4. $\sin^2\dfrac{\theta}{2} = \dfrac{\sin\theta\tan\frac{\theta}{2}}{2}$

5. $\tan\theta\cdot\tan\dfrac{\theta}{2} = \sec\theta - 1$

6. $\cos 4\theta = 1 - 8\sin^2\theta\cos^2\theta$

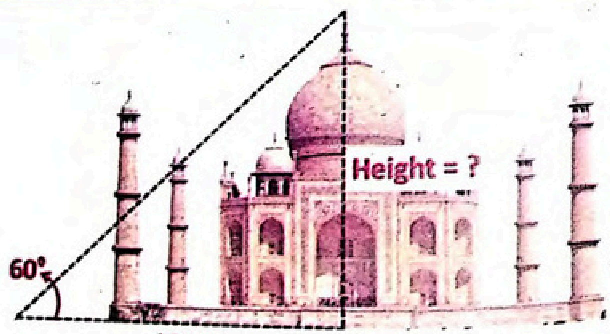
7. $\sin 6x\sin x + \cos 4x\cos 3x = \cos 3x\cos 2x$

8. Prove that $\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{2}\cos 2\theta$

9. Prove that i) $\dfrac{\sin^2(\pi+\theta)\tan\left(\frac{3\pi}{2}+\theta\right)}{\cot^2\left(\frac{3\pi}{2}-\theta\right)\cos^2(\pi-\theta)\operatorname{cosec}(2\pi-\theta)} = \cos\theta$

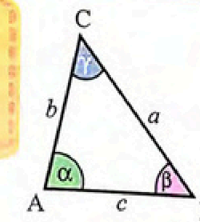
ii) $\dfrac{\cos(90^0+\theta)\sec(-\theta)\tan(180^0-\theta)}{\sec(360^0-\theta)\sin(180^0+\theta)\cot(90^0-\theta)} = -1$

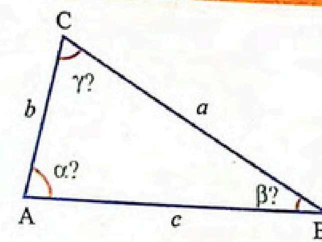
Real Life Applications of Trigonometry



UNIT 11

APPLICATION OF TRIGONOMETRY





STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Solve right angled triangle when measures of
  - two sides are given,
  - one side and one angle are given.
- Define an oblique triangle and prove
  - the law of cosines,
  - the law of sines,
  - the law of tangents, and deduce respective half angle formulae.
- Apply above laws to solve oblique triangles.
- Derive the formulae to find the area of a triangle in terms of the measures of
  - two sides and their included angle,
  - one side and two angles,
  - three sides (Hero's formula)
- Define circum-circle, in-circle and escribed-circle.
- Derive the formulae to find
  - circum-radius,
  - in-radius,
  - escribed-radii, and apply them to deduce different identities.

11.1 Introduction

Trigonometry has an enormous variety of application. It is used extensively in a number of academic fields, primarily mathematics, science and engineering.

Trigonometry, in ancient times, was often used in the measurement of heights and distances of objects which could not be otherwise measured. For example, trigonometry was used to find the distance of stars from the earth. Even today, in spite of more accurate methods being available, trigonometry is often used for making quick and simple calculations regarding heights and distance of far-off objects.

One of the important uses of trigonometry is solving triangles. Every triangle has three sides and three angles, which are called the elements (or parts) of the triangle. We say that a triangle is solved when all six elements are known and listed. Typically three elements, in which one is side, will be given and it will be our task to find the other three elements using trigonometric laws and definitions.

As shown in figure 11.1 we use standard lettering for naming the sides and angles of a right triangle, side a is opposite to angle A, side b is opposite to angle B, where a and b are the legs, and side c, the hypotenuse, is opposite to angle C, the right angle.

A triangle is usually labeled as shown in figure 11.1

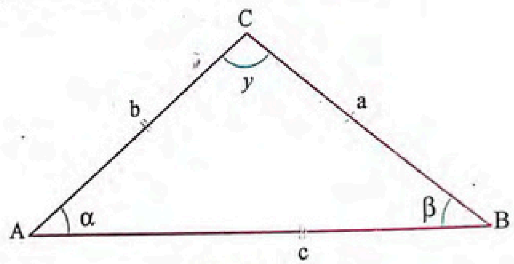


Figure 11.1

The vertices are labeled A, B, C with sides opposite to these vertices are denoted by a, b, c respectively and the measure of three angles are usually denoted by α, β and γ respectively.

We begin with, using the trigonometric functions to solve right angled triangles. Later we will learn how to solve triangles that are not necessarily right angled triangles. We will also derive formulae for finding the areas of such triangles.

11.1.1 Solution of Right Angled Triangles

We can solve a right angled triangle provided that either measure of (i) two sides are given or (ii) one acute angle and one side are given. We consider the cases as follows:

Case-I: When measure of two sides are given

Example 1: Solve the right angled triangle ABC, in which $a = 15$, $c = 17$ and $\gamma = 90°$.

Solution: From figure 11.2, we have

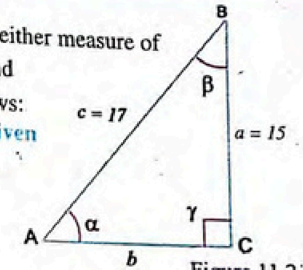


Figure 11.2

$$\sin\alpha = \frac{a}{c} = \frac{15}{17} = 0.882$$

$$\Rightarrow \quad \alpha = \sin^{-1}(0.882) = 61.89°$$

since $\alpha + \beta = 90°$

$$\Rightarrow \quad \beta = 90° - \alpha$$
$$= 90° - 61.89° = 28.11°$$

Now $\cos\alpha = \dfrac{b}{c}$

$$\Rightarrow \quad b = c\cos\alpha = 17\cos(61.89°)$$
$$= 17(0.471)$$
$$= 8$$

Note

The side b can also be found by using Pythagorean Theorem $c^2 = a^2 + b^2$ or $b^2 = c^2 - a^2 = \sqrt{(17)^2 - (15)^2}$ so that $b = 8$.

Case-II: When measure of one angle and one side are given

Example 2: Solve the right angled triangle ABC, in which $b = 12$, $\alpha = 70°$ and $\gamma = 90$

Solution: From figure 11.3, we have

$$\tan 70° = \frac{a}{12}$$

or $a = 12\tan 70°$
$= 12\,(2.747)$
$= 32.97$ ft.

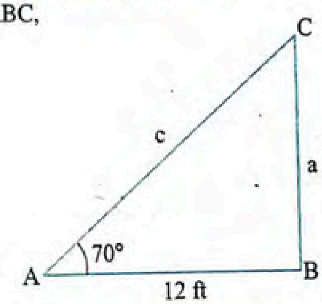


Figure 11.3

To find the length c of the ladder we have

$$\cos 70° = \frac{12}{c}$$

or $c = 12\sec 70°$
$= 12\,(2.92)$
$= 35.088$ ft.

Example 3: The angle of elevation of a tree from a point on the ground 42m from its base is 33°. Find the height of the tree?

Solution: Let the angle of elevation $= \theta$
and height of the tree $= h$

Then $\tan\theta = \frac{h}{42}$ $\Rightarrow$ $\tan 33° = \frac{h}{42}$

$\Rightarrow$ $h = 42 \tan 33°$
≈ 27.28

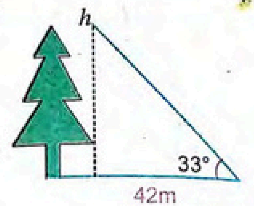


Figure 11.4

The tree is 27m tall.

