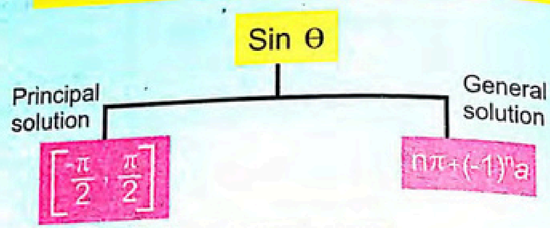
UNIT 12

Graph of Trigonometric and Inverse Trigonometric Functions And Solutions of Trignomotric Equations



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Find the domain and range of the trigonometric functions.
- Define even and odd functions.
- Discuss the periodicity of trigonometric functions.
- Find the maximum and minimum value of a given function of the type:
  - $a+b\sin\theta$,
  - $a+b\cos\theta$,
  - $a+b\sin(c\theta+d)$,
  - $a+b\cos(c\theta+d)$,
  - the reciprocals of above, where a, b, c and d are real numbers.
- Recognize the shapes of the graphs of sine, cosine and tangent for all angles.
- Draw the graphs of the six basic trigonometric functions within the domain from -2π to 2π.
- Guess the graphs of $\sin 2\theta$, $\cos 2\theta$, $\sin \theta/2$, $\cos \theta/2$ etc. without actually drawing them.
- Define periodic, even/odd and translation properties of the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$, i.e., $\sin\theta$ has
  - periodic property $\sin(\theta \pm 2\pi) = \sin\theta$,
  - odd property $\sin(-\theta) = -\sin\theta$,
  - translation property $\begin{cases}\sin(\theta-\pi) = -\sin\theta \\ \sin(\pi-\theta) = \sin\theta\end{cases}$
- Deduce $\sin(\theta + 2k\pi) = \sin\theta$ where k is an integer.
- Solve trigonometric equations of the type $\sin\theta = k$, $\cos\theta = k$ and $\tan\theta = k$, using periodic, even/odd and translation properties.

- Solve graphically the trigonometric equations of the type:
  - $\sin\theta = \theta/2$,
  - $\cos\theta = \theta$,
  - $\tan\theta = 2\theta$ when $-\pi/2 \le \theta \le \pi/2$
- Define the inverse trigonometric functions and their domain and range.
- Find domains and ranges of
  - principal trigonometric functions,
  - inverse trigonometric functions.
- Draw the graphs of inverse trigonometric functions.
- Prove the addition and subtraction formulae of inverse trigonometric functions.
- Apply addition and subtraction formulae of inverse trigonometric functions to verify related identities.
- Solve trigonometric equations and check their roots by substitution in the given trigonometric equations so as to discard extraneous roots.
- Use the periods of trigonometric functions to find the solution of general trigonometric equations.

12 Introduction

Trigonometric functions are usually defined either with the help of a unit circle or right angled triangles. We will also study their properties with a special emphasis on their graphs. Rest of the unit is concerned with inverse trigonometric functions and solutions of trigonometric equations.

12.1 Trigonometric functions

We know that the domain of the function defined by the equation $y=f(x)$ is the set of all those values of x for which the function attains finite definite values, and the range is the set of all those values which y attains. So far the functions we have studied all had subsets of real numbers as their domain and range. But the domains of trigonometric functions are the set of angles, rather than real numbers. We can however, make the domains of the trigonometric function, subsets of real numbers, by defining them on the unit circle, that is a circle whose radius is 1.

Let θ be a central angle of the unit circle and P(x, y) be the point as shown in the Figure 12.1 then $r = OP = 1 = \sqrt{x^2+y^2}$, and the six trigonometric ratios also called **trigonometric functions** or **circular functions** of θ are defined as follows:

sine $\theta = \frac{y}{1} = y$

cosine $\theta = \frac{x}{1} = x$

tangent $\theta = \frac{y}{x}$ $(x \neq 0)$

cosecant $\theta = \frac{1}{y}$ $(y \neq 0)$

secant $\theta = \frac{1}{x}$ $(x \neq 0)$

cotangent $\theta = \frac{x}{y}$ $(y \neq 0)$

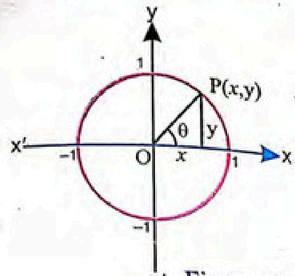


Figure 12.1

The trigonometric functions are abbreviated as follows:

(i) Sine θ as sin θ
(ii) Cosine θ as cos θ
(iii) Tangent θ as tan θ
(iv) Cosecant θ as cosec θ
(v) Secant θ as sec θ
(vi) Cotangent θ as cot θ

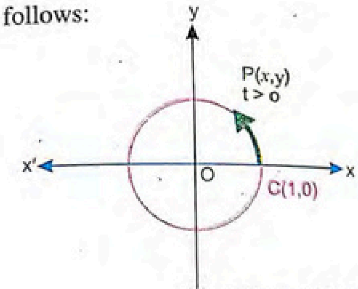


Figure 12.2

It can be seen that

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Since any real number can represent the length of exactly one Arc on the unit circle. If t is a positive number, we can find the Arc of length t by measuring a distance t in counter clockwise direction along an Arc of the unit circle beginning at C(1,0). So we get Arc CP of length t.

If t is a negative number, we can find the Arc of length t, by measuring a distance t in a clockwise direction along an Arc of the unit circle beginning at the point C(1,0).

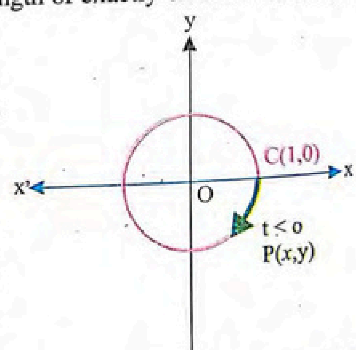


Figure 12.3

In each case, we get a unique point

P(x,y) that corresponds to the real number t. We also know that if s is an Arc which subtends an angle θ at the centre of circle with radius r, we have $s = r\theta$ (θ in radians).

Let s = t and r = 1 then above equation reduces to $t = \theta$ or $\theta = t$.

Thus we obtain $\sin\theta = \sin t$, $\operatorname{cosec}\theta = \operatorname{cosec} t$
$\cos\theta = \cos t$, $\sec\theta = \sec t$
$\tan\theta = \tan t$, $\cot\theta = \cot t$.

where θ is the angle measured in radians and t is a real number.

Thus we can think of each trigonometric expression as being either a trigonometric function of an angle measured in radians or as a trigonometric function of a real number t.

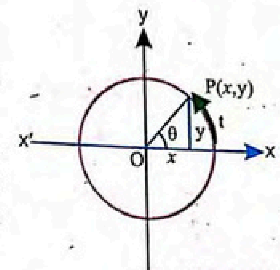


Figure 12.4

Thus the trigonometric functions can be thought of as functions that have domains and ranges that are subsets of real numbers.

12.1.1 Domain and Range of Trigonometric Functions

(a) Domain and Range of Sine and Cosine Functions

Refer to Figure 12.1, $\sin\theta = y$ $\cos\theta = x$

Domain of sine and cosine is the set of real numbers R. Since point P(x,y) is on the unit circle

$\therefore$ $-1 \leq y \leq 1$ and $-1 \leq x \leq 1$ or $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$.

Thus the range of sine and cosine functions are [−1, 1].

(b) Domain and range of tangent and cotangent functions

Refer to Figure 12.1. $\tan\theta = \frac{y}{x}$ $x \neq 0$.

When $x \neq 0$, then terminal side OP cannot coincide with OY or OY'; in other words

$$\theta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots \quad \text{or} \quad \theta \neq (2n+1)\frac{\pi}{2}; n \in Z$$

Therefore for the tangent function.

Domain $= R - \left\{ t \mid t = (2n+1)\frac{\pi}{2};\ n \in Z \right\}$ and Range = R (the set of real numbers)

Since $\cot\theta = \frac{x}{y}$, $y \neq 0$.

when $y \neq 0$ then terminal side OP does not coincide with OX or OX′ in other words

$\theta \neq 0, \pm\pi, \pm 2\pi, \ldots\ldots$ or $\theta \neq n\pi, n \in Z$.

Hence in case of cotangent function Domain $= R-\{t | t = n\pi, n \in Z\}$

Range = R (The set of real numbers)

(c) Domain and Range of Secant and Cosecant functions

Refer to Figure.12.1. $\operatorname{cosec}\theta = \frac{1}{y}$, $y \neq 0$

If $y \neq 0$ then as seen in the case of $\cot\theta$, $\theta \neq n\pi; n \in Z$.

Domain of cosec function $= R-\{t | t = n\pi; n \in Z\}$.

Since $|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} = 1$ (Figure 12.1)

Hence $|y| \leq 1$. or $\frac{1}{|y|} \geq 1$.

Thus either $\frac{1}{y} \geq 1$ or $\frac{1}{y} \leq -1$ that is $\operatorname{cosec}\theta \geq 1$ or $\operatorname{cosec}\theta \leq -1$.

That is $\operatorname{cosec}\theta$ attains all values except those which lie between −1 and 1.

Hence Range of cosec function $= R- \{t | -1 < t < 1\}$.Now $\sec\theta = \frac{1}{x}$, $x \neq 0$. Then as seen in the case of $\tan\theta$. $\theta \neq (2n+1)\frac{\pi}{2}$, $n \in Z$.

Domain of secant function $= R - \{t | t = (2n+1)\frac{\pi}{2}, n \in Z\}$.

Also $|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2} = 1$

$|x| \leq 1$ or $\frac{1}{|x|} \geq 1$.

Thus either $\frac{1}{x} \geq 1$ or $\frac{1}{x} \leq -1$ that is $\sec\theta \geq 1$ or $\sec\theta \leq -1$.

That is $\sec\theta$ attains all values except those which lie between −1 and 1.

Range of secant function $= R- \{t | -1 < t < 1\}$.

We now give table of the domain and range of trigonometric functions; which are written in words as well as in symbolic notations:

Function	Domain	Range
$y = \sin x$	All real numbers; $-\infty < x < \infty$	$-1 \leq$ real numbers ≤ 1 $-1 \leq y \leq 1$
$y = \cos x$	All real numbers; $-\infty < x < \infty$	$-1 \leq$ real numbers ≤ 1 $-1 \leq y \leq 1$
$y = \tan x$	All real numbers except $(2n+1)\frac{\pi}{2}$, $n \in Z$. $-\infty < x < \infty$, $x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$.	all real numbers; $-\infty < y < \infty$
$y = \cot x$	All real numbers except $n\pi$, $n \in Z$. $-\infty < x < \infty$, $x \neq n\pi$, $n \in Z$.	all real numbers; $-\infty < y < \infty$
$y = \sec x$	All real numbers except $(2n+1)\frac{\pi}{2}$, $n \in Z$. $-\infty < x < \infty$, $x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$.	all real numbers ≤ -1 or ≥ 1 $y \geq 1$ or $y \leq -1$
$y = \operatorname{cosec} x$	All real numbers except $n\pi$, $n \in Z$. $-\infty < x < \infty$, $x \neq n\pi$, $n \in Z$.	all real numbers ≤ -1 or ≥ 1 $y \geq 1$ or $y \leq -1$

Example 1: Find the domain of each of the following functions.

(i) $\sec 3x$ (ii) $\tan\frac{1}{5}x$ (iii) $\operatorname{cosec}\frac{1}{2}x$

Solution (i) We know that the domain of sec t is $-\infty < t < \infty$, $t \neq (2n+1)\frac{\pi}{2}$, $n \in Z$.

If $t = 3x$, then dom sec $3x$ is $-\infty < 3x < \infty$, $3x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

or $-\infty < x < \infty$, $x \neq (2n+1)\frac{\pi}{6}$, $n \in Z$

$\therefore$ Dom sec $3x = R-\{x | x = (2n+1)\frac{\pi}{6};\ n \in Z\}$

(ii) Domain tant is $-\infty < t < \infty$, $t \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

if $t=\frac{1}{5}x$ then dom $\tan\frac{1}{5}x$ is

$-\infty<\frac{1}{5}x<\infty,\quad \frac{1}{5}x\neq(2n+1)\frac{\pi}{2},\ n\in Z$

or $\quad -\infty<x<\infty,\ x\neq\frac{5}{2}(2n+1)\cdot\pi,\ n\in Z.$

dom $\tan\frac{1}{5}x=R-\{x|x=\frac{5}{2}(2n+1)\pi,\ n\in Z\}$

(iii) Dom cosec t is $-\infty<t<\infty,\quad t\neq n\pi,\ n\in Z$

Let $t=\frac{1}{2}x$ then dom cosec $\frac{1}{2}x$ is $-\infty<\frac{1}{2}x<\infty,\quad \frac{1}{2}x\neq n\pi,\ n\in Z$

or $-\infty<x<\infty,\quad x\neq 2n\pi,\ n\in Z\ \therefore$ dom cosec $\frac{1}{2}x=R-\{x|x=2n\pi,\ n\in Z\}$

Example 2: Find the range of each function.

(i) $\cos 3x$ (ii) $3\tan 2x$ (iii) $2\text{ cosec }\frac{1}{3}x$

Solution: (i) We know that for all $t\in$ dom cos t, $-1\le\cos t\le 1$

Let $t=3x$ then $-1\le\cos 3x\le 1.$

Hence range $\cos 3x$ is the closed interval $[-1, 1]$

(ii) Since for all $t\in$ dom tan t, $-\infty<\tan t<\infty$

Let $t=2x$ then $-\infty<\tan 2x<\infty$

Hence $-\infty<3\tan 2x<\infty$. Thus Range of $3\tan 2x$ is R.

(iii) Since for all $t\in$ dom cosec t

cosec $t\le -1$ or cosec $t\ge 1$

Let $t=\frac{1}{3}x$

Then cosec $\frac{1}{3}x\le -1$ or cosec $\frac{1}{3}x\ge 1.$

Hence $2\text{ cosec }\frac{1}{3}x\le -2$ or $2\text{ cosec }\frac{1}{3}x\ge 2.$

Hence range of $2\text{ cosec }\frac{1}{3}x=R-\{p|-2<p<2\}.$

12.1.2 Even and Odd Functions

Even Functions

A function f is **even** if for every x in the domain, $f(x)=f(-x)$.

Even functions are symmetric about the y-axis. For each point (x, y) on the graph, the point $(-x, y)$ is also on the graph.

The following are the graphs of even functions.

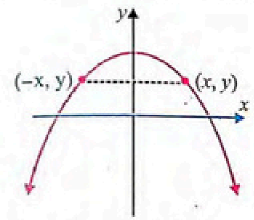


Figure 12.5

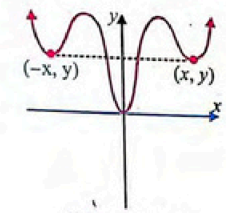


Figure 12.6

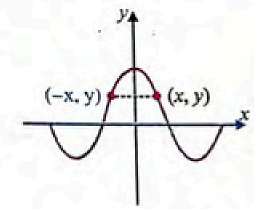


Figure 12.7

Notice that for any point (x, y) on each graph, the point $(-x, y)$ also lies on the graph. Therefore, for any x value in the domain, $f(x)=f(-x)$.

Odd Functions

A function f is **odd** if for every x in the domain, $-f(x)=f(-x)$.

Odd functions are symmetric about the origin. For each point (x, y) on the graph, the point $(-x, -y)$ is also on the graph.

The following are the graphs of odd functions.

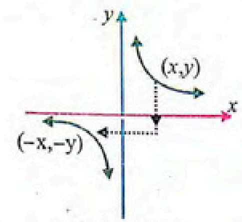


Figure 12.8

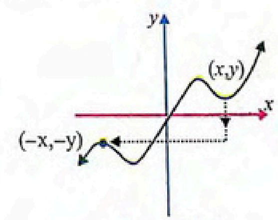


Figure 12.9

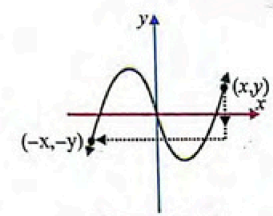


Figure 12.10

Notice that for any point (x, y) on each graph, the point $(-x, -y)$ also lies on the graph. Therefore, for any x value in the domain, $f(x)=-f(-x)$ or equivalently $-f(x)=f(-x)$.

Example 3: Prove that: (i) $f(x)=x^2$ is an even function.
(ii) $f(x)=x^3$ is an odd function.

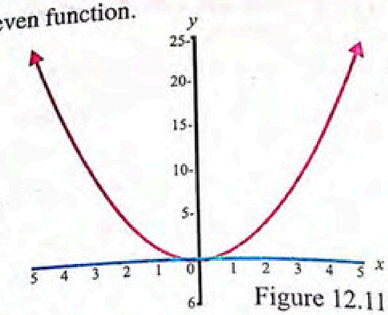


Figure 12.11

Solution: (i) $f(x)=x^2$

$f(-x)=(-x)^2$
$=x^2$
$=f(x)$
$\Rightarrow f(x)$ is even.

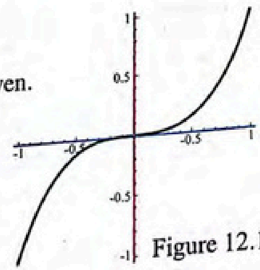


Figure 12.12

(ii) $f(x)=x^3$
$f(-x)=(-x)^3$
$=-x^3$
$=-f(x)$
$\Rightarrow f(x)$ is odd

One of the important properties of the trigonometric functions is that of being either even or odd.

We know from trigonometry that:

$\sin(-\theta)=-\sin\theta,$	$\cos(-\theta)=\cos\theta,$	$\tan(-\theta)=-\tan\theta$
$\operatorname{cosec}(-\theta)=-\operatorname{cosec}\theta,$	$\sec(-\theta)=\sec\theta,$	$\cot(-\theta)=-\cot\theta$

Thus sinθ, cosecθ, tanθ, cotθ are odd functions and cosθ and secθ are even functions.

Example 4: Is the function $f(x)=\sin x-\cos x$ even, odd, or neither?

$f(-x)=\sin(-x)-\cos(-x)$
$=-\sin x-\cos x$
$=-(\sin x)+\cos x$

Because $-(\sin x+\cos x)\neq-(\sin x-\cos x)$

And $-(\sin x+\cos x)\neq\sin x-\cos x$

the function is neither even nor odd.

Remember

The sum of an odd function and an even function is neither even nor odd.

12.1.3 Periodicity of Trigonometric Functions

Periodic Function

A function f is said to be periodic if there exists a positive constant p such that. $f(x+p)=f(x)$ for all x in the domain of f. The smallest such positive number p is called the period of the function..

All the six trigonometric functions are periodic functions, because they repeat their values after their periods. This behavior of trigonometric functions is called **periodicity**.

Note

If $f(x)$ is a periodic function then $af(x)$ and $f(x)+b$ are also periodic functions and the periods of all these functions are the same. Can you say why?

Theorem 1: Show that the period of sinθ is 2π.

Proof: If p is the period of sinθ, then

$\sin(\theta+p)=\sin\theta$ (1)

for all $\theta\in$ dom sinθ.

Since $0\in$ dom $\sin\theta=R$, put $\theta=0$ in (1), we have

$\sin p=\sin 0=0$

Thus possible values of p are $0,\pm\pi,\ \pm2\pi,\ldots\ldots$

The first smallest positive value of $p=\pi$, for which $\sin(\theta+\pi)=-\sin\theta$ which contradicts (1). Therefore π is not the period of sinθ

Next put $p=2\pi$ then $\sin(\theta+2\pi)=\sin\theta$

Hence 2π is the period of sinθ.

Theorem 2: Show that the period of cosθ is 2π.

Proof: If p is the period of cosθ, then

$\cos(\theta+p)=\cos\theta$ (1)

for all $\theta\in$ domcosθ

Since $0\in$ dom$\cos\theta=R$, put $\theta=0$ in (1), we have

$\cos p=\cos 0=0$

Thus possible values of p are $0,\pm2\pi,\ \pm4\pi,\ldots\ldots$

The first smallest positive value of $p=2\pi$, for which $\cos(\theta+2\pi)=\cos\theta$.

Hence 2π is the period of cosθ.

Theorem 3: Show that the period of $\tan\theta$ is π.

Proof: If p is the period of $\tan\theta$, then

$\tan(\theta + p) = \tan\theta$ (1)

for all $\theta \in$ dom $\tan\theta$

Put $\theta = 0$ in (1), we have

$\tan p = \tan 0 = 0$

Thus possible values of p are $0, \pm\pi, \pm 2\pi, \ldots\ldots$

The first smallest positive value of $p = \pi$,

for which $\tan(\theta+\pi) = \tan\theta$.

Hence π is the period of $\tan\theta$.

Remember

We may easily find out for the reciprocals of $\sin\theta$, $\cos\theta$ and $\tan\theta$, i.e. $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$ that

(i) 2π is the period of $\operatorname{cosec}\theta$

(ii) 2π is the period of $\sec\theta$

(iii) π is the period of $\cot\theta$

Example 5: Find period of $5\sin x$.

Solution: We know that period of sine function is 2π.

$\therefore \quad \sin x = \sin(x + 2\pi)$

$\Rightarrow \quad 5\sin x = 5\sin(x + 2\pi)$

It means that when x is increased by 2π, values of $5\sin x$ repeats, hence period of $5\sin x$ is the same as that of $\sin x$.

Thus if f is a trigonometric function, period of cf (c constant) is the same as that of f.

Example 6: Find period of $\cos 6x$.

Solution: We know that period of cosine function is 2π.

$\therefore \quad \cos 6x = \cos(6x+2\pi)$

$= \cos 6(x+\frac{2\pi}{6})$.

when x is increased by $\frac{2\pi}{6}$, value of $\cos 6x$ remains the same; hence period of $\cos 6x$ is $\frac{2\pi}{6}$ or $\frac{\pi}{3}$.

Thus period of $\cos 6x$ is equal to the period of $\cos x$ divided by 6.

This result holds for other trigonometric functions also

Thus, if f is a trigonometric function, then for any constant k, period of $f(kx) = \dfrac{\text{Period of } f(x)}{k}$

Example 7: Find period of each function.

(i) $\frac{1}{2}\tan 3x$ (ii) $3\sec\frac{x}{3}$

Solution: (i) Period of $\frac{1}{2}\tan 3x = \dfrac{\text{Period of } \tan x}{3}$

$= \frac{\pi}{3}$ ($\because$ period of $\tan x$ is π)

(ii) Period of $3\sec\frac{x}{3} = \dfrac{\text{Period of } \sec x}{\frac{1}{3}}$

$= \dfrac{2\pi}{\frac{1}{3}}$ ($\because$ period of $\sec x$ is 2π)

$= 3(2\pi) = 6\pi$.

12.1.4 Maximum and minimum values of certain trigonometric functions

In this section we are concerned with finding the maximum and minimum value of a function of the type:

(i) $a+b\sin\theta$ (ii) $a+b\cos\theta$

(iii) $a+b\sin(c\theta+d)$ (iv) $a+b\cos(c\theta+d)$

and the reciprocals of the above, where a, b, c and d are real numbers.

Before doing so, we recall that the term a in the above functions allows for a vertical shift in the graph of the functions. The term b in the functions allows for amplitude variation of the functions.

Now to find the maximum and minimum for sine and cosine functions we only need to remember that the maximum and minimum for both $\sin\theta$ and $\cos\theta$ are 1 and -1 respectively.

Consider types (i) and (ii) above. These functions reach its maximum when both $\sin\theta$ and $\cos\theta$ are at the maximum i.e. $\sin\theta = 1$ and $\cos\theta = 1$.

So the maximum of $a+b\sin\theta = a+|b|$ (maximum of $\sin\theta$)

$= a+|b|(1)$

$= a+|b|$ (1)

Similarly, the maximum of $a+b\cos\theta = a+|b|$ (2)

These functions reach its minimum when both $\sin\theta$ and $\cos\theta$ are at the minimum i.e. $\sin\theta = -1$ and $\cos\theta = -1$. So the minimum of $a+b\sin\theta = a+|b|$ (minimum of $\sin\theta$)

$= a + |b|(-1)$

$= a - |b|$ (3)

Similarly, the minimum of $a + b\cos\theta = a - |b|$ (4)

Consider types (iii) and (iv). In these types the values of c and d do not matter because they do not affect the amplitude of the function, so we treat these two types in similar way as (i) and (ii).

So the maximum value of $a + b\sin(c\theta + d) = a + |b|$ (5)

and the maximum value of $a + b\cos(c\theta + d) = a + |b|$ (6)

The minimum value of $a + b\sin(c\theta + d) = a - |b|$ (7)

and the minimum value of $a + b\sin(c\theta + d) = a - |b|$ (8)

Thus, we conclude that, if M and m respectively denote the maximum value and minimum value of the function, then we have the following formulas.

