Unit 9

CONICS-II

By the end of this unit, the students will be able to:

9.1 **Parabola**
- i. Define a parabola and its elements (i.e. focus, directrix, eccentricity, vertex, axis, focal chord and latus rectum).
- ii. General form of an equation of a parabola.
- iii. Standard equations of parabola, sketch their graphs and find their elements.
- iv. Find the equation of a parabola with the following given elements:
  - focus and vertex,
  - focus and directrix,
  - vertex and directrix.
- v. Recognize tangent and normal to a parabola.
- vi. Find the condition when a line is tangent to a parabola at a point and hence write the equation of a tangent line in slope form.
- vii. Find the equation of a tangent and a normal to a parabola at a point.
- viii. Solve suspension and reflection problems related to parabola.

9.2 **Ellipse**
- i. Define ellips and its elements (i.e. centre, foci, vertices, covertices, directories, major and minor axes, eccentricity, focal chord and latus rectum).
- ii. Explain that circle is a special case of an ellipse.
- iii. Derive the standard form of equation of an ellipse and identify its elements.
- iv. Find the equation of an ellipse with the following given elements.
  - major and minor axes,
  - two points,
  - foci, vertices or lengths of a latus rectum,
  - foci, minor axes or length of a latus rectum.
- v. Convert a given equation to the standard form of equation of an ellipse, find its elements and draw the graph.
- vi. Recognize tangent and normal to an ellipse.
- vii. Find points of intersection of an ellipse with a line including the condition of tangency.
- viii. Find the equation of a tangent in slope form.
- ix. Fine the equation of a tangent and a normal, to an ellipse at a point.

9.3 **Hyperbola**
- i. Define hyperbola and its elements (i.e. centre, foci, vertices, directrices, transverse and conjugate axes, eccentricity, focal chord and latus rectum).
- ii. Derive the standard form of equation of a hyperbola and identify its elements.
- iii. Find the equation of a hyperbola with the following given elements:
  - transverse and conjugate axes with centre at origin,
  - two points,
  - eccentricity, Latera recta and transverse axes,
  - focus, eccentricity and centre,
  - focus, centre and directrix.
- iv. Convert a given equation to the standard form of equation of a hyperbola, find its elements and sketch the graph.
- v. Recognize tangent and normal to a hyperbola.
- vi. Find,
  - points of intersection of a hyperbola with a line including the condition of tangency,
  - the equation of tangent in slope form.
- vii. Find the equation of a tangent and a normal to a hyperbola at a point.

9.4 **Translation and rotation of axes**
- i. Define translation and rotation of axes and demonstrate through examples.
- ii. Find the equations of transformation for
  - translation of axes,
  - rotation of axes.
- iii. Find the transformed equation by using translation or rotation of axes.
- iv. Find new origin and new axes referred to old origin and old axes.
- v. Find the angle through which the axes be rotated about the origin so that the product term xy is removed form the transformed equation.

Introduction

In our previous unit of this book we have learnt that a conic section (or simply a conic) is a curve obtained as the intersection of the surface of a cone with a plane. In this unit we will study in details about the three types of conic sections that are parabola, hyperbola and the ellipse. The circle is a type of ellipse and same time considered to be the fourth type of conic section. We have already discussed in details about tangent and normal in previous section.

9.1 Parabola

When you kick a soccer ball (or shoot an arrow, fire a missile or throw a stone) it arcs up into the air and comes down again ...

A parabola is a curve where any point is at an **equal distance** from:

- a fixed point (the **focus**), and
- a fixed straight line (the **directrix**)

et a piece of paper, draw a straight line on it, then make a big dot for the focus (not on the line!).

Now play around with some measurements until you have another dot that is exactly the same distance from the focus and the straight line.

Keep going until you have lots of little dots, then join the little dots and you will have a parabola!

In our study of quadratic functions, the graph of the general form of the quadratic equation $y = ax^2 + bx + c$ (1)

(with $a \neq 0$) is a parabola that opens upward if $a > 0$ and downward if $a < 0$.

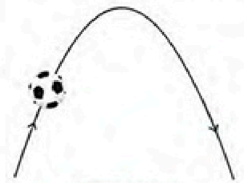


Figure 9.1(a)

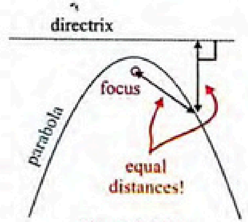


Figure 9.1(b)

Remember

The graph of a quadratic equation is always parabola. But all parabolas can not be represented by quadratic equation, because all parabolas are not graphs of the functions.

(i) Parabola and its elements (i.e focus, directrix, eccentricity, vertex, axis, focal chord and latus rectum

The **parabola** is the set of all points P in the plane such that the distance from a fixed point F **(focus)** and the distance from a fixed straight line **(directrix)** to a point are **equidistant**.

The line through the focus perpendicular to the directrix is called the **principal axis** of the parabola, and the point where the axis intersects the parabola is called the **vertex**. The line segment AB that passes through the focus perpendicular to the axis and with endpoints on the parabola is called the **focal chord** or its **latus rectum**. These terminologies are shown in the Figure 9.2.

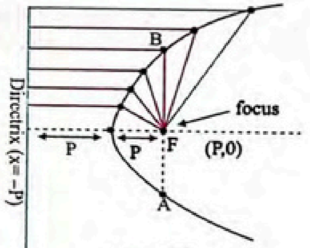


Figure 9.2

(ii) General form of an equation of a parabola

To obtain the general form of the parabola, let us assume a focus with coordinates F(p,0) and a directrix $x=-p$, (p is any positive number) parallel to the y–axis. If P(x, y) is any point on the curve and $p_1(-p,y)$ is a point on the directrix $x=-p$, then by the definition of parabola Figure 9.3.

$$\frac{\text{distance from P } to \text{ F}}{\text{distance from P } to \text{ P}_1}=e=1, \quad \text{for parabola } e=1$$

distance from $\text{P}(x,y)$ *to* P_1 = distance from $\text{P}(x,y)$ *to* F

$$d(\text{P},\text{P}_1)=d(\text{P},\text{F})$$

$$\sqrt{(x+p)^2+0}=\sqrt{(x-p)^2+(y-0)^2}, \qquad (2)$$

$$(x+p)^2=(x-p)^2+y^2, \qquad \text{by squaring}$$

$$x^2+2px+p^2=x^2-2px+p^2+y^2 \Rightarrow 4px=y^2 \qquad (3)$$

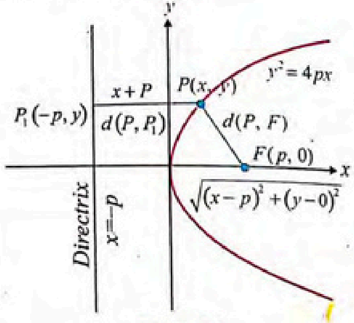


Figure 9.3

The result (3) is the **standard form** obtained from the **general form** of the equation of a parabola with vertex at V(0,0), focus F(p,0) and directrix $x=-p$. The parabola is symmetric with respect to the positive x–*axis* if $p>0$ and symmetric with respect to the negative x–*axis* if $p<0$. The vertex V(0,0) of the parabola is on the principal axis of symmetry midway between the focus and the directrix.

(iii) Standard equations of parabola, sketch their graphs and find their element

a. Standard equations of parabola

"The standard form of the equation of a parabola that is symmetric with respect to the x–axis, with vertex V(0,0), focus F(p, 0) and directrix the line x = –p is:

$$y^2=4px \qquad (4)$$

"The standard form of an equation of a parabola that is symmetric with respect to the y–axis, with vertex V(0,0), focus F(0, p) and directrix the line y = –p is:

$$x^2=4py \qquad (5)$$

The parabolas that have their vertex at the origin and open upward, downward, to the left and to the right are summarized in the following table:

Parabola	Curve	Focus	Directrix	Vertex
$x^2=4py$	up, if $p>0$ down, if $p<0$	$F(0,p)$ $F(0,p)$	$y=-p$ $y=-p$	V(0,0) V(0,0)
$y^2=4px$	right, if $p>0$ left, if $p<0$	$F(p,0)$ $F(p,0)$	$x=-p$ $x=-p$	V(0,0) V(0,0)

b. Graphing standard form of a parabola

Here, we will find and plot the parabola by inspection and count out units from the vertex in the appropriate direction as determined by the form of the equation. Finally, it is shown in the problem set that the length of the focal chord (latus rectum) is |4p|. This number could be used in determination of the width of the parabola. This approach is employed in the following examples.

Example 1 Graph the parabola $y^2-8x=0$ and indicate the vertex, focus, directrix and the focal chord.

Solution Rewrite the given parabola in the standard form

$$y^2=8x \qquad (6)$$

and is compared with the standard form of the parabola (3) to obtain:

$$8=4p \Rightarrow p=2$$

Since $p>0$, the parabola opens to the right. The vertex is V(0,0), the focus is F(2,0), the directrix is the line $x=-2$ and the length of the focal chord is $4p=4(2)=8$. The line of symmetry is the positive x–*axis*. This is shown in the Figure 9.4.

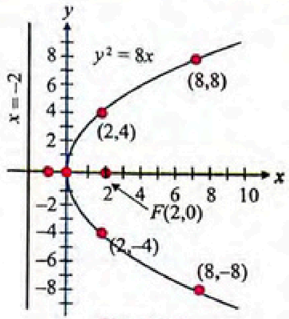


Figure 9.4

Example 2 Graph the parabola $x^2+y=0$ and indicate the vertex, focus, directrix and the focal chord.

Solution Rewrite the given parabola in the standard form

$$x^2=-y \qquad (7)$$

and is compared with the standard form of the parabola (5) to obtain :

$$-1=4p \Rightarrow p=-\frac{1}{4}$$

Since $p<0$, the parabola opens downward. The vertex is V(0,0), the focus is $\text{F}\left(0,\frac{-1}{4}\right)$, the directrix is the line $y=\frac{1}{4}$ and the length of the focal chord is $4p=4\left(\frac{-1}{4}\right)=-1$. The line of symmetry is the negative y–*axis*. This is shown in the Figure 9.5.

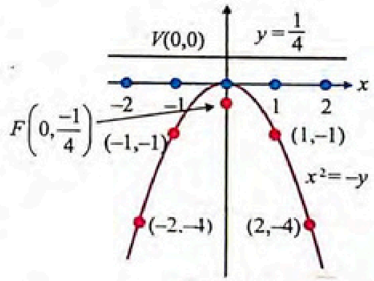


Figure 9.5

(iv) The equation of a parabola with the given elements

- **focus and vertex**
- **focus and directrix**
- **fertex and directrix**

Example 3 Find an equation of parabola with

(*a*). Focus F(0, –2) and directrix $y=2$.

(*b*). Focus $\left(\frac{5}{8},0\right)$ and vertex (0,0).

(*c*). Vertex V(0,0) and directrix $x=\frac{1}{2}$.

Solution

a. By inspection, the value of p is $p=-2$ that satisfies the directrix $y=2$. This gives the equation of parabola $x^2=4py=4(-2)y=-8y$, that opens downward ($p<0$) and the line of symmetry is the negative y–axis. This is shown in the Figure 9.6.

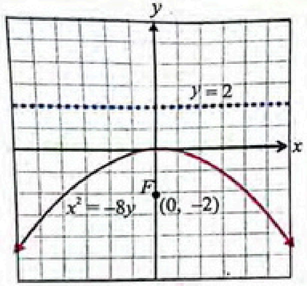


Figure 9.6

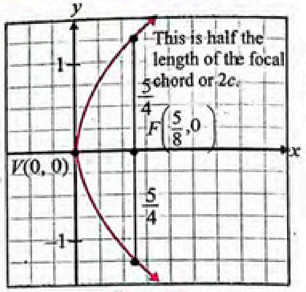


Figure 9.7

b. By inspection, the value of p is $p=\frac{5}{8}$ that satisfies the directrix $x=-\frac{5}{8}$. This gives the equation of parabola $y^2=4px=4\left(\frac{5}{8}\right)x=\frac{5}{2}x$, that opens right ($p>0$) and the line of symmetry is the positive *x–axis*. This is shown in the Figure 9.7.

c. By inspection, the value of p is $p=-\frac{1}{2}$ that satisfies the directrix $x=\frac{1}{2}$. This gives the equation of parabola $y^2=4px=4\left(-\frac{1}{2}\right)x=-2x$, that opens left ($p<0$) and the line of symmetry is the negative x–axis. This is shown in the Figure 9.8.

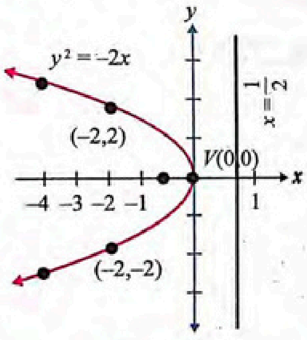


Figure 9.8

(v) Recognition of tangent and normal to a parabola

A line which is parallel to the axis of a parabola intersects the parabola in only one (finite) point; all other lines will cut the parabola in two real and distinct points, real and coincident points, or complex conjugate points. *"A line which meets a parabola in two coincident points is called a tangent."* A tangent to any curve at a point P is the limiting position of a secant line, cutting the curve in two points P and Q as $Q\to P$. The normal can easily be shown in the subsection of this section.

(vi) The condition at which a line is tangent to parabola at a point

The line is tangent to parabola, when the line intersects the parabola in **two real and coincident** points. The given parabola and line

$$y^2=4px \qquad (8)$$

$$y=mx+c \qquad (9)$$

develops a system of nonlinear equations:

$$\left.\begin{array}{l} y^2=4px \\ y=mx+c \end{array}\right\} \qquad (10)$$

The solution set $\{x,y\}$ of nonlinear system of equations (10) exists only, if the curves of the system (10) are intersecting. That set of points of intersection $\{x,y\}$ (a solution set) can be found by solving the nonlinear system (10) simultaneously.