Example 4: From point B, the top of a light house 120 ft above the sea, the angle of depression of a boat at point A is 5°. How far is it from the light house to the boat?

Solution: Since the angle of depression is the acute angle formed by the line of sight and the horizontal line passing through the position of sighting. Figure 11.5 indicates the situation. The angle A must also be 5° in measure. We have

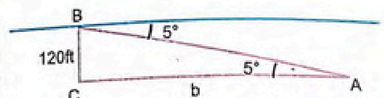


Figure 11.5

$\cot A = \frac{b}{120}$ or $\cot 5° = \frac{b}{120}$

$\Rightarrow b = 120\ (11.43) = 1372$ ft, approx.

Example 5: From the two successive positions on a straight road 1000 meter apart, a man observes that the angle of elevation of the top of a building directly ahead of him is 12°10′ and 42°35′. How high is the building?

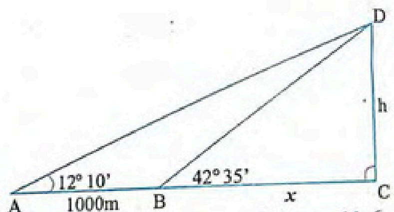


Figure 11.6

Solution: Let A and B be the two successive positions of a man on the road such that $|\overline{AB}| = 1000$m. CD denote the height h of the building and let $BC = x$

In ΔACD we have $\tan 12°10' = \frac{CD}{AC} = \frac{h}{AB+BC} = \frac{h}{x+1000}$

or $x + 1000 = h \cot 12°10' = 4.6382\,h$ (1)

In ΔBCD we have $\tan 42°35' = \frac{h}{x}$

$\Rightarrow x = h \cot 42°35' = 1.088\,h$ (2)

From (1),(2) $1.088h + 1000 = 4.6382\,h$

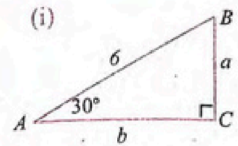
$\Rightarrow h = 281.67\text{m} \simeq 282$ m Now from (2) we get

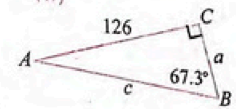
$x = 306.8 \simeq 307$m

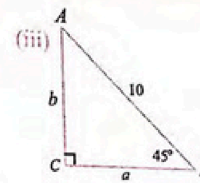
Not For Sale

EXERCISE 11.1

1. Solve the following right triangles.

(i)


(ii)


(iii)


2. Solve right triangles ABC in which $\gamma = 90°$ and
(i) $a = 14$, $\beta = 28°$ (ii) $b = 8.9$, $\beta = 21.5°$ (iii) $b = 14$, $c = 450$

3. The angle of elevation of the top of a post from a point on level ground 38m away is 33.23°. Find the height of the post.

4. A masjid minar 82 meters high casts a shadow 62 meters long. Find the angle of elevation of the sun at that moment.

5. The angle of depression of a boat 65.7m from the base of a cliff is 28.9°. How high is the cliff?

6. From the top of a cliff 52m high the angles of depression of two ships due east of it are 36° and 24° respectively. Find the distance between the ships.

7. Two masts are 20m and 12m high. If the line joining their tops makes an angle of 35° with the horizontal; find their distance apart.

8. The measure of the angle of elevation of a kite is 35°. The string of the kite is 340 meters long. If the sag in the string is 10 meters, find the height of the kite.

9. A parachutist is descending vertically. How far does the parachutist fall as the angle of elevation changes from 50° to 30° which is observed from a point 100m away from the feet of a parachutist where he touches the ground.

10. An isosceles triangle has a vertical angle of 108° and a base 20 cm long. Calculate its altitude.

11.1.2 Oblique Triangles

If none of the angle of a triangle is right angle, the triangle is called oblique triangle. In Figure 11.7 both triangles are oblique triangles.

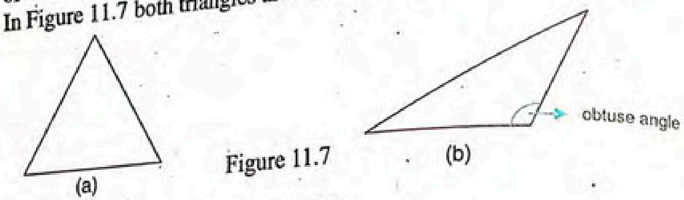


Figure 11.7

We see that an oblique triangle has either

(i) three acute angles (figure 11.7(a)) or

(ii) two acute angles and one obtuse angle (figure 11.7(b))

In the last section we solved right angled triangles, however, in this section we will solve oblique triangles. Given three elements of a triangle we will be asked to find the remaining three elements. Thus, we have the following five possibilities:

When three parts of a triangle including at least one side are known, the triangle is uniquely determined. The five cases of oblique triangles are

1. A.A.S: Given two angles and the side opposite to one of them
2. A.S.A: Given two angles and the included side
3. S.S.A: Given two sides and the angle opposite to one of them
4. S.A.S: Given two sides and the included angle
5. S.S.S: Given the three sides

In case of (S.S.A) there is not always a unique solution. It is possible to have no solution for the angle, one solution for the angle, or two solutions——an angle and its supplement.

In order to solve the above cases of oblique triangles, we develop special mathematical tools called the law of cosines, the law of sines the law of tangents.

(a) The Law of Cosines

In this section, we will derive the law of cosines and we use it to solve the case 5 of oblique triangles..

Theorem (Law of Cosines) In any triangle with usual labelling

(i) $a^2 = b^2 + c^2 - 2bc\cos\alpha$

(ii) $b^2 = c^2 + a^2 - 2ca\cos\beta$

(iii) $c^2 = a^2 + b^2 - 2ab\cos\gamma$

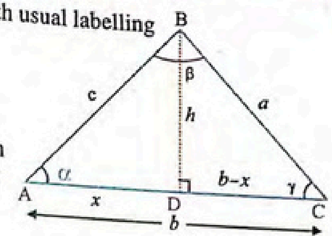


Figure 11.8

Proof: Case 1: All the angles are acute, α is an acute angle in figure 11.8. If h is the altitude of vertex B, then in ΔBCD, we have,

$$a^2 = h^2 + (b-x)^2 \qquad (1)$$

In ΔBAD we have

$$\cos\alpha = \frac{x}{c}$$

$$\therefore x = c\cos\alpha \qquad (2)$$

and $c^2 = x^2 + h^2$ (3)

Put (2) and (3) in (1)

$$a^2 = (c^2 - x^2) + (b^2 - 2bx + x^2)$$

$$= b^2 + c^2 - 2bc\cos\alpha$$

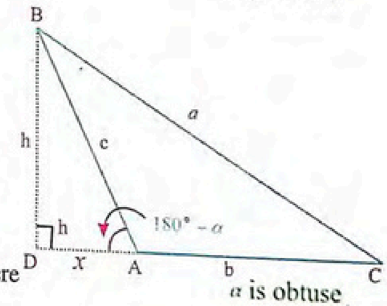


Figure 11.9

Case2: One angle is obtuse. α is obtuse here

In ΔBCD $a^2 = h^2 + (b+x)^2$

giving $a^2 = h^2 + b^2 + x^2 + 2bx$ (1)

In ΔBAD, $\cos(180° - \alpha) = \frac{x}{c}$

$$\therefore x = c\cos(180° - \alpha) = -c\cos\alpha \qquad (2)$$

and $c^2 = h^2 + x^2$ (3)

Put (2) and (3) into (1)

$$a^2 = (c^2 - x^2) + b^2 + x^2 + 2b(x) = b^2 + c^2 + 2b(-c\cos\alpha) = b^2 + c^2 - 2bc\cos\alpha$$

In both the triangles, we obtained $a^2 = b^2 + c^2 - 2bc\cos\alpha$.