$M = a + |b|$ and $m = a - |b|$

Let M' and m' be the maximum value and minimum value of the reciprocals of the above functions, then clearly for $m > 0$, $M > 0$ and $m < 0$, $M < 0$

$M' = \frac{1}{m}$ and $m' = \frac{1}{M}$

and for $m < 0$, $M > 0$

$M' = \frac{1}{M}$ and $m' = \frac{1}{m}$

Example 8: Find maximum and minimum values of the functions.

(i) $y = 1 + 2\sin\theta$ (ii) $y = 3 + 2\cos(3\theta - 2)$ (iii) $y = \frac{1}{1 + 3\sin(2\theta - 15)}$

Solution: (i) Here $a = 1$ and $b = 2$

$\therefore$ the maximum value of $y = M = a + |b|$

$= 1 + |2| = 1 + 2 = 3$

and the minimum value of $y = m = a - |b|$

$= 1 - |2| = 1 - 2 = -1$

(ii) Here $a = 3$ and $b = 2$

$\therefore M = a + |b| = 3 + |2| = 5$ and $m = a - |b| = 3 - |2| = 1$

(iii) Let $y' = 1 + 3\sin(2\theta - 15)$

Then $M = 1 + |3| = 4$ and $m = 1 - |3| = -2$

If M' and m' are the maximum value and minimum value of y respectively, then

$M' = \frac{1}{M} = \frac{1}{4}$ and $m' = \frac{1}{m} = \frac{1}{-2}$

Since $m < 0$ and $M > 0$

EXERCISE 12.1

1. Find the domain, range and period of each of the following function:

(i) $3\sin 3x$ (ii) $\tan\frac{1}{2}x$ (iii) $\operatorname{cosec} 2x$

(iv) $\cos 4x$ (v) $6\sec 2x$ (vi) $\frac{7}{2}\cot\frac{2\pi x}{3}$

(vii) $-\frac{1}{4}\tan x$ (viii) $\frac{1}{2}\operatorname{cosec} x$ (ix) $\sec\frac{\pi}{4}x$

2. Find maximum and minimum of each of the following functions:

(i) $y = -2 + \frac{1}{2}\sin\left(\frac{1}{3}\theta + 2\right)$ (ii) $y = 5 - 4\sin(\theta + 30)$

(iii) $y = \frac{1}{19 - 10\sin(3\theta - 45)}$ (iv) $y = \frac{1}{4\cos 2\pi\theta}$

12.2 Graphs of Trigonometric Functions

The graph of a real valued function is the set of points in the cartesian plane, whose co-ordinates are the ordered pairs, belonging to the given function. For example to graph a function y=f(x), we give a number of values to x, which belong to the domain of the function, and find the corresponding values of y, which satisfy the equation y=f(x). We plot these ordered pairs (x, y), join them by smooth curves or line segments, the diagram so formed is the graph of the function.

In case of trigonometric functions the points are joined by smooth curves. Since trigonometric functions are periodic, it is sufficient to draw graph over a period. This information can be used to extend the graph to the right and the left, because the graph will be identical over those values of x which form the period.

12.2.1 The shapes of the graphs of sine, cosine and tangent

$y = \sin x$ and **$y = \cos x$** look pretty similar; in fact the main difference is that the sine graph starts at (0,0) and the cosine at (0,1).

Both of these graphs repeat every 360 degrees, and the cosine graph is essentially a transformation of the sin graph—it's been translated along the x-axis by 90 degrees. Thinking about the fact that $\sin x = \cos(90 - x)$ and $\cos x = \sin(90 - x)$, **$y = \tan x$** crosses the x-axis at 0, and has an asymptote at 90. This graph repeats every 180 degrees.

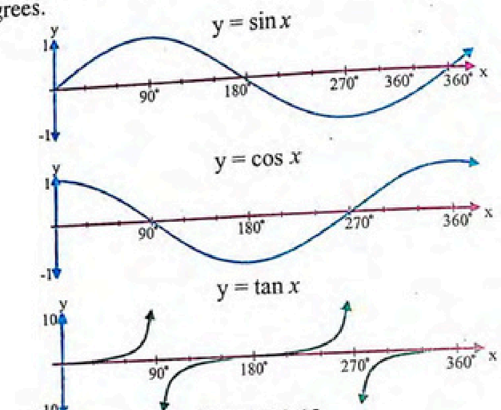


Figure 12.13

12.2.2 Graphs of six basic trigonometric functions

(a) Graph of $y=\sin x, -2\pi \le x \le 2\pi$

Since $\sin x$ is periodic function of period 2π, whose domain is R, it is sufficient to draw a detailed graph over the interval $[0, 2\pi]$; portions x of the graph over the intervals $[-2\pi,0]$, $[0,2\pi]$, $[2\pi, 4\pi]$ and so on will be identical.

Suitable values of x, and the corresponding values of y, satisfying $y=\sin x$ are given below in the form of a table.

Values of y for different angles x, can be found by use of trigonometric identities.

x	0 0°	$\frac{\pi}{6}$ 30°	$\frac{\pi}{3}$ 60°	$\frac{\pi}{2}$ 90°	$\frac{2\pi}{3}$ 120°	$\frac{5\pi}{6}$ 150°	π 180°	$\frac{7\pi}{6}$ 210°	$\frac{4\pi}{3}$ 240°	$\frac{3\pi}{2}$ 270°	$\frac{5\pi}{3}$ 300°	$\frac{11\pi}{6}$ 330°	2π 360°
y	0	0.5	0.87	1	0.87	0.5	0	–0.5	–0.87	–1	–0.87	–0.5	0

Take a set of rectangular axes, choosing a convenient length for 30° on the x–axis and a convenient length as a unit on the y–axis. We plot the points (x, y) to get the following graph of $y=\sin x$ in the interval $(0, 2\pi)$

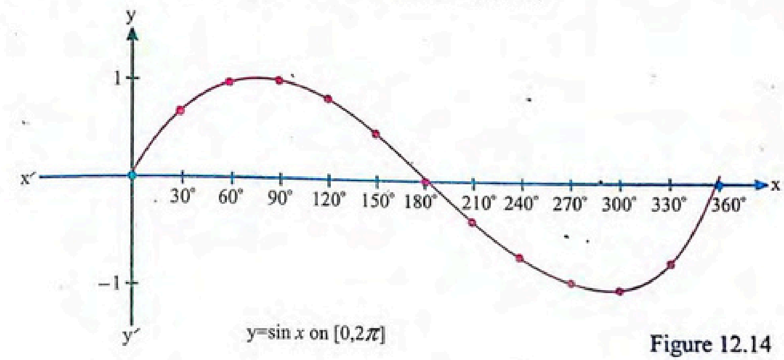


Figure 12.14

We note for all values of x, $-1 \le \sin x \le 1$.

We often call the graph of $y = \sin x$, a sine wave and the graph in the interval $[0, 2\pi]$ a cycle. Extended graph of $\sin x$ which is the repetition of the graph in figure 12.14 is given in figure 12.15.

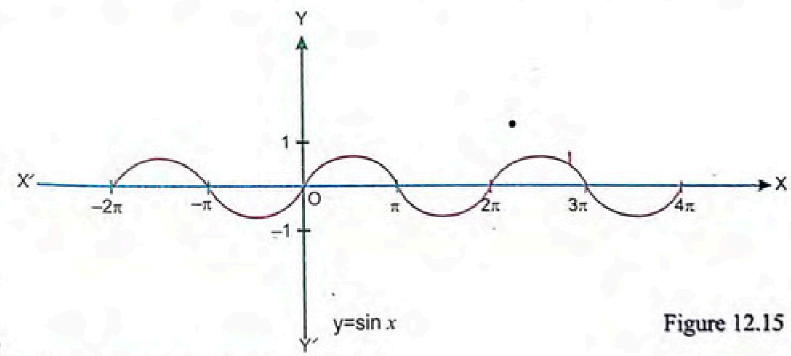


Figure 12.15

(b) Graph of $y = \cos x, -2\pi \le x \le 2\pi$.

The cosine function also has a period of 2π, and its range is $[-1,1]$. Values of (x, y), satisfying $y=\cos x$ are given below in the form of a table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1

Plotting the points (x, y); the graph of y=cos*x* on the interval $[0, 2\pi]$, is shown in figure 12.16.

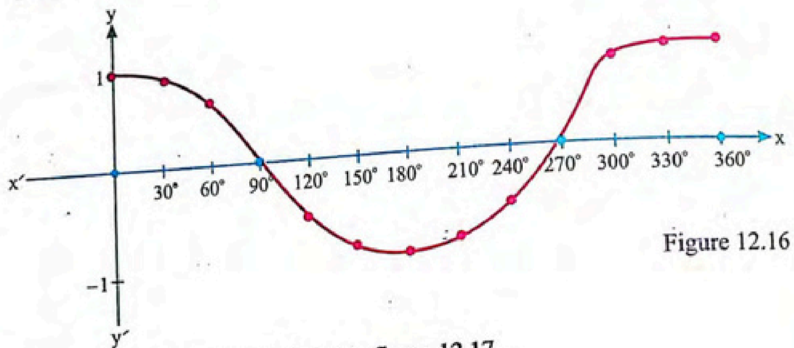


Figure 12.16

Extended graph of y=cos*x* is shown in figure 12.17.

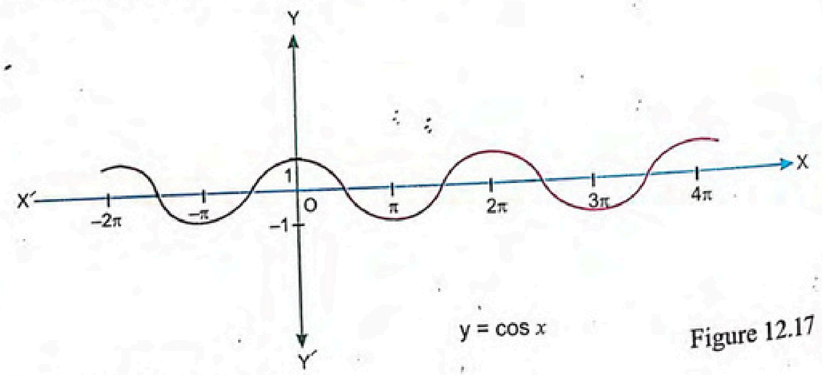


Figure 12.17

(c) Graph of y=tan *x*, $0 \le x \le \pi$

The period of tan *x* is π and the domain is the set $R - \{x \mid x = (2n+1)\frac{\pi}{2},\ n \in Z\}$.

When $x = (2n+1)\frac{\pi}{2}$, $n \in Z$ or $x = \pm 90°, \pm 270°, \ldots\ldots$; the tangent function is not

defined, at these values of *x*, it becomes very large, in other words it approaches $\pm\infty$.

In the interval $[0, \pi]$ as we approach 90° from the left, the tangent becomes larger positively, that is, it tends to $+\infty$; and when we approach 90° from the right, it becomes larger negatively, that is it tends to $-\infty$. Table of values (*x*, y) satisfying y=tan *x* on $[0, \pi]$ are given in the below table. The graph is shown in figure 12.18.

x	0°	30°	60°	90°	120°	150°	180°
y	0	0.58	1.73	$-\infty$ ∞	−1.73	−0.58	0°

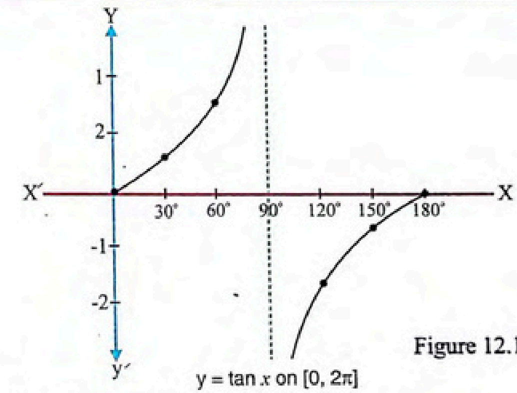


Figure 12.18

Extended graph of y = tan *x* is given in figure 12.19.

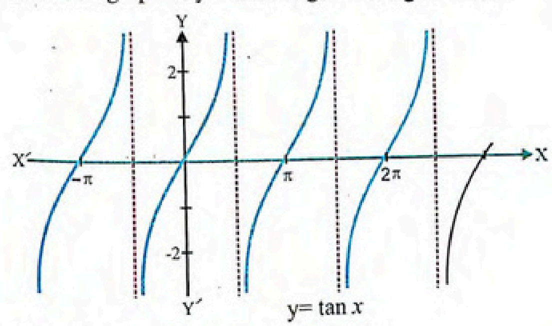


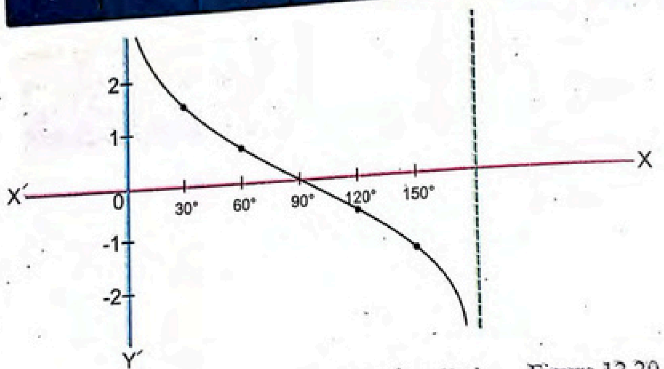
Figure 12.19

(d) Graph of y=cot *x*, $-\pi \le x \le \pi$.

The period of cotangent is also π.

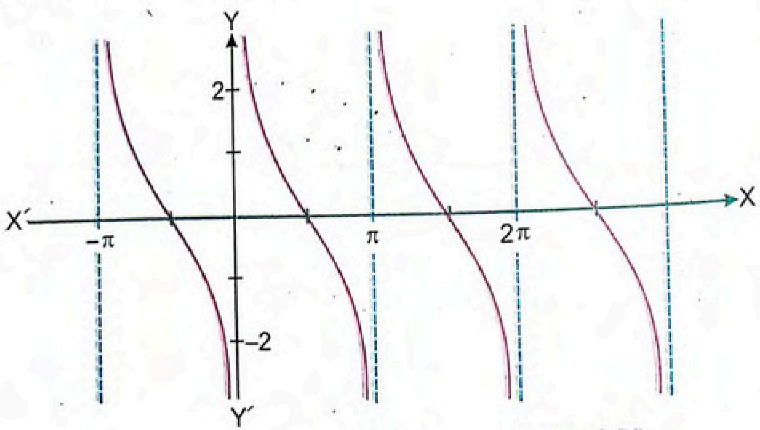
Table of values (x, y) satisfying $y = \cot x$ on $[0, \pi]$ is given below, while graph is shown in figure 12.20.

x	0°	30°	60°	90°	120°	150°	180°
y	∞	1.73	0.58	0	−0.58	−1.73	−∞



y = cot x on [0, π] Figure 12.20

Extended graph of $y = \cot x$ is given below in figure 12.21, which is the repetition of the graph given in figure 12.20.

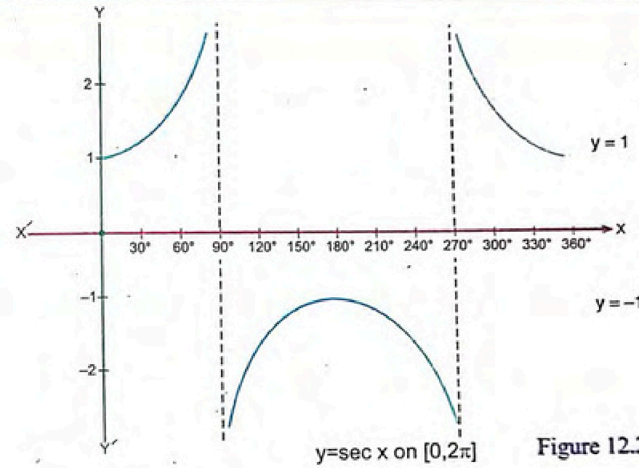


y = cot x Figure 12.21

(e) Graph of $y = \sec x, -2\pi \le x \le 2\pi$.

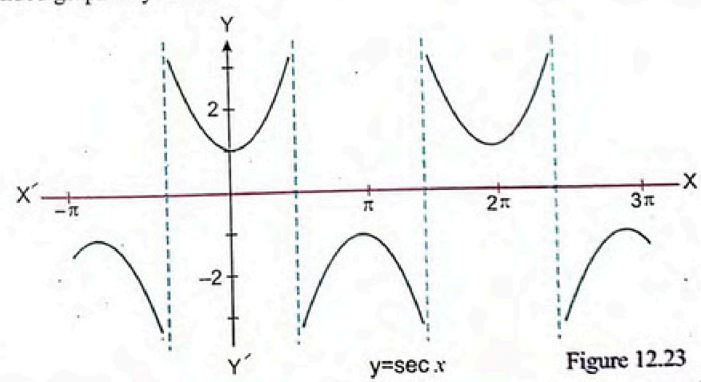
We know that period of secant is 2π, table of values (x, y) for $y = \sec x$ on $[0, 2\pi]$ is given below. Graph is shown in figure 12.22.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	1	1.15	2	∞	−2	−1.15	−1	−1.15	−2	−∞	2	1.15	1



y=sec x on [0,2π] Figure 12.22

Extended graph of $y = \sec x$

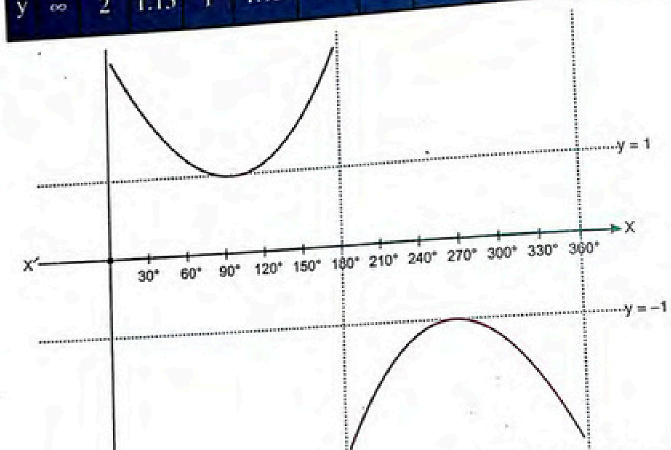


y=sec x Figure 12.23

(f) Graph of y = cosec x, $-2\pi \le x \le 2\pi$

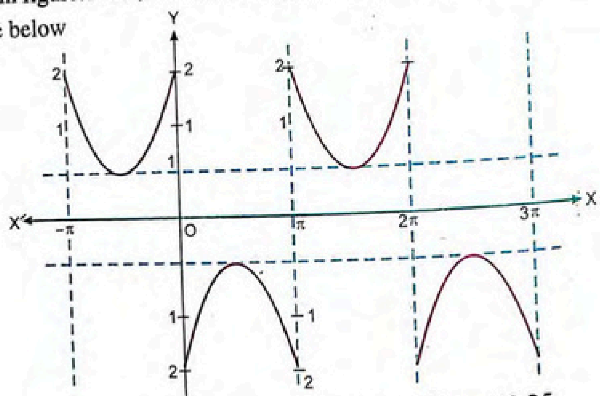
Period of cosec is 2π, table of values (x, y) satisfying y=cosec x on $[0, 2\pi]$ is as follows:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	∞	2	1.15	1	1.15	2	$-\infty$	−2	−1.15	−1	−1.15	−2	∞



y=cosec x on [0, 2π] Figure 12.24

Repeating the graph in figure.12.24, the extended graph of y=cosec x is obtained as given in the figure below



y = cosec x Figure 12.25

12.2.3 Graphs of sinAθ and cosAθ where A is a positive constant.

In figure 12.26 the graph of y = sinθ is shown.

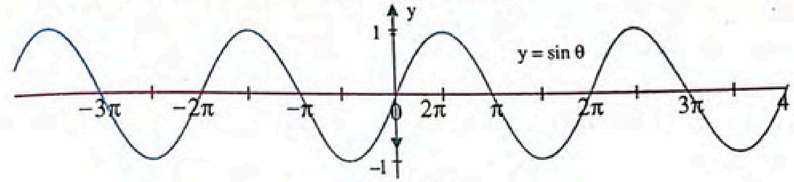


Figure 12.26

We see that the graph of y = sinθ has period 2π, so the constant A in y = sin Aθ indicates the number of periods in the interval of length 2π. If y = sinθ, we notice that A = 1. This means that there is only 1 period in that interval.

For example, if A = 2, then

$$y = \sin 2\theta$$

means that there are 2 periods in an interval of length 2π as shown in figure 12.27. The graph of y = sin2θ is the compressed version of the graph of y = sinθ in the x–direction.

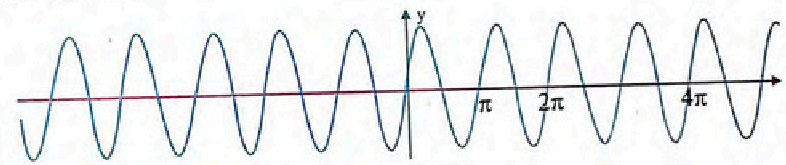


Figure 12.27

If A = 3, then y = sin3θ indicates that there are 3 periods in the interval of length 2π as shown in figure 12.28. The graph of y = sin3θ is more compressed version of the graph of y = sin θ as compared to the graph of y = sin2θ.

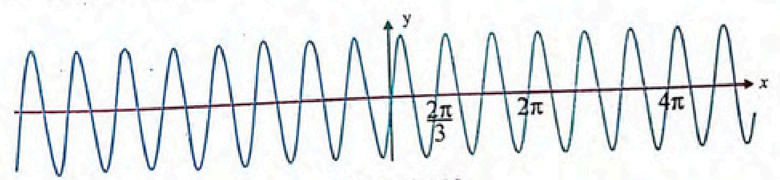


Figure 12.28

On the other hand, if $A = \frac{1}{2}$, then $y = \sin\frac{1}{2}\theta$ means that there is only half a period in the interval of length 2π as shown in **figure 12.29.** The graph of $y = \sin\frac{1}{2}\theta$ is the expanded version of the graph of $y = \sin\theta$ in the x–direction.

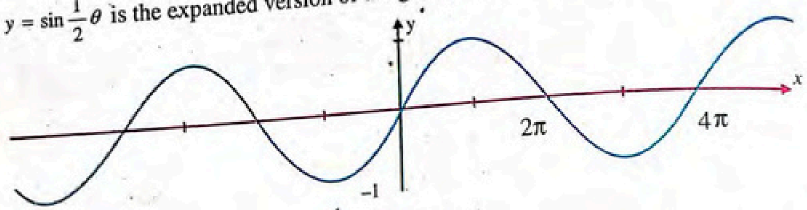


Figure 12.29

Similarly, multiplying θ by a positive constant has the geometric effect of compressing or expanding the graph of y = cosθ in the x–direction.

Thus, multiplying θ by a number greater than 1 compresses the graph of sinθ or cosθ in the x–direction and shortens its period. Multiplying θ by a positive number less than 1 expands the graph and lengthens its period. In this case the period is given by $\boxed{\text{Period} = \frac{2\pi}{A}}$

Example 8: Without drawing, guess the graph of $\cos\frac{1}{3}\theta$. Also find its period, frequency and amplitude.

Solution: Here $A = \frac{1}{3} < 1$, so the graph of $\cos\frac{1}{3}\theta$ is an expanded version of the graph of cosθ. Also in an interval of length 2π, there is one third of a period.

We have $\quad \text{Period} = \frac{2\pi}{A}$

$\therefore \quad \text{Period of } \cos\frac{1}{3}\theta = \frac{2\pi}{\frac{1}{3}} = 6\pi$

$\text{Frequency} = \frac{A}{2\pi}$

$\therefore \text{ Frequency of } \cos\frac{1}{3}\theta = \frac{1}{6\pi}$

$\text{Amplitude of } \cos\frac{1}{3}\theta = 1$

Did You Know

(i) The period is also called the **wave length.**

(ii) The reciprocal of the period is called the frequency of the functions. Thus

$\boxed{\text{Frequency} = \frac{A}{2\pi}}$

(iii) The maximum distance between the graph of the sine or cosine and the horizontal axis is called the **amplitude** of the function. Thus, the functions y = sinθ and y = cosθ have amplitude 1. In general, the amplitude of a periodic function is half of the difference between the maximum and minimum values.

12.2.4 Periodic, Even/Odd and Translation Properties of the Graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$

In **section 12.2,** we draw the graphs of all six trigonometric functions. If we examine the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$, we observe that they have many symmetry properties.