The line (9) is used in parabola (8) to obtain the quadratic equation in *x*:

$$(mx+c)^2=4px$$

$$m^2x^2+2mcx+c^2=4px$$

$$m^2x^2+2x(mc-2p)+c^2=0 \qquad (11)$$

The equation (11) being a quadratic equation in *x*, gives a set of two values x_1 and x_2 of *x*, which will be used in a line (9) to obtain a set of two y values y_1 and y_2.

Thus, a solution set $\{(x_1,y_1),(x_2,y_2)\}$ of the system (10) is of course a set of points of intersection of the system (10).

The points of intersection of the system (10) are real, coincident or imaginary, according as the roots of the quadratic equation (11) are real, coincident or imaginary or according as the discriminant of the quadratic equation (11):

$$Disc=4(mc-2p)^2-4m^2c^2>0, \quad \textit{real and different}$$

$$Disc=4(mc-2p)^2-4m^2c^2=0, \quad \textit{real and coincident}$$

$$Disc=4(mc-2p)^2-4m^2c^2<0, \quad \textit{imaginary}$$

Example 4 For what condition the tangent line $4x-y-4=0$ intersects the parabola $x^2=y$?

Solution The equations of the line and parabola are:

$$4x-y-4=0 \qquad (12)$$

$$y=4x-4$$

$$x^2=y \qquad (13)$$

The line (12) is used in parabola (13) to obtain the *y*–coordinates of the points of intersection:

$$x^2=y$$

$x^2=4x-4$ Put the value of *y* from equation (12)

$$x^2-4x+4=0\Rightarrow(x-2)^2=0\Rightarrow x=2,2$$

The *x*–coordinates are used in the line (12) to obtain the *y*–coordinates $y=4,4$

Thus, the set of two points of intersection (2,4) and (2,4) are real and coincident and the tangent line $4x-y-4=0$ is of course intersecting the parabola (13) at two coincident points (2,4) and (2,4).

a. The Equation of a tangent line in slope-form

If m is the slope of the tangent to parabola $y^2=4px$ (14)

then the equation of that tangent line is of the form $y=mx+c$ (15)

Here *c* is to be calculated from the fact that the line (15) is tangent to parabola (14). The line (15) is used in parabola (14) to obtain the quadratic equation in *x*:

$$y^2=4px$$

$$(mx+c)^2=4px$$

$$m^2x^2+c^2+2mcx=4px$$

$$m^2x^2+2(mc-2p)x+c^2=0 \qquad (16)$$

If the line (15) touches the parabola (14), then the quadratic equation (16) has coincident roots for which the discriminant of the quadratic equation (16) equals zero:

$$4(mc-2p)^2-4(m^2)(c^2)=0$$

$$4m^2c^2+16p^2-16mcp-4m^2c^2=0$$

$$16p^2-16mcp=0$$

$$16p(p-mc)=0 \Rightarrow p-mc=0 \Rightarrow c=\frac{p}{m} \qquad (17)$$

The equation (17) represents the **condition of tangency**. The value of c from equation (17) is used in the line (15) to obtain the required equation of tangent:

$$y=mx+\left[\frac{p}{m}\right] \qquad (18)$$

Remember

- the equation of any tangent to parabola $y^2=4px$ in the slope–form is:

  $y=mx+\left[\frac{p}{m}\right]$
- the line y=mx+c should touch the parabola $y^2=4px$ under condition:

  $y=mx+c=mx+\left[\frac{p}{m}\right]$ (19) $c=\frac{p}{m}, y^2=4px$
- the **condition of tangency** in case of parabola $x^2=4py$ and line $y=mx+c$ is:

  $y=mx+c=mx-pm^2,\ c=-pm^2,\ x^2=4py$ (20)

Example 5 For what value of c, the line $x-y+c=0$ will touch the parabola $x^2=8y$? Use that value of c to find the tangent line that should touch the given parabola.

Solution The value of c at which the line $x-y+c=0$ will touch the given parabola through result (20) is:

$$c=-pm^2=-2(1)=-2$$
$$x^2=4py=4(2)y$$
$$p=2, m=1$$

Here m is the slope of the line $x-y+c=0$, which is $m=1$. The required tangent line that should touch the parabola through (20) is: $y=mx+c$

$$=x-2 \Rightarrow x-y-2=0$$

(vii) The equation of a tangent and a normal to a parabola at a point

a. Equation of tangent to a parabola at a point

Let the equation of the tangent line at a point $p(x_1,y_1)$ to parabola $y^2=4px$ be:

$$y-y_1=m_1(x-x_1) \qquad (21)$$

Here m_1 is the slope of the tangent line to parabola $y^2=4px$ at a point $p(x_1,y_1)$ that can be found by differentiating $y^2=4px$ with respect to x:

$$y^2=4px$$
$$2y\frac{dy}{dx}=4p \Rightarrow \frac{dy}{dx}=\frac{2p}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{2p}{y_1}=m_1,\ say \qquad (22)$$

The substitution of (22) in (21) is giving the equation of the tangent line at a point $p(x_1,y_1)$ to parabola $y^2=4px$:

$$y-y_1=m_1(x-x_1)$$
$$y-y_1=\frac{2p}{y_1}(x-x_1)$$

$$yy_1-y_1^2=2px-2px_1$$
$$yy_1-4px_1=2px-2px_1,\quad y_1^2=4px_1$$
$$yy_1=2px+2px_1 \Rightarrow yy_1=2p(x+x_1) \qquad (23)$$

Note

- the equation of the tangent line at a point $p(x_1,y_1)$ to parabola $x^2=4py$ is:

  $$xx_1=2p(y+y_1) \qquad (24)$$
- if the tangent line $y=mx+\left[\frac{p}{m}\right]$ to parabola $y^2=4px$ is identical to $yy_1=2p(x+x_1)$, then the coefficients of like terms of $y=mx+\left[\frac{p}{m}\right]$ and $yy_1=2p(x+x_1)$ are compared to obtain the contact point:

  $$mx=\frac{2px}{y_1} \Rightarrow y_1=\frac{2p}{m}$$
  $$\frac{2px_1}{y_1}=\frac{p}{m} \Rightarrow 2x_1=\frac{y_1}{m} \Rightarrow 2x_1=\frac{2p}{m^2} \Rightarrow x_1=\frac{p}{m^2}$$

  Thus, the contact point is $p(x_1,y_1)=\left(\frac{p}{m^2},\frac{2p}{m}\right)$ in case of parabola $y^2=4px$. (25)
- if the tangent line $y=mx-pm^2$ to parabola $x^2=4py$ is identical to $xx_1=2p(y+y_1)$, then the coefficients of like terms of $y=mx-pm^2$ and $xx_1=2p(y+y_1)$ are compared to obtain the point of contact:

  $$mx=\frac{xx_1}{2p} \Rightarrow x_1=2pm$$
  $$-pm^2=-y_1 \Rightarrow y_1=pm^2$$

  Thus, the contact point is $p(x_1,y_1)=\left(2pm,pm^2\right)$ in case of parabola $x^2=4py$. (26)

Example 6 Find the equation of tangent line at a point $p(2,-4)$ to parabola $y^2=8x$. Show that $p(2,-4)$ is the point of contact in between the required tangent line and the given parabola.

Solution Result (23) is used to obtain the tangent line to the given parabola:

$$yy_1=2p(x+x_1)$$
$$y(-4)=2(2)(x+2) \qquad \therefore p(x_1,y_1)=(2,-4),\ 4p=8$$
$$-4y=4x+8 \Rightarrow 4x+4y+8=0 \Rightarrow x+y+2=0$$

The point of contact through result (25) is:

$p(x_1,y_1)=\left(\frac{p}{m^2},\frac{2p}{m}\right)=(2,-4)$, $p=2$, $m=-1$ is the slope of the tangent line $x+y+2=0$

b. The Equation of a normal line to parabola at a point

The equation of the normal line at a point $p(x_1,y_1)$ to parabola $y^2=4px$ is:

$$y-y_1=m_2(x-x_1) \qquad (27)$$

Here m_2 is the slope of the normal line to parabola $y^2=4px$ at a point $p(x_1,y_1)$ that can be found by differentiating $y^2=4px$ with respect to x:

$$y^2=4px$$

$$2y\frac{dy}{dx}=4p$$

$$\frac{dy}{dx}=\frac{2p}{y}\Rightarrow\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{2p}{y_1}=m_1\Rightarrow m_2=\frac{-1}{m_1}=\frac{-y_1}{2p},\ say \quad (28)$$

The substitution of (28) in (27) is giving the normal equation at a point $p(x_1,y_1)$ to parabola $y^2=4px$:

$$y-y_1=m_2(x-x_1)$$

$$y-y_1=\frac{-y_1}{2p}(x-x_1) \quad (29) \quad \because m_2=\frac{-y_1}{2p}$$

Example 7 Find thė normal equation at a point $p(2,-4)$ to parabola $y^2=8x$.

Solution Result (29) is used to obtain the normal line to the given parabola:

$$y-y_1=\frac{-y_1}{2p}(x-x_1)$$

$$y-(-4)=\frac{4}{2(2)}(x-2),\quad P(x_1,y_1)=(2,-4), 4p=8$$

$$y+4=x-2\Rightarrow x-2-y-4=0\Rightarrow x-y-6=0$$

(viii) Suspension and reflection problems related to parabola

The parabola is more than just a geometric concept. It has many uses in the physical world that are listed under:

1. Projectiles in the air, such as a ball, or a missile, or water sprayed from a hose, describe a parabolic path when acted on only by gravity.
2. Many arches of bridges or buildings are parabolic in shape. With this shape, the arch can support the structure above it.
3. Rotating a parabola about its line of symmetry, creates a bowl type surface called a paraboloid of revolution. A paraboloid has an important reflection property. Any ray or wave that originates at the focus and strikes the surface of the paraboloid is reflected parallel to the line of symmetry. See Figure 9.9.

This forms the basic design of the reflectors for automobile headlights, flashlights, searchlights, telescopes, etc. This is also an excellent collecting device and is the basic design of TV, radar, and radio antennas.

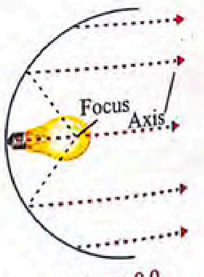


Figure 9.9

Example 8 The cables of a bridge form a parabolic arc. The low point of the cable is 10ft above the roadway midway between two towers. The distance between the towers is 400 ft. The cable is attached to the towers 50ft above the roadway. Determine the equation of the parabola that describes the path of the cable. This is shown in the Figure 9.10:

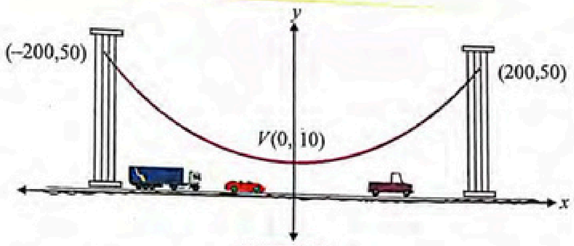


Figure 9.10

Solution The parabola is formed by the cable between the two towers. The low point on the cable is midway between the towers, and 10ft above the roadway. In order to write an equation, locate the *x–axis* and the *y–axis* in the *xy–plane*. Select the roadway as the *x–axis* and the line perpendicular to the roadway through the lowest point of the tower as the *y–axis*. The parabola opens up with the vertex at the point (0,10). Two other points on the parabola are (200,50) and (–200,50). The standard form for this equation is:

$$(x-h)^2=4p(y-k) \quad (30)$$

The vertex V(0,10) and a point on the curve $(x, y)=(200,50)$ are used in (30) to obtain p:

$$(x-h)^2=4p(y-k),\ translate\ h\ units\ on\ the\ x-axis,\ k\ units\ on\ the\ y-axis$$

$$(200-0)^2=4p(50-10)),\quad V(h,k)=V(0,10),(x,y)=(200,50)$$

$$40000=160p\Rightarrow p=250$$

The substitution of $V(h, k)=(0,10)$ and $p=250$ in equation (30) is giving the parabolic equation

$$(x-0)^2=4(250)(y-10)$$

$$x^2=1000(y-10)\Rightarrow x^2-1000y+10000=0$$

that describes the path of the cable.

Example 9 A radar antenna is constructed so that a cross section along its axis is a parabola with the receiver at the focus. Find the focus if the antenna is 12 *m* across and its depth is 4 *m*. Find the equation of parabola that described the radar antenna. This is shown in the Figure 9.11.

Solution The parabola is formed by the radar antenna. In order to write an equation, locate the *x–axis* and the *y–axis* in the *xy–plane*. The axis of symmetry is the positive *x–axis*. The parabola opens to the right with the vertex at the origin $V(0,0)$. The other point on the parabola is (4, 6). The standard form for this equation is:

$$y^2=4px$$

$$36=4p(4),\quad (x,y)=(4,6) \quad (31)$$

$$p=\frac{36}{16}=\frac{9}{4}$$

Thus, the parabolic equation that describes the radar antenna is obtained by putting $p=\frac{9}{4}$ in (31):

$$y^2=4px=4\left(\frac{9}{4}\right)x=9x$$

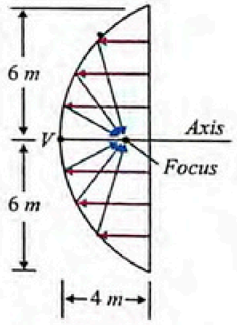


Figure 9.11

The focus is $F\left(\frac{9}{4},0\right)$ which is $\frac{9}{4}$ *m* from the vertex $V(0,0)$.

