Unit - 7

Hypothesis Testing

After studying this unit, the students will be able to

- Describe statistical hypothesis and hypothesis testing.
- Differentiate between null and alternative hypothesis, simple and composite hypothesis.
- Formulate null and alternative hypothesis.
- Recognize the elements involved in hypothesis testing: test statistic, rejection and non-rejection regions, critical values, one tailed test, two tailed test, type-I and type-II errors, level of significance, decision rule and conclusion.
- Apply the test of hypothesis about mean of a normal population (known and unknown standard deviation)
- Apply the test of hypothesis about population proportion (large sample).
- Apply the test of hypothesis about the difference between means of two normal populations (known/unknown standard deviations).
- Apply the test of hypothesis about the difference between proportions of two populations (large samples).

7.1 Introduction

The second important phase of statistical inference is the hypothesis testing. Here decisions are made about the population parameters on the basis of sample information. For example, hypothesis testing procedure can decide whether (i) a new medicine is really effective in curing a particular disease (ii) one educational procedure is better than other etc. The difference between estimation and hypothesis testing is that in estimation parameter values are unknown and are obtained from sample information while in hypothesis testing parameters values are hypothesized and are checked on the basis of sample information whether they are true or false.

Hypothesis is an unproved claim or assertion or assumption which acts as a starting point in a research irrespective of its probable truthfulness or falsity. Hypothesis may be non-statistical or statistical. A statistical hypothesis is a testable claim about one or more parameters of empirical distributions.

7.1.1 Hypothesis testing

A statistical method that uses sample data to accept or reject a hypothesis about a parameter is called hypothesis testing.

7.1.2 Principle steps involved in hypothesis testing

Hypothesis testing procedure mainly involves the following six steps:

Step i.	State the null and alternative hypothesis.
Step ii.	Choose an appropriate level of significance.
Step iii.	Decide about test statistic.
Step iv.	Compute test statistic value.
Step v.	Obtain critical values for test statistic from respective tables.
Step vi.	Make a decision.

7.1.3 Definitions of key terms

- **Null hypothesis**

It is a claim about parameter. It is tested for possible rejection under the assumption that it is true. It is denoted by H_0.

- **Alternative hypothesis**

It is also a claim about parameter which is accepted if H_0 is rejected. It is denoted by H_1. It may be directional or non-directional.

- **Simple hypothesis**

When all parameters of a distribution are well specified under the hypothesis it is called simple hypothesis. For example in case of normal distribution if H_0: $\mu = 5, \sigma^2 = 2.5$, it is simple hypothesis because all parameters values are well specified under H_0.

- **Composite hypothesis**

When all parameters of a distribution are not well specified under the hypothesis it is called composite hypothesis. For example for normal distribution H_0: $\mu = 5, \sigma^2 \leq 2.5$ or H_0: $\mu < 5, \sigma^2 = 2.5$ or H_0: $\mu = 5$, σ unknown etc. are composite hypothesis because in the first two cases values are not well specified and in the third case all parameters are not considered. The concept of simple and composite hypothesis is applicable to both H_0 and H_1.

- **Type-I and Type-II errors**

Two types of errors are possible in making a decision about a hypothesis.

(1) Type-I error

"To reject a null hypothesis when it is true, it is called type-I error"

Examples

i. If a medicine is given to a few patients of a particular disease to cure them and the medicine is curing the disease, but it is claimed that it has no effect or has an adverse effect and hence discontinued. This is Type-I error.
ii. To fail an intelligent student.
iii. To punish an innocent person.
iv. When a good player is not played in the cricket match.
v. To reject a good item.

The probability of committing type-I error is denoted by α i.e. $\alpha = p(\text{reject } H_0/H_0 \text{ is true})$

(2) Type-II error

"To accept a null hypothesis when it is false, it is called type-II error."

Examples

i. If the medicine has adverse effect and is claimed to have good effect and the treatment is continued. This is type-II error.
ii. To pass a dull student.
iii. To release a guilty person.
iv. To allow a non-deserving player in the cricket match.
v. To accept a bad item.

The probability of committing type-II error is denoted by β i.e. $\beta = p(\text{accept } H_0/H_0 \text{ is false})$

Remember that

(i) Type-I error is more serious than type-II error.

(ii) α and β have an inverse relation i.e. if one increases the other decreases and vice-versa.

(iii) Both α and β can be decreased by increasing n.

- **Significance level**

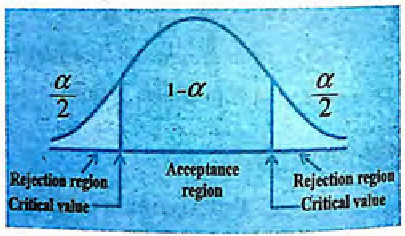
The probability of type-I error which we are ready to tolerate in making decision about H_0 is called significance level (S.L). It is denoted by α i.e. $\alpha = P(\text{reject } H_0/H_0 \text{ is true})$. The higher the significance level we use for testing a hypothesis, the higher the probability of rejecting H_0 when it is true. Usually, it is suggested to keep $\alpha \leq 0.05$.

- **Test statistic and test of significance**

A function of sample data is called statistic. When a statistic is used to test a hypothesis, it is called test statistic or decision rule. It helps us in the decision whether to accept or reject H_0. The choice of test statistic depends on the hypothesis under question. Commonly used test statistics are Z, t, χ^2 and F.

The value of test statistic is said to be statistically significant if it falls in the rejection region and as a result H_0 is rejected. On the other hand, value of test statistic is said to be statistically insignificant if it falls in the acceptance region and as a result H_0 is accepted.

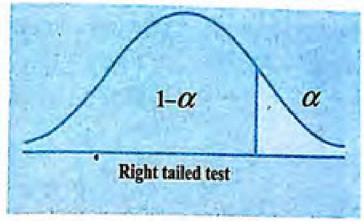
- **Acceptance and rejection regions**

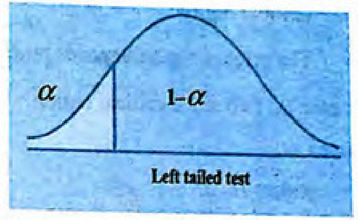


All possible values of a test statistic are divided into two mutually exclusive groups. The group of values which lead to the acceptance of H_0 is called acceptance region (AR) for the test. The group of values which would lead us to rejection of H_0 is called rejection region (RR) or critical region for the test. The values which separate the (AR) and (RR) are called critical values and are obtained from the tables of respective test statistics.

