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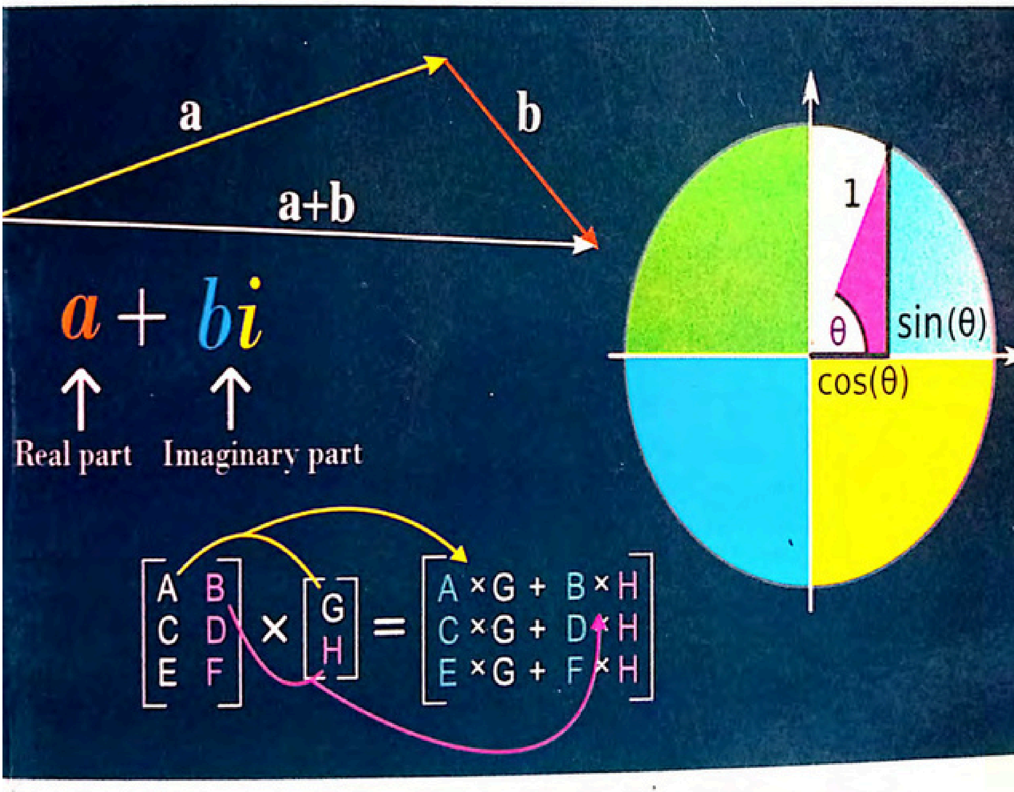
A TEXTBOOK OF
MATHEMATICS
FOR GRADE XI



Khyber Pakhtunkhwa Textbook Board,
Peshawar

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Authors:
Prof. Dr. Gulzar Ali Khan (Retired)
Prof. Dr. Islam Noor (Retired)
Prof. Dr. Muhammad Shah

Review Supervision:
Gohar Ali Khan, Director Curriculum and Teachers Education, Khyber Pakhtunkhwa, Abbottabad

Focal Person:
Zulfiqar Khan, Additional Director, Directorate of Curriculum and Teachers Education, Khyber Pakhtunkhwa, Abbottabad

Re-reviewers:
Nadeem Sultan Principal GHS Pawa Abbottabad
Majid Khan AP Maths GPGC Mandian Abbottabad
Muhammad Nasim SS GHSS Muslim Abad Abbottabad
Shakeel Iqbal Subject Specialist TBB KP, Peshawar
Mukhtiar Muhammad Subject Specialist DCTE, Abbottabad

Editor:
Shakeel Iqbal Subject Specialist Mathematics,
Khyber Pakhtunkhwa Textbook Board, Peshawar

Printing Supervision:
Rashid Khan Paindakhel, Chairman
Saeedur Rehman, Member (E&P),
Khyber Pakhtunkhwa Textbook Board, Peshawar

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A Textbook of MATHEMATICS

For Grade XI

Khyber Pakhtunkhwa Textbook Board, Peshawar

CONTENTS

UNIT 1 COMPLEX NUMBERS

$a + bi$

↑ Real part ↑ Imaginary part

STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Recall complex number z represented by an expression of the form $z=a+ib$ or of the form (a,b) where a and b are real numbers and $i=\sqrt{-1}$
- Recognize a as real part of z and b as imaginary part of z
- Know the condition for equality of complex numbers
- Carry out basic operations on complex numbers
- Define $\overline{z} = a-ib$ as the complex conjugate of $z=a+ib$
- Define $|z|=\sqrt{a^2+b^2}$ as the absolute value or modulus of a complex number $z=a+ib$
- Describe algebraic properties of complex numbers (e.g. commutative, associative and distributive) with respect to '+' and '×'
- Know additive identity and multiplicative identity for the set of complex numbers
- Find additive inverse and multiplicative inverse of a complex number z
- Demonstrate the following properties
  - $|z|=|-z|=|\overline{z}|=|-\overline{z}|$
  - $\overline{\overline{z}}=z$, $z\overline{z}=|z|^2$, $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$,
  - $\overline{z_1 z_2}=\overline{z_1}\,\overline{z_2}$, $\overline{\left(\dfrac{z_1}{z_2}\right)}=\dfrac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$.
- Find real and imaginary parts of the following type of complex numbers
  - $(x+iy)^n$,
  - $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$, $x_2+iy_2 \neq 0$, where n = ± 1, and ± 2
- Solve the simultaneous linear equations with complex coefficients. For example, $\begin{cases} 5z-(3+i)w=7-i, \\ (2-i)z+2iw=-1+i \end{cases}$
- Write the polynomial p(z) as a product of linear factors. For example, $z^2+a^2=(z+ia)(z-ia)$, $z^3-3z^2+z+5=(z+1)(z-2-i)(z-2+i)$
- Solve quadratic equation of the form $pz^2+qz+r=0$ by completing squares, where p, q, r are real numbers and z a complex number. For example: Solve $z^2-2z+5=0 \Rightarrow (z-1-2i)(z-1+2i)=0 \Rightarrow z=1+2i, 1-2i$

1.1 Introduction

In our previous class we learnt that besides the real numbers, there are other numbers called complex numbers. Such numbers play a very important role in mathematics and other branches of science. The use of complex numbers is indispensable in Physics, Aeronautical and Electrical Engineering especially in the analysis of Electric circuits.

1.1.1 Complex Numbers

In 1832, Gauss (1777-1855), a German mathematician gave the concept of **complex numbers** as numbers of the form $a+bi$, where a and b are real numbers. The number a is called the real part of $a+bi$ and the number b is called the **imaginary part** of $a+bi$.

For example, the complex number $-3+2i$ has the real part $a = -3$ and the imaginary part $b = 2$.

In $a + bi$, if $b = 0$, then $a + bi = a + 0i = a$ is a real number. Thus every real number a can be written as a complex number by choosing $b = 0$. If $a = 0$ and $b \neq 0$, then $a + bi = 0 + bi = bi$ is called a pure imaginary number.

For example, $\frac{1}{4}i$ and $-i$ are pure imaginary numbers. Usually, the complex number $a + bi$ is denoted by $z = a + bi$

Accordingly, $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i, \ldots$

The set of all complex numbers is denoted by C, that is $C = \{a + bi : a, b \in R\}$.

