vector whose magnitude is k times that of a. Thus, if

(i) $k = 0$, then ka is the zero vector

(ii) $k > 0$, then a and ka are in the same direction

(iii) $k < 0$, then a and ka are in the opposite direction

For illustration, see **(Figure 3.15).**

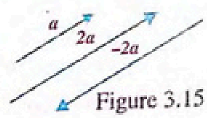


Figure 3.15

Example 4: For the vectors a and b given in **(Figure 3.16** (a)**)**, draw the vector

(i) $2a+b$ (ii) $a-b$ (iii) $a-2b$

Solution: The vectors are shown in **(Figure 3.16** (b)**.)**

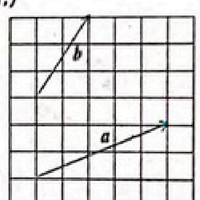


Figure 3.16 (a)

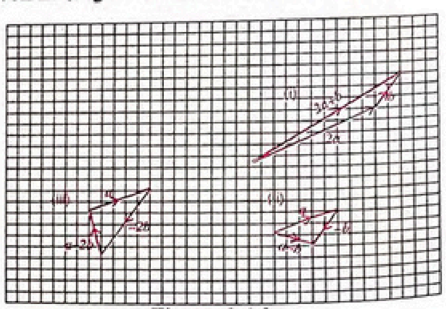


Figure 3.16 (b)

Draw vector $2a$ and from the head of $2a$ draw b. Then use head–to–tail rule to obtain $2a + b$.

(i) Draw a followed by $-b$, use triangle law of addition of vectors to obtain $a - b$.

(ii) Draw a followed by $-2b$, use triangle law of addition of vectors to obtain $a-2b$.

Example 5: In ΔABC, $\overrightarrow{AB} = a$, $\overrightarrow{AC} = b$ and D is the midpoint of AB (Figure 3.17). State in terms of a, b. (i) $\overrightarrow{AD}$ (ii) $\overrightarrow{DC}$ (iii) $\overrightarrow{CD}$

Solution:

i) $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}a$

(ii) $\overrightarrow{DC} = \overrightarrow{AC} - \overrightarrow{AD} = b - \frac{1}{2}a$

(iii) $\overrightarrow{CD} = -\overrightarrow{DC} = \frac{1}{2}a - b$

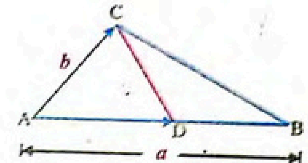


Figure 3.17

Theorem. For any vector a,

(i) The zero vector o has the property that $o + a = a + o = a$

(ii) The negative vector $-a$ of a has the property $a + (-a) = a - a = 0$

Proof. **(i) easy.**

If $\overrightarrow{OA} = a$, we have, according to the definition of the multiplication of vectors by scalars, $\overrightarrow{AO} = (-1)a$ Thus, $a + (-1)a = \overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO} = 0$

(ii) On account of this property, the vector (-1) **a** is called the negative of the vector **a**, and we write $-a = (-1)a$

So that the relation $\quad a + (-1)a = 0,$

may also be re-written as $a + (-a) = 0$

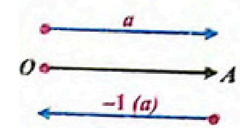
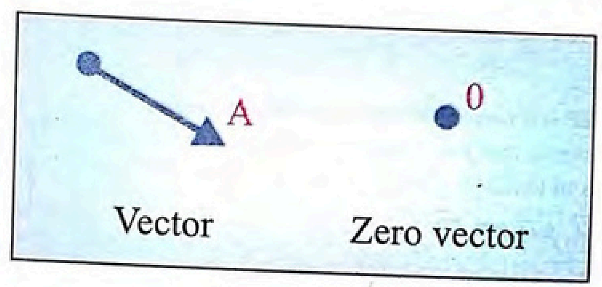


Figure 3.18

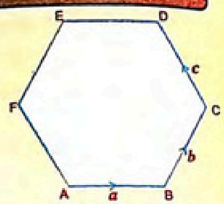


EXERCISE 3.1

1. ABCDEF is a regular hexagon $\overrightarrow{AB} = a$, $\overrightarrow{BC} = b$ and $\overrightarrow{CD} = c$, state the following vectors as scalar multiple of a, b or c.

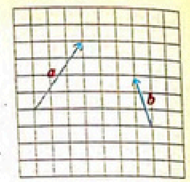
   (i) $\overrightarrow{DE}$ (ii) $\overrightarrow{EF}$ (iii) $\overrightarrow{FA}$ (iv) $\overrightarrow{AD}$ (v) $\overrightarrow{BE}$

   Hint: In a regular hexagon main diagonal $\overrightarrow{AD}$ is double the side $\overline{BC}$ and parallel to it.



2. Given the vectors a and b as in Figure, draw the vectors:

   (i) $a + 2b$ (ii) $2a - b$ (iii) $3a - 2b$



3. In ΔOPQ, $\overrightarrow{OP} = p$, $\overrightarrow{OQ} = q$, R is the midpoint of $\overline{OP}$ and S lies on $\overline{OQ}$ such that $|\overline{OS}| = 3|\overline{SQ}|$. State in terms of p and q.

   (i) $\overrightarrow{OR}$ (ii) $\overrightarrow{PQ}$ (iii) $\overrightarrow{OS}$ (iv) $\overrightarrow{RS}$

4. OACB is a parallelogram with $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$, $\overline{AC}$ is extended to D where $|AC| = 2|CD|$. Find in terms of a and b

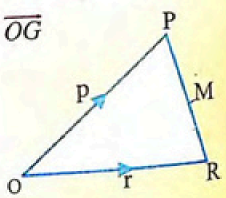
   (i) $\overrightarrow{AD}$ (ii) $\overrightarrow{OD}$ (iii) $\overrightarrow{BD}$

5. OAB is a triangle with $\overrightarrow{OA} = a$, $\overrightarrow{OB} = b$. M is the midpoint of OA and G lies on $\overline{MB}$ such that $|\overline{MG}| = \frac{1}{2}|\overline{GB}|$. State in terms of a and b

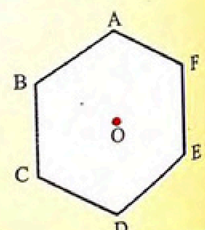
   (i) $\overrightarrow{OM}$ (ii) $\overrightarrow{MB}$ (iii) $\overrightarrow{MG}$ (iv) $\overrightarrow{OG}$

6. In Δ OPR, the mid-point of PR is M. If $\overrightarrow{OP} = \text{p}$ and $\overrightarrow{OR} = \text{r}$, find in terms of p and r.

   (i) $\overrightarrow{PR}$ (ii) $\overrightarrow{PM}$ (iii) $\overrightarrow{OM}$



7. ABCDEF is a regular hexagon and O is its centre. The vectors x and y are such that $\overrightarrow{AB} = x$ and $\overrightarrow{BC} = y$. Express in terms of x and y the vectors $\overrightarrow{AC}$, $\overrightarrow{AO}$, $\overrightarrow{CD}$ and $\overrightarrow{BF}$.



3.1.4 Representation of a vector in a cartesian plane

We recall from our previous class that a **rectangular coordinate system** consists of two lines xx′ and yy′ drawn at right angle to each other as shown in(Figure 3.20),are known as **coordinate axes.** Their point of intersection is called **origin** and is denoted by O. The rectangular coordinate system is also called as **Cartesian coordinate system.**

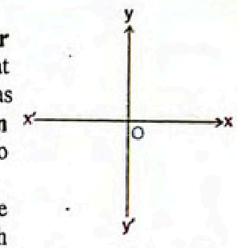


Figure 3.20

The horizontal line is called **x–axis** with positive direction to the right and the vertical line is called **y–axis** with positive direction upward. If P is a point in plane, it has two coordinates, one along x–axis and the other along y–axis. If the distances along x–axis and y–axis are determined by a and b respectively, then the point P is assigned an ordered pair of real numbers as (a,b) or P (a,b) as shown in(Figure 3.21).We call a and b the **x–coordinate** and **y–coordinate** of P.

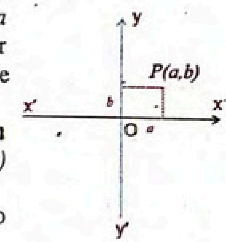


Figure 3.21

The set $\mathbb{R}^2 = \{(a,b): a,b \in \mathbb{R}\}$ is called the **Cartesian plane.** Thus an element $(a,b) \in \mathbb{R}^2$ represents a point $P(a,b)$ which is uniquely determined by its coordinates a and b.

In this section, we use rectangular coordinate system to represent a vector in the plane.

Let i denote the unit vector whose direction is along the positive x–axis and let j denote the unit vector whose direction is along the positive y–axis. Then every vector $\overrightarrow{OP}$ in the plane can be written uniquely in terms of the vectors i and j as $\overrightarrow{OP} = r = xi + yj$ where x and y are scalar. See (Figure 3.22).

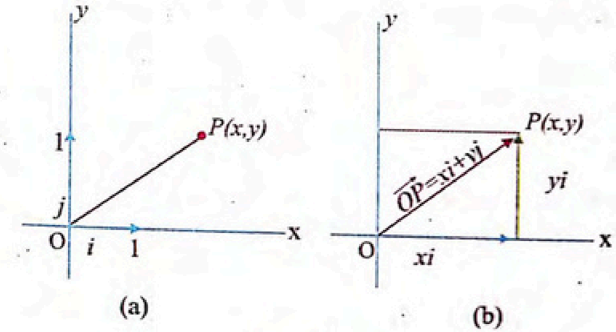


Figure 3.22

The vector $\overrightarrow{OP}$ is called the **position vector** of the point P. Thus, the position vector of any point P(x,y) is the vector $\overrightarrow{OP}$ whose initial point is the origin 'O' and whose terminal point is P.

Component of a Vector

In the representation of the position vector to any point P(x,y) in the plane as $\overrightarrow{OP} = r = xi + yj$, the scalars x and y are called the **components** of the vector r. The component in i–direction is x, while the component in j–direction is y.

For example, if P(5,–4) be a point in the plane. Then the vector r represented by the position vector to the point P(5,–4) is

$$r = xi + yj = 5i + (-4)j.$$

Thus, the i–direction component is 5 and the j–direction component is –4.

Theorem: If a and b are position vectors of points A and B respectively,

then $\overrightarrow{AB} = b - a$

Proof: If a and b are position vectors of the points A and B respectively, then

$a = \overrightarrow{OA}$ and $b = \overrightarrow{OB}$ (Figure 3.23)

Using triangle law of vector additions, we have

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\Rightarrow \quad \overrightarrow{AB} = b - a$$

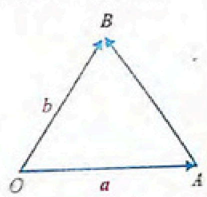


Figure 3.23

Example 6: Find the vector $\overrightarrow{AB}$ from the point A (–4,6) to the point B (6,8).

Solution: The position vectors of A and B are $\overrightarrow{OA} = -4i + 6j$ and $\overrightarrow{OB} = 6i + 8j$.

Therefore by the above theorem

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (6i + 8j) - (-4i + 6j) = 6i + 8j + 4i - 6j = 10i + 2j$$

Vectors with initial point not at the origin

We defined the component of a vector to be the coordinates of its terminal point when its initial point is at the origin. Now we will find the components of a vector whose initial point is not at the origin.

Suppose $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are two points in the plane. Suppose $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ be the position vectors of P_1 and P_2 as shown in (Figure 3.24).

