UNIT 2

MATRICES AND DETERMINANTS

$$\begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \times \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} A\times G + B\times H \\ C\times G + D\times H \\ E\times G + F\times H \end{bmatrix}$$

STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Recall the concept of
  - a matrix and its notation,
  - order of a matrix,
  - Equality of two matrices.
- Define row matrix, column matrix, square matrix, rectangular matrix, zero/null matrix, identity matrix, scalar matrix, diagonal matrix, upper and lower triangular matrix, transpose of a matrix, symmetric matrix and skew-symmetric matrix.
- Carryout scalar multiplication, addition/subtraction of matrices, multiplication of matrices with real and complex entries.
- Show that commutative property
  - holds under addition.
  - does not hold under multiplication, in general.
- Verify that $(AB)^t = B^t A^t$
- Describe determinant of a square matrix, minor and cofactor of an element of a matrix.
- Evaluate determinant of a square matrix using cofactors.
- Define singular and non-singular matrices.
- Know the adjoint of a square matrix.
- Use adjoint method to calculate inverse of a square matrix.
- Verify the result $(AB)^{-1} = B^{-1}A^{-1}$.
- State and prove the properties of determinants.
- Evaluate the determinant without expansion (i.e. using properties of determinants).
- Know the row and column operations on matrices.
- Define echelon and reduced echelon form of a matrix.
- Reduce a matrix to its echelon and reduced echelon form.
- Recognize the rank of a matrix.
- Use row operations to find the inverse and the rank of a matrix.
- Distinguish between homogeneous and non-homogeneous linear equations in 2 and 3 unknowns.
- Solve a system of three homogeneous linear equations in three unknowns.

- Define a consistent and inconsistent system of linear equations and demonstrate through examples.
- Solve a system of 3 by 3 non-homogeneous linear equations using:
  - matrix inversion method,
  - Gauss elimination method (echelon form),
  - Gauss-Jordan method (reduced echelon form),
  - Cramer's rule.

2.1 Introduction

The concept of matrices is a highly useful tool which is not only used in mathematics but also in all branches of science, engineering and the business world. Now-a-days matrices and matrix methods have widespread applications in the operation of high speed computers.

2.1.1 (a) Concept of a matrix and its notation

In previous class we have taken a simple example for the concept of a matrix. Here we take a bit more tricky example.

Suppose three colleges A,B,C take part in an inter-colleges debate competition, where any participant can speak in either of the four languages English, Urdu, Pashto or Hindko. College A consists of 3 participants in English, 2 in Urdu, 3 in Pashto and 1 in Hindko, College B consists of 2 participants in English, 3 in Urdu, 1 in Pashto and 2 in Hindko, College C consists of 4 participants in English, 2 in Urdu, 2 in Pashto and 1 in Hindko.

The information given in the above example, can be put in a compact way in a tabular form as follows:

Name of the School	Number of speakers (language wise)			
	English	Urdu	Pashto	Hindko
A	3	2	3	1
B	2	3	1	2
C	4	2	2	1

Now we write the data given in the above arrangement in a capital or small brackets without any top or left heading as shown.

$$\begin{bmatrix} 3 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{bmatrix} \quad or \quad \begin{pmatrix} 3 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{pmatrix}$$

This array of numbers gives all the information needed which we call a matrix. Thus a **matrix** is a rectangular array of numbers enclosed in large square brackets or parenthesis. Unless otherwise specified, all numbers in a matrix array will be real.

For example,

$$\text{rows}\begin{matrix}\rightarrow\\\rightarrow\\\rightarrow\end{matrix}\begin{bmatrix}3 & 2 & 3 & 1\\2 & 3 & 1 & 2\\4 & 2 & 2 & 1\end{bmatrix} \quad or \quad \text{rows}\begin{matrix}\rightarrow\\\rightarrow\\\rightarrow\end{matrix}\begin{pmatrix}3 & 2 & 3 & 1\\2 & 3 & 1 & 2\\4 & 2 & 2 & 1\end{pmatrix}$$

$$\begin{matrix}\uparrow & \uparrow & \uparrow & \uparrow\\ & \text{columns} & & \end{matrix} \qquad \begin{matrix}\uparrow & \uparrow & \uparrow & \uparrow\\ & \text{columns} & & \end{matrix}$$

represents matrix. However, throughout we will use square brackets to denote matrices.

In the above matrix the horizontal lines of numbers are called rows and the vertical lines of numbers are called **columns**. Each number in the array is called an **element** or an **entry** of the matrix.

The above matrix has three rows and four columns.

We are now ready to give the general definition of a matrix as follows:

A matrix is a rectangular array of mn elements a_{ij}; $i = 1,2,3,\ldots,m$ $j=1,2,\ldots,n$ arranged in m rows and n columns. In writing down matrices, it is usual to denote the matrix by a capital single letter A (say) such that

$$A=\begin{bmatrix}a_{11} & a_{12}\ldots a_{1n}\\a_{21} & a_{22}\ldots a_{2n}\\\vdots & \vdots \quad \vdots\\a_{m1} & a_{m2}\ldots a_{mn}\end{bmatrix}$$

(b) Order of a matrix

The **order** of a matrix is given by the number of rows followed by the number of columns, if the matrix A has m rows and n columns, and so is said to be of order $m \times n$ (read as m by n matrix).

For simplicity and to convey the idea, the matrix A is an $m \times n$ matrix, unless otherwise specified.

In the matrix A, the ith row and the jth column are represented as follows:

$$A=\text{ith row}\rightarrow\begin{bmatrix}a_{11} & a_{12}\cdots & a_{1j} & \cdots a_{1n}\\a_{21} & a_{22}\cdots & a_{2j} & \cdots a_{2n}\\\vdots & & & \\a_{i1} & a_{i2}\cdots & a_{ij} & \cdots a_{in}\\\vdots & & & \\a_{m1} & a_{m2}\cdots & a_{mj} & \cdots a_{mn}\end{bmatrix}\quad (a_{1j} \text{ in the } j\text{th column}\downarrow)$$

The elements of the ith row of A are $a_{i1}, a_{i2},\ldots,a_{ij},\ldots,a_{in}$ and the elements of the jth column of A are $a_{1j}, a_{2j},\ldots,a_{ij},\ldots,a_{mj}$. We see that the element a_{ij} occurs in the ith row and jth column of A. The elements in the ith row and jth column will usually be referred to as the (i,j)th element because of the two subscripts i and j.

We may also write the matrix A as

$A=[a_{ij}]_{m\times n}$ or $A=[a_{ij}]$; $i=1,2,\ldots,m$; $j=1,2,\ldots,n$, where a_{ij} is the (i,j)th elements of A.

Did You Know

A matrix is merely a table of numbers. Apart from being a convenient way of recording certain types of numerical values, it has no particular value in itself.

(c) Equality of two matrices

Two matrices $A=[a_{ij}]$ and $B = [b_{ij}]$ of the same order are said to be equal when their corresponding elements are equal i.e. $a_{ij} = b_{ij}$ for all i and j where $i=1,2,\ldots,m$; $j=1,2,\ldots,n$

For example, if

$$A=\begin{bmatrix}p & q\\r & s\end{bmatrix} \text{ and } B=\begin{bmatrix}p & q\\r & s\end{bmatrix} \text{ then } A = B.$$

2.1.2 Types of Matrices

(a) Row Matrix or Row Vector

A matrix with only one row i.e. a 1xn matrix of the form $[a_{11}\ a_{12}\ldots\ a_{1n}]$ is called row matrix or a row vector. For example, $[-1\ \ -2\ \ -3]$ is a row matrix having three columns.

(b) Column matrix or Column vector

A matrix with only one column i.e. an $m \times 1$ matrix of the form $\begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix}$ is called a column matrix or a column vector.

For example, $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is a column matrix having four rows.

(c) Square matrix

If the number of rows and columns in a matrix are equal i.e. if m=n, then the matrix of order m×n is called a square matrix of order n or m.

For example, $A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}$

is a square matrix of order n and [a], $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 5 \\ 3 & 6 & 2 \end{bmatrix}$ are square matrices of order 1, 2 and 3 respectively.

The diagonal of the square matrix **A** containing the elements $a_{11}, a_{22}, \ldots, a_{nn}$ is called the **principal diagonal** of A. It is also termed as the **leading diagonal** or **main diagonal** of the matrix **A**.

(d) Rectangular matrix

If the number of rows and columns in a matrix A are not equal, i.e. if $m \neq n$, the matrix is called a rectangular matrix of order m × n.

For example, $\begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -1 & 2 \\ 3 & 5 & -4 & 3 \end{bmatrix}$

are rectangular matrices of order 2×3 and 3×4 respectively.

(e) Diagonal Matrix

A square matrix is called a diagonal matrix if all its non-diagonal elements are zero.

Thus, the square matrix $[a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.

For example, [2], $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

are diagonal matrices.

(f) Scalar matrix

A square matrix is called a scalar matrix, if its non-diagonal elements are zero and diagonal elements are equal.

Thus, the square matrix $[a_{ij}]$ is a scalar matrix if

$$a_{ij} = \begin{cases} k & for\, i = j \\ 0 & for\, i \neq j \end{cases}$$

For example $\begin{bmatrix} k & 0 & 0 & \ldots & 0 \\ 0 & k & 0 & \ldots & 0 \\ 0 & 0 & k & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & k \end{bmatrix}$ is a general scalar matrix of order n.

$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are scalar matrices of order 2 and 3 respectively.

(g) Unit matrix or Identity matrix

A square matrix is called a unit matrix if its non-diagonal elements are zero and diagonal elements are all equal to one (unity).

Thus, the square matrix $[a_{ij}]$ is a unit matrix if

$$a_{ij} = \begin{cases} 1 & for\ i = j \\ 0 & for\, i \neq j \end{cases}$$

Such a matrix is denoted by

$$I_n = \begin{bmatrix} 1 & 0 \ldots\ldots & 0 \\ 0 & 1 \ldots\ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \ldots\ldots & 1 \end{bmatrix},$$

We have unit matrices of different order such as

$$I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad I_4=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\text{ and so on.}$$

(h) Zero matrix or Null matrix

A matrix all whose elements are zero is called a zero matrix or null matrix. If it has m rows and n columns, we denote it by $O_{m\times n}$ or simply by O if there is no ambiguity about its number of rows and number of columns.