In this sections we are concerned with the periodic, even/odd and translation properties of the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

1. Symmetry properties of the graph of Sin θ

The graph of $\sin\theta$ is reproduced in **figure 12.30.**

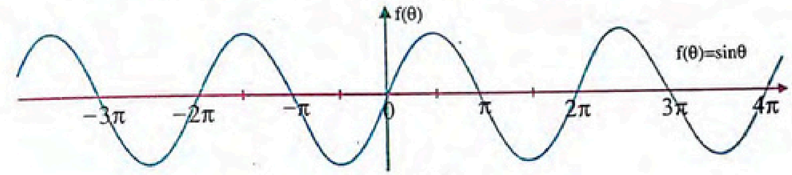


Figure 12.30

(a) Periodic Properties

We see that the graph of $\sin\theta$ keeps repeating itself after a period of 2π units. Therefore $\quad \sin(\theta \pm 2\pi) = \sin\theta$

This property possessing by $\sin\theta$ is called the **periodic property.**

(b) Even/Odd Property

The graph $\sin\theta$ is symmetrical about the origin. This means that if we replace θ by $-\theta$, the graph is changed. Therefore

$$\sin(-\theta) = -\sin\theta$$

This shows that $\sin\theta$ is an odd function which is in conformity with the results in theorem of **section 12.1.2.** This property possessing by $\sin\theta$ is called the **odd property.**

(c) Translation Property

If in **figure 12.30,** θ is decreased or increased by π, then the sign of $f(\theta)$ is changed. Therefore $\quad \sin(\theta - \pi) = -\sin\theta$

This property possessing by $\sin\theta$ is called the **translation property.**

Using the odd and translation properties, we have

$\sin(\pi-\theta) = \sin[-(\theta-\pi)] = -\sin(\theta-\pi) = -(-\sin\theta) = \sin\theta$

i.e. $\sin(\pi-\theta) = \sin\theta$

Thus, the graph of $\sin\theta$ possesses the following properties:

- Periodic property: $\sin(\theta \pm 2\pi) = \sin\theta$
- Odd Property: $\sin(-\theta) = -\sin\theta$
- Translation Property: $\begin{cases}\sin(\theta-\pi) = -\sin\theta \\ \sin(\pi-\theta) = \sin\theta\end{cases}$

2. **Symmetry Properties of the Graphs of $\cos\theta$**

The graph of $\cos\theta$ is reproduced in figure 12.31.

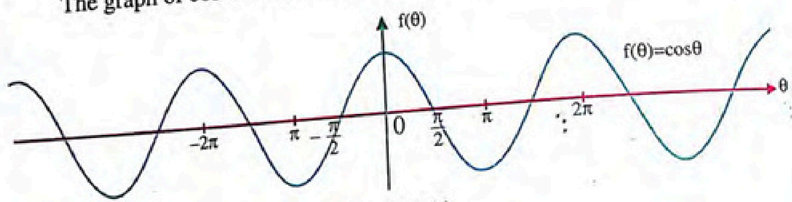


Figure 12.31

(a) **Periodic Properties**

Like $\sin\theta$, the graph of $\cos\theta$ also repeats itself after a period of 2π.

Therefore $\cos(\theta \pm 2\pi) = \cos\theta$

This property possessing by $\cos\theta$ is called the **periodic property.**

(b) **Even/Odd Property**

The graph of $\cos\theta$ is symmetrical about the y–axis. This means that if we replace θ by $-\theta$, the graph is unchanged. Therefore $\cos(-\theta) = \cos\theta$

This shows that $\cos\theta$ is an even function which is also in conformity with the results in theorem of section 12.1.2. This property possessing by $\cos\theta$ is called the **even property.**

(c) **Translation Property**

If in figure 12.31, θ is decreased or increased by π unit, then the sign of $f(\theta)$ is changed. Therefore $\cos(\theta-\pi) = -\cos\theta$

This property possessing by $\cos\theta$ is called the **translation property.**

Also $\cos(\pi-\theta) = \cos[-(\pi-\theta)] = \cos(\pi-\theta) = -\cos\theta$

i.e. $\cos(\pi-\theta) = -\cos\theta$

Thus, the graph of $\cos\theta$ possesses the following properties:

- Periodic property: $\cos(\theta \pm 2\pi) = \cos\theta$
- Even Property: $\cos(-\theta) = \cos\theta$
- Translation Property: $\begin{cases}\cos(\theta-\pi) = -\cos\theta \\ \cos(\pi-\theta) = -\cos\theta\end{cases}$

Symmetry properties of the graph of tanθ

The graph of tanθ is shown in **Figure 12.32.**

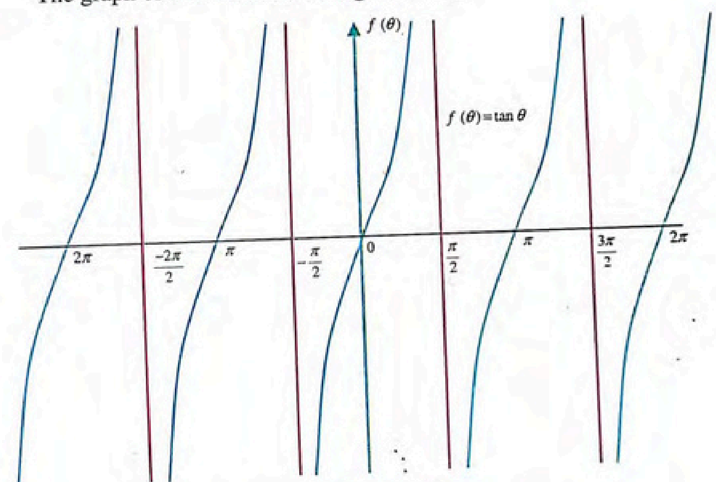


Figure 12.32

The symmetry properties of the graph of tanθ can be obtained in similar fashion as in the case of sinθ and cosθ. However, it is pertinent to note that in the present case the period of tanθ is π. Therefore, the translation property of the graph of tanθ equals its periodic property.

The properties of the graph of tanθ are given as below:

- Periodic Property: $\tan(\theta \pm \pi) = \tan\theta$
- Odd Property: $\tan(-\theta) = -\tan\theta$
- Translation Property: $\begin{cases} \tan(\theta - \pi) = \tan\theta \\ \tan(\pi - \theta) = -\tan\theta \end{cases}$

Example 9: Use the symmetric and periodic properties of the cosine, to establish the following identity. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

Solution:
By translating the graph of cosθ by $\frac{\pi}{2}$ units in the direction of the positive θ–axis the graph of cosθ becomes the graph of sinθ

That is $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$

But the cosine is an even function, so $\cos\left(\frac{\pi}{2} - \theta\right) = \cos\left(\theta - \frac{\pi}{2}\right)$

Thus, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

EXERCISE 12.2

1. Draw the graph of the following functions in the indicated interval.

   (i). $y = 2\sin x \quad 0 \le x \le 2\pi$ (ii). $y = \cos 2x \quad 0 \le x \le 2\pi$

   (iii) $y = -4 + \sin x \quad 0 \le x \le \pi$ (iv) $y = -\cot x \quad -\pi \le x \le \pi$

   (v) $y = 2\,\mathrm{cosec}\,2x \quad 0 \le x \le 2\pi$ (vi) $y = \sec\frac{x}{2} \quad \pi \le x \le 2\pi$

2. Without drawing, guess the graph of each of the following functions. Also find its period, frequency and amplitude.

   (i) $y = \cos 2\theta$ (ii) $y = \sin 6\theta$

   (iii) $y = \sin \pi\theta$ (iv) $y = \cos\frac{\pi}{2}\theta$

3. Use the symmetric and periodic properties of the sine, cosine and tangent functions to establish the following identities.

   (i) $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$ (ii) $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

   (iii) $\sin(\pi - \theta) = \sin\theta$ (iv) $\cos(\pi - \theta) = -\cos\theta$

   (v) $\tan(\pi - \theta) = -\tan\theta$ (vi) $\tan(2\pi - \theta) = -\tan\theta$

4. For any integer k, deduce that

   (i) $\sin(\theta + 2k\pi) = \sin\theta$ (ii) $\cos(\theta + 2k\pi) = \cos\theta$

   (iii) $\tan(\theta + 2k\pi) = \tan\theta$ (iv) $\cot(\theta + 2k\pi) = \cot\theta$

   (v) $\sec(\theta + 2k\pi) = \sec\theta$ (vi) $cosec(\theta + 2k\pi) = cosec\theta$

12.3 Solution/graphical solution of trigonometric equations

An equation involving trigonometric functions is called **a trigonometric equation.**

There is no general procedure for solving all trigonometric equations. However, we can solve many trigonometric equations by means of algebraic methods such as rearranging equations, factoring, squaring and taking roots and by using the basic trigonometric identities already proved in earlier units.

12.3.1 Solution of trigonometric functions of the type sinθ = k, cosθ = k and tanθ = k

The simplest trigonometric equations are of the form

$$\sin\theta = k \qquad (1)$$
$$\cos\theta = k \qquad (2)$$
$$\tan\theta = k \qquad (3)$$

> **Did You Know**
> An **identity** is an equation which is true for all values of the variable.

where k is a constant.

In this section, we are concerned with to solve these equations, using periodic, even/odd and translation properties.

In section 12.1.3, we noticed that the sine functions and cosine functions are periodic and both have period 2π, i.e. they repeat their values every 2π units. Thus, if we want to find all solutions of (1) and (2) then we simply add and subtract integer multiple of 2π to the solutions in the interval 0 ≤θ< 2π. We also noticed that tangent function is also periodic having π as its period. Thus to find all solutions of equation (3), we add and subtract integer multiple of π to the solutions in the interval $0 \leq \theta < \pi$.

Thus, to find all solutions of such trigonometric equations, first of all find the solution over the interval whose length is equal to its periods and then find the formula for all solutions of the equations.

Example 11: Solve the equation $\sin\theta = \frac{1}{2}$.

Solution: We have $\sin\frac{\pi}{6} = \frac{1}{2}$, so the reference angle is $\theta = \frac{\pi}{6}$. Since sine is positive in quadrant I and quadrant II, so the equation has two solutions in the interval $0 \leq \theta < 2\pi$, one in quadrant I and the other in quadrant II i.e.

$$\theta = \frac{\pi}{6} \ \ or \ \ \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Now to find all solutions of the equation, we add and subtract integer multiples of 2π to the solutions $\frac{\pi}{6}$ or $\frac{5\pi}{6}$.

$$\therefore \quad \theta = \frac{\pi}{6}, \frac{\pi}{6}+2\pi, \frac{\pi}{6}-2\pi, \frac{\pi}{6}+4\pi, \frac{\pi}{6}-4\pi, \ldots\ldots\ldots$$

$$or \quad \theta = \frac{5\pi}{6}, \frac{5\pi}{6}+2\pi, \frac{5\pi}{6}-2\pi, \frac{5\pi}{6}+4\pi, \frac{5\pi}{6}-4\pi, \ldots\ldots\ldots$$

These solutions can be written compactly as follows:

$$\theta = \frac{\pi}{6}+2\pi n \ \ or \ \ \theta = \frac{5\pi}{6}+2\pi n \ \ for \ n = 0, \pm 1, \pm 2, \ldots\ldots\ldots$$

Example 12: Solve the equation $\cos\theta = \frac{1}{2}$.

Solution: We have $\cos\frac{\pi}{3} = \frac{1}{2}$, so the reference angle is $\theta = \frac{\pi}{3}$. Thus, the equation has two solutions in the interval $0 \leq \theta < 2\pi$, one in quadrant I and the other in quadrant II i.e. $\theta = \frac{\pi}{3}$ or $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

To find all solutions of the equation, we add and subtract integer multiples of 2π to the solutions $\frac{\pi}{3}$ or $\frac{5\pi}{3}$.

$$\therefore \quad \theta = \frac{\pi}{3}, \frac{\pi}{3}+2\pi, \frac{\pi}{3}-2\pi, \frac{\pi}{3}+4\pi, \frac{\pi}{3}-4\pi, \ldots\ldots\ldots$$

$$or \quad \theta = \frac{5\pi}{3}, \frac{5\pi}{3}+2\pi, \frac{5\pi}{3}-2\pi, \frac{5\pi}{3}+4\pi, \frac{5\pi}{3}-4\pi, \ldots\ldots\ldots$$

Thus, $\theta = \frac{\pi}{3}+2\pi n$ or $\theta = \frac{5\pi}{3}+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots\ldots\ldots$ are all solutions of the equation.

Example 13: Solve the equation $\tan\theta = -\frac{\sqrt{3}}{3}$.

Solution: We have $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$, so the reference angle is $\theta = -\frac{\pi}{6}$. The tanθ is negative in the quadrant II and quadrant IV, however, in the interval $0 \leq \theta < \pi$, the equation has one solution in the quadrant II i.e. $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Thus, all solutions of the equation are given by

$$\theta = \frac{5\pi}{6}+2\pi n \ \text{for} \ n = 0, \pm 1, \pm 2, \ldots\ldots\ldots$$

12.3.2 Graphical Solution of some Trigonometric Equations

Recall that the graph of a function is the set of all points whose coordinates satisfy that function. If the graph of two functions intersects, then the coordinates of their intersection points represent a pair of numbers which satisfy both functions. The points of intersection are called the solutions of the given functions. These facts can be used to solve trigonometric equations by graphing. In this section, however, we are concerned with the graphical solution of trigonometric equation of the type:

- $\sin\theta = \frac{\theta}{2}$
- $\cos\theta = \theta$
- $\tan\theta = 2\theta$

in the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The method of graphical solution of such equations is illustrated through the following example.

Example 14: Use graph to find the solution of the equation $\cos\theta - \theta = 0$ in the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Solution: The equation $\cos\theta - \theta = 0$ can be written as $\cos\theta = \theta$

Let $y = \cos\theta$ and $y = \theta$

If we draw the graphs of these two functions on the same set of coordinate axes, then their intersection point (if any) must be the solution of the given equation.

We construct the tables of values of the two functions as follows:

$y = \cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
Y	0	-0.71	1	0.71	0

,

$y = \theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

θ	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$
Y	-0.79	0	0.79

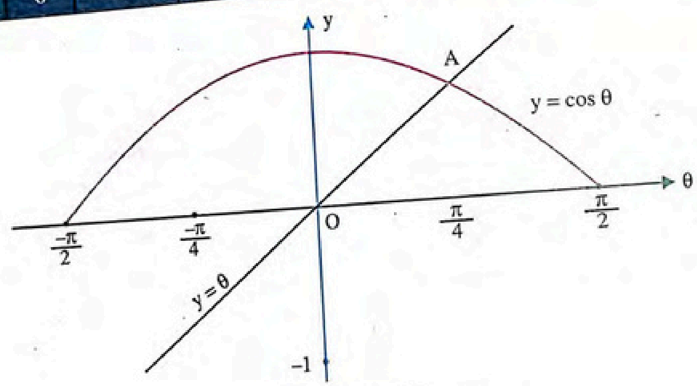


Figure 12.33

We see that the graphs intersect at point A. The point lies about midway between 0 and $\frac{\pi}{2}$. Thus, we estimate this solution as $\theta = \frac{\pi}{4}$.

Verification. Substituting the value of θ in the original equation, we obtain

$$\cos\left(\frac{\pi}{4}\right) - \frac{\pi}{4} = 0$$

$$\Rightarrow 0.71 - \frac{3.14}{4} = 0 \qquad \left(\because \pi = \frac{22}{7} = 3.14 \text{ (Approx.)}\right)$$

$$\Rightarrow 0.71 - 0.79 = 0$$

$$\Rightarrow -0.08 = 0$$

This agreement seems quite good for a graphical approximation.

Note

(1) If the graphs intersect at more than one point, the other solutions of the equation may similarly be estimated.

(2) We could have estimated the solution as coordinate pair (θ, y). However, the variable y does not appear in the original equation. Hence, we are interested only in values of the angle that satisfy the equation.

(3) A process of successive trial and error with use of trigonometric tables or scientific calculator would give the x–coordinate of the intersecting point as accurately as desired.

EXERCISE 12.3

1. Find all solutions of the trigonometric functions graphically.

(i) $\sin\theta = \frac{\sqrt{2}}{2}$ (ii) $\cos\theta = -\frac{\sqrt{3}}{2}$ (iii) $\tan\theta = \sqrt{3}$

(iv) $\cos\theta = \frac{1}{2}$ (v) $\tan\theta = -1$ (vi) $\sin\theta = -\frac{1}{2}$

12.4 Inverse Trigonometric Functions

12.4.1 Inverse trigonometric functions and their domain and range

We know that if f: $x \to y$ is one to one and onto, then there exists a unique function g: $y \to x$ such that $g(y) = x$, where $x \in X$ is such that $y = f(x)$. Thus, the function g: $y \to x$ such that g(y) = x … domain of g = range of f and range of g = domain of f. The function g is called the inverse of f and is denoted by f^{-1}.

Thus, $f(x) = y \Rightarrow f^{-1}(y) = x$

(a) The Inverse Sine Function

Reproducing the graph of the sine function

$\{(x, y) \mid y = \sin x \in R\}$

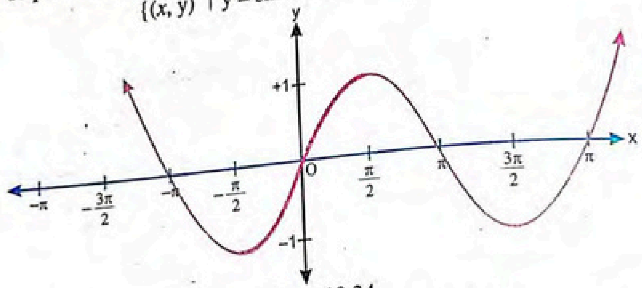


Figure 12.34

It follows from the horizontal line test that any line y = b, where b lies between –1 and +1 intersects the graph of y = sin x infinitely many times. Hence the function is not one to one. However, if we restrict the domain of y=sin x to the Interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the restricted function y = sin x, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ represented by bold portion of the curve in Figure 12.34 is one-to-one and hence will have an inverse. This new function with domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and range [–1, 1] is sometimes called **principal sine function** and is denoted by Sin$x$ (with capital S).

The inverse sine function denoted by Sin^{-1} is the inverse of the principal sine function and defined by:

$y = \text{Sin}^{-1}x$ if and only if $x = \text{Sin}\, y$, $-1 \le x \le 1$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

That graph of $y = \text{Sin}^{-1}x$ can be obtained by reflecting the restricted portion of y = Sin x about the line y = x. The reflected graph of $y = \text{Sin}^{-1}x$ is illustrated in bold portion.

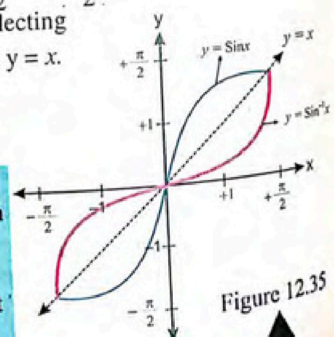


Figure 12.35

Note

Here $y = \text{Sin}^{-1}x$ means that y is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (both inclusive) whose sine is x.

The superscript –1 that appears in $y = \text{Sin}^{-1}x$ is not an exponent i.e. $\text{Sin}^{-1}x \ne \frac{1}{\text{Sin}x}$.

Inverse Relation of General Sine Functions

Generally $y = \sin^{-1}x$ gives the relation defined by

$y = \sin^{-1}x$ if and only if $x = \sin y$,

for $-1 \le x \le 1$, $y \in \mathbb{R}$

The various values obtained for a particular x represent the angles for which $x = \sin y$ and are called the inverse values of general sine functions. Since the domain of sin$x$ is not restricted, $\sin^{-1}x$ is not itself a function. This can be proved by vertical line test.

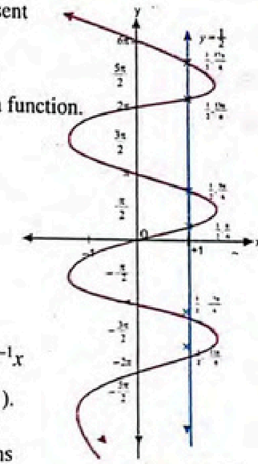


Figure 12.36

Example 15: Find the values of

(i) $\sin^{-1}(\frac{1}{2})$ (ii) $\text{Sin}^{-1}(\frac{1}{2})$

Solution (i): **Figure 12.36** shows the graph of $y = \sin^{-1}x$ for $y \in R$. The line $x = \frac{1}{2}$ cuts the graph at more than one point showing that $\sin^{-1}x$ is not a function. However the intersection of $y = \sin^{-1}x$ and $x = \frac{1}{2}$ provides the various values of $\sin^{-1}(\frac{1}{2})$.

Hence from the graph in **Figure 12.36** the solutions of $y = \sin^{-1}(\frac{1}{2})$ are

$y = \frac{\pi}{6} + 2k\pi, k \in Z$ or $y = \frac{5\pi}{6} + 2k\pi, k \in z$

i.e. $y \in \{\frac{\pi}{6} + 2k\pi, k \in z\} \cup \{\frac{5\pi}{6} + 2k\pi, k \in Z\}$

(ii) Only one of the above numerous values satisfies the equation $y = \text{Sin}^{-1}(\frac{1}{2})$; that is, the value which lies in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Looking at the graph again $\frac{\pi}{6}$ is such a value. Hence $y = \text{Sin}^{-1}(\frac{1}{2}) \Rightarrow y = \frac{\pi}{6}$.

The notation Arc sin (with capital A) is sometimes used for Sin^{-1}.

Example 16: Find the exact values of the following inverse functions

(i) $\text{Sin}^{-1}(1)$ (ii) $\text{Sin}^{-1}(\frac{\sqrt{3}}{2})$ (iii) $\text{Sin}^{-1}(-\frac{1}{2})$

after evaluating their corresponding inverse relations.

Solution: (i) Let $y = \text{Sin}^{-1}(1)$, we seek for the general sine function sin(y), the values of y whose sine is 1. They are $\{\frac{\pi}{2}+2n\pi,\ n\in Z\}$.

Out of these values, the solution of

$y = \text{Sin}^{-1}(1)$ is $\frac{\pi}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

(ii) Similarly the solutions of $y = \sin^{-1}(\frac{\sqrt{3}}{2})$ are

$\{\frac{\pi}{3}+2n\pi,\ n\in Z\} \cup \{2\frac{\pi}{3}+2n\pi,\ n\in Z\}$

However, $y = \text{Sin}^{-1}(\frac{\sqrt{3}}{2})$, where $y\in[-\frac{\pi}{2}, \frac{\pi}{2}]$ is only $y = \frac{\pi}{3}$.

(iii) The general solutions of $y = \sin^{-1}(-\frac{1}{2})$ are

$y \in \{-\frac{\pi}{6}+2n\pi,\ n\in Z\} \cup \{-\frac{7\pi}{6}+2n\pi\}$

However $y = \text{Sin}^{-1}(-\frac{1}{2})$ yields $y = -\frac{\pi}{6}$.

Important Results. The relationship $ff^{-1}(y) = y$ and $ff^{-1}(x) = x$ that hold for every inverse functions gives us the following important results.

$\text{Sin}^{-1}(\text{Sin}\, y) = y$ if $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\text{Sin}(\text{Sin}^{-1} x) = x$ if $-1 \le x \le 1$

Example 17: Find (i) $\text{Sin}^{-1}\left[\tan(\frac{3\pi}{2})\right]$ (ii) $\text{Sin}^{-1}\left[\tan\frac{\pi}{3}\right]$

Solution: (i) We know that $\tan\frac{3\pi}{2} = -1$

Let $y = \text{Sin}^{-1}\left[\tan\frac{3\pi}{4}\right]$, then $y = \text{Sin}^{-1}(-1)$

By definition $\text{Sin}\, y = -1$, if $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Thus y is an angle in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is -1, it follows $y = -\frac{\pi}{2}$

(ii) Let $y = \text{Sin}^{-1}\left[\tan\frac{\pi}{3}\right]$ As $\tan\frac{\pi}{3} = \sqrt{3}$ and $\sqrt{3} \notin [-1, 1]$

Hence no values of y exist which satisfies $y = \text{Sin}^{-1}(\sqrt{3})$

Thus the solution set of $y = \text{Sin}^{-1}\left[\tan\frac{\pi}{3}\right]$ is empty.

(b) The Inverse Cosine Function

In figure 12.37 we reproduce the graph of the function

$\{(x, y) \mid y = \cos x,\ x\in \mathbb{R}, -1 \le y \le 1\}$

Because every horizontal line y = b, where b lies between −1 and +1 intersects the graph of y = cos*x* at infinitely many points, it follows that cosine function is not one–to–one.

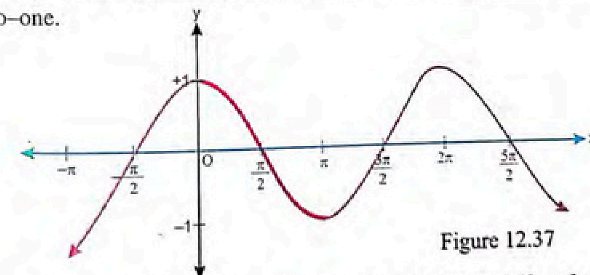


Figure 12.37

However if we restrict the domain of y = cos*x* to the interval $[0, \pi]$ as illustrated by the bold portion of the curve in figure 12.37, we obtain a decreasing function that takes on all the values of the cosine function one and only once. This new function is called the **principal Cosine function** and is denoted by Cos*x* (capital C).