Complex numbers may also be defined as ordered pairs of real numbers. Thus a **complex number** z is an ordered pair (a, b) of real numbers a and b, written as $z = (a, b)$. The first component a is called the real part of z and the second component b is called the imaginary part of z denoted by $Re(z)$ and $Im(z)$ respectively i.e. $Re(z) = a$ and $Im(z) = b$.

The ordered pair (0,1) is called the imaginary unit and is denoted by $i = (0,1)$.

The set of all ordered pairs of real numbers is the set of complex numbers denoted by **C**, that is $C = \{(a, b) : a, b \text{ are real numbers}\}$

$\mathbb{C} = \mathbb{R} \times \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers.

Since $i = \sqrt{-1}$, a simple consequence of the definition of i is that all powers of i may be expressed in terms of ± 1 and $\pm i$.

For example, $i^1 = i$, $i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = (i^2)^2 = 1$ and if we continue in this way to obtain higher powers of i, we obtain the values $1, i, -1$ or $-i$. Similarly, for negative powers, we have

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{i^2 \cdot i} = \frac{1}{-i} = \frac{i}{-i \cdot i} = i$$

$$i^{-4} = \frac{1}{i^4} = \frac{1}{(i^2)^2} = \frac{1}{(-1)^2} = 1$$

Note

In Example 1(c) we see that 0 can be expressed as a sum of a real and an imaginary number and hence is a complex number. Such a complex number whose real and imaginary parts are zero, is called **zero complex number.**

Similarly in Example 1(d), 1 can be expressed as a complex number with real part 1 and imaginary part 0. The complex number 1 is called the **unit complex number.**

Example 1: Write the following complex numbers in ordered pair form.

(a) 6 (b) $5i$ (c) 0 (d) 1 (e) $3-\sqrt{-9}$

Solution:

(a) $6 = 6 + 0i = (6,0)$ (b) $5i = 0 + 5i = (0,5)$

(c) $0 = 0 + 0i = (0,0)$ (d) $1 = 1 + 0i = (1,0)$

(e) $3-\sqrt{-9} = 3-i\sqrt{9} = 3-3i = (3,-3)$

Example 2: Find the value of

$$\frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1$$

Solution: Given expression

$$= \frac{i^{10}(i^{582}+i^{580}+i^{578}+i^{576}+i^{574})}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1 = i^{10}-1$$

$$= (i^2)^5 - 1 = (-1)^5 - 1 = -1-1 = -2$$

1.1.2 Equality of Complex Numbers

Two complex numbers are said to be equal if and only if their corresponding real parts and imaginary parts are equal. i.e. $a + ib = c + id \Leftrightarrow a = c$ and $b = d$

i.e. $z_1 = z_2 \Leftrightarrow Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$

Illustration: If $x+iy=3-4i$, then $x=3$ and $y=-4$

1.1.3 Conjugate of a complex number

The conjugate of a complex number $z=x+iy$ is denoted by $\overline{z}$, and is defined as $\overline{z}=x-iy$

Illustration: (i) Let $z=5-3i$, then $\overline{z}=5+3i$

(ii) Let $z=2=2+0i$, then $\overline{z}=2-0i=2$

(iii) Let $z=3i=0+3i$, then $\overline{z}=0-3i=-3i$

1.1.4 Basic algebraic operation on complex numbers

(i) Addition

For two complex numbers, $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$, their sum is defined as:

$$z=z_1+z_2=(a_1+a_2)+i(b_1+b_2)$$

Illustration: If $z_1=4+5i$ and $z_2=2-3i$, then $z_1+z_2=(4+2)+(5-3)i=6+2i$

Example 3: Add the complex numbers $z_1=3+4i$ and $z_2=2-7i$

Solution: $z_1+z_2=(3+4i)+(2-7i)$

$=(3+2)+(4-7)i=5-3i$

(ii) Subtraction

For two complex numbers $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$, the subtraction of z_2 from z_1 is defined as:

$$z_1-z_2=(a_1-a_2)+i(b_1-b_2)$$

Illustration: If $z_1=1-i$ and $z_2=5+2i$, then

$z_1-z_2=(1-i)-(5+2i)=(1-i)+(-5-2i)$

$=(1-5)+i(-1-2)=-4-3i$

Remember

For any two real numbers a and b, $\sqrt{a}\sqrt{b}=\sqrt{ab}$ is true only when at least one of a and b is either zero or positive.
If both a and b are positive real numbers, then the calculation $\sqrt{-a}\sqrt{-b}=\sqrt{(-a)(-b)}=\sqrt{ab}$ *is wrong*. The correct calculation is

$\sqrt{-a}\sqrt{-b}=\left(\sqrt{-1}\sqrt{a}\right)\left(\sqrt{-1}\sqrt{b}\right)$

$=\left(i\sqrt{a}\right)\left(i\sqrt{b}\right)$

$=i^2(\sqrt{a}\sqrt{b})=(-1)\left(\sqrt{ab}\right)=-\sqrt{ab}$

Thus, the calculation $\sqrt{-2}\sqrt{-3}=\sqrt{(-2)(-3)}=\sqrt{6}$ is wrong. The correct result is

$\sqrt{-2}\sqrt{-3}=\left(i\sqrt{2}\right)\left(i\sqrt{3}\right)$

$=i^2\left(\sqrt{2}\sqrt{3}\right)=-\sqrt{6}$

Example 4: Let $z_1=2+4i$ and $z_2=1-3i$. Compute z_1-3z_2

Solution: Putting values of z_1 and z_2 in the given expression,

$z_1-3z_2=(2+4i)-3(1-3i)$

$=2+4i-3+9i=-1+13i$

(iii) Multiplication

Multiplication of two complex numbers $z_1=a+ib$ and $z_2=c+id$ is defined as

$$z_1z_2=(a+ib)(c+id)=(ac-bd)+i(ad+bc)$$

Illustration: If $z_1=4+3i$ and $z_2=3-2i$, then

$z_1z_2=(4+3i)(3-2i)$

$=[4\times3-3\times(-2)]+i[4\times(-2)+3\times3]=18+i$

Example 5: Find the product of $2-3i$ and $7+5i$.

Solution: $(2-3i)(7+5i)=2(7+5i)-3i(7+5i)$

$=14+10i-21i-15i^2$

$=14-11i-15(-1)$ $\quad(\because i^2=-1)$

$=14-11i+15=29-11i$

(iv) Division

The division of one complex number by another complex number can not be carried out, because the denominator consists of two independent terms. This difficulty can be overcome by multiplying the numerator and denominator by the conjugate of the complex number in the denominator. This process is called **rationalization**.

We have $\dfrac{z_1}{z_2}=\dfrac{a+bi}{c+di}=\dfrac{a+bi}{c+di}\times\dfrac{c-di}{c-di}$ (By rationalization)

$$=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{ac-adi+bci-bdi^2}{c^2+d^2}$$

$$=\frac{(ac+bd)-(ad-bc)i}{c^2+d^2}\quad(\because i^2=-1)$$

$=\dfrac{ac+bd}{c^2+d^2}+\dfrac{bc-ad}{c^2+d^2}i$. Thus $\dfrac{z_1}{z_2}=\dfrac{a+bi}{c+di}=\dfrac{ac+bd}{c^2+d^2}+\dfrac{bc-ad}{c^2+d^2}i$

Illustration: Solve $(x+iy)(2-3i)=4+i$, where x and y are real.

Solution: We have, $(x+iy)(2-3i)=4+i$

$$\Rightarrow x+iy=\frac{4+i}{2-3i}=\frac{4+i}{2-3i}\times\frac{2+3i}{2+3i}=\frac{(8-3)+i(12+2)}{2^2-(3i)^2}=\frac{5+14i}{13}=\frac{5}{13}+\frac{14}{13}i$$

$$\therefore \quad x=\frac{5}{13} \text{ and } y=\frac{14}{13}$$

Example 6: Write $\frac{3+2i}{4-3i}$ in the form $a+bi$.