Following are some examples of zero or null matrices:

$$[0]\ ,\ [0\ \ 0\ \ 0]\ ,\ \begin{bmatrix}0\\0\\0\end{bmatrix},\ \begin{bmatrix}0&0\\0&0\end{bmatrix}\ \ and\ \ \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}.$$

(i) Transpose of a matrix

Let $A=[a_{ij}]$ be an m×n matrix. The **transpose** of A denoted by A^t, is an n×m matrix obtained by interchanging rows and columns of A. Thus $A^t=[b_{ij}]$ where $b_{ij}=a_{ji}$ for $i=1,2,\ldots,n;\ j=1,2,\ldots,m$.

For example, if A is a 3×2 matrix given by $A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\end{bmatrix}$,

then its transpose A^t is a 2×3 matrix

$$A^t=\begin{bmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\end{bmatrix}.$$

(j) Upper triangular matrix

A square matrix $A=[a_{ij}]_{m\times n}$ is said to be upper triangular matrix, if all the elements below the principal diagonal are zero that is $a_{ij}=0$ for all $i>j$.

For example, $\begin{bmatrix}2&3&1\\0&4&-2\\0&0&1\end{bmatrix}$ and $\begin{bmatrix}1&2&3&4\\0&-1&2&0\\0&0&6&3\\0&0&0&1\end{bmatrix}$ are upper triangular matrices.

(k) Lower triangular matrix

A square matrix $A=[a_{ij}]_{m\times n}$ is said to be lower triangular matrix, if all the elements above the principal diagonal are zero, that is $a_{ij}=0$ for all $i<j$.

For example, $\begin{bmatrix}2&0&0\\-4&5&0\\2&3&1\end{bmatrix}$ and $\begin{bmatrix}1&0&0&0\\2&1&0&0\\4&-2&3&0\\1&0&3&2\end{bmatrix}$ are lower triangular matrices.

(l) Triangular matrix

A square matrix A is called a triangular matrix, if it is either upper triangular or lower triangular.

Note

- It is obvious that diagonal matrices are both upper triangular and lower triangular.
- If A is triangular, then $|A|$ = product of diagonal elements.

For example,

$\begin{bmatrix}2&-3&4\\0&4&-3\\0&0&1\end{bmatrix}$ and $\begin{bmatrix}1&0&0&0\\2&0&0&0\\4&-2&3&0\\1&0&3&2\end{bmatrix}$ are triangular matrices.

The first matrix is upper triangular while the second is lower triangular.

(m) Symmetric matrix

A square matrix $A=[a_{ij}]$ of order n is said to be symmetric if $A^t=A$, that is, if $a_{ij}=a_{ji}$ for $i,j=1,2,\ldots,n$.

For example, the matrix $A=\begin{bmatrix}2&3&6\\3&1&-5\\6&-5&4\end{bmatrix}$ is symmetric, since $A^t=\begin{bmatrix}2&3&6\\3&1&-5\\6&-5&4\end{bmatrix}=A$

(n) Skew symmetric matrix

A square matrix $A=[a_{ij}]$ of order n is said to be skew symmetric (or anti symmetric), if $A^t=-A$, that is, if $a_{ij}=-a_{ji}$ for $i,j=1,2,\ldots,n$.

For elements on the principal diagonal, we have

$a_{ii}=-a_{ii}\Rightarrow 2a_{ii}=0\Rightarrow a_{ii}=0$ for $i=1,2,\ldots,n$.

Thus the elements on principal diagonal of skew symmetric matrix are zero.

For example, the matrix $A=\begin{bmatrix}0&2&3\\-2&0&-4\\-3&4&0\end{bmatrix}$ is skew symmetric,

$$\text{since } A^t = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} = -A$$

2.2 Algebra of matrices

In this section various operations of addition, subtraction, multiplication etc on matrices are defined.

2.2.1. (a) Addition of matrices

If $A=[a_{ij}]$ and $B=[b_{ij}]$ are two matrices of the same order m×n, then their sum A+B is defined as a matrix $C=[c_{ij}]$ of the same order as A and B and whose elements are obtained by adding the corresponding elements of A and B together.

Symbolically, we write C=A+B whose elements $c_{ij}=a_{ij}+b_{ij}$ for i=1,2,…,m; j=1,2,…,n.

Did You Know ?

- If the sum of two matrices is defined, we say that the two matrices are conformable for addition.
- The sum of two matrices of different order is not defined that is, they are not conformable for addition.

For example, if $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ *and* $B=\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$, *then*

$$C=A+B=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}+\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}=\begin{bmatrix} 1+3 & 2+4 & 3+5 \\ 0+1 & -1+2 & 2+3 \end{bmatrix}=\begin{bmatrix} 4 & 6 & 8 \\ 1 & 1 & 5 \end{bmatrix}$$

(b) Subtraction of matrices

If $A=[a_{ij}]$ and $B=[b_{ij}]$ are matrices of the same order m × n, then subtraction of matrices A and B is obtained by subtracting the corresponding elements of A and B respectively. The difference of A and B (or the subtraction of B from A) is a matrix D=A–B whose elements are $d_{ij}=a_{ij}-b_{ij}$; *i*= 1,2,……,m; *j*= 1,2,……,n.

If $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ then,

$$D=A-B=A+(-B) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}+\begin{bmatrix} -3 & -4 & -5 \\ -1 & -2 & -3 \end{bmatrix}$$

$$=\begin{bmatrix} 1-3 & 2-4 & 3-5 \\ 0-1 & -1-2 & 2-3 \end{bmatrix}=\begin{bmatrix} -2 & -2 & -2 \\ -1 & -3 & -1 \end{bmatrix}$$

(c) Scalar multiplication

If $A=[a_{ij}]$ is a matrix of order m×n and k is any scalar, then the scalar multiplication kA of the scalar k and matrix A is defined as a matrix each of whose element is the product of k and the corresponding elements of A i.e.

$kA = k[a_{ij}]=[ka_{ij}]$; *i*=1,2,……,m; *j*= 1,2,……,n.

For example, if $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

and k is any scalar, then

$$kA = k\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} k & 2k \\ 3k & 4k \end{bmatrix}.$$

Note !

- Clearly kA is a matrix of the same order as the given matrix A.
- A + A = 2A, A + A + A = 3A and in general, if n is a positive integer, then $\underbrace{A+A+\ldots+A}_{n-times} = nA$

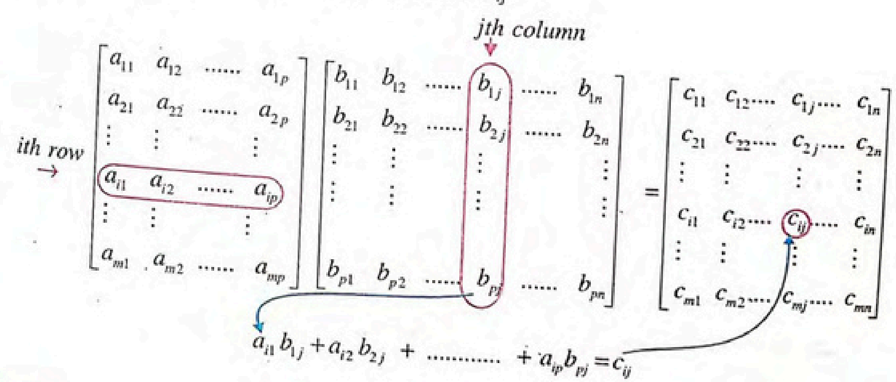
(d) Multiplication of matrices

Two matrices A and B are said to be conformable for multiplication giving the product AB, if the number of columns in A is equal to the number of rows in **B**.

Suppose $A=[a_{ij}]$ is matrix of order m×p and $B=[b_{ij}]$ is a matrix of order p×n. Then their product AB is a matrix $C=[c_{ij}]$ of order m×n with elements c_{ij} defined as the sum of the product of the corresponding elements of the ith row of A and the jth column of **B** i.e.

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots\ldots+a_{ip}b_{pj}=\sum_{k=1}^{p}a_{ik}b_{kj}$$

The following illustrates the expression for c_{ij}



For example if $A=\begin{bmatrix}3&1&2\\2&1&3\end{bmatrix}$ and $B=\begin{bmatrix}1&2\\3&1\\2&3\end{bmatrix}$ are two matrices of order 2×3 and 3×2 respectively. Then the product C = AB is 2×2 matrix defined by

$$C=AB=\begin{bmatrix}3&1&2\\2&1&3\end{bmatrix}\begin{bmatrix}1&2\\3&1\\2&3\end{bmatrix}=\begin{bmatrix}3\times1+1\times3+2\times2&3\times2+1\times1+2\times3\\2\times1+1\times3+3\times2&2\times2+1\times1+3\times3\end{bmatrix}=\begin{bmatrix}10&13\\11&14\end{bmatrix}$$

The matrices A and B are also conformable for the product D=BA defined as

$$D=BA=\begin{bmatrix}1&2\\3&1\\2&3\end{bmatrix}\begin{bmatrix}3&1&2\\2&1&3\end{bmatrix}=\begin{bmatrix}7&3&8\\11&4&9\\12&5&13\end{bmatrix}$$

C and D are matrices of order 2×2 and 3×3 respectively.

2.2.2 Commutative property

Let $A=\begin{bmatrix}1&-2&3\\3&2&-1\end{bmatrix}$ and $B=\begin{bmatrix}2&3\\-1&2\\4&-5\end{bmatrix}$. Find AB and BA and show that AB ≠ BA.

Here, A is a 2x3 matrix and B is a 3x2 matrix. So, AB exists and it is of order 2×2

We have, $AB=\begin{bmatrix}1&-2&3\\3&2&-1\end{bmatrix}\begin{bmatrix}2&3\\-1&2\\4&-5\end{bmatrix}$

$$=\begin{bmatrix}2+2+12&3-4-15\\6-2-4&9+4+5\end{bmatrix}=\begin{bmatrix}16&-16\\0&18\end{bmatrix}$$

Again, B is a 3x2 matrix and A is a 2x3 matrix. So BA exists and it is of order 3x3

Now, $BA=\begin{bmatrix}2&3\\-1&2\\4&-5\end{bmatrix}\begin{bmatrix}1&-2&3\\3&2&-1\end{bmatrix}$

$$\Rightarrow BA=\begin{bmatrix}2+9&-4+6&6-3\\-1+6&2+4&-3-2\\4-15&-8-10&12+5\end{bmatrix}=\begin{bmatrix}11&2&3\\5&6&-5\\-11&-18&17\end{bmatrix}$$

Clearly, AB ≠ BA

However, commutative property w.r.t. addition clearly holds if both matrices are conformable for addition and is explained below:

Commutative property w.r.t. addition, i.e., A + B = B + A.