Then $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

$$= (x_2 i + y_2 j) - (x_1 i + y_1 j)$$
$$= (x_2 - x_1)i + (y_2 - y_1)j$$

Thus the i–component is $x_2 - x_1$ and the j–component is $y_2 - y_1$

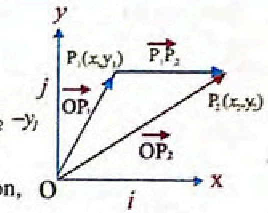


Figure 3.24

3.1.5 Algebra of Vectors

In this section we define addition, subtraction, scalar multiplication, and so on, for vectors in plane.

Equal Vectors

Two vectors $u = x_1 i + y_1 j$ and $v = x_2 i + y_2 j$ are said to be equal if and only if they have the same components that is

$u = v$ if and only if $x_1 = x_2$ and $y_1 = y_2$

Example 7: If $u = 2i + yj$ and $v = xi - j$, then find x and y.

Solution: $u = v$ or $2i + yj = xi - j$

By comparison we have $x = 2$ and $y = -1$

Addition of Vectors

If $u = x_1 i + y_1 j$ and $v = x_2 i + y_2 j$ are two vectors, then their addition, denoted by $u + v$, is defined as $u + v = (x_1 + x_2)\, i + (y_1 + y_2)j$

Thus, to add two vectors, we add their corresponding components.

Scalar Multiplication

The multiplication of the vector $u = xi + yj$ by a scalar k, that is ku is defined as

$$ku = k(xi + yj) = (kx)i + (ky)j$$

Negative of a Vector

If $u = xi + yj$ is a vector, then negative of u, denoted by $-u$, is defined as

$$-u = -(xi + yj) = -xi - yj$$

Thus, if we take $k = -1$ in the definition of scalar multiplication, we obtain $-u$ that is the negative of the vector u.

Subtraction of Vector

If $u = x_1 i + y_1 j$ and $v = x_2 i + y_2 j$ are two vectors, then their difference, denoted by $u - v$, is defined as $u - v = (x_1 - x_2)\, i + (y_1 - y_2)j$

Thus, to subtract two vectors, we subtract their corresponding components.

Example 8: If $u = 3i + 4j$ and $v = 4i - 5j$,

Find (i) $u + v$ (ii) $2u$ (iii) $-v$ (iv) $2u - 3v$

Solution:

(i) $u + v = (3i + 4j) + (4i - 5j) = (3 + 4)i + [4 + (-5)]j = 7i - j$

(ii) $2u = 2(3i + 4j) = (2 \cdot 3)i + (2 \cdot 4)j = 6i + 8j$

(iii) $-v = -(4i - 5j) = -4i - (-5)j = -4i + 5j$

(iv) $2u - 3v = 2(3i + 4j) - 3(4i - 5j) = 6i + 8j - 12i + 15j = -6i + 23j$

Zero Vector or Null Vector

The zero vector or null vector is denoted by O and is defined as $0 = 0i + 0j$

Magnitude of a Vector

If $u = xi + yj$ is a vector, its **magnitude** or **norm** or **length** is denoted by $|u|$ and is defined as

$$|u| = \sqrt{x^2 + y^2}$$

Example 9: If $u = 2i - 3j$, then find $|u|$.

Solution: $|u| = \sqrt{(2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

Unit Vector

If the magnitude of the given vector $u = xi + yj$ is 1, it is called a unit vector. That is, u is a unit vector if $|u| = 1$

Properties of Magnitude of a Vector

Theorem If $u = xi + yj$ is a vector and k is a scalar, then

(i) $|u| \geq 0$ (ii) $|u| = 0$ if and only if $u = 0$ (zero vector)

(iii) $|-u| = |u|$ (iv) $|ku| = |k|\,|u|$

Proof.

(i) $|u| = \sqrt{x^2 + y^2} \geq 0$ for all x and y.

(ii) $|u| = \sqrt{x^2 + y^2} = 0$ if and only if $x = 0$ and $y = 0$
if and only if $u = 0i + 0j$
if and only if $u = o$ (zero vector)

(iii) $|-u| = |-xi - yj| = \sqrt{(-x)^2 + (-y)^2} = \sqrt{x^2 + y^2} = |u|$

(iv) $|ku| = |(kx)i + (ky)j| = \sqrt{(kx)^2 + (ky)^2} = \sqrt{k^2(x^2 + y^2)}$
$= \sqrt{k^2}\,\sqrt{x^2 + y^2} = |k|\,|u|$

3.1.6 A Unit Vector in the direction of another Vector

If $u = xi + yj$ is a vector with magnitude $|u| \neq 0$, then $\frac{u}{|u|}$ is a unit vector whose direction is the same as that of u. It is usual to denote a unit vector in the direction of vector u by $\hat{u}$.

Clearly any vector u can be written in terms of unit vector as $u = |u|\,\hat{u}$

Hence a unit vector in the direction of u is given by

$$\hat{u} = \frac{u}{|u|} \Rightarrow \quad \hat{u} = \frac{xi + yj}{\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}i + \frac{y}{\sqrt{x^2 + y^2}}j.$$

Example 10: Find a unit vector in the same direction as the vector $3i-2j$

Solution: Let $u = 3i - 2j$

Then $|u| = \sqrt{(3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$

Since $\hat{u} = \frac{u}{|u|}$, so $\hat{u} = \frac{3i - 2j}{\sqrt{13}} = \frac{3}{\sqrt{13}}i - \frac{2}{\sqrt{13}}j.$

Note

The vector $\hat{u}$ is in fact a unit vector, because by property (iv) of magnitude of a vector

$|\hat{u}| = |\frac{u}{|u|}| = \frac{|u|}{|u|} = 1$

Notation for Vectors in Coordinate System

Sometimes we use the notation $[x,y]$ or $<x,y>$ for the vector $r = xi + yj$ which has its initial point at the origin of the rectangular coordinate system. The terminal point of r will have coordinates of the form (x,y). We call these coordinates the components of r. In this notation, the unit vectors i and j are given by $i = [1,0]$, $j = [0,1]$. If $r_1 = [x_1, y_1]$ and $r_2 = [x_2, y_2]$ are vectors and k any scalar, then addition and scalar multiplication are defined as $r_1 + r_2 = [x_1, y_1] + [x_2, y_2] = [x_1 + x_2,\ y_1 + y_2]$ and $kr_1 = k[x_1, y_1] = [kx_1, ky_1]$

Using the definition of addition and scalar multiplication, the vector $r = xi + yj$ can be written as

$$r = xi + yj = x[1,0] + y[0,1] = [x,0] + [0,y] = [x,y]$$

Thus $r = xi + yj = [x,y]$

3.1.7 Ratio Formula

Theorem: Let a and b be the position vectors of the points A and B respectively. If C divides AB internally in the ratio $p{:}q$, then the position vector c of C is given by $c = \frac{qa + pb}{q + p}$

Proof: If C divides the line segment $\overrightarrow{AB}$ internally in the ratio $p{:}q$, then $\frac{\overline{AC}}{\overline{CB}} = \frac{p}{q}$ as shown in the **(Figure 3.25)**.

Hence $q\,\overrightarrow{AC} = p\,\overrightarrow{CB} \Rightarrow q(c - a) = p(b - c)$

$\Rightarrow qc - qa = pb - pc \Rightarrow qc + pc = qa + pb$

$\Rightarrow (q + p)c = qa + pb \Rightarrow c = \frac{qa + pb}{q + p}$

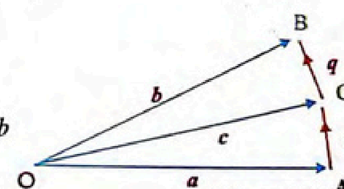


Figure 3.25

Corollary: If p : q = 1 : 1, then C is the midpoint of AB and its position vector c is given by $c = \frac{a + b}{2}$

Example 11: Find the position vector of the point dividing the join of point A with position vector $2i-3j$ and point B with position vector $3i+2j$ in the ratio 4:3

Solution: Suppose that the position vectors of the points A and B are a and b respectively. Then $a = 2i - 3j$ and $b = 3i + 2j$

Suppose that c is the position vector of the point C that divides the segment AB in ratio 4:3.

Then by ratio theorem (theorem above)

$$c = \frac{3a+4b}{3+4} = \frac{3(2i-3j)+4(3i+2j)}{7} = \frac{6i-9j+12i+8j}{7} = \frac{18i-j}{7} = \frac{18}{7}i - \frac{1}{7}j$$

3.1.8 Application to Geometry

In this section, we shall use vectors to prove some basic theorems of geometry.

Theorem: Prove that the straight line joining the midpoints of the two sides of a triangle is parallel to the third side and equal to one half of it.

Proof: Let OAB be a triangle and D,E be the midpoints of sides OA and OB respectively (see **Figure 3.26**)

Let $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, then

$\overrightarrow{OD} = \frac{a}{2}$, $\overrightarrow{OE} = \frac{b}{2}$ $\therefore$ D & E are the mid-points of $\overrightarrow{OA}$ & $\overrightarrow{OB}$ respectively

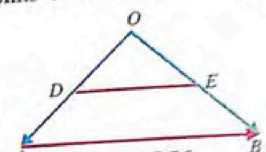


Figure 3.26

Now $\overrightarrow{DE} = \overrightarrow{DO} + \overrightarrow{OE} = -\overrightarrow{OD} + \overrightarrow{OE}$

$$= \frac{-a}{2} + \frac{b}{2} = \frac{b-a}{2} \qquad (1)$$

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -a + b = b - a \qquad (2)$$

Therefore from (1) and (2), we have

$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AB}$ Hence $\overrightarrow{DE} \parallel \overrightarrow{AB}$ and $\overrightarrow{DE}$ is equal to one half of $\overrightarrow{AB}$

Theorem: The diagonals of a parallelogram bisect each other.

Proof: Let the vertices of the parallelogram be O,A,B and C (See **Figure 3.27**)

Let $\boldsymbol{a}$, $\boldsymbol{b}$ be the position vectors of A and B respectively.

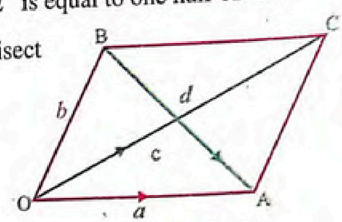


Figure 3.27

Then $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$.

By addition of vectors, we have $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = a + b$

The midpoint of the diagonal $\overrightarrow{OC}$ has the position vector

$$c = \frac{\overrightarrow{OC}}{2} = \frac{a+b}{2} \qquad (1)$$

Again by addition of vectors, we have $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = b - a$

The midpoint of the diagonal $\overrightarrow{AB}$ has the position vector

$$d = \overrightarrow{OA} + \frac{\overrightarrow{AB}}{2} = a + \frac{b-a}{2} = \frac{2a+b-a}{2} = \frac{a+b}{2} \qquad (2)$$

From (1) and (2), we have $c = d$.

This shows that the midpoints of the diagonal $\overrightarrow{OC}$ and $\overrightarrow{AB}$ are the same.

Thus the diagonals of the parallelogram bisect each other.

3.2 Vectors in Space

In section 3.1.4 we discussed vectors in the plane.

In this section, we again consider vectors, but vectors in space.

3.2.1 Rectangular coordinate system in space

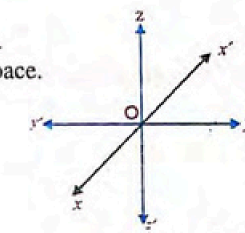


Figure 3.28

In space, a rectangular coordinate system (or Cartesian coordinate system) consists of three mutually perpendicular lines through a common point O. The point O is called **origin** and the mutually perpendicular coordinate lines xox', yoy' and zoz' are respectively x–, y– and z–axis (Figure 3.28). The positive x–axis points towards the reader, the y–axis to the right and z–axis points upwards.