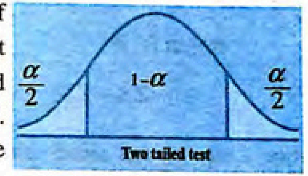
- **One-tailed and two-tailed test**

Three types of symbols [>, <, ≠] are used in the alternative hypothesis H_1. When the symbols [>, <] are used in H_1, it means that the whole rejection region of size α lies only on one tail of the concerned sampling distribution as shown below,





It is called one-sided rejection region and the test is called one-sided or one tailed test. On the other hand when the symbol [≠] is used in H_1, it means that the whole rejection region of size α is equally divided; $\alpha/2$ lies in the right tail and $\alpha/2$ lies in the left tail of the concerned sampling distribution, as shown in the figure. It is called two sided rejection region and the test is called two-tailed test.



7.2 General procedure for testing hypothesis about Mean (μ) of a Normal population [when σ^2 is known]

i. Generally H_0 and H_1 can be set in three different possible forms as follow:

a) $H_0: \mu = \mu_0$	b) $H_0: \mu \geq \mu_0$	c) $H_0: \mu \leq \mu_0$
$H_1: \mu \neq \mu_0$	$H_1: \mu < \mu_0$	$H_1: \mu > \mu_0$
[Two tailed test]	[Left tailed test]	[Right tailed test]

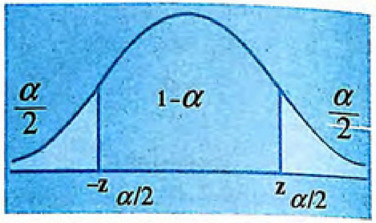
ii. $\alpha = 0.01$ or 0.05 etc.

iii. Test statistic to be used is $Z = \dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}}$, which follows N(0,1) when H_0 is true.

iv. Calculation of Z value from the given data.

v. Critical region:

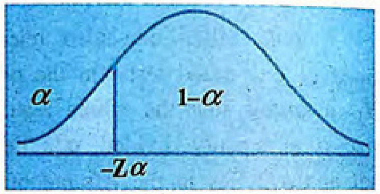
The critical region depends on H_1.



For case (a) two sided critical region is used as;

Reject H_0 if

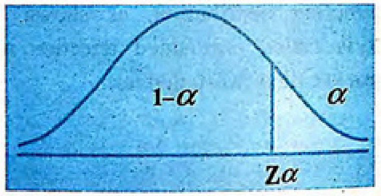
$$Z \le -Z_{\alpha/2} \text{ or } Z \ge Z_{\alpha/2}.$$



For case (b) one sided critical region is used as;

Reject H_0 if

$$Z \le -Z_{\alpha}$$



For Case (c) one sided critical region is used as;

Reject H_0 if

$$Z \ge Z_{\alpha}$$

Where Z is computed value of Z-test and $Z_{\alpha/2}$ or Z_{α} is table value.

vi. Conclusion:

If the computed value of Z falls in the acceptance region, accept H_0, otherwise reject H_0. Remember that acceptance of a hypothesis does not mean that it is really true.

We may interpret it as that sample data is in the support of H_0 that is why we are accepting H_0.

Example 7.1

An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed with a mean of 812 hours and a standard deviation of 40 hours. Test the hypothesis that $\mu = 812$ hours against the alternative $\mu \ne 812$ hours if a random sample of 36 bulbs has an average life of 800 hours. Use a 5% level of significance.

Solution:

i. Null hypothesis H_0: $\mu = 812$

Alternative hypothesis H_1: $\mu \ne 812$

ii. Significance level $\alpha = 0.05$

iii. Test statistic to be used here is;

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

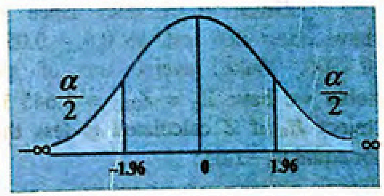
iv. Calculation: Given $n = 36$, $\bar{X} = 800$, $\sigma = 40$

Hence $Z = \dfrac{800 - 812}{40 / \sqrt{36}} = \dfrac{-12}{40/6} = \dfrac{-12 \times 6}{40} = -1.8$

v. Critical region:

Here the test is two sided, so $\alpha/2 = 0.025$ Now $0.5 - 0.025 = 0.4750$ search this value in the body of the area table of standard normal distribution which correspond to $Z_{\alpha/2} = Z_{0.05} = 1.96$.

Thus reject H_0 if $Z \le -1.96$ or $Z \ge 1.96$.



vi. Conclusion:

As computed value of $Z = -1.8$ falls in the acceptance region, therefore, we accept H_0 i.e. mean life of the bulbs produced by this firm is equal to 812 hours.

Example 7.2

A random sample of 25 hens from a normal population showed that the average laying is 250 eggs per year. The company claims that the average laying is 285 eggs per year with a standard deviation of 25 eggs per year. Test the claim of the company against the alternative that average laying is less than 285 eggs at $\alpha = 0.05$

Solution:

i. $H_0: \mu = 285$

$H_1: \mu < 285$

ii. $\alpha = 0.05$

iii. Test statistic: As $n = 25$ is small but σ is known, therefore, we use test statistic for μ as $Z = \dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}}$

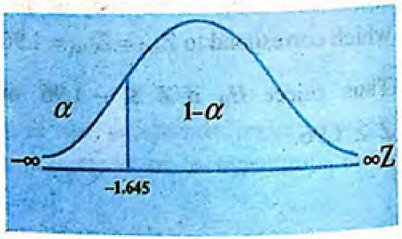
iv. Computation:

Given $n = 25$, $\bar{X} = 250$, $\sigma = 25$

$$Z = \frac{250 - 285}{25/\sqrt{25}} = \frac{-35}{25/5} = \frac{-35}{5} = -7$$

v. Critical region:

Given $\alpha = 0.05$ since we have one tailed test, so $0.5 - 0.05 = 0.4500$. Make inverse use of area table we have $Z_\alpha = Z_{0.05} = 1.645$ i.e. reject H_0 if Z calculated is less than or equal to -1.645.



vi. Conclusion:

Since our calculated value of Z lies in the rejection region, therefore, we reject H_0 and accept H_1.

Example 7.3

Past records show that the average score of students in statistics is 57 with standard deviation 10. A new method of teaching is employed and a random sample of 70 students is selected. The sample average is 60. Can we conclude on the basis of these results, at 5% level of significance, that the average score has increased?