By considering angles B and C in a similar manner, we can prove that

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

By rearranging the formula we can express the cosine of the angles in terms of three lengths sides of the triangle.

Did You Know

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$
$$\cos\beta = \frac{a^2+c^2-b^2}{2ac}$$
$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab}$$

S.S.S. and S.A.S. possibilities could be tackled by using cosine law. However in S.A.S., where two sides and included angle is given, it is necessary that the given angle must be less than 180°.

Example 6: (SSS): What is the smallest angle of a triangle whose sides measure 25, 18 and 21ft?

Solution: If γ represent the smallest angle, then c (the side opposite) γ must be the smallest side, so c = 18. Then $\cos\gamma = \frac{a^2+b^2-c^2}{2ab} = \frac{(25)^2+(21)^2-(18)^2}{2(25)(21)} = 0.707$

$$\Rightarrow \gamma = \cos^{-1}(0.707) = 45°$$

Example 7: (S.A.S.): Find c where $a = 52$, b = 28.3, $\gamma = 38.5°$

Solution: γ is the angle included between a and b.

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$= (52)^2 + (28.3)^2 - 2(52)(28.3)\cos 38.5°$$
$$\Rightarrow c^2 \simeq 918.355$$
$$\Rightarrow c \simeq 30.30 \text{ unit}$$

Example 8: A body is acted upon by the forces 10N and 20N making an angle $25°35'$ with each other. Find the magnitude of the resultant of the forces.

Solution:

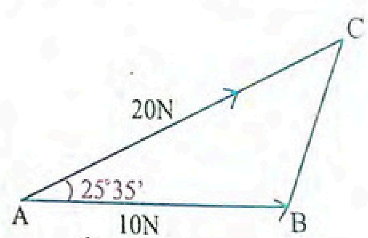


Figure 11.10 (a)

The forces of 10N and 20N are represented by sides of parallelogram.

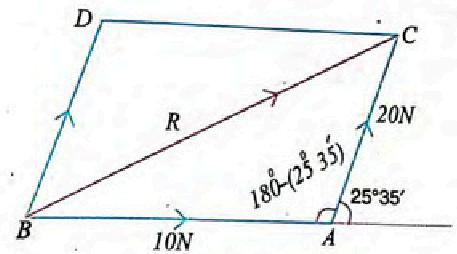


Figure 11.10 (b)

The resultant R is the diagonal of parallelogram ABCD. Hence

$$R^2 = (10)^2 + (20)^2 - 2\times10\times20\cos(180° - 25°35')$$
$$= 860.78\,N^2 \Rightarrow R = 29.3\text{ N}.$$

Example 9: An equilateral triangle is inscribed in a circle of radius 5cm. Find the perimeter of the triangle.

Solution: Let O be the centre of the circle. Join O with vertices B and C.

In the equilateral triangle ABC, we have

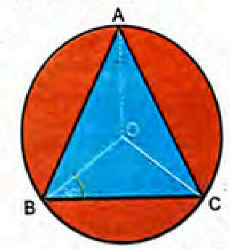


Figure 11.11

$$\angle BOC = \angle AOC = \angle AOB = \frac{1}{3}(360°) = 120°$$
$$|\overline{OB}| = |\overline{OC}| = |\overline{OA}| = 5cm$$

Using cosine law

$$|\overline{BC}|^2 = |\overline{OB}|^2 + |\overline{OC}|^2 - 2\times|\overline{OB}||\overline{OC}|\cos\angle BOC$$
$$= 5^2 + 5^2 - 2\times5\times5\cos120°$$
$$= \sqrt{75}\text{ .Each side is } \sqrt{75}\text{ cm.}$$

Hence perimeter of $\Delta ABC = \sqrt{75} + \sqrt{75} + \sqrt{75} = 3\sqrt{75} = 15\sqrt{3}cm$

(b) The Law of sines

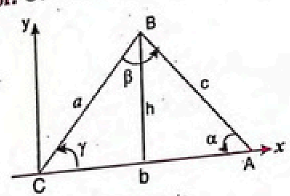
In the last section we discussed the two possibilities of solving oblique triangles SSS, SAS.

In this section we will consider the fourth case ASA or AAS which is one case because knowing any two angles and one side means knowing all the three angles and one side. The law of cosine does not work where at least two sides are needed. We state and prove the law of sines for this purpose.

Theorem: In any Δ ABC with usual labelling

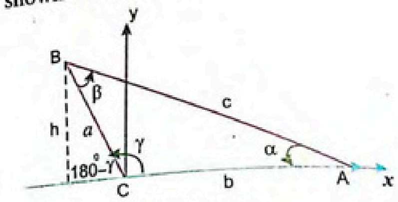
$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$

Proof: Consider any oblique triangle as shown in figure 11.12



(i) γ is acute

Figure 11.12(a)



(ii) γ is obtuse

Figure 11.12(b)

Let h = height of the triangle with base $\overline{CA}$. Then in figure 11.12 (a)

$$\sin\alpha=\frac{h}{c} \text{ and } \sin\gamma=\frac{h}{a} \qquad \text{(Solving for h)}$$

$$h=c\sin\alpha \text{ and } h=a\sin\gamma$$

Thus $c\sin\alpha=a\sin\gamma \Rightarrow \dfrac{\sin\alpha}{a}=\dfrac{\sin\gamma}{c}$ (1)

In figure 11.12 (b) $h=a\sin(180^\circ-\gamma)=a\sin\gamma$ and $h=c\sin\alpha$

Hence $c\sin\alpha=a\sin\gamma$

Similarly if we draw perpendiculars from the other two vertices on opposite sides of ΔABC we get

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b} \qquad (2)$$

and
$$\frac{\sin\beta}{b}=\frac{\sin\gamma}{c} \qquad (3)$$

Combining (1), (2) and (3) we have

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$

or equivalently,

$$\boxed{\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}}$$

These equations give **the law of sines.**

Example 10: For a triangle ABC, given $a=30$, $b=70$ $\beta=85°$. Find α.

Solution: Using law of sine

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}$$

$$\Rightarrow \quad \sin\alpha=30\times\frac{\sin 85^\circ}{70}=0.4269 \Rightarrow \alpha=25^\circ16'25''$$

Example 11: From a point A the angle of elevation of the top C of a tower is 28°. From a second point B, which is 2200 ft closer to the base of the tower, the angle of elevation of the top is 66°. What is the height h of the tower?

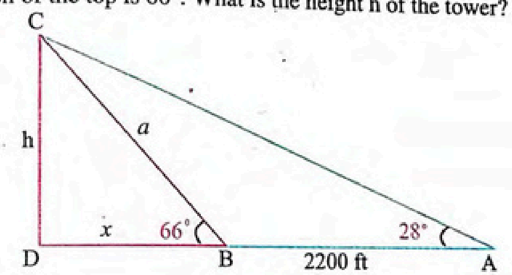


Figure 11.13

Solution: For ΔABC, AB = 2200, $\angle$ABC = 180°–66° = 114° and $\angle$ BCA=38°.

Applying the law of sines to Δ ABC, we have

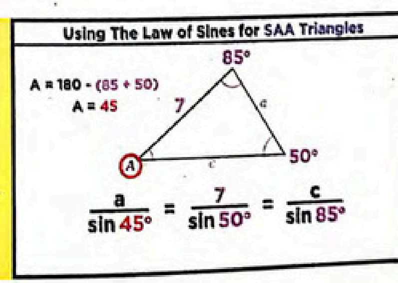
$$\frac{\sin 38^\circ}{2200}=\frac{\sin 28^\circ}{a}$$

Thus $a=1678$ ft, approximately.

