Unit 8 CONICS-I

By the end of this unit, the students will be able to:

8.1 **Introduction**

i. Define conics and demonstrate members of its family i.e. circle, parabola, ellipse and hyperbola.

8.2 **Circle**

i. Define circle and derive its equation in standard form i.e. $(x-h)^2 + (y-k)^2 = r^2$.

ii. Recognize general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and find its centre and radius.

iii. Find the equation of a circle passing through

- three non-collinear points,
- two points and having its centre on a given line,
- two points and equation of tangent at one of these points is known,
- two points and touching a given line.

8.3 **Tangent and Normal**

i. Find the condition when a line intersects the circle.

ii. Find the condition when a line touches the circle.

iii. Find the equation of a tangent to a circle in slope form.

iv. Find the equations of a tangent and a normal to a circle at a point.

v. Find the length of tangent to a circle from a given external point.

vi. Prove that two tangent drawn to a circle from an external point are equal in length.

8.4 **Properties of circle**

i. Prove analytically the following properties of a circle.

- Perpendicular from the centre of a circle on a chord bisects the chord.
- Perpendicular bisector of any chord of a circle passes through the centre of the circle.
- Line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.
- Congruent chords of a circle are equidistant from its centre and its converse.
- Measure of the central angle of a minor arc is double the measure of the angle subtended by the corresponding major arc.
- An angle in a semi-circle is a right angle.
- The perpendicular at the outer end of a radial segment is tangent to the circle.
- The tangent to a circle at any point of the circle is perpendicular to the radial segment at that point.

Introduction

In geometry conic section is also called a conic. Any curve produced by the intersection of plane and right circle cone depending on the angle of plane and right circular cone. The 'conic section' comes from the fact that the principle type of conic section known as ellipses, hyperbolas and parabola are generated by cutting a cone with a plane. Most modern textbooks of calculus depart from this geometrical approach, instead conic section are defined as some types of loci and studied through analytic geometry. In this unit we will study an approach to conic section which are defined as the intersection of two cones. Then the vertices of two cones becomes the inherent foci of the conic section and directrix exists associated with each of the inherent foci. At the end of this unit we will learn about the properties through real life examples.

8.1 Conics and members of its family

A conic section (or simply conic) is a curve obtained as the intersection of the surface of a cone with a plane. The three types of conic sections are the hyperbola, the parabola, and the ellipse. The circle is type of ellipse, and is sometimes considered to be a fourth type of conic section.

Conic sections can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called nappes. One nappe is what most people mean by "cone," and has the shape of a party hat. Conic sections are generated by the intersection of a plane with a cone. If the plane is parallel to the axis of revolution (the y-axis), then the conic section is a hyperbola. If the plane is parallel to the generating line, the conic section is a parabola. If the plane is perpendicular to the axis of revolution, the conic section is a circle. If the plane intersects one nappe at an angle to the axis (other than 90^0), then the conic section is an ellipse.

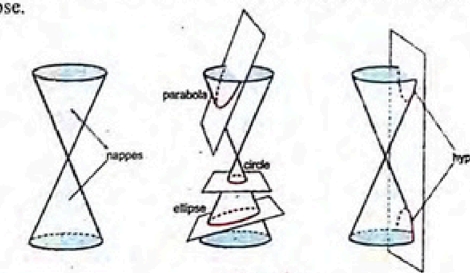


Figure 8.1

In this unit, we look at one of these conic sections that are the circles.

8.2 Circle

A circle is a shape that has a continuous and constant curve. Though it is always curving; it has an algebraic expression that describes its nature.

8.2.1 Circle and its equation in standard form

*"The set of all points in the plane in such a way that its distances from a fixed point in that plane **(called the center)** is equal to a fixed distance **(called the radius)** of the circle."*

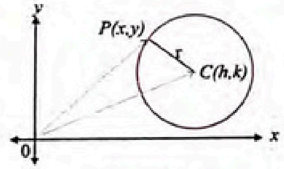


Figure 8.2

Derivation of Circle Equation: This definition helps us in developing a standard form of the equation of a circle.

Let C (h, k) be a fixed point at the center of the circle and r is the radius of the circle and $P(x, y)$ is any one of the collection of points on the circumference of the circle that gives the distance from the fixed point $C(h, k)$ which is called the radius of a circle. The position vectors of P and C relative to origin are respectively. $OP = (x, y)$, $OC = (h, k)$ (i)

From the Figure 8.2, the distance from the center C to point P is the fixed distance equals the radius of the circle: $OC + CP = OP$

$\Rightarrow \quad CP = OP - OC$

$\Rightarrow \quad CP = (x, y) - (h, k) = (x-h, y-k)$

$|CP| = \sqrt{(x-h)^2 + (y-k)^2}$, distance formula

$|CP|^2 = \left(\sqrt{(x-h)^2 + (y-k)^2}\right)^2$, squaring both sides

$r^2 = (x-h)^2 + (y-k)^2$, $|CP|^2 = (CP)^2 = r^2$

The standard form of a circle with radius r and center (h, k) is: $(x-h)^2+(h-k)^2=r^2$ (ii)

If the center of the circle is at the origin $(h, k)=(0,0)$, then the circle equation (ii) becomes:

$x^2+y^2=r^2$ (iii)

Example 1 Determine the equation of a circle with center at $(-2,1)$ and radius $r=3$.

Solution The standard form of the equation of a circle is $(x-h)^2+(y-k)^2=r^2$

Where (h, k) are the coordinates of the centre and r is the radius.

In this equation $h=-2$, $k=1$ and $r=3$

substitute these values into the standard equation.