Solution:

i. $H_0: \mu = 57$

$H_1: \mu > 57$

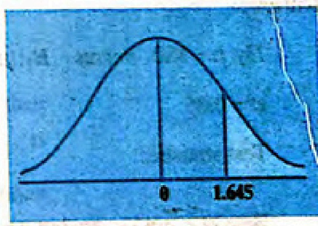
ii. $\alpha = 5\% = 0.05$

iii. Test statistic: since σ is known, therefore, test statistic for μ in this case is

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

iv. Calculation: Given that $n = 70$, $\bar{X} = 60$, $\sigma = 10$, therefore,

$$Z = \frac{60 - 57}{10/\sqrt{70}} = \frac{(3)\sqrt{70}}{10} = 2.51$$



v. Critical region:

As we see in H_1, we need to use one-sided test to the right. Here $\alpha = 0.05$ (from the area table of S.N.D, we have $Z_\alpha = Z_{0.05} = 1.645$).

Hence reject H_0 if $Z \geq 1.645$.

vi. Conclusion:

We see that $Z = 2.51$ lies in the rejection region, so we reject H_0 and conclude that the average score has increased.

7.2.1 Procedure for hypothesis testing about μ when σ is unknown [large sample]

When σ is unknown but $n \geq 30$, then central limit theorem allows us to consider sampling distribution of $\bar{x}$ as approximately normal with mean μ and S.E $= \frac{S}{\sqrt{n}}$. Hence the test statistic is $Z = \frac{\bar{X} - \mu}{S/\sqrt{n}}$ i.e. only σ is replaced by S. All other steps of hypothesis testing procedure are exactly the same as when σ^2 is known.

Example 7.4

The daily yield for a local chemical plant has average 880 tons for the last several years. The quality control manager would like to know whether this average has changed in recent months. She randomly selects 50 days from the computer database and computes the average and standard deviation of the $n = 50$ yields as $\bar{X} = 871$ tons and $S = 21$ tons respectively. Test the appropriate hypothesis using $\alpha = 0.05$

Solution:

i. $H_0: \mu = 880$ versus $H_1: \mu \neq 880$

ii. $\alpha = 0.05$

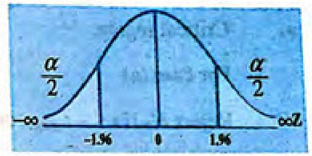
iii. Test statistic:

Here σ is not known but $n = 50$ is large, so we use $Z = \frac{\bar{X} - \mu}{S/\sqrt{n}}$ as test statistic for μ.

iv. Calculation:

$$Z = \frac{871 - 880}{21/\sqrt{50}} = -3.03$$

v. Rejection region:

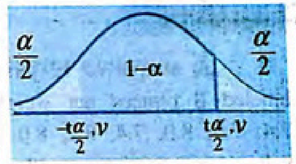


For two-tailed test we use $\alpha/2 = 0.025$ so $0.5 - 0.025 = 0.4750$ which corresponds to $Z_{\alpha/2} = 1.96$ in the area table of S.N.D. We will reject if $Z \leq -1.96$ or $Z \geq 1.96$

vi. Conclusion:

Since $Z = -3.03$ falls in the rejection region, therefore, we reject H_0.

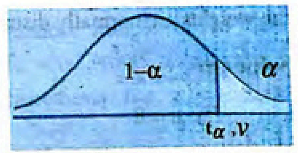
7.2.2 Procedure for hypothesis testing about μ when σ is unknown [small sample]



i. a) $H_0: \mu = \mu_0$ versus $H_1: \mu \neq \mu_0$

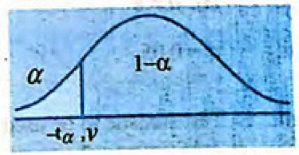
b) $H_0: \mu \leq \mu_0$ versus $H_1: \mu > \mu_0$

c) $H_0: \mu \geq \mu_0$ versus $H_1: \mu < \mu_0$



ii. Choose $\alpha = 0.01$ or 0.05 etc.

iii. Test statistic:



When σ is unknown and $n < 30$, then test statistic to be used for μ is $t = \frac{\bar{X} - \mu}{s/\sqrt{n}}$ which has t-distribution with $v = n - 1$ degrees of freedom, where as

$$s=\sqrt{\frac{1}{n-1}\sum(x-\bar{x})^2}$$

iv. **Calculation of t-value.**

v. **Critical region:**

For case (a)

Reject H_0 if $t \le -t_{\alpha/2,(v)}$ or $t \ge t_{\alpha/2,(v)}$

For case (b)

Reject H_0 if $t \ge t_{\alpha,(v)}$

For case (c)

Reject H_0 if $t \le -t_{\alpha,(v)}$

Where t is computed value of t-statistic and $t_{\alpha/2,(v)}$ or $t_{\alpha,(v)}$ is table value.

vi. **Conclusion:**

Example 7.5

A sample of 12 jars of butter was taken form a lot, each jar being labeled 8 ounces net weight. The individual weights in ounces are 7.3, 7.4, 7.5, 8.0, 7.4, 8.2, 8.0, 7.6, 7.6, 7.5, 7.5, and 7.7. Test whether these values are consistent with a population mean of 8 ounces. Assume that the weights are normally distributed.

Solution:

i. $H_0: \mu = 8$

$H_1: \mu \ne 8$

ii. $\alpha = 0.05$

iii. Test statistic to be used in this case is $t=\frac{\bar{X}-\mu}{s/\sqrt{n}}$ with $v = n-1$ d.f

iv. Calculation:

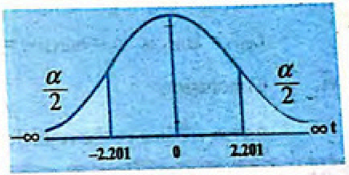
Given $n = 12, \sum x = 91.7, \sum x^2 = 701.61,\ \bar{X}=\frac{\sum x}{n}=\frac{91.7}{12}=7.64$

$$s=\sqrt{\frac{1}{n-1}\left[\sum x^2-\frac{(\sum x)^2}{n}\right]}=\sqrt{\frac{1}{12-1}\left[701.61-\frac{(91.7)^2}{12}\right]}$$

$$=\sqrt{\frac{1}{11}[0.8692]}=0.28$$

Hence $t=\frac{7.64-8}{0.28/\sqrt{12}}=\frac{(-0.36)\sqrt{12}}{0.28}=-4.45$

v. Critical region:



Reject H_0 if $t \le -t_{\alpha/2,(v)}$ or $t \ge t_{\alpha/2,(v)}$ whereas from t-table 6.1, we have

$t_{\alpha/2,(v)} = t_{0.05,(12-1)} = t_{0.025,(11)} = 2.201$

vi. Conclusion:

As $t = -4.45$ falls in the rejection region, therefore, we reject H_0.