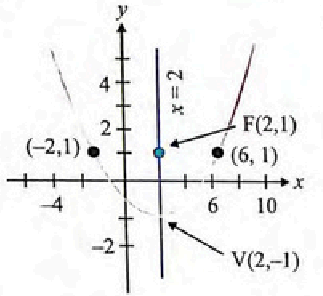
Exercise 9.1

1. In each case, sketch the parabola represented by the equation, indicate the vertex, the focus, the end points of the focal chord (latus rectum) and the axis of symmetry:

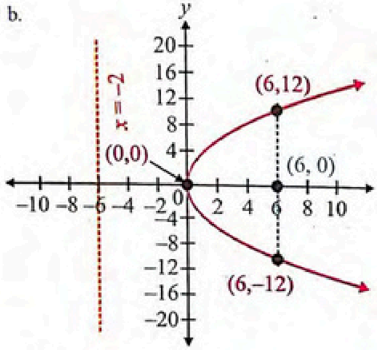
   a. $x^2 = 2y$ b. $y^2 = -3(x+1)$ c. $(y-3)^2 = x$

2. In each case, determine the equation of graphed parabola:

   a.



   b.



3. In each case, write the equation of parabola through the given information:

   a. Focus at $F(0,3)$, directrix $y=-3$. b. Focus at $F(4,0)$, directrix $x=-4$.

   c. Vertex at $V(0,0)$, x–*axis* is the line of symmetry, passes through $(3,6)$.

   d. Vertex at $V(0,0)$, y–*axis* is the line of symmetry, passes through $(-12,-3)$.

   e. Line of symmetry is vertical, passes through $(-3,4)$, vertex at $V(5,1)$.

   f. Line of symmetry is horizontal, passes through $(7,9)$, vertex at $V(3,-7)$.

4. Find the equation of the set of all points with distances from $(4,3)$ that equal their distances from $(-2,1)$.

5. Find an equation for a parabola whose focal chord has length 6, if it is known that the parabola has focus $(4,-2)$ and its directrix is parallel to the y–*axis*.

6. In each case, find the points of intersection in between the line and the parabola:

   a. $y^2+3x=-8,\ x-y+2=0$ b. $x^2=2y,\ x-y-2=0$

7. For what value of c,

   a. the line $x-y+c=0$ will touch the parabola $y^2=9x$?

   b. the line $x-y+c=0$ will touch the parabola $x^2=\frac{2}{3}y$?

8. In each case, find the tangent equation and normal equation

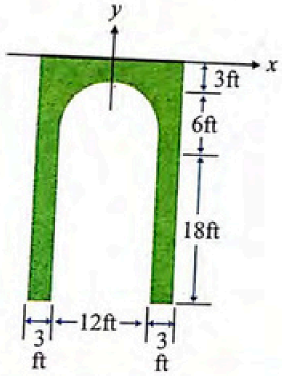
   a. at a point $(3,6)$ to parabola $y^2=12x$. b. at a point $\left(\frac{1}{2},\frac{-1}{3}\right)$ to parabola $x^2=\frac{-3y}{4}$.

9. Find the tangent equation

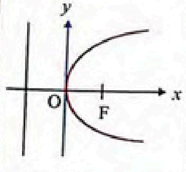
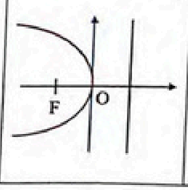
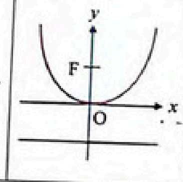
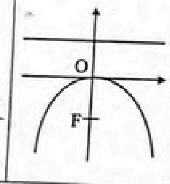
   a. to parabola $y^2=x$, which makes an angle of 135° with the x–*axis*.

   b. to parabola $x^2=y$ which makes an angle of 60° with the x–*axis*.

10. Find the equation of the parabolic portion of the archway, if parabolic archway has the dimensions shown in the figure below:



Summary of standard Parabola

Equation	$y^2=4ax$	$y^2=-4ax$	$x^2=4ay$	$x^2=-4ay$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Directrix	$x=-a$	$x=a$	$y=-a$	$y=a$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Axis	$y=0$	$y=0$	$x=0$	$x=0$
Latus rectum	$x=a$	$x=-a$	$y=a$	$y=-a$
Graph				

9.2 Ellipse

In shape and in format, the ellipse is different from the parabola. Although the parabola is open at one end, the ellipse is entirely closed. The parabola has one focus and one vertex, while the ellipse has two foci (plural of focus) and two vertices.

(i) Ellipse and its elements

Ellipse

The second type of conic is called an ellipse, and is defined as follows.

An ellipse is the set of all points in a plane, the sum of whose distances from two distinct fixed points (foci) is constant.

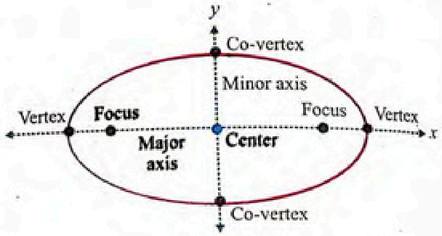


Figure 9.12

- **Center**– It is the point where major and minor axis intersects each other. The midpoint of the connecting two foci line segment is the center.
- **Focus** – There are two focal points on the major axis which defines the ellipse. These are at the same distance to the both sides from the center.
- **Major Axis** – It is the lengthiest diameter of the ellipse. It has the end points on the widest part of the ellipse and passes through the center.
- **Minor Axis** – It is the shortest diameter of the ellipse. It is the perpendicular bisector of the major axis. It has the end points on the narrow part of the ellipse and passes through the center.
- **Vertices** – The four points where the major and minor axis touches the ellipse are the vertices. The end points of major axis are generally called **Vertex** and the end points of minor axis are generally called **Co-vertex**.
- **Chord** – It is a line segment that has both the end points on the ellipse. Major axis is also the chord which is the longest one in an ellipse.

Eccentricity of an Ellipse

Eccentricity is the factor related to conic sections which shows how circular the conic section is. More eccentricity means less spherical and less eccentricity means more spherical. It is denoted by "e".

The eccentricity of an ellipse is showed by the ratio of the distance between the two foci, to the size of the major axis,

$$e = \frac{c}{a}.$$

where $e =$ Eccentricity, $c =$ The distance from the center to any one of the foci and $a =$ The semi major axis.

The eccentricity of an ellipse is between 0 and 1 ($0 < e < 1$). If the eccentricity is zero the foci match with the center point and become a circle. If the eccentricity moves toward 1, the ellipse gets a more stretched shape.

Directrix of an ellipse

Directrix is the line which is parallel to the minor axis of the ellipse and related to both the foci of the ellipse.

Latus rectum of an Ellipse

It is the line parallel to directrix and passes through any of the focus of an ellipse. It is denoted by "$2l$". In an ellipse, latus rectum is $2b^2/a$ (where a is one half of the major diameter and b is the half of the minor diameter).

The half of latus rectum till its intersection point with the major axis is the semi latus rectum. It is denoted by "l".

Note: General form of the ellipse

If $P(x, y)$ is any point on the ellipse, then the distances from the two foci $F_1(-c,0)$ and $F_2(c,0)$ to the point $P(x, y)$ are the following:

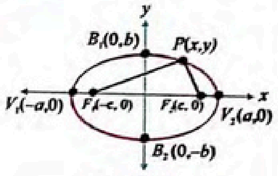


Figure 9.13

$$d(F_1,P) = (x+c, y-0)$$
$$|d(F_1,P)| = \sqrt{(x+c)^2+(y)^2}$$
$$d(F_2,P) = (x-c, y-0)$$
$$|d(F_2,P)| = \sqrt{(x-c)^2+(y)^2}$$

By definition of an ellipse , the general form of an ellipse is:

$$|d(F_1,P)| + |d(F_2,P)| = 2a \qquad (32)$$
$$\sqrt{(x+c)^2+(y)^2} + \sqrt{(x-c)^2+y^2} = 2a$$
$$\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$$

Squaring both sides to obtain

$$(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$$
$$4a\sqrt{(x-c)^2+y^2} = 4a^2-4cx,$$
$$a\sqrt{(x-c)^2+y^2} = a^2-cx \qquad (33)$$

Again squaring to obtain

$$a^2[(x-c)^2+y^2] = a^4-2a^2cx+c^2x^2$$
$$a^2\left(x^2-2cx+c^2+y^2\right) = a^4-2a^2cx+c^2x^2$$
$$a^2x^2-2a^2cx+a^2c^2+a^2y^2 = a^4-2a^2cx+c^2x^2$$
$$a^2x^2-c^2x^2+a^2y^2 = a^4-a^2c^2$$
$$(a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2)$$
$$b^2x^2+a^2y^2 = a^2b^2. \qquad a^2-c^2=b^2,\ a>0, b>0$$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \qquad \textit{divide out by } a^2b^2 \qquad (34)$$

(ii) Circle is a special case of an ellipse

The relative shape of an ellipse can be determined by its eccentricity e. The distance from the center of the ellipse to a foci is c, and the distance from the center to a vertex is a. The eccentricity is given by the equation:

$$e = \frac{\text{distance from center to focus}}{\text{distance from center to vertex}} = \frac{c}{a} \qquad (35)$$

The eccentricity of all ellipses are in a range between 0 and 1 $(0 < e < 1)$. This is shown in the Figure 9.14.

An ellipse with an eccentricity close to 1 is long and thin, and the foci are relatively far apart. If the eccentricity is small, close to 0, then the ellipse resembles a circle. It can be shown that the circle is a special case of the ellipse when $e = 0$.

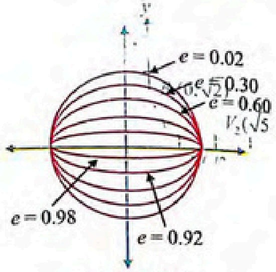


Figure 9.14

(iii) Standard form of equation of an ellipse

The standard form of the equation of an ellipse with center at the origin, length of the semimajor axis a, length of the semiminor axis b and major axis along the x–axis is shown in the Figure 9.15.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (36)$$

If we replace the foci on the y–axis, center at the origin, and pick any point $P(x, y)$ on the plane, then we can develop the equation of the vertical ellipse given in the following definition.

"The standard form of the equation of an ellipse with center at the origin, length of the semimajor axis a, length of the semiminor axis b and major axis along the y–axis is shown in the Figure 9.16.

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad (37)"$$

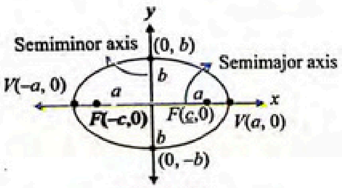


Figure 9.15

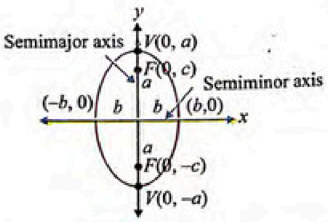


Figure 9.16

Graphing Ellipse: In order to sketch an ellipse, it is required to plot the center, the intercepts $\pm a$ on the major axis and $\pm b$ on the minor axis.

First, rewrite the equation of the ellipse in the standard form, so that there is a "1" on the right and the numerator coefficients of the square terms are also 1. The center is at (0,0) and plot the intercepts on the x–axis and y–axis. For the x–intercepts, plot ± the square root of the number a^2; for the y–intercepts, plot ± the square root of the number b^2, finally, and draw the ellipse using these intercepts. The longer axis is called the **major axis**. If this larger axis is horizontal, then the ellipse is called **horizontal**, and if the major axis is vertical, the ellipse is then called **vertical**.

The orientation of the ellipse equation with center $C(0,0)$, vertices/end points of the major axis and the end points of the semiminor axis are summarized in the box:

Orientation	Foci	Vertices/Semimajor axis	Semiminor axis
Horizontal:	$(-c,0), (c,0)$	$(-a,0), (a,0)$	$(0,b), (0,-b)$
Vertical:	$(0,c), (0,-c)$	$(0,-a), (0,a)$	$(b,0), (-b,0)$
	$c^2 = a^2 - b^2$	$a > b > 0$	

Example 10 Determine the vertices, end points of the minor axis and the coordinates of foci of the ellipse $9x^2 + 4y^2 = 36$. Sketch the ellipse.

Solution Rewrite the ellipse equation in the standard form:

$$9x^2 + 4y^2 = 36$$

$$\frac{9x^2}{36} + \frac{4y^2}{36} = 1, \quad \textit{divide out by } 36$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad (38\text{-a})$$

The equation (38-a) is related to the vertical standard form ellipse (37). The center of the ellipse is at the origin, but the vertices of the major axis are on the y–axis, since the larger numerical value is under y^2. Thus, $a^2 = 9$ or $a = 3$, and $b^2 = 4$ or $b = 2$ and $c^2 = a^2 - b^2 = 9 - 4 = 5$ or $c = \pm\sqrt{5}$.

The coordinates of the center, vertices/end points of the major axis, end points of the minor axis and the foci are the following:

$C(0,0)$ / center
$V_1(0,3), V_2(0,-3)$ end points of the major axis
$B_1(2,0), B_2(-2,0)$ end points of the minor axis
$F_1(0,\sqrt{5}), F_2(0,-\sqrt{5})$ foci

For some points on the ellipse,

when $y = 1$, then $\frac{x^2}{4} + \frac{1}{9} = 1 \Rightarrow x = \pm 4\frac{\sqrt{2}}{3}$

when $y = 2$, then $\frac{x^2}{4} + \frac{4}{9} = 1 \Rightarrow x = \pm 2\frac{\sqrt{5}}{3}$

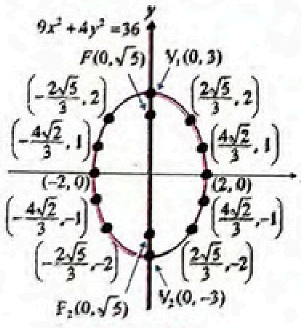


Figure 9.17

The ellipse is symmetrical with respect to the major axis, minor axis. The center, vertices, foci, and the points

$$\left(\frac{4\sqrt{2}}{3}, -1\right), \left(\frac{-4\sqrt{2}}{3}, 1\right), \left(\frac{-4\sqrt{2}}{3}, -1\right), \left(\frac{2\sqrt{5}}{3}, -2\right), \left(\frac{-2\sqrt{5}}{3}, 2\right), \left(\frac{-2\sqrt{5}}{3}, -2\right)$$

are labeled to obtain the graph of the given ellipse in Figure 9.17.