$\Rightarrow (x-(-2))^2+(y-1)^2=(3)^2$

$\Rightarrow (x+2)^2+(y-1)^2=9$ is the equation

8.2.2 General form of an equation of a circle

The arrangement of the general equation of the second degree in x and y that may represent a circle through the following procedure:

The general equation of the second degree in variables x and y is:

$ax^2+2hxy+by^2+2gx+2fy+c=0$ (i)

Divide out both sides of equation (i) by a to obtain:

$x^2+\frac{2h}{a}xy+\frac{b}{a}y^2+\frac{2g}{a}x+\frac{2f}{a}y+\frac{c}{a}=0$ (ii)

$x^2+\frac{2h}{a}xy+\frac{b}{a}y^2+2g_1x+2f_1y+c_1=0,$ $g_1=\frac{g}{a}, f_1=\frac{f}{a}, c_1=\frac{c}{a}$

The rearranged equation (ii) of the general equation of the second degree (i) in x and y gives the general equation of a circle if and only if $\frac{b}{a}=1$ and $\frac{2h}{a}=0$:

$x^2+y^2+2g_1x+2f_1y+c_1=0$

$x^2+y^2+2gx+2fy+c=0$ (iii) $g_1=g,\ f_1=f, c_1=c$

$x^2+2gx+g^2-g^2+y^2+2fy+f^2-f^2+c=0$ Adding and subtracting g^2 and f^2 in equation (iii)

$(x+g)^2+(y+f)^2=g^2+f^2-c$

$[x-(-g)]^2+[y-(-f)]^2=\left(\sqrt{g^2+f^2-c}\right)^2$

The locus of a point (x, y) which moves in such a way that its distance from a fixed point $(-g,-f)$ is constant and equals $\sqrt{g^2+f^2-c}$. This of course represents a circle.

For general equation of circle $x^2+y^2+2gx+2fy+c=0$

- **Center and Radius:** The coordinates of the center are $(-g,-f)$ and the radius is $r=\sqrt{(-g)^2+(-f)^2-c}$.
- **Independent Constant:** The general equation contains three independent constants g, f and c. They can be determined from the three independent conditions.
- **Nature of the Circle:**

If $g^2+f^2-c>0$, then, the circle is real and different from zero.

If $g^2+f^2-c=0$, then, the circle shrinks into a point $(-g,-f)$. It is called point circle.

If $g^2+f^2-c<0$, then, the circle is imaginary or virtual.

- The coefficients of x^2 is equal to the coefficients of y^2, and there is no term containing $x\,y$ and the square of the radius $r^2\ge 0$.

Example 2 Find the center and radius of a circle $45x^2+45y^2-60y+36x+19=0$.

Solution The given circle equation is rearranged to obtain:

$x^2+y^2-\frac{4}{3}x+\frac{4}{5}y+\frac{19}{45}=0,$ (i) Dividing by 45

The circle equation (i) is compared to the general form of a an equation of circle to obtain the values of g, f and c: $2g=\frac{-4}{3}\Rightarrow g=\frac{-2}{3},\ 2f=\frac{4}{4}\Rightarrow f=\frac{2}{5},\ c=\frac{19}{45}$

The center and radius of the given circle are therefore:

$(-g,-f)=\left(\frac{2}{3},\frac{-2}{5}\right), r=\sqrt{\frac{4}{9}+\frac{4}{25}-\frac{19}{45}}=\sqrt{\frac{41}{225}}=\frac{\sqrt{41}}{15}$

8.2.3 The equation of a circle passing through

A. three non-collinear points
B. two points and having its centre on a given line
C. two points and equation of tangent at one of these points is known
D. two points and touching a given line

A. Equation passing through three non-collinear points

Consider the general equation a circle is given by $x^2+y^2+2gx+2fy+c=0$

If the given circle is passing through three non-collinear points, say, $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ then these points must satisfy the general equation of a circle. Now put the above three points in the given equation of a circle, i.e.:

$x_1^2+y_1^2+2gx_1+2fy_1+c=0$ (i)

$x_2^2+y_2^2+2gx_2+2fy_2+c=0$ (ii)

$x_3^2+y_3^2+2gx_3+2fy_3+c=0$ (iii)

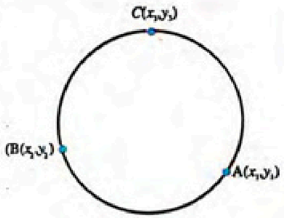


Figure 8.3

Example 3 Find the equation of a circle which passes through the three points A(1,0), B(0,−6) and C(3,4)

Solution The required equation of a circle is $x^2+y^2+2gx+2fy+c=0$ (i) which passes through the three points A(1,0), B(0,−6) and C(3,4), which gives a system of three linea equations in three unknowns g, f and c:

$$\begin{aligned} 1+2g+c=0 &\Rightarrow 2g+c=-1 \\ 36-12f+c=0 &\Rightarrow -12f+c=-36 \\ 25+6g+8f+c=0 &\Rightarrow 6g+8f+c=-25 \end{aligned} \quad \text{(ii)}$$

The system of three linear equation (ii) in matrix form $\begin{pmatrix} 2 & 0 & 1 \\ 0 & -12 & 1 \\ 6 & 8 & 1 \end{pmatrix}\begin{pmatrix} g \\ f \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ -36 \\ -25 \end{pmatrix}$, $Ax = b$

whose augmented matrix is: $A|b = \left(\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & -12 & 1 & -36 \\ 6 & 8 & 1 & -25 \end{array}\right)$

Reduce this augmented matrix in an echelon form to obtain:

$$R\left(\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & -12 & 1 & -36 \\ 0 & 8 & -2 & -22 \end{array}\right), -3R_{31} = R_3 + (-3)R_1$$

$$R\left(\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & -12 & 1 & -36 \\ 0 & 0 & \frac{-4}{3} & -46 \end{array}\right), \frac{2}{3}R_{32} = R_3 + \frac{2}{3}R_2$$

$$\left.\begin{array}{r} 2g + c = -1 \\ -12f + c = -36 \\ \left(\frac{-4}{3}\right)c = -46 \end{array}\right| \quad \text{(iii)}$$

Third equation of the system (iii) is giving $c = \frac{69}{2}$ which is used in second and first equations to obtain the values of $f = \frac{47}{8}$ and $g = \frac{-71}{4}$.