Example 7.6

A manufacturing company making automobile tire claims that the average life of its product is 35000 miles. A random sample of 16 tires was selected and it was found that the mean life was 34000 miles with a standard deviation s = 2000 miles. Test the hypothesis H_0: $\mu = 35000$ against the alternative H_1: $\mu < 35000$ at $\alpha = 0.05$.

Solution:

i. H_0: $\mu = 35000$

H_1: $\mu < 35000$

ii. $\alpha = 0.05$

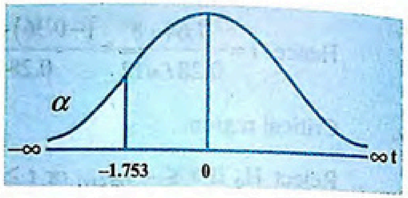
iii. Test statistic:

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}}, \quad v = n - 1 \text{ d.f}$$

iv. Calculation:

$$t = \frac{34000 - 35000}{2000/\sqrt{16}} = \frac{-1000}{2000/4} = -2$$

v. Critical region:



Reject H_0 if $t \leq -t_{\alpha,(v)}$

Whereas as from t-table

$t_{\alpha,(v)} = t_{0.05,(16-1)} = t_{0.05,(15)} = 1.753$

vi. Conclusion

As $t = -2$ lies in the rejection region, so we reject H_0

7.2.3 Procedure for testing hypothesis about difference between two populations means ($\mu_1 - \mu_2$) when σ_1^2 and σ_2^2 are known

The samples are randomly and independently selected from the two normal populations. The formal testing procedure is explained below:

i. (a) $H_0: \mu_1 = \mu_2$ or $\mu_1 - \mu_2 = 0$

$H_1: \mu_1 \neq \mu_2$ or $\mu_1 - \mu_2 \neq 0$

(b) $H_0: \mu_1 \leq \mu_2$ or $\mu_1 - \mu_2 \leq 0$

$H_1: \mu_1 > \mu_2$ or $\mu_1 - \mu_2 > 0$

(c) $H_0: \mu_1 \geq \mu_2$ or $\mu_1 - \mu_2 \geq 0$

$H_1: \mu_1 < \mu_2$ or $\mu_1 - \mu_2 < 0$

ii. Level of significance is decided as 0.01 or 0.05

iii. Test statistic to be used in this case is

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1) \text{ under } H_0.$$

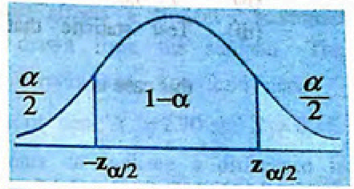
iv. Z-statistic is computed from the given data.

v. Critical region:

For case (a) in H_1

Reject H_0 if

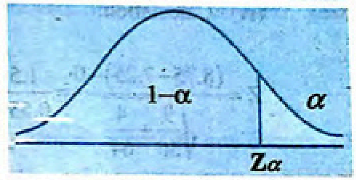
$Z \leq -Z_{\alpha/2}$ or $Z \geq Z_{\alpha/2}$



For case (b)

Reject H_0 if

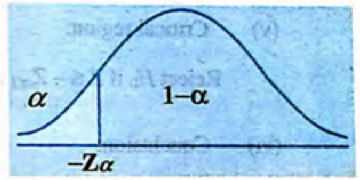
$Z \geq Z_\alpha$



For case (c)

Reject H_0 if

$Z \leq -Z_\alpha$



vi. Conclusion:

H_0 is rejected when computed value of Z falls in the rejection region.

Example 7.7

To see the effects of a certain sleeping pills on male and female, two independent samples were taken and the following data were recorded.

	male	female
sample size	$n_1 = 36$	$n_2 = 64$
sample mean	$\bar{x}_1 = 8.75$	$\bar{x}_2 = 7.25$
population variance	$\sigma_1^2 = 9$	$\sigma_2^2 = 4$

Test H_0: $\mu_1 = \mu_2$ against H_1: $\mu_1 \neq \mu_2$ at 5% significance level.

Solution:

(i) H_0: $\mu_1 = \mu_2$ means that $\mu_1 - \mu_2 = 0$

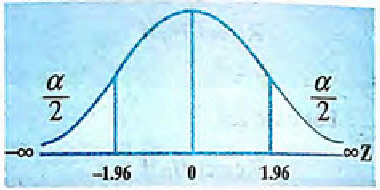
H_1: $\mu_1 \neq \mu_2$

(ii) $\alpha = 0.05$

(iii) Test statistic that summarizes the sample information in this case is $$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

(iv) Calculation:

$$Z = \frac{(8.75 - 7.25) - 0}{\sqrt{\frac{9}{36} + \frac{4}{64}}} = \frac{1.5}{0.559} = 2.683$$



(v) Critical region:

Reject H_0 if $Z \leq -Z_{\alpha/2}$ or $Z \geq Z_{\alpha/2}$ whereas $Z_{\alpha/2} = Z_{0.025} = 1.96$

(vi) Conclusion:

As Z = 2.683 falls in the rejection region, therefore, we reject H_0 and conclude that effect of sleeping pill on male and female is different.

7.2.4 Procedure for testing hypothesis about ($\mu_1 - \mu_2$) when σ_1^2 and σ_2^2 are unknown but sample sizes are large.

When σ_1^2 and σ_2^2 are unknown, then for $n_1, n_2 \geq 30$, they are estimated by S_1^2 and S_2^2 respectively. The test statistic Z in this case is

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \sim N(0,1) \text{ approximately.}$$

Rest of the procedure for testing H_0 is same.

Example 7.8

To determine whether car ownership affects a student's academic achievement, two random samples were drawn from the students. The grade point average for the $n_1 = 100$ car owners had $\bar{X}_1 = 2.54$ and $S_1^2 = 0.40$ while for the $n_2 = 100$ non-owners of cars, $\bar{X}_2 = 2.70$ and $S_2^2 = 0.36$.

Do the data present sufficient evidence to indicate a difference in the mean achievements between car owners and non-owners of cars? Test using $\alpha = 0.05$.