Now for Δ BDC, we have

$$\sin 66^\circ=\frac{h}{a}=\frac{h}{1678}$$

Thus $h=1678\sin 66^\circ=1533$ ft, approximately.



Example 12: Solve the triangle in which

$\alpha = 38^\circ, \beta = 121^\circ$ *and* $a = 20$

Solution: Since $\alpha+\beta+\gamma = 180^\circ$

$\gamma = 180^\circ - 121^\circ - 38^\circ = 21^\circ$ Use the law of sines to get b

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha}$$

$$b = \frac{a.\sin\beta}{\sin\alpha} \quad \Rightarrow \quad b = 20 \times \frac{\sin 121^\circ}{\sin 38^\circ} = 28 \text{ approximately.}$$

Use the law of sines again but this time using α , γ to get

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} \Rightarrow c = \frac{a}{\sin\alpha} \times \sin\gamma = 20 \times \frac{\sin 21^\circ}{\sin 38^\circ} = 11.6 \approx 12 \text{ approximately.}$$

(c) The Law of Tangents

Theorem: In any triangle ABC, show that

(i) $\dfrac{a+b}{a-b} = \dfrac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}$ (ii) $\dfrac{b+c}{b-c} = \dfrac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\frac{1}{2}(\beta-\gamma)}$

(iii) $\dfrac{c+a}{c-a} = \dfrac{\tan\frac{1}{2}(\gamma+\alpha)}{\tan\frac{1}{2}(\gamma-\alpha)}$

Proof: By the law of sines in any triangle ABC

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = \text{D (say)}$$

We have

$a = \text{D}\sin\alpha$ and $b = \text{D}\sin\beta$

$a + b = \text{D}(\sin\alpha + \sin\beta)$ (1)

$a - b = \text{D}(\sin\alpha - \sin\beta)$ (2)

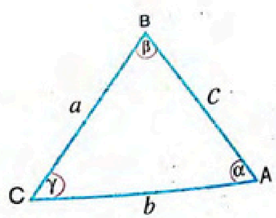


Figure 11.14

From (1) and (2)

$$\frac{a+b}{a-b} = \frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta}$$

Using the formulae

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

and $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

we get

$$\frac{a+b}{a-b} = \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}} \quad \Rightarrow \quad \boxed{\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}}$$

Similarly $\boxed{\dfrac{b+c}{b-c} = \dfrac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\frac{1}{2}(\beta-\gamma)}}$ and $\boxed{\dfrac{c+a}{c-a} = \dfrac{\tan\frac{1}{2}(\gamma+\alpha)}{\tan\frac{1}{2}(\gamma-\alpha)}}$

These three relations are known as the law of tangents. Note that the interchange of lengths a, b result in the interchange of angles α , β. Hence if b>a then it is better to use the tangent formula in the form.

$$\frac{b+a}{b-a} = \frac{\tan\frac{1}{2}(\beta+\alpha)}{\tan\frac{1}{2}(\beta-\alpha)}$$

Example 13: Use the law of tangents to solve the triangle ABC in which $a = 925$, $c = 432$ and $\beta = 42^\circ 30'$.

Solution: $\dfrac{a-c}{a+c} = \dfrac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$

But $\alpha + \gamma = 180^\circ - \beta = 137^\circ 30' \Rightarrow \frac{1}{2}(\alpha+\gamma) = 68^\circ 45'$

Hence $\dfrac{925-432}{925+432} = \dfrac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan(68^\circ 45')}$

Therefore $\tan\frac{1}{2}(\alpha-\gamma) = \frac{493}{1357} \times 2.5715 = 0.93 \Rightarrow \frac{1}{2}(\alpha-\gamma) = 43^\circ 3' \Rightarrow \alpha-\gamma = 86^\circ 6'$

Now $\alpha + \gamma = 137^\circ 30'$

$\alpha - \gamma = 86^\circ 6'$

By addition $2\alpha = 223^\circ 36'$

$\alpha = 111^\circ 48'$

By subtraction $2\gamma = 51^\circ 24'$

$\gamma = 25^\circ 42'$

To find b, we use the law of sines

$$\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

$$b=\frac{c}{\sin\gamma}\sin\beta=\frac{432}{\sin 25°42'}\sin 42°30' = 673$$

(d) Half Angle Formulae

The half angle formulae are very useful to solve a triangle when the measures of three sides of a triangle are given and no angle is known. These formulae could be derived using the law of cosines.

(i) The cosine of Half the Angle in Terms of Sides

Theorem: In any triangle ABC, show that

$$\cos\frac{\alpha}{2}=\sqrt{\frac{S(S-a)}{bc}} \qquad \cos\frac{\beta}{2}=\sqrt{\frac{S(S-b)}{ac}}$$

$$\cos\frac{\gamma}{2}=\sqrt{\frac{S(S-c)}{ab}} \text{ where } S=\frac{1}{2}(a+b+c)$$

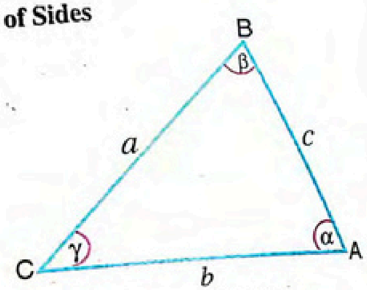


Figure 11.15

Proof: Let $S=\frac{1}{2}(a+b+c)$

Using the law of cosines

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}$$

But $\cos\alpha = 2\cos^2\frac{\alpha}{2}-1$

Hence $2\cos^2\frac{\alpha}{2}-1=\frac{b^2+c^2-a^2}{2bc}$

$$\Rightarrow \quad 2\cos^2\frac{\alpha}{2}=\frac{b^2+c^2-a^2}{2bc}+1=\frac{(b+c)^2-a^2}{2bc}$$

The numerator being difference of two squares, can be written as

$$2\cos^2\frac{\alpha}{2}=\frac{[(b+c)+a][(b+c)-a]}{2bc}$$

Since $a+b+c=2S$

and $b+c-a=2S-2a=2(S-a)$.

Hence $2\cos^2\frac{\alpha}{2}=\frac{(2S)\times 2(S-a)}{2bc}$

$$\Rightarrow \quad \cos^2\frac{\alpha}{2}=\frac{S(S-a)}{bc} \text{ or } \quad \cos\frac{\alpha}{2}=\pm\sqrt{\frac{S(S-a)}{bc}}$$

As $\alpha \angle 180°$, $\frac{\alpha}{2}$ is a measure of acute angle, the value of $\cos\frac{\alpha}{2}$ will be positive.

Hence $\boxed{\cos\frac{\alpha}{2}=\sqrt{\frac{S(S-a)}{bc}}}$ (1)

Similarly we can prove

$$\boxed{\cos\frac{\beta}{2}=\sqrt{\frac{S(S-b)}{ac}}}$$

$$\boxed{\cos\frac{\gamma}{2}=\sqrt{\frac{S(S-c)}{ab}}}$$

(ii) The sines of Half the Angle in Terms of Sides

Theorem: In any triangle ABC, show that

$$\sin\frac{\alpha}{2}=\sqrt{\frac{(S-b)(S-c)}{bc}}$$

$$\sin\frac{\beta}{2}=\sqrt{\frac{(S-c)(S-a)}{ac}}$$

$$\sin\frac{\gamma}{2}=\sqrt{\frac{(S-a)(S-b)}{ab}}, \text{ where } S=\frac{1}{2}(a+b+c)$$

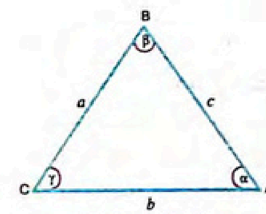


Figure 11.16

Proof: $\cos\alpha = 1-2\sin^2\frac{\alpha}{2}$

Hence $2\sin^2\frac{\alpha}{2}=1-\cos\alpha$

$$=1-\frac{b^2+c^2-a^2}{2bc}=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(a-b+c)(a+b-c)}{2bc}$$

Since $a+b+c=2S$

so $a-b+c=2S-2b=2(S-b)$ and $a+b-c=2S-2c=2(S-c)$.

