Unit - 02

Random Variables and Probability Distributions

After studying this unit, the students will be able to

- Define random variable and differentiate between discrete and continuous random variables with real life examples.
- Describe probability distribution of a discrete random variable
- Find probability distribution of a discrete random variable
- Recognize probability mass function and its properties
- Describe and find the probability distribution of a function of discrete random variable.
- Define and find the expected value of a discrete random variable.
- Find the expected value of a linear function of a discrete random variable.
- Describe and verify properties of expected value of a discrete random variable.
- Apply the properties of expected value of a discrete random variable.
- Define variance and standard deviation of a discrete random variable
- Find mean, variance and standard deviation of a discrete random variable.
- Define and find variance and standard deviation of a linear function of a discrete random variable.
- Describe, verify and apply the properties of variance and standard deviation of a discrete random variable.
- Define probability distribution and probability density function of a continuous random variable.
- Define and expected value, variance and standard deviation of a continuous random variable.
- Describe the properties about the expected value and variance for the sum/difference of two independent random variable x and y also apply these properties to solve real life problems.

2.1 Random variable

Probability distributions, which shall be studied in the coming units, are very easy methods for calculation of probabilities in respective situations but they need quantification of the outcomes of a random experiment. For this purpose, sample space of a random experiment is expressed in numerical form according to a characteristic of interest. This numerical presentation of the sample space is termed as random variable (r.v). It is denoted by English letters X, Y, Z or $X_1, X_2, X_3\ldots$

2.1.1 Definition of random variable

A random variable is a numerical description of a random experiment.

2.1.2 Types of random variable

Random variable can be classified as follow:

(i) Discrete random variable

A variable which takes jumping values or isolated values is called discrete random variable. For example, number of rotten tomatoes in a crate, number of children per house in a street etc. Its probability distribution is called discrete probability distribution.

(ii) Continuous random variable

A variable which takes any value between two limits, $[a, b]$, $a<b$, is called continuous random variable. For example, life of a mobile set, speed of a car etc. It is written as $a \leq X \leq b$. Its probability distribution is called continuous probability distribution.

2.2 Probability distribution of a discrete random variable

If all possible values of a random variable along with their respective probabilities are shown in tabular form and sum of probabilities is equal to one, then it is called probability distribution. For example, discrete probability distribution of X is presented as

X	p(x)
x_1 x_2 . . . x_n	$p(x_1)$ $p(x_2)$ . . . $p(x_n)$
Total	$\sum_{i=1}^{n} p(x_i)=1$

Example 2.1

Three children were born in a government hospital on Sunday. Find the probability distribution for the number of girls.

Solution:

Here $n(S) = 2^n = 2^3 = 8$

S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}

X: the number of girls

= {0, 1, 1, 2, 1, 2, 2, 3}

Probability distribution for X

X	p(x)
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$
Total	1

Example 2.2

A fair die is rolled once. Find the probability distribution for up turned faces.

Solution:

When a die is rolled we may get 1 or 2 or 3 or 4 or 5 or 6 each with same probability 1/6 of its occurrence because the die is fair. It can be shown in tabular form, called probability distribution as follows:

X	1	2	3	4	5	6	Total
p(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

2.2.1 Probability mass function

If probability distribution of a random variable X is expressed in a mathematical form or formula, then it is called probability mass function or probability function. It is denoted by $p(x_i)$ and is presented as:

$$p(x_i)=\begin{cases} p(X=x_i), & i=1, 2, ..., n \\ 0 & otherwise \end{cases}$$

Remember that the objective of probability distribution and probability function is same. Some writers make no distinction and they use probability distribution and probability function interchangeably.

2.2.2 Properties of discrete probability distribution

A probability distribution and probability mass function $p(x_i)$ must satisfy the following two properties:

(i) $0 \le p(x_i) \le 1$

(ii) $\sum_{i=1}^{n} p(x_i)=1$

First property means that answer of probability must be within the range 0 to 1 (inclusive) and second property means that the sum of probabilities for all possible values of a random variable must be equal to one.

Example 2.3

A discrete random variable has the probability function

$$p(x)=\begin{cases} {}^{3}C_{x}\left(\frac{1}{2}\right)^{3} & for \quad x=0,1,2,3 \\ 0 & otherwise \end{cases}$$

i) Compute probabilities for all values of X.
ii) Check that this is a probability mass function.
iii) Find the probability distribution of X.

Solution:

i) Given $p(x) = {}^{3}C_{x}\left(\frac{1}{2}\right)^{3}, x=0,1,2,3$

Putting values of x, we get

$$p(0) = {}^{3}C_{0}\left(\frac{1}{2}\right)^{3} = (1)\left(\frac{1}{8}\right) = \frac{1}{8}$$

$$p(1) = {}^{3}C_{1}\left(\frac{1}{2}\right)^{3} = (3)\left(\frac{1}{8}\right) = \frac{3}{8}$$

$$p(2) = {}^{3}C_{2}\left(\frac{1}{2}\right)^{3} = (3)\left(\frac{1}{8}\right) = \frac{3}{8}$$

$$p(3) = {}^{3}C_{3}\left(\frac{1}{2}\right)^{3} = (1)\left(\frac{1}{8}\right) = \frac{1}{8}$$

ii) All probabilities are lying between 0 and 1, and

$$\sum_{x=0}^{3} p(x_i) = p(0)+p(1)+p(2)+p(3) = \frac{1}{8}+\frac{3}{8}+\frac{3}{8}+\frac{1}{8} = 1$$

Since both properties are satisfied, therefore, the given formula is a probability mass function or probability function.

iii) The probability distribution of X

X	0	1	2	3	Total
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

Example 2.4

Suppose a discrete probability distribution of random variable X is given in the following table:

X	-1	0	1
$p(x)$	$3c$	$3c$	$6c$

Find (i) the value of c (ii) $P(X=0)$ (iii) $P(X<0)$ (iv) $P(X \geq -1)$

Solution:

(i) As $\sum_{x=-1}^{1} p(x) = 1$

or $p(-1)+p(0)+p(1) = 1$

$$3c + 3c + 6c = 1$$

$$12c = 1 \qquad \Rightarrow c = \frac{1}{12}$$

Putting the value of c in the given probability distribution, we get

X	-1	0	1	Total
p(x)	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{6}{12}$	1

(ii) We see that $P(X=0)=\frac{3}{12}$

(iii) $P(X<0)=P(X=-1)=\frac{3}{12}$

(iv) $P(X\geq -1)=P(X=-1)+P(X=0)+P(X=1)$

$$=\frac{3}{12}+\frac{3}{12}+\frac{6}{12}=\frac{12}{12}=1$$

Example 2.5

Find the value of k so that the function can serve as a probability function of the random variable Y

$p(y)=\begin{cases} ky \;, & y=1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases}$. Also find (i) $P(Y=5)$ (ii) $P(Y>3)$

Solution:

As we know that

$$\sum_{y=1}^{5} p(y)=1$$

$$\sum_{y=1}^{5} ky=1$$

$$k\sum_{y=1}^{5} y=1$$

$$k[1+2+3+4+5]=1$$

$$k(15)=1 \qquad \Rightarrow k=\frac{1}{15}$$

Put value of k in the given function we get

$$p(y)=\begin{cases} \frac{1}{15}y \;, & y=1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases}$$

(i) Putting $Y=5$ in the formula, we have $P(Y=5)=\frac{5}{15}$

(ii) $P(Y>3)=P(Y=4)+P(Y=5)=\frac{4}{15}+\frac{5}{15}=\frac{9}{15}$

Example 2.6

A pair of fair dice is rolled. Find (i) the probability distribution for the sum of dots. (ii) Using the probability distribution, compute the probabilities of (a) sum of dots is equal to 7 (b) sum of dots is less than 6 (c) sum of dots is greater than or equal to 2 but less than 5 (d) sum of dots is 5 or 10.

Solution:

(i) $n(S)=6^n=6^2=36$

The sample space for the experiment is:

$$S=\begin{Bmatrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{Bmatrix}$$

Let X is a random variable denoting the sum of dots on the upper faces of the two dice. Its probability distribution is:

X	2	3	4	5	6	7	8	9	10	11	12	Total
p(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1

This distribution can also be shown by the formula called probability function as

$$p(x)=\begin{cases}\dfrac{6-|7-x|}{36}, & x=2,3,4,\dots 12\\ 0 & \text{otherwise}\end{cases}$$

(ii) Now using the probability distribution, we get:

a) $P(X=7)=\frac{6}{36}$

b) $P(X<6)=P(X=5)+P(X=4)+P(X=3)+P(X=2)$

$$=\frac{4}{36}+\frac{3}{36}+\frac{2}{36}+\frac{1}{36}=\frac{10}{36}.$$

c) $P(2\le X<5)=P(X=2)+P(X=3)+P(X=4)$

$$=\frac{1}{36}+\frac{2}{36}+\frac{3}{36}=\frac{6}{36}$$

d) $P(X=5 \text{ or } 10)=P(X=5)+P(X=10),$

$$=\frac{4}{36}+\frac{3}{36}=\frac{7}{36}$$

2.2.3 Probability distribution of a function of discrete random variable

If X is a random variable, then its functions like $X^2, \frac{1}{X}$, aX + b etc. are also random variables and thus have probability distributions. In case of two random variables, say X and Y, their functions like $X+Y$, $X-Y$, aX+bY etc. are also random variables where a and b are any two non-zero constants. The function of a random variable is usually denoted by H(X).

Example 2.7

If a discrete random variable X has the following probability distribution.

X	−2	2	1
p(x)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Find probability distribution for X^2, $2X+4$.

Solution:

The probability distribution of the random variable X^2 is

$H(X)=X^2$	$p(x)$
4	$\frac{1}{3}$
4	$\frac{1}{2}$
1	$\frac{1}{6}$
Total	1

The probability distribution of the random variable $2X + 4$ is

$H(X) = 2X+4$	$p(x)$
0	$\frac{1}{3}$
8	$\frac{1}{2}$
6	$\frac{1}{6}$
Total	1

2.2.4 Mathematical expectation of a random variable

Hope you have understood random variable and its presentation methods like probability distribution and probability function. Now we want to study its properties like mean, variance and standard deviation etc.

2.2.5 Definition of mathematical expectation or expected value of a random variable

Mathematical expectation or expected value of a random variable is defined as "the mean of a random variable over a very large number of trails". If X is a discrete random variable having the following probability distribution

X	$p(x)$	$X\,p(x)$
x_1	$p(x_1)$	$x_1\,p(x_1)$
x_2	$p(x_2)$	$x_2\,p(x_2)$
.	.	.
.	.	.
.	.	.
x_n	$p(x_n)$	$x_n\,p(x_n)$
Total	1	$\sum_{i=1}^{n} x_i p(x_i)$

The expected value of X is denoted by $E(X)$ and its formula is given by $E(X)=\sum_{i=1}^{n} x_i p(x_i)$, provided it exists.

Example 2.8

A random variable X has the following probability distribution

X	0	1	2	3
$p(x)$	1/7	3/7	2/7	1/7

Find the mean or expected value of X, i.e. $E(X)$.

Solution:

X	$p(x)$	$X\,p(x)$
0	1/7	0
1	3/7	3/7
2	2/7	4/7
3	1/7	3/7
Total	1	10/7

$$\text{Mean} = E(X) = \sum_{x=0}^{3} x_i\, p(x_i) = \frac{10}{7} = 1.43$$

Example 2.9

A discrete random variable can have the values $x_1 = 3$, $x_2 = 8$, and $x_3 = 10$ and the respective probabilities are 0.2, 0.7 and 0.1. Determine the mean.