Example 11 Determine the vertices, end points of the minor axis and the coordinates of foci of the ellipse $2x^2 + 5y^2 = 10$. Sketch the ellipse.

Solution Rewrite the ellipse equation in the standard form:

$$2x^2 + 5y^2 = 10$$

$$\frac{2x^2}{10} + \frac{5y^2}{10} = 1, \quad \textit{divide out by } 10$$

$$\frac{x^2}{5} + \frac{y^2}{2} = 1 \Rightarrow \frac{x^2}{(\sqrt{5})^2} + \frac{y^2}{(\sqrt{2})^2} = 1 \qquad (38\text{-b})$$

The ellipse (38-b) is related to the horizontal standard form ellipse (36). The center of the ellipse is at the origin, but the vertices of the major axis are on the x–axis, since the large numerical value is under x^2. Thus, $a^2=5$ or $a=\sqrt{5}$ and $b^2=2$ or $b=\sqrt{2}$ and $c^2=a^2-b^2=5-2=3$ *or* $c=\pm\sqrt{3}$.

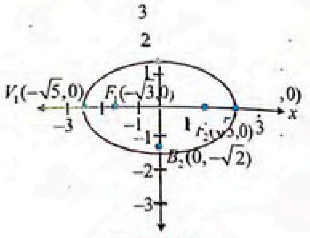


Figure 9.18

The coordinates of the center, vertices/end points of the major axis, end points of the minor axis and the foci are the following:

$C(0,0)$	center
$V_1(-\sqrt{5},0), V_2(\sqrt{5},0)$	end points of the major axis
$B_1(0,\sqrt{2}), B_2(0,-\sqrt{2})$	end points of the minor axis
$F_1(-\sqrt{3},0), F_2(\sqrt{3},0)$	foci

(iv) Equation of an ellipse through its elements

Example 12 Find an equation for the ellipse with foci $F_1(-1,0)$ and $F_2(1,0)$ and vertices $V_1(-2,0)$ and $V_2(2,0)$.

Solution By inspection, the center of the ellipse is at C(0,0) and the distance from the center to the vertex is $a=2$; and the distance to a focus is $c=1$. The value of b is obtained by inserting a and c in the equation:

$$b^2=a^2-c^2=4-1=3 \Rightarrow b=\pm\sqrt{3}$$

The values of a and b are used in the horizontal standard form ellipse (36) to obtain

$$\frac{x^2}{4}+\frac{y^2}{3}=1 \qquad (39)$$

(v) Standard form of equation of an ellipse

The standard form of the equation of an ellipse with center at C(h, k), length of the semimajor axis a and semiminor axis b, and major axis parallel to the x–*axis* is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,\ a>b \qquad (40)$$

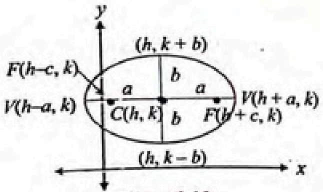


Figure 9.19

The standard form of the equation of an ellipse with center at C(h, k), length of the semimajor axis a and semiminor axis b, and major axis parallel to the y–*axis* is:

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,\ a>b \qquad (41)$$

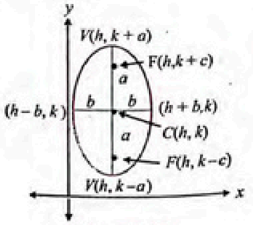


Figure 9.20

Example 13 Graph the ellipse whose equation is $4x^2+25y^2-8x+100y+4=0$. Indicate the center, vertices, foci and the end points of the minor axis.

Solution Rewrite the given ellipse equation to the standard form by completing square:

$$4x^2-8x+25y^2+100y=-4$$

$$4(x^2-2x+1)+25(y^2+4y)=0$$

Add and subtract 100 to obtain

$$4\left(x^2-2x+1\right)+25\left(y^2+4y+4\right)=100$$

$$4(x-1)^2+25(y+2)^2=100$$

$$\frac{4(x-1)^2}{100}+\frac{25(y+2)^2}{100}=1 \Rightarrow \frac{(x-1)^2}{25}+\frac{(y+2)^2}{4}=1 \qquad (42)$$

The given ellipse (42) with substitution $X=x-h=x-1$ and $Y=y-k=y+2$, $h=1$, $k=-2$ gives the translated ellipse in the XY–plane:

$$\frac{X^2}{25}+\frac{Y^2}{4}=1 \qquad (43)$$

The center of the ellipse is at the origin. The major axis is horizontal and the vertices are on the *x–axis*. Thus, $a=5$, $b=2$ and $c=\pm\sqrt{21}=\pm4.58$.

The coordinates of the center, vertices/end points of the major axis, end points of the minor axis and the foci of the translated ellipse (43) are the following:

$C(0,0)$	center
$V_1(-5,0), V_2(5,0)$	end points of the major axis
$B_1(0,2), B_2(0,-2)$	end points of the minor axis
$F_1(-4.58,0), F_2(4.58,0),\ \pm\sqrt{21}=\pm4.58$	foci

The coordinates of the center, vertices/end points of the major axis, the end points of the minor axis and foci of the given ellipse (42) are the following:

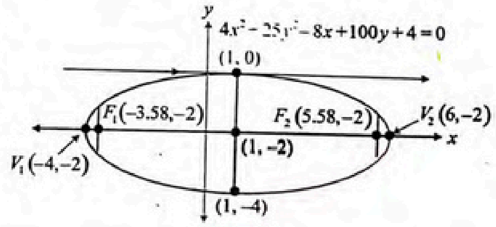


Figure 9.21

- The coordinates of the center $C(0,0)$ of the translated ellipse are $X=0$, $Y=0$. Put $X=0$ and $Y=0$ in (43) to obtain the coordinates of the center of the given ellipse (42):

  $X=x-1 \Rightarrow 0=x-1 \Rightarrow x=1$ and $Y=y+2 \Rightarrow 0=y+2 \Rightarrow y=-2$

  The center of the given ellipse (42) is $C(1,-2)$.
- The coordinates of the vertices $V_1(-5,0)$, $V_2(5,0)$ of the translated ellipse are $X=-5$, $Y=0$ (in case of V_1) and $X=-5$, $Y=0$ (in case of V_2). Put $X=-5$ and $Y=0$ in (43) to obtain the coordinates of the vertex V_1 of the given ellipse (42):

  $X=x-1 \Rightarrow -5=x-1 \Rightarrow x=-4$ and $Y=y+2 \Rightarrow 0=y+2 \Rightarrow y=-2$

The vertex V_1 of the given ellipse (42) is $V_1(-4,-2)$ and the vertex V_2 of the given ellipse (42) is of course $V_2(6,-2)$.

- The coordinates of the foci $F_1(-4.58,0), F_2(4.58,0)$ of the translated ellipse are $X=-4.58$, $Y=0$ (in case of F_1) and $X=4.58$, $Y=0$ (in case of F_2). Put $X=-4.58$ and $Y=0$ in (43) to obtain the coordinates of the focus F_1 of the given ellipse (42):

  $X=x-1 \Rightarrow -4.58=x-1 \Rightarrow x=-3.58$ and $Y=y+2 \Rightarrow 0=y+2 \Rightarrow y=-2$

  The focus F_1 of the given ellipse (42) is $F_1(-3.58,-2)$ and the focus F_2 of the given ellipse (42) is of course $F_2(5.58,-2)$.

  The graph of the ellipse is shown in the Figure 9.21.

The orientation of the ellipse equation with center $C(h, k)$ are summarized in the boxes:

Orientation	Foci	Vertices/End Points of Major Axis	End Points of Minor Axis
Horizontal	$F_1(h-c,k), F_2(h+c,k)$	$V_1(h-a,k), V_2(h+a,k)$	$B_1(h,k+b), B_2(h,k-b)$
Vertical	$F_1(h,k+c), F_2(h,k-c)$	$V_1(h,k+a), V_2(h,k-a)$	$B_1(h+b,k), B_2(h-b,k)$

Note that $b^2=a^2-c^2$ or $c^2=a^2-b^2$ with $a>b>0$.

Example 14 Find the equation of the ellipse with vertices at (–1, 2) and (7, 2) and with 2 as the length of the semiminor axis.

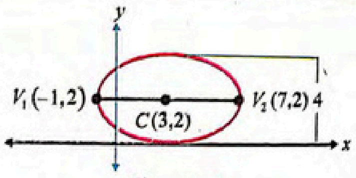


Figure 9.22

Solution With the vertices of the ellipse are at $V_1(-1,2)$ and $V_2(7,2)$, the center is at the midpoint of the line segment V_1V_2 joining these vertices. The midpoint (3, 2) of the line segment V_1V_2 is the center $C(h, k)=C(3, 2)$. This is shown in the Figure 9.22.

The distance from the center C(3, 2) to either vertex is $a=4$ units. The semiminor axis has a length of $b=2$. From the Figure 9.22, we see that the major axis is parallel to the x-axis. The horizontal standard form of the ellipse (40) is used to obtain the required ellipse equation:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-3)^2}{16}+\frac{(y-2)^2}{4}=1,\quad C(h,k)=(3,2), a=4, b=2 \qquad (44)$$

(vi) Recognition of tangent and normal to an ellipse

A. Tangent to an ellipse

"A line that interacts the ellipse at a point is known as tangent at the ellipse" in the Figure 9.23, the line LM is tangent to the ellipse which is intersecting the ellipse at point 'P' as shown in Figure 9.23.

B. Normal to an ellipse

Normal to an ellipse is a line perpendicular to the tangent to curve through the point of contact. Line QR is normal to the ellipse which is perpendicular to the tangent LM at point 'P', as shown in Figure 9.23.

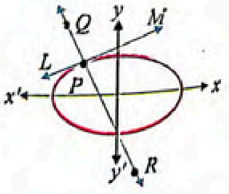


Figure 9.23

(vii) Point of Intersection of an ellipse and a line

The given line and ellipse

$$y=mx+c \qquad (45)$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (46)$$

develops a system of nonlinear equations:

$$\left.\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\\ y=mx+c\end{aligned}\right\} \qquad (47)$$

Note

The angle between tangent to ellipse and normal is always a right angle.

The solution set $\{x,y\}$ of nonlinear system of equations (47) exists only, if the curves of the system (47) are intersecting. That set of points of intersection $\{x,y\}$ (a solution set) can be found by solving the nonlinear system (47) simultaneously.

The line (45) is used in ellipse (46) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

$$x^2\left(a^2m^2+b^2\right)+2a^2mcx+a^2\left(c^2-b^2\right)=0 \qquad (48)$$

The equation (48) being a quadratic equation in x, gives a set of two values x_1 and x_2 of x, which will be used in a line (45) to obtain a set of two values y_1 and y_2 of y.

Thus, a solution set $\{(x_1,y_1),(x_2,y_2)\}$ of the system (47) is of course a set of points of intersection of the system (47).

The points of intersection of the system (47) are real, coincident or imaginary, according as the roots of the quadratic equation (48) are real, coincident or imaginary, according as the discriminant of the quadratic equation (48)

$$Disc=4a^4m^2c^2-4\left(a^2m^2+b^2\right)\left(a^2\right)\left(c^2-b^2\right)>0, \text{ real and different}$$

$$Disc=4a^4m^2c^2-4\left(a^2m^2+b^2\right)\left(a^2\right)\left(c^2-b^2\right)=0, \text{ real and coincident}$$

$$Disc=4a^4m^2c^2-4\left(a^2m^2+b^2\right))\left(a^2\right)\left(c^2-b^2\right)<0, \text{ imaginary}$$

Example 15 Find the points of intersection of the line $2x-y-2=0$ and the ellipse $4x^2+9y^2=36$.

Solution The equations of the line and ellipse are:

$$2x-y-2=0 \qquad (49)$$

$$y=2x-2$$

$$4x^2+9y^2=36 \qquad (50)$$

The line (49) is used in an ellipse (50) to obtain the x–coordinates of the points of intersection:

$$4x^2+9(2x-2)^2=36$$

$$4x^2+9(4x^2+4-8x)-36=0$$

$$40x^2-72x=0$$

$$\Rightarrow x=0,\ x=\frac{9}{5}$$

The x–coordinates are used in the line (49) to obtain the y–coordinates: $x = 0, \frac{9}{5}$ give $y = -2, \frac{8}{5}$

Thus, the set of two points of intersection (0,–2) and $\left(\frac{9}{5}, \frac{8}{5}\right)$ are real and distant and the line $2x - y - 2 = 0$ intersects the ellipse (50) at points (0,–2) and $\left(\frac{9}{5}, \frac{8}{5}\right)$.

(viii) The equation of a tangent line in slope-form

If m is the slope of the tangent line to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (51)

then the equation of that tangent line is of the form $y = mx + c$ (52)

Here c is to be calculated from the fact that the line (52) is tangent to ellipse (51).

