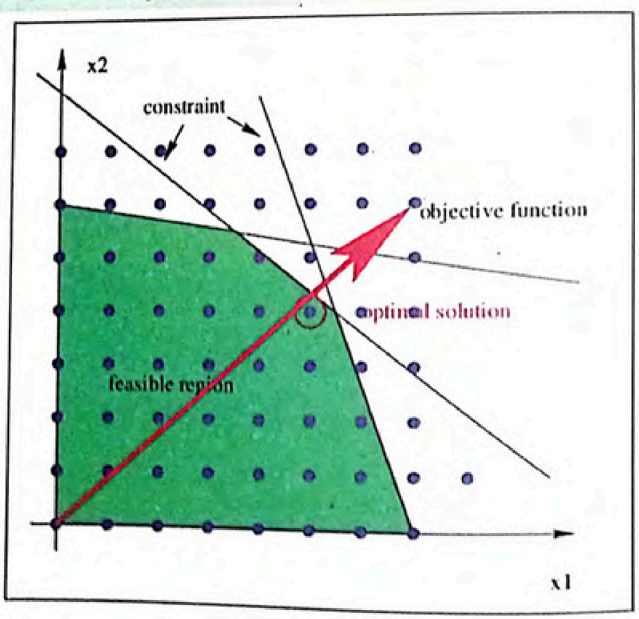
3. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains atleast 10 units of vitamin A, 12 units of vitamins B and 8 units of vitamin C. The vitamin content of one kg, food is given below:

| Food | Vitamin A | Vitamin B | Vitamin C |
|---|---|---|---|
| X | 1 | 2 | 3 |
| Y | 2 | 2 | 1 |

One kg of food X costs Rs.16 and one kg of food Y costs Rs.20. Find the least cost of the mixture which will produce the required diet.

4. Find the maximum and minimum values of the function $Z = 5x + 10y$ subject to the constraints

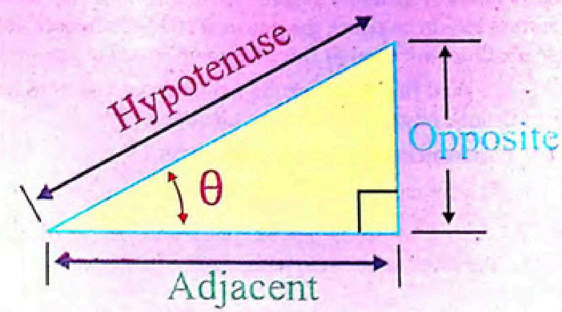
$$x + 2y > 120$$
$$x + y > 60$$
$$x - 2y > 0$$
$$x, y > 0$$






# UNIT 10

# TRIGONOMETRIC IDENTITIES OF SUM AND DIFFERENCE OF ANGLES




STUDENTS' LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Use distance formula to establish fundamental law of trigonometry
  - $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, and deduce that
  - $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$,
  - $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$,
  - $\tan(\alpha\pm\beta) = \dfrac{\tan\alpha\pm\tan\beta}{1\pm\tan\alpha\tan\beta}$
- Define allied angles
- Use fundamental law and its deductions to derive trigonometric ratios of allied angles
- Express $a\sin\theta + b\cos\theta$ in the form $r\sin(\theta+\varphi)$ where $a = r\cos\varphi$ and $b = r\sin\varphi$
- Derive double angle, half angle and triple angle identities from fundamental law and its deductions.
- Express the product (of sines and cosines) as sums or differences (of sines and cosines)
- Express the sums or differences (of sines and cosines) as products (of sines and cosines)

## 10.1 Introduction

In the previous class some basic trigonometric identities have been proved and applied to show different trigonometric relations. This unit is a continuation of derivations of different trigonometric identities. These identities play an important role in calculus, the physical and life sciences and economics, where these identities are used to simplify complicated expressions.

We shall first establish the fundamental law of trigonometry so as to be able to deduce other trigonometric identities.

### 10.1.1 Fundamental law of trigonometry

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$.......(1)

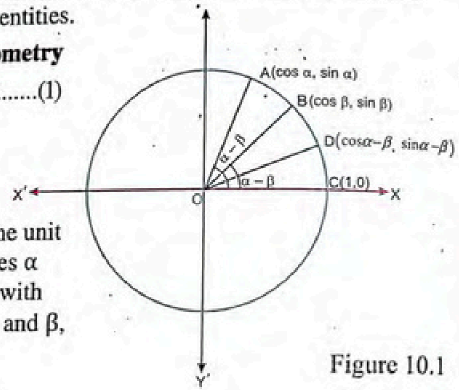



Figure 10.1

Consider the given unit circle with center at O.

To establish the identity (1), we use the unit circle shown in **Figure 10.1**. The angles α and β are drawn in standard position, with $\overline{OA}$ and $\overline{OB}$ as the terminal sides of α and β, respectively.

The coordinates of A are (cosα, sinα),

The coordinates of B are(cosβ, sinβ).

The angle (α−β) is formed by the terminal sides of the angles α and β. An angle equal in measure to angle (α−β) is placed in standard position in the same figure ($\angle COD$).

From geometry, if two central angles of a circle have the same measure, then the respective chords are also equal in measure. Thus the chords $\overline{AB}$ and $\overline{CD}$ are equal in length. Using the distance formula, we can calculate the lengths of the chords $\overline{AB}$ and $\overline{CD}$.

The length of a line segment with end points $(x_1, y_1)$ and $(x_2, y_2)$ is given by the following distance formula

$$d=|P_1P_2|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

We apply this formula to the chords $\overline{AB}$ and $\overline{CD}$.

As $|\overline{AB}|=|\overline{CD}|$, so by distance formula,

$$\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2}=\sqrt{[\cos(\alpha-\beta)-1]^2+[\sin(\alpha-\beta)]^2}$$





Squaring each side of the equation and simplifying, we obtain

$$(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=\left[\cos(\alpha-\beta)-1\right]^2+\left[\sin(\alpha-\beta)\right]^2$$

$$\Rightarrow \cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta$$

$$=\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)$$

$$\Rightarrow \cos^2\alpha+\sin^2\alpha+\cos^2\beta+\sin^2\beta-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$$

$$=\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta)+1-2\cos(\alpha-\beta)$$

Simplifying by using $\sin^2\theta+\cos^2\theta=1$, we have

$$2-2\sin\alpha\sin\beta-2\cos\alpha\cos\beta=2-2\cos(\alpha-\beta).$$

Solving for cos(α−β), it gives us

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

We refer to (1) as fundamental law of trigonometry.

### 10.1.2. Deductions from the fundamental law of trigonometry

The following can be deduced from the fundamental law of trigonometry which are useful and play a significant role in proving the other trigonometry identities.