Solution:

By definition $E(X) = \sum_{i=1}^{n} x_i p(x_i)$

$$= x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3)$$

$$= 3(0.2) + 8(0.7) + 10(0.1)$$

$$E(X) = 0.6 + 5.6 + 1 = 7.2$$

2.2.6 Mathematical expectation of a function of discrete random variable

If function of a random variable is denoted by $H(X)$, then expected value of the function is denoted by $E[H(X)]$ and its formula is given by $E[H(X)] = \Sigma H(x_i)p(x_i)$.

Example 2.10

A discrete random variable X has the probability distribution:

X	-2	1	3
$p(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$

Find (i) $E(X^2)$ (ii) $E(2X+5)$

Solution:

Since X^2 and $(2X+5)$ are functions of the given random variable X, therefore, we first find their probability distributions and then the means as follows:

X	$p(x)$	$H(X)=X^2$	$H(X)=2X+5$	$X^2p(x)$	$(2X+5)p(x)$
-2	$\frac{1}{3}$	4	1	$\frac{4}{3} = 1.33$	$\frac{1}{3} = 0.33$
1	$\frac{1}{6}$	1	7	$\frac{1}{6} = 0.17$	$\frac{7}{6} = 1.17$
3	$\frac{1}{2}$	9	11	$\frac{9}{2} = 4.5$	$\frac{11}{2} = 5.5$
Total	1	-	-	6	7

Now $\quad E\{H(X)\} = \Sigma H(x_i)\, p(x_i)$

(i) $\quad E\{X^2\} = \Sigma x_i^2\, p(x_i) = 6$

(ii) $\quad E\{2X+5\} = \Sigma(2x_i+5)\, p(x_i) = 7$

Example 2.11

The probability function of a discrete random variable Y is given by

$$p(y) = \begin{cases} \overset{3}{C}_{y}\left(\frac{1}{2}\right)^{y}\left(\frac{1}{2}\right)^{3-y}, & y = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(Y)$ and $E(Y^2)$.

Solution:

By definition

$$E(Y) = \sum_{y=0}^{3} y_i\, p(y_i)$$

$$= \sum_{y=0}^{3} y \overset{3}{C}_{y}\left(\frac{1}{2}\right)^{y}\left(\frac{1}{2}\right)^{3-y}$$

$$= 0\overset{3}{C}_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{3-0} + 1\overset{3}{C}_{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{3-1} + 2\overset{3}{C}_{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{3-2} + 3\overset{3}{C}_{3}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{3-3}$$

$$= 0 + 3\left(\frac{1}{8}\right) + 6\left(\frac{1}{8}\right) + 3\left(\frac{1}{8}\right) = \frac{12}{8} = 1.5$$

$$E(Y^2) = \sum_{y=0}^{3} y^2 \overset{3}{C}_{y}\left(\frac{1}{2}\right)^{y}\left(\frac{1}{2}\right)^{3-y}$$

$$= 0\overset{3}{C}_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{3-0} + 1\overset{3}{C}_{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{3-1} + 4\overset{3}{C}_{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{3-2} + 9\overset{3}{C}_{3}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{3-3}$$

$$= 0 + 3\left(\frac{1}{8}\right) + 12\left(\frac{1}{8}\right) + 9\left(\frac{1}{8}\right) = \frac{24}{8} = 3$$

2.2.7 Properties of mathematical expectation

Expected value of a random variable satisfies the following properties.

(i) Expectation of a constant is the constant itself i.e. $E(c) = c$, where c is a constant.

Proof:

By definition

$$E(X) = \Sigma x\, p(x)$$

As the variable X is taking only a constant value c therefore

$$E(c) = \Sigma c\, p(x)$$

$$= c\Sigma p(x)$$

$$= c\,(1), \text{ as sum of probabilities} = \Sigma p(x) = 1$$

$$= c$$

(ii) If a and b are two constants, then $E[aX + b] = a\,E(X) + b$

Proof:

By definition

$$E(X) = \Sigma x\, p(x)$$

Here we have a function of the random variable X i.e. $a\,X + b$, so

$$E(aX + b) = \Sigma(ax + b)\, p(x)$$

$$= \Sigma ax\, p(x) + \Sigma b\, p(x)$$

$$= a\Sigma x\, p(x) + b\Sigma p(x)$$

$$= a\,E(X) + b\,(1)$$

$$= a\,E(X) + b$$

This property shows that expectation is changed by that constant which is added to, subtracted from, multiplied with or divided by the values of a variable i.e.

(i) $E[X + c] = E(X) + c$ (ii) $E[X - c] = E(X) - c$

(iii) $E(cX) = c\,E(X)$ (iv) $E\left(\frac{X}{c}\right) = \frac{E(X)}{c}$

where X is a random variable and c is any constant.

Example 2.12

Suppose that X is a simple discrete random variable, distributed as follows:

X	$p(x)$
2	0.50
4	0.50

Find (i) $E(X)$ (ii) $E(X+10)$ (iii) $E(X - 10)$ (iv) $E(10X)$ (v) $E\left(\frac{X}{10}\right)$

Solution:

Given

X	$p(x)$	$X\,p(x)$
2	0.50	1
4	0.50	2
Total	1	3

(i) By definition $E(X) = \Sigma x\, p(x) = 3$

(ii) By property $E(X+10) = E(X) + 10 = 3 + 10 = 13$

(iii) $E(X-10) = E(X) - 10 = 3 - 10 = -7$

(iv) $E(10X) = 10E(X) = 10(3) = 30$

(v) $E\left(\frac{X}{10}\right) = \frac{E(X)}{10} = \frac{3}{10} = 0.3$

2.2.8 Variance and standard deviation of a random variable

If X is a random variable, then its variance is denoted by $Var(X)$ or $V(X)$ and its formula is given by

$$V(X) = E[X - E(X)]^2$$

Or $V(X) = E[X^2] - [E(X)]^2$

$$S.D(X) = \sqrt{V(X)}$$

Example 2.13

Consider the following discrete probability distribution:

X	:	3	8	10
$p(x)$	:	0.2	0.7	0.1

Determine the mean, variance and standard deviation.