Solution:

$$\frac{3+2i}{4-3i}=\frac{3+2i}{4-3i}\times\frac{4+3i}{4+3i} \quad (\textit{By rationalization})$$

$$=\frac{(3+2i)(4+3i)}{(4-3i)(4+3i)}=\frac{12+9i+8i+6i^2}{16+12i-12i-9i^2}=\frac{12+17i+6(-1)}{16-9(-1)} \quad (\because i^2=-1)$$

$$=\frac{6+17i}{25}=\frac{6}{25}+\frac{17}{25}i$$

1.1.5 Absolute value or modulus of a complex number

Let $z=(a,b)=a+bi$ be a complex number. Then absolute value (or modulus) of z, denoted by $|z|$, is defined by $|z|=\sqrt{a^2+b^2}$.

In the adjoining figure P represents $a+bi$. $\overline{PM}$ is a perpendicular drawn on $\overline{OX}$. Therefore $\overline{OM}=a$ and $\overline{PM}=b$. In the right angled–triangle OMP, we have by Pythagoras theorem

$$\left|\overline{OP}\right|^2=\left|\overline{OM}\right|^2+\left|\overline{PM}\right|^2=a^2+b^2$$

$$\therefore \left|\overline{OP}\right|=\sqrt{a^2+b^2}=|z|.$$ Thus, the modulus of a complex number is the distance from the origin to the point representing the number.

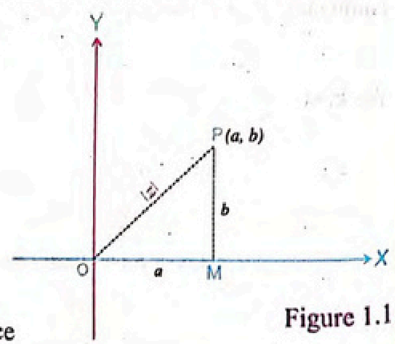


Figure 1.1

Example 7: Compute the absolute value of the given complex numbers:

(a) i (b) 3 (c) $2-5i$

Solution: (a) Let $z=i$ or $z=0+i$,

Then by definition

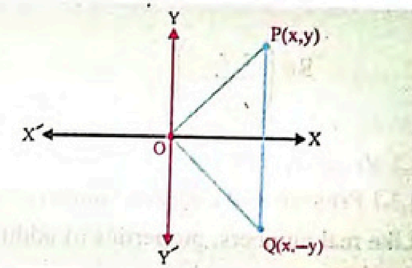
$$|z|=\sqrt{(0)^2+(1)^2}=\sqrt{1^2}=1$$

(b) Let $z=3$ or $z=3+0i$. Then $|z|=\sqrt{(3)^2+(0)^2}=\sqrt{9}=3$

(c) Let $z=2-5i$. Then $|z|=\sqrt{(2)^2+(-5)^2}=\sqrt{4+25}=\sqrt{29}$

For Your Information

The complex number $z=x+iy$ and its conjugate $\overline{z}=x-iy$ are respectively represented by the points $P(x,y)$ and $Q(x,-y)$. Geometrically, the point $Q(x,-y)$ is the mirror image of the point $P(x,y)$ on the x-axis and vice versa.



EXERCISE 1.1

1. Simplify the following

   (i) i^9+i^{19} (ii) $(-i)^{23}$ (iii) $(-1)^{\frac{-23}{2}}$ (iv) $(-1)^{\frac{15}{2}}$

2. Prove that $i^{107}+i^{112}+i^{122}+i^{153}=0$

3. Add the following complex numbers

   (i) $3(1+2i), -2(1-3i)$ (ii) $\frac{1}{2}-\frac{2}{3}i,\ \frac{1}{4}-\frac{1}{3}i$ (iii) $(\sqrt{2},1), (1,\sqrt{2})$

4. Subtract the second complex number from first

   (i) $(a,0), (2,-b)$ (ii) $(-3,\frac{1}{2}), (3,\frac{1}{2})$ (iii) $3\sqrt{3}-5\sqrt{7}i,\ \sqrt{3}+2\sqrt{7}i$

5. Multiply the following complex numbers

   (i) $8i+11,\ -7+5i$ (ii) $3i, 2(1-i)$ (iii) $\sqrt{2}+\sqrt{3}\,i,\ 2\sqrt{2}-\sqrt{3}\,i$

6. Perform the indicated division and write the answer in the form $a+bi$

   (i) $\frac{4+i}{3+5i}$ (ii) $\frac{1}{-8+i}$ (iii) $\frac{1}{7-3i}$ (iv) $\frac{6+i}{i}$

7. If $z_1 = 1+2i$ and $z_2 = 2+3i$, evaluate
(i) $|z_1+z_2|$ (ii) $|z_1z_2|$ (iii) $\left|\dfrac{z_1}{z_2}\right|$

8. Express the following in the standard form $a+ib$
(i) $\dfrac{1-2i}{2+i}+\dfrac{4-i}{3+2i}$ (ii) $\dfrac{2+\sqrt{-9}}{-5-\sqrt{-16}}$ (iii) $\dfrac{(1+i)^2}{4+3i}$

9. Find the conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ 10. Evaluate $\left[i^{18}+\left(\dfrac{1}{i}\right)^{25}\right]^3$

11. Let $z_1 = 2-i$, $z_2 = -2+i$, find
(i) $\text{Re}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$ (ii) $\text{Im}\left(\dfrac{1}{z_1\bar{z}_1}\right)$

1.2 Properties of complex numbers

1.2.1 Properties of complex numbers with respect to addition and multiplication

Like real numbers, properties of addition and multiplication also hold in complex numbers.

(i) Properties of Addition

A-1 Addition is commutative i.e. $z_1+z_2=z_2+z_1$

If $z_1 = a+bi$ and $z_2 = c+di$, then

$z_1+z_2 = (a+bi)+(c+di)$
$= (a+c)+(b+d)i$
$= (c+a)+(d+b)i$ (by commutative property for addition in $I\!R$)
$= (c+di)+(a+bi) = z_2+z_1$

Thus $z_1+z_2 = z_2+z_1$

Example 8: If $z_1 = 1+3i$ and $z_2 = 3-5i$, then $z_1+z_2 = z_2+z_1$

Solution: $z_1+z_2 = (1+3i)+(3-5i)$
$= (1+3)+(3-5)i = 4-2i$

and $z_2+z_1 = (3-5i)+(1+3i)$
$= (3+1)+(-5+3)i = 4-2i$

Hence $z_1+z_2 = z_2+z_1$

A-2 Addition is associative i.e. $z_1+(z_2+z_3) = (z_1+z_2)+z_3$

If $z_1 = a+bi$, $z_2 = c+di$ *and* $z_3 = e+fi$, *then*

$z_1+(z_2+z_3) = (a+bi)+[(c+di)+(e+fi)]$
$= (a+bi)+[(c+e)+(d+f)i]$
$= a+(c+e)+[b+(d+f)]i$
$= (a+c)+e+[(b+d)+f]i$ (by associative property for addition in $I\!R$)
$= [(a+c)+(b+d)i]+e+fi$
$= [(a+bi)+(c+di)]+e+fi$
$= (z_1+z_2)+z_3$

Thus $z_1+(z_2+z_3) = (z_1+z_2)+z_3$

Example 9: If $z_1 = 1+2i$, $z_2 = -2+3i$ and $z_3 = -3-5i$, then $z_1+(z_2+z_3) = (z_1+z_2)+z_3$