The values of $g = \frac{-71}{4}$, $f = \frac{47}{8}$ and $c = \frac{69}{2}$ are used in equation (iii) to obtain the required circle equation:

$$x^2 + y^2 + 2\left(\frac{-71}{4}\right)x + 2\left(\frac{47}{8}\right)y + \frac{69}{2} = 0$$

$$x^2 + y^2 - \frac{71}{2}x + \frac{47}{4}y + \frac{69}{2} = 0$$

$$4x^2 + 4y^2 - 142x + 47y + 138 = 0$$

B. Equation of circle passing through two points and having its centre on a given line

Consider the general equation a circle is given by

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

If the given circle is passing through two points, say $A(x_1, y_1)$ and $B(x_2, y_2)$, then these points must satisfy the general equation of a circle. Now put these two points in the given equation of a circle, i.e.:

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{(i)}$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \text{(ii)}$$

Also, the given straight line $ax + by + c_1 = 0$ passes through the center $(-g, -f)$ of the circle.

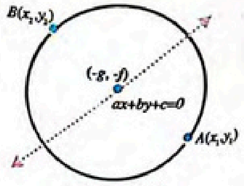


Figure 8.4

Example 4 Find the equation of a circle which passes through the points A(3,1) and B(2,2) having its center on the line $x + y - 3 = 0$.

Solution The required equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

which passes through the two points A(3,1) and B(2,2) that gives a system of two linear equations

$$\left.\begin{array}{lcl} 10 + 6g + 2f + c = 0 & \Rightarrow & 6g + 2f + c = -10 \\ 8 + 4g + 4f + c = 0 & \Rightarrow & 4g + 4f + c = -8 \end{array}\right\} \quad \text{(ii)}$$

If the center $(-g, -f)$ of the circle lies on the line $x + y - 3 = 0$, then the line $x + y - 3 = 0$ becomes:

$$-g - f - 3 = 0 \quad \Rightarrow \quad g + f = -3 \quad \text{(iii)}$$

The combination of equations (ii) and (iii) is giving the system of three linear equations in three unknown g, f and c

$$\begin{pmatrix} 6 & 2 & 1 \\ 4 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} g \\ f \\ c \end{pmatrix} = \begin{pmatrix} -10 \\ -8 \\ -3 \end{pmatrix}, \quad Ax = b$$

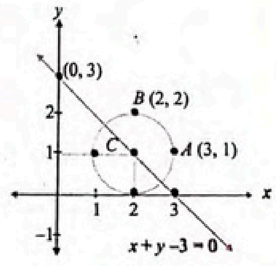


Figure 8.5

whose augmented matrix is: $A|b = \left(\begin{array}{ccc|c} 6 & 2 & 1 & -10 \\ 4 & 4 & 1 & -8 \\ 1 & 1 & 0 & -3 \end{array}\right)$

Reduce this augmented matrix in an echelon form to obtain the unknowns g, f and c:

$$R\left(\begin{array}{ccc|c} 6 & 2 & 1 & -10 \\ 0 & \frac{8}{3} & \frac{1}{3} & \frac{-4}{3} \\ 0 & \frac{2}{3} & \frac{-1}{6} & \frac{-4}{3} \end{array}\right) \text{by } R_2 - \frac{2}{3}R_1,\ R_3 - \frac{1}{6}R_1$$

$$R\left(\begin{array}{ccc|c} 6 & 2 & 1 & -10 \\ 0 & \frac{8}{3} & \frac{1}{3} & \frac{-4}{3} \\ 0 & 0 & \frac{-1}{4} & -1 \end{array}\right) \text{by} \left(R_3 - \frac{1}{4}R_2\right)$$

$$\begin{array}{r} 6g + 2f + c = -10 \\ \left(\frac{8}{3}\right)f + \left(\frac{1}{3}\right)c = -\frac{4}{3} \\ \left(\frac{-1}{4}\right)c = -1 \end{array} \quad \text{(iv)}$$

Third equation of the system (iv) is giving $c = 4$ which is used in second and first equations t obtain the values of $f = -1$ and $g = -2$.

The values of $g = -2$, $f = -1$ and $c = 4$ are used in equation (i) to obtain the required circ equation: $x^2 + y^2 - 4x - 2y + 4 = 0$

C. The equation of a circle passing through two points and equation of tangent at one of these points is known

Consider the general equation a circle is given by

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{(i)}$$

If the given circle is passing through two points, say $A(x_1,y_1)$ and $B(x_2,y_2)$, then these points must satisfy the general equation of a circle. Now put these two points in the given equation of a circle, i.e.:

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \qquad \text{(ii)}$$

$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \qquad \text{(iii)}$$

Also the given straight line $ax+by+d=0$ (iv) touches the circle at one point, as shown in the given diagram, and it is clear from the diagram that the distance of a given point from the center $(-g,-f)$ must be equal to the radius of the circle.

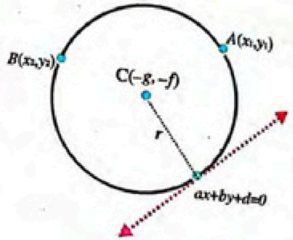


Figure 8.6

Example 5 Find the equation of a circle which passes through the two points A(0,–1) and B(3,–3) and $3x-2y-2=0$ is the tangent line on the circle at a point A(0,–1).