Solution:

Let μ_1 denote mean of car owners and μ_2 denote mean of non-owners of cars, then

i. H_0: $\mu_1 - \mu_2 = 0$
H_1: $\mu_1 - \mu_2 \neq 0$

ii. $\alpha = 0.05$

iii. Test statistics:

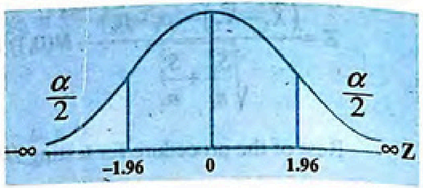
Since σ_1^2, σ_2^2 are not given but n_1, n_2 are large, so Z-statistic will be of the form $$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

iv. Computations: Substituting values into the formula we get

$$Z = \frac{(2.54 - 2.70) - 0}{\sqrt{\frac{0.40}{100} + \frac{0.36}{100}}} = -1.84$$

v. Critical region:

Reject H_0 if $Z \geq Z_{\alpha/2}$ or $Z \leq -Z_{\alpha/2}$ where $Z_{\alpha/2} = 1.96$



vi. Conclusion:

We see that $Z = -1.84$ is lying in the acceptance region. Hence H_0 cannot be rejected.

7.2.5 Procedure for testing hypothesis about $(\mu_1 - \mu_2)$ when $\sigma_1^2 = \sigma_2^2 = \sigma^2$ but unknown [small samples]

When population variances are unknown and samples are of small size then the following assumption are made to use a two-sample t-test.

(i) The sampled populations are normal.

(ii) σ_1^2 and σ_2^2 are assumed to be equal but unknown

(iii) The samples are drawn randomly.

Procedure for testing H_0 is given below:

i. (a) $H_0: \mu_1 - \mu_2 = 0$ versus $H_1: \mu_1 - \mu_2 \neq 0$

(b) $H_0: \mu_1 - \mu_2 \geq 0$ versus $H_1: \mu_1 - \mu_2 < 0$

(c) $H_0: \mu_1 - \mu_2 \leq 0$ versus $H_1: \mu_1 - \mu_2 > 0$

ii. Choose $\alpha = 0.01$ or 0.05 etc.

iii. Test statistic to be used in this case is

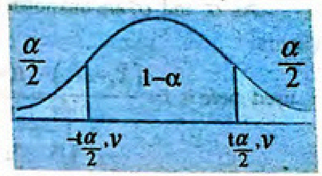
$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$,Which has t-distribution with $v = n_1 + n_2 - 2$ d.f.

iv. Computation of t-statistic value.

v. Critical region:

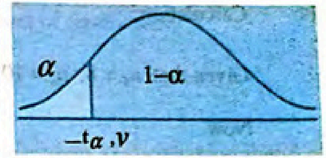
For case (a)

Reject H_0 if $t_c \leq -t_{\alpha/2,v}$ or $t \geq t_{\alpha/2,v}$



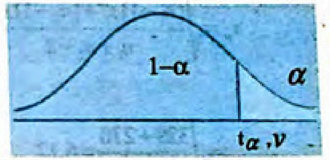
For case (b)

Reject H_0 if $t_c \leq -t_{\alpha,v}$



For case (c)

Reject H_0 if $t_c \geq t_{\alpha,v}$



vi. Conclusion.

Example 7.9

An examination was given to two classes of 8 and 10 students respectively. In the first class mean grade was 95 with a standard deviation of 6.8556, while in the second class, the mean grade was 97 with a standard deviation of 5.4772. It is assumed that the two classes of students are normally distributed having identical variances. Is there a significance difference between the mean grades? Test at $\alpha = 0.01$.

Solution:

i. $H_0: \mu_1 = \mu_2$ versus $H_1: \mu_1 \neq \mu_2$

ii. $\alpha = 0.01$

iii. Test statistic:

As σ_1^2 and σ_2^2 are unknown and n_1, n_2 are small ,therefore, test statistic to be used here is $t=\dfrac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$ with $v = n_1+ n_2 -2$ d.f

iv. Calculation:

Given that $n_1 = 8$, $n_2 = 10$, $\bar{X}_1 = 95$, $\bar{X}_2= 97$, $s_1^2= 47$, $s_2^2= 30$

Now

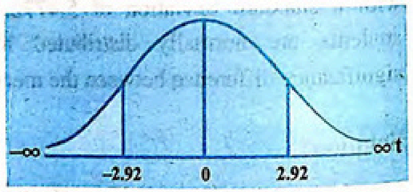
$$s_p=\sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}=\sqrt{\frac{(8-1)47+(10-1)30}{8+10-2}}$$

$$=\sqrt{\frac{329+270}{16}}=6.12$$

$$\text{Hence } t=\frac{(95-97)-0}{6.12\sqrt{\frac{1}{8}+\frac{1}{10}}}=\frac{-2}{2.930}=-0.689$$

v. Critical region:

Reject H_0 if $t \le -t_{\alpha/2,v}$ or $t \ge t_{\alpha/2,v}$, whereas $t_{\alpha/2,v} = t_{0.01/2,\,8+10-2} = t_{0.005,\,(16)} = 2.92$



vi. Conclusion:

Our calculated value of t falls in the acceptance region, so we accept H_0.

Example 7.10

The following two samples are drawn from the normally distributed population with identical but unknown variances.

Sample 1: 70 68 63 60 59 57 53 51 50 49 46 45

Sample 2: 75 72 70 50 48 43 52 50 46 45

Test the equality of means at $\alpha = 0.05$ level of significance.

Solution:

i. $H_0: \mu_1 = \mu_2$

$H_1: \mu_1 \ne \mu_2$

ii. $\alpha = 0.05$

iii t-test statistic to be used here is

$$t=\frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \text{ with } v = n_1+ n_2 -2 \text{ degrees of freedom}$$

(iv) Calculation:

For sample-1:

$n_1 = 12$, $\sum x_1 = 671$, $\sum x_1^2 = 38275$, $\bar{X}_1 = 55.92$

$$s_1^2=\frac{1}{n_1-1}\left[\sum x_1^2-\frac{(\sum x_1)^2}{n_1}\right]$$

$$\text{or } (n_1-1)s_1^2=\sum x_1^2-\frac{(\sum x_1)^2}{n_1}=38275-\frac{(671)^2}{12}=754.9167$$

For sample-2:

$n_2 = 10$, $\sum x_2 = 551$, $\sum x_2^2 = 31707$, $\bar{X}_2 = 55.1$

$$s_2^2=\frac{1}{n_2-1}\left[\sum x_2^2-\frac{(\sum x_2)^2}{n_2}\right]$$

Or $(n_2-1)s_2^2=\sum x_2^2-\frac{(\sum x_2)^2}{n_2}=31707-\frac{(551)^2}{10}=1346.9$

Now

$$s_p=\sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}=\sqrt{\frac{754.9167+1346.9}{12+10-2}}$$

$$=\sqrt{\frac{2101.8167}{20}}=10.25$$

Hence $t=\frac{(55.92-55.1)-0}{10.25\sqrt{\frac{1}{12}+\frac{1}{10}}}=\frac{0.82}{4.3888}=0.187$

v. Critical region:

We will reject H_0 if $t<t_{\alpha/2,v}$ of $t>t_{\alpha/2,v}$ where as

$t_{\alpha/2,v}=t_{0.05/2,(12+10-2)}=t_{0.025,(20)}=2.086$

vi. Conclusion:

As our computed value of t lies in the acceptance region, therefore, we accept H_0.

7.2.6 Procedure of testing hypothesis about population pro ortion p [large sample]

For large n, ($n \geq 30$), and the hypothesis testing procedure is out lined as below:

i. (a) $H_0: p = p_0$ versus $H_1: p \neq p_0$

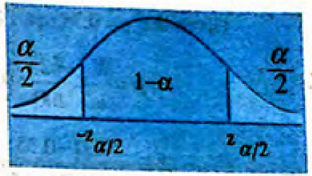
(b) $H_0: p \leq p_0$ versus $H_1: p > p_0$

(c) $H_0: p \geq p_0$ versus $H_1: p < p_0$

ii. Choose $\alpha=0.01$ or 0.05 etc.

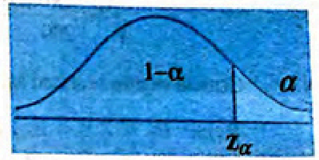
iii. Test statistic to be used in this case is

$$Z=\frac{\hat{p}-p}{\sqrt{\frac{pq}{n}}}\sim N(0,1) \text{ under } H_0 \text{ when } n\geq 30$$



iv. Calculation:

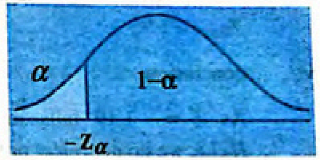
v. Critical region according to H_1 is stated as



(a) Reject H_0 if $Z\leq -Z_{\alpha/2}$ or $Z\geq Z_{\alpha/2}$

(b) Reject H_0 if $Z>Z_\alpha$

(c) Reject H_0 if $Z<-Z_\alpha$



vi. Conclusion.

Example 7.11

A producer of orange juice claims that 35 percent of all orange juice drinkers prefer its product. To test the claim, a random sample of 200 orange juice drinkers was taken at random and it was found that only 62 of them preferred the producer's brand. Test the producer's claim at five percent level of significance.

Solution:

i. $H_0: p=35\%=0.35$

$H_1: p\neq 0.35$

ii. $\alpha=0.05$

iii. Test statistic to be used here is $Z=\frac{\hat{p}-p}{\sqrt{\frac{pq}{n}}}$

iv. Calculation:

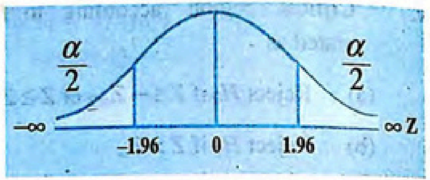
Given that p= 0.35, q= $1-p=1-0.35=0.65$, n = 200, X = 62,

$$\therefore \hat{p}=\frac{X}{n}=\frac{62}{200}=0.31,$$

$$Z=\frac{0.31-0.35}{\sqrt{\frac{(0.35)(0.65)}{200}}}=-1.19$$

v. Critical region is stated as:

Reject H_0 if $Z \le -Z_{\alpha/2}$ or $Z \ge Z_{\alpha/2}$ where $Z_{\alpha/2} = Z_{0.05/2} = Z_{0.025} = 1.96$



vi. conclusion:

As Z =− 1.19 falls in the acceptance region, therefore, we accept H_0 i.e. we do not reject producer's claim is correct.

7.2.7 Procedure for testing hypothesis about difference of two population proportions (p_1-p_2) {large sample]

The testing of hypothesis procedure in this case is given below.

i. (a) $H_0: p_1-p_2=0$ (b) $H_0: p_1-p_2\le 0$ (c) $H_0: p_1-p_2\ge 0$

$H_1: p_1-p_2\ne 0$ $H_1: p_1-p_2>0$ $H_1: p_1-p_2<0$

ii. Choose an appropriate level of significance.

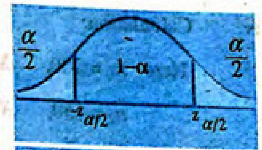
iii. Z-test is used where as

$$Z=\frac{(\hat{p}_1-\hat{p}_2)-(p_1-p_2)}{\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1}+\frac{\hat{p}_2\hat{q}_2}{n_2}}}\sim N(0,1) \text{ under } H_0.$$

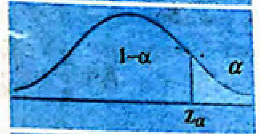
iv Calculation:

v. Critical region is constructed according to H_1 as;

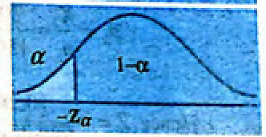
Case (a) Reject H_0 if $Z_c \le -Z_{\alpha/2}$ or $Z_c \ge Z_{\alpha/2}$ i.e.



Case (b) Reject H_0 if $Z_c \ge Z_\alpha$ i.e.



Case (c) Reject H_0 if $Z_c \le -Z_\alpha$ i.e.



v. Conclusion.

Example7.12

A soap-manufacturing factory produces two brands of soap. A sample survey was conducted and it was found that 56 users out of 200 preferred brand "A" and that 30 users out of 150 preferred brand "B". Test the hypothesis that sale of brand A is at least 10% greater than brand B at 5% level of significance.