**(i)** $\cos(-\beta)=\cos\beta$

By Fundamental Law of Trigonometry,

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Letting $\alpha=0$, we get

$\cos(0-\beta)=\cos 0\cos\beta+\sin 0\sin\beta$

$\cos(-\beta)=1\cdot\cos\beta+0\cdot\sin\beta$

$\cos(-\beta)=\cos\beta$

**(ii)** $\cos(\frac{\pi}{2}-\beta)=\sin\beta$

By Fundamental Law of Trigonometry,

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Letting $\alpha=\frac{\pi}{2}$, we get

$$\cos(\frac{\pi}{2}-\beta)=\cos\frac{\pi}{2}\cos\beta+\sin\frac{\pi}{2}\sin\beta$$

$\Rightarrow \cos(\frac{\pi}{2}-\beta)=0\cdot\cos\beta+1\cdot\sin\beta \quad \left(\because \cos\frac{\pi}{2}=0, \sin\frac{\pi}{2}=1\right)$

$\therefore \cos(\frac{\pi}{2}-\beta)=\sin\beta$

**(iii)** $\sin(\frac{\pi}{2}+\alpha)=\cos\alpha$

By identity (ii), $\cos(\frac{\pi}{2}-\beta)=\sin\beta$

Letting $\beta=\frac{\pi}{2}+\alpha$, we get

$\cos\left(\frac{\pi}{2}-\left(\frac{\pi}{2}+\alpha\right)\right)=\sin\left(\frac{\pi}{2}+\alpha\right) \Rightarrow \cos(-\alpha)=\sin\left(\frac{\pi}{2}+\alpha\right)$

$\Rightarrow \cos\alpha=\sin\left(\frac{\pi}{2}+\alpha\right) (\because \cos(-\alpha)=\cos\alpha)$

$\therefore \sin(\frac{\pi}{2}+\alpha)=\cos\alpha$

**(iv)** (a) $\cos(\frac{\pi}{2}+\alpha)=-\sin\alpha$

By Fundamental Law of Trigonometry,
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Letting $\beta=-\frac{\pi}{2}$, we get

$\cos(\alpha-(-\frac{\pi}{2}))=\cos\alpha\cos(-\frac{\pi}{2})+\sin\alpha\sin(-\frac{\pi}{2})$

$\Rightarrow \cos(\alpha+\frac{\pi}{2})=\cos\alpha\cdot 0+\sin\alpha(-1)$

$\left(\because \cos(-\frac{\pi}{2})=0, \sin(-\frac{\pi}{2})=-1\right)$

$\therefore \cos(\frac{\pi}{2}+\alpha)=-\sin\alpha$

**(iv)** (b) $\sin(-\alpha)=-\sin\alpha$

By identity (ii)

$\cos\left(\frac{\pi}{2}-\beta\right)=\sin\beta$

Setting $\beta=-\alpha$

$\cos\left(\frac{\pi}{2}+\alpha\right)=\sin(-\alpha)$

By using(iv) (a)

$-\sin\alpha=\sin(-\alpha)$

**(v)** $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

By Fundamental law of trigonometry,
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$





Replacing $\beta$ by $-\beta$, we get

$\cos(\alpha-(-\beta))=\cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)$

$\therefore \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \quad (\because \cos(-\beta)=\cos\beta, \sin(-\beta)=-\sin\beta)$

**(vi)** $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

By identity (v), $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

Replacing $\alpha$ by $\frac{\pi}{2}+\alpha$, we get

$\cos\left(\left(\frac{\pi}{2}+\alpha\right)+\beta\right)=\cos\left(\frac{\pi}{2}+\alpha\right)\cos\beta-\sin\left(\frac{\pi}{2}+\alpha\right)\sin\beta$

$\Rightarrow \cos(\frac{\pi}{2}+(\alpha+\beta))=\cos\left(\frac{\pi}{2}+\alpha\right)\cos\beta-\sin\left(\frac{\pi}{2}+\alpha\right)\sin\beta$

By using identities (iii) and (iv), we get

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

**(vii)** $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

By identity (vi), $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

Replacing $\beta$ by $-\beta$, we get

$\sin(\alpha+(-\beta))=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$

$\Rightarrow \sin(\alpha-\beta)=\sin\alpha(\cos\beta)+\cos\alpha(-\sin\beta)$

$(\because \cos(-\beta)=\cos\beta, \sin(-\beta)=-\sin\beta)$

$\therefore \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

**(viii)** $\tan(-\theta)=-\tan\theta$

$$\tan(-\theta)=\frac{\sin(-\theta)}{\cos(-\theta)}=\frac{-\sin\theta}{\cos\theta}=-\tan\theta$$

**(ix)** $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

Dividing numerator and denominator of R.H.S by $\cos\alpha\cos\beta$,

$$\tan(\alpha+\beta)=\frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}=\frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$=\frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}}$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

**(x)** $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

By identity (ix), $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

Replacing $\beta$ by $-\beta$, we get

$$\tan(\alpha+(-\beta))=\frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)}$$

$$\Rightarrow \tan(\alpha-\beta)=\frac{\tan\alpha+(-\tan\beta)}{1-\tan\alpha(-\tan\beta)}\quad(\because \tan(-\beta)=-\tan\beta)$$

$$\therefore \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

**Example 1:** Find $\tan 15°$ exactly.

**Solution:** We rewrite $15°$ as $45°-30°$ and using the identity

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$\tan 15°=\tan(45°-30°)=\frac{\tan 45°-\tan 30°}{1+\tan 45°\tan 30°}=\frac{1-\frac{1}{\sqrt{3}}}{1+1\cdot\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{3-\sqrt{3}}{3+\sqrt{3}}$$

**Example 2:** Find the exact value of: $\sin 42°\cos 12°-\cos 42°\sin 12°$.

**Solution:** Using the identity $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$$\sin 42°\cos 12°-\cos 42°\sin 12°=\sin(42°-12°)=\sin 30°=\frac{1}{2}.$$





**Example 3:** Given $\sin\alpha=\frac{12}{13}$ and $\cos\beta=\frac{3}{5}$, where $\alpha$ and $\beta$ are in the first quadrant. Find in which quadrant does $(\alpha+\beta)$ lie.

**Solution:** Given that $\alpha,\beta$ are both in the first quadrant. Since cosine is positive in the first quadrant and negative in the second quadrant, therefore, $\cos(\alpha+\beta)$ will decide the quadrant in which $(\alpha+\beta)$ lies?

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \qquad (1)$$

As $\cos^2\alpha=1-\sin^2\alpha$, putting $\sin^2\alpha=\left(\frac{12}{13}\right)^2=\frac{144}{169}$

$$\cos^2\alpha=1-\frac{144}{169}=\frac{169-144}{169}=\frac{25}{169}$$

$\cos\alpha=\pm\frac{5}{13}$. But $\cos\alpha$ is +ve in the 1st quadrant,

$$\therefore \cos\alpha=+\frac{5}{13}$$

As $\sin^2\beta=1-\cos^2\beta$, putting in it $\cos^2\beta=\frac{9}{25}$

$$\sin^2\beta=1-\frac{9}{25}=\frac{25-9}{25}=\frac{16}{25}=\pm\frac{4}{5}$$

But $\sin\beta$ is +ve in the 1st quadrant,

$$\therefore \sin\beta=\frac{4}{5}$$

Putting values of $\sin\alpha$, $\cos\alpha$, $\sin\beta$ and $\cos\beta$ in (1)

$$\cos(\alpha+\beta)=\left(\frac{5}{13}\right)\left(\frac{3}{5}\right)-\left(\frac{12}{13}\right)\left(\frac{4}{5}\right)=\left(\frac{15}{65}\right)-\left(\frac{48}{65}\right)=\frac{15-48}{65}=\frac{-33}{65}$$

Since $\cos(\alpha+\beta)$ is negative, it follows that $(\alpha+\beta)$ is in the second quadrant.