The line (52) is used in an ellipse (51) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$x^2\left(a^2m^2 + b^2\right) + 2a^2mcx + a^2\left(c^2 - b^2\right) = 0 \quad (53)$$

If the line (52) touches the ellipse (51), then the quadratic equation (53) has coincident roots for which the discriminant of the quadratic equation (53) equals zero:

$$4a^4m^2c^2 - 4\left(a^2m^2 + b^2\right)\left(a^2\right)\left(c^2 - b^2\right) = 0$$

$$a^2m^2c^2 - \left(a^2m^2 + b^2\right)\left(c^2 - b^2\right) = 0, \textit{ divide out by } 4a^2$$

$$a^2m^2c^2 - a^2m^2c^2 + a^2m^2b^2 - b^2c^2 + b^4 = 0$$

$$a^2m^2b^2 - b^2c^2 + b^4 = 0$$

$$-b^2c^2 = -\left(a^2m^2b^2 + b^4\right)$$

$$c^2 = a^2m^2 + b^2$$

$$c = \pm\sqrt{a^2m^2 + b^2} \quad (54)$$

The equation (54) is the **condition of tangency**. The value of c from equation (54) is used in the line (52) to obtain the required equation of the tangent line: $y = mx + c = mx \pm \sqrt{a^2m^2 + b^2}$ (55)

Note

- the equation of any tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the slope–form is:

  $$y = mx \pm \sqrt{a^2m^2 + b^2} \quad (56)$$

- **Condition of Tangency:** The line $y = mx + c$ should touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ under condition: $c = \pm\sqrt{a^2m^2 + b^2}$ (57)

Example 16 For what value of c, the line $2x - y + c = 0$ will touch an ellipse $\frac{x^2}{3} + \frac{y^2}{4} = 1$. Use those values of c to find the tangent lines that should touch the given ellipse.

Solution The values of c at which the line $2x - y + c = 0$ will touch the given ellipse through result (57) are: $c = \pm\sqrt{a^2m^2 + b^2} = \pm\sqrt{3(2)^2 + 4} = \pm 4$

Here $m = 2$ is the slope of the line $2x - y + c = 0$.

The required tangent lines that should touch the ellipse through result (56) is:

$$y = mx + c = 2x \pm 4, \; m = 2$$

(ix) The equation of a tangent line to ellipse at a point

The equation of a tangent line at a point $P(x_1, y_1)$ to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is: $y - y_1 = m_1(x - x_1)$ (58)

Here m_1 is the slope of the tangent line to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point $P(x_1, y_1)$ that can be found by differentiating $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to x:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-b^2x_1}{a^2y_1} = m_1 \quad (59)$$

The substitution of (59) in (58) is giving the equation of the tangent line at a point $P(x_1, y_1)$ to ellipse:

$$y - y_1 = m_1(x - x_1)$$

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = -\frac{xx_1}{a^2} + \frac{x_1^2}{a^2} \Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad (60) \quad \therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \; at \; (x_1, y_1)$$

Example 17 Find the equation of the tangent at a point $P\left(3, \frac{12}{5}\right)$ to ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

Solution Result (60) is used to obtain the tangent line to the given ellipse:

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

$$\frac{x(3)}{25} + \frac{y\left(\frac{12}{5}\right)}{9} = 1 \qquad \therefore (x_1, y_1) = \left(3, \frac{12}{5}\right), a^2 = 25, b^2 = 9$$

$$\frac{3x}{25} + \frac{12y}{45} = 1 \Rightarrow 27x + 60y - 225 = 0 \Rightarrow 9x + 20y - 75 = 0$$

(x) The equation of a normal line to ellipse at a point

The equation of a normal at a point $P(x_1, y_1)$ to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is:

$$y-y_1=m_2(x-x_1) \qquad (61)$$

Here m_2 is the slope of the normal to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ that can be found by differentiating $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to x:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-b^2x_1}{a^2y_1}=m_1$$

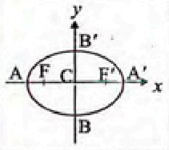
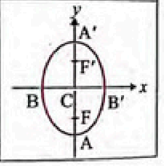
$$m_2=-\frac{1}{m_1}=\frac{a^2y_1}{b^2x_1} \qquad (62)$$

The substitution of (62) in (61) is giving the normal equation at a point $P(x_1, y_1)$ to ellipse:

$$y-y_1=m_2(x-x_1)$$

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1),\ m_2=\frac{a^2y_1}{b^2x_1}$$

$$\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{x-x_1}{\frac{x_1}{a^2}} \qquad (63)$$

Summary of standard Ellipses		
Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$ $c^2=a^2-b^2$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a<b$ $c^2=a^2-b^2$
Focus	$(\pm c, 0)$	$(0, \pm C)$
Directrices	$x=\pm\frac{c}{e^2}$	$y=\pm\frac{c}{e^2}$
Major axis	$y=0$	$x=0$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Covertices	$(0, \pm b)$	$(\pm b, 0)$
Centre	$(0,0)$	$(0,0)$
Eccentricity	$e=\frac{c}{a}<1$	$e=\frac{c}{a}<1$
Graph		

Example 18 Find the normal equation at a point $P\left(3,\frac{12}{5}\right)$ to ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Solution Result (63) is used to obtain the normal line to the given ellipse:

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}} \qquad \therefore (x_1,y_1)=\left(3,\frac{12}{5}\right), a^2=25, b^2=9$$

$$\frac{x-3}{\frac{3}{25}}=\frac{y-\frac{12}{5}}{\frac{\left(\frac{12}{5}\right)}{(9)}}$$

$$\frac{25(x-3)}{3}=\frac{3(5y-12)}{4}$$

$$100(x-3)=9(5y-12) \Rightarrow 100x-300-45y+108=0 \Rightarrow 100x-45y-192=0$$

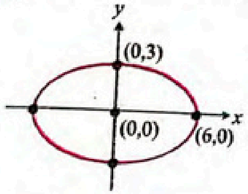
Exercise 9.2

1. In each case, sketch the ellipse represented by the equation. Indicate the center, foci, endpoints of the major axis and end points of the minor axis:

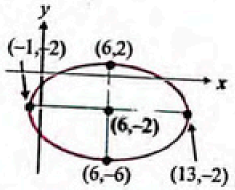
   a. $\frac{x^2}{9}+\frac{y^2}{4}=1$ b. $\frac{x^2}{16}+\frac{y^2}{25}=1$ c. $\frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1$

2. In each case, determine the equation of graphed ellipse:

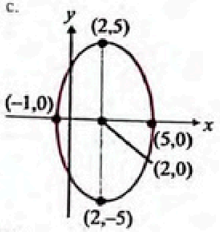
   a.



   b.



   c.



3. In each case, write the equation of ellipse through the given information:

   a. Center is at (–3, 2), $a=2$, $b=1$, major axis is horizontal.
   b. Vertices are at (4, 2) and (12,2), $b=2$.
   c. A focus is at (–2, 3), a vertex is at (6,3), length of minor axis is 6.
   d. Vertices are at (0,8) and (0,2), $c=\sqrt{5}$.

4. The shape of an ellipse depends on the eccentricity of the ellipse $e=\frac{c}{a}$. Determine

   a. the eccentricity of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.
   b. the equation of the ellipse with vertices are at (–5,0), and (5,0) and the eccentricity is $e=\frac{3}{5}$.
   c. the eccentricity of the ellipse, if the length of the semimajor axis is $a=4$ and the length of the semiminor axis is $b=2$.

5. For what value of c,

   a. the line $x-y+c=0$ will touch the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$?
   b. the line $2x-y+c=0$ will touch the ellipse $\frac{x^2}{3}+\frac{y^2}{4}=1$?
   c. the line $x+y+c=0$ will touch the ellipse $\frac{x^2}{25}+\frac{y^2}{11}=1$?

6. In each case, find the tangent equation and normal equation

   a. at a point (1,2) to ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$? b. at a point (3,6) to ellipse $\frac{x^2}{7}+\frac{y^2}{4}=1$?
   c. at a point (1,1) to ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$?

7. Find the tangent equation

   a. to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ which is perpendicular to the line $9x+8y-36=0$.
   b. to the ellipse $\frac{x^2}{7}+\frac{y^2}{4}=1$ which is parallel to the line $6x+21y-14=0$.

9.3 Hyperbola

The last of the conic sections to be considered has a definition similar to that of the ellipse. In the previous section, the ellipse was expressed in terms of the **sum** of two distances being a constant. Exactly, the hyperbola is expressed in terms of the **difference** of two distances being a constant.

(i) Hyperbola and its center, foci, vertices, eccentricity, focal chord, transverse and conjugate axes, eccentricity, focal chord and latus rectum

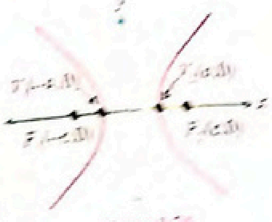


Figure 9.24

A **hyperbola** is the set of all points in the plane such that the difference of the distances from two fixed points (foci) to a point on the hyperbola hold off constant quantity.

The two fixed points $F_1(-c,0)$ and $F_2(c,0)$ are called the **foci** that lie on the x-axis at a distance c on each side of the origin. The hyperbola crosses the x-axis at two points $V_1(-a,0)$ and $V_2(a,0)$ that are at a distance a on each side of the origin called **vertices**. The **transverse axis** (major axis) of the hyperbola coincides with the x-axis, while the **conjugate axis** (minor axis) of the hyperbola coincides with the y-axis. This is shown in the Figure 9.24.

To check the definition, the absolute value of the difference of the distances from the two foci $F_1(-c,0)$, $F_2(c,0)$ to the point $V_2(a,0) = P$, say, is

$$|d(F_1,V_2) - d(F_2,V_2)| = |(a-(-c)) - (c-a)| = 2a$$

Since a is a measured distance and is always positive, the constant specified in the definition of a hyperbola equals 2a.

(ii) General form of the hyperbola

If P(x, y) is any point on the hyperbola, then the distances from the two foci $F_1(-c,0)$ and $F_2(c,0)$ to the point P(x, y) are the following:

$$F_1P = (x-(-c),\ y-0)$$

$$d(F_1,P) = \sqrt{(x+c)^2+y^2}$$

$$F_2P = (x-c,\ y-0)$$

$$d(F_2,P) = \sqrt{(x-c)^2+y^2}$$

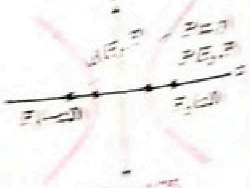


Figure 9.25

The definition of hyperbola is used to obtain the general form equation:

$$d(F_1,P) - d(F_2,P) = 2a$$

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = 2a$$

$$\sqrt{(x+c)^2+y^2} = 2a + \sqrt{(x-c)^2+y^2}$$

Squaring both sides, we obtain

$$(x+c)^2+y^2 = 4a^2 + 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$$

$$4a\sqrt{(x-c)^2+y^2} = 4cx - 4a^2$$

$$a\sqrt{(x-c)^2+y^2} = cx - a^2$$

Again squaring, we obtain

$$a^2(x-c)^2 + a^2y^2 = c^2x^2 - 2a^2cx + a^4$$

$$a^2(x^2-2cx+c^2) + a^2y^2 = c^2x^2 - 2a^2cx + a^4$$

$$a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 = c^2x^2 - 2a^2cx + a^4$$

$$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$$

$$-x^2(c^2-a^2) + a^2y^2 = -a^2(c^2-a^2)$$

$$-b^2x^2 + a^2y^2 = -a^2b^2$$

$$b^2x^2 - a^2y^2 = a^2b^2 \qquad \because c^2 - a^2 = b^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{Dividing by } a^2b^2$$

Notice that $b^2 = a^2 - c^2$ for ellipse and that $b^2 = c^2 - a^2$ for the hyperbola. For ellipse, it is necessary that $c^2 < a^2$, but for the hyperbola, there is no restriction on the relative sizes for a and b but c is still greater than a for the hyperbola.

(iii) Standard form of the equation of hyperbola

The standard form of the equation of a hyperbola with center at the origin and the x-axis as the transverse axis is shown in the Figure 9.26.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

If we replace the foci on the y-axis, center at the origin, and pick any point P(x, y) on the plane, then we can develop the equation of the vertical hyperbola given below (standard form of vertical hyperbola).

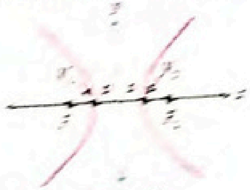


Figure 9.26

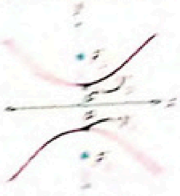


Figure 9.27

Standard form of vertical Hyperbola: The standard form of the equation of a hyperbola with center at the origin and the y-axis as the transverse axis is shown in the Figure 9.27.

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Graphing standard form Hyperbola: As with the other conics, we shall sketch a hyperbola by determining some information about the curve directly from the equation by inspection. The end points of the transverse axis are the vertices of the hyperbola at $V_1(-a,0)$ and $V_2(a,0)$. The axis has a length of 2a. The conjugate axis coincides with the y-axis and has its end points at (0, b) and (0, −b). If we set x = 0, then there is no real number y such that $y = \pm\sqrt{-b^2}$ is a complex conjugate. The equation is why should we be concerned about the conjugate axis or the length 2b? The significance of b is determined by solving the standard form of hyperbola for y:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$-\frac{y^2}{b^2}=-\frac{x^2}{a^2}+1$$

$$y^2=b^2\left(\frac{x^2-a^2}{a^2}\right)=\frac{b^2}{a^2}(x^2-a^2)=\frac{b^2x^2}{a^2}\left(1-\frac{a^2}{x^2}\right)$$

$$y=\pm\frac{bx}{a}\sqrt{1-\frac{a^2}{x^2}} \qquad (68)$$

Let us examine the fraction $\frac{a^2}{x^2}$ (a is constant). If we substitute larger and larger values for x, then the fraction $\frac{a^2}{x^2}$ becomes smaller and smaller. In fact, the fraction eventually gets very close to zero. Thus, for large values of x, the term $1-\frac{a^2}{x^2}$ approaches 1. Therefore, for large values of x, the y values approaches the value $\pm\frac{b}{a}x$, and the value of the hyperbola gets closer and closer to the lines:

$$y=\pm\frac{b}{a}x.$$

These lines are called the **asymptotes** of the hyperbola. As x takes on values that are greater distances from the center of the hyperbola, the values of y (of the hyperbola) become closer and closer to the asymptotes even though they never actually reach the corresponding y–values of the asymptotes. Since these lines are easy to graph, the asymptotes are valuable aids in sketching the hyperbola.