Solution Let C (h, k) be the center of the required circle. If A(0,–1) and B(3,–3) are the two points lie on the circle, then the square of the distance from C to A equals the square of the distance from C to B:

$$|CA|^2=|CB|^2, CA=(h-0,k+1), CB=(h-3,k+3)$$

$$(h-0)^2+(k+1)^2=(h-3)^2+(k+3)^2$$

$$h^2+k^2+2k+1=h^2-6h+9+k^2+9+6k$$

$$6h-4k-17=0 \qquad \text{(i)}$$

The slope of CA is

$$m_1=\frac{-1-k}{0-h}=\frac{k+1}{h}$$

and the slope of the tangent line $3x-2y-2=0$ is

$$3x-2y-2=0$$

$$-2y=-3x+2 \Rightarrow y=\frac{3}{2}x-1, m_2=\frac{3}{2}$$

If CA is perpendicular to the tangent line $3x-2y-2=0$, then the product of their slopes equals to (–1):

i.e. $m_1 m_2=-1$

$$\left(\frac{k+1}{h}\right)\left(\frac{3}{2}\right)=-1 \qquad \text{(ii)}$$

$$3k+3=-2h \Rightarrow 2h+3k+3=0$$

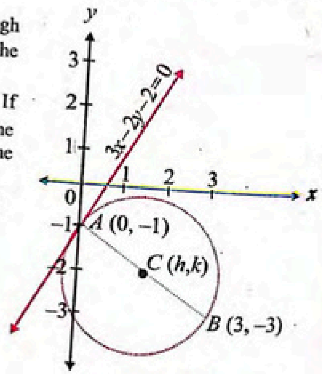


Figure 8.7

The equations (i) and (ii) are solved to obtain the values of $k=-2$ and $h=\frac{3}{2}$.

The required circle with center $(h, k)=\left(\frac{3}{2},-2\right)$ and radius

$$r=|CA|=|CB|=\sqrt{4^2+(k+1)^2}=\sqrt{\left(\frac{9}{4}\right)+1}=\frac{\sqrt{13}}{2} \text{ is:}$$

$$(x-h)^2+(y-k)^2=r^2$$

$$\left(x-\frac{3}{2}\right)^2+(y+2)^2=\frac{13}{4} \Rightarrow x^2+y^2-3x+4y+3=0$$

D. The equation of a circle passing through two points and touching a given line

Example 6 Find the equation of a circle which passes through the two points $A(0,0)$ and $B(4,0)$ and is touching a line $3x+4y+4=0$.

Solution Let $C(h, k)$ be the center of the required circle. If $A(0,0)$ and $B(4,0)$ are the two points lie on the circle, then the radius of the circle from center C to point A equals the radius of the circle from the center C to point B:

$$|CA|^2=|CB|^2, CA=(0-4,0-1), CB=(4-h,0-k)$$

$$\left(\sqrt{h^2+k^2}\right)^2=\left(\sqrt{(4-h)^2+k^2}\right)^2$$

$$h^2+k^2=16+h^2-8h+k^2 \Rightarrow 8h=16 \Rightarrow h=2$$

The radius of the required circle is $r=|CA|=\sqrt{4+k^2}$ and the center is C(2, k).

For the values of k, the perpendicular distance from the center (2, k) on the line $3x+4y+4=0$ equals the radius of the circle:

$$\frac{3(2)+4(k)+4}{\sqrt{9+16}}=\sqrt{4+k^2}$$

$$\frac{4k+10}{5}=\sqrt{4+k^2}$$

$$4k+10=5\sqrt{4+k^2}, \text{ squaring both sides}$$

$$16k^2+100+80k=25(4+k^2)$$

$$-9k^2+80k=0 \Rightarrow -k(9k-80)=0 \Rightarrow k=0, k=\frac{80}{9}$$

The coordinates of the center are (2,0) and $\left(2, \frac{80}{9}\right)$ and the radii are $r=\sqrt{4+0}=2$ and $r=\sqrt{4+\frac{80^2}{9^2}}=\sqrt{\frac{6724}{81}}=\frac{82}{9}$.

The equations of the circles with the above centers and radii are the following:

$$(x-2)^2+y^2=4,$$

$$(x-2)^2+\left(y-\frac{80}{9}\right)^2=\frac{6724}{81}$$

Exercise 8.1

1. In each case, find an equation of a circle, when the center and radius are the following:
   a. $(0,0)$, $r=4$
   b. $(3,2)$, $r=1$
   c. $(-4,-3)$, $r=4$
   d. $(-a,-b)$, $r=a+b$
2. In each case, determine the equation of a circle using the given information:
   a. C(0,0), tangent to the line $x=-5$
   b. C(0,0), tangent to the line $y=6$
   c. C(6,– 6), circumference passes through the origin.
   d. C(– 9,– 6), circumference passes through the point (– 20,8).
   e. C(– 5,4), tangent to the x-axis.
   f. C(5,3), tangent to the y-axis.
3. In each case, find the center C(–g,–f) and radius $r=\sqrt{g^2+f^2-c}$ of the following:
   a. $x^2+y^2-8x-6y+9=0$
   b. $4x^2+4y^2+16x-12y-7=0$
   c. $x^2+y^2+4x-6y+13=0$
   d. $x^2+y^2-x-8y+18=0$
4. In each case, find an equation of a circle which passes through the three points:
   a. $(-3,0),(5,4),(6,-3)$
   b. $(7,-1),(5,3),(-4,6)$
   c. $(1,2),(3,-4),(5,-6)$
5. In each case, find an equation of a circle which
   a. contains the point $(2,6),(6,4)$ and has its center on the line $3x+2y-1=0$.
   b. contains the point $(4,1),(6,5)$ and has its center on the line $4x+y-16=0$.
6. Find an equation of a circle which passes through the points
   a. $(0,0),(0,3)$ and the line $4x-5y=0$ is tangent to it at (0,0).
   b. $(0,-1),(3,0)$ and the line $3x+y=9$ is tangent to it at (3,0).
7. Find an equation of a circle that is concentric to circle
   a. $2x^2+2y^2+16x-7y=0$ and is tangent to the y-axis.
   b. $x^2+y^2-8x+4=0$ and is tangent to the line $x+2y+6=0$.
   c. $x^2+y^2+6x-10y+33=0$ and is touching the x-axis.
8. Find equation of circle which passes through origin and whose intercepts are on the coordinate axis are:
   a. 3 and 4
   b. 2 and 4

8.3 Tangents and Normal

If a secant PQ of a circle is moved upward about one of its points of intersection P, then the second point of intersection Q is moving gradually along the curve that tends to coincide with P. The limiting position PT of PQ is then called the **tangent** to the circle at the point P.