Solution:

Let p_1 denote the proportion of brand-A users and p_2 denote the proportion of brand-B users.

i. $H_0: p_1-p_2\ge 0.10$

$H_1: p_1-p_2<0.10$

ii. $\alpha=0.05$

iii. Test statistic:

From H_0 we see that $p_1 \ne p_2$ and they are unknown therefore they are estimated by $\hat{p}_1$ and $\hat{p}_2$ respectively. The test statistic to be used is;

$$Z=\frac{(\hat{p}_1-\hat{p}_2)-(p_1-p_2)}{\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1}+\frac{\hat{p}_2\hat{q}_2}{n_2}}}\sim N(0,1) \text{ under } H_0.$$

iv. **Calculation:**

Here $n_1 = 200, \quad X_1 = 56$

$n_2 = 150, \quad X_2 = 30$

So

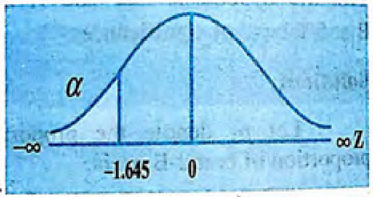
$$\hat{p}_1 = \frac{X_1}{n_1} = \frac{56}{200} = 0.28\ ,\ \hat{q}_1 = 1 - \hat{p}_1 = 1 - 0.28 = 0.72$$

$$\hat{p}_2 = \frac{X_2}{n_2} = \frac{30}{150} = 0.2\ ,\ \hat{q}_2 = 1 - \hat{p}_2 = 1 - 0.2 = 0.8$$

$$\text{Hence } Z = \frac{(0.28-0.2)-0.10}{\sqrt{\frac{(0.28)(0.72)}{200}+\frac{(0.2)(0.8)}{150}}} = \frac{-0.02}{0.0455} = -0.44$$

v. **Critical region:**

Reject H_0 if Z-calculated value is less than $-Z_\alpha = -1.645$



vi. **Decision:**

As computed value of Z falls in the acceptance region, so we accept H_0.

Example 7.13

The records of a hospital show that 52 men in a sample of 1000 men versus 23 women in a sample of 1000 women were admitted because of heart disease. Do these data present sufficient evidence to indicate a higher rate of heart disease among men admitted to the hospital? Use $\alpha = 0.05$.

Solution:

Let p_1 denote the proportion of men with heart disease and p_2 denote the proportion of women with heart disease.

i $H_0: p_1 \leq p_2$ or $p_1 - p_2 \leq 0$

$H_1: p_1 > p_2$ or $p_1 - p_2 > 0$

ii $\alpha = 0.05$

iii Test statistic:

From H_0 we see that $p_1 = p_2$ but unknown. So it is better to use a pooled estimate for the unknown value of the proportion. The estimated test-statistic to be employed here is $Z = \dfrac{(\hat{p}_1-\hat{p}_2)-(p_1-p_2)}{\sqrt{\hat{p}_c\hat{q}_c\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$

iv. **Calculation:**

Given $n_1 = 1000, X_1 = 52,\ \hat{p}_1 = \frac{X_1}{n_1} = \frac{52}{1000} = 0.052$,

$n_2 = 1000, X_2 = 23,\ \hat{p}_2 = \frac{X_2}{n_2} = \frac{23}{1000} = 0.023$

$$\text{Now } \hat{p}_c = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1+n_2} = \frac{52+23}{1000+1000} = \frac{75}{2000} = 0.0375$$

And $\hat{q}_c = 1 - \hat{p}_c = 1 - 0.0375 = 0.9625$

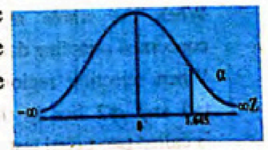
$$\text{Hence } Z = \frac{(0.052-0.023)-0}{\sqrt{(0.0375)(0.9625)\left(\frac{1}{1000}+\frac{1}{1000}\right)}} = 3.41$$

v. **Critical region:**

Reject H_0 if $Z > Z_\alpha$, whereas $Z_\alpha = Z_{0.05} = 1.645$

vi. **Conclusion:**

$Z = 3.41$ falls in the rejection region, so we reject H_0. This observed data indicates that the percentage of men entering the hospital because of heart disease is higher than that of women.



Key points

- Hypothesis is an unproved claim or assertion or assumption which acts as a starting point in a research irrespective of its probable truthfulness or falsity.
- A statistical hypothesis is a testable claim about one or more parameter(s) of empirical distributions.
- A statistical method that uses sample data to accept or reject a hypothesis about a parameter is called hypothesis testing.
- Null hypothesis is a claim about parameter. It is tested for possible rejection under the assumption it is true. It is denoted by H_0
- Alternative hypothesis is also a claim about parameter which is accepted if H_0 is rejected. It is tested for possible acceptance under the assumption it is false. It is denoted by H_1.
- When all parameters of a distribution are well specified, it is called simple hypothesis.
- When all parameters of a distribution are not well specified, it is called composite hypothesis.
- "Reject a null hypothesis when it is true" is called Type-I error.
- "Accept a null hypothesis when it is false" is called Type-II error.
- The probability of Type-I error which we are ready to tolerate in making decision about H_0 is called significance level
- A function of sample data is called statistic. When a statistic is used to test a hypothesis it is called test statistic or decision rule.
- The group of values which would lead us to acceptance of H_0 or the part where $(1-\alpha)$ lies is called acceptance region (AR) for the test.
- The group of values which would lead us to rejection of H_0 or the part where (α) lies is called rejection region (RR) or critical region for the test.
- The values which separate the (AR) and (RR) are called critical values.
- When the whole rejection region of size α lies on one tail of the concerned sampling distribution, it is called one-tailed test.
- When rejection region of size α is equally divided, $\alpha/2$ lies in the right tail and $\alpha/2$ lies in the left tail of the concerned sampling distribution, it is called two-tailed test.

Exercise

7.1 Write "T" for true and "F" for false in the following statement.

i. A statistical hypothesis is a statement about the value of a statistic.
ii. The null hypothesis is framed for possible rejection.
iii. In statistical inference "accept H_0" means that there is insufficient information to reject H_0.
iv. An upper-tailed test occurs when H_0: $\mu \geq \mu_0$ and H_1: $\mu < \mu_0$.
v. If a null hypothesis H_0: $\mu = 50$ is rejected at 1% level of significance, it will also be rejected a 5% level of significance.
vi. When a true null hypothesis has been rejected, we say that Type-I error might have been committed.
vii. α and β have an inverse relationship.
viii. H_0: $\mu < 5$ is a simple null hypothesis.
ix. When the sample size n increases the probability of rejecting a true hypothesis decreases.
x. Confidence interval estimate of a parameter is the same thing as testing of hypothesis about the population parameter.