- For horizontal standard form hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the asymptotes are the lines:

$$y=\pm\frac{b}{a}x \qquad (69)$$

- For vertical standard form hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, the asymptotes are the lines

$$y=\pm\frac{a}{b}x \qquad (70)$$

Asymptotes:

If a straight line cuts a hyperbola in two points at an infinite distance from the origin and is itself at a finite distance from the origin is then called the **asymptotes**.

The orientation of the hyperbola with center C(0,0), vertices and foci are summarized in the box:

Orientation	Foci	Vertices
Horizontal	$F_1(-c,0), F_2(c,0)$	$V_1(-a,0), V_2(a,0)$
Vertical	$F_1(0,c), F_2(0,-c)$	$V_1(0,a), V_2(0,-a)$

Note that $c^2=a^2+b^2, c>0$.

Example 19 Sketch the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$, with center at the origin and the transverse axis is at the x–axis. Determine the vertices and foci of the hyperbola.

Solution The given hyperbola equation is in the standard form of the equation of a hyperbola with transverse axis along the x–axis. This tells us that $a^2=9, a=\pm3$ and $b^2=16, b=\pm4$. The vertices of the hyperbola are $V_1(-3,0)$ and $V_2(3,0)$. The value of c for foci can be found by using the formula:

$c^2=a^2+b^2=9+16=25, c=\pm5$

The foci are therefore $F_1(-5,0)$ and $F_2(5,0)$. The asymptotes are the lines

$$y=\frac{b}{a}x=\frac{4}{3}x, \quad y=-\frac{b}{a}x=-\frac{4}{3}x$$

For sketching the hyperbola, the end points of the conjugate axis (0,–4) and (0,4) are located, then draw the lines through the points (0,–4) and (0,4) parallel to the x–axis. Similarly, draw the lines through the end points of the transverse axis $V_1(-3,0)$ and $V_2(3,0)$ parallel to y–axis to complete the rectangle. The resultant rectangle and the extended diagonals of the rectangle are the asymptotes of the hyperbola. The sketch of the hyperbola is shown in the Figure 9.28.

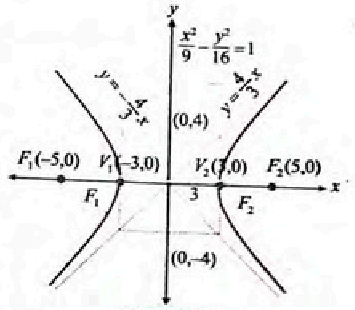


Figure 9.28

Example 20 Sketch the hyperbola $16y^2-9x^2=144$, with center at the origin and the transverse axis is at the y–axis. Determine the vertices and foci of the hyperbola.

Solution Rewrite the given hyperbola in the standard form of the hyperbola (67):

$$16y^2-9x^2=144$$

$$\frac{16y^2}{144}-\frac{9x^2}{144}=1 \Rightarrow \frac{y^2}{9}-\frac{x^2}{16}=1 \qquad (71)$$

If the transverse axis is along the y–axis, then select $a^2=9, a=\pm3$ and $b^2=16, b=\pm4$. The vertices of the hyperbola are $V_1(0,3)$ and $V_2(0,-3)$. The end points of the conjugate axis are (–4, 0) and (4, 0). The value of c for foci can be found by using the formula:

$$c^2=a^2+b^2=9+16=25, c=\pm5$$

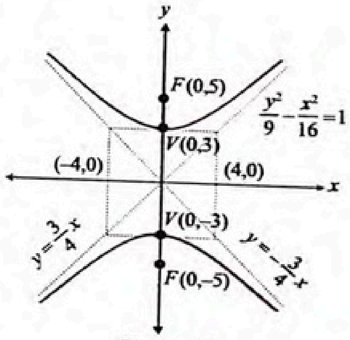


Figure 9.29

The foci are therefore $F_1(0,5)$ and $F_2(0,-5)$. The asymptotes are the lines

$$y=\frac{a}{b}x=\frac{3}{4}x, \quad y=-\frac{a}{b}x=-\frac{3}{4}x$$

Sketch the rectangle formed by the points $(0,\pm3)$ and $(\pm4,0)$ and then sketch the asymptotes using the diagonals of the rectangle. With the asymptotes, vertices and foci, it is easy to sketch the hyperbola, as shown in the Figure 9.29.

(iv) Equation of hyperbola through its elements

Example 21 Find the vertices, foci, eccentricity and the asymptotes of the hyperbola $16x^2-9y^2=144$.

Solution Rewrite the given hyperbola in the standard form:

$$16x^2-9y^2=144$$

$$\frac{16x^2}{144}-\frac{9y^2}{144}=1 \Rightarrow \frac{x^2}{9}-\frac{y^2}{16}=1$$

If the transverse axis is the x–axis, then select $a^2=9,\ a=3$ and $b^2=16,\ b=4$. The value of c is obtained by formula $c=\sqrt{a^2+b^2}=\sqrt{9+16}=\pm 5$.

The vertices, foci, eccentricity and asymptotes of the given hyperbola are the following:

$V_1(a,0)=V_1(3,0),\ V_2(-a,0)=V_2(-3,0)$	vertices
$F_1(c,0)=F_1(5,0),\ F_2(-c,0)=F_2(-5,0)$	foci
$e=\frac{c}{a}=\frac{5}{3}>1$	eccentricity
$y=\pm\frac{b}{a}x=\pm\frac{4}{3}x$	asymptotes

Example 22 Find the equation of hyperbola, when one focus is at $(0,6)$, center is at $C(0,0)$ and the eccentricity is 3.

Solution Focus (0,6) gives $c=6$. This indicates that the transverse axis is the y–axis. The eccentricity 3 is giving the value of a:

$$e=\frac{c}{a}\Rightarrow 3=\frac{6}{a}\Rightarrow a=2$$

These values of a and c is used in the formula to obtain the value of b:

$$c^2=a^2+b^2$$

$$b^2=c^2-a^2=36-4=32$$

The required hyperbola equation is:

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

$$\frac{y^2}{4}-\frac{x^2}{32}=1,\ a^2=4,\ b^2=32$$

(v) Standard form of equation of Hyperbola

If any point (h, k) on the plane is selected as the center of the hyperbola and a major axis parallel to the x–axis or y–axis is selected, then with the geometrical definition, a new set of equations for hyperbola can be derived through translation of axes.

Translation of hyperbola horizontally:

The standard form of the equation of a hyperbola with center at $C(h, k)$, vertices at $V_1(h+a,k)$ and $V_2(h-a,k)$, foci at $F_1(h-c,k)$ and $F_2(h+c,k)$ is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \qquad (72)$$

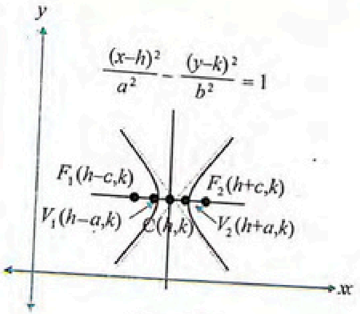


Figure 9.30

Translation of hyperbola vertically:

The standard form of the equation of a hyperbola with center at $C(h, k)$, vertices at $V_1(h,k+a)$ and $V_2(h,k-a)$, foci at $F_1(h,k+c)$ and $F_2(h,k-c)$ is:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \qquad (73)$$

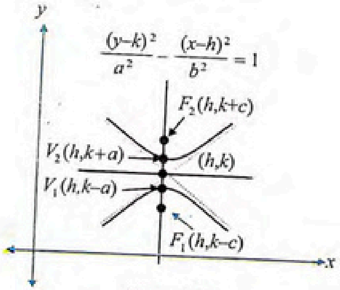


Figure 9.31

The orientation of the hyperbola equation with center $C(h, k)$ are summarized in the box:

Orientation	Foci	Vertices
Horizontal	$F_1(h-c,k), F_2(h+c,k)$	$V_1(h-a,k), V_2(h+a,k)$
Vertical	$F_1(h,k+c), F_2(h,k-c)$	$V_1(h,k+a), V_2(h,k-a)$

Note that $c^2=a^2+b^2$, $c>0$.

Example 23 Sketch the hyperbola

$$\frac{(x-1)^2}{144}-\frac{(y+2)^2}{25}=1$$

with center at $C(h, k)$ and the transverse axis is the x–axis. Determine the vertices and foci of the hyperbola.

Solution The given hyperbola is:

$$\frac{(x-1)^2}{144}-\frac{(y+2)^2}{25}=1 \qquad (74)$$

The given hyperbola (74) with substitution
$X=x-h=x-1$ and $Y=y-k=y+2,\ h=1,\ k=-2$ gives the new hyperbola in the XY–system:

$$\frac{X^2}{144}-\frac{Y^2}{25}=1 \qquad (75)$$

The hyperbola (75) has a transverse axis on the x–axis that give $a^2=144, a=12$ and $b^2=25, b=5$.

The value of c is obtained by formula:

$$c=\sqrt{a^2+b^2}=\sqrt{144+25}=13$$

The coordinates of the vertices and foci of the new hyperbola in the XY–system are the following:

$V_1(-a,0)=V_1(-12,0), V_2(a,0)=V_2(12,0)$ vertices

$F_1(-c,0)=F_1(-13,0), F_2(c,0)=F_2(13,0)$ foci

The asymptotes in the XY–system are the lines:

$$Y=\frac{b}{a}X=\frac{5}{12}X, \qquad Y=-\frac{b}{a}X=-\frac{5}{12}X$$

The coordinates of the vertices and foci of the given hyperbola (74) in the xy–system are the following:

$V_1(h+a,k)=V_1(13,-2), V_2(h-a,k)=V_2(-11,-2)$ vertices

$F_1(h-c,k)=F_1(-12,-2), F_2(h+c,k)=V_2(14,-2)$ foci

The asymptotes in the XY–system can be converted in the xy–system to obtain:

$$Y=\frac{b}{a}X$$

$$=\frac{5}{12}X,\ a=12, b=5$$

$$(y-k)=\frac{5}{12}(x-h),\ X=x-h, Y=y-k,\ h=1, k=-2$$

$$y+2=\frac{5}{12}(x-1),$$

$$12y+24=5x-5 \Rightarrow 12y-5x+29=0$$

$$Y=-\frac{b}{a}X$$

$$=-\frac{5}{12}X$$

$$(y-k)=-\frac{5}{12}(x-h)$$

$$y+2=-\frac{5}{12}(x-1) \Rightarrow 12y+24=-5x+5 \Rightarrow 12y+5x+19=0$$

Having sketched the asymptotes and the vertices, we can sketch the hyperbola as shown in the Figure 9.32 .

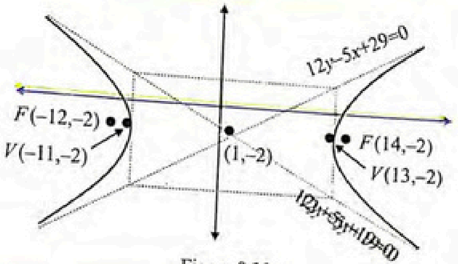


Figure 9.32

- For translating the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ horizontally, the asymptotes are the lines:

$$(y-k)=\pm\frac{b}{a}(x-h) \qquad (76)$$

- For translating the hyperbola $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ vertically, the asymptotes are the lines:

$$(y-k)=\pm\frac{a}{b}(x-h) \qquad (77)$$

(vi) Recognition of tangent and normal to hyperbola

A line which intersects a hyperbola in two coincident points is a tangent. For the hyperbola, there will be two tangents [real and distanct, coincident (with an asymptote), or complex] with a given slope. The formulation for tangents to hyperbola will be discussed in the succeeding sections.

(vii) Point of intersection of hyperbola with a line including the condition of tangency

The given line and hyperbola

$$y=mx+c \qquad (78)$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (79)$$

develops a system of nonlinear equations:

$$\left.\begin{aligned}\frac{x^2}{a^2}-\frac{y^2}{b^2}&=1\\ y&=mx+c\end{aligned}\right\} \qquad (80)$$

The solution set $\{x,y\}$ of nonlinear system of equations (80) exists only, if the curves of the system (80) are intersecting. That set of points of intersection $\{x,y\}$ (a solution set) can be found by solving the nonlinear system (80) simultaneously.

The line (78) is used in hyperbola (79) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

$$x^2\left(b^2-a^2m^2\right)-2a^2mcx+a^2\left(c^2+b^2\right)=0 \qquad (81)$$

The equation being a quadratic equation in x, gives a set of two values x_1 and x_2 of x, which will be used in a line to obtain a set of two values y_1 and y_2 of y.

The solution set $\{(x_1, y_1), (x_2, y_2)\}$ of the system is of course a set of points of intersection of the system.

The points of intersection of the system are real, coincident or imaginary, according as the roots of the quadratic equation are real, coincident or imaginary, or according as the discriminant of the quadratic equation :

$$Disc = 4a^4m^2c^2 + 4(b^2 - a^2m^2)(a^2)(c^2 + b^2) > 0, \text{ real and different}$$

$$Disc = 4a^4m^2c^2 + 4(b^2 - a^2m^2)(a^2)(c^2 + b^2) = 0, \text{ real and coincident}$$

$$Disc = 4a^4m^2c^2 + 4(b^2 - a^2m^2)(a^2)(c^2 + b^2) < 0, \text{ imaginary}$$

Example 24 Find the points of intersection of the line $x - y - 1 = 0$ and the hyperbola $4x^2 - y^2 = 4$.