## 10.2 Trigonometric ratios of allied angles

**10.2.1** The angles of measure $\frac{\pi}{2}\pm\theta, \pi\pm\theta, \frac{3\pi}{2}\pm\theta, 2\pi\pm\theta$ are called **allied angles.**

Thus the angles which are connected with basic angles of measure $\theta$ by a right angle or its multiple are known as allied angles.

### 10.2.2 Derivation of trigonometric ratios of allied angles

All the following trigonometric ratios of allied angles can be derived from the fundamental theorem of trigonometry and thus has been left for the students as an exercise.

i. $\sin(\frac{\pi}{2}-\theta)=\cos\theta$ , $\sin(\frac{\pi}{2}+\theta)=\cos\theta$

ii. $\cos(\frac{\pi}{2}-\theta)=\sin\theta$ , $\cos(\frac{\pi}{2}+\theta)=-\sin\theta$

iii. $\tan(\frac{\pi}{2}-\theta)=\cot\theta$ , $\tan(\frac{\pi}{2}+\theta)=-\cot\theta$

iv. $\sin(\pi-\theta)=\sin\theta$ , $\cos(\pi-\theta)=-\cos\theta$

v. $\sin(\pi+\theta)=-\sin\theta$ , $\cos(\pi+\theta)=-\cos\theta$

vi. $\tan(\pi-\theta)=-\tan\theta$ , $\tan(\pi+\theta)=\tan\theta$

vii. $\sin(\frac{3\pi}{2}+\theta)=-\cos\theta$ , $\cos(\frac{3\pi}{2}-\theta)=-\sin\theta$

viii. $\sin(\frac{3\pi}{2}+\theta)=-\cos\theta$ , $\cos(\frac{3\pi}{2}+\theta)=\sin\theta$

ix. $\tan(\frac{3\pi}{2}+\theta)=\cot\theta$ , $\tan(\frac{3\pi}{2}+\theta)=-\cot\theta$

x. $\sin(2\pi-\theta)=-\sin\theta$ , $\cos(2\pi-\theta)=\cos\theta$

xi. $\sin(2\pi+\theta)=\sin\theta$ , $\cos(2\pi+\theta)=\cos\theta$

xii. $\tan(2\pi-\theta)=-\tan\theta$ , $\tan(2\pi+\theta)=\tan\theta$

Note: 1. The above results also apply to the reciprocals of sine, cosine and tangent. These results are to be applied frequently in the study of trigonometry.

2. They can be obtained by using the following two-steps procedure:

a)

| | | |
|---|---|---|
| First quadrant | $(0, \pi/2)$ | All are +ve |
| Second quadrant | $(\pi/2, \pi)$ | sin is +ve |
| Third quadrant | $(\pi, 3\pi/2)$ | tan is +ve |
| Fourth quadrant | $(3\pi/2, 2\pi)$ | cos is +ve |





b) If we have $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ in the formula, the formula changes sine to cosine and cosine to sine, tangent to cotangent and cotangent to tangent, secant to cosecant and cosecant to secant. If we have $\pi$ or $2\pi$ in the formula, the function does not change.

**Example 4:** Simplify each expression, given that $0 < x < \pi/2$.

(i) $\sin(\pi/2 + x)$ (ii) $\cos(\pi/2 + x)$ (iii) $\tan(3\pi/2 + x)$

(iv) $\cot(2\pi - x)$ (v) $\sin(\pi + x)$ (vi) $\cos(2\pi + x)$

**Solution:** (i) $(\pi/2 + x)$ is in the second quadrant, so $\sin(\pi/2 + x) = \cos x$

(ii) $(\pi/2 + x)$ is in the second quadrant, so $\cos(\pi/2 + x) = -\sin x$

(iii) $(3\pi/2 + x)$ is in the fourth quadrant, so $\tan(3\pi/2 + x) = -\cot x$

(iv) $(2\pi - x)$ is in the fourth quadrant, so $\cot(2\pi - x) = -\sin x$

(v) $(\pi + x)$ is in the third quadrant, so $\sin(\pi + x) = -\cot x$

(vi) $(2\pi + x)$ is in the first quadrant, so $\cos(2\pi + x) = \cos x$

**Example 5: Simplify** $\frac{\cos(90^\circ+x)+\sin(270^\circ-x)+\sin(180^\circ-x)}{\cos(-x)-\cos(360^\circ-x)+\sin(90^\circ+x)}$

**Solution:** $\frac{\cos(90^\circ+x)+\sin(270^\circ-x)+\sin(180^\circ-x)}{\cos(-x)-\cos(360^\circ-x)+\sin(90^\circ+x)}$

$$=\frac{-\sin x-\cos x+\sin x}{\cos x-\cos x+\cos x}=\frac{-\cos x}{\cos x}=-1$$

**Example 6:** If $\alpha, \beta, \gamma$ are the angles of $\Delta$ ABC. Prove that

i) $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

ii) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=-1$

**Solution:** As $\alpha, \beta, \gamma$ are the angles of $\Delta ABC$ $\therefore \alpha + \beta + \gamma = 180^0$

i) $\alpha + \beta = 180^0 - \gamma$

$\tan(\alpha + \beta) = \tan(180^0 - \gamma)$ $\Rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\tan\gamma$

$\Rightarrow \tan\alpha + \tan\beta = -\tan\gamma + \tan\alpha\tan\beta\tan\gamma$

$\therefore \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

ii) As $\alpha + \beta + \gamma = 180^0 \Rightarrow \frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=90^0$

So $\frac{\alpha}{2}+\frac{\beta}{2}=90^0-\frac{\gamma}{2}$ $\therefore \tan(\frac{\alpha}{2}+\frac{\beta}{2})=\tan\left(90^0-\frac{\gamma}{2}\right)$

$$\frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}=\cot\frac{\gamma}{2}=\frac{1}{\tan\frac{\gamma}{2}} \Rightarrow \tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$$

### 10.2.3 Writing $a\sin\theta + b\cos\theta$ in the form $r\sin(\theta+\phi)$ where $a = r\cos\phi$ and $b= r\sin\phi$

Writing $a\sin\theta + b\cos\theta$ in the Form $r\sin(\theta+\phi)$.

Let P(a, b) be a coordinate point in the plane and let $\theta$ be the angle with initial side x-axis and terminal side the ray $\overline{OP}$ as shown in Figure 10.2.