Solution: $z_1+(z_2+z_3) = (1+2i)+[(-2+3i)+(-3-5i)]$
$= (1+2i)+[(-2-3)+(3-5)i] = (1+2i)+(-5-2i)$
$= (1-5)+(2-2)i = -4+0i$

$(z_1+z_2)+z_3 = [(1+2i)+(-2+3i)]+(-3-5i)$
$= [(1-2)+(2+3)i]+(-3-5i) = (-1+5i)+(-3-5i)$
$= (-1-3)+(5-5)i = -4+0i$

Hence $z_1+(z_2+z_3) = (z_1+z_2)+z_3$

(ii) Properties of Multiplication

M-1 Multiplication is commutative i.e. $z_1z_2 = z_2z_1$

If $z_1 = a+bi$ and $z_2 = c+di$, then $z_1z_2 = (a+bi).(c+di)$
$= (ac-bd)+(ad+bc)i$ (by definition of multiplication of complex numbers)

and $z_2z_1 = (c+di)(a+bi) = (ca-db)+(cb+da)i$
$= (ac-bd)+(ad+bc)i$ (by commutative properties of multiplication and addition of real numbers). Thus, $z_1z_2 = z_2z_1$

Example 10: If $z_1 = 2-3i$ and $z_2 = -1+2i$, then $z_1z_2 = z_2z_1$

Solution: $z_1z_2 = (2-3i)(-1+2i) = 2(-1+2i)-3i(-1+2i)$
$= -2+4i+3i-6i^2 = -2+7i+6$ $(\because i^2=-1)$
$= 4+7i$

and $z_2z_1 = (-1+2i)(2-3i) = -1(2-3i)+2i(2-3i)$
$= -2+3i+4i-6i^2 = -2+7i+6$ $(\because i^2=-1)$
$= 4+7i$

Hence $z_1z_2 = z_2z_1$

M-2 Multiplication is associative i.e. $z_1(z_2 z_3)=(z_1 z_2)z_3$

If $z_1 = a + bi$, $z_2 = c + di$ and $z_3 = e + fi$, then

$$z_1(z_2 z_3) = (a + bi)[(c + di)(e + fi)] = (a + bi)[(ce - df) + (cf + de)i]$$
$$= [a(ce - df) - b(cf + de)] + [a(cf + de) + b(ce - df)]i$$

and
$$(z_1 z_2)z_3 = [(a + bi)(c + di)](e + fi) = [(ac - bd) + (ad + bc)i](e + fi)$$
$$= [(ac - bd)e - (ad + bc)f] + [(ac - bd)f + (ad + bc)e]i$$
$$= [(ace - adf - (bde + bcf)] + [(acf + ade) + (bce - bdf)]i$$
$$= [a(ce - df) - b(cf + de)] + [a(cf + de) + b(ce - df)]i$$

Thus, $z_1(z_2 z_3) = (z_1 z_2) z_3$

Example 11: If $z_1 = 1 - i$, $z_2 = -1 + 2i$ and $z_3 = 2 - 3i$, then $z_1(z_2 z_3) = (z_1 z_2)z_3$

Solution: We have

$$z_1(z_2 z_3) = (1 - i)[(-1 + 2i)(2 - 3i)] = (1 - i)[-1(2 - 3i) + 2i(2 - 3i)]$$
$$= (1 - i)(-2 + 3i + 4i - 6i^2) = (1 - i)(-2 + 7i + 6)$$
$$= (1 - i)(4 + 7i) = 1(4 + 7i) - i(4 + 7i)$$
$$= 4 + 7i - 4i - 7i^2 = 4 + 3i + 7 = 11 + 3i$$

and
$$(z_1 z_2)z_3 = [(1 - i)(-1 + 2i)](2 - 3i) = [1(-1 + 2i) - i(-1 + 2i)](2 - 3i)$$
$$= (-1 + 2i + i - 2i^2)(2 - 3i) = (-1 + 3i + 2)(2 - 3i)$$
$$= (1 + 3i)(2 - 3i) = 1(2 - 3i) + 3i(2 - 3i)$$
$$= 2 - 3i + 6i - 9i^2 = 2 + 3i + 9 = 11 + 3i$$

Hence $z_1(z_2 z_3) = (z_1 z_2)z_3$

(iii) Multiplication–Addition Property (The Distributive Property)

This property is more explicitly stated as follows:

M–A. Multiplication is distributive over addition i.e. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

If $z_1 = a + bi$, and $z_2 = c + di$ and $z_3 = e + fi$, then

$$z_1(z_2 + z_3) = (a + bi)[(c + di) + (e + fi)]$$
$$= (a + bi)[(c + e) + (d + f)i]$$
$$= [a(c + e) - b(d + f)] + [a(d + f) + b(c + e)]i$$

and
$$z_1 z_2 + z_1 z_3 = (a + bi)(c + di) + (a + bi)(e + fi)$$
$$= [(ac - bd) + (ad + bc)i] + [(ae - bf) + (af + be)i]$$
$$= [(ac + ae) + (-bd - bf)] + [(ad + af)i + (bc + be)i]$$
$$= [a(c + e) - b(d + f)] + [a(d + f) + b(c + e)]i$$

Thus, $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Example 12: If $z_1 = -1 + 2i$, $z_2 = 3 + 4i$ and $z_3 = -2 + 5i$, then $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Solution: We have

$$z_1(z_2 + z_3) = (-1 + 2i)[(3 + 4i) + (-2 + 5i)]$$
$$= (-1 + 2i)[3 + 4i - 2 + 5i]$$
$$= (-1 + 2i)(1 + 9i) = -1 - 9i + 2i + 18i^2$$
$$= -1 - 7i - 18 = -19 - 7i \quad (\because i^2 = -1)$$

and
$$z_1 z_2 + z_1 z_3 = (-1 + 2i)(3 + 4i) + (-1 + 2i)(-2 + 5i)$$
$$= -3 - 4i + 6i + 8i^2 + 2 - 5i - 4i + 10i^2$$
$$= -1 - 7i - 8 - 10 = -19 - 7i$$

Hence $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Note

According to the commutative property for multiplication $iy = yi$. Hence we can write $z = x + iy$ instead of $z = x + yi$

1.2.2 Additive identity and multiplicative identity of complex numbers

A complex number $c + di$ is called the **additive identity** of the complex number $a + bi$ if $(a + bi) + (c + di) = (c + di) + (a + bi) = a + bi$

Let $a + bi$ be any complex number and $c + di = 0 + 0i$ be the zero complex number. Then

$$(a + bi) + (0 + 0i) = (a + 0) + (b + 0)i \quad \text{(by definition of addition)}$$
$$= a + bi$$

Similarly $(0 + 0i) + (a + bi) = a + bi$

Thus the additive identity in **C** is the zero complex number i.e. $0 + 0i$

A complex number $c + di$ is called the **multiplicative identity** of the complex number $a + bi$ if $(a + bi)(c + di) = (c + di)(a + bi) = a + bi$

Let $a + bi$ be any complex number and $c + di = 1 + 0i$ be the unit complex number. Then

$$(a + bi)(1 + 0i) = (a \cdot 1 - b \cdot 0) + (a \cdot 0 + b \cdot 1)i \quad \text{(by definition of multiplication of complex numbers)}$$
$$= a + bi$$

Similarly $(1 + 0i)(a + bi) = a + bi$

Thus the multiplicative identity in **C** is the unit complex number $1 + 0i$.