The point of the circle at which a tangent meets the circle is called **point of contact** (Say P) of the tangent.

The **normal** at a contact point P to a circle (or conic) is the straight line PR perpendicular to the tangent PT to the circle (or conic) at that point P.

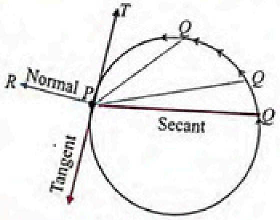


Figure 8.8

8.3.1 The condition when a line intersect the circle

The circle and the line are $x^2+y^2=a^2$ (i)

$y=mx+c$ (ii)

which develops a system of two nonlinear equations:

$$x^2+y^2=a^2$$
$$y=mx+c \quad \text{(iii)}$$

The solution set $\{(x,y)\}$ of the nonlinear system of equations (iii) exists only, if the curves of the system (iii) are intersecting. That set of points of intersection $\{(x,y)\}$ is the solution set, can be found by solving the nonlinear system (iii) simultaneously.

The line (ii) is used in a circle (i) to obtain the quadratic equation in x:

$$x^2+(mx+c)^2=a^2$$
$$x^2(1+m^2)+2mcx+(c^2-a^2)=0 \quad \text{(iv)}$$

The equation (iv) being a quadratic equation in x, gives a set of two values x_1 and x_2 of x which will be used in a line (ii) to obtain a set of two y values y_1 and y_2.

The solution set $\{(x_1,y_1),(x_2,y_2)\}$ of the system (iii) is of course a set of points of intersection of the line and circle.

The points of intersection of the system (iii) are real, coincident or imaginary, according as the roots of the quadratic equation (iv) are real, coincident or imaginary or according as the discriminate of the quadratic equation (iv):

$$disc=4m^2c^2-4\left[(1+m^2)(c^2-a^2)\right]>0, \quad \text{real and distant}$$
$$disc=4m^2c^2-4\left[(1+m^2)(c^2-a^2)\right]=0, \quad \text{coincident}$$
$$disc=4m^2c^2-4\left[(1+m^2)(c^2-a^2)\right]<0, \quad \text{imaginary}$$

Example 7 Find the points of intersection of the line $3x-4y+20=0$ and the circle $x^2+y^2=25$.

Solution The equations of the line and circle are: $3x-4y+20=0 \Rightarrow y=\frac{3}{4}x+5$ (i)

$x^2+y^2=25$ (ii)

The line (i) is used in a circle (ii) to obtain the x-coordinates of the points of intersection:

$$x^2+\left(\frac{3}{4}x+5\right)^2=25$$
$$x^2+\frac{9}{16}x^2+25+\frac{30}{4}x=25$$

$$x^2+\frac{9}{16}x^2+\frac{30}{4}x=0 \Rightarrow 25x^2+120x=0 \Rightarrow x=0,\frac{-24}{5}$$

The x-coordinates $x=0, \frac{-24}{5}$ are used in the line(i) to obtain the y-coordinates:

$x=0$ gives $y=5$

$x=\frac{-24}{5}$ gives $y=\frac{3}{4}\left(\frac{-24}{5}\right)+5=\frac{-72+100}{20}=\frac{28}{20}=\frac{7}{5}$

Thus, the points of intersection (0, 5) and $\left(\frac{-24}{5},\frac{7}{5}\right)$ are real and distant.

8.3.2 Condition when a line touches the circle $x^2+y^2=a^2$

To determine the position of a line with respect to the circle, we need to find its distance from centre of a circle and compare it with radius then:

i. If distance is less than the radius, the line will intersect at two points.
ii. If the distance is equal to the radius, then the line will touch the circle.
iii. If the distance is greater than the radius, the line will completely outside the circle.

Let AB be the straight line $y=mx+c$ that intersects the circle $x^2+y^2=a^2$ at points P and Q respectively.

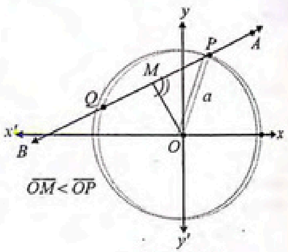


Figure: 8.9

Join $\overline{OP}$ and put it by $\overline{OP}$ = a, which is the radius of a given circle. Draw $\overline{OM}$ perpendicular on $\overline{PQ}$. If $\overline{OM}$ is perpendicular to $\overline{PQ}$, then, the perpendicular distance $\overline{OM}$ from O(0,0) on a secant line $mx-y+c=0$ (line $\overline{PQ}$) is: $\overline{OM}=\frac{|m(0)-(0)+c|}{\sqrt{m^2+1}}=\frac{c}{\sqrt{1+m^2}}$

From the right-angled triangle OMP, it is known that:

$$|\overline{OP}|^2=|\overline{OM}|^2+|\overline{MP}|^2$$

$$|\overline{MP}|^2=|\overline{OP}|^2-|\overline{OM}|^2$$

$$=a^2-\frac{c^2}{1+m^2}=\frac{a^2(1+m^2)-c^2}{1+m^2}$$

$$|\overline{MP}|=\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$$

The secant line $\overline{PQ}$ is 2 times of $\overline{OM}$, and the length of the intercept $\overline{PQ}$ is therefore:

$$|\overline{PQ}|=2|\overline{OM}|=2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}} \qquad \text{(i)}$$

Condition of Tangency: The line $y=mx+c$ touches the circle $x^2+y^2=a^2$, if the length of the intercept $\overline{PQ}$ is zero: $2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}=0$

$$2\frac{\sqrt{a^2(1+m^2)-c^2}}{\sqrt{1+m^2}}=0$$

$$\sqrt{a^2(1+m^2)-c^2}=0, \text{ squaring both sides}$$

$$a^2(1+m^2)-c^2=0 \Rightarrow c^2=a^2(1+m^2) \Rightarrow c=\pm a\sqrt{1+m^2} \qquad \text{(ii)}$$

The equation (ii) is the required condition at which the line $y=mx+c$ touches the circle $x^2+y^2=a^2$.