Solution The equations of the line and hyperbola are :

$$x - y - 1 = 0 \qquad (82)$$

$$y = x - 1$$

$$4x^2 - y^2 = 4 \qquad (83)$$

The line (82) is used in hyperbola (83) to obtain the x–coordinates of the points of intersection:

$$4x^2 - (x - 1)^2 = 4$$

$$4x^2 - (x^2 + 1 - 2x) - 4 = 0 \Rightarrow 3x^2 + 2x - 5 = 0 \Rightarrow x = 1, -\frac{5}{3}$$

The x–coordinates are used in the line to obtain the y–coordinates:

$$x = 1, -\frac{5}{3} \text{ give } y = 0, -\frac{8}{3}$$

Thus, the set of two points of intersection (1,0) and $\left(-\frac{5}{3}, -\frac{8}{3}\right)$ are real and distinct and the line $x - y - 1 = 0$ intersects the hyperbola (83) at points (1,0) and $\left(-\frac{5}{3}, -\frac{8}{3}\right)$.

(viii) Equation of a tangent line in slope-form

If m is the slope of the tangent line to hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (84)$$

then the equation of that tangent line is of the form

$$y = mx + c \qquad (85)$$

Here c is to be calculated from the fact that the line (85) is tangent to hyperbola (84).
The line (85) is used in hyperbola (84) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$$

$$x^2(b^2 - a^2m^2) - 2a^2mcx + a^2(c^2 + b^2) = 0 \qquad (86)$$

If the line (85) touches the hyperbola (84), then the quadratic equation (86) has coincident roots for which the discriminant of the quadratic equation (86) is going to be zero:

$$4a^4m^2c^2 + 4(b^2 - a^2m^2)(a^2)(c^2 + b^2) = 0$$

$$a^2m^2c^2 + (b^2 - a^2m^2)(c^2 + b^2) = 0, \textit{ divide out by } 4a^2$$

$$a^2m^2c^2 - a^2m^2c^2 - a^2m^2b^2 + b^2c^2 + b^4 = 0$$

$$-a^2m^2b^2 + b^2c^2 + b^4 = 0$$

$$-a^2m^2 + c^2 + b^2 = 0$$

$$c^2 = (a^2m^2 - b^2)$$

$$c^2 = a^2m^2 - b^2$$

$$c = \pm\sqrt{a^2m^2 - b^2} \qquad (87)$$

The equation (87) is the **condition of tangency**. The value of c from equation (87) is used in the line (85) to obtain the required equation of the tangent line:

$$y = mx + c = mx \pm \sqrt{a^2m^2 - b^2} \qquad (88)$$

Remember

- the equation of any tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in the slope–form is:
  $y = mx \pm \sqrt{a^2m^2 + b^2}$ (89)
- the line $y = mx + c$ should touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ under condition:
  $c = \pm\sqrt{a^2m^2 - b^2}$ (90)

Example 25 For what value of c, the line $y = \frac{5}{2}x + c$ will touch the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$. Use those values of c to find the tangent lines that should touch the given hyperbola.

Solution The values of c at which the line $y = \frac{5}{2}x + c$ will touch the given hyperbola through result (87) are:

$$c = \pm\sqrt{a^2m^2 - b^2} = \pm\sqrt{4\left(\frac{25}{4}\right) - 9} = \pm 4 \qquad \therefore a^2 = 4, b^2 = 9, m = \frac{5}{2}$$

Here $m = \frac{5}{2}$ is the slope of the line $y = \frac{5}{2}x + c$. The required tangent lines that should touch the hyperbola through result (88) is:

$$y = mx + c = \frac{5}{2}x \pm 4 \qquad \therefore m = \frac{5}{2}$$

(ix) Equation of a tangent line to hyperbola at a point

The equation of the tangent at a point $P(x_1, y_1)$ to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is:

$$y - y_1 = m_1(x - x_1)$$

Here m_1 is the slope of the tangent line to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ that can be found by differentiating $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with respect to x:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{differentiate w.r.t. } x$$

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{b^2x_1}{a^2y_1}=m_1, say \qquad (92)$$

The substitution of (92) in (91) is giving the equation of the tangent line at a point $P(x_1, y_1)$ to hyperbola:

$$y-y_1=m_1(x-x_1)$$

$$y-y_1=\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=\frac{xx_1}{a^2}-\frac{x_1^2}{a^2} \quad \Rightarrow \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$$

$$\Rightarrow \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \qquad (93) \qquad \therefore \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1\, at\, (x_1, y_1)$$

Example 26 Find the equation of the tangent at a point $P\left(5,\frac{16}{9}\right)$ to hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$.

Solution Result (93) is used to obtain the tangent line to the given hyperbola:

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1, \quad a^2=9, b^2=16$$

$$\frac{x(5)}{9}-\frac{y\left(\frac{16}{9}\right)}{16}=1 \qquad \therefore (x_1, y_1)=\left(5,\frac{16}{9}\right)$$

$$\frac{5x}{9}-\frac{16y}{144}=1$$

$$80x-16y=144 \Rightarrow 80x-16y-144=0 \Rightarrow 5x-y-9=0$$

(x) Equation of a normal line to hyperbola at a point

The equation of the normal at a point $P(x_1, y_1)$ to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is:

$$y-y_1=m_2(x-x_1) \qquad (94)$$

Here m_2 is the slope of the normal to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, at a point $P(x_1, y_1)$ that can be found by differentiating $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with respect to x:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{b^2x_1}{a^2y_1}=m_1,$$

$$m_2=-\frac{1}{m_1}=\frac{-a^2y_1}{b^2x_1}, say \qquad (95)$$

The substitution of (95) in (94) is giving the normal equation at a point $P(x_1, y_1)$ to hyperbola:

$$y-y_1=m_2(x-x_1)$$

$$y-y_1=\frac{-a^2y_1}{b^2x_1}(x-x_1) \qquad \therefore m_2=-\frac{a^2y_1}{b^2x_1}$$

$$\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{-(x-x_1)}{\frac{x_1}{a^2}} \qquad (96)$$

Example 27 Find the normal equation at a point $P\left(5,\frac{16}{9}\right)$ to hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$.

Solution Result (96) is used to obtain the normal line to the given hyperbola:

$$\frac{-(x-x_1)}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}, \quad a^2=9, b^2=16$$

$$\frac{-(x-5)}{\frac{5}{9}}=\frac{y-\frac{16}{9}}{\frac{\left(\frac{16}{9}\right)}{16}} \qquad \therefore (x_1, y_1)=\left(5,\frac{16}{9}\right)$$

$$\frac{-9(x-5)}{5}=\frac{(9y-16)}{1}$$

$$\Rightarrow -9(x-5)(1)=5(9y-16)$$

$$-9x+45=45y-80$$

$$9x+45y=125$$

Exercise 9.3

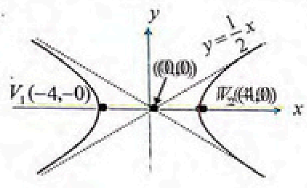
1. In each case, sketch the hyperbola represented by the equation. Indicate the center, vertices, foci and the equations of the asymptotes:

a. $\frac{x^2}{4}-\frac{y^2}{9}=1$ b. $\frac{y^2}{25}-\frac{x^2}{4}=1$

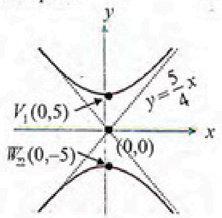
c. $\frac{(x-2)^2}{9}-\frac{(y-3)^2}{16}=1$ d. $\frac{(y+1)^2}{16}-\frac{(x+3)^2}{25}=1$

2. In each case, determine the equation of graphed ellipse:

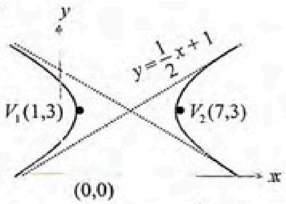
a.



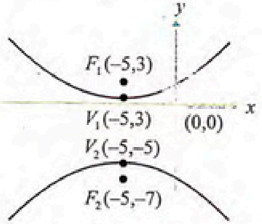
b.



c.



d.



3. In each case, write the equation of hyperbola through the given information:

a. Foci are at (0,3) and (0,–3), one vertex is at (0,–2).

b. Vertices are at (5,0) and (–5,0), one focus is at (–7,0).

c. Transverse axis is the x–axis, asymptotes are the lines $y=3x$ and $y=-3x$.

d. Foci are at (5,0) and (–5,0), eccentricity is 5/3.

e. Vertices are at (3,–1) and (–1,–1), asymptotes are the lines $y=(9/4)x-(13/4)$ and $y=(-9/4)x+(5/4)$.

4. Determine the path of a point that moves so that the difference of its distances from

a. the points (–5,0) and (5,0) is 8.

b. the points (0,–13) and (0,13) is 10.

5. Write the equation of the hyperbola

a. with vertices at (2,–2), (–4,–2) and that passes through the point with coordinates (5,1).

b. with vertices at (–3,1), (–3,3) and that passes through the point with coordinates (0,4).

6. In each case, sketch the rectangular hyperbola and identify the vertices, the foci and the asymptotes:

a. $(x+1)^2-(y-2)^2=1$ b. $\frac{(x-3)^2}{4}-\frac{(y+1)^2}{4}=1$

7. In each case, find the points of intersection in between the line and the hyperbola:

a. $xy=4,\ y=x-3$ b. $\frac{y^2}{1}-\frac{x^2}{4}=1,\ x+4y-4=0$

8. For what value of c,

a. the line $y=x+c$ will touch the hyperbola $\frac{x^2}{16}-\frac{y^2}{4}=1$?

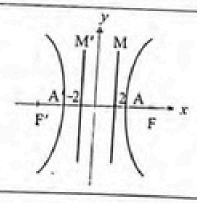
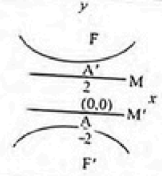
b. the line $y=-x+c$ will touch the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$?

9. In each case, find the tangent equation and normal equation

a. at a point $\left(-\sqrt{13},\frac{9}{2}\right)$ to hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$?

b. at a point $\left(\frac{16}{3},5\right)$ to hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$?

Summary of standard Hyperbolas

Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $c^2=a^2+b^2$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ $c^2=a^2+b^2$
Focus	(±C,0)	(0,±C)
Directrices	$x=\pm\frac{c}{e^2}$	$y=\pm\frac{c}{e^2}$
Major axis	$y=0$	$x=0$
Vertices	$(\pm a,0)$	$(0,\pm a)$
Covertices	$(0,\pm b)$	$(\pm b,0)$
Centre	(0,0)	(0,0)
Eccentricity	$e=\frac{c}{a}>1$	$e=\frac{c}{a}>1$
Graph		

9.4 Translation and Rotation of Axes

If the coordinates of a point or the equation of a curve be given with reference to a system of axes, rectangular or oblique, then the coordinates of the same point or the equation of the same curve can be obtained with reference to another system of axes, rectangular or oblique. The process of so changing the coordinates of a point or the equation of a curve is called the **transformation of coordinates.**

(i) Translation and rotation of axes

In general, we come across to define three types of change of axes that are the following:

a. **Translation of Axes:** This will be used in changing the origin only and the new axes are parallel to the old ones.

b. **Rotation of Axes:** This will be used in changing the directions of the axes without changing the origin of the system.

c. **General Transformation:** This will be used, when the change of the direction and the origin of the axes both come together.

The relationship between the two sets of coordinate axes is called the **translation of axes.**

The rotational relationship between the two sets of coordinate axes is called the **rotation of axes.**

(ii) Equations of transformation for translation of axes

If O(0,0) is the old origin of the set of old rectangular coordinate axes *ox* and *oy*, then the coordinates of a point P with respect to the old axes are $P(x, y)$.

If $O(h, k)$ is the new origin of the set of new rectangular coordinate axes OX and OY parallel to the old rectangular coordinate axes, then the coordinates of a point P with respect to the new axes are P(X, Y).

If PM and ON are perpendicular to old coordinate axis *ox*, where PM intersects the new coordinate axis OX at M_1, then, the following assumptions

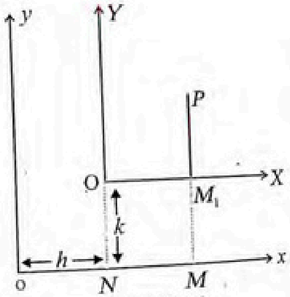


Figure 9.33

$oN = h,\ NO = k,\ oM = x,\ MP = y$ also $OM_1 = X,\ M_1P = Y$

through old rectangular coordinate axes

$$x = oM = oN + NM = oN + OM_1 = h + X$$

$$y = MP = MM_1 + M_1P = NO + M_1P = k + Y$$

develops a set of rectangular coordinate axes in terms of new coordinates *X*, *Y* by means of the relation:

$$x = X + h,\quad y = Y + k \qquad (97)$$

The set of equations (97) are the equations of **transformation for translation of axes**. By making this substitution in a given equation, a new equation of the same graph is obtained, referred now to the new translated axes.

(iii) Equations of transformation for rotation of axes

If ox and oy is the set of old rectangular coordinate axes, then the set of new rectangular coordinate axes OX and OY is obtained by rotating the old rectangular coordinates through an angle $\theta, 0 < \theta < 90^0$ in anti-clockwise direction.

If the coordinates of a point P with respect to the old axes are $P(x, y)$, then the coordinates of a point P with respect to the new axes are P(X, Y.)