1.2.3 Additive inverse and multiplicative inverse of complex numbers

A complex number $c + di$ is called the **additive inverse** of the complex number $a + bi$ if $(a + bi) + (c + di) = 0 + 0i$ i.e. the additive identity.

We have $(a + bi) + (c + di) = 0 + 0i \Rightarrow (a + c) + (b + d)i = 0 + 0i$

$\Rightarrow a + c = 0$ and $b + d = 0 \Rightarrow c = -a$ and $d = -b$

so that $c + di = -a - bi$, Thus the additive inverse of $a + bi$ is $-a - bi$.

Example 13: Find additive inverse of $5-3i$

Solution:

Let $z=5-3i$

$\therefore -z=-(5-3i)=-5+3i$

Thus the additive inverse of $5-3i$ is $-5+3i$.

Multiplicative inverse A complex number $c+di$ is called the **multiplicative inverse** of the complex number $a+bi$ if $(a+bi)(c+di)=1+0i$ i.e. the multiplicative identity.

We have $(a+bi)(c+di)=1+0i \Rightarrow (ac-bd)+(ad+bc)i=1+0i$

$\Rightarrow \quad ac-bd=1$ (i)

and $\quad ad+bc=0$ (ii) From (ii), we have

$ad=-bc$ or $d=-\frac{bc}{a}$ (iii) Putting the value of d in (i), we get

$$ac+b\left(\frac{bc}{a}\right)=1 \Rightarrow \frac{a^2c+b^2c}{a}=1$$

$$\Rightarrow (a^2+b^2)c=a \Rightarrow c=\frac{a}{a^2+b^2} \quad \text{(iv)}$$

Putting the value of c in (iii), we get

$$d=-\frac{b\cdot a}{a(a^2+b^2)} \Rightarrow d=-\frac{b}{a^2+b^2} \quad \text{(v)}$$

From (iv) and (v), we have

$$c+di=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$$

Thus the multiplicative inverse of $a+bi$ is

$$\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$$

Did You Know

The complex numbers possess all the properties that real numbers possess except for the order relation, that is, we can not say that one complex number is greater than the other complex number.

Example 14: Find multiplicative inverse of $-2-3i$

Solution: Let $z=-2-3i$ Here $a=2, b=-3$

$$\therefore z^{-1}=\left(\frac{a}{a^2+b^2},-\frac{b}{a^2+b^2}\right)=\left(\frac{-2}{(-2)^2+(-3)^2},-\frac{-3}{(-2)^2+(-3)^2}\right)=\left(\frac{-2}{13},\frac{3}{13}\right)=\frac{-2}{13}+\frac{3}{13}i$$

Thus $-\frac{2}{13}+\frac{3}{13}i$ is the multiplicative inverse of $-2-3i$.

1.2.4 Some properties of the conjugate and modulus of complex numbers

In the following theorem we prove some properties pertaining to conjugation and modulus of complex numbers.

Theorem: For all z_1, z_2, z_3 in **C**

(a) $|z|=|-z|=|\bar{z}|=|-\bar{z}|$ (b) $\bar{\bar{z}}=z$ (c) $z\bar{z}=|z|^2$

(d) $\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2$ (e) $\overline{z_1z_2}=\bar{z}_1\bar{z}_2$ (f) $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}$, $z_2\neq 0$

Proof (a) Let $z=a+bi$. Then $-z=-a-bi$, $\bar{z}=a-bi$ and $-\bar{z}=-a+bi$

Therefore by definition $\quad |z|=\sqrt{a^2+b^2}$ (i)

$|-z|=\sqrt{(-a)^2+(-b)^2}=\sqrt{a^2+b^2}$ (ii)

$|\bar{z}|=\sqrt{(a)^2+(-b)^2}=\sqrt{a^2+b^2}$ (iii)

$|-\bar{z}|=\sqrt{(-a)^2+(b)^2}=\sqrt{a^2+b^2}$ (iv)

Equation (i), (ii), (iii) and (iv) yield that

$$|z|=|-z|=|\bar{z}|=|-\bar{z}|$$

(b) Let $z=a+bi$, then $\bar{z}=a-bi$, and so

$\bar{\bar{z}}=a+bi=z$

Thus $\bar{\bar{z}}=z$

(c) Let $z=a+bi$. Then $\bar{z}=a-bi$

Therefore $z\bar{z}=(a+bi)(a-bi)$

$=a^2-abi+bai-b^2i^2$

$=a^2-(-1)b^2 \quad (\because i^2=-1)$

$=a^2+b^2$

$=|z|^2 \quad (\because |z|=\sqrt{a^2+b^2})$

Thus $z\bar{z}=|z|^2$

(d) Let $z_1=a+bi$ and $z_2=c+di$

Then $\bar{z}_1=a-bi$, $\bar{z}_2=c-di$ and

$z_1+z_2=(a+bi)+(c+di)$

$=(a+c)+(b+d)i$

Therefore $\overline{z_1+z_2}=(a+c)-(b+d)i$

$= (a - bi) + (c - di) = \overline{z_1} + \overline{z_2}$

Thus $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(e) Let $z_1 = a + bi$ and $z_2 = c + di$

Then $\overline{z_1 z_2} = \overline{(a+bi)(c+di)}$

$= \overline{(ac-bd)+(ad+bc)i}$ (i)

$= (ac - bd) - (ad + bc)\, i$

and $\overline{z_1}\,\overline{z_2} = \overline{(a+bi)}\;\overline{(c+di)} = (a - bi)(c - di)$

$= (ac - bd) + (-ad - bc)\, i$ (ii)

$= (ac - bd) - (ad + bc)\, i$

Thus from equations (i) and (ii), we have

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

(f) Let $z_1 = a + bi$ and $z_2 = c + di$

Then $\dfrac{z_1}{z_2} = \dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di} \times \dfrac{c-di}{c-di}$ (by rationalization)

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

$$\therefore \quad \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i}$$

$$= \frac{ac+bd}{c^2+d^2} - \frac{bc-ad}{c^2+d^2}i \quad \text{(i)}$$

and $\dfrac{\overline{z_1}}{\overline{z_2}} = \dfrac{\overline{a+bi}}{\overline{c+di}} = \dfrac{a-bi}{c-di} = \dfrac{a-bi}{c-di} \times \dfrac{c+di}{c+di}$ (by rationalization)

$$= \frac{(ac+bd)-(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} - \frac{bc-ad}{c^2+d^2}i \quad \text{(ii)}$$

Thus from equations (i) and (ii), we have

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

1.2.5. Real and imaginary parts of the complex number of the form

(i) $(x + iy)^n$ (ii) $\left(\dfrac{x_1 + iy_1}{x_2 + iy_2}\right)^n$, $x_2 + iy_2 \neq 0$ where $n = \pm 1$ and ± 2

i. Real and imaginary parts of $(x + iy)^n$ where $n = \pm 1$ and ± 2

when $n = 1$, $(x + iy)^n$ reduces to $x + iy$

Therefore, real part $= x$ and imaginary part $= y$

When $n = -1$, $(x + iy)^n$ reduces to $(x + iy)^{-1}$

We have, $(x + iy)^{-1} = \dfrac{1}{(x+iy)} = \dfrac{1}{(x+iy)} \times \dfrac{x-iy}{x-iy}$ (by rationalization).

$$= \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Therefore real part $= \dfrac{x}{x^2+y^2}$ and imaginary part $= \dfrac{-y}{x^2+y^2}$

When $n = 2$, $(x + iy)^n$ reduces to $(x + iy)^2$,

we have $(x + iy)^2 = x^2 + 2ixy + i^2y^2$

$= x^2 + 2ixy - y^2$ $(\because i^2 = -1)$

$= (x^2 - y^2) + 2ixy$

Therefore real part $= x^2 - y^2$ and imaginary part $= 2xy$

When $n = -2$, $(x + iy)^n$ reduces to $(x + iy)^{-2}$

We have, $(x + iy)^{-2} = \dfrac{1}{(x+iy)^2}$

$$= \frac{1}{(x+iy)^2} \times \frac{(x-iy)^2}{(x-iy)^2} = \frac{x^2-y^2-2ixy}{(x+iy)^2(x-iy)^2}$$

$$= \frac{x^2-y^2-2ixy}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} - i\frac{2xy}{(x^2+y^2)^2}$$

Therefore real part $= \dfrac{x^2-y^2}{(x^2+y^2)^2}$ and imaginary part $= \dfrac{-2xy}{(x^2+y^2)^2}$

Example 15: Find the real and imaginary parts of the following complex numbers.