Let $A=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$ and $B=\begin{bmatrix}j&k&l\\m&n&o\\p&q&r\end{bmatrix}$ be two 3 x 3 square matrices.

$$\text{Then } A+B=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}+\begin{bmatrix}j&k&l\\m&n&o\\p&q&r\end{bmatrix}=\begin{bmatrix}a+j&b+k&c+l\\d+m&e+n&f+o\\g+p&h+q&i+r\end{bmatrix} \quad (1)$$

$$\text{and } B+A=\begin{bmatrix}j&k&l\\m&n&o\\p&q&r\end{bmatrix}+\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$$

$$=\begin{bmatrix}j+a&k+b&l+c\\m+d&n+e&o+f\\p+g&q+h&r+i\end{bmatrix}=\begin{bmatrix}a+j&b+k&c+l\\d+m&e+n&f+o\\g+p&h+q&i+r\end{bmatrix} \quad (2)$$

Since addition is commutative in $\mathbb{R}$. From (1) and (2), we have A + B = B + A

Example 1: If $A=\begin{bmatrix}3&2\\4&-1\\6&1\end{bmatrix}$ and $B=\begin{bmatrix}2&5\\-1&4\\0&3\end{bmatrix}$, then show that $(A+B)^t=A^t+B^t$.

Solution: Since

$$A+B=\begin{bmatrix}3&2\\4&-1\\6&1\end{bmatrix}+\begin{bmatrix}2&5\\-1&4\\0&3\end{bmatrix}=\begin{bmatrix}3+2&2+5\\4-1&-1+4\\6+0&1+3\end{bmatrix}=\begin{bmatrix}5&7\\3&3\\6&4\end{bmatrix}, \text{So } (A+B)^t=\begin{bmatrix}5&3&6\\7&3&4\end{bmatrix} \quad (1)$$

Now $A^t=\begin{bmatrix}3&4&6\\2&-1&1\end{bmatrix}$, $B^t=\begin{bmatrix}2&-1&0\\5&4&3\end{bmatrix}$,

$$\therefore\ A^t + B^t = \begin{bmatrix}3 & 4 & 6\\ 2 & -1 & 1\end{bmatrix} + \begin{bmatrix}2 & -1 & 0\\ 5 & 4 & 3\end{bmatrix} = \begin{bmatrix}3+2 & 4-1 & 6+0\\ 2+5 & -1+4 & 1+3\end{bmatrix} = \begin{bmatrix}5 & 3 & 6\\ 7 & 3 & 4\end{bmatrix} \quad (2)$$

From (1) and (2), we have $(A+B)^t = A^t + B^t$.

2.2.3 Verification of $(AB)^t = B^t A^t$

Example 2: *If* $A = \begin{bmatrix}-1\\ 2\\ 3\end{bmatrix}$ and $B = \begin{bmatrix}-2 & -1 & -4\end{bmatrix}$, verify $(AB)^t = B^t A^t$

Solution: $A = \begin{bmatrix}-1\\ 2\\ 3\end{bmatrix}$ and $B = \begin{bmatrix}-2 & -1 & -4\end{bmatrix}$

$$\therefore AB = \begin{bmatrix}-1\\ 2\\ 3\end{bmatrix}\begin{bmatrix}-2 & -1 & -4\end{bmatrix} = \begin{bmatrix}2 & 1 & 4\\ -4 & -2 & -8\\ -6 & -3 & -12\end{bmatrix}$$

$$\Rightarrow (AB)^t = \begin{bmatrix}2 & -4 & -6\\ 1 & -2 & -3\\ 4 & -8 & -12\end{bmatrix} \quad \text{(i)}$$

$$\text{Also, } B^t A^t = \begin{bmatrix}-2 & -1 & -4\end{bmatrix}^t \begin{bmatrix}-1\\ 2\\ 3\end{bmatrix}^t = \begin{bmatrix}-2\\ -1\\ -4\end{bmatrix}\begin{bmatrix}-1 & 2 & 3\end{bmatrix} = \begin{bmatrix}2 & -4 & -6\\ 1 & -2 & -3\\ 4 & -8 & -12\end{bmatrix} \quad \text{(ii)}$$

From (i) and (ii), we observe that $(AB)^t = B^t A^t$

EXERCISE 2.1

1. Express the following as a single matrix.

(i) $\begin{bmatrix}1 & 2 & 4\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 2 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}2\\ 4\\ 6\end{bmatrix}$ (ii) $\begin{bmatrix}1 & -2 & 3\end{bmatrix}\begin{bmatrix}2 & -1 & 5\\ 0 & 2 & 4\\ -7 & 5 & 0\end{bmatrix} - \begin{bmatrix}2 & -5 & 7\end{bmatrix}$

(iii) $\begin{bmatrix}7 & 1 & 2\\ 9 & 2 & 1\end{bmatrix}\begin{bmatrix}3\\ 4\\ 5\end{bmatrix} + 2\begin{bmatrix}4\\ 2\end{bmatrix}$ (iv) $\left\{\begin{bmatrix}1 & 3\\ -1 & -4\end{bmatrix} + \begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}\right\}\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}$

2. Let $A = \begin{bmatrix}2 & -5 & 1\\ 3 & 0 & -4\end{bmatrix}$, $B = \begin{bmatrix}1 & -2 & -3\\ 0 & -1 & 5\end{bmatrix}$ and $C = \begin{bmatrix}0 & 1 & -2\\ 0 & -1 & -1\end{bmatrix}$.

Find $2A + 3B - 4C$.

3. (i) if $A = \begin{bmatrix}x & y & z\end{bmatrix}$, $B = \begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}$ and $C = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$, verify that $(AB)C = A(BC)$

(ii) If $A = \begin{bmatrix}1 & 3\\ -1 & 4\end{bmatrix}$, $B = \begin{bmatrix}2 & 1 & -3\\ 0 & 4 & 2\end{bmatrix}$ and $C = \begin{bmatrix}3 & -1 & 5\\ 2 & 1 & 0\end{bmatrix}$, verify that:

(a) $A(B+C) = AB + AC$ (b) $A(B-C) = AB - AC$

4. Let $A = \begin{bmatrix}1 & 4 & 4\\ 4 & 1 & 4\\ 4 & 4 & 1\end{bmatrix}$, show that $\frac{1}{3}A^2 - 2A - 9I = O$.

5. Matrix $A = \begin{bmatrix}0 & 2b & -2\\ 3 & 1 & 3\\ 3a & 3 & -1\end{bmatrix}$ is given to be symmetric, find values of a and b.

6. Solve the following matrix equations for X.

(i) $X - 3A = 2B$, if $A = \begin{bmatrix}1 & 0 & 3\\ -2 & 2 & 1\end{bmatrix}$ and $B = \begin{bmatrix}2 & 1 & 1\\ 3 & -1 & 4\end{bmatrix}$

(ii) $2(X - A) = B$, if $A = \begin{bmatrix}1 & 2 & 2\\ 3 & -1 & 2\end{bmatrix}$, $B = \begin{bmatrix}4 & 6 & 2\\ 0 & -4 & 2\end{bmatrix}$

7. *If* $A = \begin{bmatrix}1 & 0 & -1 & 2\\ 3 & 1 & 2 & 5\\ 0 & -2 & 1 & 6\end{bmatrix}$ and $B = \begin{bmatrix}2 & -1 & 3 & 1\\ 1 & 3 & -1 & 4\\ 3 & 1 & 2 & -1\end{bmatrix}$,

then show that $(A+B)^t = A^t + B^t$.

8. Let $A=\begin{bmatrix}1 & 2 & 0\\ 3 & -1 & 4\end{bmatrix}$. Show that

(i) $(A^t)^t = A$ (ii) $AA^t \neq A^tA$.

9. Verify that $(AB)^t = B^t A^t$ if

(i) $A=\begin{bmatrix}2 & -1 & 3\\ 1 & 0 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 2\\ 2 & 2\\ 3 & 0\end{bmatrix}$ (ii) $A=\begin{bmatrix}1 & 2 & 0\\ -1 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\ 2 & 3\\ 1 & -2\end{bmatrix}$

10. Let $A=\begin{bmatrix}1 & -3 & 4\\ -3 & 2 & -5\\ 4 & -5 & 0\end{bmatrix}$ and $B=\begin{bmatrix}5 & 6 & 7\\ 6 & -8 & 3\\ 7 & 3 & 1\end{bmatrix}$

Verify that A and B are symmetric. Also verify that A + B is symmetric.

11. Let $A=\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 3\\ 2 & -3 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & -6 & 11\\ 6 & 0 & -7\\ -11 & 7 & 0\end{bmatrix}$

Verify that A + B is skew-symmetric.

12. If $A=\begin{bmatrix}3 & 2 & 1\\ 4 & 5 & 6\\ -2 & 3 & 4\end{bmatrix}$, then verify that

(i) $A + A^t$ is symmetric (ii) $A - A^t$ is skew-symmetric.

13. If A is a square matrix of order 3, then show that:

(i) $A + A^t$ is symmetric (ii) $A - A^t$ is skew-symmetric.

2.3. Determinants

Consider a square matrix A of order n given by

$$A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix} \quad (1)$$

The associated determinant of A is denoted by

$$|A|=\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix} \quad (2)$$

Some determinants of higher order can be evaluated only after much tedious calculations. The more calculation is involved, the greater the chance of error. Our aim in this section is to describe a procedure for evaluating the determinants of order $n \geq 3$. However, this procedure will be greatly simplified by the introduction of the following.

2.3.1. Minor and Cofactor of an element of a matrix or its determinants

(i) **Minor of an Element** Let A be a square matrix of order n (as defined in (1) above). The minor of the element a_{ij} of A, denoted by M_{ij}, is the determinant of $(n-1)\times(n-1)$ matrix obtained by crossing out the ith row and jth column of A (or |A|).

If $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$, then

minor of $a_{11} = M_{11} = \begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}$ obtained as $\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$,

minor of $a_{23} = M_{23} = \begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix}$ obtained as $\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$ and so on.

Remember

From the formula $A_{ij} = (-1)^{i+j} M_{ij}$ it is clear that if the sum i+j is an even integer, then the cofactor equals the minor. On the other hand, if the sum i+j is odd, the cofactor is equal to the negative of the minor. The signs accompanying the minors may be best remembered by the rule of alternating signs with +'s on the main diagonals.

$$\begin{bmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{bmatrix}$$

Example 3: Let $A=\begin{bmatrix}1&2&3\\6&5&4\\7&8&9\end{bmatrix}$. Find the minors M_{11}, M_{12}, M_{13} and M_{22} of the matrix A.