If PM and PN are perpendicular to Ox and OX and NN_1 and NM_1 are perpendicular to Ox and PM, then, the following assumptions

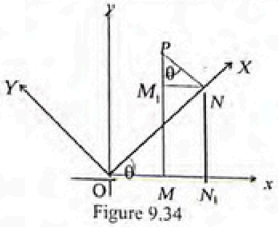


Figure 9.34

$$OM = x,\ MP = y;\ ON = X,\ NP = Y,\ \angle M_1PN = \theta$$

through old rectangular coordinate axes

$x = OM$

$= ON_1 - MN_1 = ON_1 - M_1N = ON\cos\theta - NP\sin\theta = X\cos\theta - Y\sin\theta$

$y = MP$

$= MM_1 + M_1P = N_1N + M_1P = ON\sin\theta + NP\cos\theta = X\sin\theta + Y\cos\theta$

develops a set of rectangular coordinate axes in terms of new coordinates X, Y by means of the relation:

$$x = X\cos\theta - Y\sin\theta,\quad y = X\sin\theta + Y\cos\theta \qquad (98)$$

An equivalent form of the relations (98) is:

$$X = x\cos\theta + y\sin\theta,\quad Y = -x\sin\theta + y\cos\theta \qquad (99)$$

(iv) Transformed equations through translation and rotation of axes

Example 28 Translate to parallel axes through the point (1, –2) the conic $4x^2 + 25y^2 - 8x + 100y + 4 = 0$.

Solution Substitute $x = X + h = X + 1$ $(h = 1)$ and $y = Y + k = Y - 2$ $(k = -2)$ in the given equation

$$4x^2 + 25y^2 - 8x + 100y + 4 = 0$$

$$4(X+1)^2 + 25(Y-2)^2 - 8(X+1) + 100(Y-2) + 4 = 0$$

which yields the standard form of ellipse in XY-plane by completing the square:

$$\frac{Y^2}{25} + \frac{Y^2}{4} = 1,\ XY - System \qquad (100)$$

The standard form of the given conic (ellipse) equation in *xy*-plane is obtained by backward substitution of $X = x + 1$ and $Y = y - 2$:

$$\frac{(x-1)^2}{25} + \frac{(y+2)^2}{4} = 1,\ X = x - 1, = y + 2,\ xy - system$$

Example 29 Transform to axes inclined at an angle 45^0 to the original axes of the conic $x^2 + y^2 - 8x + 4xy - 1 = 0$.

Solution The substitution of equation (98)

$$x = X\cos\theta - Y\sin\theta,\ \theta = 45^0$$

$$= X\cos 45 - Y\sin 45 = \frac{\sqrt{2}}{2}X - \frac{\sqrt{2}}{2}Y = \frac{\sqrt{2}}{2}(X - Y) \qquad \therefore \sin 45 = \cos 45 = \frac{\sqrt{2}}{2}$$

$$y = X\sin\theta + Y\cos\theta$$

$$= X\cos 45 + Y\sin 45 = \frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}Y = \frac{\sqrt{2}}{2}(X + Y)$$

in the given conic equation

$$\frac{2}{4}(X-Y)^2 + \frac{2}{4}(X+Y)^2 - 8\frac{\sqrt{2}}{2}(X-Y) + 4\left(\frac{2}{4}\right)(X-Y)(X+Y) - 1 = 0$$

to obtain the transformed equation of hyperbola

$$3X^2-Y^2-4\sqrt{2}(X-Y)-1=0$$

that yields the given conic by substituting:

$$\left.\begin{aligned}X+Y&=\frac{2y}{\sqrt{2}}\\ X-Y&=\frac{2x}{\sqrt{2}}\end{aligned}\right\}\Rightarrow X=\frac{1}{\sqrt{2}}(x+y),\ Y=\frac{1}{\sqrt{2}}(x-y)$$

(v) New origin and new axes with respect to old origin and old axes

This is actually the general transformation (third type) which requires both translation and rotation of axes. The procedure developed is as under:

If O(0,0) is the old origin of the set of old rectangular coordinate axes ox and *oy*, then the origin of the set of rectangular coordinate axes OX and OY parallel to the old rectangular coordinate axes is O(*h*, *k*). Further, the set of new rectangular coordinate axes is OX′ and OY′ is obtained by rotating the rectangular coordinates axes OX and OY through an angle $\theta, 0<\theta<90^\circ$ in anti-clockwise direction.

If the rectangular coordinates of a point P with respect to old rectangular coordinate axes are P(*x*,*y*), then the rectangular coordinates of a point P with respect to rectangular coordinates axes OX, OY and new rectangular coordinates axes OX′, OY′ are respectively P(X,Y) and P(X′,Y′).

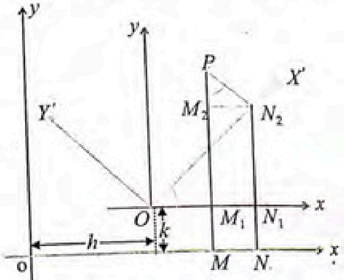


Figure 9.35

The following assumptions

$$oM=x,\ MP=y,\ OM_1=X,\ M_1P=Y,\ ON_2=X',\ N_2P=Y'$$

through old rectangular coordinate axes

$$\begin{aligned}x&=oM\\ &=oN-MN\\ &=oL+LN-MN\\ &=oL+ON_1-M_2N_2\\ &=oL+ON_2\cos\theta-N_2P\sin\theta=h+X'\cos\theta-Y'\sin\theta\end{aligned}$$

$$\begin{aligned}y&=MP\\ &=MM_1+M_1M_2+M_2P\\ &=OL+N_1N_2+M_2P=k+ON_2\sin\theta+N_2P\cos\theta\\ &=k+X'\sin\theta+Y'\cos\theta\end{aligned}$$

develops a set of rectangular coordinate axes in terms of new coordinates X′ and Y′ by means of the relation:

$$x=h+X'\cos\theta-Y'\sin\theta,\quad y=k+X'\sin\theta+Y'\cos\theta \qquad (101)$$

Example 30 Transform to new axes inclined at an angle 45° to the original axes of the conic $x^2+y^2-8x+4xy-1=0$ through (2, 3).

Solution The substitution of equation (101)

$$\begin{aligned}x&=h+X'\cos\theta-Y'\sin\theta,\ \theta=45^0\\ &=h+X'\cos45-Y'\sin45\\ &=2+\frac{\sqrt{2}}{2}X'-\frac{\sqrt{2}}{2}Y'=2+\frac{\sqrt{2}}{2}(X'-Y')\end{aligned} \qquad \therefore \sin45=\cos45=\frac{\sqrt{2}}{2}$$

$$\begin{aligned}y&=k+X'\sin\theta+Y'\cos\theta\\ &=k+X'\cos45+Y'\sin45\\ &=3+\frac{\sqrt{2}}{2}X'+\frac{\sqrt{2}}{2}Y'=3+\frac{\sqrt{2}}{2}(X'+Y')\end{aligned}$$

in the given conic equation

$$\left[2+\frac{2}{\sqrt{2}}(X'-Y')\right]^2+\left[3+\frac{2}{\sqrt{2}}(X'+Y')\right]^2-8\left[2+\frac{\sqrt{2}}{2}(X'-Y')\right]+4\left[2+\frac{\sqrt{2}}{2}(X'-Y')\right]\left[3+\frac{\sqrt{2}}{2}(X'+Y')\right]-1=0$$

to obtain the transformed equation of hyperbola

$$3X'^2-Y'^2+4\sqrt{2}(X'-Y')+7\sqrt{2}(X'+Y')+20=0$$

that yields the given conic by substituting:

$$\left.\begin{aligned}x&=2+\frac{\sqrt{2}}{2}(X'-Y')\\ y&=3+\frac{\sqrt{2}}{2}(X'+Y')\end{aligned}\right\}\Rightarrow X'=\frac{(x-2)}{\sqrt{2}}+\frac{(y-3)}{\sqrt{2}},\ Y'=\frac{(y-3)}{\sqrt{2}}-\frac{(x-2)}{\sqrt{2}}$$

(vi) Angle through which the axes be rotated about the origin so that the product term xy is removed from the transformed equation

The substitution of the equations of transformation (98)

$$x=X\cos\theta-Y\sin\theta,\quad y=X\sin\theta+Y\cos\theta$$

in the conic of the form $ax^2+2hxy+by^2$ is

$$ax^2+2hxy+by^2=a\left(X\cos\theta-Y\sin\theta\right)^2+2h(X\cos\theta-Y\sin\theta)(X\sin\theta+Y\cos\theta)+b\left(X\sin\theta+Y\cos\theta\right)^2$$

$$=\left(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta\right)X^2+\left\{-2(a-b)\sin\theta\cos\theta+2h(\cos^2\theta-\sin^2\theta)\right\}XY+\left\{a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta\right\}Y^2$$

The expression $ax^2+2hxy+by^2$ will be of the form aX^2+bY^2, if the coefficient of XY term on the right side of the above equation equals zero:

$$\left\{-2(a-b)\sin\theta\cos\theta+2h(\cos^2\theta-\sin^2\theta)\right\}=0$$

$$-(a-b)\sin2\theta+2h\cos2\theta=0$$

$$-(a-b)\sin2\theta=-2h\cos2\theta$$

$$\frac{\sin2\theta}{\cos2\theta}=\frac{2h}{a-b}$$

$$\tan2\theta=\frac{2h}{a-b}$$

$$\theta=\frac{1}{2}\tan^{-1}\frac{2h}{a-b} \qquad (102)$$

Example 31 At what angle the axes are rotated about the origin so that the transformed equation of the conic $9x^2+4y^2+12xy-x-y=0$ does not contain the term involving XY?

Solution If the axes of the given conic are rotated through an angle θ, then the angle θ can be found through result (102):

$$\theta=\frac{1}{2}\tan^{-1}\frac{2h}{a-b},\quad a=9, b=4, h=6$$
$$=\frac{1}{2}\tan^{-1}\frac{2(6)}{9-4}=\frac{1}{2}\tan^{-1}\frac{12}{5}=\frac{1}{2}\tan^{-1}(2.4)=\frac{1}{2}(67^0)\approx 34^0$$

Exercise 9.4

1. Translate to parallel axes through the
   a. point (0,2), the equation $2x-y+2=0$.
   b. point (–1,2), the conic $x^2+y^2+2x-4y+1=0$.
   c. point (3,–4), the conic $x^2+2y^2-6x+16y+39=0$.
   d. point (–2,2), the conic $x^2+y^2-3xy+10x-10y+21=0$.
2. Transform to axes inclined at an angle
   a. 45^0 to the original axes of the conic $x^2-y^2=a^2$.
   b. 90^0 to the original axes of the conic $y^2=4px$.
   c. 45^0 to the original axes of the conic $x^2+y^2+4xy-1=0$.
   d. 45^0 to the original axes of the conic $x^2-y^2-2\sqrt{2}x-10\sqrt{2}y+2=0$.
3. Transform to new axes inclined at an angle
   a. $\tan^{-1}(1/2)$ to the original axes of the conic $14x^2+11y^2-36x+48y-4xy+41=0$ through (1,–2).
   b. $\tan^{-1}(-4/3)$ to the original axes of the conic $11x^2+4y^2-20x-40y+24xy-5=0$ through (2,–1)
   c. $\tan^{-1}(3/4)$ to the original axes of the conic $3x^2+10y^2+6x+52y-24xy=0$ through (3,1).
4. At what angle the axes are rotated about the origin so that the transformed equation of the conic
   a. $11x^2+4y^2-20x-40y+24xy-5=0$ does not contain the term involving XY?
   b. $5x^2+7y^2+2\sqrt{3}xy-16=0$. does not contain the term involving XY?

Summary

Parabola:

- The eccentricity of the conic is e =c/a. The conic is
  ellipse , if e < 1
  parabola, if e = 1
  hyperbola, if e > 1
- a. The standard form of the equation of a parabola that is symmetric with respect to the x–axis, with vertex V(0,0), focus F(*p*,0) and directrix the line $x=-p$ is: $y^2=4px$
  b. The standard form of the equation of a parabola that is symmetric with respect to the y–axis, with vertex V(0,0), focus F(0,*p*) and directrix the line $y=-p$ is: $x^2=4py$
- a. The standard form of the equation of a parabola that is symmetric with respect to the line $y=k$ and with vertex V(*h*, *k*), focus F(*h* + *p*, *k*) and directrix line $x=h-p$ is: $(y-k)^2=4p(x-h)$
  b. The standard form of the equation of a parabola that is symmetric with respect to the line x=h and with vertex V(*h*, *k*), focus F(*h*, *k* + *p*) and directrix line $y=k-p$ is: $(x-h)^2=4p(y-k)$
- The equation of the tangent line at a point $P(x_1,y_1)$ to parabola is: $yy_1=2p(x+x_1)$
- The equation of any tangent to parabola $y^2=4px$ in the slope–form is: $y=mx+\frac{p}{m}$
- The line $y=mx+c$ should touch the parabola $y^2=4px$ under condition: $y=mx+\frac{p}{m},\ y^2=4py$
- The normal equation to parabola $y^2=4px$ at a point $P(x_1,y_1)$ is: $y-y_1=\frac{-y_1}{2p}(x-x_1)$

Ellipse:

- a. The standard form of the equation of an ellipse with center at the origin, length of the semimajor axis a, length of the semiminor axis b and major axis along the x–axis is: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
  b. The standard form of the equation of an ellipse with center at the origin, length of the semimajor axis a, length of the semiminor axis b and major axis along the y–axis is: $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
- a. The standard form of the equation of an ellipse with center at C(*h*, *k*), length of the semimajor axis a and semiminor axis b, and major axis parallel to the x–axis is: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$
  b. The standard form of the equation of an ellipse with center at C(*h*, *k*), length of the semimajor axis a and semiminor axis b, and major axis parallel to the y–axis is: $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$
- The equation of the tangent line to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point $P(x_1,y_1)$ is:
  $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,\quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1\ at\ (x_1,y_1)$
- The equation of any tangent line to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the slope–form is: $y=mx\pm\sqrt{a^2m^2+b^2}$
- The line $y=mx+c$ should touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ under condition: $c=\pm\sqrt{a^2m^2+b^2}$