The principal cosine function has an inverse denoted by Cos^{-1}.
The inverse cosine function Cos^{-1} is defined by

$$y = Cos^{-1}x \text{ if and only if}$$
$$x = Cos y, \text{ for } -1 \le x \le 1, \text{ and } 0 \le y \le \pi$$

This is also referred to Arc cosine function. Using general properties of inverse functions, we obtain

$$Cos\,(Cos^{-1}x) = Cos\,(Arc\,cos x) = x, \text{ if } -1 \le x \le 1,$$
$$Cos^{-1}(Cos\,y) = Arc\,cos\,(Cos\,y) = y, \text{ if } 0 \le y \le \pi.$$

The notation Arc cos$x$ (Capital A) is sometimes used instead of $Cos^{-1}x$.
The graph of the inverse cosine function can be found by reflecting the bold portion of Figure 12.38 in the line $y = x$. The resulting curve of $y = Cos^{-1}x$ is show in Figure 12.39 in bold portion.

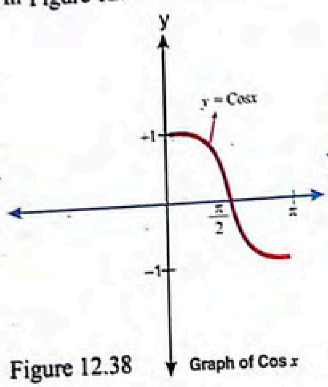
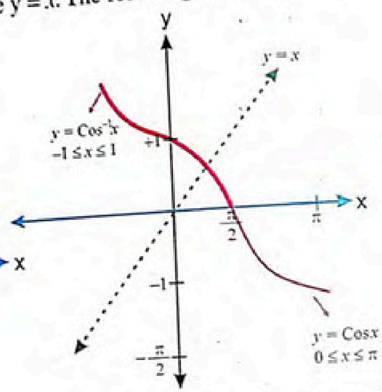


Figure 12.38 Graph of Cos x



Graph of Cos x and Arc Cos x Figure 12.39

Example 18: Find the exact values of

(i) $Cos^{-1}0$ (ii) $Cos^{-1}(\frac{1}{\sqrt{2}})$ (iii) $Cos^{-1}(-\frac{1}{2})$ (iv) $Cos^{-1}(-\frac{\sqrt{3}}{2})$

Solution: (i) Let $y = Cos^{-1}0$, we know that
$y = Arc\,cos\,(0)$ if and only if
$cos\,y = 0$ and $y \in [0, \pi]$

Consequently, $y = \frac{\pi}{2}$ and $Arc\,cos\,(0) = \frac{\pi}{2}$

(ii) Let $y = Cos^{-1}(\frac{1}{\sqrt{2}})$. We seek the angle $0 \le y \le \pi$, whose cosine equals $\frac{1}{\sqrt{2}}$.

$$Cos\,y = \frac{1}{\sqrt{2}}, 0 \le y \le \pi \quad \Rightarrow \quad y = \frac{\pi}{4}$$

Thus $Cos^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$

(iii) Let $y = Cos^{-1}(-\frac{1}{2})$, we seek the angle whose cosine equals $-\frac{1}{2}$. The reference point in the first quadrant is $\frac{\pi}{3}$.

Hence for negative sine we go to II quadrant by finding supplementary angle.

$$y = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \in [0, \pi]$$

Hence $Cos\,y = -\frac{1}{2} \Rightarrow y = \frac{2\pi}{3}$

Thus $Cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$

(iv) By definition.

$$y = Cos^{-1}(-\frac{\sqrt{3}}{2}), \text{ if and only if}$$
$$Cos y = -\frac{\sqrt{3}}{2}, \text{ and } 0 \le y \le \pi$$

The reference angle (1st quadrant) is $\frac{\pi}{6}$. But for negative cosine, y lies in the second quadrant (as $0 \le y \le \pi$).

Thus $y = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Hence $Cos^{-1}(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$

Example 19: Find (i) Arc cos (Cos2) (ii) Cos (Arc cos 0.5)
(iii) Arc cos (Cos4) (iv) Sin (Arc sin 2.463) (v) Arc cos (cos 4)
where the angles are measured in radians.

Solution: When finding these values we must pay attention to the ranges of principal trigonometric functions and their inverse functions.

(i) Since Arc cosine and Cosine (principal) are inverse function and 2 radians is between 0 and π. Hence

Arc cos (Cos2) = 2 radians

(ii) Let θ = Arc (Cos 0.5), then

Cos θ = 0.5 and by substitution

Cos (Arc cos 0.5) = Cos (θ) = 0.5

(iii) For Arc cos (cos 4), we see that cos4 (general function) has the angle 4 radians in the third quadrant and therefore cos4 is negative. The Arc cosine (inverse function) of a negative value will be a second quadrant angle.

Hence Arc cos (cos4) = Arc cos (–0.653644)

$\approx$ 2.2832 radians.

(iv) (Arc sin 2.463) is not defined, since 2.463 is not between –1 and +1.

(v) Principal function Cos4 is not defined as 4 does not lie between 0 and π.

Hence Arc cos (cos 4) does not exist.

(c) The Inverse Tangent function

The graph of tangent function shows that every horizontal line intersects the graph infinitely many times,it follows that tangent function is not one–to–one.

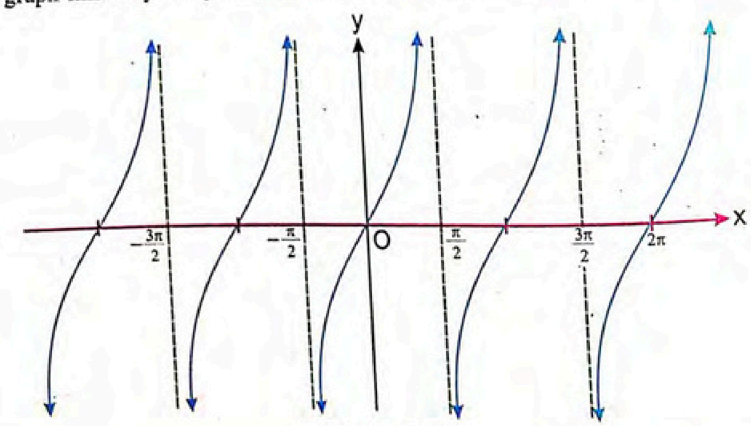


Figure 12.40

However if we restrict the domain to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, the restricted function $y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ is one–to–one and hence has an inverse. This is called the **principal tangent function** and is denoted by y = Tan$x$ (Capital T). This leads to the definition of inverse tangent function as follows.

The inverse tangent function Tan^{-1} is defined by $y = \text{Tan}^{-1} x$ if and only if

$$x = \text{Tan } y, \text{ where } -\infty < x < \infty, -\frac{\pi}{2} < y < \frac{\pi}{2}$$

The graph of $\text{Tan}^{-1} x$ can be obtained as before by reflecting the principal Tangent function in the line $y = x$ as shown below:

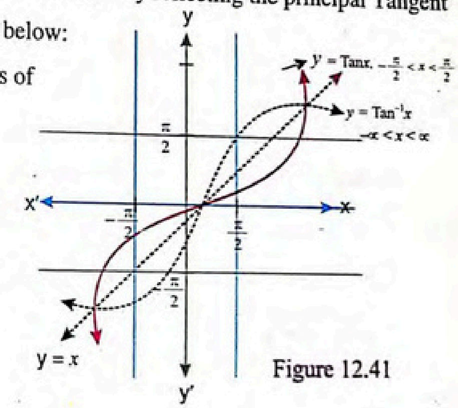


Figure 12.41

Example 20: Find the exact values of

(i) $\text{Tan}^{-1}(1)$ (ii) $\text{Tan}^{-1}(-\sqrt{3})$

(iii) $\text{Tan}(\frac{3\pi}{2})$

Solution: Let $y = \text{Tan}^{-1}(1)$.

We seek the angle y, $-\frac{\pi}{2} < y < \frac{\pi}{2}$ whose tangent equals 1, i.e.,

$\text{Tan } y = 1$, for $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Rightarrow y = \frac{\pi}{4}$

Therefore $y = \text{Tan}^{-1}(1) = \frac{\pi}{4}$

(ii) Let $y = \text{Tan}^{-1}(-\sqrt{3})$. We seek the angle y where $-\frac{\pi}{2} < y < \frac{\pi}{2}$

whose tangent $-\sqrt{3}$, that is

$\text{Tan } y = -\sqrt{3}$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$

The reference angle in the first quadrant is $\frac{\pi}{3}$. Since $\tan(-\theta) = -\tan\theta$.

Hence $y = \text{Tan}^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

(iii) $\text{Tan}(\frac{3\pi}{2})$ where Tan represents principal tangent function exists only for the angles in the range between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since $\frac{3\pi}{2} \notin (-\frac{\pi}{2}, \frac{\pi}{2})$.

Hence $\text{Tan}(\frac{3\pi}{2})$ is not defined.

Example 21: Find the exact value of

(i) $\text{Sin}(\text{Cos}^{-1}\frac{\sqrt{3}}{2})$ (ii) $\text{Cos}[\text{Tan}^{-1}(-1)]$

(iii) $\text{Sec}(\text{Sin}^{-1}\frac{1}{2})$ (iv) $\cos[\text{Tan}^{-1}(-1)]$

Solution: (i) We first find the angle, $y \in [0, \pi]$ such that

$$\text{Cos } y = \frac{\sqrt{3}}{2}$$

$$\text{or } y = \text{Cos}^{-1}(\frac{\sqrt{3}}{2}) \Rightarrow y = \frac{\pi}{6} \in [0, \pi]$$

$$\text{Now Sin}(\text{Cos}^{-1}\frac{\sqrt{3}}{2}) = \text{Sin}(y)$$

$$= \text{Sin}(\frac{\pi}{6}) = \frac{1}{2}$$

(ii) Let $y = \text{Tan}^{-1}(-1)$. We first seek the angel $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for which

$$\text{Tan } y = -1 \Rightarrow y = \frac{\pi}{4} \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

Now $\text{Cos}(\text{Tan}^{-1}(-1)) = \text{Cos}(y) = \text{Cos}(-\frac{\pi}{4})$ is not defined because Cos is the principal Cosine function whose angle must be in the interval 0 to π.

Since $-\frac{\pi}{4} \notin [0, \pi]$, hence $\text{Cos}(-\frac{\pi}{4})$ is not defined.

(iii) For $\text{Sec}(\text{Sin}^{-1}\frac{1}{2})$, let $y = \text{Sin}^{-1}\frac{1}{2} \Rightarrow \text{Sin } y = \frac{1}{2}$, then by definition y

is the angle in the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that

$$\text{Sin } y = \frac{1}{2} \Rightarrow y = \frac{\pi}{6}$$

$$\text{Hence } \text{Sec}(\text{Sin}^{-1}\frac{1}{2}) = \text{Sec}(\frac{\pi}{6}) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

(iv) As in part (ii) $\text{Tan}^{-1}(-1) = -\frac{\pi}{4}$

Here $\cos\theta$ is the general cosine function.

Hence $\cos[\text{Tan}^{-1}(-1)] = \cos(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

(d) The Remaining inverse trigonometric functions

The inverse cotangent, inverse secant and inverse cosecant are not used very widely. However, we list their definition as follows:

(i) $y = \text{Cot } x$, where $0 < x < \pi$ is called **Principal Cotangent Function** which is one–to–one and has an inverse.

$y = \text{Cot}^{-1}x$ means $x = \text{Cot } y$, where $0 < y < \pi$ and $x \in (-\infty, +\infty)$

(ii) $y = \text{Sec } x$, where $0 \le x \le \pi$, $x \ne \frac{\pi}{2}$ is called the **Principal Secant Function** which is one–to–one and has an inverse.

$y = \text{Sec}^{-1}x$ means $x = \text{Sec } y$ where

$$0 \le y \le \pi,\ y \ne \frac{\pi}{2} \text{ and } |x| \ge 1$$

(iii) $y = \text{Cosec } x$ where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}, x \ne 0$ is called the **Principal Cosecant Function**, which is one–to–one and has an inverse.

$y = \text{Cosec}^{-1}x = \text{Csc}^{-1}x$ means $x = \text{Csc}y$ where, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $|x| \ge 1$

12.4.2 Domains and ranges of principal trigonometric function and inverse trigonometric functions

For convenience, the domains and ranges of principal trigonometric functions and their inverses are listed in the following table.

Functions	Domains	Range
$y = \sin x$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$-1 \le y \le 1$
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos x$	$0 \le x \le \pi$	$-1 \le y \le 1$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$y \in IR$
$y = \tan^{-1} x$	$x \in IR$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \text{Cosec}\, x$	$x \in [-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$	$y \le -1,\ y \ge 1$
$y = \text{Cosec}^{-1} x$	$x \le -1,\ x \ge 1$	$y \in [-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$
$y = \sec x$	$x \in [0, \pi] - \{\frac{\pi}{2}\}$	$y \le -1,\ y \ge 1$
$y = \sec^{-1} x$	$x \le -1,\ x \ge 1$	$y \in [0, \pi] - \{\frac{\pi}{2}\}$
$y = \cot x$	$x \in (0, \pi)$	$y \in IR$
$y = \cot^{-1} x$	$x \in IR$	$y \in (0, \pi)$

Example 22: Evaluate: (i) Arc sec 2 (ii) Arc sec (–2) (iii) Arc tan (3.5) (iv) Arc tan (–2.3)

Solution: (i) Let θ = Arc sec2, which is an inverse function. By definition,

$\sec\theta = 2$, where $\theta \in [0,\pi] - \{\frac{\pi}{2}\}$

We Know that $\sec\frac{\pi}{3} = 2 \Rightarrow \theta = \frac{\pi}{2} \in [0, \pi]$

(ii) Let θ = Arc sec(–2). This is an inverse relation not a function. Therefore, there are infinitely many values for θ. Since $\sec\theta$ is negative, the reference angle lies both in the quadrants (II) and (III) which are $\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\theta_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

Hence, adding the multiples of periods of sec i.e. 2πn, we get

$$\left\{\theta \in \frac{2\pi}{3} + 2n\pi\ ,\ \frac{4\pi}{3} + 2n\pi,\ \text{n is any integer.}\right\}$$

(iii) Arc Tan (3.5) is an inverse function whose solution must lie in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Since exact values for Arc tan(3.5) are not known, we put a calculator in radian mode, to get

Arc tan (3.5) $\simeq$ 1.2925 rad.

(iv) By definition of inverse tangent function $-\frac{\pi}{2} <$ Arc Tan $< \frac{\pi}{2}$. Using a calculator we have, Arc tan (–2.3) $\simeq$ –1.16

Example 23: Evaluate: (i) $\tan[\cos^{-1}(-\frac{1}{2})]$ (ii) $\tan[\text{Cos}^{-1}(-\frac{1}{2})]$

(iii) $\text{Tan}[\cos^{-1}(-\frac{1}{2})]$ (iv) $\text{Tan}[\text{Cos}^{-1}(-\frac{1}{2})]$

Solution: (i) For $\cos^{-1}(-\frac{1}{2})$, we seek an angle whose cosine is $(-\frac{1}{2})$.

The reference point is $\frac{\pi}{3}$. But cosine is negative in the II and III quadrants.

Hence the required angles are $(\pi - \frac{\pi}{3})$ and $(\pi + \frac{\pi}{3})$. Adding the period

$2n\pi$ we get, $\cos^{-1}(-\frac{1}{2}) \in \{\frac{2\pi}{3} + 2n\pi\} \cup \{\frac{4\pi}{3} + 2n\pi\}, n \in Z$

Now $\tan(\frac{2\pi}{3} + 2n\pi) = \tan(\frac{2\pi}{3}) = -\sqrt{3}$ and $\tan(\frac{4\pi}{3} + 2n\pi) = \tan(\frac{4\pi}{3}) = +\sqrt{3}$

$\therefore \tan(\cos^{-1}(-\frac{1}{2})) = \tan\frac{2\pi}{3} \cup \{\tan\frac{4\pi}{3}\} = \{-\sqrt{3}, +\sqrt{3}\}$

(ii) $\text{Cos}^{-1}(-\frac{1}{2}) = \text{Arc Cos}(-\frac{1}{2}) = 2\pi \in [0, \pi]$ But $\tan(\frac{2\pi}{3}) = -\sqrt{3}$.

Therefore $\tan(\text{Cos}^{-1}(-\frac{1}{2})) = \{-\sqrt{3}\}$ only

(iii) Again $\cos^{-1}(-\frac{1}{2}) = \{\frac{2\pi}{3}+2n\pi\} \cup \{\frac{4\pi}{3}+2n\pi\}$

But since $\frac{2\pi}{3} \notin (-\frac{\pi}{2}, \frac{\pi}{2})$ and also $\frac{4\pi}{3} \notin (-\frac{\pi}{2}, \frac{\pi}{2})$

Neither $\frac{2\pi}{3}$ nor $\frac{4\pi}{3}$ could be the argument of principal tangent function. Thus

$\text{Tan}(\cos^{-1}(-\frac{1}{2}))$ does not exist.

(iv) Similarly $\text{Tan}[\text{Arc}\cos(-\frac{1}{2})] = \text{Tan}(\frac{2\pi}{3})$ is not defined.

Example 24: Evaluate

(i) $\text{Arc}\sin(\sin\frac{12\pi}{5})$ (ii) $\text{Arc}\sin(\sin\frac{12\pi}{5})$

(ii) $\text{Arc}\cos(\cos\frac{29\pi}{7})$ (iv) $\text{Arc}\cos(\cos\frac{29\pi}{7})$

Solution: (i) As $\frac{12\pi}{5} \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$, it follows that

$$\text{Arc Sin}(\sin\frac{12\pi}{5}) \neq \frac{12\pi}{5}$$

However, since $\sin(\frac{12\pi}{5}) = \sin[2\pi + \frac{2\pi}{5}] = \sin\frac{2\pi}{5}$

and $\frac{2\pi}{5} \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$, we find that

$$\text{Arc}\sin(\sin\frac{12\pi}{5}) = \text{Sin}^{-1}(\sin\frac{2\pi}{5}) = \frac{2\pi}{5}$$

(ii) By definition of inverse relation of sine function

$$\text{Arc}\sin(\sin\frac{12\pi}{5}) = \frac{12\pi}{5}$$

(iii) $\text{Arc Cos}(\cos\frac{29\pi}{7}) \neq \frac{29\pi}{7}$,

because $\frac{29\pi}{7} \notin [0, \pi]$.

But $\cos\frac{29\pi}{7} = \cos(4\pi + \frac{\pi}{7}) = \cos\frac{\pi}{7}$ and $\frac{\pi}{7} \in [0,\pi]$.

Hence $\text{Arc}\cos(\cos\frac{29\pi}{7}) = \frac{\pi}{7}$

(iv) By definition of inverse relation of general sine function, we see that

$\text{Arc}\cos(\cos\frac{29\pi}{7})$ exists when

$\text{Arc}\cos(\cos\frac{29\pi}{7}) = \frac{29\pi}{7}$

Example 25: Find the value of $\sin[\text{Tan}^{-1}(-x)]$, x being a positive number

Solution: Let $\text{Tan}^{-1}(-x) = \theta$. Then $\tan\theta = -x$, and θ lies between $-\pi/2$ and 0. If angle θ is constructed in standard position, as shown in **Figure 12.42**, then $\sin\theta$ is found to be $\frac{-x}{\sqrt{1+x^2}}$.

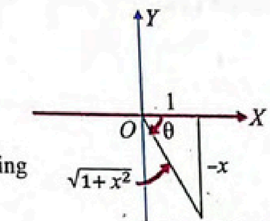


Figure 12.42

Hence, $\sin[\text{Tan}^{-1}(-x)] = \frac{-x}{\sqrt{1+x^2}}$

EXERCISE 12.4

1. Evaluate the following inverse relations of general trigonometric functions.

(i). arc $\sin(-1)$ (ii) arc $\cos(-\frac{\sqrt{2}}{2})$ (iii) arc $\tan(-\frac{\sqrt{3}}{3})$

2. Compute the following expressions

(i) $\text{arc}\cos\left[\tan\frac{3\pi}{4}\right]$ (ii) $Sin[Tan^{-1}(\frac{1}{\sqrt{3}})]$

(iii) $Sin\left[arc\cos\left(\frac{-\sqrt{3}}{2}\right)\right]$ (iv) $\tan[arc\cos(\frac{-4}{5})]$

3. Find the exact value of each expression.

(i) $\text{Cos}[Sin^{-1}\frac{\sqrt{2}}{2}]$ (ii) $\text{Tan}[Cos^{-1}\frac{\sqrt{3}}{2}]$ (iii) $\text{Sec}[Cos^{-1}\frac{1}{2}]$

(iv) $\text{Cosec}[Tan^{-1}(1)]$ (v) $\text{Sin}[Tan^{-1}(-1)]$ (vi) $\text{Sec}[Sin^{-1}(\frac{1}{2})]$

4. Simplify the given expression, taking u as a positive number.

i) $\operatorname{cosec}(\operatorname{Sin}^{-1}\frac{1}{u})$ ii) $\tan(\operatorname{Tan}^{-1}u)$

iii) $\tan(\operatorname{Cos}^{-1}\frac{1}{\sqrt{1+u^2}})$ iv) $\operatorname{Cos}^{-1}(\cos\sqrt{1-u^2})$

12.4.3 Graphs of Inverse trigonometric functions

Inverse Trigonometric Identities

It is hard to evaluate Arc secant, Arc cosecant, or Arc cotangent functions, when their exact values are not known, since most of the calculators or computers are not programmed for these functions. For this purpose we introduce the inverse of the reciprocal functions. The procedure is summarized by the following inverse identifies:

1. $\operatorname{Cosec}^{-1}x = \operatorname{Sin}^{-1}(\frac{1}{x}), x \neq 0$
2. $\operatorname{Sec}^{-1}x = \operatorname{Cos}^{-1}(\frac{1}{x}), x \neq 0$
3. $\operatorname{Cot}^{-1}x = \operatorname{Tan}^{-1}(\frac{1}{x}), x > 0$
4. $\operatorname{Cot}^{-1}x = \pi + \operatorname{Tan}^{-1}(\frac{1}{x}), x < 0$
5. $\operatorname{Sin}^{-1}(-x) = -\operatorname{Sin}^{-1}(x)$
6. $\operatorname{Cos}^{-1}(-x) = \pi - \operatorname{Cos}^{-1}(x)$
7. $\operatorname{Tan}^{-1}(-x) = -\operatorname{Tan}^{-1}(x)$
8. $\operatorname{Sin}^{-1}(x) = \frac{\pi}{2} - \operatorname{Cos}^{-1}(x)$

Proof. (1) Let $y = \operatorname{Sin}^{-1}\frac{1}{x}, x \neq 0$ (I)

$\Rightarrow \operatorname{Sin} y = \frac{1}{x}, x \neq 0 \Rightarrow \frac{1}{\operatorname{Sin} y} = x$

$\Rightarrow \operatorname{Cosec} y = x \Rightarrow y = \operatorname{Cosec}^{-1}x$ (II)

From (I) and (II) $\operatorname{Sin}^{-1}\left(\frac{1}{x}\right) = \operatorname{Cosec}^{-1}x$

Similarly (2) can be proved.

To prove (3), let

$y = \operatorname{Cot}^{-1}x$ where $x > 0$ (III)

$\Rightarrow y = \operatorname{Cot}^{-1}x, 0 < y < \pi/2$ $\Rightarrow \operatorname{Cot} x = y, 0 < y < \pi/2$

$\Rightarrow 1/\operatorname{Tan}x = y, 0 < y < \pi/2$ $\Rightarrow \operatorname{Tan}x = 1/y, 0 < y < \pi/2$

$\Rightarrow x = \operatorname{Tan}^{-1}(1/y)$ where $x > 0$ (IV)

From (III) and (IV) $\operatorname{Cot}^{-1}x = \operatorname{Tan}^{-1}(\frac{1}{x}), x > 0$

To prove (5), let

$y = -\operatorname{Sin}^{-1}(x)$ (V)

$\Rightarrow -y = \operatorname{Sin}^{-1}(x) \Rightarrow x = \operatorname{Sin}(-y)$

$\Rightarrow x = -\operatorname{Sin}(y) \Rightarrow \operatorname{Sin} y = -x$

$\Rightarrow y = \operatorname{Sin}^{-1}(-x)$ (VI)

From (V) and (VI), $\operatorname{Sin}^{-1}(-x) = -\operatorname{Sin}^{-1}(x)$

Similarly (6) and (7) can be proved. Finally to prove (8)

let $\theta = \frac{\pi}{2} - \operatorname{Cos}^{-1}x$ (VII)

i.e. $\operatorname{Cos}^{-1}x = \frac{\pi}{2} - \theta \Rightarrow x = \operatorname{Cos}(\frac{\pi}{2} - \theta)$, for $0 \leq \frac{\pi}{2} - \theta \leq \pi$

Now $0 \leq \frac{\pi}{2} - \theta \leq \pi$ implies $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and in this range for θ, $\operatorname{Sin}\theta$ exists.