Substituting these values in the above equation

$$2\sin^2\frac{\alpha}{2}=\frac{2(S-b)\times 2(S-c)}{2bc} \quad \Rightarrow \sin\frac{\alpha}{2}=\pm\sqrt{\frac{(S-b)(S-c)}{bc}}$$

Again $\sin\frac{\alpha}{2}$ is measure of an acute angle $\sin\frac{\alpha}{2}$ is always positive.

Hence $\boxed{\sin\frac{\alpha}{2}=\sqrt{\frac{(S-b)(S-c)}{bc}}}$ (2)

Similarly $\boxed{\sin\frac{\beta}{2} = \sqrt{\frac{(S-c)(S-a)}{ac}}}$ and $\boxed{\sin\frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{ab}}}$

(iii) The Tangent of Half the Angle in Terms of the Sides

Theorem: In any triangle ABC, show that

$$\tan\frac{\alpha}{2} = S\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$$

$$\tan\frac{\beta}{2} = S\sqrt{\frac{(S-c)(S-a)}{S(S-b)}}$$

$$\tan\frac{\gamma}{2} = S\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

where $S = \frac{1}{2}(a+b+c)$

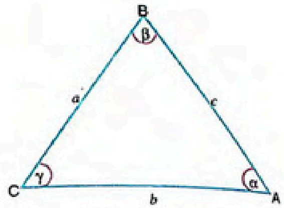


Figure 11.17

Proof: $\tan\frac{\alpha}{2} = \dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$

$\Rightarrow \quad \tan\frac{\alpha}{2} = \dfrac{\sqrt{(S-c)(S-b)/bc}}{\sqrt{S(S-a)/bc}}$ (by (1) and (2))

$\Rightarrow \quad \boxed{\tan\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}}}$ (3)

Similarly $\boxed{\tan\frac{\beta}{2} = \sqrt{\frac{(S-c)(S-a)}{S(S-b)}}}$

$\boxed{\tan\frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{S(S-c)}}}$

Now if we multiply and divide the right hand side of (3) by $\sqrt{(S-a)}$ we get

$$\tan\frac{\alpha}{2} = \frac{1}{(S-a)}\sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$$

Denoting $\sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$ by r, we get

$\boxed{\tan\frac{\alpha}{2} = \frac{r}{S-a}}$ (4)

Similarly $\boxed{\tan\frac{\beta}{2} = \frac{r}{S-b}}$ and $\boxed{\tan\frac{\gamma}{2} = \frac{r}{S-a}}$

where $S = \frac{1}{2}(a+b+c)$ and $r = \sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$

Example 14: Solve the triangle ABC with usual notation for its sides given that $a = 75$, $b = 55$ and $c = 50$

Solution: $S = \frac{1}{2}(a+b+c) = \frac{1}{2}(75+55+50)$

$= 90$

So $S - a = 90 - 75 = 15$

$S - b = 90 - 55 = 35$

$S - c = 90 - 50 = 40$

Using half angle formula

$$\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}} = \sqrt{\frac{90(15)}{(55)(50)}} = 0.700649$$

$\Rightarrow \quad \frac{\alpha}{2} = 45°31'$ or $\alpha = 91°2'$

Also $\cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}} = \sqrt{\frac{90\times35}{75\times50}} = 0.9165$

$\Rightarrow \quad \frac{\beta}{2} = 23°35'$, $\beta = 47°10'$

Hence $\gamma = 180° - (\alpha + \beta) = 180° - 138°12' = 41°48'$

EXERCISE 11.2

1. Solve the triangles with dimensions.

(i) $a = 209$, $b = 120$, $c = 241$ (ii) $a = 120$, $b = 240$, $\gamma = 32°$

(iii) $\alpha = 100°$, $c = 345$, $\gamma = 56.4°$ (iv) $a = 24.5$, $c = 43.8$, $\beta = 112°$

(v) $b = 1.6$, $c = 3.2$, $\alpha = 100°24'$ (vi) $\beta = 39°30'$, $\gamma = 34°10'$, $a = 240$

(vii) $\alpha = 35°$, $\beta = 70°$, $c = 115$ (viii) $a = 37.5$, $b = 12.4$, $\beta = 72°$

(ix) $b = 12.5$, $c = 23$, $\alpha = 38°20'$ (x) $a = 168$, $c = 319$, $\beta = 110°22'$

2. Find the angle of largest measure (Using half sine law).

(i) $a = 74$, $b = 52$ and $c = 47$

(ii) $a = 7$, $b = 9$ and $c = 7$

(iii) $a = 2.3$, $b = 1.5$ and $c = 2.7$

3. Solve the triangle for which length of three sides are given.(Using half cosine law)

   (i) $a = 9$, $b = 7$ and $c = 5$

   (ii) $a = 1.2$, $b = 9$ and $c = 10$

   (iii) $a = 6$, $b = 8$ and $c = 12$

4. One diagonal of a parallelogram is 20cm long and at one end forms angles 20° and 40° with the sides of the parallelogram. Find the length of the sides.

5. Two planes start from Karachi International Airport at the same time and fly in directions that make an angle of 127° with each other. Their speeds are 525km/h. How far apart they are at the end of 2 hours of flying time?

6. A city block is bounded by three streets. If the measure of the sides of the block are 285,375 and 396 meters, find the measure of the angles of the street make with each other.

7. The diagonal of a parallelogram meets the sides at angle of 30° and 40°. If the length of the diagonal is 30.0cm, then find the perimeter of the parallelogram.

8. Use the law of cosines to prove

   (i) $1+\cos\alpha = \dfrac{(b+c+a)(b+c-a)}{2bc}$

   (ii) $1-\cos\alpha = \dfrac{(a-b+c)(a+b-c)}{2bc}$

11.2 Areas of Triangular Regions

To find the area of a triangle ABC we discuss three cases SAS, SAA and SSS separately as follow

(a) Area of a triangle when two sides and their included angle is given.

From elementary geometry we know that the area of a triangle is equal to

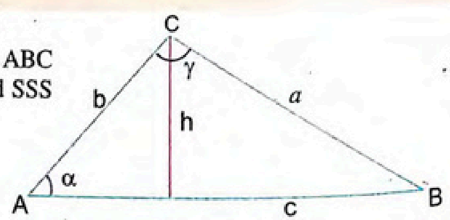


Figure 11.18

one half the product of measure of the base and measure of altitude. In figure 11.18 for the triangle ABC. Let h be the measure of altitude.

Then area Δ is given by $\Delta = \frac{1}{2}(AB)(h)$

But since $AB = c$ and $\frac{h}{b} = \sin\alpha$ or $h = b\sin\alpha$

$$\therefore \quad \Delta = \frac{1}{2}c(b\sin\alpha) = \frac{1}{2}bc\sin\alpha \qquad (1)$$

Also h can be written as $\frac{h}{a} = \sin\beta$ or $h = a\sin\beta$

So that Δ becomes, $\Delta = \frac{1}{2}(c)(a\sin\beta) = \frac{1}{2}ac\sin\beta$

Similarly by taking other sides of the triangle ABC as base

We have $\Delta = \frac{1}{2}ab\sin\gamma$

Hence the area Δ can be found by either formula

$$\Delta = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ac\sin\beta = \frac{1}{2}bc\sin\alpha$$

This shows that the area of a triangle is

"One half the product of the measure of two sides and the sine of the measure of the angle included between them."