7.2 Fill in the suitable word in the blanks.

i. A hypothesis is an ________ about the parameter of a population.
ii. The hypothesis which is under test for possible rejection is called ________ hypothesis.
iii. A hypothesis contrary to null hypothesis is known as ________.
iv. The hypothesis H_0: $\mu > \mu_0$ is a ________ hypothesis.
v. Type ________ error is more severe than type ________ error.
vi. Probability of Type-I error is called ________.
vii. Level of significance lies between ________ and ________.
viii. Critical region is also known as ________.
ix. A statistical test is a ________ to decide about H_0.
x. The number of independent values in a set of values is known as - ________.

7.3 Choose the correct answer.

i. A hypothesis under test is:
(a) Simple hypothesis (b) composite hypothesis
(c) Null hypothesis (d) alternative hypothesis

ii. Whether a test is one-sided or two-sided depends on:

(a) Alternative hypothesis (b)Composite hypothesis

(c) null hypothesis (d) Simple hypothesis

iii. A wrong decision about H_0 leads to:

(a) One kind of error (b) Two kind of error

(c) Three kind of error (d) Four kind of error.

iv. Level of significance is the probability of:

(a) Type-I error (b) Type-II error

(c) Not committing error (d) Any of the above

v. Degrees of freedom is related to:

(a) Number of observations in a set

(b) Hypothesis under test

(c) Number of independent observation in set

(d) confidence interval

vi. As compared to normal distribution, the t-distribution is:

(a) Flatter (b) More peaked

(c) Symmetric (d) Negatively skewed.

vii. Student's t-test is applicable in case of:

(a) Small samples

(b) Large samples

(c) For samples of size between 5 and 29.

(d) biased samples

viii. To test H_0: $\mu = \mu_0$ vs H_1: $\mu > \mu_0$ when the population S.D is known, the appropriate test is:

(a) *t*-test (b) Z-test

(c) Chi-square (d) *F*-test.

ix. Testing hypothesis H_0: $\mu = 10$ vs H_1: $\mu > 10$ leads to:

(a) One sided left tailed test (b) one sided right tailed test

(c) Two-tailed test (d) all of the above

x. Range of statistic-*t* is:

(a) -1 to $+1$, (b) $-\infty$ to ∞

(c) 0 to ∞ (d) 0 to 1

7.4 Define hypothesis, statistical hypothesis and testing of hypothesis?

7.5 Explain in your own words (i) Hypothesis testing (ii) Estimation. What are the principle steps involved in hypothesis testing procedure?

7.6 Differentiate between:

(i) Null and alternative hypothesis.

(ii) Simple and composite hypothesis.

(iii) Type-I and Type-II Errors.

(iv) Acceptance and rejection region.

7.7 Explain the following terms:

(a) Level of significance.

(b) Test statistic and test of significance.

(c) Small sample and large sample.

7.8 What do you mean by one-sided test and two sided test? Explain your idea with diagrams?

7.9 Describe the general procedure (steps) for testing hypothesis about mean of a normal population when population standard deviation is known and sample size is large.

7.10 A random sample of size n = 900 plants and its mean is computed which is equal to 34 cm. Can it be reasonably regarded as a random sample from a large population with mean 32 cm and standard deviation 23 cm. Testing at $\alpha = 0.05$.

7.11 Let $\overline{X} = 15$ be the mean of a random sample of 64 observations drawn from a normal population whose variance is 100. Test H_0: $\mu = 12$ versus H_1: $\mu > 12$ at 5% level of significance.

7.12 If $\overline{X} = 42.6$ is the mean of a random sample of size 36 taken from a normal population with a known standard deviation $\sigma = 5$. Test the null hypothesis $\mu = 45$ against the alternative $\mu < 45$ using $\alpha = 0.05$.

7.13 Test the hypothesis that population mean is 150 when $n = 196$, $\overline{X} = 160$, $S = 60$ using $\alpha = 0.05$.

7.14 A random sample of 49 college students showed an average IQ of $\overline{X} = 120.67$ and $S = 8.44$. Test the hypothesis that the average IQ of the college students is equal to 123 against the alternative that it is less. Test at 5% level of significance.

7.15 Explain hypothesis testing procedure about mean of a normal population when σ is unknown and sample size is small.

7.16 A random sample of size 10 is drawn from a normal population gives $\overline{X} = 20$ and $s^2 = 16$. Test the hypothesis H_0: $\mu = 19.6$ against H_1: $\mu > 19.6$ at $\alpha = 0.05$.

7.17 The nine items of a sample had the following values 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of the nine items differ significantly from an assumed population mean of 47.5? Test at 5% level of significance.

7.18 Describe the procedure for testing hypothesis about the equality of means of two normal populations when population standard deviations are known and sample sizes are large.

7.19 A random sample of size $n_1 = 25$ taken from a normal population with a standard deviation $\sigma_1 = 5.2$ has a mean $\overline{X}_1 = 81$. A second random sample of size $n_2 = 36$ taken from a different normal population with a standard deviation $\sigma_2 = 3.4$ has a mean $\overline{X}_2 = 76$. Test the hypothesis at the 0.05 level of significance that $\mu_1 - \mu_2 = 0$ against the alternative $\mu_1 - \mu_2 \neq 0$.

7.20 Independent random samples are drawn from two quantitative populations, 1 and 2 respectively. The sample data summary is shown here:

	sample-1	sample-2
sample size	36	45
sample mean	1.24	1.31
sample variance	0.0560	0.0540

Do the data present sufficient evidence to indicate that the mean for population-1 is smaller than the mean for population-2? Use $\alpha = 0.05$

7.21 Write the steps used in hypothesis testing about equality of means of two normal populations when $\sigma_1^2 = \sigma_2^2 = \sigma^2$ but unknown and sample sizes n_1, n_2 are small.

7.22 Test the hypothesis that the mean number of kilometers per liter is the same for foreign and domestic auto-mobiles on the basis of the following summary of sample data:

	n	$\overline{x}$	s^2
Foreign automobiles	8	36.5	5.29
Domestic automobiles	10	32.4	7.84

Test at 5% level of significance.

7.23 Given the following sample observations

X_1	17	27	18	25	27	29	27	23	17
X_2	16	16	20	16	21	17	15	20	

Examine the significance of difference between the two population means at $\alpha = 0.05$

7.24 Explain the general procedure for testing hypothesis about population proportion P for a large sample.

7.25 A random sample of 120 observations was selected from a binomial population, and 72 successes were observed. Do the data provide sufficient evidence to indicate that p is greater than 0.5? Use $\alpha = 0.05$.