Hence from (5) $x = \operatorname{Cos}(\frac{\pi}{2} - \theta) = \operatorname{Sin}\theta$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Now $x = \operatorname{Sin}\theta \Rightarrow \theta = \operatorname{Sin}^{-1}x$ (VIII)

Substituting θ from (VIII) in (VII), we have $\operatorname{Sin}^{-1}x = \frac{\pi}{2} - \operatorname{Cos}^{-1}x$

Example 26: Solve the equation $2\operatorname{Sin}^{-1}x - \operatorname{Cos}^{-1}x = \frac{\pi}{2}$

Solution: The given equation can be written as

$$2\operatorname{Sin}^{-1}x - \operatorname{Cos}^{-1}x + \frac{\pi}{2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

i.e. $2\operatorname{Sin}^{-1}x + \operatorname{Sin}^{-1}x = \pi$ $\left(\because \sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x\right)$

or $3\operatorname{Sin}^{-1}x = \pi$

$\Rightarrow \operatorname{Sin}^{-1}x = \frac{\pi}{3} \Rightarrow x = \operatorname{Sin}\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Example 27: Evaluate $\text{Cos}\,(\text{Sin}^{-1}\frac{4}{5} + \text{Cos}^{-1}\frac{3}{5})$ without tables or a calculator.

Solution: Let $x = \text{Sin}^{-1}\frac{4}{5}$ and $y = \text{Cos}^{-1}\frac{3}{5}$ then

$\text{Sin}\,x = \frac{4}{5}$ and $\text{Cos}\,y = \frac{3}{5}$ Where x and y are in 1st quadrant.

We have $\text{Cos}\,(x + y) = \text{Cos}\,x\,\text{Cos}\,y - \text{Sin}\,x\,\text{Sin}\,y$.

We know Sin x, but need to find Cos x, where

$$\text{Cos}\,x = \sqrt{1-\text{Sin}^2 x}$$, as Cos x is +ve in 1st quadrant

$$= \sqrt{1-\frac{16}{25}} = \frac{3}{5}$$

Again we know Cos y but need to find Sin y, where

$$\text{Sin}\,y = \sqrt{1-\text{Cos}^2 y}$$, Sin y is +ve in Quadrant 1

$$= \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$

Therefore $\text{Cos}\left[\text{Sin}^{-1}\frac{4}{5}+\text{Cos}^{-1}\frac{3}{5}\right] = \text{Cos}\,(x + y)$

$$= \text{Cos}\,x\,\text{Cos}\,y - \text{Sin}\,x\,\text{Sin}\,y$$

$$= \frac{3}{5}\times\frac{3}{5} - \frac{4}{5}\times\frac{4}{5} = -\frac{7}{25}$$

Example 28: Evaluate $\text{Sin}(\text{Arc tan}\,\frac{1}{2} - \text{Arc cos}\,\frac{4}{5})$

Solution: Let $u = \text{Arc tan}\,\frac{1}{2}$ and $v = \text{Arc cos}\,\frac{4}{5}$,

then $\text{Tan}\,u = \frac{1}{2}$ and $\text{Cos}\,v = \frac{4}{5}$.

As Tan u is +ve, and $u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ hence u must be positive i.e.

$u \in [0, \frac{\pi}{2}]$. Similarly Cosv being positive means $v \in [0, \frac{\pi}{2}]$.

We wish to find Sin (u – v). Since u and v are in the interval

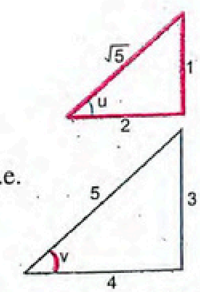


Figure 12.43

$[0, \frac{\pi}{2}]$. They can be considered as the radian measure of positive acute angles and we may construct right angled triangles for u and v as shown in **fig 12.43.** These triangles show that

$$\text{Sin}\,u = \frac{1}{\sqrt{5}}, \quad \text{Sin}\,v = \frac{3}{5}$$

$$\text{Cos}\,u = \frac{2}{\sqrt{5}}, \text{ etc.}$$

Hence,

$$\text{Sin}\,(u - v) = \text{Sin}\,u\,\text{Cos}\,v - \text{Cos}\,u\,\text{Sin}\,v$$

$$= \frac{1}{\sqrt{5}}\times\frac{4}{5} - \frac{2}{\sqrt{5}}\times\frac{3}{5} = \frac{4-6}{5\sqrt{5}} = \frac{-2}{5\sqrt{5}} = \frac{-2\sqrt{5}}{25}.$$

Example 29: Write (i) $\text{Cos}\,(\text{Sin}^{-1} x)$

(ii) $\text{Cos}\,(\sin^{-1} x)$ as an algebraic expression.

Solution: (i) To simplify, let $y = \text{Sin}^{-1} x$ Then $\text{Sin}\,y = x$ for $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

We wish to find an algebraic expression for $\text{Cos}\,(\text{Sin}^{-1} x) = \text{Cos}\,y$

Since $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, it follows that

$\text{Cos}\,y = +\sqrt{1-\text{Sin}^2 y} = \sqrt{1-x^2}$

Consequently, $\text{Cos}\,(\text{Sin}^{-1} x) = \sqrt{1-x^2}$

(ii) Let $y = \sin^{-1} x$, $-1 \le x \le 1$ which is an inverse relation

$\Rightarrow x = \sin y$ is not a principal function. Hence its argument y is any real number of the set $\mathbb{R}$. Consequently. $\text{Cos}\,y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}$

or $\text{Cos}\,(\sin^{-1}x) = \pm\sqrt{1-x^2}$

Example 30: Express tan (Arc sin x) as an algebraic expression in x if $-1 < x < 1$.

Solution: Let $y = \text{Arc sin}\,x \Rightarrow x = \text{Sin}\,y$, $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Since $\tan(-\frac{\pi}{2})$ and $\tan(\frac{\pi}{2})$ are not defined, we seek to find

$\tan y$ for $y \in (-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq [-\frac{\pi}{2}, \frac{\pi}{2}]$ for which $-1 \le x \le 1$

(i) If x is positive then $y \in (0, \frac{\pi}{2})$.

Figure 12.44 (i) shows the triangle for y.

(ii) If x is negative, then

$y \in (-\frac{\pi}{2}, 0)$ and the triangle for y is shown in **Figure 12.44 (ii)**

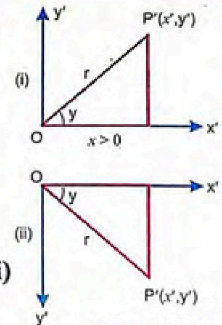


Figure 12.44

From each of the triangles, $x'^2 + y'^2 = r^2$ $\left[\sin y = \frac{y'}{r} = x'\right]$

$\Rightarrow x'^2 = r^2 - y'^2 = r^2 - r^2 x'^2 = r^2(1 - x'^2)$

$\Rightarrow$ $x' = r\sqrt{1 - x'^2}$ as x' is positive in both the cases whether

$y \in (0, \frac{\pi}{2})$ or $y \in (-\frac{\pi}{2}, 0)$

Remember

tan(Arc sin$x$) itself is positive if x is positive and negative if x is negative.

Thus $\tan y = \frac{y'}{x'} = \frac{rx}{r\sqrt{1-x^2}}$ i.e. $\tan(\text{Arc}\sin x) = \frac{x}{\sqrt{1-x^2}}$

Example 31: Verify the identify

$$\frac{1}{2}\text{Cos}^{-1}x = \text{Tan}^{-1}\sqrt{\frac{1-x}{1+x}} \quad \text{for } |x| < 1$$

Solution: Let $y = \text{Cos}^{-1}x$, we wish to show $\frac{1}{2}y = \text{Tan}^{-1}\sqrt{\frac{1-x}{1+x}}$

By half angle formula $\quad \text{Tan}\frac{y}{2} = \sqrt{\frac{1-\text{Cos}\,y}{1+\text{Cos}\,y}}$

Since $y = \text{Cos}^{-1}x$ and $|x| < 1$, it follows that $|Cosy| < 1$ and $y \in (0, \pi)$

Consequently $\frac{y}{2} \in [0, \frac{y}{2}]$ and thus $\text{Tan}\frac{y}{2} > 0$

We may drop the absolute value, obtaining $\text{Tan}\frac{y}{2} = \sqrt{\frac{1-\text{Cos}\,y}{1+\text{Cos}\,y}} = \sqrt{\frac{1-x}{1+x}}$

Thus $\frac{y}{2} = \text{Tan}^{-1}\sqrt{\frac{1-x}{1+x}}$, as required.

12.4.4 Addition and Subtraction Formulae of Inverse Trigonometric Functions

In this section we aim at to prove some important addition and subtraction formulae of inverse trigonometric functions.

1. $Sin^{-1}A + Sin^{-1}B = Sin^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})$

Let $x = Sin^{-1}A \Rightarrow Sinx = A$ and $y = Sin^{-1}B \Rightarrow Siny = B$

Since $Cos^2x + Sin^2x = 1$, so $Cosx = \pm\sqrt{1 - Sin^2x} = \pm\sqrt{1-A^2}$

For Sin x = A, domain is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in which Cosine is positive, so

$Cos\,x = \sqrt{1-A^2}$. Similarly $Cos\,y = \sqrt{1-B^2}$, We have

$\text{Sin}(x+y) = \text{Sin}\,x\,\text{Cos}\,y + \text{Cos}\,x\,\text{Sin}\,y$

$\text{Sin}(x+y) = \text{Sin}\,x\text{Cos}\,y + \text{Cos}\,x\text{Sin}\,y$

$\Rightarrow Sin(x+y) = A\sqrt{1-A^2} + B\sqrt{1-B^2} \Rightarrow x + y = Sin^{-1}\left(A\sqrt{1-A^2} + B\sqrt{1-B^2}\right)$

$\Rightarrow Sin^{-1}A + Sin^{-1}B = Sin^{-1}\left(A\sqrt{1-A^2} - B\sqrt{1-B^2}\right)$

2. $Sin^{-1}A - Sin^{-1}B = Sin^{-1}\left(A\sqrt{1-A^2} - B\sqrt{1-B^2}\right)$

Proof of this formula is similar to (1), so is left as an exercise

3. $Cos^{-1}A + Cos^{-1}B = Cos^{-1}\left(AB - \sqrt{1-A^2}\sqrt{1-B^2}\right)$

Let $x = Cos^{-1}A \Rightarrow Cosx = A$ and $y = Cos^{-1}B \Rightarrow Cos\,y = B$

We have $\sin x = \pm\sqrt{1 - Cos^2x} = \pm\sqrt{1-A^2}$

For $Cos\,x = A$, domain is $[0, \pi]$ in which Sine is positive,

So $Sin\,x = \sqrt{1-A^2}$. Similarly $Sin\,x = \sqrt{1-B^2}$

Now $Cos(x+y) = Cos\,xCos\,y - Sin\,x\,Sin\,y$

$\Rightarrow Cos(x+y) = AB - \sqrt{1-A^2}\sqrt{1-B^2} \Rightarrow x + y = Cos^{-1}\left(AB - \sqrt{1-A^2}\sqrt{1-B^2}\right)$

$\Rightarrow Cos^{-1}A + Cos^{-1}B = Cos^{-1}\left(AB - \sqrt{1-A^2}\sqrt{1-B^2}\right)$

4. $Cos^{-1}A - Cos^{-1}B = Cos^{-1}\left(AB + \sqrt{1-A^2}\sqrt{1-B^2}\right)$

Proof is left as an exercise.

5. $Tan^{-1}A + Tan^{-1}B = Tan^{-1}\dfrac{A+B}{1-AB}$

Let $x = Tan^{-1}A \Rightarrow Tan x = A$ and $y = Tan^{-1}B \Rightarrow Tan y = B$. We have

$$Tan(x+y) = \frac{Tan\,x + Tan\,y}{1 - Tan\,x\,Tan\,y} \Rightarrow Tan(x+y) = \frac{A+B}{1-AB} \Rightarrow Tan^{-1}A + Tan^{-1}B = Tan^{-1}\frac{A+B}{1-AB}$$

6. $\Rightarrow Tan^{-1}A - Tan^{-1}B = Tan^{-1}\dfrac{A-B}{1+AB}$

Proof is left as an exercise.

Example 32: Show that $2\,Tan^{-1}A = Tan^{-1}\dfrac{2A}{1-A^2}$

Solution: Put B = A in the inverse trigonometric formula

$$Tan^{-1}A + Tan^{-1}B = Tan^{-1}\frac{A+B}{1-AB}, \text{ we have}$$

$$Tan^{-1}A + Tan^{-1}A = Tan^{-1}\frac{A+A}{1-A\cdot A} = Tan^{-1}\frac{2A}{1-A^2}$$

Example 33: Show that $Tan^{-1}\dfrac{3}{4} + Tan^{-1}\dfrac{3}{5} - Tan^{-1}\dfrac{8}{19} = \dfrac{\pi}{4}$

Solution: Using addition and subtraction formulae for $\tan^{-1}$, we have

$$Tan^{-1}\frac{3}{4} + Tan^{-1}\frac{3}{5} - Tan^{-1}\frac{8}{19} = \left(Tan^{-1}\frac{3}{4} + Tan^{-1}\frac{3}{5}\right) - Tan^{-1}\frac{8}{19}$$

$$= \left(Tan^{-1}\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right) - Tan^{-1}\frac{8}{19}$$

$$= \left(Tan^{-1}\frac{\frac{15+12}{20}}{1-\frac{9}{20}}\right) - Tan^{-1}\frac{8}{19}$$

$$= Tan^{-1}\frac{\frac{27}{20}}{\frac{11}{20}} - Tan^{-1}\frac{8}{19} = Tan^{-1}\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}$$

$$= Tan^{-1}\frac{\frac{513-88}{209}}{\frac{209+216}{209}} = Tan^{-1}\frac{425}{425} = Tan^{-1}1 = \frac{\pi}{4}$$

EXERCISE 12.5

1. Find x, if

(i) $\text{Sin}^{-1}\dfrac{1}{2} = \dfrac{\pi}{2} - x$ (ii) $\text{Cos}^{-1}\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{2} - \text{Sin}^{-1}x$

2. Show that

(i) $\text{Sin}^{-1}x + \text{Cos}^{-1}x = \dfrac{\pi}{2}$ (ii) $\text{Tan}^{-1}x + \text{Tan}^{-1}\dfrac{1}{x} = \dfrac{\pi}{2}$

(iii) $\sec(\text{Arc}\tan x) = \sqrt{1+x^2}$ (iv) $\tan(\text{Sin}^{-1}x) = \dfrac{x}{\sqrt{1-x^2}}$

3. Evaluate. (i) $\text{Sin}\left[\dfrac{\pi}{2} - \text{Cos}^{-1}\dfrac{4}{5}\right]$ (ii) $\text{Sin}\left[\text{Arc}\cos\dfrac{\pi}{2} + \pi\right]$

4. Show that (i) $\text{Cos}(\text{Sin}^{-1}x - \text{Sin}^{-1}y) = \sqrt{(1-x^2)(1-y^2)} + xy$

(ii) $\text{Cos}(2\,\text{Sin}^{-1}x) = 1 - 2x^2,\ -1 \le x \le +1$

(iii) $2\,\text{Arc Cos}\,x = \text{Arc Cos}(2x^2-1),\ 0 \le x \le 1$

(iv) $\text{Cos}(\text{Arc}\tan x) = \dfrac{1}{\sqrt{1+x^2}}$ for $x \ge 0$

5. Express the following in terms of $\text{Tan}^{-1}(x)$

(i) $\text{Sin}^{-1}x$ (ii) Arc cos$x$ (iii) Arc cot x

6. Verify that:

(i) $2\,\text{Tan}^{-1}\left(\dfrac{1}{2}\right) + \text{Tan}^{-1}\left(-\dfrac{1}{7}\right) = \dfrac{\pi}{4}$

(ii) $\text{Sin}^{-1}\left(\dfrac{77}{85}\right) - \text{Sin}^{-1}\left(\dfrac{3}{5}\right) = \text{Cos}^{-1}\left(\dfrac{15}{17}\right)$

7. Express: $\frac{\pi}{4} - \text{Tan}^{-1}\left(\frac{1}{11}\right)$ as single inverse tangent

8. Prove that

(i) $3\,\text{Sin}^{-1}x = \text{Sin}^{-1}(3x-4x^3)$ (ii) $3\,\text{Cos}^{-1}x = \text{Cos}^{-1}(4x^3-3x)$

(iii) $\text{Sin}(2\,\text{Sin}^{-1}x) = 2x\sqrt{1-x^2}$ (iv) $\text{Tan}^{-1}\sqrt{x} = \frac{1}{2}\text{Cos}^{-1}\left(\frac{1-x}{1+x}\right)$

(v) $\text{Sin}^{-1}x = \text{Cos}^{-1}\sqrt{1-x^2}$, for $x \geq 0$ (vi) $\text{Tan}^{-1}a + \text{Cot}^{-1}(a+1) = \text{Tan}^{-1}(a^2+a+1)$

9. If $\text{Tan}^{-1}x + \text{Tan}^{-1}y + \text{Tan}^{-1}z = \frac{\pi}{2}$. Prove that $xy + yz + zx = 1$.

12.5 Solutions of General Trigonometric Equations

Recall equations that contain trigonometric functions are called trigonometric equations. These will generally have an infinite number of solutions due to periodicity of the trigonometric function. For example the equation $\sin\theta=0$ has the solutions: $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ which can be written as: $\theta = k\pi$, where k is an integer. In a trigonometric equation, the unknown may not be the angle itself. For example in $\cos(2x+1) = 0$, the unknown is x while the angle is $(2x+1)$ and the function is cosine. We first use the definition of inverse trigonometric function to get the angle $(2x+1)$ and then solve for x to arrive at the solution of the equation.

When a trigonometric equation contains more than one trigonometric function, trigonometric identities and algebraic formulae are used to transform such trigonometric equation to an equivalent equation that contains only one trigonometric function.

12.5.1 Techniques for Solving Trigonometric Equations

Many trigonometric equations can be solved by methods already known. The following examples illustrate by these methods.

1. Using Factorization.

Example 9: Solve $\tan^2 x + \sec x - 1 = 0$ in $[0, 2\pi)$

Solution: We have, $\tan^2 x + \sec x - 1 = 0$

$\text{Sec}^2 x - 1 + \sec x - 1 = 0$ using identity $1 + \tan^2 x = \sec^2 x$, or $\tan^2 x = \sec^2 x - 1$

$\sec^2 x + \sec x - 2 = 0$

$(\sec x + 2)(\sec x - 1) = 0$ Factorizing

$\sec x = -2$ or $\sec x = 1$ Principle of zero products

$\cos x = -1/2$ or $\cos x = 1$ Using the identity $\cos x = 1/\sec x$

$x = 2\pi/3, 4\pi/3$ or $x = 0$

All these values check. The solutions in $[0, 2\pi)$ are $0, 2\pi/3$ and $4\pi/3$

Example 34: Solve $2\sin x\cos x - \sin x = 0$

Solution: $2\sin x\cos x - \sin x = 0$ (i)

$\Rightarrow \sin x[2\cos x - 1] = 0$ (ii)

Equating each factor to zero, we get

$\sin x = 0$ (iii)

or $\cos x = \frac{1}{2}$ (iv)

The equation (iii) $\sin x = 0$ is satisfied by 0 and π giving the solution $\{2k_1\pi\} \cup \{2k_2\pi + \pi\}$, where $k_1, k_2 \in Z$.

This is all even multiples of π $\{2k_1\pi\}$ and odd multiples of π $\{(2k_2+1)\pi\}$ which can be simplified to $\{k\pi, k \in Z\}$.

The values of the x satisfying (iv) in the interval $[0, \pi]$ are: $\frac{\pi}{3}$ and $(2\pi - \frac{\pi}{3}) = \frac{5\pi}{3}$

Thus the solution of (iv) $\cos x = \frac{1}{2}$ is $\{\frac{\pi}{3} + 2k\pi\} \cup \{\frac{5\pi}{3} + 2k\pi\}, k \in Z$

Combining the two we get the general solution of the given equation (i) as

$\{k\pi\} \cup \{2k\pi + \frac{\pi}{3}\} \cup \{\frac{5\pi}{3} + 2k\pi\}$, where $k \in Z$

2. Using trigonometric identities

Example 35: Solve $4\cos^2 x + 4\sin x - 5 = 0, 0 \leq x < 2\pi$

Solution: We cannot factor and solve this quadratic equation until each term involves the same trigonometric function. If we change the $\cos^2 x$ in the first term to $1 - \sin^2 x$, we will obtain an equation that involves the sine function only.

$4\cos^2 x + 4\sin x - 5 = 0$	
$4(1 - \sin^2 x) + 4\sin x - 5 = 0$	$\cos^2 x = 1 - \sin^2 x$
$4 - 4\sin^2 x + 4\sin x - 5 = 0$	Distributive property
$-4\sin^2 x + 4\sin x - 1 = 0$	Add 4 and –5
$4\sin^2 x - 4\sin x + 1 = 0$	multiply each side by –1
$(2\sin x - 1)^2 = 0$	Factor
$2\sin x - 1 = 0$	set factor to 0
$\sin x = ½$	
$x = \pi/6, 5\pi/6$	

Example 36: Solve $\sin 2x \cos x + \cos 2x \sin x = \frac{1}{\sqrt{2}}$

Solution: We can simplify the left side by using the formula for sin(A+B)

$$\sin 2x \cos x + \cos 2x \sin x = \frac{1}{\sqrt{2}}$$

$$\sin(2x + x) = \frac{1}{\sqrt{2}}$$

$$\sin(3x) = \frac{1}{\sqrt{2}}$$

First we find all possible solutions for x:

$3x = \frac{\pi}{4} + 2k\pi$ or $3x = \frac{3\pi}{4} + 2k\pi$ k is any integer

$x = \frac{\pi}{12} + \frac{2k\pi}{3}$ or $x = \frac{\pi}{4} + \frac{2k\pi}{3}$ Divide by 3

Example 37: Solve $\sin\theta - \cos\theta = 1$, if $0 \le \theta < 2\pi$

Solution: If we separte $\sin\theta - \cos\theta$ on opposite sides of the equal sign, and then square both sides of the equation, we will be able to use an identity to write the equation in terms of one trigonometric function only.

$\sin\theta - \cos\theta = 1$	
$\sin\theta = 1 + \cos\theta$	Add $\cos\theta$ to each side
$\sin^2\theta = (1 + \cos\theta)^2$	Square each side
$\sin^2\theta = 1 + 2\cos\theta + \cos^2\theta$	Expand $(1 + \cos\theta)^2$
$1 - \cos^2\theta = 1 + 2\cos\theta + \cos^2\theta$	$\sin^2\theta = 1 - \cos^2\theta$
$0 = 2\cos\theta + 2\cos^2\theta$	Standard form
$0 = 2\cos\theta(1 + \cos\theta)$	Factorize
$2\cos\theta = 0$ or $1 + \cos\theta = 0$	Set factors to 0

$\cos\theta = 0$ or $\cos\theta = -1$

$\theta = \pi/2, 3\pi/2$ or $\theta = \pi$

We have three possible solutions, some of which may be extraneous because we squared both sides of the equation in step 2. Any time we raise both sides of an equation to an even power, we have the possibility of introducing extraneous solutions. We must check each possible solution in our original equation.

Checking $\theta = \pi/2$	Checking $\theta = \pi$	Checking $\theta = 3\pi/2$
$\sin\pi/2 - \cos\pi/2 = 1$	$\sin\pi - \cos\pi = 1$	$\sin 3\pi/2 - \cos 3\pi/2 = 1$
$1 - 0 = 1$	$0 - (-1) = 1$	$-1 - 0 = 1$
$1 = 1$, true	$1 = 1$, true	$-1 = 1$, false
$\theta = \pi/2$ is a solution	$\theta = \pi$ is a solution	$\theta = 3\pi/2$ is not a solution

3. Using Quadratic Formula

Example 38: Solve $\cos 2x = 3(\sin x - 1)$ for all eal values of x.

Solution:

$\cos 2x = 3(\sin x - 1)$	given
$1 - 2\sin^2 x = 3\sin x - 3$	double angle formula
$2\sin^2 x + 3\sin x - 4 = 0$	quadratic equation
$\sin x = \frac{-3 \pm \sqrt{9-(4)(2)(-4)}}{(2)(2)}$	use quadratic formula
$\sin x = \frac{-3 \pm \sqrt{41}}{4}$	
$\sin x = -2.351$ or $0.85-8$	

The first answer, – 2.351, is not a solution, since the sine function must range between – 1 and 1. The second answer, 0.8508, is a valid value.

$$x = \sin^{-1} 0.8508 + 2k\pi, \quad x = \pi - \sin^{-1} 0.8505 + 2k\pi$$

In radian form,

$$x = 1\text{–}0175 + 2k\pi \quad x = 2.124 + 2k\pi$$

Example 39: Find the general solution of the equation.

$$2\sin^2 x + 3\sin x - 2 = 0$$

Solution: The equation is quadratic in $\sin x$, we get

$$\sin x = \frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3 \pm 5}{4}$$

$\Rightarrow \sin x = \frac{1}{2}, -2.$

$\sin x \in [-1,1]$, it follows that $\sin x = \frac{1}{2}$ has a solution but $\sin x = -2$ has no solution because $-2 \notin [-1,1]$.