Solution:

X	$p(x)$	$X\,p(x)$	$X^2 p(x)$
3	0.2	0.6	1.8
8	0.7	5.6	44.8
10	0.1	1	10.0
Total	1	7.2	56.6

Mean $= E(X) = \Sigma x\, p(x) = 7.2$

Variance $= V(X) = E(X^2) - [E(X)]^2 = \Sigma x^2 p(x) - [7.2]^2 = 56.6 - 51.84 = 4.76$

and $S.D(X) = \sqrt{V(X)} = \sqrt{4.76} = 2.181$

2.2.9 Properties of variance and standard deviation of a random variable

Following are some of the important properties.

(i) The variance and standard deviation of a constant is equal to zero i.e. $V(c) = 0$ and $S.D(c) = 0$, where c is a constant.

Proof:

By definition $V(X) = E[X - E(X)]^2$

As the variable X is taking only a constant value c, therefore

$$V(c) = E[c - E(c)]^2$$
$$= E[c - c]^2 \quad \because E(c) = c$$
$$= 0$$
$$S.D(c) = \sqrt{0} = 0$$

(ii) Variance and S.D of a random variable are not changed by adding/ subtracting a constant to/from the values of a random variable i.e.

$$V(X \pm c) = V(X)$$

Proof:

By definition $V(X) = E[X - E(X)]^2$

As $(X + c)$ is a function of the random variable X, therefore,

$$V(X+c) = E[X + c - E(X+c)]^2$$
$$= E[X + c - E(X) - E(c)]^2$$

$= E[X + c - E(X) - c]^2 \because E(c) = c$

$= E[X - E(X)]^2$

$= V(X)$

Similarly $V(X - c) = V(X)$

$S.D(X \pm c) = S.D(X)$

(iii) If the values of a random variable are multiplied or divided by a constant, then its variance will change by the square of that constant i.e.

$$V(cX) = c^2 V(X)$$

$$V\left(\frac{X}{c}\right) = \frac{V(X)}{c^2}$$

Proof:

By definition $V(X) = E[X - E(X)]^2$

As (cX) is function of the random variable X so,

$V(cX) = E[cX - E(cX)]^2$

$= E[cX - cE(X)]^2$

$= c^2 E[X - E(X)]^2$

$= c^2 V(X)$

Similarly $V\left(\frac{X}{c}\right) = \frac{V(X)}{c^2}$

$S.D(cX) = |c|\, S.D(X)$

$S.D\left(\frac{X}{c}\right) = \frac{S.D(X)}{|c|}$

(iv) Variance and standard deviation can never be negative i.e. $V(X) \geq 0$ and $S.D(X) \geq 0$

Example 2.14

A random variable X has the probability distribution given below

X:	0	1	2	3
$p(x)$:	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{2}{10}$	$\frac{1}{10}$

Find i) $E(X)$, ii) $V(X)$, iii) $V(X+5)$, iv) $V(X-3)$, v) $V(3X)$ vi) $S.D\left(\frac{X}{3}\right)$

Solution:

X	$p(x)$	$X\,p(x)$	$X^2\,p(x)$
0	3/10	0	0
1	4/10	4/10	4/10
2	2/10	4/10	8/10
3	1/10	3/10	9/10
Total	1	11/10	21/10

(i) $E(X) = \Sigma x\, p(x) = \frac{11}{10} = 1.1$

(ii) $V(X) = E(X^2) - [E(X)]^2 = \Sigma x^2 p(x) - (1.1)^2$

$= \frac{21}{10} - 1.21$

$= 2.1 - 1.21 = 0.89$

(iii) By property of variance $V(X+5) = V(X) = 0.89$

(iv) $V(X-3) = V(X) = 0.89$

(v) $V(3X) = 3^2 V(X) = 9(0.89) = 8.01$

(vi) S.D $\frac{X}{3} = \frac{S.D(X)}{131}$ $\because$ S.D(x) = $\sqrt{0.89} = 0.94$

$= \frac{0.94}{3} = 0.314$

2.3 Probability distribution of a continuous random variable

All possible values of a continuous random variable along with probability cannot be presented in tabular form. This purpose is only achieved by formula, called probability density function of the continuous random variable.

2.3.1 Probability density function

If probability distribution of a continuous random variable is expressed by a formula, then it is called probability density function (pdf) or simply density function and is denoted by $f(x)$.

2.3.2 Properties of probability density function

A pdf must satisfy the following important properties:

(i) $f(x) \geq 0, \; a \leq x \leq b$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Example: 2.15

A continuous random variable X has the probability function given by

$$f(x) = \begin{cases} cx, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

i. Find value of the constant c.
ii. Check that the function is a p.d.f

iii. Compute

(a) $P\left[\frac{1}{4} < X < \frac{1}{2}\right]$ (b) $P\left(X > \frac{1}{2}\right)$ (c) $P\left(X = \frac{1}{2}\right)$

Solution:

(i) We know that sum of all probabilities is equal to one i.e.

$$\int_0^1 f(x).dx = 1$$

$$\int_0^1 cx\,dx = 1$$

$$c\int_0^1 x dx = 1$$

$$c\left[\frac{x2}{2}\right]_0^1 = 1$$

$$c\left(\frac{1}{2} - 0\right) = 1$$

$$\frac{c}{2} = 1 \quad \Rightarrow c = 2$$

Put the value of c in the given function, we have

$$f(x) = \begin{Bmatrix} 2x & 0 < x < 1 \\ 0 & otherwise \end{Bmatrix}$$

(ii) a) $f(x) \geq 0$ for the given range

b) $\int_x f(x)dx = 1$

Taking L.H.S

$$\int_x f(x)dx = \int_0^1 2x\,dx = 2\int_0^1 x\,dx = 2\left.\frac{x^2}{2}\right|_0^1 = \left.x^2\right|_0^1 = 1 - 0 = 1 = R.H.S$$

Since both properties are satisfied, therefore, the given function is a p.d.f.