Solution: We have

$$M_{11}=\begin{vmatrix}5&4\\8&9\end{vmatrix}=45-32=13, M_{12}=\begin{vmatrix}6&4\\7&9\end{vmatrix}=54-28=26,$$

$$M_{13}=\begin{vmatrix}6&5\\7&8\end{vmatrix}=48-35=13, M_{22}=\begin{vmatrix}1&3\\7&9\end{vmatrix}=9-21=-12.$$

(ii) Cofactor of an element

Let A be a square matrix of order n. The cofactor of the element a_{ij}, denoted by A_{ij}, is defined by $A_{ij}=(-1)^{i+j}M_{ij}$, where M_{ij} is the minor of a_{ij}.

Thus if $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$, then

$$\text{cofactor of } a_{11}=A_{11}=(-1)^{1+1}M_{11}=(-1)^2\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}$$
$$=1\times(a_{22}a_{33}-a_{23}a_{32})$$
$$=a_{22}a_{33}-a_{23}a_{32}$$

$$\text{cofactor of } a_{23}=A_{23}=(-1)^{2+3}M_{23}=(-1)^5\begin{vmatrix}a_{11}&a_{12}\\a_{31}&a_{32}\end{vmatrix}$$
$$=-1\times(a_{11}a_{32}-a_{12}a_{31})$$
$$=-(a_{11}a_{32}-a_{12}a_{31}) \text{ and so on.}$$

Example 4: Let $A=\begin{bmatrix}1&-2&5\\3&0&-1\\5&2&0\end{bmatrix}$. Find the cofactor A_{13} and A_{21}.

Solution: We have $A_{13}=(-1)^{1+3}M_{13}=(-1)^4\begin{vmatrix}3&0\\5&2\end{vmatrix}=1\times(3\times2-0\times5)=6,$

and $A_{21}=(-1)^{2+1}M_{21}=(-1)^3\begin{vmatrix}-2&5\\2&0\end{vmatrix}=-1\times(-2\times0-5\times2)=10.$

2.3.2 Determinant of a square matrix of order $n\geq3$

Let A be a square matrix of order $n(\geq3)$ given by

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots\cdots&a_{1j}&\cdots\cdots&a_{1n}\\a_{21}&a_{22}&\cdots\cdots&a_{2j}&\cdots\cdots&a_{2n}\\\vdots&\vdots&&\vdots&&\vdots\\a_{i1}&a_{i2}&\cdots\cdots&a_{ij}&\cdots\cdots&a_{in}\\\vdots&\vdots&&\vdots&&\vdots\\a_{n1}&a_{n2}&\cdots\cdots&a_{nj}&\cdots\cdots&a_{nn}\end{bmatrix} \quad (1)$$

The determinant |A| of the matrix A is defined to be the sum of the products of each element of row (or column) and its cofactor, that is

$$|A|=a_{i1}A_{i1}+a_{i2}A_{i2}+\ldots\ldots+a_{in}A_{in};\ i=1,2\ldots\ldots,n \quad (2)$$

or $$|A|=a_{1j}A_{1j}+a_{2j}A_{2j}+\ldots\ldots+a_{nj}A_{nj};\ j=1,2\ldots\ldots,n \quad (3)$$

If we put i=1 in (2), we get

$|A|=a_{11}A_{11}+a_{12}A_{12}+\ldots\ldots+a_{1n}A_{1n}$. This is called the expansion of |A| by first row (or w.r.t. first row).

Similarly, if we put j=1 in (3), we get

$|A|=a_{11}A_{11}+a_{21}A_{21}+\ldots\ldots+a_{n1}A_{n1}$. This is called the expansion of |A| by first column and so on. Thus, if A is a square matrix of order 3, that is $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$, then by (2) and (3), we have

$$|A|=a_{i1}A_{i1}+a_{i2}A_{i2}+a_{i3}A_{i3};\quad i=1,2,3 \quad (2')$$

or $$|A|=a_{1j}A_{1j}+a_{2j}A_{2j}+a_{3j}A_{3j};\quad j=1,2,3 \quad (3')$$

For example, if i=2, then by (2′), we have

$|A|=a_{21}A_{21}+a_{22}A_{22}+a_{23}A_{23}$. This can be written as

$$|A|=a_{21}(-1)^{2+1}M_{21}+a_{22}(-1)^{2+2}M_{22}+a_{23}(-1)^{2+3}M_{23}$$
$$=-a_{21}\begin{vmatrix}a_{12}&a_{13}\\a_{32}&a_{33}\end{vmatrix}+a_{22}\begin{vmatrix}a_{11}&a_{13}\\a_{31}&a_{33}\end{vmatrix}-a_{23}\begin{vmatrix}a_{11}&a_{12}\\a_{31}&a_{32}\end{vmatrix}$$

$$= -a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{23}(a_{11}a_{32} - a_{12}a_{31})$$

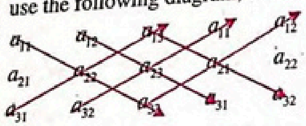
$$= -a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{22}a_{11}a_{33} - a_{22}a_{13}a_{31} - a_{23}a_{11}a_{32} + a_{23}a_{12}a_{31}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \quad (4)$$

Similarly, we can find |A| for other values of i and j.

The expansion of |A| in (4) can also be remembered by the following procedure.

Rewrite the first two columns of the matrix A after the third column and use the following diagram, if A is a 3x3 matrix.


(5)

The arrows pointing downward represent the three products having positive sign and the arrows pointing upward represent the three products having a negative sign.

Example 5: If $A = \begin{bmatrix} 3 & -1 & 2 \\ 3 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, then find $|A|$.

Solution: $|A| = \begin{vmatrix} 3 & -1 & 2 \\ 3 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$

We expand the determinant by using the elements of the first row, we have

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \quad (1)$$

But $A_{11} = (-1)^{1+1}M_{11} = M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$

$A_{12} = (-1)^{1+2}M_{12} = -M_{12} = -\begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix}$

$A_{13} = (-1)^{1+3}M_{13} = M_{13} = \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix}$

Putting these values in (1), we obtain

$$|A| = (3)\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix} + (2)\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= (3)[(1\times -1 - 0\times 0)] - (-1)[(3\times -1 - 0\times 1)] + (2)[(3\times 0 - 1\times 1)]$$

$$= -3 - 3 - 2 = -8$$

We now expand the same determinant by using elements of the third column, that is $|A| = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$ (2)

Now
$A_{13} = (-1)^{1+3}M_{13} = M_{13} = \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix}$

$A_{23} = (-1)^{2+3}M_{23} = -M_{23} = -\begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix}$

$A_{33} = (-1)^{3+3}M_{33} = M_{33} = \begin{vmatrix} 3 & -1 \\ 3 & 1 \end{vmatrix}$

Note

We get the same result, no matter which row or column is used to expand a 3×3 determinant.
The determinant of the square matrix A of order 3 in the above example can also be evaluated by the two simple methods given in (4) and (5).

Putting in (2), we get

$$|A| = (2)\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -1 \\ 3 & 1 \end{vmatrix}$$

$$= (2)(3\times 0 - 1\times 1) - 0(3\times 0 + 1\times 1) - 1(3\times 1 + 1\times 3) = -2 - 0 - 6 = -8.$$

2.3.3 Singular matrix and non-singular matrix

A square matrix A is called a singular matrix if its determinant is zero, i.e. $|A| = 0$, otherwise, it is a non-singular matrix.

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1(45-48) - 2(36-42) + 3(32-35) = 0$

Therefore, A is a singular matrix

2.3.4 Adjoint of a square matrix

Let A be a square matrix of order n. Let $\mathcal{A}$ denote the matrix obtained by replacing each element of A by its corresponding cofactor. Then $\mathcal{A}^t$ is called the adjoint of A and is usually denoted by adj A i.e. adj A = $\mathcal{A}^t$

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $\mathcal{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

and so adj A $= \tilde{A}^t = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

For example, if $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$, then $\tilde{A} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & -1 & -1 \\ 2 & 2 & -2 \end{bmatrix}$

and so adj $A = \tilde{A}^t = \begin{bmatrix} 3 & -1 & 2 \\ -1 & -1 & 2 \\ -1 & -1 & -2 \end{bmatrix}$.

2.3.5 Use adjoint method to calculate inverse of a square matrix

Let A be a square matrix of order n. If there exists a square matrix B of order n such that $AB = BA = I_n$ where I_n is the multiplicative identity matrix of order n, then B is called the **multiplicative inverse** of A and is denoted by A^{-1}.

Thus $AA^{-1} = A^{-1}A = I_n$.

It may be noted that inverse of a square matrix, if it exists, is unique. Moreover, if A is a non-singular square matrix of order n, then $A^{-1} = \frac{1}{|A|}$ adj A.

Example 6: *Let* $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ -1 & 2 & 0 \end{bmatrix}$. *Find* A^{-1}.

Solution: Since $A^{-1} = \frac{1}{|A|} adj\, A$, we need to find adj A and | A |.

First we find co-factor of every element of A.

$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} = 1\cdot(0+4) = 4,\qquad A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & -2 \\ -1 & 0 \end{vmatrix} = -1\cdot(0-2) = 2$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 1\cdot(0+1) = 1,\qquad A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 2 & 0 \end{vmatrix} = -1\cdot(0-2) = 2$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = 1\cdot(0+1) = 1,\qquad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = -1\cdot(2-2) = 0$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 1\cdot(4-1) = 3,\qquad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = -1\cdot(-2-0) = 2$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1\cdot(1+0) = 1$

So $adj\, A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

Next we find |A|.

Since $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= 1.(4) - 2(2) + 1(1)$

$= 4 - 4 + 1 = 1 \neq 0.$

Thus $A^{-1} = \frac{1}{|A|} adj\, A = 1\cdot\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

2.4 Properties of determinants

We shall state some of the useful properties of determinants which simplify the evaluation of determinants.

Property 1. If every element in a row or column of a square matrix A is zero, then |A|=0.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and every element in the first row is zero,

then

$A = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Now $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 0A_{11} + 0A_{12} + 0A_{13} = 0.$

We get the same result if every element of any other row or column is zero.

Property 2. If all elements of the corresponding rows and columns of a square matrix A are interchanged, then the determinant of the resulting matrix is equal to |A|. That is, the determinant of a square matrix and its transpose are always same.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ *and* $B = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$, then

|B| = |A|. Proof is left as an exercise.