Example 8 Find the length of the chord joining the points P and Q on the line $\frac{x}{a}+\frac{y}{b}=1$ which cuts the circle $x^2+y^2=r^2$. Show that if the line touches the circle, then $a^{-2}+b^{-2}=r^{-2}$.

Solution The slope of a given line $\frac{x}{a}+\frac{y}{b}=1$

$$\frac{y}{b}=-\frac{x}{a}+1$$

$$y=-\frac{b}{a}x+b,\ m=-\frac{b}{a}$$

If PQ is the chord of a circle $x^2+y^2=r^2$, and PQ is 2 times of MP, then the length of the chord PQ through result (i) is:

$$|\overline{PQ}|=2|\overline{OM}|=2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$$

$$=2\sqrt{\frac{r^2\left[1+\left(\frac{-b}{a}\right)^2\right]-b^2}{1+\left(\frac{-b}{a}\right)^2}}, \quad a^2=r^2,\ m=\frac{-b}{a},\ c=b$$

If the given line touches the circle $x^2+y^2=r^2$, then, the length of the chord $\overline{PQ}$ is going to be zero:

$$2\sqrt{\frac{r^2\left[1+\left(\frac{-b}{a}\right)^2\right]-b^2}{1+\left(\frac{-b}{a}\right)^2}}=0$$

$$4\frac{r^2\left[1+\left(\frac{-b}{a}\right)^2\right]-b^2}{1+\left(\frac{-b}{a}\right)^2}=0, \quad \text{squaring both sides}$$

$$r^2\left[1+\left(\frac{-b}{a}\right)^2\right]-b^2=0$$

$$b^2=r^2\left[1+\frac{b^2}{a^2}\right]$$

$$b^2=r^2\frac{a^2+b^2}{a^2}\Rightarrow\frac{a^2b^2}{a^2+b^2}=r^2\Rightarrow r^{-2}=\frac{a^2+b^2}{a^2b^2}=\frac{1}{a^2}+\frac{1}{b^2}=a^{-2}+b^{-2}$$

Example 9 Find the coordinates of the middle point of the chord which the circle $x^2+y^2+4x-2y-3=0$ cuts off on the line $x-y+2=0$.

Solution The center of the given circle is $C(-g,-f)=C(-2,1)$ and the line $x-y+2=0$ (line AB) intersects the circle at points P and Q and $M(x_1,y_1)$ is the middle point of the chord PQ. Join C and M that develops a line $\overline{CM}$ perpendicular to chord PQ.

If M lies on line AB, then, the line equation $x-y+2=0$ becomes: (i)

$$x_1-y_1+2=0$$

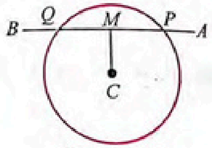


Figure 8.10

The slopes of the lines (i) and $\overline{CM}$ are respectively:

$m_2=1$, *coefficient of x*

$m_1=\frac{y_1-1}{x_1+2}$, *slop of* $\overline{CM}$

If CM is perpendicular to $\overline{AB}$, then the product of their slopes equals -1:

$$\left(\frac{y_1-1}{x_1+2}\right)(1)=-1\Rightarrow y_1-1=-x_1-2\Rightarrow x_1+y_1+1=0 \quad \text{(ii)}$$

The equations (i) and (ii) are solved to obtain the coordinates of the middle point M:

$$\left.\begin{aligned}x_1-y_1+2=0\\x_1+y_1+1=0\end{aligned}\right\} \qquad x_1=\frac{-3}{2},\ y=\frac{1}{2}$$

Thus, the coordinates of the middle point is $M\left(-\frac{3}{2},\frac{1}{2}\right)$

8.3.3 The equation of a tangent to a circle in slope form

If m is the slope of the tangent line to the circle $x^2+y^2=a^2$ (i)

then the equation of that tangent line is of the form $y=mx+c$ (ii)

Here c is to be calculated from the fact that the line (ii) is tangent to the circle (i). The line (ii) is used in circle (i) to obtain the quadratic equation in x:

$$x^2+(mx+c)^2=a^2,\quad y=mx+c$$

$$x^2(1+m^2)+2mcx+(c^2-a^2)=0 \quad \text{(iii)}$$

If the line (ii) touches the circle (i), then the quadratic equation (iii) has coincident roots for which the discriminant of the quadratic equation (iii) equals zero:

$$\text{Disc}=0$$

$$4m^2c^2-4(1+m^2)(c^2-a^2)=0$$

$$4m^2c^2=4(1+m^2)(c^2-a^2)$$

$$m^2c^2=m^2c^2+c^2-a^2-a^2m^2$$

$$-c^2=-a^2(1+m^2)\Rightarrow c=\pm a\sqrt{1+m^2} \quad \text{(iv)}$$

CONICS-I

Equation (iv) is the **condition of tangency**. The value of c from equation (iv) is used in the line (ii) to obtain the required equation of the tangent:

$$y=mx+c=mx\pm a\sqrt{1+m^2} \quad \text{(v)}$$

Remember

- The equation of any tangent to the circle $x^2+y^2=a^2$ in the slope form is: $y=mx\pm a\sqrt{1+m^2}$ (vi)
- The line $y=mx+c$ should touch the circle $x^2+y^2=a^2$ under condition: $c=\pm a\sqrt{1+m^2}$ (vii)
- The interpretation of result (v) is that the line $x+my+n=0$ should touch the circle $x^2+y^2=a^2$ under condition: $a^2(l^2+m^2)-n^2=0\Rightarrow n=\pm a\sqrt{l^2+m^2}$ (viii)
- The interpretation of result (v) that the line $x+my+n=0$ should touch the circle $x^2+y^2+2gx+2fy+c=0$ under condition:
$$(c-f^2)^2+2fg\,m+(c-g^2)m^2-2n(g+fm)+n^2=0 \quad \text{(ix)}$$
- Let $y=mx\pm a\sqrt{1+m^2}$ be a tangent to a circle (i) at a point (x_1,y_1), if the circle equation (i) is identical to $xx_1+yy_1=a^2$, then the coefficients of like terms of $y=mx\pm a\sqrt{1+m^2}$ and $xx_1+yy_1=a^2 \Rightarrow yy_1=-xx_1+a^2$ are compared to obtain the point of contact:
$$\frac{x_1}{-m}=\frac{y_1}{1}=\frac{a^2}{a\sqrt{1+m^2}}$$
$$\frac{x_1}{-m}=\frac{a}{\sqrt{1+m^2}}\Rightarrow x_1=\frac{-am}{\sqrt{1+m^2}}$$
$$\frac{y_1}{1}=\frac{a}{\sqrt{1+m^2}}\Rightarrow y_1=\frac{a}{\sqrt{1+m^2}}$$
Thus, the point of contact is $(x_1,y_1)=\left(\frac{-am}{\sqrt{1+m^2}},\frac{a}{\sqrt{1+m^2}}\right)$ (x)

Example 10 For what value of c, the line $x+y+c=0$ will touch the circle $x^2+y^2=64$? Use that value of c to find the tangent that should touch the given circle. Find also the contact point.

Solution The slope of the line $x+y+c=0$ is $m=-1$. The value of c at which the line $x+y+c=0$ will touch the given circle $x^2+y^2=64$ is: $c=\pm a\sqrt{1+m^2}$, result (vii)

$$=\pm8\sqrt{1+(-1)^2}=8\sqrt{2},\quad a=8,\quad m=-1$$

The required tangent line that should touch the given circle is:

$$y=mx\pm a\sqrt{1+m^2}=-x\pm8\sqrt{2}\text{, result (vi)}$$

The point of contact through result (x) is: $(x_1,y_1)=\left(\frac{-am}{\sqrt{1+m^2}},\frac{a}{\sqrt{1+m^2}}\right)=\left(\frac{8}{\sqrt{2}},\frac{8}{\sqrt{2}}\right)$

8.3.4 The equations of tangent and normal to a circle at a point

The equation of a circle is: $x^2+y^2+2gx+2fy+c=0$ (i)

If $A(x_1,y_1)$ is a point lying on the circle (i), then the circle (i) becomes:

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \quad \text{(ii)}$$

If r_1 and r_2 are the position vectors of A and the center C(–g, –f) of the circle relative to origin

$r_1=(x_1,\ y_1)=x_1 i+y_1 j,\quad r_2=(-g,-f)=-gi-fj$ then, from the Figure 8.11:

$$OC+CA=OA$$
$$CA=OA-OC=r_1-r_2=(x_1+g,\ y_1+f)=(x_1+g)i+(y_1+f)j$$

Let $P(x,\ y)$ be any point on the tangent line AT, whose position vector is $OP=(x,\ y)$ which gives:

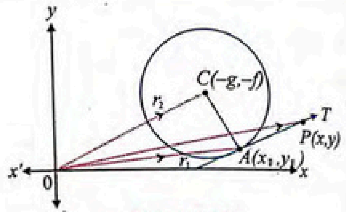


Figure 8.11

$$OA+AP=OP$$
$$AP=OP-OA$$
$$=r-r_1$$
$$=(x,\ y)-(x_1,y_1)$$
$$=(x-x_1,y-y_1)=(x-x_1)i+(y-y_1)j$$

The equation of tangent to the circle (i) is obtained if AP is perpendicular to CA for which the dot product in between the vectors AP and AC equals zero:

$$AP.CA=0$$
$$(x_1+g,y_1+f).(x-x_1,y-y_1)=0$$
$$(x_1+g)(x-x_1)+(y_1+f)(y-y_1)=0$$
$$xx_1+yy_1+gx+fy-(x_1^2+y_1^2+gx_1+fy_1)=0$$
$$xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1$$
$$=-gx_1-fy_1-c \qquad \text{result} \qquad \text{(ii)}$$
$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0 \qquad \text{(iii)}$$

The tangent equation to the circle $x^2+y^2=a^2$ at a point $A(x_1,y_1)$ through result (iii) is:

$$xx_1+yy_1=a^2 \qquad \text{(iv)}$$

The procedure for the normal equation at a point (x_1,y_1) on the circle $x^2+y^2+2gx+2fy+c=0$ is as under:

If C(– g, – f) is the center of the circle and $A(x_1,y_1)$ is a contact point, then the slope $\frac{y_1+f}{x_1+g}$ of the required normal line develops the normal line CA at $A(x_1,y_1)$:

$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1)$$
$$(y-y_1)(g+x_1)=(f+y_1)(x-x_1)$$
$$x(y_1+f)-y(x_1+g)+(gy_1-fx_1)=0 \qquad \text{(v)}$$

The normal equation to the circle $x^2+y^2=a^2$ at a point $A(x_1,y_1)$ through result (v) is:

$$xy_1-yx_1=0 \qquad \text{(vi)}$$

Example 11 Find the equations of the tangent and normal to the circle $x^2+y^2=25$ at a point (3, 4).

Solution Result (iv) is used to obtain the tangent equation to the given circle:

$$xx_1+yy_1=a^2$$
$$3x+4y=25,\quad a^2=25,\ (x_1,y_1)=(3,4)$$

Result (vi) is used to obtain the normal equation to the given circle:

$$xy_1-yx_1=0$$
$$4x-3y=0,\quad a^2=0,\ (x_1,y_1)=(3,4)$$

Example 12 Find the equations of the tangent and normal to the circle $x^2+y^2-2x+4y+3=25$ at a point (2,– 3).