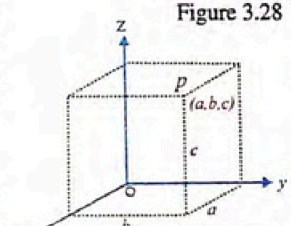


Figure 3.29

The coordinate axes, taken in pair, determine **three coordinate planes** namely the **xy–plane**, the **xz–plane** and the **yz–plane.** If the distances along x–, y– and z– axes are denoted by a, b, c, then the point P is assigned an ordered triple of real numbers as (a, b, c) or P(a, b, c) as shown in Figure 3.29. We call a, b and c the **x–coordinate, y–coordinate** and **z–coordinate** of P.Hence the point P whose coordinates are (4, 5, 6) is 4 units from O in the direction of $\overrightarrow{ox}$: 5 units from O in the direction of $\overrightarrow{oy}$. 6 units from O in the direction of oz as shown in (Figure 3.30).

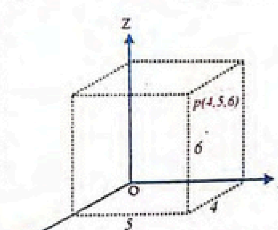


Figure 3.30

The set $\mathbb{R}^3 = \{(a, b, c): a, b, c \in \mathbb{R}\}$ is called the **three– dimensional space (or 3–dimensional space).**

3.2.2 Vectors in three dimensional space

Let i, j and k be three mutually perpendicular unit vectors in the direction of coordinate axes as follows:

i is a unit vector along positive x–axis, ox
j is a unit vector along positive y–axis, oy
k is a unit vector along positive z–axis, oz
as shown in **Figure 3.31.**

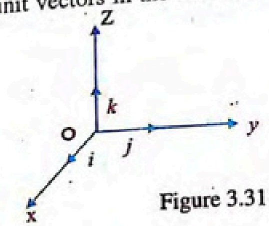


Figure 3.31

If P(x, y, z) is any point in the space, then the **position vector** $\overrightarrow{OP}$ of the point P can be written in the form.

$\overrightarrow{OP} = r = xi + yj + zk$ as shown in **Figure 3.32.**

Thus, a position vector of the point P is a vector $\overrightarrow{OP}$ whose initial point is at the origin O and whose terminal point is P.

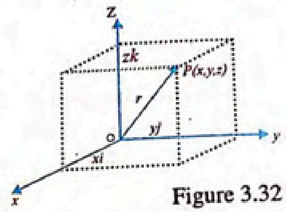


Figure 3.32

3.2.3 Component of a Vector

In the representation of the position vector to any point P(x, y, z) in the space as $\overrightarrow{OP} = r = xi + yj + zk$, the scalars x, y and z are called the **components** of r. The unit vectors i, j and k are the **unit base vectors** for this coordinate system.

Did You Know

If $\overrightarrow{P_1P_2}$ is a vector in space with initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$, Then $\overrightarrow{P_1P_2} = (x_2-x_1)i + (y_2-y_1)j + (z_2-z_1)k$. So the components of $\overrightarrow{P_1P_2}$ in i, j and k directions are $(x_2-x_1), (y_2-y_1), (z_2-z_1)$ respectively.

3.2.4 Notation for vectors in coordinate system

As in plane, we use the notation $[x, y, z]$ or $<x, y, z>$ for the vector $r = xi + yj + zk$ in space.

In this notation, the unit vectors i, j and k are given by

$$i = [1,0,0], j = [0,1,0], k = [0,0,1]$$

If $r_1 = [x_1, y_1, z_1]$ and $r_2 = [x_2, y_2, z_2]$ are vectors and α any scalar, then addition and scalar multiplication is defined as

$$r_1 + r_2 = [x_1, y_1, z_1] + [x_2, y_2, z_2] = [x_1+x_2, y_1+y_2, z_1+z_2] \text{ and}$$

$$\alpha r_1 = \alpha [x_1, y_1, z_1] = [\alpha x_1, \alpha y_1, \alpha z_1]$$

Using these definitions, the vector $r = xi+yj+zk$ can be written as

$$r = xi+yj+zk = x[1,0,0] + y[0,1,0] + z[0,0,1]$$
$$= [x,0,0] + [0,y,0] + [0,0,z] = [x,y,z]$$

Thus $r = xi + yj + zk = [x,y,z]$

3.2.5 Magnitude of a Vector

The magnitude or norm or length $|u|$ of a vector $u = xi + yj + zk$ in the space is the distance of the point $P(x, y, z)$ from the origin. That is $|u| = \sqrt{x^2 + y^2 + z^2}$

Unit Vector

If the magnitude of the vector $u = xi + yj + zk$ is 1, it is called a unit vector. That is $|u| = 1$

Example 12: If $u = 2i - j + 3k$, $v = i + j - k$, then find

(i) $u+2v$ (ii) $3u-2v$ (iii) $3(u-2v)$

(iv) $|u+v|$ (v) $|u|+|v|$ (vi) $\frac{u}{|u|}$

Solution:

(i) $u+2v = (2i-j+3k)+2(i+j-k) = 2i-j+3k+2i+2j-2k = 4i+j-k$

(ii) $3u-2v = 3(2i-j+3k)-2(i+j-k) = 6i-3j+9k-2i-2j+2k = 4i-5j+11k$

(iii) $3(u-2v) = 3[(2i-j+3k)-2(i+j-k)] = 3[2i-j+3k-2i-2j+2k]$
$= 3(0i-3j+5k) = -9j+15k$

(iv) $|u+v| = |(2i-j+3k)+(i+j-k)| = |2i-j+3k+i+j-k| = |3i+0j+2k|$
$= \sqrt{(3)^2+(0)^2+(2)^2} = \sqrt{9+4} = \sqrt{13}$

(v) $|u|+|v| = \sqrt{(2)^2+(-1)^2+(3)^2} + \sqrt{(1)^2+(1)^2+(-1)^2}$
$= \sqrt{4+1+9} + \sqrt{1+1+1} = \sqrt{14} + \sqrt{3}$

(vi) $\frac{u}{|u|} = \frac{2i-j+3k}{\sqrt{13}} = \frac{2}{\sqrt{13}}i - \frac{1}{\sqrt{13}}j + \frac{3}{\sqrt{13}}k$

3.2.6 Algebra of Vectors

In this section we define addition, subtraction scalar multiplication etc of vectors. Our definitions are the same as given for plane vectors except that in this case we consider vectors in space.

Equal Vectors

Two vectors $u = x_1i + y_1j + z_1k$ and $v = x_2i + y_2j + z_2k$ are said to be equal if and only if they have the same components.
That is $u = v$ if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$

Addition of Vectors

The addition of two vectors $u = x_1i + y_1j + z_1k$ and $v = x_2i + y_2j + z_2k$ is defined as
$u + v = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$

That is, to add two vectors, we add their corresponding components.

Scalar Multiplication

The scalar multiplication αu of a vector $u = xi + yj + zk$ by a scalar α is defined as

$\alpha u = \alpha(xi + yj + zk) = (\alpha x)i + (\alpha y)j + (\alpha z)k$

Negative of a Vector

The negative of a vector $u = xi + yj + zk$ is defined as

$$-u = -(xi + yj + zk) = -xi - yj - zk$$

Subtraction of a Vector

The difference of two vectors $u = x_1 i + y_1 j + z_1 k$ and $v = x_2 i + y_2 j + z_2 k$ is defined as $u - v = (x_1 - x_2)i + (y_1 - y_2)j + (z_1 - z_2)k$

That is, to subtract two vectors, we subtract their corresponding components.

Zero Vector or Null Vector

The zero vector or null vector O is defined as $0 = 0i + 0j + 0k$

Properties of Vectors

The following properties hold for vectors in plane as well as in space.

Let u, v and w be vectors and let α and β be scalars, then

$u+v=v+u$	(commutative property for addition)
$(u+v)+w=u+(v+w)$	(Associative property for addition)
$u+o=o+u=o$	(Identity for vector addition)
$u+(-u)=0$	(Inverse for vector addition)
$\alpha(\beta u)=(\alpha\beta)u$	(Associative property for scalar multiplication)
$\alpha(u+v)=\alpha u+\alpha v$	(Distributive property of scalar multiplication over vector addition)
$(\alpha+\beta)u=\alpha u+\beta u$	(Distributive property of vector multiplication over scalar addition)
$1u=u$	

Application to Geometry

Distance between two points in Space

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space. Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be the position vectors of A and B (Figure 3.33). Then

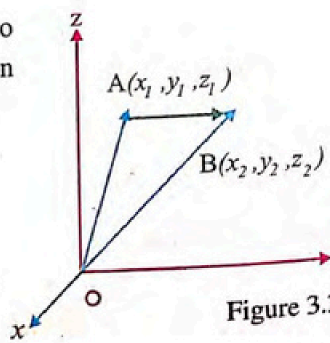


Figure 3.33

$\overrightarrow{OA} = x_1 i + y_1 j + z_1 k, \overrightarrow{OB} = x_2 i + y_2 j + z_2 k$

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

This is called the **distance formula**.

Theorem: Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space. The coordinate of point C which divides $\overrightarrow{AB}$ in the ratio $m_1:m_2$ are

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$$

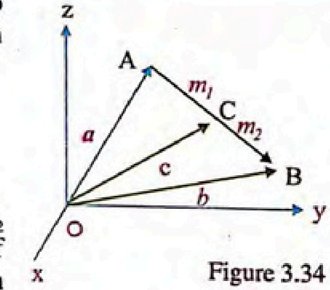


Figure 3.34

Proof. Let $C(x, y, z)$ divides $\overrightarrow{AB}$ in the ratio $m_1:m_2$ internally (Figure 3.34). If a is the position vector of A and b is the position vector of B, then the position vector c of C already found in Ratio theorem is

$$c = \frac{m_1 b + m_2 a}{m_1 + m_2}$$

$$\therefore \; xi + yj + zk = \frac{1}{m_1 + m_2}[m_1(x_2 i + y_2 j + z_2 k) + m_2(x_1 i + y_1 j + z_1 k)]$$

$$= \frac{(m_1x_2 + m_2x_1)i + (m_1y_2 + m_2y_1)j + (m_1z_2 + m_2z_1)k}{m_1 + m_2}$$

$$= \frac{m_1x_2 + m_2x_1}{m_1 + m_2}i + \frac{m_1y_2 + m_2y_1}{m_1 + m_2}j + \frac{m_1z_2 + m_2z_1}{m_1 + m_2}k$$

Comparing the corresponding components on both sides,

$$\Rightarrow \quad x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \; y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \; z = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$$

Thus $C(x,y,z) = C\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$

Did You Know ?

If λ is negative, the point C divides $\overline{AB}$ externally in the ratio $\lambda:1$

Corollary: If $\frac{m_1}{m_2} = \lambda$, then the point C divides $\overrightarrow{AB}$ in the ratio $\lambda : 1$ and

$$x = \frac{\lambda_1 x_2 + x_1}{1+\lambda}, \; y = \frac{\lambda_1 y_2 + y_1}{1+\lambda}, \; z = \frac{\lambda_1 z_2 + z_1}{1+\lambda}.$$

Theorem: Prove that the coordinates of the centroid of a triangle ABC with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

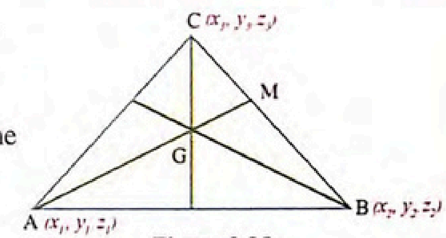


Figure 3.35

Proof: The centroid of the triangle ABC is the point G where all the three medians intersect each other in the ratio 2:1 (see Figure 3.35)

The midpoint M of BC has the coordinates $M\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$.