(i) $2-3i$ **(ii)** $(5-3i)^{-1}$ **(iii)** $(3+i)^2$ **(iv)** $(1+2i)^{-2}$

Solution:

(i) Let $z = 2-3i$ Therefore real part of $z = 2$ and imaginary part of $z = -3$

(ii) Let $z = (5-3i)^{-1}$. Here $x = 5$ and $y = -3$

Therefore, real part of $z = \dfrac{x}{x^2+y^2} = \dfrac{5}{(5)^2+(-3)^2} = \dfrac{5}{25+9} = \dfrac{5}{34}$

and imaginary part of $z = \dfrac{-y}{x^2+y^2} = \dfrac{-(-3)}{(5)^2+(-3)^2} = \dfrac{3}{25+9} = \dfrac{3}{34}$

(iii) Let $z = (3+i)^2$. Here $x = 3$ and $y = 1$

Therefore, real part of $z = x^2 - y^2 = (3)^2 - (1)^2 = 9 - 1 = 8$

imaginary part of $z = 2xy = 2(3)(1) = 6$

(iv) Let $z = (1+2i)^{-2}$. Here $x = 1$ and $y = 2$

Therefore, real part of $z = \dfrac{x^2-y^2}{(x^2+y^2)^2} = \dfrac{(1)^2-(2)^2}{[(1)^2+(2)^2]^2} = \dfrac{1-4}{(5)^2} = \dfrac{-3}{25}$

imaginary part of $z = \dfrac{-2xy}{(x^2+y^2)^2} = \dfrac{-2(1)(2)}{[(1)^2+(2)^2]} = \dfrac{-4}{(5)^2} = \dfrac{-4}{25}$

ii. Real and imaginary parts of $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$ where $n = \pm 1$ and ± 2

When $n = 1$, $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$ reduces to $\dfrac{x_1+iy_1}{x_2+iy_2}$. We have,

$$\frac{x_1+iy_1}{x_2+iy_2} = \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2} \quad \text{(By rationalization)}$$

$$= \frac{x_1x_2 - ix_1y_2 + iy_1x_2 - i_2y_1y_2}{x_2^2 - i^2y_2^2} = \frac{x_1x_2 + i(y_1x_2 - x_1y_2) + y_1y_2}{x_2^2+y_2^2} \quad (\because i^2 = -1)$$

$$= \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2+y_2^2} = \frac{x_1x_2 + y_1y_2}{x_2^2+y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2+y_2^2}$$

Therefore, real part $= \dfrac{x_1x_2 + y_1y_2}{x_2^2+y_2^2}$ and imaginary part $= \dfrac{y_1x_2 - x_1y_2}{x_2^2+y_2^2}$

When $n = -1$, $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$ reduces to $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^{-1}$

We have $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^{-1} = \dfrac{x_2+iy_2}{x_1+iy_1} = \dfrac{x_2+iy_2}{x_1+iy_1} \times \dfrac{x_1-iy_1}{x_1-iy_1}$ (by rationalization)

$$= \frac{x_2x_1 + y_2y_1}{x_1^2+y_1^2} + i\frac{y_2x_1 - x_2y_1}{x_1^2+y_1^2} \quad \text{(by routine calculation)}$$

Therefore, real part $= \dfrac{x_2x_1 + y_2y_1}{x_1^2+y_1^2}$ and imaginary part $= \dfrac{y_2x_1 - x_2y_1}{x_1^2+y_1^2}$

When $n = 2$, $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$ reduces to $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^2$. We have,

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^2 = \frac{(x_1+iy_1)^2}{(x_2+iy_2)^2} = \frac{(x_1+iy_1)^2}{(x_2+iy_2)^2} \times \frac{(x_2-iy_2)^2}{(x_2-iy_2)^2} \quad \text{(By rationalization)}$$

$$= \frac{[(x_1^2-y_1^2)+2ix_1y_1]\,[(x_2^2-y_2^2)-2ix_2y_2]}{(x_2+iy_2)^2(x_2-iy_2)^2}$$

$$= \frac{[(x_1^2-y_1^2)(x_2^2-y_2^2)+4x_1x_2y_1y_2] + 2i[x_1y_1(x_2^2-y_2^2) - x_2y_2(x_1^2-y_1^2)]}{(x_2^2+y_2^2)^2}$$

Therefore, real part $= \dfrac{(x_1^2-y_1^2)(x_2^2-y_2^2)+4x_1x_2y_1y_2}{(x_2^2+y_2^2)^2}$

imaginary part $= 2\dfrac{x_1y_1(x_2^2-y_2^2) - x_2y_2(x_1^2-y_1^2)}{(x_2^2+y_2^2)^2}$

When $n = -2$, $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$ reduces to $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^{-2}$

We have $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^{-2} = \dfrac{(x_2+iy_2)^2}{(x_1+iy_1)^2} = \dfrac{(x_2+iy_2)^2}{(x_1+iy_1)^2} \times \dfrac{(x_1-iy_1)^2}{(x_1-iy_1)^2}$

$$= \frac{[(x_2^2-y_2^2)(x_1^2-y_1^2)+4x_1x_2y_1y_2]+2i[x_2y_2(x_1^2-y_1^2)-x_1y_1(x_2^2-y_2^2)]}{(x_1^2+y_1^2)^2}$$

Therefore, real part $= \dfrac{(x_2^2-y_2^2)(x_1^2-y_1^2)+4x_1x_2y_1y_2}{(x_1^2+y_1^2)^2}$

imaginary part $= 2\dfrac{x_2y_2(x_1^2-y_1^2)-x_1y_1(x_2^2-y_2^2)}{(x_1^2+y_1^2)^2}$

EXERCISE 1.2

1. If $z_1 = 2+i$ and $z_2 = 1-i$, then verify commutative property w.r.t. addition and multiplication.
2. $z_1 = -1+i$, $z_2 = 3-2i$ and $z_3 = 2+3i$, verify associative property w.r.t. addition and multiplication.
3. $z_1 = \sqrt{3}+\sqrt{2}i$, $z_2 = \sqrt{2}-\sqrt{3}i$ and $z_3 = 2-2i$, verify distributive property of multiplication over addition.
4. Find the additive and multiplicative inverses of the following complex numbers.
   (i) $5+2i$ (ii) $(7,-9)$
5. (i) Let $z_1 = 2+4i$ and $z_2 = 1-3i$. Verify that $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$
   (ii) Let $z_1 = 2+3i$ and $z_2 = 2-3i$. Verify that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
   (iii) If $z_1 = -a-3bi$, $z_2 = 2a-3bi$, then verify that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$
6. Show that for all complex numbers z_1 and z_2
   (i) $|z_1z_2| = |z_1||z_2|$ (ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, where $z_2 \neq 0$.
7. Separate into real and imaginary parts
   (i) $\dfrac{2+3i}{5-2i}$ (ii) $\dfrac{(1+2i)^2}{1-3i}$ (iii) $\dfrac{1-i}{(1+i)^2}$
   (iv) $(2a-bi)^{-2}$ (v) $\left(\dfrac{3+4i}{4-3i}\right)^{-2}$ (vi) $\left(\dfrac{4-5i}{2+3i}\right)^2$

8. Show that
   (i) $z+\overline{z} = 2Re(z)$ (ii) $z-\overline{z} = 2i\,Im(z)$
   (iii) $z\overline{z} = [Re(z)]^2+[Im(z)]^2$ (iv) $z=\overline{z} \Rightarrow z$ is real
   (v) $\overline{z} = -z$ if and only if z is pure imaginary
9. If $z = 3+2i$, then verify that (i) $-|z| \le Re(z) \le |z|$ (ii) $-|z| \le Im(z) \le |z|$

1.3 Solution of equations

In this section we shall find solution of different equations in complex variables either with real or complex coefficients.