Property 3. If any two rows or two columns in a square matrix A are interchanged, then the determinant of the resulting matrix is –|A|. In other words, both the determinants are additive inverses of each other.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ *and* $B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the

matrix obtained by interchanging the first and second row of A, then

$$|B| = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{21}(a_{12}a_{33} - a_{13}a_{32}) - a_{22}(a_{11}a_{33} - a_{13}a_{31}) + a_{23}(a_{11}a_{32} - a_{21}a_{31})$$

$$= a_{21}a_{12}a_{33} - a_{21}a_{13}a_{32} - a_{22}a_{11}a_{33} + a_{22}a_{13}a_{31} + a_{23}a_{11}a_{32} - a_{23}a_{12}a_{31}$$

$$= -(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{21}a_{31})$$

$$= -|A|.$$

Property 4. If a square matrix A has two identical rows or two identical columns, then |A|=0

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a matrix

obtained by interchanging the first and second rows of A. Then by property (3), |B| = –|A|. But the first and second rows of A are identical, mean A=B and so

|A|=|B|. Hence |A| = –|A| or 2|A| = 0 or |A| = 0. The same result is obtained if any two columns are identical.

Property 5. If every element of a row or column of a square matrix A is multiplied by the real number k, then the determinant of the resulting matrix is k|A|.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the matrix

obtained by multiplying first row of A by k. Then

$$|B| = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}A_{11} + ka_{12}A_{12} + ka_{13}A_{13}$$

$$= k(a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13})$$

$$= k|A|.$$

A similar result is obtained if any other row or column is multiplied by k.

Property 6. If every element of a row or column of a square matrix A is the sum of two terms, then its determinant can be written as the sum of two determinants.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then, $|A| = \begin{bmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{bmatrix}$

Expanding by the first column, we have

$$|A| = (a_{11} + b_{11})A_{11} + (a_{21} + b_{21})A_{21} + (a_{31} + b_{31})A_{31}$$

$$= (a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}) + (b_{11}A_{11} + b_{21}A_{21} + b_{31}A_{31})$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

Property 7. If every element of any row or column of a square matrix is multiplied by a real number k and the resulting product is added to the corresponding elements of another row or column of the matrix, then the determinant of the resulting matrix is equal to the determinant of the original matrix.

If $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$ then $B=\begin{bmatrix}a_{11}+ka_{12}&a_{12}&a_{13}\\a_{21}+ka_{22}&a_{22}&a_{23}\\a_{31}+ka_{32}&a_{32}&a_{33}\end{bmatrix}$ is the matrix obtained by multiplying every element of the second column of A and then adding to the corresponding element of the first column of A, then

$$|B|=\begin{vmatrix}a_{11}+ka_{12}&a_{12}&a_{13}\\a_{21}+ka_{22}&a_{22}&a_{23}\\a_{31}+ka_{32}&a_{32}&a_{33}\end{vmatrix}=\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}+\begin{vmatrix}ka_{12}&a_{12}&a_{13}\\ka_{22}&a_{22}&a_{23}\\ka_{32}&a_{32}&a_{33}\end{vmatrix}$$ by property (6)

$$=\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}+k\begin{vmatrix}a_{12}&a_{12}&a_{13}\\a_{22}&a_{22}&a_{23}\\a_{32}&a_{32}&a_{33}\end{vmatrix}$$ by property (5)

$$=\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}+k(0)$$ by property (4)

$$=\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}=|A|.$$

EXERCISE 2.2

1. *If* $A=\begin{bmatrix}1&3&1\\-1&2&0\\2&0&-2\end{bmatrix}$, then find $A_{11}, A_{21}, A_{23}, A_{31}, A_{32}, A_{33}$. Also find $|A|$.

2. Without evaluating state the reasons for the following equalities.

(i) $\begin{vmatrix}1&2&0\\3&1&0\\-1&2&0\end{vmatrix}=0$ (ii) $\begin{vmatrix}1&2&3\\-8&4&-12\\2&-1&3\end{vmatrix}=0$

(iii) $\begin{vmatrix}1&3&-2\\3&-1&1\\2&1&4\end{vmatrix}=\begin{vmatrix}1&3&2\\3&-1&1\\-2&1&4\end{vmatrix}$ (iv) $\begin{vmatrix}3&2&0\\1&1&-3\\2&4&-6\end{vmatrix}=-3\begin{vmatrix}3&2&0\\1&1&1\\2&4&2\end{vmatrix}$

(v) $\begin{vmatrix}1&0&-1\\3&2&1\\1&-1&0\end{vmatrix}=-\begin{vmatrix}1&0&-1\\1&-1&0\\3&2&1\end{vmatrix}$ (vi) $\begin{vmatrix}2&0&1\\3&1&2\\1&2&2\end{vmatrix}=\begin{vmatrix}2&0&1\\5&5&6\\1&2&2\end{vmatrix}$

3. Let A be a square matrix of order 3, then verify $|A'|=|A|$.

4. Evaluate the following determinants.

(i) $\begin{vmatrix}0&1&3\\-1&2&1\\2&1&1\end{vmatrix}$ (ii) $\begin{vmatrix}3&4&-2\\2&4&-6\\-4&2&0\end{vmatrix}$

(iii) $\begin{vmatrix}3&1&2\\6&-5&4\\-9&8&-7\end{vmatrix}$ (iv) $\begin{vmatrix}2&1&-3\\1&1&0\\-2&3&4\end{vmatrix}$

5. Show that

(i) $\begin{vmatrix}a&b&c\\l&m&n\\x&y&z\end{vmatrix}=\begin{vmatrix}a&l&x\\b&m&y\\c&n&z\end{vmatrix}$ (ii) $\begin{vmatrix}a&b&c\\1-3a&2-3b&3-3c\\4&5&6\end{vmatrix}=\begin{vmatrix}a&b&c\\1&2&3\\4&5&6\end{vmatrix}$

(iii) $\begin{vmatrix}1&1&1\\a&b&c\\b+c&c+a&a+b\end{vmatrix}=0$ (iv) $\begin{vmatrix}bc&ca&ab\\a&b&c\\a^2&b^2&c^2\end{vmatrix}=\begin{vmatrix}1&1&1\\a^2&b^2&c^2\\a^3&b^3&c^3\end{vmatrix}$

6. Prove that

(i) $\begin{vmatrix}a-b&b-c&c-a\\b-c&c-a&a-b\\c-a&a-b&b-a\end{vmatrix}=0.$

(ii) $\begin{vmatrix}1&a&a^3\\1&b&b^3\\1&c&c^3\end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$

- Multiplication of a row R_i by a non-zero scalar k is denoted by kR_i.
- Adding k times R_i to R_j is expressed as R_j+kR_i.

(b) Column operations on matrices

The following three operations performed on matrices are called elementary column operations:

(i) Interchanging of any two columns i.e. $C_i \leftrightarrow C_j$.

(ii) Multiplication of a column by any non-zero scalar k i.e. kC_i

(iii) Addition of any multiple of one column to another column i.e. $C_i + kC_j$.

where C_i, C_j are any two columns and k is any non-zero scalar.

If A is an m×n matrix, then an m×n matrix B obtained from A by performing a finite number of elementary row operations on A is called row equivalent to A. Symbolically, we write $B \overset{R}{\sim} A$ to denote B is row equivalent to A.

Similarly, we can define a column equivalent matrix that is replacing the word "row" by "column" in the above definition. We write $B \overset{C}{\sim} A$ to denote B is column equivalent to A.

Example 7: Let $A=\begin{bmatrix}1 & 2\\ 3 & 5\\ -1 & -4\end{bmatrix}$. Perform the following elementary row and column operations on A.

(i) $R_3 \leftrightarrow R_1$ (ii) $C_1 \leftrightarrow C_2$ (iii) R_2+2R_1 (iv) C_2-C_1 (v) R_1-4R_3.

Solution: $A=\begin{bmatrix}1 & 2\\ 3 & 5\\ -1 & -4\end{bmatrix}$

(i) $R_3 \leftrightarrow R_1 : \begin{bmatrix}-1 & -4\\ 3 & 5\\ 1 & 2\end{bmatrix}$ (ii) $C_1 \leftrightarrow C_2 : \begin{bmatrix}2 & 1\\ 5 & 3\\ -4 & -1\end{bmatrix}$

(iii) $R_2+2R_1 : \begin{bmatrix}1 & 2\\ 3+2(1) & 5+2(2)\\ -1 & -4\end{bmatrix} = \begin{bmatrix}1 & 2\\ 5 & 9\\ -1 & -4\end{bmatrix}$

(iv) $C_2-C_1 : \begin{bmatrix}1 & 2+(-1)\\ 3 & 5+(-3)\\ -1 & -4+(-(-1))\end{bmatrix} = \begin{bmatrix}1 & 1\\ 3 & 2\\ -1 & -3\end{bmatrix}$

(v) $R_1-4R_3 : \begin{bmatrix}1+(-4(-1)) & 2+(-4(-4))\\ 3 & 5\\ -1 & -4\end{bmatrix} = \begin{bmatrix}5 & 18\\ 3 & 5\\ -1 & -4\end{bmatrix}$

2.5.2 Echelon and reduced echelon form of a matrix

(a) Echelon form of a matrix

An m×n matrix A is said to be in (row) echelon form (or an echelon matrix) if it satisfies the following properties.

(i) In each successive non-zero row, the number of zeros before the first non-zero entry of a row increases row by row,

(ii) Every non-zero row in A precedes every zero row (if there is any).

For example, the matrices $\begin{bmatrix}2 & 3 & -4 & 1\\ 0 & 1 & 5 & 3\\ 0 & 0 & 0 & 6\end{bmatrix}$ and $\begin{bmatrix}0 & 2 & 3\\ 0 & 0 & -5\\ 0 & 0 & 0\end{bmatrix}$ are in echelon form, but the matrix $\begin{bmatrix}0 & 0 & 1 & 2\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 1\end{bmatrix}$ is not in echelon form.

(b) Reduced echelon form of a matrix

An m×n matrix A is said to be in reduced (row) echelon form (or reduced echelon matrix) if it satisfies the following properties.

(i) It is in (row) echelon form,

(ii) The first non-zero entry in R_i lies in C_j is 1 and all other entries of C_j are zero.

For example, the matrices $\begin{bmatrix}0 & 1 & 0 & 2\\ 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 0 & 2 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$ are in (row) reduced echelon form but $\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are not in (row) reduced echelon form.

2.5.3 Reduce a matrix to its echelon and reduced echelon form

Example 8: Reduce $A = \begin{bmatrix} 2 & 3 & -4 \\ 3 & 1 & -1 \\ 1 & -2 & -5 \end{bmatrix}$ to echelon form and then to reduced echelon form.