The equation $\sin x = \frac{1}{2}$ is satisfied by the reference angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ in the interval $[0,2\pi]$. Thus the general solution set of the given equation is

$$\{\frac{\pi}{6} + 2n\pi\} \cup \{\frac{5\pi}{6} + 2n\pi\}, \text{ where } n \in Z$$

4. A Reduction Identity

Applications of inverse trigonometric functions are very useful in graphing to study the behavior of some wave functions and also in calculus and space sciences. It involves an identity to reduce the form of a trigonometric linear function i.e. $a\cos\theta + b\sin\theta = c$

where a, b, c are constants, either $a = 0$ and $b \neq 0$

Example 40: Solve the equation.

$$\sqrt{3}\cos\theta - \sin\theta = 0 \qquad \text{(i)}$$

Solution: Compare the given equation with the expression

$a\sin\theta + b\cos\theta$ we get, $a = -1,\ b = \sqrt{3}$

Let $-\sin\theta + \sqrt{3}\cos\theta = r\sin(\theta + \alpha)$ (ii).

We know that $r = \sqrt{a^2+b^2}$, $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

$$\Rightarrow \quad r = 2,\ \cos\alpha = -\frac{1}{2},\ \sin\alpha = \frac{\sqrt{3}}{2}$$

The reference angle for α is $\frac{\pi}{3}$ but since $\sin\alpha$ is positive and $\cos\alpha$ negative, the angle x lies in II quadrant.

Thus $\quad \alpha = (\pi - \frac{\pi}{3}) + 2n\pi = \frac{2\pi}{3} + 2n\pi,\ n \in Z \qquad$ (iii)

Substituting (iii) in (ii) gives

$$\Rightarrow \quad -\sin\theta + \sqrt{3}\cos\theta = 2\sin(\theta + \alpha)$$

$$= 2\sin(\theta + \frac{2\pi}{3}) = 0$$

$$\Rightarrow \quad \theta + \alpha = k\pi,\ k \in Z$$

$$\Rightarrow \quad \theta = k\pi - \frac{2\pi}{3} - 2n\pi,\ n \in Z$$

$$\text{or} \quad \theta = -\frac{2\pi}{3} + m\pi,\ m \in Z$$

$$\text{S.S.} = \{-\frac{2\pi}{3} + m\pi,\ m \in Z\} = \{\frac{2\pi}{3} + m\pi,\ m \in Z\}$$

EXERCISE 12.6

1. Solve each equation giving general solutions.

   (i) $\cos x = \frac{\sqrt{3}}{2}$ (ii) $\sin x = \frac{1}{2}$

   (iii) $\tan x = -\sqrt{3}$ (iv) $\cos(2\theta - \frac{\pi}{2}) = -1$

   (v) $\sec\frac{3\theta}{2} = -2$ (vi) $4\cos^2 x - 1 = 0$

2. Solve each equation. Use exact values in the given interval.

   (i) $(\sin x)(\cos x) = 0$, $0 \leq x \leq 360°$

   (ii) $(\sin x)(\cot x) = 0$, $0 \leq x \leq 2\pi$

   (iii) $(\sec x - 2)(2\sin x - 1) = 0$, $0 \leq x \leq 2\pi$

   (iv) $(\operatorname{cosec} x - 2)(2\cos x - 1) = 0$, $0 \leq x \leq 2\pi$

3. Find the solution sets of the following equations.

   (i) $\cos\theta = \sin\theta$ (ii) $\tan\theta = 2\sin\theta$ (iii) $\sin\theta = \operatorname{cosec}\theta$

   (iv) $4\cos^2(\frac{\theta}{2}) - 3 = 0$ (v) $\sin x \cos x = \frac{\sqrt{3}}{4}$ (vi) $\sin 2\theta + \sin\theta = 0$

4. Solve the following equations.

   (i) $2\sin^2 x - 3\sin x + 1 = 0$ (ii) $\cos^2 x \sin x = 2$

   (iii) $\cos^2 x - \sin^2 x = \sin x$ (iv) $\cos 2x + \cos x + 1 = 0$

   (v) $1 - \sin x = 2\cos^2 x$ (vi) $\tan^2 x = \frac{3}{2}\sec x$

   (vii) $3 - \sin x = \cos 2x$ (viii) $\sin\theta + \cos\theta = 1$

REVIEW EXERCISE 12

1. **Choose the correct option.**

(i) Solve $\sin 4x \cos x + \cos 4x \sin x = -1$ for all radian solutions.

(a) $\frac{\pi}{5}+\frac{2\pi}{5}k$ (b) $\frac{3\pi}{10}+\frac{2\pi}{5}k$ (c) $\frac{\pi}{2}+\frac{2\pi}{3}k$ (d) $\frac{\pi}{3}+\frac{2\pi}{3}k$

(ii) $\text{Tan}^{-1}\sqrt{3} - \text{Sec}^{-1}(-2)$ is equal to

(a) π (b) $-\pi/3$ (c) $\pi/3$ (d) $2\pi/3$

(iii) If $\text{Sin}^{-1}x = y$, then

(a) $0<y<\pi$ (b) $-\pi/2 \le y \le \pi/2$ (c) $0<y<\pi$ (d) $-\pi/2< y < \pi/2$

(iv) $\sin(\text{Tan}^{-1} x)$, $|x| < 1$ is equal to

(a) $\frac{x}{\sqrt{1-x^2}}$ (b) $\frac{1}{\sqrt{1-x^2}}$ (c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{x}{\sqrt{1+x^2}}$

(v) $\text{Tan}^{-1}\left(\frac{x}{y}\right) - \text{Tan}^{-1}\left(\frac{x-y}{x+y}\right)$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

2. Find the period of each function.

(i) $-2\text{cosec}\,\pi x$ (ii) $6\tan \pi x$ (iii) $\frac{9}{5}\cos\left(-\frac{3\pi}{2}x\right)$

3. Solve the following equations.

(i) $\sin 2x = \cos x$ (ii) $\sin^2 x + \cos x = 1$ (iii) $\text{coec}\,x = \sqrt{3} + \cot x$

4. Prove the following.

(i) $2\,\text{Tan}^{-1}\frac{2}{3} = \text{Sin}^{-1}\frac{12}{13}$ (ii) $\text{Tan}^{-1}\frac{3}{4} + \text{Tan}^{-1}\frac{3}{5} - \text{Tan}^{-1}\frac{8}{19} = \frac{\pi}{4}$

(iii) $\text{Sin}^{-1}\frac{4}{5} + \text{Sin}^{-1}\frac{5}{13} - \text{Sin}^{-1}\frac{16}{65} = \frac{\pi}{2}$ (iv) $\text{Tan}^{-1}\frac{1}{11} + \text{Tan}^{-1}\frac{5}{6} = \text{Tan}^{-1}\frac{1}{3}\ \text{Tan}^{-1}\frac{1}{2}$

5. Prove the following.

(i) $\text{Cos}^{-1}A - \text{Cos}^{-1}B = \text{Cos}^{-1}(AB + \sqrt{A^2 + B^2})$

(ii) $\text{Tan}^{-1}A - \text{Tan}^{-1}B = \text{Tan}^{-1}(A-B)$

Answers

EXERCISE 1.1

1. (i) 0 (ii) i (iii) i (iv) $-i$

3. (i) $1+12i$ (ii) $\frac{3}{4}-i$ (iii) $(\sqrt{2}+1)+(\sqrt{2}+1)i$

4. (i) $(a-2)+bi$ (ii) $-6+0i$ (iii) $2\sqrt{3}-7\sqrt{7}\,i$

5. (i) $-117-i$ (ii) $6+6i$ (iii) $7+\sqrt{6}\,i$

6. (i) $\frac{1}{2}-\frac{1}{2}i$ (ii) $\frac{-8}{65}-\frac{1}{65}i$ (iii) $\frac{7}{58}+\frac{3}{58}i$ (iv) $1-6i$

7. (i) $\sqrt{34}$ (ii) $\sqrt{65}$ (iii) $\frac{8}{13}+\frac{1}{13}i$

8. (i) $\frac{6}{13}-\frac{24}{13}i$ (ii) $-\frac{22}{41}-\frac{7}{41}i$ (iii) $\frac{6}{25}+\frac{8}{25}i$

9. $\frac{63}{25}+\frac{16}{25}i$

10. $2-2i$

11. (i) -2 (ii) 0

EXERCISE 1.2

4. (i) $-5-2i, \frac{5}{29}-\frac{2}{29}i$ (ii) $(-7, 9), \left(\frac{7}{130}, \frac{9}{130}\right)$

7. (i) $\frac{4}{29}+\frac{19}{29}i$, *Real part* $x=\frac{4}{29}$, *Imaginary part* $y=\frac{19}{29}$

(ii) $-\frac{3}{2}-\frac{1}{2}i$, *Real part* $x=-\frac{3}{2}$, *Imaginary part* $y=-\frac{1}{2}$

(iii) $-\frac{1}{2}-\frac{1}{2}i$, *Real part* $x=-\frac{1}{2}$, *Imaginary part* $y=\frac{1}{2}$

iv) $\frac{4a^2-b^2}{(4a^2+b^2)^2}+\frac{4abi}{(4a^2+b^2)^2}$, Real part $=\frac{4a^2-b^2}{(4a^2+b^2)^2}$ *and imaginary part* $=\frac{-4ab}{(4a^2+b^2)^2}$

v) -1, Real part $x=-1$ *and imaginary part* $y=0$

vi) $\frac{533}{169}+\frac{308}{169}i$, Real part $=\frac{533}{169}$ *and imaginary part* $=\frac{308}{169}$

EXERCISE 1.3

1. i) $z=-2+9i,\ w=2-6i$ ii) $z=4-i,\ w=1-i$ iii) $z=1,\ w=3-2i$

2. i) $(z+2)(z-1+3i)(z-1-3i)$ ii) $(\sqrt{3}z+\sqrt{7}i)(\sqrt{3}z-\sqrt{7}i)$

iii) $(z+2i)(z-2i)$ iv) $(z-2)(z+i)(z-i)$

4. Yes, it is a solution 5. i) $-\frac{1}{2}\pm\frac{1}{2}i$ ii) $\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$

iii) $1\pm\sqrt{1-i}$ iv) $\pm 2i$ 6. i) $\pm\frac{1}{2}\pm\frac{1}{2}i\sqrt{3}$ ii) $-2, 1\pm i\sqrt{3}$

iii) $0, \frac{3}{2}\pm\frac{1}{2}i\sqrt{3}$ iv) $1, \frac{-1}{2}\pm\frac{1}{2}i\sqrt{3}$

REVIEW EXERCISE 1

1. i) b ii) c iii) a iv) d v) a vi) b vii) b

3. i) $6+10i$ ii) $-4-4i$ iii) $-16+22i$ iv) $\frac{13}{37}+\frac{4}{37}i$

4. $\sqrt{2}$ 5. 2 6. $\frac{3}{25}+\frac{4}{25}i$ 7. $-i$ 8. $z=1\pm i$

EXERCISE 2.1

1. i) $[98]$ ii) $[-21\ \ 5\ \ -10]$ iii) $\begin{bmatrix}43\\44\end{bmatrix}$ iv) $\begin{bmatrix}6 & 16 & 26\\-8 & -18 & -28\end{bmatrix}$

2. $\begin{bmatrix}7 & -20 & 1\\6 & 1 & 11\end{bmatrix}$ 5. $a=-\frac{2}{3},\ b=\frac{3}{2}$

6. i) $X=\begin{bmatrix}7 & 2 & 11\\0 & 4 & 11\end{bmatrix}$ ii) $X=\begin{bmatrix}3 & 5 & 3\\3 & -3 & 3\end{bmatrix}$

EXERCISE 2.2

1. $A_{11}=-4, A_{21}=6, A_{23}=6, A_{31}=-2, A_{32}=-1, A_{33}=5, |A|=-14$

4. i) -12 ii) 92 iii) 21 iv) -11 7. i) -2 ii) 0

11. i) Singular ii) Non-singular iii) Singular 12. $\lambda=0, \pm\sqrt{2}$

13. i) $x=-1$ ii) $x=0,-1$ iii) $x=0,-9$

15. $\begin{bmatrix}\frac{15}{8} & \frac{-10}{8} & \frac{-2}{8}\\ \frac{7}{8} & \frac{-2}{8} & \frac{-2}{8}\\ \frac{-3}{8} & \frac{2}{8} & \frac{2}{8}\end{bmatrix}$

EXERCISE 2.3

1. i) $\begin{bmatrix}1 & 3 & -1\\0 & -5 & 6\\0 & 0 & -8\end{bmatrix}$ ii) $\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & -1\end{bmatrix}$ iii) $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

iv) $\begin{bmatrix}1 & 0 & -2\\0 & 1 & 5\\0 & 0 & -1\end{bmatrix}$ 2. i) $\frac{1}{49}\begin{bmatrix}3 & 16 & -5\\-6 & 17 & 10\\5 & -6 & 8\end{bmatrix}$

(ii) $\begin{bmatrix} -\frac{17}{2} & \frac{31}{2} & -11 \\ -\frac{5}{2} & \frac{9}{2} & -3 \\ 4 & -7 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$ (iv) $\begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

3. (i) 3 (ii) 2 4. 2

EXERCISE 2.4

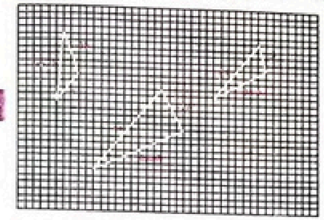
1. (i) $x=3,\ y=1,\ z=2$ (ii) $x=1,\ y=\frac{2}{3},\ z=\frac{-2}{3}$
2. (i) $x=-1,\ y=3,\ z=2$ (ii) $x=6,\ y=-2,\ z=4$
3. (i) $x=-2,\ y=1,\ z=3$ (ii) $x=2,\ y=-2,\ z=3$
4. (i) Trivial solution (ii) $x_1=-t,\ x_2=-t,\ x_3=t$
5. $\lambda=1,\ x_1=\frac{1}{3}t;\ x_2=\frac{-2}{3}t,\ x_3=t$

REVIEW EXERCISE 2

1. (i) c (ii) d (iii) c (iv) b (v) d (vi) c
2. $\begin{bmatrix} -193 \\ 232 \\ -78 \end{bmatrix}$ 4. -58 8. 5I 9. $x=2,\ y=1,\ z=1$

EXERCISE 3.1

1. (i) $-a$ (ii) $-b$ (iii) $-c$ (iv) $2b$ (v) $2c$

2. 

3. (i) $\frac{1}{2}p$ (ii) $q-p$ (iii) $\frac{3}{4}q$ (iv) $\frac{3}{4}q-\frac{1}{2}p$
4. (i) $\frac{3}{2}b$ (ii) $a+\frac{3}{2}b$ (iii) $a+\frac{1}{2}b$
5. (i) $\frac{1}{2}a$ (ii) $b-\frac{1}{2}a$ (iii) $\frac{1}{3}b-\frac{1}{6}a$ (iv) $\frac{1}{3}a+\frac{1}{3}b$
6. (i) $\bar{r}-\bar{p}$ (ii) $\frac{1}{2}\bar{r}-\frac{1}{2}\bar{p}$ (iii) $\frac{1}{2}\bar{p}+\frac{1}{2}\bar{r}$
7. $\vec{x}+\vec{y},\ \vec{y},\ \vec{y}-\vec{x},\ \vec{y}-2\vec{x}$

EXERCISE 3.2

1. (i) $-\vec{i}+\vec{j}$ (ii) $13\vec{i}-21\vec{j}$ (iii) $10\vec{i}-16\vec{j}$ (iv) $\sqrt{5}$ (v) $\sqrt{34}-\sqrt{13}$ (vi) $\sqrt{34}/\sqrt{13}$
2. (i) $\vec{i}$ (ii) $\frac{3}{5}\vec{i}-\frac{4}{5}\vec{j}$ (iii) $\frac{1}{\sqrt{6}}\bar{i}+\frac{1}{\sqrt{6}}\bar{j}-\frac{2}{\sqrt{6}}\bar{k}$ (iv) $\frac{\sqrt{3}}{2}\vec{i}-\frac{1}{2}\vec{j}$
3. $p=-4,\ q=1$ 4. $-2\pm\sqrt{21}$
5. Length of $\overline{AB}=2\sqrt{29}$, unit vector in the direction of $\overline{AB}=\frac{5}{\sqrt{29}}i+\frac{2}{\sqrt{29}}j$
6. $-1,-4$ 7. (i) components; 3, -3, magnitude $3\sqrt{2}$ (ii) components; 4, 2, magnitude $2\sqrt{5}$ (iii) components; 3, -2, 1, magnitude $\sqrt{14}$ (iv) components; $-1,-6,-3$, magnitude $\sqrt{46}$
8. (i) $Q(-1,1)$

ii $P(-5,6)$ iii $Q(1,2,-5)$ iv $P(1,2,8)$ **9.** $2\bar{i}-4\bar{j}+4\bar{k}$

10. Internally $-\frac{1}{3}\bar{i}+\frac{4}{3}\bar{j}+\frac{1}{3}\bar{k}$, Externally $-3\bar{i}+3\bar{k}$ **11.** i $\frac{8}{7}\bar{i}+\frac{27}{7}\bar{j}$

ii $6\bar{i}+17\bar{j}$ **12.** No real value of α **13.** $z=-3$

EXERCISE 3.3

1. i -4 ii 15 iii 11 iv -12 v 29

2. $\frac{4}{13}\bar{i}+\frac{3}{13}\bar{j}-\frac{12}{13}\bar{k}$

3. i 90^0 ii 73^0 *(approximately)* iii 99^0 *(approximately)*

5. 4 **6.** i $m=\frac{26}{27}$ ii $m=\frac{27}{38}$

7. i $\frac{7}{15},\frac{14}{17}$ ii $\frac{4}{\sqrt{3}},\frac{12}{\sqrt{14}}$ **8.** $\frac{1}{2}$ **9.** work = 6 units **10.** 12 units

EXERCISE 3.4

1. i $3i$ ii $-3i-2j$ iii $-i+36j+22k$

3. i $\frac{1}{5\sqrt{3}}(-i+7j+5k)$ ii $\frac{1}{\sqrt{803}}(-25i+3j+13k)$

4. i $-19i-2j+9k$ ii $-3i+6j+3k$ iii $38i+4j-18k$

5. i 15 ii $\sqrt{19}$

6. i $-6\bar{i}+\bar{j}+4\bar{k}$ ii $22\bar{i}+3\bar{j}-12\bar{k}$ **8.** i $\frac{-5\bar{i}-\bar{j}+3\bar{k}}{\sqrt{35}}$

ii $10\frac{5\bar{i}+6\bar{j}+2\bar{k}}{\sqrt{65}}$ **9.** i $\sqrt{110}$ ii $\frac{\sqrt{321}}{2}$

EXERCISE 3.5

1. -10 **2.** 57 **5.** i 14 ii 9 iii 25

6. *Yes, they lie in a plane.* **7.** i $\frac{53}{7}$ ii 1 iii $\frac{12}{5}$

8. i 3 ii $\frac{5}{6}$ **9.** i 1 ii -1

REVIEW EXERCISE 3

1. i a ii a iii d iv b v b vi a vii a viii c

2. $\lambda=-9, \mu=27$ **3.** $\frac{11\bar{i}-3\bar{j}}{\sqrt{130}}$ **4.** 0 **5.** $\frac{8}{7}$

6. $-\frac{21}{2}$ **7.** $\theta=\frac{\pi}{3}$ **8.** $\frac{7}{2}$ *square units.* **9.** $\sqrt{3336}$ *square units.*

EXCERCISE 4.1

1. i Finite ii Infinite iii Infinite iv Finite

2. i $1, 3, 6, 10, \ldots$ ii $4,-8,16,-32$ iii $1,\frac{1}{3},\frac{1}{9},\frac{1}{27}$ iv $0,0,1,4,\ldots$

3. i $\frac{n}{n+1}$ ii $(-1)^{n+1}\,2n$ iii $(-1)^{n+1}$ **4.** i $3, 2, 3, 2, 3$ ii $3,3,\frac{3}{2},\frac{1}{2},\frac{1}{8}$

5. i $-1+1+3+5+7+9$ ii $-1+2-4+8-16$ iii $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots$

iv $1+\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^3+\cdots$ **6.** i $1,5,10,10,5,1,0,0,0,\ldots$

ii $1,6,15,20,15,6,1,0,0,0,\cdots$ iii $1,8,28,56,70,56,28,8,1,0,0,0$

EXCERCISE 4.2

1. 44 **2.** 38 **3.** n=26 **4.** 21 **5.** $\log(ab^{n-1})$

6. $k=\frac{7}{2}$, the sequence is 14, 19, 24, …… **7.** $\frac{5}{3}, 2, \frac{7}{3}, \ldots\ldots$

9. 3m **10.** 15135 **11.** 45 hours **12.** \$ 18500 **13.** (i) 15

(ii) $\frac{7}{24}$ (iii) -111 (iv) a^2+b^2 **14.** (i) $14\frac{3}{4}, 23\frac{1}{2}, 32\frac{1}{4}$

(ii) 20, 23, 26, 29 **15.** 0 **16.** $\frac{11}{2}, 6, \frac{13}{2}, 7, \frac{15}{2}$ **17.** 8

EXCERCISE 4.3

1. (i) 20^{th} term: -29, sum: -200 (ii) 11^{th} term: $-\frac{1}{3}$, sum: $\frac{44}{3}$

2. (i) $a_{17}=50$; $S_{17}=442$ (ii) $a_{21}=60$; $d=5; n=21$

(iii) $n=9$; $a_9=57$ (iv) $n=15$; $d=\frac{2}{7}$; $a_1=0$ **3.** 12375

4. 8, 12, 16 ; 16, 12, 8 **5.** 2, 4, 6, 8 ; 8, 6, 4, 2 **6.** -21

7. $n(3n-4)$ **9.** 21978 **10.** Rs.280, Rs.260, Rs.240, Rs.220

11. 576 feet **12.** Rs.465 **13.** 3140 **14.** $\frac{66}{17}, \frac{115}{17}, \ldots, \frac{801}{17}$

EXERCISE 4.4

1. (i) 5, 15, 45, 135, 405 (ii) $8, -4, 2, -1, \frac{1}{2}$ (iii) $-\frac{9}{16}, \frac{3}{8}, -\frac{1}{4}, \frac{1}{6}, -\frac{1}{9}$

(iv) $\frac{x}{y}, -1, \frac{y}{x}, -\frac{y^2}{x^2}, \frac{y^3}{x^3}$ **2.** 3 ; $r=\pm 3$ **3.** $-2^{-\frac{3}{2}}$ **4.** 11

5. $x=13$; 20,10,5 **8.** (i) 2.92 or $-$ 2.92 (ii) 36 or -36 (iii) $\pm\sqrt{x^2-y^2}$

(iv) Does not exit **9.** (i) $\frac{16}{3}$, 8, 12, 18, 27 (ii) $-7, \frac{7}{2}, -\frac{7}{4}, \frac{7}{8}, -\frac{7}{16}, \frac{7}{32}$

10. 49,1 **12.** $n=-\frac{1}{2}$

EXERCISE 4.5

1. (i) $3(2^{10}-1)$ (ii) $\frac{255}{7}$ (iii) 2032 (iv) $\frac{64}{65}$ (v) $4+2\sqrt{2}$ (vi) $-\frac{463}{192}$

2. (i) $n=7, S_7=43$ (ii) $n=9, a_1=256, S_9=511$ (iii) $a_1=3, n=6$

3. $4, 2, 1, \frac{1}{2}, \frac{1}{4}$; $S=8$ **4.** (i) $\frac{8}{9}$ (ii) $\frac{18}{11}$ (iii) $\frac{97}{45}$ (iv) $\frac{41}{333}$

5. 3 **7.** $1-\frac{1}{2^n}$ **8.** 18, 12, 8 or 8, 12, 18 **9.** 2 **10.** 7

12. $a_1=\frac{24}{5}, r=\frac{1}{5}$; $\frac{24}{5}+\frac{24}{5^2}+\ldots$ **14.** $29\frac{27}{32}\,ft$

15. Rs. 16384; Rs. 1073741823 **16.** $\frac{3}{2}$

EXERCISE 4.6

1. (i) $\frac{1}{26}$ (ii) $\frac{2}{13}$ (iii) 1 **2.** $-\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7} -\frac{1}{9}, -\frac{1}{11}$ **3.** $-\frac{1}{18}$