(b) Area of a triangle when the measure of one side and measure of two angles is given (SAA).

If in the formula $\frac{1}{2}ac\sin\beta$ of the area of a triangle one of the sides say c is not known we can replace it from the law of sines.

We have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow c = \frac{a\sin\gamma}{\sin\alpha}$$

So that the area is now given by

$$\Delta = \frac{1}{2}ac\sin\beta = \frac{1}{2}a\left(\frac{a\sin\gamma}{\sin\alpha}\right)\times\sin\beta$$

$$\Delta = \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha} \qquad (2)$$

Similarly we have $\Delta = \frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta} = \frac{1}{2}\frac{c^2\sin\alpha\sin\beta}{\sin\gamma}$

(c) Area of a triangle when measures of all the sides of a triangle are given.

We know that the area Δ is given by

$$\Delta = \frac{1}{2}bc\sin\alpha = \frac{1}{2}bc\times 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = bc\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

Using half angle formulae

$$\Delta = bc\sqrt{\frac{(S-b)(S-c)}{bc}}\times\sqrt{\frac{S(S-a)}{bc}} = \sqrt{S(S-a)(S-b)(S-c)} \qquad (3)$$

This formula is known as **Hero's formula** (alternatively known as Heron's formula).

We now find the area of a triangle by using the above mentioned formula.

Example 15: Find the area of the Δ ABC where $\alpha = 18.4°$, b = 154ft and c = 211ft.

Solution: $\Delta = \frac{1}{2}bc\sin\alpha = \frac{1}{2}(154)(211)(\sin 18.4°) = 5128.349$

To two decimal places the area is 5128.35 square feet.

Example 16: Find the area of a triangle with angles 20°, 50° and 110° if the side opposite the 50° angle is 24 inches long.

Solution: Let $\alpha = 20°$, $\beta = 50°$ $\gamma = 110°$

Now b is given which is 24 inches

Hence the area Δ is

$$\Delta = \frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta}$$

$$= \frac{1}{2}(24)^2\frac{\sin 20°\sin 110°}{\sin 50°} = 120.83 \text{ square inches.}$$

Example 17: Find the area of a triangle having sides of 43ft, 89ft and 120ft.

Solution: Since three sides (but none of the angles) are known, we need Hero's formula to find area.

Let $a = 43$, b = 89 and c = 120, then

$$S = \frac{1}{2}(43+89+120) = 126$$

$$\Delta = \sqrt{126(126-43)(126-89)(126-120)} \simeq 1523.70$$

To two decimal places the area is 1523 square ft.

Example 18: What is the vertex angle of an isosceles triangle whose equal sides are 13ft long if the area is 50 ft^2.

Solution: Area Δ ABC = $\frac{1}{2}ab\sin c$

$$50 = \frac{1}{2}(13)(13)(\sin c)$$

$$\sin c = \frac{100}{169} = 0.5917$$

$$c = \sin^{-1}(0.5917) = 36.3°$$

$$= 36°18'$$

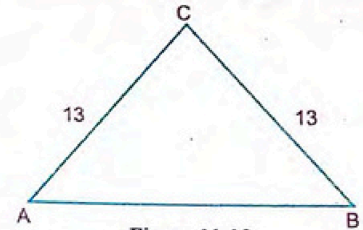


Figure 11.19

EXERCISE 11.3

1. Find the area of the triangle ABC in each case

(i)	$a = 15$	b = 80	$\gamma = 38°$
(ii)	b = 14	c = 12	$\alpha = 82°$
(iii)	$a = 30$	$\beta = 50°$	$\gamma = 100°$
(iv)	b = 40	$\alpha = 50°$	$\gamma = 60°$
(v)	$a = 7.0$	b = 8.0	c = 2.0
(vi)	$a = 11$	b = 9.0	c = 8.0
(vii)	b = 414	c = 485	$\alpha = 49°47'$
(viii)	$a = 32$	$\beta = 47°24'$	$\gamma = 70°16'$
(ix)	b = 47	$\alpha = 60°25'$	$\gamma = 41°35'$
(x)	c = 57	$\alpha = 23°24'$	$\beta = 71°36'$
(xi)	$a = 925$	c = 433	$\beta = 42°17'$
(xii)	$a = 92$	b = 71	$\gamma = 56°44'$

2. The area of a triangle is 121.34. If $\alpha = 32°25'$, $\beta = 65°65'$ then find c and angle γ.

3. One side of a triangular garden is 30 m. If its two corner angles are $22\frac{1°}{2}$ and $112\frac{1°}{2}$, find the cost of planting the grass at the rate of Rs. 5 per square meter.

4. A new home owner has a triangular-shaped back yard. Two of the three sides measure 53 ft and 42 ft and form an included angle of 135°. To determine the

amount of fertilizer and grass seed to be purchased, the owner has to know the area of the yard. Find the area of the yard to the nearest square foot.

11.3 Circles Connected with Triangles

11.3.1 (a) Circumcircle: A circle passing through the vertices of any triangle is called the circumcircle. The measure of radius of this circle called **circumradius** and is denoted by R. The center of this circle is called **circumcenter.**

The circumcenter is the point where the right bisectors of its sides meet each other.

(b) Incircle: A circle drawn inside a triangle and touching its sides is called the incircle associated with the triangle. Its radius is called **inradius** and its center is called **incenter.**

The student knows from elementary geometry that incenter is the point at which internal bisectors of the angles of a triangle meet each other.

(c) Escribed Circles: A circle, which touches one side of a triangle externally, and the other two sides internally when produced is called **escribed circle** or **ex-circle** or **e-circle.**

There are three such circles, touching the sides a, b and c externally. Each circle is associated with the side of the triangle it touches externally. The measure of the radius of the circle opposite to the vertex (touching side externally) is denoted by r_1 and measures of the radius of the circles opposite to the vertices B and C are denoted by r_2 and r_3 respectively. The centres of these circles called **ex-centres** are similarly denoted by I_1, I_2 and I_3.

The ex-centre I_1 with respect to the vertex A is the point of intersection of the external bisectors of angles B and C and internal bisector of angle A.

11.3.2 (a) To find circumradius for any triangle ABC

(i) To find R, the circumradius of a triangle ABC in terms of measure of a side and its opposite angle.

Let O be the circumcenter of the triangle ABC. Join B and O and produce it to meet the circle at D. Join C and D.

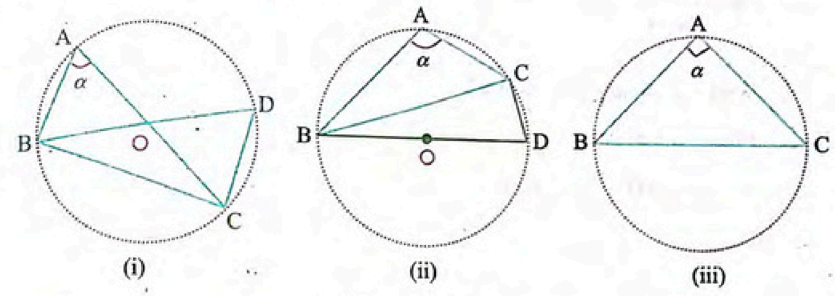


Figure 11.20

Figure 11.20 (i), (ii) and (iii) depicts the cases where measure of angle α is acute, obtuse and right angle respectively.