The point G dividing $\overline{AM}$ in the ratio AG:GM = 2:1 has the coordinates

$$\left(\frac{\frac{2(x_2+x_3)}{2}+1\cdot x_1}{2+1}, \frac{\frac{2(y_2+y_3)}{2}+1\cdot y_1}{2+1}, \frac{\frac{2(z_2+z_3)}{2}+1\cdot z_1}{2+1}\right)$$

$$=\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Example 13: Find the length of the median through O of the triangle OAB, where A is the point (2, 7, –1) and B is the point (4, 1, 2).

Solution: Let OAB be a triangle as shown in **(Figure 3.36)**.

The coordinates of M the midpoint of AB are

$$\left(\frac{2+4}{2}, \frac{7+1}{2}, \frac{-1+2}{2}\right) = (3, 4, \frac{1}{2})$$

So the length of $\overline{OM}$ is

$$|\overline{OM}| = \sqrt{(3)^2+(4)^2+(\frac{1}{2})^2} = \frac{\sqrt{101}}{2}$$

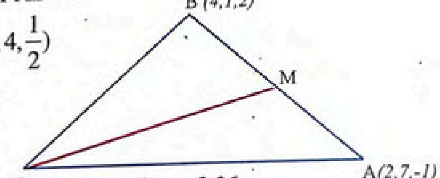


Figure 3.36

EXERCISE 3.2

1. If $a = 3i - 5j$ and $b = -2i + 3j$, then find
   (i) $a+2b$ (ii) $3a-2b$ (iii) $2(a-b)$
   (iv) $|a+b|$ (v) $|a|-|b|$ (vi) $\frac{|a|}{|b|}$
2. Find the unit vector having the same direction as the vector given below.
   (i) $3i$ (ii) $3i - 4j$ (iii) $i+j-2k$ (iv) $\frac{\sqrt{3}}{2}i - \frac{1}{2}j$
3. If $r = i - 9j$, $a = i + 2j$ and $b = 5i - j$, determine the real numbers p and q such that $r = pa + qb$.
4. If $p = 2i - j$ and $q = xi + 3j$, then find the value of x such that $|p + q| = 5$.
5. Find the length of the vector $\overrightarrow{AB}$ from the point A(–3,5) to B (7,9). Also find a unit vector in the direction of $\overrightarrow{AB}$.
6. If ABCD is a parallelogram such that the coordinates of the vertices A, B and C are respectively given by (–2,–3), (1, 4) and (0, 5). Find the coordinates of the vertex D.

7. Find the components and the magnitude of $\overrightarrow{PQ}$
   i. P (–1,2), Q(2, –1). ii. P (–2,1), Q(2,3).
   iii. P (–1,1,2), Q:(2, –1,3). iv. P(2,4,6), Q(1, –2, 3).
8. Find the initial point P or the terminal point Q whichever is missing:
   i. $\overrightarrow{PQ}$ = [–2, 3], P (1, –2). ii. $\overrightarrow{PQ}$ = [4, –5], Q(–1, 1).
   iii. $\overrightarrow{PQ}$ = [–1,3, –2], P(2, –1, –3). iv. $\overrightarrow{PQ}$ = [2, –3, –4], Q(3, –1,4).
9. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 4\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c} = \hat{i}-2\hat{j}+\hat{k}$, find a vector of magnitude 6 units which is parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.
10. Find the position vector of a point R which divides the line joining the points whose position vectors are $P(\hat{\imath}+2\hat{\jmath}-\hat{k})$ and $Q(-\hat{\imath}+\hat{\jmath}+\hat{k})$ in the ratio 2:1 internally and externally.
11. Find the position vectors of the point of division of the line segments joining
    (i) Point C with position vector $5j$ and point D with position vector $4i+j$ in the ratio 2 : 5 internally.
    (ii) Point E with position vector $2i-3j$ and point F with position vector $3i+2j$ in the ratio 4 : 3 externally.
12. Find α, so that $|\alpha\hat{\imath}+(\alpha+1)\hat{\jmath}+2\hat{k}| = 3$
13. If $\vec{u} = 2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{v} = -\hat{i}+3\hat{j}-\hat{k}$ and $\vec{w} = \hat{i}+6\hat{j}+z\hat{k}$ represent the sides of a triangle. Find the value of z.
14. The position vectors of the points A, B, C and D are $2\hat{i}-\hat{j}+\hat{k}$, $3\hat{i}+\hat{j}$, $2\hat{i}+4\hat{j}-2\hat{k}$ and $-\hat{i}-2\hat{j}+\hat{k}$ respectively. Show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

3.5 Dot or Scalar Product

3.5.1 The **dot** or **scalar product** of two vectors a, b denoted by $a\cdot b$, is defined as $a\cdot b = |a||b|\cos\theta$ where θ is the angle between the vectors a and b (Figure 3.37).

For example, if $|a| = 2$, $|b| = 4$, $\theta = 60°$, then $a\cdot b = 3\times4\cos60° = 12\times\frac{1}{2} = 6.$

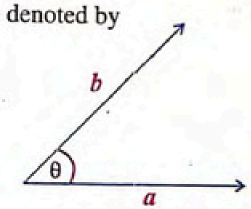


Figure 3.37

This will be negative if $\frac{\pi}{2} < \theta < \pi$ as $\cos\theta$ is negative, and $|a|$, $|b|$ are always positive.

3.5.2 Immediate consequences of the definition of Dot Product

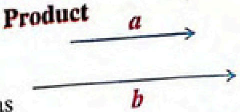


Figure 3.38

(i) **Parallel vectors**

If a and b are parallel but in the same direction as shown in **(Figure 3.38)**, then $\theta = 0°$.

In this case $a \cdot b = |a||b|\cos 0° = |a||b|$

If a and b are parallel but in opposite direction as shown in Figure (3.39) then $\theta = 180°$. In this case $a \cdot b = |a||b| \cos 180° = -|a||b|$

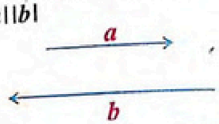


Figure 3:39

In the special case when $a = b$, then

$$a \cdot a = |a||a|\cos 0° = |a||a| = |a|^2$$

Hence $\quad |a| = \sqrt{a \cdot a}$

(ii) **Orthogonal vectors**

If a and b are orthogonal vectors, then $\theta = 90°$ and $\cos 90° = 0$

$\therefore\ a \cdot b = |a|\,|b| \cos 90° = 0$

Hence the condition for orthogonality of two vectors is $a \cdot b = 0$

3.5.3 Scalar product of unit vectors *i, j* and *k*

$$i \cdot i = |i|\,|i| \cos 0° = 1\ ,\ i \cdot j = |i|\,|j| \cos 90° = 0$$
$$j \cdot j = |j|\,|j| \cos 0° = 1\ ,\ j \cdot k = |j|\,|k| \cos 90° = 0$$
$$k \cdot k = |k|\,|k| \cos 0° = 1\ ,\ k \cdot i = |k|\,|i| \cos 90° = 0$$

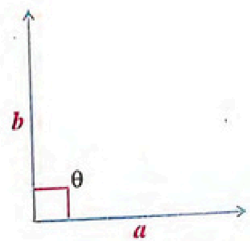


Figure 3.40

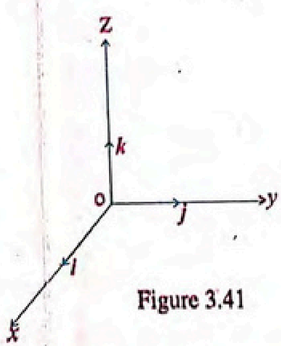


Figure 3.41

Remember

The dot product is always a number (scalar). We sometimes refer to it as the **scalar product** or **inner product**.

Thus, $\quad i \cdot i = j \cdot j = k \cdot k = 1 \quad and \quad i \cdot j = j \cdot k = k \cdot i = 0$

3.5.4 Expression of Dot Product in Terms of Components

Let $a = x_1 i + y_1 j + z_1 k$ and $b = x_2\, i + y_2\, j + z_2\, k$ be two vectors in space. Then using the properties of dot product, we have

Remember

If $a = x_1 i + y_1 j$ and $b = x_2 i + y_2 j$ are vectors in the plane, then $\quad a \cdot b = x_1 x_2 + y_1 y_2$

$$a \cdot b = (x_1\, i + y_1\, j + z_1\, k) \cdot (x_2\, i + y_2\, j + z_2\, k)$$
$$= x_1 x_2\,(i \cdot i) + x_1\, y_2\,(i \cdot j) + x_1\, z_2\,(i \cdot k) + y_1 x_2\,(j \cdot i) + y_1 y_2\,(j \cdot j) + y_1 z_2\,(j \cdot k)$$
$$+\ z_1 x_2\,(k \cdot i) + z_1 y_2\,(k \cdot j) + z_1 z_2\,(k \cdot k) = x_1 x_2 + y_1 y_2 + z_1 z_2$$
$$\therefore \quad a \cdot b = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Thus, dot product of two vectors is the sum of the product of their corresponding components.

Example 14: If $a = 2i - 3j + 4k$ and $b = i + 3j - 2k$, then find $a.b$ in terms of their components.

Solution:
$$a \cdot b = (2i - 3j + 4k) \cdot (i + 3j - 2k)$$
$$= (2)(1) + (-3)(3) + (4)(-2) = -15$$

3.5.5 Commutative and Distributive Properties of Dot Product

Theorem: If a, b, and c are vectors and α any scalar, then

(a) Dot product is commutative i.e. $a \cdot b = b \cdot a$

(b) Dot product is distributive over vector addition i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$

Proof:

(a) Let $a = x_1\, i + y_1\, j + z_1\, k$ and $b = x_2\, i + y_2\, j + z_2\, k$

Using the properties of dot product and scalars, we have

$$a \cdot b = (x_1 i + y_1 j + z_1 k) \cdot (x_2\, i + y_2\, j + z_2\, k)$$
$$= x_1 x_2 + y_1 y_2 + z_1 z_2 = x_2 x_1 + y_2 y_1 + z_2 z_1$$
$$= b \cdot a$$

Thus, $\quad a \cdot b = b \cdot a$

(b) Let $\quad c = x_3 i + y_3 j + z_3 k$, then

$$a \cdot (b+c) = (x_1 i + y_1 j + z_1 k) \cdot [(x_2 i + y_2 j + z_2 k) + (x_3 i + y_3 j + z_3 k)]$$
$$= (x_1 i + y_1 j + z_1 k) \cdot [(x_2 + x_3) i + (y_2 + y_3) j + (z_2 + z_3) k]$$
$$= x_1 (x_2 + x_3) + y_1 (y_2 + y_3) + z_1 (z_2 + z_3)$$
$$= x_1 x_2 + x_1 x_3 + y_1 y_2 + y_1 y_3 + z_1 z_2 + z_1 z_3$$
$$= (x_1 x_2 + y_1 y_2 + z_1 z_2) + (x_1 x_3 + y_1 y_3 + z_1 z_3) = a.b + a.c$$

Thus, $\quad a \cdot (b+c) = a \cdot b + a \cdot c$

3.5.6 Direction Angles and Direction Cosines of Vectors

Let $r = xi+yj+zk$ be a non–zero vector. Let α, β and γ be the angles which the vector r makes with the positive directions of the coordinate axes where each of these angles lies between 0 and π i.e. $0 \le \alpha, \beta, \gamma \le \pi$.