(iii) (a) $P\left(\frac{1}{4}<X<\frac{1}{2}\right)=\int_{1/4}^{1/2} f(x)\,dx=\int_{1/4}^{1/2} 2x\,dx=2\int_{1/4}^{1/2} x\,dx$

$$=2\left(\frac{x^2}{2}\right)\Big|_{1/4}^{1/2}=x^2\Big|_{1/4}^{1/2}=\frac{1}{4}-\frac{1}{16}=\frac{4-1}{16}=\frac{3}{16}$$

(b) $P\left(X>\frac{1}{2}\right)=\int_{\frac{1}{2}}^{1} f(x)\,dx=\int_{\frac{1}{2}}^{1} 2x\,dx=2\int_{\frac{1}{2}}^{1} x\,dx=2\frac{x^2}{2}\Big|_{1/2}^{1}$

$$=x^2\Big|_{1/2}^{1}=1-\frac{1}{4}=\frac{4-1}{4}=\frac{3}{4}$$

(c) $P\left(X=\frac{1}{2}\right)=$ 0, because in continuous case point probability is always equal to zero.

2.3.3 Expectation, variance and standard deviation of a continuous random variable

If X is a continuous random variable, then expected value of X is given by $E(X)=\int_x x f(x)\,dx$, provided that it exists.

Example 2.16

Let X be a continuous random variable with pdf given by

$$f(x)=\begin{cases}\frac{x}{2} & 0\le x\le 2\\ 0 & otherwise\end{cases}.$$

Find the mean, variance and standard deviation of X.

Solution:

(i) By definition

$$\text{Mean}=E(X)=\int_x xf(x)dx$$

$$=\int_0^2 x\left(\frac{x}{2}\right)dx=\frac{1}{2}\int_0^2 x^2dx=\frac{1}{2}\left(\frac{x^3}{3}\right|_0^2$$

$$=\left(\frac{x^3}{6}\right|_0^2=\frac{8}{6}-0=\frac{4}{3}=1.333$$

(ii) By definition

$$V(X)=E(X^2)-[E(X)]^2$$

$$=\int_0^2 x^2f(x)\,dx-\left(\frac{4}{3}\right)^2$$

$$=\int_0^2 x^2\left(\frac{x}{2}\right)dx-\frac{16}{9}$$

$$=\int_0^2\frac{x^3}{2}dx-\frac{16}{9}=\frac{x^4}{8}\Big|_0^2-\frac{16}{9}=\left(\frac{16}{8}-0\right)-\frac{16}{9}$$

$$=2-1.78=0.22$$

(iii) $S.D(X)=\sqrt{0.22}=0.47$

Example 2.17

A random variable X has the density function given by

$$f(x)=\begin{cases}(2-2x), & 0\le x\le 1\\ 0 & otherwise\end{cases}$$

Find (i) $E(X)$ (ii) $E(X^2)$ (iii) $E(2X)$ (iv) $E(2X-1)$

(v) variance and standard deviation of X

Solution:

(i) $E(X) = \int_x x f(x)\, dx = \int_0^1 x\,(2-2x)\, dx = \int_0^1 (2x-2x^2)\, dx$

$$= \int_0^1 2x\, dx - \int_0^1 2x^2\, dx = 2\int_0^1 x\, d_x - 2\int_0^1 x^2 dx$$

$$= x^2\Big|_0^1 - 2\,\frac{x^3}{3}\Big|_0^1$$

$$= (1-0) - \left(\frac{2}{3} - 0\right) = 1 - \frac{2}{3} = \frac{3-2}{3} = \frac{1}{3}$$

(ii) $E(X^2) = \int_x x^2 f(x)\, dx = \int_0^1 x^2(2-2x)\, dx = \int_0^1 (2x^2-2x^3)\, dx$

$$= 2\int_0^1 x^2 dx - 2\int_0^1 x^3 dx = 2\left(\frac{x^3}{3}\Big|_0^1\right) - 2\left(\frac{x^4}{4}\Big|_0^1\right)$$

$$= 2\left(\frac{1}{3} - 0\right) - 2\left(\frac{1}{4} - 0\right) = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$$

(iii) $E(2X) = 2E(X)$, (By property)

$= 2(1/3) = 2/3$

(iv) $E(2X-1) = 2E(X) - 1 = 2(\frac{1}{3}) - 1 = -1/3$

(v) Variance $= V(X) = E(X^2) - [E(X)]^2$

$$= \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9} = \frac{3-2}{18} = \frac{1}{18} = 0.056$$

$\text{S.D}(X) = \sqrt{0.056} = 0.24$

2.4 Two independent random variables

This is an extension of the one random variable case. The problem will be to recognize the old ideas behind the new names.

2.4.1 Joint distributions

The distribution of two or more random variables is called joint distribution. Specifically;

The distribution of one random variable is called univariate probability distribution.

The distribution of two random variables is called bivariate probability distribution.

The distribution of three random variables is called trivariate probability distribution.

The distribution of many random variables is called multivariate probability distribution.