1.3.1 Solution of simultaneous linear equations with complex coefficients

Consider the following equation

$$pz + qw = r \qquad (1)$$

where p, q and r are complex numbers. The equation (1) is called a **linear equation** in two complex variables (or unknown) z and w.

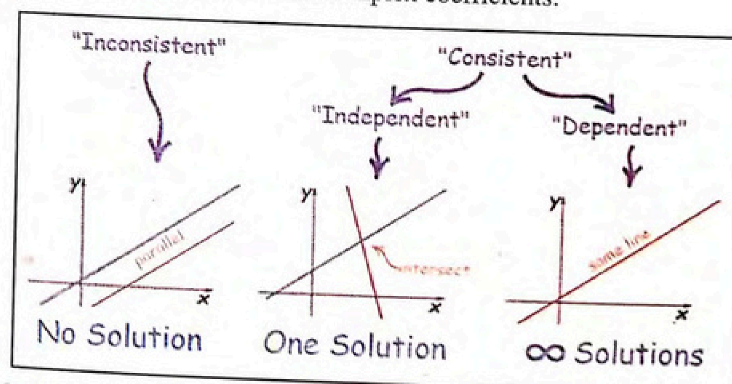
$$\left.\begin{aligned} p_1 z + q_1 w &= r_1 \\ p_2 z + q_2 w &= r_2 \end{aligned}\right\} \qquad (2)$$

These two equations together form a system of linear equations in two variables z and w. The linear equations in two variables are also called **simultaneous linear equations**.

For example

$$\left.\begin{aligned} 5z-(3+i)w &= 7-i \\ (2-i)z+2iw &= -1+i \end{aligned}\right\} \qquad (3)$$

is a system of linear equations with complex coefficients.



Not For Sale

A **solution** of a system in two variables z and w is an ordered pair (z,w) such that both the equations in the system are satisfied. For example consider system (3). The ordered pair (z,w) where $z = 1+i$ and $w = 2i$ is a solution of (3) because if we replace z by $1+i$ and w by $2i$, then both the equations are satisfied. The process of finding all solutions of the system of equations is called **solving** the system.

Here we shall find solution of a system of two equations with complex co-efficient in two variables z and w. The simple rule for solving such system of equations is the "method of elimination and substitution".

Step-1 If necessary multiply each equation by a constant so that the co-efficient of one variable in each equation is the same.

Step-2 Add or subtract the resulting equations to eliminate one variable, thus getting an equation in one variable.

Step-3 Solve the equation in one variable obtained in Step-2.

Step-4 Substitute the known value of one variable in either of the original equations in step-1 and solve for the other variable.

Step-5 Writing together the corresponding values of the variables in the form of ordered pairs gives solution of the system.

Example 16: Solve the simultaneous linear equations with complex coefficients.

$$5z-(3+i)w = 7-i$$
$$(2-i)z+2iw = -1+i$$

Solution: Given that $5z-(3+i)w = 7-i \quad (1)$

$$(2-i)z+2iw = -1+i \quad (2)$$

Multiplying equation (1) by $(2-i)$ we have

$$5(2-i)z-(3+i)(2-i)w = (7-i)(2-i)$$
$$\Rightarrow 5(2-i)z-(6-3i+2i-i^2)w = 14-7i-2i+i^2$$
$$\Rightarrow 5(2-i)z-(6-i+1)w = 14-9i-1 \quad (\because i^2=-1)$$
$$\Rightarrow 5(2-i)z-(7-i)w = 13-9i \quad (3)$$

Multiplying equation (2) by 5, we have

$$5(2-i)z+10iw = -5+5i \quad (4)$$

Subtracting equation (3) from equation (4), we have

$$\begin{array}{rcl} 5(2-i)z+10iw & = & -5+5i \\ +5(2-i)z-(7-i)w & = & +13-9i \\ - \quad\quad + & & - \quad + \\ \hline 10iw+(7-i)w & = & -18+14i \end{array}$$

$$\Rightarrow (7+9i)w = -18+14i \quad \Rightarrow w = \frac{-18+14i}{7+9i}$$
$$\Rightarrow w = \frac{-18+14i}{7+9i} \times \frac{7-9i}{7-9i} \quad \text{(By Rationalization)}$$
$$\Rightarrow w = \frac{260i}{130} = 2i$$

Substituting the value of w in (1), we have

$$5z-(3+i)(2i) = 7-i \Rightarrow 5z-(6i+2i^2) = 7-i$$
$$\Rightarrow 5z-(6i-2) = 7-i \Rightarrow 5z = 7-i+6i-2$$
$$\Rightarrow 5z = 5+5i \Rightarrow z = \frac{5+5i}{5} = 1+i$$

Thus (z,w) where $z = 1+i$ and $w = 2i$ is the solution of the simultaneous linear equations.

1.3.2 Expression of the polynomial P(z) as a product of linear factors

Recall that an expression of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots\dots + a_1 x + a_0,\ a_n \neq 0$$

where n is a positive integer or zero and the coefficients $a_n, a_{n-1}, \dots, a_1$ and a_0 are constants that are either to be real or complex numbers, is a polynomial of **degree n**.

For example, $2x+3$, $3x^2+2x+1$ and $5x^3-6x^2+5x-1$ are polynomials of degree 1, 2 and 3 respectively.

Here we are concerned with finding the linear factors of the following two types of polynomials.

(i) $P(z) = z^2 + a^2$, where a is a real number.

(ii) $P(z) = az^3+bz^2+cz+d$ where a, b, c and d are real numbers.

In factorizing polynomials of type (i) we simply use the fact that $i^2 = -1$ so that to find linear factors.

For example, $P(z) = z^2+a^2 = z^2 - i^2a^2 = (z+ia)(z-ia)$. However, in factorizing polynomials of type (ii), we use the factor theorem which has already been proved in the previous class and stated below.

The factor theorem: Let $P(x)$ be any polynomial. Then $x-a$ is a factor of $P(x)$ if and only if $P(a) = 0$

The method for factorizing the polynomials of type (ii) into linear factors is explained through the following example.

Example 17: Factorize the polynomial $P(z) = z^3 + 5z^2 + 19z - 25$ into linear factors.