Now measure of $\overline{BD}$ is the diameter of circumcircle. Hence

$$\overline{BD} = 2R$$
$$m\overline{BC} = a$$

In figure (i) m $\angle BDC = \alpha < \frac{\pi}{2}$

Because α and $\angle BDC$ are angels in the same area of circle made by chord $\overline{BC}$.

Hence $\frac{m\overline{BC}}{m\overline{BD}} = \sin\angle BDC = \sin\alpha$

So $\frac{a}{2R} = \sin\alpha \qquad (1)$

In figure (ii) $\angle BDC$ and $\angle\alpha$ are supplementary angles because they are made by the same chord $\overline{BC}$ in two opposite arcs $\widehat{BAC}$ and $\widehat{BDC}$.

Hence

$$\frac{m\overline{BC}}{m\overline{BD}} = \sin\angle BDC = \sin(\pi-\alpha)$$
$$= \sin\alpha \qquad (2)$$

In figure (iii) $\alpha = \frac{\pi}{2}$

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Hence in this case

$$\frac{m\overline{BC}}{m\overline{BD}} = 1 = \sin\frac{\pi}{2} = \sin\alpha \text{ i.e. } \frac{a}{2R} = \sin\alpha \qquad (3)$$

Hence all the three situations lead to the conclusion that

$$2R = \frac{a}{\sin\alpha} \quad \text{or} \quad R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$$

(ii) Circumradius in terms of the measurements of sides of a triangle

We have already proved that

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 2\sqrt{\frac{(S-b)(S-c)}{bc}} \times \sqrt{\frac{S(S-a)}{bc}}$$

$$= \frac{2\sqrt{S(S-a)(S-b)(S-c)}}{bc} = \frac{2\Delta}{bc}$$

$$R = \frac{a}{2\sin\alpha} = \frac{abc}{4\Delta}$$

where $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$

Example 19: Find the circumscribing radius for a triangle whose sides are 3,5 and 6.

Solution: $S = \frac{a+b+c}{2} = \frac{3+5+6}{2} = 7$

$$R = \frac{abc}{4\sqrt{S(S-a)(S-b)(S-c)}} = \frac{3\times5\times6}{4\sqrt{7(4)(2)(1)}} = \frac{90}{4\sqrt{56}} = \frac{45}{2\sqrt{56}} = 3 \text{ (approx.)}$$

(b) To find inradius r for any triangle ABC

We shall prove $\quad r = \sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$

where $S = \frac{1}{2}(a+b+c)$ is the half perimeter.

Let the internal bisectors of a triangle ABC meet at the point O which is the incenter. Join O with vertices A, B and C. We obtain three triangles OAB, OBC and OCA. The altitude OF, OD and OE respectively of these triangles is a radius of the inscribed circle. The bases of these triangles are sides of the original triangle. Then from figure 11.21

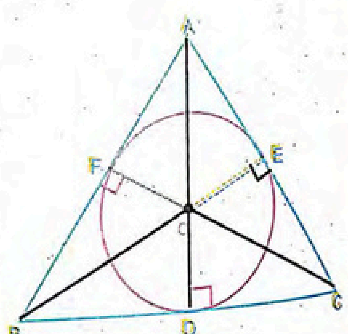


Figure 11.21

Area Δ ABC = Area Δ AOB + Area Δ BOC + Area Δ AOC

$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = r\,\frac{(a+b+c)}{2} = rS$$

To obtain r the radius of inscribed circle, we divide both sides by S

Hence $\quad r = \frac{Area\ \Delta ABC}{S}$

If we write Δ for the area of triangle ABC then

$$r = \frac{\Delta}{S} = \frac{\sqrt{S(S-a)(S-b)(S-c)}}{S}$$

$$= \sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$$

Example 20: Find the radius of the circle inscribed in a triangle whose sides are 7, 24 and 25.

Solution: We must first calculate the half perimeter S.

$$S = \frac{a+b+c}{2} = \frac{7+24+25}{2} = \frac{56}{2} = 28$$

Then $\quad r = \sqrt{\frac{(28-7)(28-24)(28-25)}{28}} = \sqrt{\frac{21\times4\times3}{28}} = \sqrt{9} = 3$

Example 21: Prove that in any triangle ABC $r = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

Solution: R.H.S $= 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

$$= \frac{4(abc)}{4\Delta}\sqrt{\frac{(S-b)(S-c)}{bc}} \times \sqrt{\frac{(S-a)(S-c)}{ac}} \times \sqrt{\frac{(S-a)(S-b)}{ab}}$$

$$= \frac{1}{\Delta}(a\,b\,c)\sqrt{\frac{(S-a)^2(S-b)^2(S-c)^2}{a^2b^2c^2}} = \frac{1}{\Delta}(a\,b\,c) \times \frac{(S-a)(S-b)(S-c)}{abc}$$

$$= \frac{1}{S\Delta} \times S(S-a)(S-b)(S-c) = \frac{1}{S\Delta} \times \Delta^2 = \frac{\Delta}{S} = r$$

as $\quad \Delta^2 = S(S-a)(S-b)(S-c)$.

(c) To find the Radius of e-circle of a triangle

Let O be the e-center opposite to the vertex A as shown in Figure 11.22

Let L, M and N be the points at which the e-circle touches the side $\overline{BC}$ externally and touches the sides AB, AC when produced respectively.

Then from elementary geometry OL, OM and ON are perpendiculars to the side $\overline{BC}$ and sides AB, AC

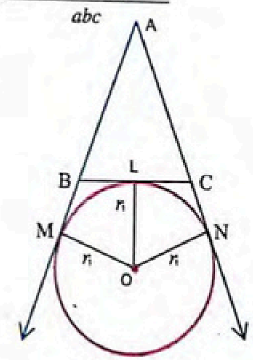


Figure 11.22

(when produced) respectively. Join the e-center O with A, B and C.

Clearly $m\overline{OL} = m\overline{OM} = m\overline{ON} = r_1$

Area ΔABC = Area ΔAOB + Area ΔAOC – Area ΔBOC

$$= \frac{1}{2}c\,r_1 + \frac{1}{2}b\,r_1 - \frac{1}{2}a\,r_1 = \frac{1}{2}r_1(c+b-a)$$

$$= \frac{1}{2}r_1(2S-2a) \text{ where } S = \frac{a+b+c}{2}$$

Thus the area Δ of triangle ABC is

$$\Delta = r_1(S-a) \text{ or } \boxed{r_1 = \frac{\Delta}{S-a}}$$

Similarly $\boxed{r_2 = \frac{\Delta}{S-b}}$ if the e-circle touches side b directly but sides a, c when produced.

The e-radius r_3 of escribed circle associated with vertex C is given by $\boxed{r_3 = \frac{\Delta}{S-c}}$

Example 22: Find R, r, r_1, r_2 and r_3 for the triangle with measures of the sides 5, 12 and 13.