Solution: $\begin{bmatrix} 2 & 3 & -4 \\ 3 & 1 & -1 \\ 1 & -2 & -5 \end{bmatrix}$ $\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 3 & 1 & -1 \\ 2 & 3 & -4 \end{bmatrix}$ by $R_1 \leftrightarrow R_3$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 7 & 14 \\ 2 & 3 & -4 \end{bmatrix}$ by $R_2 - 3R_1$ $\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 7 & 14 \\ 0 & 7 & 6 \end{bmatrix}$ by $R_3 - 2R_1$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 2 \\ 0 & 7 & 6 \end{bmatrix}$ by $\frac{1}{7} R_2$ $\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{bmatrix}$ by $R_3 - 7R_2$ (1)

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ by $-\frac{1}{8} R_3$ $\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ by $R_1 + 2R_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by $R_1 + R_3$ and $R_2 - 2R_3$ (2)

The matrices in (1) and (2) are in echelon form and reduced echelon form of the given matrix A respectively.

2.5.4 Rank of a Matrix

Let A be a non-zero matrix. The rank of the matrix A is the number of non-zero rows in its (row) echelon form.

2.5.5 Using elementary row operation (ERO) to find the inverse and the rank of a matrix

(a) To find inverse of a matrix

Let A be a non-singular matrix. If we perform successive elementary row operations on the matrix [A | I], which reduce A to I and I to the resulting matrix B i.e. if [A | I] is reduced to [I | B], then B is the inverse of A written as A^{-1}. Similarly, if we perform successive elementary column operation on the matrix [A | I], which reduces A to I and I to the resulting matrix C, then C is the inverse of A written as A^{-1}.

Example 9: Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 4 & 2 \\ -1 & 2 & -2 \end{bmatrix}$

Solution: Since $\begin{vmatrix} 2 & 3 & 1 \\ 5 & 4 & 2 \\ -1 & 2 & -2 \end{vmatrix}$

$= 2\begin{vmatrix} 4 & 2 \\ 2 & -2 \end{vmatrix} - 3\begin{vmatrix} 5 & 2 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} 5 & 4 \\ -1 & 2 \end{vmatrix}$ (expanding by first row)

$= 2(-8-4) - 3(-10+2) + (10+4) = -24 + 24 + 14 = 14 \neq 0.$

So A is non-singular and A^{-1} exists.

Now $\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 5 & 4 & 2 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$ $\underset{\sim}{R} \left[\begin{array}{ccc|ccc} -1 & 2 & -2 & 0 & 0 & 1 \\ 5 & 4 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right]$ by $R_1 \leftrightarrow R_3$

$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 0 & 0 & -1 \\ 5 & 4 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right]$ by $(-1)R_1$

$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 0 & 0 & -1 \\ 0 & 14 & -8 & 0 & 1 & 5 \\ 0 & 7 & 23 & 1 & 0 & 2 \end{array}\right]$ by $R_2 - 5R_1$ and $R_3 - 2R_1$

$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 0 & 0 & -1 \\ 0 & 1 & \frac{-4}{7} & 0 & \frac{1}{14} & \frac{5}{14} \\ 0 & 7 & -3 & -1 & 0 & -2 \end{array}\right]$ by $\frac{1}{14} R_2$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{-6}{7} & 0 & \frac{-1}{7} & \frac{-2}{7} \\ 0 & 1 & \frac{-4}{7} & 0 & \frac{1}{14} & \frac{5}{14} \\ 0 & 0 & -3 & -1 & \frac{-1}{6} & \frac{-7}{6} \end{array}\right] \text{ by } R_1-2R_2 \text{ and } R_3-7R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-6}{7} & \frac{-4}{7} & \frac{-29}{7} \\ 0 & 1 & 0 & \frac{2}{3} & \frac{-5}{8} & \frac{-1}{18} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{-1}{18} & \frac{7}{18} \end{array}\right] \text{ by } R_1+\frac{6}{7}R_3 \text{ and } R_2+\frac{4}{7}R_3$$

$$\text{Thus } A^{-1} = \begin{bmatrix} \frac{-6}{7} & \frac{-4}{7} & \frac{-29}{7} \\ \frac{2}{3} & \frac{-5}{8} & \frac{-1}{18} \\ \frac{1}{3} & \frac{-1}{18} & \frac{7}{18} \end{bmatrix}$$

(b) To find rank of a matrix

Example 10: Find the rank of $A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

Solution:

$$A=\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ by } R_1 \leftrightarrow R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \text{ by } R_2-4R_1 \text{ and } R_3-7R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ by } R_3-2R_2$$

The last matrix is the echelon form of A having 2 non-zero rows. Hence the rank of A is 2.

EXERCISE 2.3

1. Reduce each of the following matrices to the indicated form

(i) $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 3 & 4 & -5 \end{bmatrix}$ Echelon form (ii) $\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ Reduced echelon form

(iii) $\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 2 \\ 4 & 1 & 7 \end{bmatrix}$ Reduced echelon form (iv) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$ Echelon form

2. Find the inverses of the following matrices by using elementary row operation.

(i) $\begin{bmatrix} 4 & -2 & 5 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & 4 \\ -1 & 5 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

3. Find the ranks of each of the following matrices.

(i) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 & -4 \\ 0 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix}$

4. Find

RANK OF MATRIX

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

4×4 matrix

2.6 System of linear equations

2.6.1 Homogeneous and non-homogeneous linear equations

Consider the equation $ax + by = k$ (1)

where $a \neq 0$, $b \neq 0$ and $k \neq 0$. The equation (1) is called a non-homogeneous linear equation in two variables (or unknowns) x and y.

Now consider the following two non-homogeneous linear equations in two variables x and y.

$$\left.\begin{aligned} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{aligned}\right\} \qquad (2)$$

These two equations together form a system of **non-homogeneous linear equations** in two variables x and y.

If we take $k = 0$ in equations (1), then it takes the form $ax + by = 0$ (3)

and is called a homogeneous linear equation in two variables x and y. If we take $k_1 = k_2 = 0$ in (2), then

$$\left.\begin{aligned} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{aligned}\right\} \qquad (4)$$

is called a system of **homogeneous linear equations** in the variables x and y.

Similarly, the following equations

$ax + by + cz = k$, where $a \neq 0, b \neq 0, c \neq 0$, and $k \neq 0$ (5)

is called a **non-homogeneous linear equation** in three variables x, y and z and the following three non-homogeneous linear equations in three variables x, y and z.

$$\left.\begin{aligned} a_1x + b_1y + c_1z = k_1 \\ a_2x + b_2y + c_2z = k_2 \\ a_3x + b_3y + c_3z = k_3 \end{aligned}\right\} \qquad (6)$$

together form a system of non-homogeneous linear equations in three variables x, y and z.

If we take $k = 0$ in (5), then $ax + by + cz = 0$ (7)

is called a homogeneous equation in three variables x, y and z.

If we take $k_1 = k_2 = k_3 = 0$ in (6) then

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{aligned}\right\} \qquad (8)$$

is called system of homogeneous linear equations in three variables x, y and z.

An order triple (t_1, t_2, t_3) is called a **solution** of system (6) if the equations are true for $x = t_1$, $y = t_2$ and $z = t_3$. The **solution set** is denoted by $S = \{(t_1, t_2, t_3)\}$.

In the case of system (8), we see that it is always true for $x = t_1 = 0$, $y = t_2 = 0$ and $z = t_3 = 0$, so the order triple $(t_1, t_2, t_3) = (0,0,0)$ is a solution of the system. Such a solution is called the **trivial** (or **zero**) **solution** and any other solution, if it exists, other than trivial solution is called a **non-trivial** (or **non-zero**) solution of the system. Consider system (6). Since

$$\begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

then system (6) may be written as a single matrix equation

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \qquad (9)$$

or $AX = B$ (10)

where, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$

Did You Know ?

In writing the augmented matrix of a linear system, we enter zero whenever a variable is missing in equation, since the coefficient of the variable is zero.

A is called the **matrix of coefficients**, X is the column vector of variables and B is the column vector of constants. If we adjoin the column vector B of the constants to the matrix A on the right separated by a bar **or a vertical line**, that is

$$[A|B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \end{array}\right],$$

the new matrix so obtained is called **augmented matrix** of the given system.

2.6.2 Solution of three homogeneous linear equations in three unknowns

Consider the following system of three homogeneous linear equations in three unknowns x_1, x_2, x_3.

$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3=0 \quad (i)\\ a_{21}x_1+a_{22}x_2+a_{23}x_3=0 \quad (ii)\\ a_{31}x_1+a_{32}x_2+a_{33}x_3=0 \quad (iii)\end{aligned}\right\} \quad (1)$$

which is equivalent to the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \text{ or simply } AX = O,$$

where $A=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$, $X=\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}$ and $O=\begin{bmatrix}0\\0\\0\end{bmatrix}$

If $|A|\neq 0$, then A is non-singular and A^{-1} exists.

We have $A^{-1}(AX)=A^{-1}0 \Rightarrow (A^{-1}A)X=O \Rightarrow IX=O \Rightarrow X=O$, that is

$$\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

or $x_1=0, x_2=0$ and $x_3=0$. This shows that the system has only trivial solution.

Thus, we may conclude "A system AX = O of three homogeneous linear equations in three variables has a trivial solution if A is non-singular i.e. $|A|\neq 0$".

Next we find the condition under which the system (1) has a non–trivial solution.

Multiplying equations (i), (ii) and (iii) of the system by the cofactors A_{11}, A_{21} and A_{31} of the corresponding elements a_{11}, a_{21} and a_{31} and then adding them up, we get

$$(a_{11}A_{11}+a_{21}A_{21}+a_{31}A_{31})x_1+(a_{12}A_{11}+a_{22}A_{21}+a_{32}A_{31})x_2+$$
$$(a_{13}A_{11}+a_{23}A_{21}+a_{33}A_{31})x_3=0.$$

From this, we have $|A|x_1=0$. Likewise, we can have $|A|x_2=0$ and $|A|x_3=0$. The system (1) has a non-trivial solution if at least one of the variable x_1, x_2 and x_3 is

different from zero. Suppose $x_1\neq 0$, then $|A|x_1=0 \Rightarrow |A|=0$. Thus, we may conclude: "A system AX = O of three homogeneous linear equations in three variables has a non-trivial solution if A is singular i.e. $|A|=0$".