2.4.2 Bivariate probability distribution

If all possible values of two random variables along with their joint probabilities are presented in tabular form then, it is called bivariate probability distribution. Suppose X and Y are two discrete random variables, where X has "m" values and Y has "n" values, then bivariate probability distribution of X and Y is given by

$X \backslash Y$	y_1	y_2	...	y_j	...	y_n	$p(x)$
x_1	$p(x_1,y_1)$	$p(x_1,y_2)$	...	$p(x_1,y_j)$	...	$p(x_1,y_n)$	$p(x_1)$
x_2	$p(x_2,y_1)$	$p(x_2,y_2)$	...	$p(x_2,y_j)$	...	$p(x_2,y_n)$	$p(x_2)$
$\vdots$				$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_i	$p(x_i,y_1)$	$p(x_i,y_2)$	...	$p(x_i,y_j)$	...	$p(x_i,y_n)$	$p(x_i)$
$\vdots$				$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	$p(x_m,y_1)$	$p(x_m,y_2)$	...	$p(x_m,y_j)$	...	$p(x_m,y_n)$	$p(x_m)$
$p(y)$	$p(y_1)$	$p(y_2)$	...	$p(y_j)$	...	$p(y_n)$	1

Marginal probability distributions

From the bivariate probability distribution we can obtain univariate probability distributions as:

(i) The marginal probability distribution of X

X	$p(x)$
x_1	$p(x_1)$
x_2	$p(x_2)$
⋮	⋮
x_i	$p(x_i)$
⋮	⋮
x_m	$p(x_m)$
Total	1

(ii) The marginal probability distribution of Y

Y	$p(y)$
y_1	$p(y_1)$
y_2	$p(y_2)$
⋮	⋮
y_j	$p(y_j)$
⋮	⋮
y_n	$p(y_n)$
Total	1

2.4.3 Bivariate probability function

If all possible values of two random variables with their associated probabilities are shown by a formula, then it is called bivariate probability function. It is denoted by $p(x_i, y_j)$ (discrete case) and $f(x_i, y_j)$ (continuous case) ; for all x_i and y_j.

2.4.4 Independence of two random variables

Two random variables X and Y are statistically independent if and only if their bivariate probability function can be expressed as the product of the marginal probability functions i.e.

$$f(x_i, y_j) = f(x_i)\, f(y_j)$$

2.4.5 Mathematical expectation of two random variables

If X and Y are two random variables then mean or expected value of their sum or difference is denoted by $E(X \pm Y)$ and is defined as:

$$E(X \pm Y) = \sum_i \sum_j (x_i \pm y_j)\, p(x_i, y_j) \quad \text{(discrete random variables case)}$$

$$= \int_x \int_y (x_i \pm y_j)\, f(x_i, y_j)\, dx_i\, dy_j \quad \text{(continuous random variables case)}$$

Expected value of their product is

$$E(XY) = \sum_j \sum_i (x_i y_j)\, p(x_i, y_j) \quad \text{(for discrete case)}$$

$$= \int_x \int_y (x_i y_j)\, f(x_i, y_j)\, dx_i\, dy_j \quad \text{(for continuous case)}$$

2.4.6 Properties of mathematical expectation of two random variables

Let X and Y are two independent random variables and then they jointly have the following properties:

(i) The expected value of the sum of two random variables is equal to the sum of the expected values of the individual random variables i.e.

$$E(X+Y) = E(X) + E(Y)$$

Proof:

By definition

$$E(X+Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j)\, p(x_i, y_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i\, p(x_i, y_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} y_j\, p(x_i, y_j) \qquad (1)$$

Consider $\sum_{i=1}^{m} \sum_{j=1}^{n} x_i\, p(x_i, y_j)$

$$= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} p(x_i, y_j)$$

$= \sum_{i=1}^{m} x_i [p(x_i, y_1) + p(x_i, y_2) + \dots + p(x_i, y_n)]$

$= \sum_{i=1}^{m} x_i [p(x_i)\, p(y_1) + p(x_i)\, p(y_2) + \dots + p(x_i)\, p(y_n)]$ (as X and Y are independent)

$= \sum_{i=1}^{m} x_i\, p(x_i)\, [p(y_1) + p(y_2) + \dots + p(y_n)]$

$= \sum_{i=1}^{m} x_i\, p(x_i)\, (1)$ (as sum of probabilities is equal to one.)

$= \sum_{i=1}^{m} x_i\, p(x_i)$

$= E(X)$ (2)

Again consider $\sum_{i=1}^{m}\sum_{j=1}^{n} y_j\, p(x_i, y_j)$

$= \sum_{i=1}^{n} y_j \sum_{j=1}^{m} p(x_i, y_j)$

$= \sum_{j=1}^{n} y_j [p(x_i, y_j) + p\, x_2, y_j) + \dots + p(x_m, y_j)]$

$= \sum_{j=1}^{n} y_j [p(x_1)\, p(y_j) + p(x_2)\, p(y_j) + \dots + p(x_m) p(y_j)]$

$= \sum_{j=1}^{n} y_j\, p(y_j)\, [p(x_1) + (p\, x_2) + \dots + p(x_m)]$

$= \sum_{j=1}^{n} y_j p(y_j)\, (1)$

$= \sum_{j=1}^{n} y_j\, p(y_j)$

$= E(Y)$ (3)

Put equation (2) and (3) in equation (1), we get $E(X+Y) = E(X) + E(Y)$

(ii) The expected value of the product of two independent random variables is equal to the product of their individual expected values i.e. $E(X\,Y) = E(X)\, E(Y)$

Proof:

By definition

$E(X\,Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} (x_i, y_j)\, p(x_i, y_j)$

$= \sum_{i=1}^{m}\sum_{j=1}^{n} x_i\, y_j\, p(x_i)\, p(y_j)$ (as X and Y are independent)

$= \sum_{i=1}^{m} x_i\, p(x_i) \sum_{j=1}^{n} y_j\, p(y_j)$

$= E(X)\, E(Y)$

Example 2.18

Let X and Y are two discrete random variables with the following joint probability distribution:

X \ Y	1	3	5
2	0.10	0.20	0.10
4	0.15	0.30	0.15

Compute; E(X), E(Y), E(X +Y), E (2X–3Y) and E(X Y).