Solution:

Let $a = 5$, b = 12 and c = 13

$$S = \frac{1}{2}(5+12+13) = 15$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{15\times10\times3\times2} = 30$$

$$R = \frac{(abc)}{4\Delta} = \frac{5\times12\times13}{4\times30} = 6.5$$

$$r = \frac{\Delta}{S} = \frac{30}{15} = 2$$

$$r_1 = \frac{\Delta}{S-a} = \frac{30}{15-5} = 3$$

$$r_2 = \frac{\Delta}{S-b} = \frac{30}{15-12} = 10$$

$$r_3 = \frac{\Delta}{S-c} = \frac{30}{15-13} = 15$$

Example 23: Prove that for any equilateral triangle $r : R : r_1 = 1 : 2 : 3$

Solution: Let the measure of each side of the triangle be denoted by c.

$$\therefore S = \frac{c+c+c}{2} = \frac{3c}{2}$$

Area of the triangle is given by

$$\Delta = \sqrt{S(S-c)^3} = \sqrt{\frac{3c}{2}\left(\frac{3c}{2}-c\right)^3} = \sqrt{\frac{3c}{2}\times\frac{c^3}{8}} = \frac{\sqrt{3}c^2}{4}$$

$$R = \frac{a.b.c}{4\Delta} = \frac{c^3}{\frac{4\sqrt{3}}{4}c^2} = \frac{c}{\sqrt{3}}$$

$$r = \frac{\Delta}{S} = \frac{\sqrt{3}c^2}{4}\times\frac{2}{3c} = \frac{c}{2\sqrt{3}}$$

Now $r_1 = \dfrac{\Delta}{S-a} = \dfrac{\frac{\sqrt{3}}{4}c^2}{\frac{3c}{2}-c} = \dfrac{\sqrt{3}c^2}{4}\times\dfrac{2}{c} = \dfrac{\sqrt{3}c}{2}$

Hence, $r : R : r_1 = \dfrac{c}{2\sqrt{3}} : \dfrac{c}{\sqrt{3}} : \dfrac{\sqrt{3}c}{2}$

$$= \frac{c}{2\sqrt{3}}\times\frac{\sqrt{3}}{c} : \frac{c}{\sqrt{3}}\times\frac{\sqrt{3}}{c} : \frac{\sqrt{3}c}{2}\times\frac{\sqrt{3}}{c} = \frac{1}{2} : 1 : \frac{3}{2} = 1:2:3$$

Example 24: Find the area of the inscribed circle of the triangle whose sides measure 7, 8 and 9 unit.

Solution: Here $S = \dfrac{7+8+9}{2} = 12$

Area of triangle with sides 7, 8 and 9.

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{12\times5\times4\times3} = 26.83 \text{ unit}^2$$

$$r = \frac{\Delta}{S} = \frac{26.83}{12} = 2.24 \text{ unit}$$

Area of inscribed circle $= \pi r^2 = (3.1416)(2.24)^2 = 15.76 \text{ unit}^2$

EXERCISE 11.4

1. Compute the in-radius (r) and circum-radius (R) of the triangles whose sides are given;
   (i) 3, 5, 6 (ii) 21, 20, 29
2. Find the area of the inscribed circle of the triangle with measures of the sides 55m, 25m and 70m.
3. The measures of the sides of a triangle are 20, 25 and 30 decimeter. Find the radius of the described circles
   (i) Opposite to larger side (ii) Opposite to smaller side

4. Show that (i) $\sqrt{r r_1 r_2 r_3} = \Delta$ (ii) $abc(\sin\alpha+\sin\beta+\sin\gamma)=4\Delta s.$
(iii) $r_1 r_2 r_3 = r s^2$

5. Prove that for any triangle ABC
(i) $r_1+r_2+r_3-r=4R$ (ii) $r_1r_2+r_2r_3+r_3r_1=s^2$
(iii) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$

6. Show that
(i) $r_1=S\tan\frac{\alpha}{2}$ (ii) $r_2=S\tan\frac{\beta}{2}$ (iii) $r_3=S\tan\frac{\gamma}{2}$

7. The sides of a triangle are in the ratio 3:7:8. The radius of the inscribed circle is 2m. Find the sides of the triangle.

REVIEW EXCERCISE 11

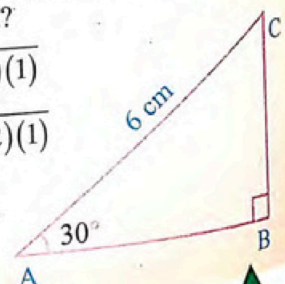
1. Choose the correct option.

(i) In right triangle ABC, find b if a = 2, c = 5, and $r = 90^0$
(a) 7 (b) 3 (c) $\sqrt{21}$ (d) $\sqrt{29}$

(ii) An escalator in a department store makes an angle of 45^0 with the ground. How long is the escalator if it carries people a vertical distance of 24 feet?
(a) $12\sqrt{2}$ ft (b) $24\sqrt{2}$ ft (c) $8\sqrt{3}$ ft (d) 48 ft

(iii) If in an isosceles triangle, 'a' is the length of the base and 'b' the length of one of the equal sides, then its area is
(a) $\frac{a}{4}\sqrt{4b^2-a^2}$ (b) $\frac{b}{4}\sqrt{4b^2-a^2}$ (c) $\frac{a+b}{4}\sqrt{a^2-b^2}$ (d) $\frac{a-b}{4}\sqrt{b^2-a^2}$

(iv) If Heron's formula is used to find the area of triangle ABC having a=3 meters, b = 5 meters, and c = 6 meters, which of the following shows the correct way to set up the formula?
(a) $\Delta=7\sqrt{(10)(12)(13)}$ (b) $\Delta=\sqrt{(4)(2)(1)}$
(c) $\Delta=\sqrt{7(3)(5)(6)}$ (d) $\Delta=\sqrt{7(4)(2)(1)}$



(v) In the adjoining figure, the length of $\overline{BC}$ is
(a) $2\sqrt{3}$ cm (b) $3\sqrt{3}$ cm
(c) $4\sqrt{3}$ cm (d) 3 cm

(vi) If the angle of depression of an object from a 75 m high tower is 30^0, then the distance of the object from the tower is
(a) $25\sqrt{3}$ m (b) $50\sqrt{3}$ m (c) $75\sqrt{3}$ m (d) 150 m

(vii) The point of Concurrency of the right bisectors of the sides of a triangleis called
(a) In-Centre (b) Orthocenter (c) Circumcentre (d) Centroid

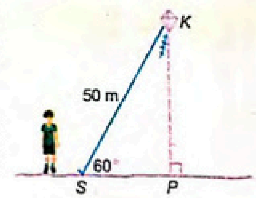
(viii) With usual notations $rr_1r_2r_3=$
(a) Δ (b) Δ^2 (c) $\frac{abc}{\Delta}$ (d) $\frac{\Delta}{abc}$

2. Solve the triangles.
(i) $a=0.7$, $c=0.8$, $\beta=141°30'$ (ii) $a=34$, $b=23$, $c=58$
(iii) $a=15.6$, $b=18$, $\gamma=35°10'$ (iv) $a=48$, $b=32$, $\gamma=57°$
(v) $b=35$, $c=37$, $\alpha=23°25'$ (vi) $a=58.4$, $\beta=37.2°$, $\gamma=100°$
(vii) $c=13.6$, $\alpha=30°24'$, $\beta=72°6'$

3. Find the measure of the smallest angle of the triangle whose sides have lengths
(i) 4.3, 5.1 and 6.3 (ii) 3, 4.2 and 3.8

4. Find the measure of the largest angle of the triangle whose sides have lengths
(i) 2.9, 3.3 and 4.1 (ii) 6.0, 8 and 9.4

5. The sides of a parallelogram are 25cm and 35cm long and one of its angles is 36°. Find the lengths of its diagonals.



6. A man is flying a kite. He has let out 50 m of string, and he notices that the string makes an angle of 60° with the ground. How high is the kite?

7. A robin on a branch 40ft up in a tree spots a worm at an angle of depression of 14°. From a branch 15ft above the robin, a crow spots the same worm at an angle of depression of 19°. How far is each bird from the worm?

8. The angle of elevation of a building is 48° from A and 61° from B. If AB is 20 m, find the height of the building.