Example 11: Show that the following system has a trivial solution.

$$\begin{aligned} 2x+y-z&=0 \quad (i)\\ x+y-z&=0 \quad (ii)\\ x+2y+2z&=0 \quad (iii)\end{aligned}$$

Solution: Since

$$|A|=\begin{vmatrix}2&1&-1\\1&1&-1\\1&2&2\end{vmatrix}=\begin{vmatrix}1&0&0\\1&1&-1\\1&2&2\end{vmatrix}=1\begin{vmatrix}1&-1\\2&2\end{vmatrix}=2+2=4\neq 0,$$ the system has a trivial solution. Subtracting equation (ii) from (i), we get $x=0$. Subtracting equation (iii) from (ii), we have $y=3z$. Putting $x=0$ and $y=3z$ in equation (i) we obtain $z=0$, and therefore from $y=3z$, we get $y=0$. Thus $x=0, y=0, z=0$ and the system has only trivial solution.

Example 12: Show that the system ha non-trivial solution

$$\begin{aligned} x+y+2z&=0 \quad (i)\\ -2x+y-z&=0 \quad (ii)\\ -x+5y+4z&=0 \quad (iii)\end{aligned}$$

Solution: Since

$$|A|=\begin{vmatrix}1&1&-2\\-2&1&-1\\-1&5&4\end{vmatrix}=\begin{vmatrix}1&0&0\\-2&3&3\\-1&6&6\end{vmatrix}=1\begin{vmatrix}3&3\\6&6\end{vmatrix}=18-18=0$$

Thus the given system has a non-trivial solution.

Adding 2 times equation (i) to (ii) we have $y=-z$

Subtracting equation (ii) from (i), we get $x=-z$ putting $x=-z=y$ in equation (iii) we have $-(-z)+5(-z)+4z=0$ which is true for any value t of z. We get that $x=-t$, $y=-t$ and $z=t$ satisfy equations (i), (ii) and (iii) for any real value of t. Thus the given system has infinitely many solutions.

Example 13: For what value of λ the system has a non-trivial solution. Solve the system for the value of λ.

$$\begin{aligned} x-y+2z&=0\\ 2x+y+\lambda z&=0\\ 3x+y+2z&=0\end{aligned}$$

Solution: First we find the value of λ. We have $A=\begin{bmatrix}1 & -1 & 2\\ 2 & 1 & \lambda\\ 3 & 1 & 2\end{bmatrix}$,

So $$|A|=\begin{vmatrix}1 & -1 & 2\\ 2 & 1 & \lambda\\ 3 & 1 & 2\end{vmatrix}=\begin{vmatrix}1 & 0 & 0\\ 2 & 3 & \lambda-4\\ 3 & 4 & -4\end{vmatrix}=1.\begin{vmatrix}3 & \lambda-4\\ 4 & -4\end{vmatrix}=-12-4(\lambda-4)=4-4\lambda.$$

We know that the system has non-trivial solution if $|A|=0$, that is $4-4\lambda=0$ or $\lambda=1$. Substituting the value of λ into the system, we have

$$x-y+2z=0$$
$$2x+y+z=0$$
$$3x+y+2z=0$$

Now solving the first two equations, we get $x=-z$, $y=3$. Putting these values in the third equation, we obtain $-3z+z+2z=0$ which is true for any value t of z. We see that $x=-t$, $y=t$ and $z=t$ satisfy all the three equations of the system for any real value of t. Thus the given system has infinitely many solutions for $\lambda=1$.

2.6.3 Consistency and inconsistency of a system

(a) A system of linear equations is said to be consistent if the system has only one (i.e. unique) solution or it has infinitely many solutions.

(b) A system of linear equations is said to be inconsistent if the system has no solution.

Consider the following three systems of linear equations in three variables.

$$\left.\begin{aligned}2x+2y-z&=4\\ x-2y+z&=2\\ x+y&=0\end{aligned}\right\}\qquad \text{(I)}$$

$$\left.\begin{aligned}x-2y+z&=2\\ -x-y+2z&=1\\ x-5y+4z&=5\end{aligned}\right\}\qquad \text{(II)}$$

$$\left.\begin{aligned}x-2y+3z&=1\\ -2x+5y-4z&=-2\\ x-4y-z&=5\end{aligned}\right\}\qquad \text{(III)}$$

We solve these systems now by performing the elementary row operations on the augmented matrices of these systems to reduce them to (row) echelon form.

(i) Consider system (I). the augmented matrix of the systems is

$$[A|B]=\left[\begin{array}{ccc|c}2 & 2 & -1 & 4\\ 1 & -2 & 1 & 2\\ 1 & 1 & 0 & 0\end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c}1 & -2 & 1 & 2\\ 2 & 2 & -1 & 4\\ 1 & 1 & 0 & 0\end{array}\right] \text{ by } R_1 \leftrightarrow R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & -2 & 1 & 2\\ 0 & 6 & -3 & 0\\ 1 & 1 & 0 & 0\end{array}\right] \text{ by } R_2-2R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & -2 & 1 & 2\\ 0 & 6 & -3 & 6\\ 0 & 3 & -1 & -2\end{array}\right] \text{ by } R_3-R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & -2 & 1 & 2\\ 0 & 6 & -3 & 0\\ 0 & -6 & 2 & 4\end{array}\right] \text{ by } -2R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & -2 & 1 & 2\\ 0 & 6 & -3 & 0\\ 0 & 0 & -1 & 4\end{array}\right] \text{ by } R_3+R_2$$

The system (I) is reduced to **equivalent system**,

$x-2y+z=2$ (i)
$0+6y-3z=0$ (ii)
$0+0-z=4$ (iii)

Remember ?
Two systems of equations are said to be equivalent if they have the same solution set.

The system is now in triangular form. In this form the system can be easily solved. By equation (iii) we get $z=-4$.
Substituting the value of z in equation (ii) we get $y=-2$.
Now substituting the values of y and z in equation (i), we get $x=2$. Thus the solution of the system is $x=2$, $y=-2$ and $z=-4$. Since the system has a solution,

so it is consistent.

(ii) Consider system (II). The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ -1 & -1 & 2 & 1 \\ 1 & -5 & 4 & 5 \end{array}\right]$$

then $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ -1 & -1 & 2 & 1 \\ 1 & -5 & 4 & 5 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -3 & 3 & 3 \\ 0 & -3 & 3 & 3 \end{array}\right]$ by $R_2 + R_1$ and $R_3 - R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ by } R_3 - R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ by } \frac{1}{3} R_2.$$

The system (II) is reduced to the equivalent system

$x - 2y + z = 2$ (i)

$-y + z = 1$ (ii)

$0z = 0$ (iii)

Equation (iii) is obviously satisfied for all choices of z. Equations (i) and (ii) yield

$x = 2 + 2y - z$ (iv)

$y = z - 1$ (v)

Since z is arbitrary, from equations (iv) and (v) we can find infinitely many values of x and y. This is equivalent to saying that the system has infinitely many solutions. Thus the system is consistent.

(3) Consider system (III). The augmented matrix of the system is

$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ -2 & 5 & -4 & -2 \\ 1 & -4 & -1 & 5 \end{array}\right]$ then $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ -2 & 5 & -4 & -2 \\ 1 & -4 & -1 & 5 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -4 & 4 \end{array}\right]$

by $R_2 + 2R_1$ and $R_3 - R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right] \text{ by } R_3 + 2R_2$$

The system (III) is reduced to the equivalent system

$x - 2y + 3z = 1$ (i)

$y + 2z = 0$ (ii)

$0z = 4$ (iii)

We see that the equation (iii) has no solution. Therefore, this system of equations has no solution. Hence the system is inconsistent.

From the above, we note that the system of linear equations may have no solution, have only one solution, or have infinitely many solutions.

2.6.4 Solution of a non-homogeneous linear equations

A system of non-homogeneous linear equations may be solved by using the following methods.

(a) Matrix Inversion Method i.e. $AX=B \Rightarrow X=A^{-1}B$

(b) Gauss Elimination Method (echelon form)

(c) Gauss–Jordan Method (reduced echelon form)

(d) Cramer's Rule.

(a) Matrix Inversion Method

Consider the following system of three non-homogeneous linear equations in three variables x_1, x_2 and x_3.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = k_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = k_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = k_3 \end{array}\right\}$$

This system is equivalent to the matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \text{ or } AX = B, \text{ where}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}.$$

If A is non-singular, then A^{-1} exists. We have

$AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B.$

Thus the matrix of variables is now determined as the product of $A^{-1}B$.

The method discussed above for finding the solution of a system of non–homogenous linear equations is known as matrix inversion method.

Example 14: Solve the system of equations by matrix **inversion** method

$$x_1 - 2x_2 + x_3 = 2$$
$$2x_1 + 2x_2 - x_3 = 4$$
$$x_1 + x_2 = 0$$

Solution: Since

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 1 & -2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 1 & -3 & 1 \end{vmatrix} = 1.\begin{vmatrix} 0 & -1 \\ -3 & 1 \end{vmatrix} = 3 \neq 0,$$

So, A^{-1} exists.

$$A^{-1} = \frac{1}{|A|} adj\, A = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 3 \\ 0 & -3 & 6 \end{bmatrix}$$

Did You Know

The matrix inversion method for solving a system of non-homogeneous linear equations is applicable only when the coefficient matrix A is non-singular i.e. $|A| \neq 0$.

But $X = A^{-1}B$, *so* $X = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 3 \\ 0 & -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 1\times2+1\times4+0\times0 \\ -1\times2-1\times4+3\times0 \\ 0\times2-3\times4+6\times0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ -6 \\ -12 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix},$$

that is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$. Thus $x_1 = 2, x_2 = -2$ *and* $x_3 = -4$.
Which is the solution of the given system.

(b) Gauss elimination method (Echelon form)

We are already familiar with the method of reducing the augmented matrix of a system of non-homogeneous linear equations to echelon form. We now apply this method to find the solution of a system of non-homogeneous linear equations. The procedure is called Gauss Elimination Method (Echelon Form).

Example 15: Solve the following system by the method of echelon form.

$$\left.\begin{aligned} 2x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + x_3 &= 2 \\ x_1 + x_2 &= 0 \end{aligned}\right\}$$

Solution: The augmented matrix of the given system is

$\left[\begin{array}{ccc|c} 2 & 2 & -1 & 4 \\ 1 & -2 & 1 & 2 \\ 1 & 1 & 0 & 0 \end{array}\right]$. By 2.6.3 (i) the echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & -1 & 4 \end{array}\right].$$

From R_3, we have $x_3 = -4$.

From R_2, we have $6x_2 - 3x_3 = 0$

Substituting $x_3 = -4$, in this equation we get $x_2 = -2$.