Solution:

Adding rows and columns to find $p(x)$ and $p(y)$ shown in the following table.

X \ Y	1	3	5	$p(x)$
2	0.10	0.20	0.10	0.40
4	0.15	0.30	0.15	0.60
$p(y)$	0.25	0.50	0.25	1

By definition

$$E(X) = \Sigma x\, p(x) = 2 \times 0.40 + 4 \times 0.60 = 0.80 + 2.40 = 3.2$$

$$E(Y) = \Sigma y\, p(y) = 1 \times 0.25 + 3 \times 0.50 + 5 \times 0.25$$

$$= 0.25 + 1.50 + 1.25 = 3$$

Now by property

$$E(X + Y) = E(X) + E(Y) = 3.2 + 3 = 6.8$$

$$E(2X - 3Y) = E(2X) - E(3Y)$$

$$= 2E(X) - 3E(Y)$$

$$= 2(3.2) - 3(3) = 6.4 - 9 = -2.6$$

Since X and Y are independent, therefore,

$$E(XY) = E(X)\,E(Y) = (3.2)(3) = 9.6$$

2.4.7 Variance of the sum or difference of two independent random variables

If X and Y are two independent random variables, then variance of their sum is given by the formula:

$$V(X+Y) = E[X+Y-E(X+Y)]^2$$

$$= E[X+Y-E(X)-E(Y)]^2$$

$$= E[\{X-E(X)\} + \{Y-E(Y)\}]^2$$

$$= E[\{X-E(X)\}^2 + \{Y-E(Y)\}^2 + 2\{X-E(X)\}\{Y-E(Y)\}]$$

$$= E[X-E(X)]^2 + E[Y-E(Y)]^2 + 2E\{X-E(X)\}\{Y-E(Y)\}$$

$$= V(X) + V(Y) + 2\,\mathrm{cov}(X, Y)$$

When X and Y are independent then $\mathrm{cov}(X, Y) = 0$

$\therefore$ $V(X+Y) = V(X) + V(Y)$, Similarly

$V(X-Y) = V(X) + V(Y)$

Example 2.19

Consider the following joint probability distribution of two independent random variables X and Y.

X \ Y	0	1	2
0	0.10	.20	.10
1	0.15	.30	.15

Find: (i) $V(X)$, (ii) $V(Y)$, (iii) $V(X+Y)$, (iv) $V(X-Y)$

Solution:

X \ Y	0	1	2	p(x)
0	0.10	.20	.10	.40
1	0.15	.30	.15	.60
p(y)	0.25	.50	.25	1

(i) $V(X) = E(X)^2 - [E(X)]^2$

$$E(X) = \Sigma x\, p(x) = 0\,(.40) + 1\,(.60) = 0 + .60 = 0.60$$

$$E(X^2) = \Sigma x^2\, p(x) = 0^2\,(.40) + 1^2\,(.60) = 0 + .60 = 0.60$$

$$\therefore V(X) = 0.60 - (0.60)^2 = 0.60 - 0.36 = 0.24$$

(ii) $V(Y) = E(Y^2) - [E(Y)]^2$

$$E(Y) = \Sigma y\, p(y) = 0(.25) + 1(.50) + 2(.25) = 0 + 0.50 + 0.5 = 1$$

$$E(Y^2) = \Sigma y^2\, p(y) = 0^2\,(.25) + 1^2\,(.50) + 2^2\,(.25) = 0 + .50 + 1 = 1.50$$

$$\therefore V(Y) = 1.50 - (1)^2 = 1.50 - 1 = 0.50$$

Since X and Y are independent, therefore,

(iii) $V(X+Y) = V(X) + V(Y) = 0.24 + 0.50 = 0.74$

(iv) $V(X-Y) = V(X) + V(Y) = 0.24 + 0.50 = 0.74$

Key points

- A variable that is itself a function of the results of a random experiment is called random variable
- A variable which takes jumping values or isolated values is called discrete random variable
- A variable which takes any value between two limits $[a, b]$, $a < b$ is called continuous random variable
- If all possible values of a random variable along with their respective probabilities are shown in tabular form and sum of probabilities is equal to one, then such tabular form is called probability distribution
- If all possible values of a random variable along with their respective probabilities are shown by a formula, then it is called probability function or probability mass function. It is denoted by $p(x_i)$
- The mean of a random variable is called mathematical expectation.
- $E(c) = c$,where c is a constant
- Mean : $E(X) = \Sigma x\, p(x)$ where X is a discrete random variable
- Variance : $V(X) = E[X^2] - [E(X)]^2$
- The distribution of two or more random variables is called joint distribution.
- If $f(x_i, y_j) = f(x_i)\, f(y_j)$, then X and Y are independent
- $E(X+Y) = E(X) + E(Y)$ and $E(X-Y) = E(X) - E(Y)$
- $E(X\,Y) = E(X)\,E(Y)$, if X and Y are independent.
- $V(X+Y) = V(X) + V(Y)$ and $V(X-Y) = V(X) + V(Y)$, if X and Y are independent.

Exercise

2.1 Read the following statements carefully and write T for true and F for false statement.

i. Random variable is also called chance variable.
ii. Discrete random variable takes every value between two limits
iii. The sum of probabilities in a probability distribution must be one.
iv. The probability function and pdf can be negative.
v. Mathematical expectation of a random variable is also called average or mean of the random variable.
vi. Expected value of a constant is zero.
vii. Variance of a constant is zero.
viii. $E(X-Y) = E(X) - E(Y)$
ix. $V(X-Y) = V(X) - V(Y)$
x. If $f(x, y) = f(x)\,f(y)$, it means X and Y are independent.