The angles α, β and γ are called the **direction angles** of the vector r (see Figure 3.42).

Referring to the figure, we have three right triangles OAP, OBP and OCP. Then

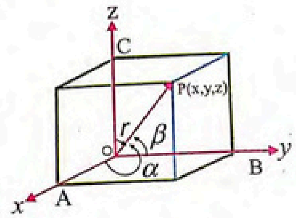


Figure 3.42

$$\cos\alpha = \frac{x}{|r|} = \frac{x}{\sqrt{x^2+y^2+z^2}} \text{ in right triangle OAP}$$

$$\cos\beta = \frac{y}{|r|} = \frac{y}{\sqrt{x^2+y^2+z^2}} \text{ in right triangle OBP}$$

$$\cos\gamma = \frac{z}{|r|} = \frac{z}{\sqrt{x^2+y^2+z^2}} \text{ in right triangle OCP}$$

The numbers $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **direction cosines** of the vector r. The direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are usually denoted by l, m and n respectively.

Theorem: If α, β and γ are the direction angles of a vector r, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Proof: By the definition of direction cosines of the vector r, we have,

$$\cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}}, \cos\beta = \frac{y}{\sqrt{x^2+y^2+z^2}} \text{ and } \cos\gamma = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{x^2+y^2+z^2} + \frac{y^2}{x^2+y^2+z^2} + \frac{z^2}{x^2+y^2+z^2}$$

$$= \frac{x^2+y^2+z^2}{x^2+y^2+z^2} = 1$$

Using symbols ℓ, m and n, we may write the above result in the form $\ell^2 + m^2 + n^2 = 1$

Example 15: Find the direction cosines of the vector from P(4,8,–3) to Q(–1,6,2)

Solution: We know that for any two points P and Q we have $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

Here $\overrightarrow{OQ} = -i+6j+2k$, $\overrightarrow{OP} = 4i+8j-3k$

$\overrightarrow{PQ} = (-i+6j+2k)-(4i+8j-3k) = -5i-2j+5k$

Since $|\overrightarrow{PQ}| = \sqrt{x^2+y^2+z^2}$

So $|\overrightarrow{PQ}| = \sqrt{(-5)^2+(-2)^2+(5)^2} = \sqrt{54} = 3\sqrt{6}$

Hence direction cosines of the vector $\overrightarrow{PQ}$ are

$$\cos\alpha = \frac{x}{|\overrightarrow{PQ}|} = \frac{-5}{3\sqrt{6}}, \cos\beta = \frac{y}{|\overrightarrow{PQ}|} = \frac{-2}{3\sqrt{6}} \text{ and } \cos\gamma = \frac{z}{|\overrightarrow{PQ}|} = \frac{5}{3\sqrt{6}}$$

3.5.7 Direction Numbers or Direction Ratios

The position vector $\overrightarrow{OP}$ of the point $P(x,y,z)$ in term of unit vectors i, j and k is given as

$$\overrightarrow{OP} = r = xi+yj+zk$$

If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of r, and p is a positive constant, then the numbers $p\cos\alpha$, $p\cos\beta$ and $p\cos\gamma$ are called the **direction numbers** or **direction ratios** of the vector r. The direction numbers are used to specify the direction of the vector r.

Since $x = |r|\cos\alpha$, $y = |r|\cos\beta$ and $z = |r|\cos\gamma$ where $|r|$ is the length of the vector r, so x, y and z are direction numbers of the vector r. Therefore the coordinates of $P(x,y,z)$ may be written as $(|r|\cos\alpha, |r|\cos\beta, |r|\cos\gamma)$

Hence $\overrightarrow{OP} = r = |r|\cos\alpha\, i + |r|\cos\beta\, j + |r|\cos r\, k$

$\Rightarrow \overrightarrow{OP} = |r|(\cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k)$ or

$\overrightarrow{OP} = |r|\,(li+mj+nk)$

Did You Know

From the above, we obtain the components of r from the direction cosines multiplying by $|r|$.
Conversely, dividing the components by $|r|$ gives the direction cosines.

Example 16: Find the direction numbers and direction cosines of the point P(2,–3,6).

Solution: The direction numbers are 2, –3, 6.

Since $\overrightarrow{OP} = r = 2i - 3j + 6k$, $|\overrightarrow{OP}| = 7$, therefore the direction cosines are

$$\ell = \frac{x}{|\overrightarrow{OP}|} = \frac{2}{7}, m = \frac{y}{|\overrightarrow{OP}|} = \frac{-3}{7} \text{ and } n = \frac{z}{|\overrightarrow{OP}|} = \frac{6}{7}$$

Example 17: Find the coordinates of P, if $\overrightarrow{OP}$ is of length 6 units in the direction of $\overrightarrow{OR}$ where R is the point (2, –1, 4)

Solution: We have $\overrightarrow{OR} = 2i - j + 4k \quad \therefore \quad |\overrightarrow{OR}| = \sqrt{21}$

The direction cosines of $\overrightarrow{OR}$ are

$$\ell=\frac{2}{\sqrt{21}},\ m=\frac{-1}{\sqrt{21}},\ n=\frac{4}{\sqrt{21}}$$

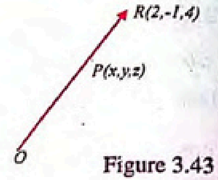


Figure 3.43

The coordinates of P are $(|r|\ell, |r|m, |r|n)$

where $|r|=|\overrightarrow{OP}|=6$.

Therefore $|r|\ell=\frac{12}{\sqrt{21}}$, $|r|m=\frac{-6}{\sqrt{21}}$, $|r|n=\frac{24}{\sqrt{21}}$

Hence the coordinates of P are $\left(\frac{12}{\sqrt{21}}, \frac{-6}{\sqrt{21}}, \frac{24}{\sqrt{21}}\right)$.

Example 18: A vector v has inclination 60° to $\overrightarrow{ox}$, 45° to $\overrightarrow{oy}$.

Find its inclination to $\overrightarrow{oz}$. If $|v|=12$, express v as $xi + yj + zk$.

Solution: Here $l=\cos 60°=\frac{1}{2}$, $m=\cos 45° = \frac{1}{\sqrt{2}}$

Let $n = \cos\gamma$, where γ is inclination to $\overrightarrow{oz}$.

Since $l^2 + m^2 + n^2 = 1$

So $n^2 = 1 - l^2 - m^2 \Rightarrow n^2 = 1-\frac{1}{4}-\frac{1}{2}=\frac{1}{4}$

$\Rightarrow n = \pm\frac{1}{2}$ $\therefore \cos\gamma = \pm\frac{1}{2}$

This shows that v is inclined to $\overrightarrow{oz}$ either at 60° or 120°.

Now ℓi, mj, nk are components of $\hat{v}$, so $\hat{v} = \frac{1}{2}i+\frac{1}{\sqrt{2}}j+\frac{1}{2}k$

But $v = |v|\hat{v} = 12\left(\frac{1}{2}i+\frac{1}{\sqrt{2}}j+\frac{1}{2}k\right)=6i+6\sqrt{2}j+6k$

3.5.8 Angle between two Vectors

One use of the dot product is to calculate the angle between two vectors.

(i) Let a and b be the two vectors. Then by definition of dot product

$a\cdot b = |a||b|\cos\theta$ where $0 \le \theta \le \pi$

$\therefore \cos\theta = \frac{a\cdot b}{|a||b|}$

i.e. the cosine of the angle between two vectors is their dot product divided by the product of their moduli.

(ii) if $a = x_1i + y_1j + z_1k$ and $b = x_2i + y_2j + z_2k$

$a\cdot b = x_1x_2 + y_1y_2 + z_1z_2$

$|a| = \sqrt{x_1^2+y_1^2+z_1^2}$ and $|b| = \sqrt{x_2^2+y_2^2+z_2^2}$ since by (i) above

$\cos\theta = \frac{a\cdot b}{|a||b|}$ $\therefore \cos\theta = \frac{x_1x_2+y_1y_2+z_1z_2}{\sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}}$

(iii) $a.b = |a||b|\cos\theta$ $\cos\theta = \frac{a\cdot b}{|a||b|}=\frac{a}{|a|}\cdot\frac{b}{|b|}=\hat{a}.\hat{b}$

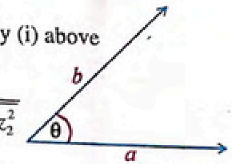


Figure 3.44

Example 19: Find the angle between the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ where $\overrightarrow{OP} = 2i + j$,

$\overrightarrow{OQ} = -3i + 2j$

Solution: Let θ be the angle between the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$

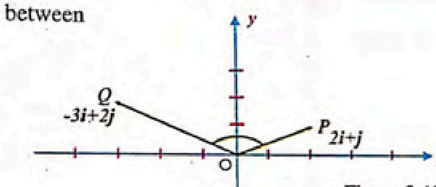


Figure 3.45

Then $\cos\theta = \frac{\overrightarrow{OP}.\overrightarrow{OQ}}{|\overrightarrow{OP}||\overrightarrow{OQ}|}$

$=\frac{(-3i+2j).(2i+j)}{\sqrt{3^2+2^2}\sqrt{2^2+1^2}}$

$\Rightarrow \cos\theta = \frac{-6+2}{\sqrt{13}\sqrt{5}} = -0.4961$

$\Rightarrow \theta = 119.74°$

Example 20: Find the value of t such that the vectors $2i - j + 2k$ and $3i + 2tj$ are orthogonal.

Solution: Let $a = 2i - j + 2k$ and $b = 3i + 2tj$. If a and b are orthogonal, then $a\cdot b=0$

$\therefore 2(3) + (-1)(2t) + 2(0) = 0 \Rightarrow -2t = -6$ or $t = 3$

3.5.9 Projection of one Vector on another

Let a and b be two vectors and θ be the angle between them as shown in (Figure 3.46) , $0 \le \theta \le \pi$

$\overrightarrow{AC}$ is perpendicular to $\overrightarrow{OB}$. Then $\overrightarrow{OC}$ is called the **projection** of a on b.

From Δ OCA, we have

$$\frac{|\overrightarrow{OC}|}{|\overrightarrow{OA}|} = \cos\theta$$

$$\Rightarrow |\overrightarrow{OC}| = |\overrightarrow{OA}|\cos\theta = |a|\cos\theta \qquad (1)$$

By definition of angle between vectors

$$\cos\theta = \frac{a.b}{|a||b|} \qquad (2)$$

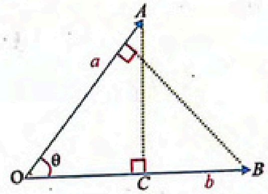


Figure 3.46

Using (1) and (2), we have $|\overrightarrow{OC}| = \frac{a\cdot b}{|b|}$

This gives that the projection of a on b is $\frac{\vec{a}\cdot\vec{b}}{|b|}$. Similarly the projection of b on a is $\frac{a.b}{|a|}$

Example 21: Find the projection of the vector $a = i-2j+k$ to the vector $b = 4i-4j+7k$.