4. (i) $A=2.925, H=2.91, G=\pm 2.92$ (ii) $A=-111, H=-11.68, G=\pm 36$

(iii) $A=x, G=\pm\sqrt{x^2-y^2}, H=\frac{x^2-y^2}{x}$ **5.** -1 **6.** 16, 24

7. $\frac{35}{23}, \frac{35}{31}, \frac{35}{39}, \frac{35}{47}$ **9.** 4 and 12 ; 12 and 4 **10.** 3 and 12 ; 12 and 3

REVIEW EXERCISE 4

1. (i) c (ii) b (iii) a (iv) b (v) d (vi) d (vii) b (viii) b

(ix) b (x) d (xi) c **2.** (i) $t_n=9-n$

(ii) The progression is 8, 7, 6, 5,… (iii) $t_{10}=-1$ (iv) $S_n=\frac{n}{2}(17-n), S_{10}=35$

3. n = 18 or 19 4. 5,11,17,.... and nth term is 6n–1 5. 156375

6. $\frac{5}{3}$ 8. 18 9. 2, 6 and 18 (or) 18, 6 and 2 10. 25 11. 196.875 feet

EXERCISE 5.1

1. (i) $\frac{n}{3}(4n^2-1)$ (ii) $\frac{n(n+1)^2(n+2)}{12}$ (iii) $\frac{2n(n+1)(2n+1)}{3}$

(iv) $n(2n^3-n)$ (v) $n(16n^3-16n^2-2n+3)$ 2. $33\times100\times101$

3. 50×3333 4. $\frac{n(n+1)^2}{2}$ 5. $\frac{n}{6}(2n^2+3n+7)$

6. $\frac{n(n+1)(2n+1)(n+3)}{4}$ 7. $\frac{n(n+1)(n+8)(n+9)}{4}$ 8. $\frac{n}{3}(32n^2+54n+25)$

9. (i) $\frac{n}{2}(n+1)(n^2+3n+1)$ (ii) $4^{n+1}-4-n(n+1)(n^2-n-1)$

EXERCISE 5.2

1. (i) $2+(n-1)2^{n+1}$ (ii) $\frac{1-(3n-2)x^n}{(1-x)}+\frac{3x(1-x^{n-1})}{(1-x)^2}$

(iii) $\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}$ (iv) $2+4(1-\frac{1}{2^{n-1}})-\frac{2n-1}{2^{n-1}}$ (v) $\frac{1-(6n-5)(-x)^n}{1+x}-\frac{6x[1-(-x)^{n-1}]}{(1+x)^2}$

2. (i) $\frac{1+7x}{(1-x)^2}+\frac{8x^2}{(1-x)^3}$ (ii) $\frac{9}{2}$ 3. $n2^{n-1}$ 4. 9 5. $\frac{1}{4}$

EXERCISE 5.3

1. $3n^2+1;\frac{1}{2}.n(2n^2+3n+3)$ 2. $3n^2+n;n(n+1)^2$

3. $n^2+3n;\frac{1}{3}.n(n+1)(n+5)$ 4. $3^{n-1}+2;\frac{1}{2}(3^n-1)+2n$

5. $3(2^n-1);3(2^{n+1}-n-2)$ 6. $5^{n-1}+27;\frac{5^n-1}{4}+27n$

EXERCISE 5.4

1. (i) $\frac{n}{n+1}$ (ii) $\frac{n}{2n+1}$ (iii) $\frac{1}{6}$ (iv) $\frac{1}{36}$

2. $\frac{n}{2(3n+2)}$ 3. $\frac{n-1}{n}$ 4. $\frac{n}{4(n+4)}$

REVIEW EXERCISE 5

1. (i) b (ii) c (iii) c (iv) b (v) c (vi) c (vii) a (viii) a

2. $\frac{n(n+1)(n+2)}{3}$ 3. $\frac{1}{4}n(n+1)(n+4)(n+5)$ 4. $\frac{1}{12}$ 5. $\frac{5}{1-x}+\frac{7x(1-x^{n-1})}{(1-x)^2}$

6. $\frac{n}{n+1}$ 7. (i) $\frac{n(n^3+4n^2+4n-1)}{2}$ (ii) $\frac{n(n+1)(3n^2+5n+1)}{6}$

8. (i) $\frac{n^2(n+1)^2}{4}+\frac{3}{2}(3^n-1)$ (ii) $\frac{n(n+1)(4n+11)}{6}$

(iii) $\frac{1}{12}n(n+1)[3n^2+23n+34]$ (iv) $\frac{n}{3}(4n^2-1)$ 9. (i) $\frac{n}{3}(n^2+3n+5)$

(ii) $2n+\frac{3}{4}(3^n-1)$ 10. $2(1-\frac{1}{2^n})$, $2(n-1)+\frac{1}{2^{n-1}}$

EXERCISE 6.1

1. (i) 4200 (ii) $\frac{5}{16}$ (iii) $\frac{1}{(n+1)n}$ (iv) 252 2. (i) $\frac{19!}{13!}$

(ii) $2^6\cdot6!$ (iii) $\frac{(n+1)!}{(n-2)!}$ (iv) $\frac{(n+2)!}{3(n-1)!}$ 4. (i) 6 (ii) 9

EXERCISE 6.2

1. (i) 720 (ii) 380 (iii) 3,360 2. (i) 11 (ii) 9 (iii) 5

4. 40320 5. 5040 6. 120, number of even numbers is 48 7. (i) 125

(ii) 60 **8.** 2880 **9.** 1956 **10.** (i) 3360 (ii) 5040 **11.** 125

12. (i) 6720 (ii) 151200 (iii) 50400 (iv) 180 **13.** 37800 (i) 15120

(ii) 3360 (iii) 5040 (iv) 22680 (v) 7560 (vi) 30240 **14.** 12 **15.** 120

EXERCISE 6.3

1. (i) 9 (ii) 8 (iii) 4 , 5 **2.** 7 ; 4 **3.** 5 **5.** (i) 66 (ii) 220 **6.** 9

7. 190 **8.** 35 **9.** (i) 525 (ii) 1287 (iii) 1281 (iv) 231

EXERCISE 6.4

1. (i) $\frac{1}{6}$ (ii) 0 (iii) 1 (iv) $\frac{1}{3}$ (v) $\frac{1}{2}$ **2.** (i) $\frac{4}{91}$ (ii) $\frac{4}{555}$

3. (i) $\frac{1}{256}$ (ii) $\frac{1}{32}$ (iii) $\frac{7}{64}$ (iv) $\frac{37}{256}$ **4.** (i) $\frac{1}{8}$

(ii) $\frac{3}{8}$ (iii) $\frac{3}{8}$ (iv) $\frac{7}{8}$ (v) $\frac{1}{2}$ (vi) $\frac{1}{8}$ **5.** (i) $\frac{10}{21}$ (ii) $\frac{5}{21}$

6. (i) $\frac{1}{13}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{4}$ (iv) $\frac{3}{13}$ (v) $\frac{51}{52}$ **7.** (i) $\frac{1}{12}$ (ii) $\frac{5}{18}$

(iii) $\frac{5}{12}$ (iv) $\frac{1}{12}$ (v) $\frac{1}{6}$ (vi) $\frac{1}{9}$ (vii) $\frac{1}{2}$ (viii) $\frac{1}{2}$ (ix) $\frac{1}{3}$ (x) $\frac{5}{12}$

EXERCISE 6.5

1. $\frac{3}{10}$ **2.** (i) $\frac{1}{8}$ (ii) $\frac{1}{4}$ **3.** 0.1 **4.** $\frac{17}{30}$ **5.** $\frac{1}{9}$ **6.** $\frac{1}{3}$

7. $\frac{9}{13}$ **8.** $\frac{7}{9}$ **9.** (i) $\frac{1}{35}$ (ii) $\frac{2}{7}$ (iii) $\frac{24}{35}$ (iv) $\frac{11}{35}$ **10.** $\frac{316}{435}$

REVIEW EXERCISE 6

1. (i) a (ii) c (iii) a (iv) d (v) a (vi) d (vii) c (viii) b (ix) c

(x) d **2.** (i) $r = n - 1$ (ii) ${}^{r}C_5 = 56$ **3.** $r = 41$ **4.** 27720

5. (i) 72 (ii) 24 **6.** $\frac{2}{5}$ **7.** (i) 0.32 (ii) 0.64 (iii) 0.98

8. 1680 **9.** 360 **10.** $\frac{2}{n-1}$ **11.** $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}$

EXERCISE 7.2

1. (i) $x^8 - \frac{4x^6}{y} + \frac{6x^4}{y^2} - \frac{4x^2}{y^3} + \frac{1}{y^4}$

(ii) $1 + 7xy + 21x^2y^2 + 35x^3y^3 + 35x^4y^4 + 21x^5y^5 + 7x^6y^6 + x^7y^7$

(iii) $\frac{1}{\sqrt{y}}\left[y^3 + 5y^2 + 10y + 10 + \frac{5}{y} + \frac{1}{y^2}\right]$

2. (i) $560a^3$ (ii) $-{}^{10}C_7\, 2^{-3}.3^7.x^3y^{-7}$ (iii) $6x^3$ **3.** (i) 2268 (ii) 405

(iii) $-{}^{21}C_7$ **4.** (i) -1140 (ii) $2^4.{}^8c_4$ (iii) $-{}^9c_3.\frac{2^6}{27}$

5. (i) $70a^4b^4$ (ii) $\frac{15309}{8}x^{13}$ *and* $\frac{5103}{16}x^{14}$ (iii) $-252x^{10}y^5$

6. There is no constant term. **7.** (i) 724 (ii) $24\sqrt{2}$

(iii) $2a^5 + 20a^3b^2 + 10ab^4$ **8.** T_4 = -885735 **9.** $T_6 = -{}^{20}c_4\, 12^6\, 4^4$

EXERCISE 7.3

1. (i) $1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5}{16}x^3 + \ldots$ (ii) $1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \ldots$

(iii) $4 + 4x - x^2 + \frac{2}{3}x^3 + \ldots$ **2.** (i) 5.099 (ii) 1.001 (iii) 5.01330

3. $1 - x + \frac{1}{2}x^2 - \frac{1}{2}x^3$ **6.** $1 - \frac{3}{x} + \frac{9}{2x^2}$ **7.** 2, $\frac{3}{16}$ **9.** 4n

10. (i) $\sqrt{\frac{2}{3}}$ (ii) $\left(\frac{24}{7}\right)^{\frac{15}{7}}$

REVIEW EXERCISE 7

1. (i) a (ii) b (iii) a (iv) a (v) b (vi) c (vii) d (viii) a

2. $90720x^{12}y^4$ 3. -35840 4. $a = 4$ 5. 840 6. 0.951

EXCERCISE 8.1

1. (i) 1, –1, 5, 29 (ii) –3 or 2 (iii) x^2+3x+1 (iv) $2x+1+h$

2. (i) $f(6)=40,\ g(-1)=\frac{2}{3},\ h(4)=12,\ k\left(\frac{1}{2}\right)=\frac{5}{4}$ (ii) 7

3. (i) $\frac{1}{5}$ (ii) 3,5 (iii) $0, \pm 1$ (iv) $1, \pm\sqrt{5}$

4. (i) Domain f = R Range f = R
(ii) Domain f = R – (–4, 4) Range f = [0, ∞)

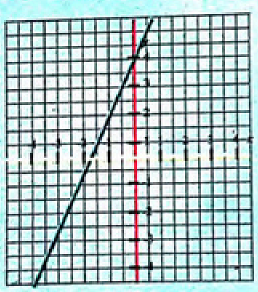
5. (i) $\frac{x+3}{2}$ (ii) $3x+15$ (iii) $2-5x$ (iv) $\frac{1}{2}(x-4)^2$

6. (i) $f^{-1}(x)=\sqrt[3]{x+2}$ (ii) $\sqrt[3]{5}$

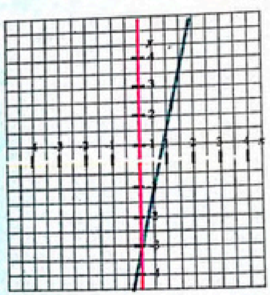
7. (i) $Dom(f)=R-\{3\}$, $Range(f)=R-\{1\}$ (ii) $Dom(f^{-1})=R-\{1\}$, $Range(f)=R-\{3\}$

EXERCISE 8.2

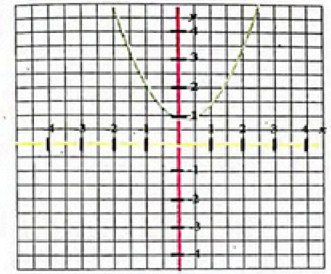
1. (i)



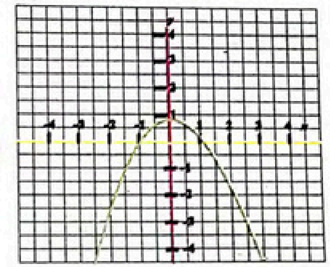
(ii)



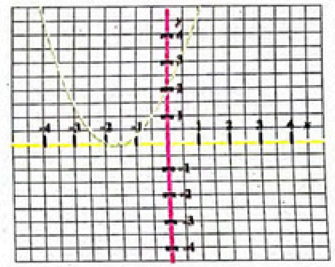
2. (i)



(ii)



(iii)



By changing the values of b in the quadratic function, the axis of symmetry of the graph moves in the x-direction

3. (i) Vertex: (0, 0), y – intercept: 0, x – intercept : 0, Axis : 0, opens upward

(ii) Vertex: (0, 8), y – intercept: 8, x – intercepts: ±2, Axis: 0, opens downward

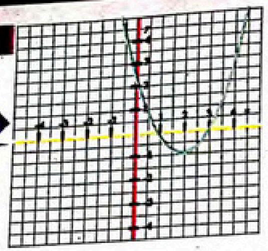
(iii) Vertex: (3, 4), y – intercept: – 5, x – intercepts: 5 and 1, Axis: 3, opens downward

(iv) Vertex: $(-1, -\frac{7}{2})$, y – intercept: –3, x – intercept: 1, Axis: 1, –3, opens upward

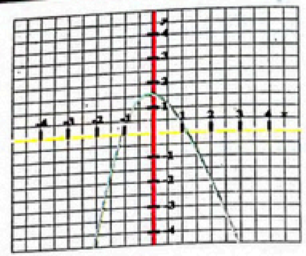
4. (i) ----(e) (ii) ----(c) (iii) ----(b) (iv) ----(a) (v) ----(f) (vi) ----(d)

EXERCISE 8.3

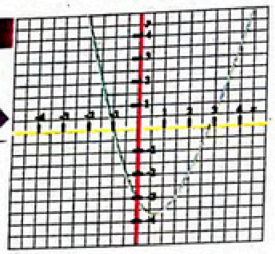
1\. (i)



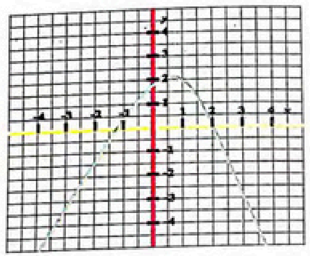
(ii)



2\. (i)



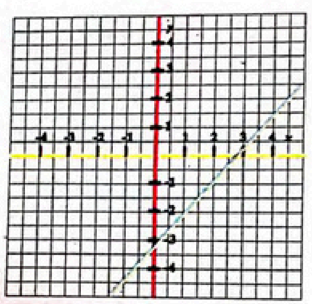
(ii)



3\. $y = x^2 - 7x + 12$

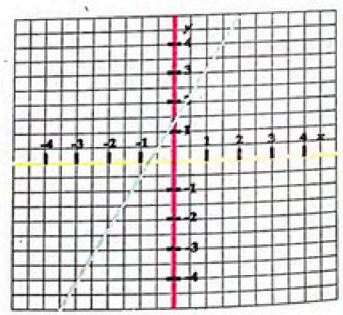
4\. (i) $y = -\frac{1}{2}x^2 - x + \frac{15}{2}$ (ii) $y = -\frac{1}{6}x^2 + \frac{1}{2}x + \frac{35}{3}$

5\. (i)



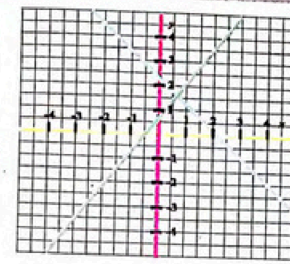
(3, 0), (0, −3)

(ii)



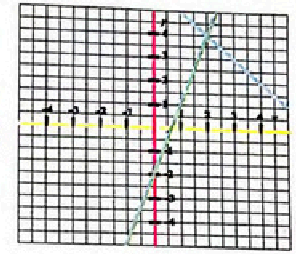
$\left(-\frac{1}{2}, 0\right)$, (0,1)

6\. (i)



$\left(\frac{1}{3}, \frac{5}{3}\right)$

(ii)

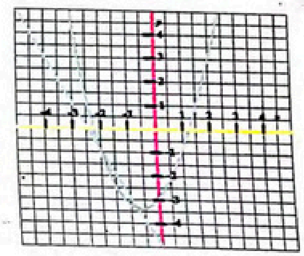


(2, 4)

7\. (i)

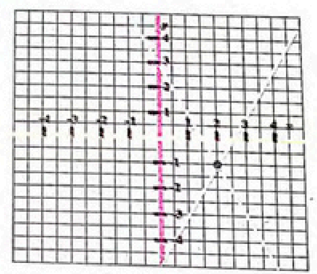
(1, 3), (−2, 0)

(ii)



(−1, −3), (−2, −1)

8\.



(2, −2)

9\. Air speed = 5 km/min, velocity of the wind = 1 km/ min.

REVIEW EXERCISE 8

1\. (i) d (ii) b (iii) c (iv) b (v) a (vi) b (vii) d (viii) c

2\. Domain $f = \left[-2\sqrt{3}, -\sqrt{3}\right] \cup \left[\sqrt{3}, 2\sqrt{3}\right]$

3\. $f(x) = 3x^2 - 2x + 5$

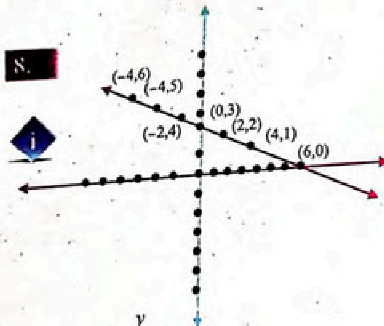
1. (i) $[2, \infty)$ (ii) R (iii) 14 (iv) 10

5. (i) 32 (ii) 82.4

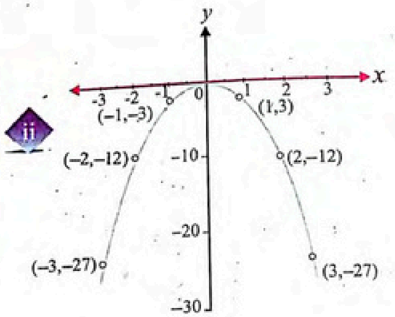
6. (i) 2 (ii) $\frac{4}{3}$

7. $a = 2, b = -2y$

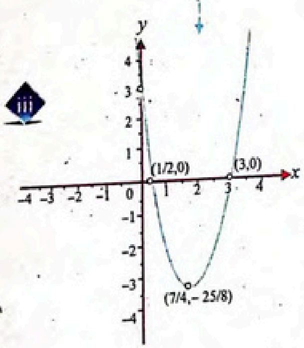
8. (i)



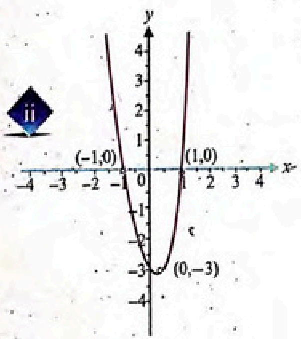
(ii)



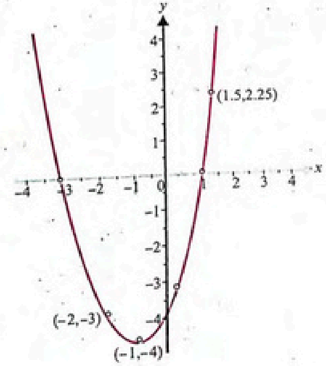
(iii)



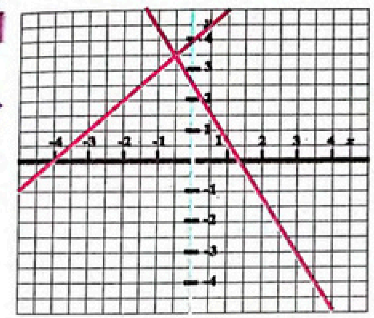
(ii)



9. (i)

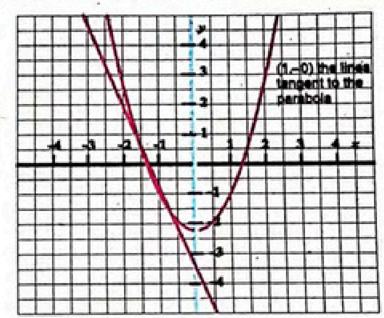


10. (i)



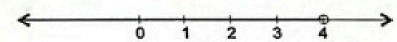
$\left(-\frac{1}{3}, \frac{11}{3}\right)$

(ii)

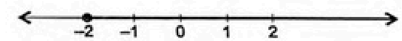


EXERCISE 9.1

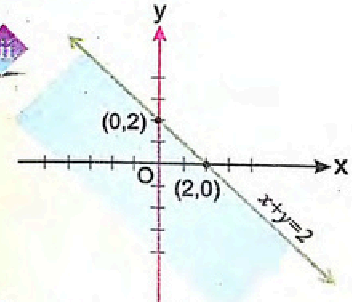
1. (i) $x < 4$



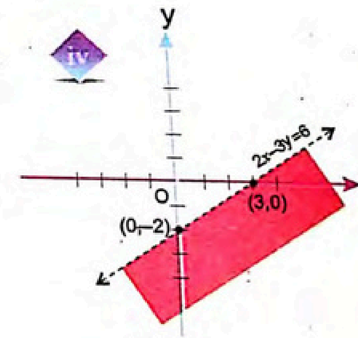
(ii) $x \geq -2$



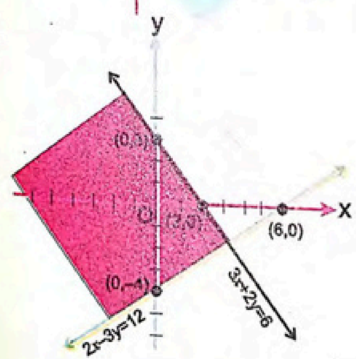
(iii)



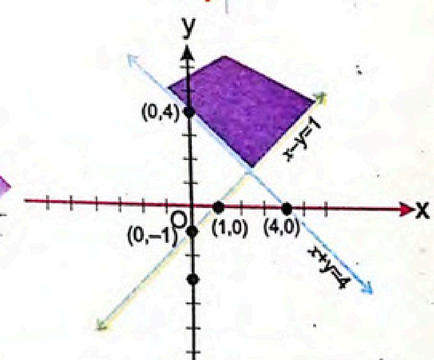
(iv)



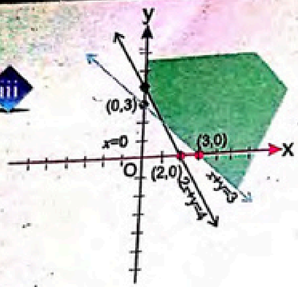
2. (i)



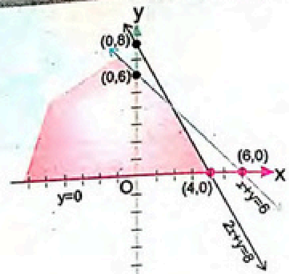
(ii)



iii

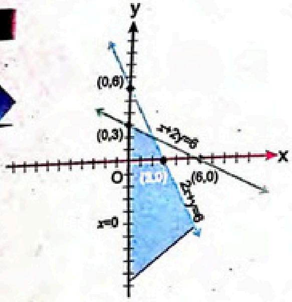


iv



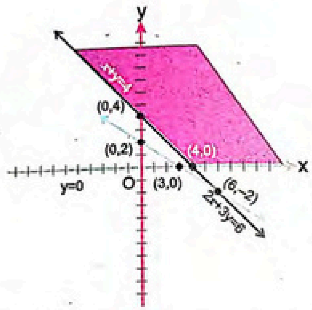
3.

i



(2,2), (0,6), (0,3); unbounded

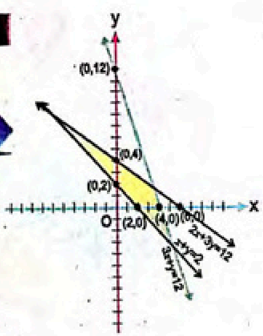
ii



(6,–2), (3,0), (4,0); unbounded

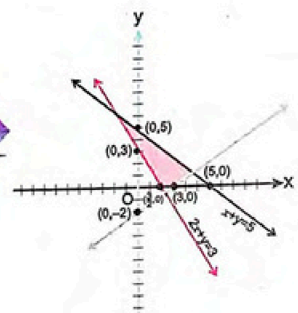
4.

i



$\left(\frac{24}{7}, \frac{12}{7}\right)$, (-6,8), (5, – 3); bounded

ii

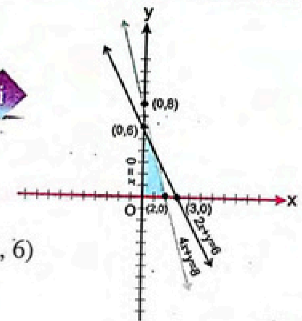


(-2, 7), $\left(\frac{5}{3}, \frac{1}{3}\right)$, $\left(\frac{7}{2}, \frac{3}{2}\right)$; bounded

EXERCISE 9.2

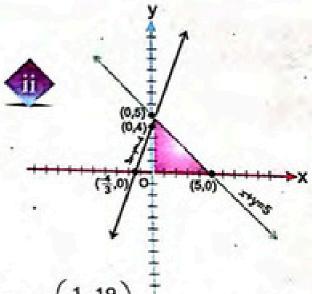
1.

i



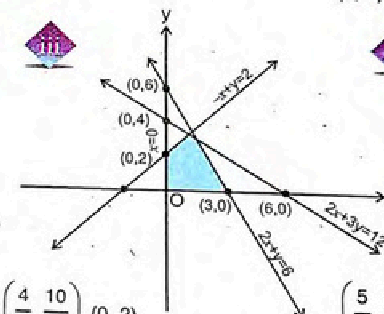
(0, 0), (2, 0), (1, 4), 0, 6)

ii



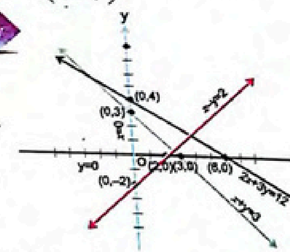
(0, 0), (5,0) $\left(\frac{1}{4}, \frac{19}{4}\right)$, (0, 4) ; bounded

iii



(0, 0), (3,0), $\left(\frac{3}{2}, 3\right)$, $\left(\frac{4}{3}, \frac{10}{3}\right)$, (0, 2)

iv



$\left(\frac{5}{2}, \frac{1}{2}\right)$, $\left(\frac{18}{5}, \frac{8}{5}\right)$, (0, 4), (0, 3)

2. i. Maximum value is 12 at the corner point (6, 0).

ii. Maximum value is 20 at the corner point (0, 4).