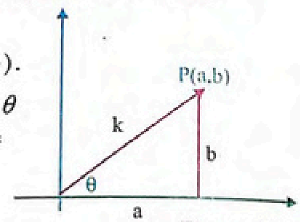


Figure 10.2

We can express $a\sin\theta + b\cos\theta$ in the form $r\sin(\theta+\phi)$ where $r=\sqrt{a^2+b^2}$ and $\phi$ is given by the equations $r\cos\phi = a$ and $r\sin\phi = b$.

The method is explained through the following example.

**Example 7:** Express $5\sin\theta + 12\cos\theta$ in the form $r\sin(\theta+\phi)$, where the terminal side of the angle of measure $\phi$ is in the 1st quadrant.

**Solution:** Identifying $5\sin\theta + 12\cos\theta$ with $r\sin(\theta+\phi)$ gives

$$5\sin\theta + 12\cos\theta = r\cos\phi\sin\theta + r\sin\phi\cos\theta \quad (1)$$

so $5 = r\cos\phi$ and $12 = r\sin\phi$

$$\therefore\ r=\sqrt{a^2+b^2} = \sqrt{(5)^2+(12)^2} = \sqrt{25+144} = \sqrt{169} = 13$$

and $r\cos\phi = 5 \Rightarrow 13\cos\phi = 5 \Rightarrow \cos\phi = \frac{5}{13}$,

$r\sin\phi = 12 \Rightarrow 13\sin\phi = 12 \Rightarrow \sin\phi = \frac{12}{13}$.

Thus, from (1) we get

$$5\sin\theta + 12\cos\theta = 13\left(\frac{5}{13}\sin\theta+\frac{12}{13}\cos\theta\right)$$

$$= 13\left(\sin\theta\frac{5}{13}+\cos\theta\frac{12}{13}\right) = r(\sin\theta\cos\phi+\cos\theta\sin\phi)$$

$$= r\sin(\theta+\phi) \quad \text{where } \sin\phi = \frac{12}{13}, \cos\phi=\frac{5}{13} \text{ and } r = 13$$





## EXERCISE 10.1

1. Write each of the following as a trigonometric function of a single angle.
(i) $\sin 37^\circ\cos 22^\circ + \cos 37^\circ \sin 22^\circ$ (ii) $\cos 83^\circ\cos 53^\circ + \sin 83^\circ\sin 53^\circ$
(iii) $\cos 19^\circ\cos 5^\circ - \sin 19^\circ\sin 5^\circ$ (iv) $\sin 40^\circ\cos 15^\circ - \cos 40^\circ\sin 15^\circ$
(v) $\frac{\tan 20^\circ+\tan 32^\circ}{1-\tan 20^\circ\tan 32^\circ}$ (vi) $\frac{\tan 35^\circ-\tan 12^\circ}{1+\tan 35^\circ\tan 12^\circ}$

2. Evaluate each of the following exactly.
(i) $\sin\frac{\pi}{12}$ (ii) $\tan 75^\circ$ (iii) $\tan 105^\circ$ (iv) $\tan\frac{5\pi}{12}$ (v) $\cos 15^\circ$ (vi) $\sin\frac{7\pi}{12}$

3. If $\sin u = \frac{3}{5}$ and $\sin v = \frac{4}{5}$ and u and v are between 0 and $\frac{\pi}{2}$, evaluate each of the following exactly.
(i) $\cos(u+v)$ (ii) $\tan(u-v)$ (iii) $\sin(u-v)$ (iv) $\cos(u-v)$

4. If $\sin\alpha = -\frac{4}{5}$ and $\cos\beta = -\frac{12}{13}$, $\alpha$ in Quadrant III and $\beta$ in Quadrant II, find the exact value of:
(i) $\sin(\alpha-\beta)$ (ii) $\cos(\alpha+\beta)$ (iii) $\tan(\alpha+\beta)$

5. If $\tan\alpha = \frac{3}{4}$, $\sec\beta = \frac{13}{5}$, and neither the terminal side of the angle of measure $\alpha$ nor $\beta$ in the first quadrant, then find:
(i) $\sin(\alpha+\beta)$ (ii) $\cos(\alpha+\beta)$ (iii) $\tan(\alpha+\beta)$

6. Show that:
(i) $\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = 1 - 2\sin^2\frac{\alpha}{2}$
(ii) $\sin(\alpha+\beta)\sin(\alpha-\beta) = \cos^2\beta - \cos^2\alpha$

7. Show that: (i) $\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$ (ii) $\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

8. Prove that: (i) $\tan\left(\frac{\pi}{4}+\theta\right) = \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$ (ii) $\tan\left(\frac{\pi}{4}-\theta\right) = \frac{1-\tan\theta}{1+\tan\theta}$
(iii) $\frac{\tan(\alpha+\beta)}{\cot(\alpha-\beta)} = \frac{\tan^2\alpha-\tan^2\beta}{1-\tan^2\alpha\tan^2\beta}$ (iv) $\frac{1-\tan\theta\tan\phi}{1+\tan\theta\tan\phi} = \frac{\cos(\theta+\phi)}{\cos(\theta-\phi)}$

9. Prove that: $\frac{\sin\theta}{\sec 4\theta}+\frac{\cos\theta}{\operatorname{cosec}4\theta}=\sin 5\theta$

10. Show that: $\frac{\sin(180^\circ-\alpha)\cos(270^\circ-\alpha)}{\sin(180^\circ+\alpha)\cos(270^\circ+\alpha)}=1$

11. If α, β, γ are the angles of a triangle ABC, show that
$\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

12. If $\alpha+\beta+\gamma=180^\circ$, show that $\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1$

13. Express each of the following in the form $r\sin(\theta+\phi)$ where terminal ray of $\theta$ and $\phi$ are in the first quadrant.
(i) $4\sin\theta+3\cos\theta$. (ii) $15\sin\theta+8\cos\theta$.
(iii) $2\sin\theta-5\cos\theta$. (iv) $\sin\theta+\cos\theta$.

## 10.3 Double, Half and Triple Angle Identities

In this section we derive formulae/identities for $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$ for $\sin\frac{1}{2}\theta$, $\cos\frac{1}{2}\theta$ and $\tan\frac{1}{2}\theta$ and for $\sin 3\theta$, $\cos 3\theta$ and $\tan 3\theta$ called double angle, half angle and triple angle formulae respectively.

### 10.3.1 Double Angle Identities

We know that. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ (1)

and $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ (2) Putting $\beta=\alpha$ in (1).