Solution: In factorizing the given polynomial $P(z)$ into linear factors, we use the factor theorem. To do so, we note that $z = 1$ is a root of $P(z)$, since

$$P(1) = (1)^3 + 5(1)^2 + 19(1) - 25 = 1 + 5 + 19 - 25 = 0$$

By factor theorem $z - 1$ is a factor of $P(z)$. We therefore arrange the terms in such a way that we can find a common factor $z - 1$ as follows:

$$\begin{aligned}
P(z) &= z^3 + 5z^2 + 19z - 25\\
&= (z^3 - 1) + (5z^2 + 19z - 24)\\
&= (z-1)(z^2 + z + 1) + (5z^2 - 5z + 24z - 24) \quad \because a^3 - b^3 = (a-b)(a^2 + ab + b^2)\\
&= (z-1)(z^2 + z + 1) + (5z^2 - 5z) + (24z - 24)\\
&= (z-1)(z^2 + z + 1) + 5z(z-1) + 24(z-1)\\
&= (z-1)[(z^2 + z + 1) + 5z + 24)] = (z-1)(z^2 + 6z + 25)\\
&= (z-1)(z^2 + 6z + 9 + 16) = (z-1)[(z^2 + 6z + 9) + 16]\\
&= (z-1)[(z^2 + 6z + 9) - (-16)]\\
&= (z-1)[(z+3)^2 - (4i)^2] \qquad (\because i^2 = -1)\\
&= (z-1)[(z+3) + 4i][(z+3) - 4i] \qquad (\because a^2 - b^2 = (a+b)(a-b))\\
&= (z-1)(z + 3 + 4i)(z + 3 - 4i)
\end{aligned}$$

1.3.3 Quadratic equation of the form $pz^2 + qz + r = 0$

Consider the quadratic equation of the form

$$pz^2 + qz + r = 0 \qquad (1)$$

where p, q, r are real numbers $p \neq 0$ and z is a complex variable.

We see that $z^2 - z + 3 = 0$, $3z^2 - 4z + 2 = 0$, $5z^2 + 6z = 0$, $z^2 - 3 = 0$, $2z^2 = 3z - 1$ and $z^2 = 0$ are all examples of quadratic equation in the variable z. Equation (1) is called the standard form of the quadratic equation.

Solution of quadratic equations

Recall that all those values of z for which the given equation is true are called **solutions** or **roots** of the equation, and the set of all solutions is called **solution set.**

For example, $z^2 + 4 = 0$ or $z^2 - (2i)^2 = 0$ is true only for $z = 2i$ or $z = -2i$, hence $z = 2i$ and $z = -2i$ are the solutions or roots of the given quadratic equation and $\{2i, -2i\}$ is the **solution set.**

To find the solutions of equations of the form (1), we use a method known as "completing the square" which is described as follows:

Step-1 Write the quadratic equation in its standard form.

Step-2 Divide both sides of the equation by the coefficient of z^2 if it is other than 1.

Step-3 Shift the constant term to the right hand side of the equation.

Step-4 Add a number which is the square of half of the coefficient of z to both sides of the equation.

Step-5 Write the left hand side of the equation as a perfect square and simplify the right hand side.

Step-6 Take square root of both sides of the equation and solve the resulting equation to find the solutions of the equation.

The method is explained in the following example.

Example 18: Solve the quadratic equation $z^2 + 6z + 25 = 0$

Solution: We have

$$\begin{aligned}
& z^2 + 6z + 25 = 0 && \text{(Step-1)}\\
\Rightarrow\ & z^2 + 6z = -25 && \text{(Step-2 and Step-3)}\\
\Rightarrow\ & z^2 + 6z + 9 = -25 + 9 && \text{(Step-4)}\\
\Rightarrow\ & (z+3)^2 = -16 && \text{(Step-5)}\\
\Rightarrow\ & (z+3)^2 = (4i)^2 &&\\
\Rightarrow\ & z + 3 = \pm 2i && \text{(Step-6)}\\
\Rightarrow\ & z = -3 + 2i \text{ or } z = -3 - 2i &&
\end{aligned}$$

Did You Know ?

The coefficient of z^2 must not be zero otherwise it becomes linear

Thus the solutions of given equation are $-3 + 2i$, $-3 - 2i$ and solution set is $\{-3 + 2i, -3 - 2i\}$

Example 19: Solve the equation $z^2 + z + 1 = 0$

Solution: According to the quadratic formula, the answer is

$$z = \frac{-1 \pm \sqrt{1^2 - 4}}{2} = -1 \pm i\frac{\sqrt{3}}{2}$$

EXERCISE 1.3

1. Solve the simultaneous linear equations with complex coefficients.
   (i) $z-4w=3i$, $2z+3w=11-5i$ (ii) $z+w=3i$, $2z+3w=2$ (iii) $3z+(2+i)w=11-i$, $(2-i)z-w=-1+i$
2. Factorize the polynomials $P(z)$ into linear factors.
   (i) $P(z)=z^3+6z+20$ (ii) $P(z)=3z^2+7$
   (iii) $P(z)=z^2+4$ (iv) $P(z)=z^3-2z^2+z-2$
3. Show that each $z_1=-1+i$ and $z_2=-1-i$ satisfies the equation $z^2+2z+2=0$
4. Determine whether $1+2i$ is a solution of $z^2-2z+5=0$
5. Find all solutions to the following equations
   (i) $z^2+z+3=0$ (ii) $z^2-1=z$ (iii) $z^2-2z+i=0$ (iv) $z^2+4=0$
6. Find the solutions to the following equations
   (i) $z^4+z^2+1=0$ (ii) $z^3=-8$ (iii) $(z-1)^3=-1$ (iv) $z^3=1$

REVIEW EXERCISE 1

1. Choose the correct option.

(i) $\left(\frac{2i}{1+i}\right)^2$

(a) i (b) $2i$ (c) $1-i$ (d) $1-2i$

(ii) Divide $\frac{5+2i}{4-3i}$

(a) $-\frac{7}{25}+\frac{26}{25}i$ (b) $\frac{5}{4}-\frac{2}{3}i$ (c) $\frac{14}{25}+\frac{23}{25}i$ (d) $\frac{26}{7}+\frac{23}{7}i$

(iii) $i^{57}+\frac{1}{i^{25}}$ when simplified has the value

(a) 0 (b) $2i$ (c) $-2i$ (d) 2

(iv) $1+i^2+i^4+i^6+\dots\dots+i^{2n}$ is

(a) Positive (b) negative (c) 0 (d) cannot be determined

(v) If $z=x+iy$ and $\left|\frac{z-5i}{z+5i}\right|=1$ then z lies on

(a) X-axis (b) Y-axis (c) line $y=5$ (d) None of these

(vi) The multiplicative inverse of $z=3-2i$, is

(a) $\frac{1}{3}(3+2i)$ (b) $\frac{1}{13}(3+2i)$ (c) $\frac{1}{13}(3-2i)$ (d) $\frac{1}{4}(3-2i)$

(vii) If $(x+iy)(2-3i)=4+i$, then

(a) $x=-14/13,\ y=5/13$ (b) $x=5/13,\ y=14/13$

(c) $x=14/13,\ y=5/13$ (d) $x=5/13,\ y=-14/13$

2. Show that $i^n+i^{n+1}+i^{n+2}+i^{n+3}=0,\ \forall\, n\in N$
3. Express the following complex numbers in the form $x+iy$.
   (i) $(1+3i)+(5+7i)$ (ii) $(1+3i)-(5+7i)$ (iii) $(1+3i)(5+7i)$ (iv) $\frac{1+3i}{5+7i}$
4. If $z_1=2-i$, $z_2=1+i$, find $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|$.
5. Find the modulus of $\frac{1+i}{1-i}-\frac{1-i}{1+i}$.
6. Find the conjugate of $\frac{1}{3+4i}$.
7. Find the multiplicative inverse of $z=\frac{3i+2}{3-2i}$.
8. Solve the quadratic equation $z+\frac{2}{z}=2$.