Solution: The projection of a on $b = \frac{a.b}{|b|}$

Now $a\cdot b = (i-2j+k).(4i-4j+7k) = (1)(4)+(-2)(-4)+(1)(7) = 19$

And $|b| = \sqrt{(4)^2+(-4)^2+(7)^2} = \sqrt{16+16+49} = \sqrt{81} = 9$

$\therefore$ the projection of a on $b = \frac{a.b}{|b|} = \frac{19}{9}$

3.5.10 Work done by a constant force

If a constant force F acts on an object during any interval of time and the object undergoes a displacement S, then the work done on the object by the force F is defined as

$$W = \vec{F}.\vec{S}$$

or $W = FS\cos\theta$, where θ is the angle between the directions of $\vec{F}$ and $\vec{S}$, as in (Figure 3.47)

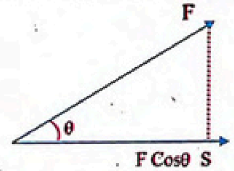


Figure 3.47

Example 22: Find the work done in moving an object along a vector $9i-j+k$ if the applied force is $3i+2j+k$.

Solution: Here $\vec{F} = 3i+2j+k$

$\vec{S} = 9i-j+k$

$$\therefore W = \vec{F}\cdot\vec{S} = (3i+2j+k)\ (9i-j+k)$$
$$= 3(9)+3(-1)+1(1)$$
$$= 27-2+1$$
$$= 26$$

Hence work done = 26 units

EXERCISE 3.3

1. If $a = 3i+4j-k$, $b = i-j+3k$ and $c = 2i+j-5k$ then find
   (i) $a\cdot b$ (ii) $a\cdot c$ (iii) $a\cdot(b+c)$
   (iv) $(2a+3b)\cdot c$ (v) $(a-b)\cdot c$
2. Write a unit vector in the direction of the sum of the vectors
   $\vec{a} = 2\hat{i}+2\hat{j}-5\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}-7\hat{k}$.
3. Find the angles between the following pairs of vectors:
   (i) $i-j+k$, $-i+j+2k$ (ii) $3i+4j$, $2j-5k$ (iii) $2i-3k$, $i+j+k$
4. Show that $i+7j+3k$ is perpendicular to both $i-j+2k$ and $2i+j-3k$.
5. Let $a = i+2j+k$ and $b = 2i+j-k$. Find a vector that is orthogonal to both a and b.
6. Let $a = i+3j-4k$ and $b = 2i-3j+5k$. Find the value of m so that $a+mb$ is orthogonal to (i) a (ii) b.
7. Given the vectors $\vec{a}$ and $\vec{b}$ as follows:
   (i) $a = -\frac{3}{2}j+\frac{4}{5}k$, $b = i-2j-2k$ (ii) $a = -3i+j+2k$, $b = -i-j+5k$
   Find in each case the projection of **a** on **b** and of **b** on **a**.
8. What is the cosine of the angle which the vector $\sqrt{2}\hat{i}+\hat{j}+\hat{k}$ makes with y – axis?
9. A force $F = 2i+3j+k$ acts through a displacement $S = 2i+j-k$. Find the work done.
10. Find the work done by the force $\vec{F} = 2i+3j+k$ in the displacement of an object from a point A(–2,1,2) to the point B(5,0,3).
11. (i) Show that the vectors $3i-2j+k$, $i-3j+5k$ and $2i+j-4k$ form a right triangle. (ii) Show that the set of points $P = (1,0,1)$, $Q = (1,1,1)$ and $R = (1,1,0)$ form a right isosceles triangle.
12. Prove that the angle in a semicircle is a right angle.
13. Prove that perpendicular bisectors of the sides of a triangle are concurrent.

3.6 The Cross or Vector Product of two Vectors

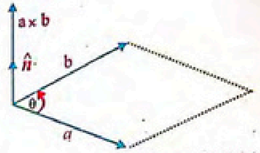


Figure 3.48(a)

In section 3.5 we noticed that dot product of two vectors in plane or in space gives a scalar. However, in this section we shall see that there is another product known as cross or vector product, which gives the result as vector in three dimensional space.

3.6.1 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non–zero vectors. The **cross** or **vector product** of $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted as $\boldsymbol{a} \times \boldsymbol{b}$, is defined by

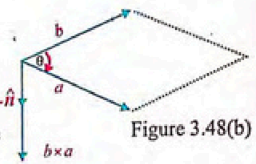


Figure 3.48(b)

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta\, \hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the plane determined by $\boldsymbol{a}$ and $\boldsymbol{b}$ (See Figure 3.48 (a))

The direction of $\hat{n}$ is determined by the right hand rule

"Join the tails of $\boldsymbol{a}$, $\boldsymbol{b}$, stretch the fingers of your right hand along the direction of first vector $\boldsymbol{a}$ and curl them towards the second vector $\boldsymbol{b}$ through smaller angle θ between $\boldsymbol{a}$ and $\boldsymbol{b}$ ($0<\theta<180°$), then the erected thumb will show the direction of $\hat{n}$ or $\boldsymbol{a} \times \boldsymbol{b}$."

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are as shown in (Figure 3.48 (a)). and the plane containing $\boldsymbol{a}$, $\boldsymbol{b}$ represents upper surface of a table then $\boldsymbol{a} \times \boldsymbol{b}$ is directed above the table.

Clearly, the direction of $\boldsymbol{b} \times \boldsymbol{a}$ by stretching fingers along $\boldsymbol{b}$ and curling towards $\boldsymbol{a}$ gives the direction of the thumb of right hand downwards (under the table) direction from the plane (see Figure 3.48(b)).

Hence $\boldsymbol{b} \times \boldsymbol{a} = -|\boldsymbol{b}|\,|\boldsymbol{a}| \sin\theta\, \hat{n}$

where $\hat{n}$ is a unit vector perpendicular to the plane directed upward.

In Figure 3.48(b) $\boldsymbol{b} \times \boldsymbol{a}$ is the scalar multiple of $-\hat{n}$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors, then the length of $\boldsymbol{a} \times \boldsymbol{b}$ is given by $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$

3.6.2 Immediate consequences of the definition of Cross Product

(i) Since $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$, hence vector product is not commutative i.e. $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{b} \times \boldsymbol{a}$.

(ii) **Parallel Vectors.** If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel but in the opposite direction as shown in Figure 3.49(a), then $\theta = 180°$.

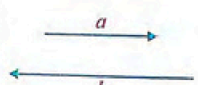


Figure 3.49(a)

In this case $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin 180°\, \hat{n} = 0$

($\because$ sin 180°=0)

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel but in the same direction as shown in Figure 3.49(b), then $\theta = 0°$

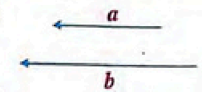


Figure 3.49(b)

In this case $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin 0°\, \hat{n} = 0$ ($\because \sin 0° = 0$)

Hence in either case $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$

If $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$, then either at least one of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ is zero or $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

In particular $\boldsymbol{a} \times \boldsymbol{0} = \boldsymbol{0}$ for all vectors $\boldsymbol{a}$.

3.6.3 Expressing Cross Product in terms of components

Let $\boldsymbol{a} = x_1 i + y_1 j + z_1 k$ and $\boldsymbol{b} = x_2 i + y_2 j + z_2 k$ be two vectors in space. Then using the properties of cross product, we have

$$a \times b = (x_1 i + y_1 j + z_1 k) \times (x_2 i + y_2 j + z_2 k)$$

$$= x_1x_2(i\times i) + x_1y_2(i\times j) + x_1z_2(i\times k) + y_1x_2(j\times i) + y_1y_2(j\times j) + y_1z_2(j\times k)$$
$$+ z_1x_2(k\times i) + z_1y_2(k\times j) + z_1z_2(k\times k)$$

$$= x_1x_2(0) + x_1y_2(k) + x_1z_2(-j) + y_1x_2(-k) + y_1y_2(0) + y_1z_2(i) + z_1x_2(j) + z_1y_2(-i) + z_1z_2(0)$$

$$= (y_1z_2 - z_1y_2)i - (x_1z_2 - z_1x_2)j + (x_1y_2 - y_1x_2)k \qquad (1)$$

The expansion of 3 ×3 determinant

$$\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - z_1y_2)i - (x_1z_2 - z_1x_2)j + (x_1y_2 - y_1x_2)k \qquad (2)$$

From (1) and (2), we have $\quad a \times b = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$

3.6.4 Application to Geometry

Theorem: Prove that the magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ represents the area of a parallelogram with adjacent sides $\boldsymbol{a}$ and $\boldsymbol{b}$.

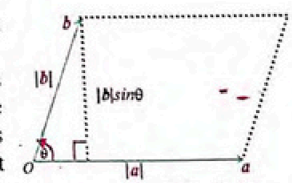


Figure 3.50

Proof: Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non–zero vectors representing the two adjacent sides of the parallelogram and θ be the angle between them as shown in Figure 3.50. We know from geometry that

Area of parallelogram = base × altitude

$= |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta = |\boldsymbol{a} \times \boldsymbol{b}|$

Thus Area of parallelogram = $|\boldsymbol{a} \times \boldsymbol{b}|$

Theorem: Prove that the area of a triangle equals $\frac{1}{2}\ |\boldsymbol{a}\times\boldsymbol{b}|$.

Proof: From **Figure 3.51**, we have that

area of triangle = $\frac{1}{2}$ (area of parallelogram).

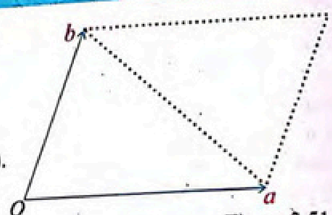


Figure 3.51

By above theorem

Area of parallelogram = $|\boldsymbol{a}\times\boldsymbol{b}|$ $\quad\therefore\quad$ Area of triangle = $\frac{1}{2}\ |\boldsymbol{a}\times\boldsymbol{b}|$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors along the two adjacent sides of the triangle.

Example 23: Find the area of the triangle whose vertices are A(2,2,0), B(–1,0,2) and C(0,4,3).

Solution: Let $\overline{AB}$ and $\overline{AC}$ be the adjacent sides of the parallelogram determined, so the required area of the triangle is half the area of the parallelogram, that is

$$\text{Area of the triangle} = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

Since $\overrightarrow{AB} = (-1,0,2)-(2,2,0) = (-3,-2,2)$ and

$\overrightarrow{AC} = (0,4,3)-(2,2,0) = (-2,2,3)$,

$$\text{so } \overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix} = -10i + 5j - 10k$$

$$\Rightarrow \left|\overline{AB}\times\overline{AC}\right| = \sqrt{(-10)^2+(5)^2+(-10)^2} = \sqrt{225} = 15$$

$$\therefore \quad \text{Area of the triangle} = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{15}{2}$$

3.6.5. Scalar triple product of i,j and k

By applying the definition of cross product to unit vectors $\boldsymbol{i},\boldsymbol{j}$ and $\boldsymbol{k}$, we have

(a) $\quad i\times i = |i|\,|i|\,\sin 0^\circ\,\hat{n} = 0$

$\quad j\times j = |j|\,|j|\,\sin 0^\circ\,\hat{n} = 0$

$\quad k\times k = |k|\,|k|\,\sin 0^\circ\,\hat{n} = 0$

(b) $\quad i\times j = |i|\,|j|\,\sin 90^\circ k = k$

$\quad j\times k = |j|\,|k|\,\sin 90^\circ i = i$

$\quad k\times i = |k|\,|i|\,\sin 90^\circ j = j$

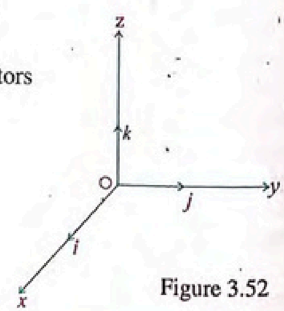


Figure 3.52

(c) $\quad j\times i = -(i\times j) = -k$

$\quad k\times j = -(j\times k) = -i$

$\quad i\times k = -(k\times i) = -j$

Thus

$$i\times i = j\times j = k\times k = 0$$
$$i\times j = k,\ j\times k = i,\ k\times i = j$$
$$j\times i = -k,\ k\times j = -i,\ i\times k = -j$$

For convenience we arrange unit vectors $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ in clockwise order as shown in **Figure 3.53**. Then the cross product of any two consecutive vectors is the remaining third vector with a plus sign or a minus sign according as the order of the product is clockwise or anticlockwise.