$\sin(\alpha+\alpha)=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha$

$\sin 2\alpha=2\sin\alpha\cos\alpha$ (3)

Now putting $\beta=\alpha$ in (2)

$\cos(\alpha+\alpha)=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha$

$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$ (4)

Putting $\cos^2\alpha=1-\sin^2\alpha$ in (4) $(\because \sin^2\alpha+\cos^2\alpha=1)$

$\cos 2\alpha=1-\sin^2\alpha-\sin^2\alpha$

$\cos 2\alpha=1-2\sin^2\alpha$ (5)





Now putting $\sin^2\alpha=1-\cos^2\alpha$ in (4)

$\cos 2\alpha=\cos^2\alpha-(1-\cos^2\alpha)$

$\cos 2\alpha=\cos^2-1+\cos^2\alpha$

$\cos 2\alpha=2\cos^2\alpha-1$ (6)

We also know that $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ Putting $\beta=\alpha$

$\tan(\alpha+\alpha)=\frac{\tan\alpha+\tan\alpha}{1-\tan\alpha.\tan\alpha}$

$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$ (7)

**Example 8:** Given that $\tan\theta=-\frac{3}{4}$ *and* $\theta$ is in the quadrant II, find each of the following.

i) $\cos 2\theta$ ii) $\cos 2\theta$

iii) $\tan 2\theta$ iv) The quadrant in which $2\theta$ lies

**Soultion:** By drawing a refernce triangle as shown, we find that

$\sin\theta=\frac{3}{5}$ And $\cos\theta=\frac{4}{5}$

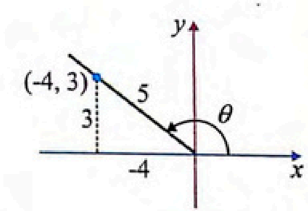


Figure 10.3

Thus we have the following.

i) $\sin 2\theta=2\sin\theta\cos\theta=2\cdot\frac{3}{5}\cdot\left(-\frac{4}{5}\right)=-\frac{24}{25}$

ii) $\cos 2\theta=\cos^2\theta-\sin^2\theta=\left(-\frac{4}{5}\right)^2-\left(\frac{3}{5}\right)^2=\frac{16}{25}-\frac{9}{25}=\frac{7}{25}$

iii) $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)}=\frac{-\frac{3}{2}}{1-\frac{9}{16}}=-\frac{3}{2}\cdot\frac{16}{7}=-\frac{24}{7}$

Note that $\tan 2\theta$ could have been found more easily in this case simply as following:

$$\tan 2\theta=\frac{\sin 2\theta}{\cos 2\theta}=\frac{-\frac{24}{25}}{\frac{7}{25}}=-\frac{24}{7}.$$

iv) Since $\sin 2\theta$ is negative and $\cos 2\theta$ is positive, we know that $2\theta$ is in quadrant IV.

### 10.3.2 Half Angle Identities

We have $\cos 2\alpha = 1-2\sin^2\alpha \Rightarrow \quad 2\sin^2\alpha = 1-\cos 2\alpha$

$$\Rightarrow \quad \sin^2\alpha=\frac{1-\cos 2\alpha}{2}$$

$$\sin\alpha=\pm\sqrt{\frac{1-\cos 2\alpha}{2}} \qquad (8)$$

Now putting $\alpha=\frac{\theta}{2}$ in (8).

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\left(2.\frac{\theta}{2}\right)}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}} \qquad (9)$$

If $\frac{\theta}{2}$ lies in the first or second quadrant then we will write the identity (9) with the positive sign i.e. $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$

If $\frac{\theta}{2}$ lies in the 3rd or 4th quadrant, we will write the identity (9) with the negative sign i.e.

$$\sin\frac{\theta}{2}=-\sqrt{\frac{1-\cos\theta}{2}}$$

Also we know that

$$\cos 2\alpha = 2\cos^2\alpha-1 \quad \Rightarrow \quad 2\cos^2\alpha = 1+\cos 2\alpha$$

$$\Rightarrow \quad \cos^2\alpha=\frac{1+\cos 2\alpha}{2} \quad \Rightarrow \quad \cos\alpha=\pm\sqrt{\frac{1+\cos 2\alpha}{2}}. \qquad \text{Putting } \alpha=\frac{\theta}{2}$$





$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos(2\cdot\frac{\theta}{2})}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}} \qquad (10)$$

From (9) and (10), we have $\tan\frac{\theta}{2}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$.

$$\tan\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad (11)$$

**Example 9:** Find tan (π/8) exactly.

**Solution:**

$$\tan\frac{\pi}{8}=\tan\left(\frac{\frac{\pi}{4}}{2}\right)=\sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}}=\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}}=\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}=\sqrt{\frac{\sqrt{2}(\sqrt{2}-1)}{\sqrt{2}(\sqrt{2}+1)}}=\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$$

$$=\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}}}=\sqrt{\frac{4-2\sqrt{2}-2\sqrt{2}+2}{4-2}}=\sqrt{\frac{6-4\sqrt{2}}{2}}=\sqrt{3-2\sqrt{2}}$$

The identities that we have developed are also useful for simplifying trigonometric expressions.

**Example 10:** Simplify each of the following.

a) $\dfrac{\sin x\cos x}{\frac{1}{2}\cos 2x}$ b) $2\sin^2\frac{x}{2}+\cos x$

**Solution:** a) $\dfrac{\sin x\cos x}{\frac{1}{2}\cos 2x}=\dfrac{2}{2}\cdot\dfrac{\sin x\cos x}{\frac{1}{2}\cos 2x}=\dfrac{2\sin x\cos x}{\cos 2x}$

$$=\frac{\sin 2x}{\cos 2x}=\tan 2x \qquad (\text{using } \sin 2x=2\sin x\cos x)$$

b) $2\sin^2\frac{x}{2}+\cos x=2\left(\frac{1-\cos x}{2}\right)+\cos x\left(\text{using } \sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}},\ or\ \sin^2\frac{x}{2}=\frac{1-\cos x}{2}\right)$

$$=1-\cos x+\cos x=1.$$

### 10.3.3. Triple Angle Identities

We have $\sin 3\alpha = \sin(2\alpha+\alpha) = \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha$

$= 2\sin\alpha\cos\alpha\cos\alpha + (1-2\sin^2\alpha)\sin\alpha$ (By (3) and (5))

$= 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha$

$= 2\sin\alpha(1-\sin^2\alpha) + \sin\alpha - 2\sin^3\alpha$ $(\because \sin^2 + \cos^2\alpha = 1)$

$= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha = 3\sin\alpha - 4\sin^3\alpha$

(12)

$\therefore \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$\cos 3\alpha = \cos(2\alpha+\alpha)$

$= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha$ (by (3) and (6))

$= (2\cos^2\alpha - 1)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha$

$= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha$

$= 2\cos^3\alpha - \cos\alpha - 2(1-\cos^2\alpha)\cos\alpha$ $(\because \sin^2 + \cos^2\alpha = 1)$

$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha = 4\cos^3\alpha - 3\cos\alpha$

(13)

$\therefore \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$$\tan 3\alpha = \tan(2\alpha+\alpha) = \frac{\tan 2\alpha + \tan\alpha}{1-\tan 2\alpha\tan\alpha} = \frac{\dfrac{2\tan\alpha}{1-\tan^2\alpha} + \tan\alpha}{1-\dfrac{2\tan\alpha}{1-\tan^2\alpha}.\tan\alpha} \quad \text{(By (7))}$$

$$= \frac{\dfrac{2\tan\alpha + \tan\alpha(1-\tan^2\alpha)}{1-\tan^2\alpha}}{\dfrac{1-\tan^2\alpha - 2\tan^2\alpha}{1-\tan^2\alpha}} = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$$

$$\therefore \tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha} \quad (14)$$