2.2 Fill in the blanks.

i. A discrete variable can take a ________ number of values within its range or an infinite number of values that are countable.
ii. The probability function $p(x)$ cannot exceed________
iii. if $p(x) = \frac{x}{21}$ for $x = 1, 2, 3, 4, 5, 6$ then $P(X = 2 \text{ or } 3) =$________
iv. A variable whose value are obtained from the outcomes of a random experiment is called ________
v. Random variables are classified into ________
vi. The distribution of two or more random variables is called________
vii. The value of the expression $\sum_i \sum_j p(x_i, y_j)$ is always________
viii. If X and Y are two independent variables then $E(X\,Y) =$________
ix. If X and Y are independent random variables then $S.D(X-Y) =$____
x. If $E(X) = \frac{2}{3}$, $E(X^2) = \frac{8}{9}$ then $S.D(X) =$ ________

2.3 Select the correct answer out of the given ones.

(i) The height of persons in a country is

(a) discrete random variable

(b) continuous random variable

(c) both discrete and continuous

(d) neither discrete nor continuous

(ii) The outcomes of tossing a coin three times is a variable of the type

(a) continuous (b) discrete

(c) neither discrete not continuous

(d) both discrete and continuous

(iii) If $f(x) = kx, 0 < x < 1$, then value of k is

(a) $\frac{1}{3}$ (b) 2 (c) $\frac{1}{2}$ (d) 1

(iv) If $V(X) = 2$ then $V(2X + 5)$ is equal to

(a) 4 (b) 2 (c) 8 (d) 6

(v) If $E(X) = 4$ then $E[3X + 10]$ is

(a) 10 (b) 13 (c) 4 (d) 22

(vi) If $E(X) = \frac{2}{3}$, $E(X^2) = \frac{8}{9}$ then S.D of X is

(a) $\frac{4}{9}$ (b) $\frac{9}{4}$ (c) $\frac{2}{3}$ (d) $\frac{2}{9}$

(vii) If $f(x, y) = f(x)\,f(y)$, it means variables are

(a) correlated (b) dependent

(c) independent (d) associated

(viii) If X and Y are independent random variables then $V(X-Y) =$

(a) $V(X) - V(Y)$ (b) $V(X) + V(Y)$

(c) $V(X)\,V(Y)$ (d) $V(X + Y)$

(ix) continuous probability distributions give

(a) interval probability (b) point probability

(c) negative probability (d) zero probability

(x) if X is a random variable having its pdf, the $E(X)$ is called

(a) median (b) geometric mean

(c) mode (d) arithmetic mean

2.4 Describe the concept of a random variable and give its examples.

2.5 Explain different types of a random variable with examples.

2.6 Classify each of the following random variables as either discrete or continuous:

i. Time to failure for an electronic system.

ii. The height of a person.

iii. The number of questions asked in an oral examination.

iv. Temperature at a place.

v. The maximum breaking strength 250 kg of a wire.

vi. The number of fatal traffic accident per month on the motor way.

vii. The life time of a mobile set.

viii. The amount of rainfall at Islamabad during different months of 2016.

ix. The number of admitted patients in a hospital in a year.

x. The number of Mosque per village in a district.

2.7 Define probability distribution, probability function and probability density function. What are the two basic properties of all probability functions?

2.8 Find probability distribution for the number of heads when 2 balanced coins are tossed.

2.9 In a family of three children find the probability distribution for the number of girls.

2.10 A random variable has the following probability distribution.

X	4	6	7	10
p(x)	0.2	0.4	c	0.1

(i) Find the value of c. (ii) $P(X<7)$ (iii) $P(X\geq 6)$

2.11 A coin is tossed three times in succession. Find the probability distribution for the number of heads minus number of tails.

2.12 A pair of fair dice is rolled once. Find the probability distribution for the difference of dots.

2.13 Check whether the following is a probability distribution or not? If not, then why?

X	0	10	15	25	50
p(x)	0.1	0.3	0.4	0.3	0.1

2.14 A discrete random variable has the probability mass function:

$$p(x)=\begin{cases}\frac{{}^{3}C_{x}}{8}, & x=0,1,2,3\\ 0, & elsewhere\end{cases}$$

Find (i) $P(X=2)$ (ii) $P(X>1)$ (iii) $P(0<X<3)$

2.15 (a) Define p.d.f and state its properties.

(b) The density function of a continuous random variable X is:

$$f(x)=\begin{bmatrix}Ax^2, & 0\leq x\leq 1\\ 0, & elsewhere\end{bmatrix}$$

Find (i) value of the constant A (ii) $P(X\leq 0.5)$ (ii) $P(0.2<X<0.3)$.

2.16 Discuss mathematical expectation. Write some of its important properties.

2.17 For the following discrete probability distribution of X

X	−2	1	2
p(x)	1/3	1/6	1/2

Find (i) $E(X)$ (ii) $E(X^2)$ (iii) $E(2X+5)$

2.18 Find the mean, variance and standard deviation for the following probability distribution:

Y	−1	0	1	2	3
p(y)	0.125	0.5	0.2	0.05	0.125

2.19 Let Y be a random variable with probability distribution as follows:

Y	1	2	3	4	5
p(y)	0.125	0.45	0.25	0.05	0.125

Find (i) Expected value

(ii) Variance

(iii) S.D for the random variable Y

2.20 A variable X has the p.d.f $f(x)=\begin{bmatrix}kx^3(1-x), & 0\leq x\leq 1\\ 0, & otherwise\end{bmatrix}$

Find (i) The value of k (ii) $E(X)$ (iii) Variance (iv) S.D