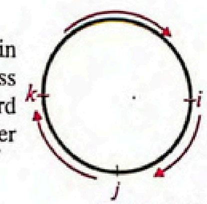


Figure 3.53

3.6.6 Anticommutative Property

Theorem: If $\boldsymbol{a}$, $\boldsymbol{b}$ are vectors, then

$$\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$$

Proof:

This property has already been proved geometrically. Analytically we prove it as follows.

Let $\boldsymbol{a} = x_1 i + y_1 j + z_1 k$, $\boldsymbol{b} = x_2 i + y_2 j + z_2 k$.

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = -\begin{vmatrix} i & j & k \\ x_2 & y_2 & z_2 \\ x_1 & y_1 & z_1 \end{vmatrix} \text{ (by interchanging the rows of the determinant)}$$

$$= -\boldsymbol{b}\times\boldsymbol{a}$$

Thus $\quad \boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$

If $\boldsymbol{a} = 0$ or $\boldsymbol{b} = 0$ or $\sin\theta = 0$, then clearly $\boldsymbol{a}\times\boldsymbol{b} = 0$

3.6.7 Distributive Property

Theorem: If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors, then

(i) $\quad (\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{c} + \boldsymbol{b}\times\boldsymbol{c}$ $\qquad$ (ii) $\quad \boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\times\boldsymbol{b} + \boldsymbol{a}\times\boldsymbol{c}$

Proof:

(i) Let $\boldsymbol{a} = x_1 i + y_1 j + z_1 k$, $\boldsymbol{b} = x_2 i + y_2 j + z_2 k$ and $\boldsymbol{c} = x_3 i + y_3 j + z_3 k$, then $\boldsymbol{a}+\boldsymbol{b} = (x_1+x_2)i + (y_1+y_2)j + (z_1+z_2)k$ and so

$$(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c} = \begin{vmatrix} i & j & k \\ x_1+x_2 & y_1+y_2 & z_1+z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \boldsymbol{a}\times\boldsymbol{c} + \boldsymbol{b}\times\boldsymbol{c}$$

Thus $(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{c} + \boldsymbol{b}\times\boldsymbol{c}$

(ii) $\quad$ Proof is similar to (i) above

3.6.8 Angle between two vectors

One use of the cross product is to calculate the angle between two vectors.

(i) Let a and b be the two vectors. Then by definition of cross product

$$|a \times b| = |a|\,|b| \sin\theta \text{ where } 0 \le \theta \le \pi$$

$\therefore \sin\theta = \dfrac{|a \times b|}{|a||b|}$ i.e. the sine of the angle between the two vectors is the modulus of their cross product divided by the product of their moduli.

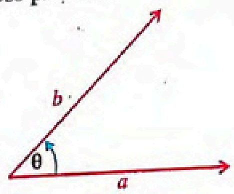


Figure 3.54

Hence $\theta = \sin^{-1} \dfrac{|a \times b|}{|a||b|}$

Example 24: For the vectors $a = 2i + 5j + 3k$, $b = 3i + 3j + 6k$, and $c = 2i + 7j + k$, find

(i) $(a - b) \times (c - a)$

(ii) a unit vector perpendicular to both a and b

(iii) $\sin\theta$ where θ is the angle between a and b.

Solution: (i) $a - b = (2i + 5j + 3k) - (3i + 3j + 6k) = -i + 2j - 3k$

$c - a = (2i + 7j + 4k) - (2i + 5j + 3k) = 2j + k$

$$\therefore (a - b) \times (c - a) = \begin{vmatrix} i & j & k \\ -1 & 2 & -3 \\ 0 & 2 & 1 \end{vmatrix} = (2 + 6)\, i - (-1 - 0)j + (-2 - 0)k = 8i + j - 2k$$

(ii) Let $\hat{n}$ be the required unit vector orthogonal to both a and b, then

$$\hat{n} = \frac{a \times b}{|a \times b|} \qquad (1)$$

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & 5 & 3 \\ 3 & 3 & 6 \end{vmatrix} = (30 - 9)i - (12 - 9)j + (6 - 15)k = 21i - 3j - 9k,$$

$$|a \times b| = \sqrt{(21)^2 + (-3)^2 + (-9)^2} = \sqrt{531} = 3\sqrt{59}$$

Putting in (1), we have

$$\hat{n} = \frac{21i - 3j - 9k}{3\sqrt{59}} = \frac{7i - j - 3k}{\sqrt{59}}$$

(iii) Since $|a \times b| = |a|\,|b| \sin\theta$ where θ is the angle between a, b, we have

$$\sin\theta = \frac{|a \times b|}{|a||b|} \therefore \sin\theta = \frac{3\sqrt{59}}{\sqrt{38}\sqrt{54}} = \frac{3\sqrt{59}}{\sqrt{38 \times 54}} = \frac{3\sqrt{59}}{3\sqrt{228}} \Rightarrow \sin\theta = \frac{\sqrt{59}}{\sqrt{228}}$$

3.6.9 Moment of Force

The moment M of a force F about a point P is defined as the product $M = |F|\, d$, where d is the (perpendicular) distance between P and the line of action L of F as shown in (**Figure 3.55**)

If r is the vector from P to any point Q on L, then

$$d = |r| \sin\theta$$

$$\therefore \quad M = |F| d = |r||F| \sin\theta$$

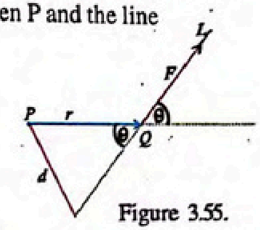


Figure 3.55.

Since θ is the angle between r and F, so $M = r \times F$

The vector $M = r \times F$ is called the **moment vector** or **vector moment** of $\overline{F}$ about Q.

Example 25: Find the moment about a point A(2,1,1) of the force $\overline{F} = 7i + 4j - 3k$ applied at (1,–2,3)

Solution: If r is the position vector of the point P of application relative to the point A about which the moment is calculated, then moment M is given by $M = r \times F$

where $r = \overrightarrow{AP} = (2i + j + k) - (i - 2j + 3k) = -(i + 3j - 2k)$. Hence,

$$M = r \times F = \begin{vmatrix} i & j & k \\ 1 & 3 & -2 \\ 7 & 4 & -3 \end{vmatrix} = -[(-9 + 8)i + (-14 + 3)j + (4 - 21)k] = i + 11j + 17k$$

EXERCISE 3.4

1. Find the following cross products.
   (i) $j \times (2j + 3k)$ (ii) $(2i - 3j) \times k$ (iii) $(2i - 3j + 5k) \times (6i + 2j - 3k)$
2. Show in two different ways that the vectors $\vec{a}$ and $\vec{b}$ are parallel:
   (i) $\vec{a} = -i + 2j - 3k$, $\vec{b} = 2i - 4j + 6k$ (ii) $\vec{a} = 3i + 6j - 9k$, $\vec{b} = i + 2j - 3k$
3. Find a unit vector that is orthogonal to the given two vectors:
   (i) $\vec{a} = i - 2j + 3k$, $\vec{b} = 2i + j - k$ (ii) $\vec{a} = 3i - j + 6k$, $\vec{b} = i + 4j + k$
4. If $\vec{a} = 3i - 6j + 5k$, $\vec{b} = 2i - j + 4k$, $\vec{c} = i + j - k$, compute
   (i) $\vec{a} \times \vec{b}$ (ii) $\vec{b} \times \vec{c}$ (iii) $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$
5. Use the vector product to compute the area of the triangle with the given vertices: (i) P: (–2,–3), Q: (3,2), R: (–1,–8)
   (ii) P: (–2, –1,3), Q: (1,2, –1), R: (4,3, –3)
6. A force $F = 3i - 2j + 5k$ acts on a particle at (1, –2, 2). Find the moment or torque of the force about (i) the origin; (ii) the point (1,2,1).
7. If $A + B + C = 0$, show that $A \times B = B \times C = C \times A$.

8. (i) Find a unit vector perpendicular to both $\vec{a} = i + j + 2k$, and $\vec{b} = -2i + j - 3k$
   (ii) Find a vector of magnitude 10 and perpendicular to both $\vec{a} = 2i - 3j + 4k, \vec{b} = 4i - 2j - 4k$.
9. Find the area of a parallelogram whose diagonals are:
   (i) $\vec{a} = 4i + j - 2k$ and $\vec{b} = -2i + 3j + 4k$
   (ii) $\vec{a} = 3i + 2j - 2k$ and $\vec{b} = i - 3j + 4k$

3.7 Scalar Triple Product of Vectors

3.7.1 Let *a, b* and *c* be three vectors. The **scalar triple product** of the vectors ***a, b*** and ***c*** is defined by $a\cdot(b \times c)$ or $(a \times b)\cdot c$

The use of parenthesis with $a \times b$ is not important, as the only other alternative given to the expression $a \times b\cdot c$, namely $a \times (b\cdot c)$ is meaningless. The scalar triple product $a\cdot b \times c$ is usually denoted by $[a\ b\ c]$.

3.7.2 Expression of Scalar Triple Product in Terms of Components

Let $a = x_1 i + y_1 j + z_1 k$, $b = x_2 i + y_2 j + z_2 k$ and $c = x_3 i + y_3 j + z_3 k$ be vectors, then

$$b \times c = \begin{vmatrix} i & j & k \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Rightarrow b \times c = (y_2 z_3 - y_3 z_2)i - (x_2 z_3 - x_3 z_2)j + (x_2 y_3 - x_3 y_2)k$$

$$\therefore \quad a\cdot(b \times c) = x_1(y_2 z_3 - y_3 z_2) - y_1(x_2 z_3 - x_3 z_2) + z_1(x_2 y_3 - x_3 y_2)$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Thus $a\cdot(b \times c) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

which is called the **determinantal form** for scalar triple product of vectors ***a, b*** and ***c***.

Theorem: For any vectors *a, b* and *c*, $a\cdot(b \times c) = b\cdot(c \times a) = c\cdot(a \times b)$

Proof: Let $a = x_1 i + y_1 j + z_1 k$, $b = x_2 i + y_2 j + z_2 k$ and $c = x_3 i + y_3 i + z_3 k$, then by determinantal form for scalar triple product of vectors ***a, b*** and ***c***, we have

$$a\cdot(b \times c) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad (1)$$

Similarly $b\cdot(c \times a) = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \end{vmatrix} = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \end{vmatrix}$

$$\Rightarrow \quad b\cdot(c \times a) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad (2)$$

and $c\cdot(a \times b) = \begin{vmatrix} x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \end{vmatrix}$

$$\Rightarrow \quad c\cdot(a \times b) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad (3)$$

From (1), (2) and (3), we have

$$a\cdot(b \times c) = b\cdot(c \times a) = c\cdot(a \times b)$$

By virtue of above theorem $[a\ b\ c] = [b\ c\ a] = [c\ a\ b]$

3.7.3. Scalar triple product of *i, j* and *k*

Theorem: Let *i, j* and *k* be the unit vectors. Prove that

(a) $i\cdot j \times k = j\cdot k \times i = k\cdot i \times j = 1$ and (b) $i\cdot k \times j = j\cdot i \times k = k\cdot j \times i = -1$

Proof: The proof is simple, so it is left for students.