**Example 11:** Prove the identity $\dfrac{\sin 2x}{\sin x} - \dfrac{\cos 2x}{\cos x} = \sec x$.

**Solution:** $\dfrac{\sin 2x}{\sin x} - \dfrac{\cos 2x}{\cos x} = \dfrac{2\sin x\cos x}{\sin x} - \dfrac{\cos^2 x - \sin^2 x}{\cos x}$ (using double – angle idnetities)





$= 2\cos x - \dfrac{\cos^2 x - \sin^2 x}{\cos x}$ (simplifying)

$= \dfrac{2\cos^2 x - \cos^2 x + \sin^2 x}{\cos x}$ (taking LCM and simplifying)

$= \dfrac{\cos^2 x + \sin^2 x}{\cos x} = \dfrac{1}{\cos x} = \sec x$

We started with the left side and obtained the right side, so the proof is complete.

**Example 12:** Prove the identity

$$\sin^2 x\tan^2 x = \tan^2 x - \sin^2 x.$$

**Solution:** For this proof, we are going to work with each side separately. We try to obtain the same expression on each side.

$\sin^2 x\tan^2 x = \sin^2 x\left(\dfrac{\sin^2 x}{\cos^2 x}\right) = \dfrac{\sin^4 x}{\cos^2 x}$ ......(1)

$\tan^2 x - \sin^2 x = \dfrac{\sin^2 x}{\cos^2 x} - \sin^2 x$ $\left(\because \tan x = \dfrac{\sin x}{\cos x}\right)$

$= \dfrac{\sin^2 x - \sin^2 x\cos^2 x}{\cos^2 x}$ (Taking LCM)

$= \dfrac{\sin^2 x(1-\cos^2 x)}{\cos^2 x}$ (Factoring)

$= \dfrac{\sin^2 x\sin^2 x}{\cos^2 x}$ (Recalling the identity $1-\cos^2 x = \sin^2 x$)

$= \dfrac{\sin^4 x}{\cos^2 x}$.......(2)

We have obtained the same expression from each side, so the proof is complete.

**Example 13:** Find the exact value of cos 105°.

**Solution:** Because $105° = \frac{1}{2}(210°)$ we can find cos 105° by using the half-angle identity for cos $\alpha/2$ with $\alpha = 210°$. The angle $\alpha/2 = 105°$ lies in Quadrant II, and the cosine function is negative in Quadrant II. Thus cos 105° < 0, and we must select the minus sign that precedes the radical in $\cos\alpha = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$ to produce the correct result.

$$\cos 105^\circ = -\sqrt{\frac{1+\cos 210^\circ}{2}} = -\sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}} = -\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{3}}{4}}$$

**Example 14:** Show that,

$$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

**Solution:** L.H.S $= \sin^4\theta = (\sin^2\theta)^2$

$$= \left(\frac{1-\cos 2\theta}{2}\right)^2 \qquad \left(\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{1-2\cos 2\theta+\cos^2 2\theta}{4} = \frac{1}{4}[1-2\cos 2\theta+\cos^2 2\theta]$$

$$= \frac{1}{4}\left[1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right] \qquad \left(\because \cos^2 2\theta = \frac{1+\cos 4\theta}{2}\right)$$

$$= \frac{1}{4}\left[\frac{2-4\cos 2\theta+1+\cos 4\theta}{2}\right] = \frac{1}{8}[3-4\cos 2\theta+\cos 4\theta]$$

$$= \frac{3}{8} - \frac{4}{8}\cos 2\theta + \frac{1}{8}\cos 4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta = \text{R.H.S.}$$

$$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

**Example 15:** Prove the following identities.

(i) $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$ (ii) $\sin 4\theta = 8\sin\theta\cos^3\theta - 4\sin\theta\cos\theta$

**Solution:** (i) $\text{R.H.S} = \dfrac{2\tan\theta}{1+\tan^2\theta} = 2\dfrac{\tan\theta}{\sec^2\theta}$ $\quad (\because \tan^2\alpha + 1 = \sec^2\alpha)$

$$= 2\tan\theta\cos^2\theta \qquad \left(\because \cos\alpha = \frac{1}{\sec\alpha}\right)$$





$$= 2\cdot\frac{\sin\theta}{\cos\theta}\cos^2\theta$$

$$= 2\sin\theta\cos\theta$$

$$= \sin 2\theta$$

(ii) $\sin 4\theta = 8\sin\theta\cos^3\theta - 4\sin\theta\cos\theta$

$\text{L.H.S} = \sin 4\theta = \sin[2(2\theta)]$

$= 2\sin 2\theta\cos 2\theta$ (Use $\sin 2\alpha = 2\sin\alpha\cos\alpha$, with $\alpha = 2\theta$)

$= 2(2\sin\theta\cos\theta)(2\cos^2\theta - 1) = 4\sin\theta\cos\theta(2\cos^2\theta - 1)$

$= 8\sin\theta\cos^3\theta - 4\sin\theta\cos\theta = \text{R.H.S}$

## EXERCISE 10.2

1. Find the values of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$, given $\tan\theta = -\frac{1}{5}$, $\theta$ in quadrant II.
2. If $\sin\theta = \frac{5}{13}$ and terminal ray of $\theta$ is in the second quadrant, then find
   (i) $\sin 2\theta$ (ii) $\cos 2\theta$ (iii) $\tan 2\theta$
3. If $\sin\theta = \frac{4}{5}$ and terminal ray of $\theta$ is in the second quadrant, then find
   (i) $\sin 2\theta$ (ii) $\cos\frac{\theta}{2}$
4. If $\cos\theta = -\frac{3}{7}$ and terminal ray of $\theta$ is in 3rd quadrant, then find $\sin\frac{\theta}{2}$.
5. Use double angle identities to evaluate exactly.
   (i) $\sin\frac{2\pi}{3}$ (ii) $\cos\frac{2\pi}{3}$
6. Use the half angle identities to evaluate exactly.
   (i) $\cos 15^\circ$ (ii) $\tan 67.5^\circ$ (iii) $\sin 112.5^\circ$
   (iv) $\cos\frac{\pi}{8}$ (v) $\tan 75^\circ$ (vi) $\sin\frac{5\pi}{12}$

7. Prove the following identities:

(i) $\cos^4\theta-\sin^4\theta=\dfrac{1}{\sec 2\theta}$ (ii) $\tan\dfrac{\theta}{2}+\cot\dfrac{\theta}{2}=\dfrac{2}{\sin\theta}$

(iii) $\dfrac{1+\cos 2\theta}{1+\cos 2\theta}=\cot^2\theta$ (iv) $\operatorname{cosec}2\theta-\cot2\theta=\tan\theta$