3. i. Maximum value is 50 at the corner point (10, 0).
Minimum value is 4 at the corner point (0,2).

ii. Maximum value is 84 at the corner point (0, 4).
Minimum value is 7 at the corner point (1, 0).

4. Maximum profit of Rs. 1140 if 16 bicycles of model A and 10 bicycles of model B are produced.

5. Maximum profit of Rs. 14000 if 200 units of product A and 400 units of product B are produced.

6. Maximum profit of Rs. 1760 if 8 lamps of model L_1 and 24 lamps of model L_2 are produced

REVIEW EXERCISE 9

1. (i) a (ii) a (iii) c (iv) c (v) d (vi) a

2. Maximum value of z is 24 at two different corner points $\left(\frac{24}{7},\frac{24}{7}\right)$ and $(5,\frac{4}{3})$

3. Rs: 112, when X = 2 kg, Y = 4 kg

4. Maximum value of Z is 600 at A (120,0) and R (60,30)

EXERCISE 10.1

1. (i) $\sin 59^0$ (ii) $\cos 30^0$ (iii) $\cos 24^0$ (iv) $\sin 25^0$ (v) $\tan 52^0$ (vi) $\tan 23^0$

2. (i) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (ii) $2+\sqrt{3}$ (iii) $-2-\sqrt{3}$ (iv) $2+\sqrt{3}$ (v) $\frac{\sqrt{6}+\sqrt{2}}{4}$ (vi) $\frac{\sqrt{6}+\sqrt{2}}{4}$

3. (i) 0 (ii) $-\frac{7}{24}$ (iii) $-\frac{7}{25}$ (iv) $\frac{24}{25}$

4. (i) $\frac{63}{65}$ (ii) $\frac{56}{65}$ (iii) $\frac{33}{56}$

5. (i) $\frac{33}{65}$ (ii) $\frac{-56}{65}$ (iii) $\frac{-33}{56}$

13. (i) $\gamma\sin(\theta+\phi)$ where $\sin\phi=\frac{3}{5}$, $\cos\phi=\frac{4}{5}$ and $r=5$

(ii) $\gamma\sin(\theta+\phi)$ where $\sin\phi=\frac{8}{17}$, $\cos\phi=\frac{15}{17}$ and $r=17$

(iii) $\gamma\sin(\theta+\phi)$ where $\sin\phi=\frac{-5}{\sqrt{29}}$, $\cos\phi=\frac{2}{\sqrt{29}}$ and $r=\sqrt{29}$

(iv) $\gamma\sin(\theta+\phi)$ where $\sin\phi=\frac{1}{\sqrt{2}}$, $\cos\phi=\frac{1}{\sqrt{2}}$ and $\gamma=\sqrt{2}$

EXERCISE 10.2

1. $-\frac{5}{13},\frac{12}{13},-\frac{5}{12}$

2. (i) $\frac{-120}{169}$ (ii) $\frac{119}{169}$ (iii) $\frac{-120}{119}$

3. (i) $\frac{-24}{25}$

(ii) $\frac{1}{\sqrt{5}}$

4. $\frac{\sqrt{5}}{\sqrt{7}}$

5. (i) $\frac{\sqrt{3}}{2}$ (ii) $\frac{-1}{2}$

6. (i) $\frac{\sqrt{2+\sqrt{3}}}{2}$ (ii) $\sqrt{3+2\sqrt{2}}$ (iii) $\frac{\sqrt{2+\sqrt{2}}}{2}$ (iv) $\frac{\sqrt{2+\sqrt{2}}}{2}$ (v) $\sqrt{7+4\sqrt{3}}$ (vi) $\frac{\sqrt{2+\sqrt{3}}}{2}$

8. $\frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta$

EXERCISE 10.3

1. (i) $\cos 5x-\cos 7x$ (ii) $\frac{1}{2}\left[\sin 178^0-\sin 66^0\right]$ (iii) $\frac{1}{2}$ (sin A + sinB) (iv) $\frac{1}{2}$ (cos P + cos Q)

2. (i) $2\sin 40^0\cos 3^0$ (ii) $-2\sin 59^0\sin 23^0$ (iii) $2\cos\frac{p}{2}\sin\frac{Q}{2}$ (iv) $2\cos\frac{A}{2}\cos\frac{B}{2}$

REVIEW EXERCISE 10

1. (i) b (ii) c (iii) b (iv) d (v) d (vi) d (vii) a (viii) c

EXERCISE 11.1

1. (i) a=3, $b=3\sqrt{3}$, $\beta=60^0$ (ii) a=52.7, c=136.6, $\alpha=22.7^0$ (iii) $a=5\sqrt{2}$, $b=5\sqrt{2}$, $\alpha=45^0$

2. (i) $\alpha=62°$, b = 7.44, c = 15.86 (ii) $\alpha=68.5°$, a = 22.59, c = 24.28 (iii) $\alpha=88.22°$, $\beta=1.78°$, a = 449.78

3. 24.89m 4. 52.9° 5. 36.3m 6. 45.3m 7. 11.43m

8. 189.3m 9. 61.4 feet 10. 7.265cm

EXERCISE 11.2

1. (i) $\alpha=60°$, $\beta=30°$, $\gamma=90°$ (ii) $\alpha=25°$, $\beta=123°$, c = 152 (iii) a = 408, b = 166, $\beta=23.6°$ (iv) $\alpha=23°$, $\gamma=45°$, b = 57.6

v. $a = 3.83,\ \beta = 24.3°,\ \gamma = 55.3°$ vi. $\alpha = 106°20',\ b = 159,\ c = 140$

vii. $a = 68,\ b = 112,\ \gamma = 75°$ viii. No triangle possible

ix. $a = 15.31,\ \beta = 30°26',\ \gamma = 111°14'$

x. $b = 409.00,\ \alpha = 22°39',\ \gamma = 46°59'$ **2.** i. $\alpha = 96°37'$

ii. $\beta = 80°0'38''$ iii. $\gamma = 87°55'$ **3.** i. $\alpha = 95.7°,\ \beta = 50.7°,\ \gamma = 33.6°$

ii. $\alpha = 4.0°,\ \beta = 31.6°,\ \gamma = 144.4°$ iii. $\alpha = 26.4°,\ \beta = 36.4°,\ \gamma = 117.2°$

4. 7.9cm, 14.8cm **5.** 1879km apart

6. $\alpha = 43°17',\ \beta = 64°26',\ \gamma = 72°17'$ **7.** 72.9cm

EXERCISE 11.3

The answers are in square units.

1. i. 369.42 ii. 83 iii. 680 iv. 564.7 v. 6.4 vi. 35.5 vii. 76662

viii. 400.5 ix. 651.7 x. 614.5 xi. 134736.6 xii. 2730.7

2. $c = 22.24,\ \gamma = 82°8'$ **3.** Rs.1125 **4.** 787 ft^2

EXERCISE 11.4

1. i. $R = 3.0,\ r = 1.07$ ii. $R = 14.5,\ r = 6$ **2.** $r = 8.16$m, Area = 209 m^2

3. i. 33.07dm ii. 14.17dm **7.** $3\sqrt{3}, 7\sqrt{3}, 8\sqrt{3}$

REVIEW EXERCISE 11

1. i. c ii. b iii. a iv. d v. d vi. c vii. c viii. b

2. i. $b = 1.42,\ \alpha = 17.2°,\ \gamma = 21.3°$ ii. No triangle possible.

iii. $\alpha = 59°43',\ \beta = 85°7',\ c = 10.4$ iv. $c = 40.68,\ \alpha = 81°43',\ \beta = 41°17'$

v. $a = 14.74,\ \beta = 70°39',\ \gamma = 85°56'$ vi. $\alpha = 42.8°,\ b = 52,\ c = 84.7$

vii. $\gamma = 77.5°,\ a = 7.05,\ b = 13.3$ **3.** i. 42.7° ii. 43.7° **4.** i. 82.5

ii. 83.03° **5.** 57.1cm, 20.84 cm **6.** $25\sqrt{3}$m **7.** 165.3 ft **8.** 57.8 m

EXERCISE 12.1

1. i. $R,\ [-3,3],\ \frac{2\pi}{3}$ ii. $R-\{x \mid x=(2n+1)\pi;\ n\in Z\}, R, 2\pi$

iii. $R-\left\{x \mid x=n\frac{\pi}{2};\ n\in Z\right\}, R-(-1,1), \pi$ iv. $R, \{y \mid -1\le y\le 1,\ y\in R\}, \frac{\pi}{2}$

v. $R-\left\{x \mid x=(2n+1)\frac{\pi}{4};\ n\in Z\right\}, R-\{y \mid -6\le y\le 6,\ y\in R\}, \pi$

vi. $R-\left\{x \mid x=\frac{3n}{2};\ n\in Z\right\}, R, \frac{3}{2}$ vii. $R-\left\{x \mid x=(2n+1)\frac{\pi}{2};\ n\in Z\right\}, R, \pi$

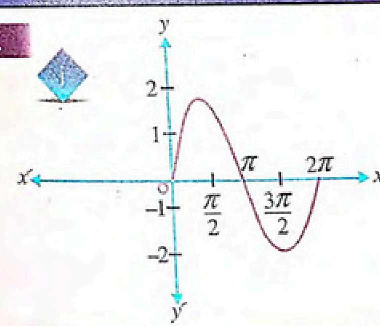
viii. $R-\{x \mid x=n\pi;\ n\in Z\}, R-\left(-\frac{1}{2},\frac{1}{2}\right), 2\pi$

ix. $R-\{x \mid x=4n+2,\ n\in Z\}, R-(-1,1), 8$ **2.** i. $-\frac{3}{2},\ -\frac{5}{2}$

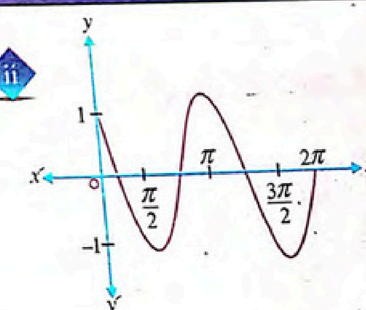
ii. 9, 1 iii. $\frac{1}{9},\ \frac{1}{29}$ iv. $\frac{1}{4},\ -\frac{1}{4}$

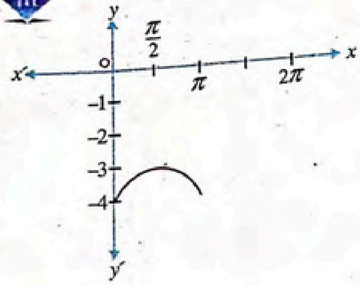
EXERCISE 12.2

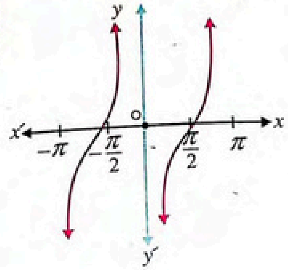
1. i.

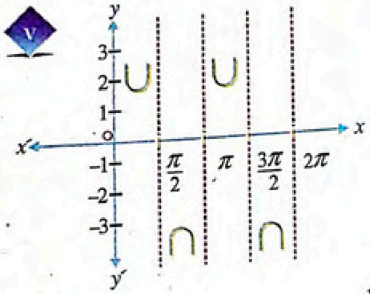


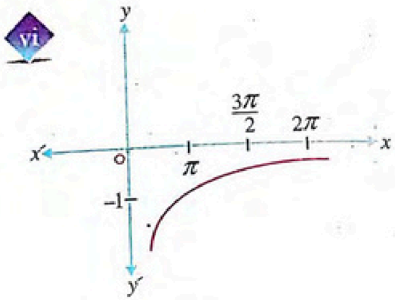
ii.



iii


iv


v


vi


2. i Period π, Frequency $\frac{1}{\pi}$, Amplitude 1

ii Period $\frac{\pi}{3}$, Frequency $\frac{3}{\pi}$, Amplitude 1

iii Period 2, Frequency $\frac{1}{2}$, Amplitude 1

iv Period 4, Frequency $\frac{1}{4}$, Amplitude 1

EXERCISE 12.3

1. i $\theta=\frac{\pi}{4}+2n\pi$ or $\theta=\frac{3\pi}{4}+2n\pi; n\in Z$

ii $\theta=\frac{5\pi}{6}+2n\pi$ or $\theta=\frac{7\pi}{6}+2n\pi; n\in Z$ iii $\theta=\frac{\pi}{3}+n\pi; n\in Z$

iv $\theta=\frac{\pi}{3}+2n\pi; n\in Z$ v $\theta=\frac{3\pi}{4}+\pi n;\ n\in Z$

vi $\theta=\frac{7\pi}{6}+2n\pi$ or $\theta=\frac{11\pi}{6}+2n\pi,\ n\in Z$

EXERCISE 12.4

1. i $\left\{-\frac{\pi}{2}+2n\pi, n\in Z\right\}\cup\left\{\frac{3\pi}{2}-2n\pi, n\in Z\right\}$

ii $\left\{\frac{3\pi}{4}+2m, m\in Z\right\}\cup\left\{\frac{5\pi}{4}+2m\pi, m\in Z\right\}$

iii $\left\{-\frac{\pi}{6}+2n\pi, n\in Z\right\}\cup\left\{\frac{5\pi}{6}+n\pi, n\in Z\right\}$

2. i $\pi+2n\pi,\ n\in Z$ ii $\frac{1}{2}$ iii $\left\{-\frac{1}{2},\frac{1}{2}\right\}$ iv $\frac{-3}{4}$

3. i $\frac{\sqrt{2}}{2}$ ii $\frac{\sqrt{3}}{3}$ iii 2 iv $\sqrt{2}$ v $\frac{-\sqrt{2}}{2}$ vi $2\frac{\sqrt{3}}{3}$

4. i u ii u iii u iv $\sqrt{1-u^2}$

EXERCISE 12.5

1. i $\frac{\pi}{3}$ ii $\frac{\sqrt{3}}{2}$ 3. i $\frac{4}{5}$ ii Does not exist.

5. i $\text{Sin}^{-1}x=\text{Tan}^{-1}\frac{x}{\sqrt{1-x^2}}$, $-1<x<1$

ii $\text{Cos}^{-1}x=\text{Tan}^{-1}\frac{\sqrt{1-x^2}}{x}$, $0<x<1$

iii $\text{Cot}^{-1}x=\text{Tan}^{-1}\left(\frac{1}{x}\right)$, $0<x<\infty$ 7. $\text{Tan}^{-1}\frac{5}{6}$

EXERCISE 12.6

1. i $\left\{\frac{\pi}{6}+2k\pi\right\}\cup\left\{\frac{11\pi}{6}+2k\pi\right\}, k\in Z$

ii $\left\{\frac{\pi}{6}+2k\pi\right\} \cup \left\{\frac{5\pi}{6}+k\pi\right\}$, k is any integer. **iii** $\left\{\frac{2\pi}{3}+k\pi\right\}, k \in Z$

iv $\theta = \frac{3\pi}{4}+k\pi$, k any integer.

v $\theta \in \left\{\frac{4\pi}{9}+\frac{4}{3}k\pi, k \in Z\right\} \cup \left\{\frac{8\pi}{9}+\frac{4}{3}k\pi, k \in Z\right\}$

vi $\left\{\frac{\pi}{3}+2n\pi, n \in Z\right\} \cup \left\{\frac{5\pi}{3}+2n\pi, n \in Z\right\} \cup \left\{\frac{2\pi}{3}+2n\pi, n \in Z\right\} \cup \left\{\frac{4\pi}{3}+2n\pi, n \in Z\right\}$

2. **i** 0°, 90°, 180°, 270° **ii** $\frac{\pi}{2}, \frac{3\pi}{2}$ **iii** $\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}$

iv $\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{5\pi}{6}$ **3.** **i** $\left\{\frac{\pi}{4}+n\pi\right\} \cup \left\{\frac{5\pi}{4}+n\pi\right\}$

ii $\{k\pi\} \cup \left\{\frac{5\pi}{3}+2k\pi\right\} \cup \left\{\frac{\pi}{3}+2k\pi\right\}, k \in Z$

iii $\left\{\frac{\pi}{2}+2k\pi\right\} \cup \left\{\frac{3\pi}{2}+2k\pi\right\}$, k any integer.

iv $\left\{\frac{\pi}{3}+4k\pi\right\} \cup \left\{\frac{11\pi}{3}+4k\pi\right\} \cup \left\{\frac{5\pi}{3}+4k\pi\right\} \cup \left\{\frac{7\pi}{3}+4k\pi\right\}, k \in Z$

v $\left\{\frac{\pi}{6}+n\pi\right\} \cup \left\{\frac{\pi}{3}+n\pi\right\}, n \in Z$

vi $\{m\pi\} \cup \left\{\frac{2\pi}{3}+2m\pi\right\} \cup \left\{\frac{4\pi}{3}+2m\pi\right\}$, m is any integer.

4. **i** $\left\{\frac{\pi}{6}+2k\pi\right\} \cup \left\{\frac{5\pi}{6}+2k\pi\right\} \cup \left\{\frac{\pi}{2}+2k\pi\right\}$, k is any integer.

ii No real solution.

iii $\left\{\frac{\pi}{6}+2n\pi\right\} \cup \left\{\frac{5\pi}{6}+2n\pi\right\} \cup \left\{\frac{3\pi}{2}+2n\pi\right\}$, n is any integer.

iv $\left\{\frac{\pi}{2}+2n\pi\right\} \cup \left\{\frac{2\pi}{3}+2n\pi\right\} \cup \left\{\frac{4\pi}{3}+2n\pi\right\}$, n is any integer.

v $\left\{\frac{\pi}{2}+2m\pi\right\} \cup \left\{\frac{7\pi}{6}+2m\pi\right\} \cup \left\{\frac{11\pi}{6}+2m\pi\right\}$, m is any integer.

vi $\left\{\frac{\pi}{3}+2k\pi\right\} \cup \left\{\frac{5\pi}{3}+2k\pi\right\}$, k is any integer.

vii No real solution. **viii** $\left\{\frac{\pi}{2}+2k\pi\right\} \cup \{2k\pi\}$, k is any integer.

REVIEW EXERCISE 12

1. **i** b **ii** d **iii** b **iv** d **v** c **2.** **i** 2 **ii** 1 **iii** $-\frac{4}{3}$

3. **i** $\left\{\frac{\pi}{2}+2k\pi\right\} \cup \left\{\frac{3\pi}{2}+2k\pi\right\} \cup \left\{\frac{\pi}{6}+2k\pi\right\} \cup \left\{\frac{5\pi}{6}+2k\pi\right\}, k \varepsilon Z$

ii $\left\{2k\pi+\frac{\pi}{2}\right\} \cup \left\{2k\pi+\frac{3\pi}{2}\right\} \cup \{2k\pi\}, k \varepsilon Z$ **iii** $\left\{2k\pi+\frac{2\pi}{3}\right\}, k \varepsilon Z$

The Authors

Prof. Dr. Gulzar Ali Khan (Retired)

Gulzar Ali Khan received his B.Sc. from Government Postgraduate College, Bannu in 1975, and M.Sc. from Gomal University, Dera Ismail Khan and Leeds University, U.K. in 1977 and 1981 respectively, and his Ph.D. from Birmingham University, U.K. in 1986 all in mathematics.

Dr. Khan taught at Gomal University from 1986 to 2000. He joined the University of Peshawar in 2001, He remained Chairman of the Department of Mathematics. He has published many research papers in his field of interest. Dr. Khan has been a member of the Advisory Committee, Ministry of Education, Government of Pakistan, Islamabad in the subject of Mathematics. He has also been a member of the National Review Committee (NRC) on mathematics. Dr. Khan is now retired.

Prof. Dr. Islam Noor (Retired)

Dr. Islam Noor is a retired Professor of Mathematics of the University of Peshawar. He obtained M.Sc. degree from the same University in 1972. He received M.S & Ph.D degrees from *Temple University (U.S.A)* in 1982 & 1984 respectively.

Dr. Noor has published numerous papers in reputed national & internal journals. He was a member of HEC curriculum committee for revising B.Sc, 4 years B.S and M.Sc. programs. He remained Chairman of the Department of Mathematics twice. He has organized seminars at Bara Gali campus, University of Peshawar for college teachers to cope with the changes made in the syllabus of F.Sc. & B.Sc.

Prof. Dr. Muhammad Shah

Dr. Muhammad Shah did his M.Sc from University of Peshawar in 1998. He did his Ph.D in 2012 from Quaid-i-Azam University Islamabad in computational Algebra. In 2010, he was awarded IRSIP scholarship of HEC for Birmingham University, UK.

Dr. Shah has been teaching Mathematics at postgraduate level since 2002 in different colleges of Khyber Pakhtunkhwa. He has been the Subject Specialist of Mathematics and Computer Science of Khyber Pakhtunkhwa Textbook Board since 2014 to 2017. He is the editor and reviewer of several textbooks of Mathematics and Computer Science.

مومن تو آپس میں بھائی بھائی ہیں۔
پس اپنے دو بھائیوں میں صلح کرادیا کرو
اور اللہ سے ڈرتے رہو تاکہ تم پر رحم
کیا جائے۔
(سورۃ الحجرات: ۱۰)

Not For Sale

Khyber Pakhtunkhwa Textbook Board
Phase - V, Hayatabad Peshawar
Phone: 091-9217714-15 Fax: 091-9217163, Website: www.kptbb.gov.pk

TEXTBOOK FEEDBACK FORM

S#	Statement	Remarks
1.	Overall the book is interesting and user friendly.	
2.	The language and content of the book is age / grade appropriate and the content is free of grammatical and punctuation errors.	
3.	Content is supported with examples from real life / culture.	
4.	Contents / texts are authentic and updated.	
5.	Pictures / diagrams / graphs / illustrations are informative, relevant and clear if not, then identify them.	
6.	Activities, projects and additional work is suggested for reinforcement of concepts.	
7.	Assessment achievements are thought provoking and comprise cognitive, psychomotor and effective skills.	
8.	The textbook is easy to be covered within academic year.	

Page No.	Observation/comments	Suggested amendment along with rationale

General Remarks: ______________________

____

Name & Address: ______________________

Contact No. & Email: ______________________

قومی ترانہ

پاک سَر زمین شاد باد   کِشورِ حسِین شاد باد
تُو نشانِ عزمِ عالی شان   ارضِ پاکستان
مرکزِ یقین شاد باد
پاک سَر زمین کا نِظام   قوّتِ اُخوّتِ عوام
قوم مُلک سلطنت   پائندہ تابندہ باد
شاد باد منزلِ مُراد
پَرچمِ ستارہ و ہلال   رہبرِ ترقّی و کمال
ترجمانِ ماضی شانِ حال   جانِ استِقبال
سایۂ خدائے ذُوالجلال

Code No.	Quantity	Printer
PSP/WPP-19-71-2122(0)	22,483	Watermark Printing & Packages, Peshawar.