3.7.4 Dot and cross are inter-changeable in scalar triple product

Theorem: The positions of dot and cross in the scalar triple product can be interchanged.

Proof: Let $a = x_1 i + y_1 j + z_1 k$, and $b = x_2 i + y_2 j + z_2 k$ and $c = x_3 i + y_3 i + z_3 k$ be any three vectors. Then

$$a\cdot(b \times c) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad (1)$$

By definition

$$a \times b = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1 z_2 - z_1 y_2)i - (x_1 z_2 - z_1 x_2)j + (x_1 y_2 - y_1 x_2)k$$

$$\therefore \quad (a \times b)\cdot c = (y_1 z_2 - z_1 y_2)x_3 - (x_1 z_2 - z_1 x_2)y_3 + (x_1 y_2 - y_1 x_2)z_3 \qquad (2)$$

$$= \begin{vmatrix} x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

From (1) and (2), we have $a\cdot(b \times c) = (a \times b)\cdot c$

This shows that the position of dot and cross in the scalar triple product can be interchanged.

3.7.5 (a) The Volume of the Parallelepiped

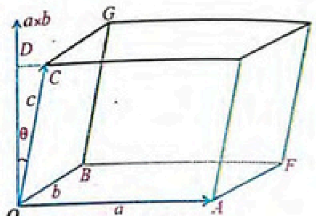


Figure 3.56

Let us consider the parallelepiped with a, b and c as co–terminal edges are shown in (Figure 3.56).

Then $a = \overrightarrow{OA}$, $b = \overrightarrow{OB}$, $c = \overrightarrow{OC}$

Let $a \times b = d$. Then by definition of cross product d is perpendicular to the plane containing a and b and geometrically represents the area of the parallelogram OAFB given by $|a \times b|$. The parallelogram is regarded as base for the parallelepiped. If θ is the angle between the vectors d and c, then $|\overrightarrow{OD}| = |c| \cos\theta$ being the projection of c on d represents the height of the parallelepiped. Then from elementary geometry, we know that the volume v of the parallelepiped is the area of the base multiplied by height.

Hence volume of parallelepiped = (Area of parallelogram) (Height)

$$\Rightarrow v = |a \times b|\,|c| \cos\theta$$

$$\Rightarrow v = (a \times b)\cdot c$$

The scalar triple product will be positive if θ is acute and c lies on the same side of the plane which contains a and b.

As $|b \times c|$ represents the area of the other side OCGB of parallelepiped, hence

$$v = a\cdot(b \times c)$$

Therefore $\quad v = a\cdot(b \times c) = (a \times b)\cdot c$

This shows that $a\cdot(b \times c)$ or $(a \times b)\cdot c$ is the volume of the parallelepiped with a, b, and c as the co–terminal edges.

3.7.5 (b) Volume of Tetrahedron A tetrahedron is determined by three edge vectors a, b, c as shown in (Figure 3.57).

The volume of a tetrahedron with a, b, c as its co–terminal edges is given by

$$v = \frac{1}{6}\,[a\ b\ c] = \frac{1}{6}\,\{a\cdot(b \times c)\}$$

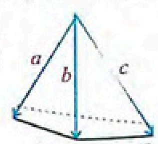


Figure 3.57

3.7.6 Properties of Scalar Triple Product

(i) $a\cdot b \times c$ being the volume of a parallelepiped with a, b, c as co–terminal edges, hence the evaluation of the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$ gives the volume of the parallelepiped as discussed earlier.

(ii) If two of the three vectors are equal, then the value of the scalar triple product is zero because for any two identical rows, the determinant vanishes.

(iii) $[a\ b\ c] = 0$ if and only if the three vectors a, b, c are coplanar.

Example 26: Find the volume of the parallelepiped determined by

$$a = 2i + 3k, \quad b = 6j - 2k, \quad \text{and} \quad c = -3i + 3j$$

Solution: Let v be the volume of the given parallelepiped.

Then $$V = a\cdot b \times c = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 6 & -2 \\ -3 & 3 & 0 \end{vmatrix} = 2(0+6) - 0(0-6) + 3(0+18)$$

$$= 12 - 0 + 54 = 66$$

Example 27: Find the volume of tetrahedron with a, b, c as adjacent edges where

$$a = i + 2k,\ b = 4i + 6j + 2k \ \text{ and } \ c = 3i + 3j - 6k$$

Solution: Let V be the volume of tetrahedron.

Then

$$V = \frac{1}{6}\begin{vmatrix} 1 & 0 & 2 \\ 4 & 6 & 2 \\ 3 & 3 & -6 \end{vmatrix} = \frac{1}{6}\,[(-36-6)-0(-24-6)+2(12-18)]$$

$$= \frac{1}{6}\,(-42-12) = \frac{-54}{6} = 9$$

We ignore the minus sign, because volume is always non–negative.

Did You Know

Two or more vectors are said to be coplanar if they lie in the same plane or parallel to the same plane otherwise non–coplanar. Non–coplanar vectors lie in three–dimensional space.

Example 29: Show that the points A(4, –2,1), B(5,1,6), C(2, 2, –5), D(3, 5, 0) are coplanar.

Solution:

Let $a = \overline{AB} = (5-4)i+(1+2)j+(6-1)k = i+3j+5k$

$b = \overline{BC} = (2-5)i+(2-1)j+(-5-6)k = -3i+j-11k$

$c = \overline{CD} = (3-2)i+(5-2)j+(0+5)k = i+3j+5k$

The four points are coplanar if the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ are coplanar. We have

$$[a\ b\ c] = \begin{vmatrix} 1 & 3 & 5 \\ -3 & 1 & -11 \\ 1 & 3 & 5 \end{vmatrix} = 1(5+33)-3(-15+11)-5(-9-1) = 38+12-5=0$$

Hence the four points are coplanar.

EXERCISE 3.5

1. Find $\bar{a}.(\bar{b}\times\bar{c})$, if $\bar{a}=2\hat{i}+\hat{j}+3\hat{k}$, $\bar{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\bar{c}=3\hat{i}+\hat{j}+2\hat{k}$
2. Find the volume of the parallelepiped whose edges are represented by $a=3i+j-k$, $b=2i-3j+k$, $c=i-3j-4k$
3. For the vectors $a=3i+2k$, $b=i+2j+k$, $c=-j+4k$ verify that $a\cdot b\times c=b\cdot c\times a=c\cdot a\times b$ but $a\cdot b\times c=-c\times b\cdot a$
4. Verify that the triple product of $i-j$, $j-k$, and $k-i$ is zero.
5. Let $a=a_1i+a_2j+a_3k$ and $b=b_1i+b_2j+b_3k$. Find $a\times b$ and prove that
   (i) $a\times b$ is orthogonal to both a and b (use dot product) (ii) Find $(a\times b)^2$
   (iii) Find $(a\cdot b)^2$, $|a|^2$, $|b|^2$
   (iv) Show that $|a\times b|^2=(a\cdot a)(b\cdot b)-(a\cdot b)^2$
6. Do the points (4,–2,1), (5, 1, 6), (2, 2, –5) and (3, 5, 0) lie in a plane?
7. For what values of c the following vectors are coplanar?
   (i) $u=i+2j+3k$, $v=2i-3j+4k$, $w=3i+j+ck$
   (ii) $u=i+j-k$, $v=i-2j+k$, $w=ci+j-ck$
   (iii) $u=i+j+2k$, $v=2i+3j+k$, $w=ci+2j+6k$
8. Find the volume of tetrahedron with the following
   (i) Vectors as coterminous edges $a=i+2j+3k$, $b=4i+5j+6k$, $c=7j+8k$
   (ii) Points A(2,3,1), B(–1,–2,0), C(0,2,–5), D(0, 1,–2) as vertices.
9. (i) Write the value of $(\hat{\imath}\times\hat{\jmath}).\hat{k}+\hat{\imath}.\hat{\jmath}$ (ii) Write the value of $(\hat{k}\times\hat{\jmath}).\hat{\imath}+\hat{\jmath}.\hat{k}$

REVIEW EXERCISE 3

1. Choose the correct option.

(i) The value of $\vec{i}\cdot(\vec{j}\times\vec{k})+\vec{j}\cdot(\vec{i}\times\vec{k})+\vec{k}\cdot(\vec{i}\times\vec{j})$
(a) 0 (b) –1 (c) 1 (d) 3

(ii) The vector $3\bar{i}+5\bar{j}+2\bar{k}$, $2\bar{i}-3\bar{j}-5\bar{k}$ and $5\bar{i}+2\bar{j}-3\bar{k}$ form the sides of a triangle which is
(a) Equilateral (b) isosceles, but not right-angled
(c) Right-angled, but not isosceles (d) right-angled and isosceles

(iii) The two vectors $\bar{a}=2\bar{i}+\bar{j}+3\bar{k}$, $\bar{b}=4\bar{i}-\lambda\bar{j}+6\bar{k}$ are parallel if $\lambda=$
(a) 2 (b) –3 (c) 3 (d) –2

(iv) If $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$, then
(a) $\vec{a}$ is parallel to $\vec{b}$ (b) $\vec{a}\perp\vec{b}$ (c) $|\vec{a}|=|\vec{b}|$ (d) None of these

(v) The projection of the vector $2\bar{i}+3\bar{j}-2\bar{k}$ on the vector $\bar{i}+2\bar{j}+3\bar{k}$ is
(a) $\frac{1}{\sqrt{14}}$ (b) $\frac{2}{\sqrt{14}}$ (c) $\frac{3}{\sqrt{14}}$ (d) None of these

(vi) Find non-zero scalars α, β for which $\alpha(\vec{a}+2\vec{b})-\beta\vec{a}+(4\vec{b}-\vec{a})=0$ for all vectors $\vec{a}$ and $\vec{b}$.
(a) $\alpha=-2,\beta=-3$ (b) $\alpha=2,\ \beta=-3$
(c) $\alpha=1,\ \beta=-3$ (d) $\alpha=-2,\ \beta=3$

(vii) If a, b, c are position vectors of the vertices of a Δ ABC, then $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=$
(a) 0 (b) 2a (c) 2b (d) 3c

(viii) If θ be the angle between any two vector $\vec{a}$ and $\vec{b}$, then $|\vec{a}\cdot\vec{b}|=|\vec{a}\times\vec{b}|$, when θ is equal to
(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

2. Find λ and μ if $(\bar{i}+3\bar{j}+9\bar{k})\times(3\bar{i}-\lambda\bar{j}+\mu\bar{k})=\bar{0}$.
3. If $\bar{a}=9\bar{i}-\bar{j}+\bar{k}$ and $\bar{b}=2\bar{i}-2\bar{j}-\bar{k}$, then find a unit vector parallel to the vector $\bar{a}+\bar{b}$.
4. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, find $(\vec{r}\times\hat{i}).(\vec{r}\times\hat{j})+xy$.
5. If $\vec{a}=7\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$.
6. Find λ, if the vectors $\vec{a}=\hat{i}+3\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}-\hat{k}$ and $\vec{c}=\lambda\hat{j}+3\hat{k}$ are coplanar.
7. Vector $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=\sqrt{3}$, $|\vec{b}|=\frac{2}{3}$ and $(\vec{a}\times\vec{b})$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$.
8. Find the area of a triangle whose vertices are (0,0,2), (–1,3,2), (1,0,4).
9. Find the area of the parallelogram with vertices A(1,2,–3), B(5,8,1), C (4,–2,2), D(0,–8,–2).
10. Prove that in any triangle ABC
    (i) $a^2=b^2+c^2-2bc\cos A$ (Cosine Law)
    (ii) $a=b\cos C+c\cos B$ (Projection Law)