(v) $\dfrac{\sin 3\beta}{\sin\beta}-\dfrac{\cos 3\beta}{\cos\beta}=2$ (vi) $\dfrac{\sin 3\theta}{\cos\theta}+\dfrac{\cos 3\theta}{\sin\theta}=2\cot 2\theta$

(vii) $\dfrac{\cos^3\theta-\sin^3\theta}{\cos\theta-\sin\theta}=\dfrac{2+\sin 2\theta}{2}$ (viii) $\dfrac{2\cos^3\theta}{1-\sin\theta}=2\cos\theta+\sin 2\theta$

(ix) $\cot 2\theta=\dfrac{1}{2}\left(\cot\theta-\dfrac{1}{\cot\theta}\right)$

(x) $\dfrac{\sin\alpha+\cos\alpha}{\cos\alpha-\sin\alpha}+\dfrac{\sin\alpha-\cos\alpha}{\cos\alpha+\sin\alpha}=2\tan 2\alpha$

(xi) $\tan\dfrac{\alpha}{2}=\dfrac{\sin\alpha}{1+\cos\alpha}$ (xii) $\dfrac{\operatorname{cosec}\theta-\cot\theta}{1+\cos\theta}=\operatorname{cosec}\theta\tan^2\dfrac{\theta}{2}$

(xiii) $\cos^2\dfrac{\theta}{2}=\dfrac{1-\cos^2\theta}{2-2\cos\theta}$ (xiv) $\dfrac{1-\tan^2\dfrac{\alpha}{2}}{1+\tan^2\dfrac{\alpha}{2}}=\cos\alpha$

(xv) $\sin2\theta-4\sin^3\theta\cos\theta=\sin2\theta\cos2\theta$

8. Write $\cos^4\theta$ in terms of the first power of one or more cosine functions,

9. Prove the following identities:

(i) $\sin4\theta=8\sin\theta\cos^3\theta-4\sin\theta\cos\theta$ (ii) $\cot4\theta=\dfrac{1-\tan^2 2\theta}{2\tan 2\theta}$

(iii) $\cot3\theta=\dfrac{\cot^3\theta-3\cot\theta}{3\cot^2\theta-1}$

## 10.4 Sum, Difference and Product of sine and cosine

### 10.4.1 Converting Product to Sums or Differences

We know that

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ (1)

and $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ (2)

Adding (1) and (2) we get





$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$

$\therefore 2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$ (3)

Now Subtracting (2) from (1)

$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta$

$\therefore 2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$ (4)

We also know that;

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ (5)

and $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ (6)

Adding (5) and (6) we have,

$\therefore 2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$ (7)

Subtracting (6) from (5) we get,

$-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$

$\therefore 2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$ (8)

So, by converting products into sums or differences we get the following four identities:

$$2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$$
$$2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$$
$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$$
$$2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$$

These identities are usually called the **Product–to–Sum formulae.**

**Example 16:** Write the product $2\sin 5\theta\cos 3\theta$ as a sum or difference of sine and cosine.

**Solution:** Using the identity $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

We have,

$2\sin 5\theta\cos 3\theta=\sin(5\theta+3\theta)+\sin(5\theta-3\theta)=\sin 8\theta+\sin 2\theta$

**Example 17:** Express $\sin10\theta\cos 4\theta$ as a sum or difference.

**Solution:** Using the identity $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

We have, $\sin\alpha\cos\beta=\dfrac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

$\sin10\theta\cos 4\theta=\dfrac{1}{2}[\sin(10\theta+4\theta)+\sin(10\theta-4\theta)]=\dfrac{1}{2}(\sin 14\theta+\sin 6\theta)$

**Example 18:** Write the product $2\cos 45^\circ \cos 15^\circ$ as a sum or difference.

**Solution:** Using the identity $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

We have,

$\therefore \quad 2\cos 45^\circ \cos 15^\circ = \cos(45^\circ+15^\circ) + \cos(45^\circ - 15^\circ)$

$= \cos 60^\circ + \cos 30^\circ$

### 10.4.2 Converting Sums or Differences to Products

Let $\alpha+\beta = \theta$ .......... (1)

$\alpha-\beta = \phi$ .......... (2)

Adding (1) and (2), we have

$$\alpha = \frac{\theta+\phi}{2}$$

Subtracting (2) from (1), we have

$$\beta = \frac{\theta-\phi}{2}$$

Substituting $\alpha = \frac{\theta+\phi}{2}$ and $\beta = \frac{\theta-\phi}{2}$ in the four identities of section 10.4.1,we get

$$\sin\theta+\sin\phi = 2\sin\frac{\theta+\phi}{2}.\cos\frac{\theta-\phi}{2}$$

$$\sin\theta-\sin\phi = 2\cos\frac{\theta+\phi}{2}.\sin\frac{\theta-\phi}{2}$$

$$\cos\theta+\cos\phi = 2\cos\frac{\theta+\phi}{2}.\cos\frac{\theta-\phi}{2}$$

$$\cos\theta-\cos\phi = -2\sin\frac{\theta+\phi}{2}.\sin\frac{\theta-\phi}{2}$$

These identities are usually called the **sum–to–product formulae.**

**Example 19:** Convert the sum $\sin 16^\circ + \sin 12^\circ$ into product.

**Solution:** We know that, $\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$





$\therefore \quad \sin 16^\circ + \sin 12^\circ = 2\sin\frac{16^\circ+12^\circ}{2}\cos\frac{16^\circ-12^\circ}{2} = 2\sin\frac{28^\circ}{2}\cos\frac{4^\circ}{2}$

$= 2\sin 14^\circ \cos 4^\circ$

**Example 20:** Express $\cos 4\theta - \cos 2\theta$ as a product.

**Solution:** We have $\cos\theta - \cos\phi = -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$

$\therefore \cos 4\theta - \cos 2\theta = -2\sin\frac{4\theta+2\theta}{2}\sin\frac{4\theta-2\theta}{2}$

$= -2\sin\frac{6\theta}{2}\sin\frac{2\theta}{2} = -2\sin 3\theta \sin\theta$

**Example 21:** Show that $\frac{\cos\alpha-\cos\beta}{\sin\alpha+\sin\beta} = -\tan\frac{1}{2}(\alpha-\beta)$

**Solution:** L.H.S= $\frac{\cos\alpha-\cos\beta}{\sin\alpha+\sin\beta} = \frac{-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} = \frac{-\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}} = -\tan\frac{\alpha-\beta}{2}$

$= -\tan\frac{1}{2}(\alpha-\beta) =$ R.H.S

**Example 22:** Show that $\sin 5\theta + 2\sin 3\theta + \sin\theta = 4\sin 3\theta\cos^2\theta$

**Solution:**

L.H.S $= \sin 5\theta + 2\sin 3\theta + \sin\theta = (\sin 5\theta + \sin 3\theta) + (\sin 3\theta + \sin\theta)$

$= 2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right) + 2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)$

$= 2\sin\frac{8\theta}{2}\cos\frac{2\theta}{2} + 2\sin\frac{4\theta}{2}\cos\frac{2\theta}{2} = 2\sin 4\theta\cos\theta + 2\sin 2\theta\cos\theta$

$= 2\cos\theta(\sin 4\theta + \sin 2\theta) = 2\cos\theta\left[2\sin\left(\frac{4\theta+2\theta}{2}\right)\cos\left(\frac{4\theta-2\theta}{2}\right)\right]$

$= 2\cos\theta(2\sin 3\theta\cos\theta) = 4\sin 3\theta\cos^2\theta =$ R.H.S.