From R_1, we have $x_1 - 2x_2 + x_3 = 2$

Now putting $x_2 = -2$ and $x_3 = -4$ we obtain $x_1 = 2$

Thus $x_1 = 2$, $x_2 = -2$, $x_3 = -4$ is the solution of the given system.

(c) Gauss–Jordan Method (Reduced Echelon Form)

Consider system of equations in example 14 above and the echelon form

$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix}$ of its augmented matrix.

We reduce the matrix $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & -1 & 4 \end{array}\right]$ to reduced (row) echelon form, that is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & -1 & 4 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 \end{array}\right] \text{by } \frac{1}{6}R_2 \text{ and } (-1)R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{array}\right] \text{by } R_1 - R_3 \text{ and } R_2 + \frac{1}{2}R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{array}\right] \text{by } R_1 + 2R_2$$

The equivalent system in the reduced (row) echelon form is

$x_1 = 2,\ x_2 = -2,\ x_3 = -4.$

which is the solution of the given system. The procedure illustrated above of transforming a system of non–homogeneous linear equations into an equivalent system in the reduced (row) **echelon form** is called the Gauss–Jordan Method (reduced echelon form).

(d) Cramer's Rule

Consider the following system of three non-homogeneous linear equations in three variables.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = k_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = k_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = k_3 \end{array}\right\} \quad (1)$$

which is equivalent to the matrix equation

$$AX = B \qquad (2)$$

Did You Know

Like matrix inversion method, the Cramer's rule is also applicable only when $|A| \neq 0$ Cramer's rule is simpler than matrix method for finding solution of the given system.

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}.$

If $|A| \neq 0$, then A^{-1} exists and (2) can be written as $X = A^{-1}B$.

Since $A^{-1} = \frac{1}{|A|} adj\, A$, we have $X = A^{-1}B = \left(\frac{1}{|A|} adj\, A\right) B$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_{11}k_1 + A_{21}k_2 + A_{31}k_3 \\ A_{12}k_1 + A_{22}k_2 + A_{32}k_3 \\ A_{13}k_1 + A_{23}k_2 + A_{33}k_3 \end{bmatrix}$$

that is
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \dfrac{A_{11}k_1 + A_{21}k_2 + A_{31}k_3}{|A|} \\ \dfrac{A_{12}k_1 + A_{22}k_2 + A_{32}k_3}{|A|} \\ \dfrac{A_{13}k_1 + A_{23}k_2 + A_{33}k_3}{|A|} \end{bmatrix}$$

Thus
$$x_1 = \frac{k_1A_{11} + k_2A_{21} + k_3A_{31}}{|A|} = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{|A|},$$

$$x_2 = \frac{k_1A_{12} + k_2A_{22} + k_3A_{32}}{|A|} = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{|A|},$$

$$x_3 = \frac{k_1A_{13} + k_2A_{23} + k_3A_{33}}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{|A|}.$$

This method of finding the solution of the system is called Cramer's Rule.

Example 16: Use Cramer's rule to solve the following system.

$$\left.\begin{aligned} x_1-2x_2+x_3 &= 2\\ 2x_1+2x_2-x_3 &= 4\\ x_1+x_2 &= 0 \end{aligned}\right\};$$

Solution: We have $|A|=\begin{vmatrix}1 & -2 & 1\\ 2 & 2 & -1\\ 1 & 1 & 0\end{vmatrix}=3\neq 0$

Now $x_1=\dfrac{\begin{vmatrix}k_1 & a_{12} & a_{13}\\ k_2 & a_{22} & a_{23}\\ k_3 & a_{32} & a_{33}\end{vmatrix}}{|A|}=-\dfrac{\begin{vmatrix}2 & -2 & 1\\ 4 & 2 & -1\\ 0 & 1 & 0\end{vmatrix}}{3}$ (Expanding by third row)

$$=-\frac{\begin{vmatrix}2 & 1\\ 4 & -1\end{vmatrix}}{3} \quad =\frac{6}{3}=2,$$

$x_2=\dfrac{\begin{vmatrix}a_{11} & k_1 & a_{13}\\ a_{21} & k_2 & a_{23}\\ a_{31} & k_3 & a_{33}\end{vmatrix}}{|A|}=\dfrac{\begin{vmatrix}1 & 2 & 1\\ 2 & 4 & -1\\ 1 & 0 & 0\end{vmatrix}}{3}$. (Expanding by third row)

$$=\frac{\begin{vmatrix}2 & 1\\ 4 & -1\end{vmatrix}}{3} \quad =\frac{-6}{3}=-2,$$

and $x_3=\dfrac{\begin{vmatrix}a_{11} & a_{12} & k_1\\ a_{21} & a_{22} & k_2\\ a_{31} & a_{32} & k_3\end{vmatrix}}{|A|}=\dfrac{\begin{vmatrix}1 & -2 & 2\\ 2 & 2 & 4\\ 1 & 1 & 0\end{vmatrix}}{3}=-\dfrac{\begin{vmatrix}1 & -3 & 2\\ 2 & 0 & 4\\ 1 & 0 & 0\end{vmatrix}}{3}$ (Expanding by 2nd Column)

$$=3\frac{\begin{vmatrix}2 & 4\\ 1 & 0\end{vmatrix}}{3} \quad =-4$$

Thus $x_1=2, x_2=-2$ *and* $x_3=-4$ is the solution of the given system.

> **Note**
> We observe that the solution of the given system obtained by any of the above four methods are the same.

EXERCISE 2.4

1. Solve the following system of equations by matrix inversion method.

(i) $4x-3y+z=11$, $2x+y-4z=-1$, $x+2y-2z=1$ (ii) $x+y+z=1$, $x+y-2z=3$, $2x+y+z=2$

2. Solve the following system of equations by the Gauss elimination method and Gauss–Jordan method.

(i) $x-y+4z=4$, $2x+2y-z=2$, $3x-2y+3z=-3$ (ii) $2x+4y-z=0$, $x-2y-2z=2$, $-5x-8y+3z=-2$

3. Use Cramer's rule to solve the following system of equations.

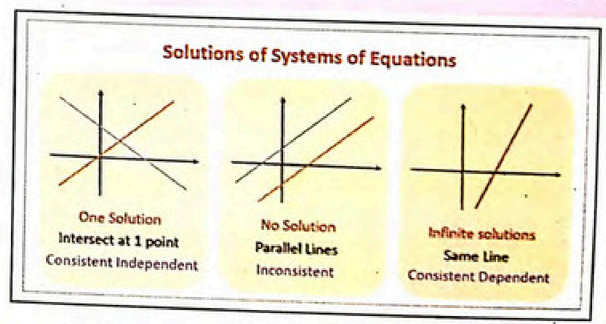
(i) $x-2y=-4$, $3x+y=-5$, $2x+z=-1$ (ii) $x-y+2z=10$, $2x+y-2z=-4$, $3x+y+z=7$

4. Solve the following system of homogeneous equations.

(i) $x_1-x_2+x_3=0$, $x_1+2x_2-x_3=0$, $2x_1+x_2+3x_3=0$ (ii) $x_1+x_2+2x_3=0$, $-2x_1+x_2-x_3=0$, $-x_1+5x_2+4x_3=0$

5. For what value of λ, the following system of homogeneous equations has a non-trivial solution. Solve the system.

$$x_1+5x_2+3x_3=0$$
$$5x_1+x_2-\lambda x_3=0$$
$$x_1+2x_2+\lambda x_3=0$$



REVIEW EXERCISE 2

1. Choose the correct options

(i) If $\begin{vmatrix}7a-5b & 3c\\ -1 & 2\end{vmatrix}=0$, then which one of the following is correct?

(a) $14a+3c=5b$ (b) $14a-3c=5b$
(c) $14a+3c=10b$ (d) $14a+10b=3c$

(ii) If $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ and A_{ij} is the cofactor of a_{ij} in A. Then the value of $|A|$ is given by

(a) $a_{11}A_{31}+a_{12}A_{32}+a_{13}A_{33}$ (b) $a_{11}A_{11}+a_{12}A_{21}+a_{13}A_{31}$
(c) $a_{21}A_{11}+a_{22}A_{12}+a_{23}A_{13}$ (d) $a_{11}A_{11}+a_{21}A_{21}+a_{31}A_{31}$

(iii) If $A=\begin{bmatrix}\alpha & 2\\ 2 & \alpha\end{bmatrix}$ and $|A^3|=125$ then the value of α is

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 5

(iv) If $|A|=47$, then find $|A^t|$

(a) -47 (b) 47 (c) 0 (d) Cannot be determined

(v) If det (A) = 5, then find det (15A) where A is of order 2x2.

(a) 225 (b) 75 (c) 375 (d) 1125

(vi) If $A=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$, then find A^n, (where $n\in N$)

(a) $\begin{bmatrix}3n & 0\\ 0 & 3n\end{bmatrix}$ (b) $\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$ (c) $3^n\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ (d) I_{2x2}

2. Compute the product $\begin{bmatrix}-5 & 1\\ 6 & -1\\ -2 & 0\end{bmatrix}\begin{bmatrix}4 & -5 & -1\\ 5 & 6 & 1\end{bmatrix}\begin{bmatrix}5\\ -4\\ 1\end{bmatrix}$.

3. Prove that $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$ satisfies $A^2-4A-5I=0$.

4. If $A=\begin{bmatrix}2 & 1 & 3\\ 0 & 4 & 6\\ 7 & 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 0 & -1\\ 2 & 0 & 3\\ -1 & 2 & 4\end{bmatrix}$. Find $|2A-B^2|$

5. Using properties of determinants, prove that

$$\begin{vmatrix}a^2+2a & 2a+1 & 1\\ 2a+1 & a+2 & 1\\ 3 & 3 & 1\end{vmatrix}=(a-1)^3$$

6. If $A=\begin{bmatrix}3 & 1 & -1\\ 0 & 1 & 2\end{bmatrix}$, then show that AA^t and A^tA are both symmetric.

7. If $A=\begin{bmatrix}4 & 3 & 3\\ -1 & 0 & -1\\ -4 & -4 & -3\end{bmatrix}$, prove that $A^{-1}=A$.

8. If $A=\begin{bmatrix}4 & 3\\ -2 & 1\end{bmatrix}$, *then find* $A+10A^{-1}$

9. Solve the system
$$\begin{aligned}x+y+z&=4\\ 2x-3y+z&=2\\ -x+2y-z&=-1\end{aligned}$$

by using the following methods:

(i) Matrix Inversion (ii) Gauss Elimination
(iii) Gauss Jordan (iv) Cramer's Rule