**Example 23:** Show that $\left(\frac{\sin 3\theta+\sin\theta}{\sin 3\theta-\sin\theta}\right)\left(\frac{\cos\theta+\cos\theta}{\cos 3\theta-\cos\theta}\right) = -\cot^2\theta$

**Solution:**

$L.H.S = \left(\frac{\sin 3\theta+\sin\theta}{\sin 3\theta-\sin\theta}\right)\left(\frac{\cos 3\theta+\cos\theta}{\cos 3\theta-\cos\theta}\right) = \left(\frac{2\sin 2\theta\cos\theta}{2\sin\theta\cos 2\theta}\right)\left(\frac{2\cos 2\theta\cos\theta}{-2\sin 2\theta\sin\theta}\right)$

$$=\left(\frac{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)}{2\cos\left(\frac{3\theta+\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right)}\right)\left(\frac{2\cos\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)}{2\sin\left(\frac{3\theta+\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right)}\right)=-\frac{\cos^2\theta}{\sin^2\theta}=-\cot^2\theta.$$

**Example 24:** Show that $\cos 20^0 \cos 40^0 \cos 80^0 = 1/8$

**Solution:**

L.H.S. $= \cos 20^0 \cos 40^0 \cos 80^0 = ¼[4 \cos 20^0 \cos 40^0 \cos 80^0]$

$= ¼ [(2 \cos 40^0 \cos 20^0)\ 2 \cos 80^0] = ¼ [(\cos 60^0 + \cos 20^0)\ \ 2 \cos 80^0]$

$= ¼ [(1/2 + \cos 20^0)\ 2 \cos 80^0] = ¼ [\cos 80^0 + 2\cos 80^0 \cos 20^0]$

$= ¼ [\cos 80^0 + \cos 100^0 + \cos 60^0] = ¼ [\ \cos 80^0 + \cos(180^0 - 80^0) + \cos 60^0]$

$= ¼ [\cos 80^0 + \cos 100^0 + ½]$ $\because \cos(180^0 - \theta) = -\cos\theta$

$= ¼ [\cos 80^0 - \cos 80^0 + ½]$

$= ¼ [1/2] = 1/8 =$ R.H.S

## EXERCISE 10.3

1. Express the following products as sums or differences.
   (i) $2\sin 6x \sin x$ (ii) $\sin 55^o \cos 123^o$
   (iii) $\sin\frac{A+B}{2}\cos\frac{A-B}{2}$ (iv) $\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$
2. Convert the following sums or differences to products:
   (i) $\sin 37^{\circ} + \sin 43^o$ (ii) $\cos 36^o - \cos 82^o$
   (iii) $\sin\frac{P+Q}{2} - \sin\frac{P-Q}{2}$ (iv) $\cos\frac{A+B}{2} + \cos\frac{A-B}{2}$
3. Prove the following.
   (i) $\frac{\cos 75^\circ + \cos 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \sqrt{3}$ (ii) $\frac{\sin 38^\circ - \cos 68^\circ}{\cos 68^\circ + \sin 38^\circ} = \sqrt{3}\tan 8^\circ$
4. Prove the following identities:
   (i) $\frac{\sin\alpha + \sin 9\alpha}{\cos\alpha + \cos 9\alpha} = \tan 5\alpha$ (ii) $\frac{\cos\beta + \cos 3\beta + \cos 5\beta}{\sin\beta + \sin 3\beta + \sin 5\beta} = \cot 3\beta.$
   (iii) $\sin 2\theta + \sin 4\theta + \sin 6\theta = 4\cos\theta\cos 2\theta\sin 3\theta$
   (iv) $\sin 5\theta + \sin\theta + 2\sin 3\theta = 4\sin 3\theta\cos^2\theta$
   (v) $\sin 3\theta + \sin 5\theta + \sin 7\theta + \sin 9\theta = 4\cos\theta\sin 6\theta\cos 2\theta$





   (vi) $\cos\beta + \cos 2\beta + \cos 5\beta = \cos 2\beta\ (1+2\cos 3\beta)$

5. Prove that (i) $\cos 20^0 \cos 40^0 \cos 60^0 \cos 80^0 = \frac{1}{16}$
   (ii) $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9} = \frac{3}{16}$ (iii) $\sin 10^0 \sin 30^0 \sin 50^0 \sin 70^0 = \frac{1}{16}$

## REVIEW EXERCISE 10

1. **Choose the correct option**
   (i) $\cos 50^0\ 50' \cos 9^0\ 10' - \sin 50^0\ 50' \sin 9^0\ 10' =$
   (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{\sqrt{3}}{2}$
   (ii) If $\tan 15^0 = 2-\sqrt{3}$, then the value of $\cot^2 75^0$ is
   (a) $7+\sqrt{3}$ (b) $7-2\sqrt{3}$ (c) $7-4\sqrt{3}$ (d) $7+4\sqrt{3}$
   (iii) If $\tan(\alpha+\beta) = 1/2$ and $\tan\alpha = 1/3$, then $\tan\beta =$
   (a) 1/6 (b) 1/7 (c) 1 (d) 7/6
   (iv) $\sin\theta\cos(90^0 - \theta) + \cos\theta\sin(90^0 - \theta) =$ _____
   (a) $-1$ (b) 2 (c) 0 (d) 1
   (v) Simplified expression of $(\sec\theta + \tan\theta)(1 - \sin\theta)$ is
   (a) $\sin^2\theta$ (b) $\cos^2\theta$ (c) $\tan^2\theta$ (d) $\cos\theta$
   (vi) $\sin\left(x - \frac{\pi}{2}\right) = ?$
   (a) $\sin x$ (b) $-\sin x$ (c) $\cos x$ (d) $-\cos x$
   (vii) A point is in Quadrant-III and on the unit circle. If its x-coordinate is $-\frac{4}{5}$, what is the y-coordinate of the point?
   (a) 3 / 5 (b) –3 / 5 (c) –2 / 5 (d) 5 / 3
   (viii) Which of the following is an identity?
   (a) sin (a) cos (a) = (1/2) sin(2a) (b) sina + cosa = 1
   (c) sin(–a) = sin a (d) tana = cosa / sina

**Prove the following identities:**

2. $\frac{2\sin\theta\sin 2\theta}{\cos\theta + \cos 3\theta} = \tan 2\theta\tan\theta$ 3. $\frac{\sin 10a - \sin 4a}{\sin 4a + \sin 2a} = \frac{\cos 7a}{\cos a}$