Solution Result (iii) is used to obtain the tangent line to the given circle:

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$
$$2x-3y+(-1)(x+2)+(2)(y-3)+3=0 \qquad 2g=-2,\ 2f=4,\ c=3$$
$$2x-3y-x-2+2y-6+3=0,\ (x_1,y_1)=(2,-3)$$
$$x-y-5=0$$

Result (v) is used to obtain the normal line to the given circle:

$$x(y_1+f)-y(x_1+g)+(gy_1-fx_1)=0$$
$$x(-3+2)-y(2-1)+(3-4)=0,\ 2g=-2,\quad 2f=4,\quad c=3$$
$$-x-y-1=0$$
$$x+y+1=0$$

8.3.5 Length of a tangents to a circle from a given external point

The procedure for finding the length of the tangent drawn from the external point $P(x_1,y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$ is as under:

Let $P(x_1,y_1)$ be the given external point and PT be one of the two tangents drawn from point P to the circle

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{(i)}$$

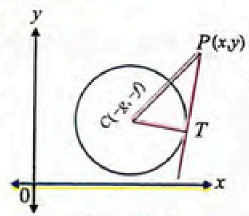


Figure 8.12

Join $\overline{CP}$ and $\overline{CT}$. C(– g, – f) is the center of the circle (i) and $\overline{CT}=\sqrt{g^2+f^2-c}$ is the radius of the circle (i).

From the right-angled triangle PTC, the length of the tangent PT drawn from point P to the given circle is:

$$|\overline{PC}|^2=|\overline{CT}|^2+|\overline{PT}|^2$$
$$|\overline{PT}|^2=|\overline{PC}|^2-|\overline{CT}|^2$$
$$=(x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)$$
$$=x_1^2+y_1^2+2gx_1+2fy_1+c$$
$$|\overline{PT}|=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c} \qquad \text{(ii)}$$

Note

- The length of the tangent drawn from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is:

  $|\overline{PT}| = \sqrt{x_1^2 + y_1^2 - a^2}$ (iii)

- The lengths of the two tangents drawn from the point $P(x_1, y_1)$ on the given circle are equal.

Example 13 Find the length of the tangent drawn from the point P(3,4) on the circles

(a). $x^2 + y^2 = 9$

(b). $x^2 + y^2 - 2x - y - 3 = 0$

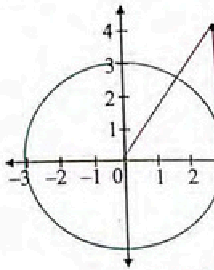


Figure 8.13

Solution

a. If $|\overline{PT}|$ is the tangent drawn from the point P(3,4) on the given circle, then, the length of the tangent PT on the given circle through result (iii) is:

$$|\overline{PT}| = \sqrt{x_1^2 + y_1^2 - 9}$$

$$= \sqrt{9+16-9} = \sqrt{16} = 4 \qquad (x_1, y_1) = (3,4),\ a^2 = 9$$

b. If PT is the tangent drawn from the point P(3,4) on the given circle, then, the length of the tangent PT on the given circle through result (ii) is:

$$|PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2f\,y_1 + c}, \qquad 2g = -2,\ 2f = -1,\ c = -3, (x_1, y_1) = (3,4)$$

$$= \sqrt{9+16-2(3)-(4)-3} = \sqrt{12}$$

8.3.6 Two tangents drawn to a circle from an external point are equal in length

If $y = mx \pm a\sqrt{1+m^2}$ is any tangent to the circle $x^2 + y^2 = a^2$, then the tangent line that passes through the point (x_1, y_1) is $y_1 = mx_1 + a\sqrt{1+m^2}$

$$y_1 - mx_1 = a\sqrt{1+m^2} \qquad \text{(i)}$$

Taking square on both sides of equation (i) $(y_1 - mx_1)^2 = a^2(1+m^2)$

that gives the quadratic equation in m: $m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - a^2) = 0$ (ii)

This quadratic equation (ii) gives two values of m that two values of m represent the slopes of the required two tangents on the given circle.

The tangents are real and different, real and coincident or imaginary or according as the discriminant of the quadratic equation (ii):

$disc = 4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) > 0,$ *real and different*

$disc = 4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) = 0,$ *real and coincident*

$disc = 4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) < 0,$ *imaginary*

or according as

$x_1^2 + y_1^2 - a^2 > 0,$ *real and different*

$x_1^2 + y_1^2 - a^2 = 0,$ *real and coincident*

$x_1^2 + y_1^2 - a^2 < 0,$ *imaginary*

or according as the point $P(x_1, y_1)$ lies outside, on, or inside the circle $x^2 + y^2 = a^2$.

In general, two tangent can also be drawn from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2f\,y + c = 0$.

Example 14 Find the equations of the tangents drawn from the point (6,4) to the circle $x^2 + y^2 = 16$.

Solution If $y = mx \pm c = mx \pm a\sqrt{1+m^2}$ is any tangent to the circle $x^2 + y^2 = 16$, then the number of tangents through result (ii)

$$m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - a^2) = 0$$

$$m^2(36-16) - 2m(6)(4) + (16-16) = 0,\ (x_1, y_1) = (6,4),\ a^2 = 16$$

$$20m^2 - 48m = 0$$

$$m(20m - 48) = 0 \Rightarrow m = 0, \frac{12}{5}$$

can be found by putting m = 0 and $m = \frac{12}{5}$ in $y = mx \pm a\sqrt{1+m^2}$:

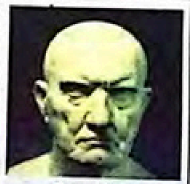
a. $y = mx \pm a\sqrt{1+m^2} = (0)x \pm 4\left(\sqrt{1+0}\right) = \pm 4 = 4,\ a = 4,\ m = 0$

b. $y = \left(\frac{12}{5}\right)x \pm 4\left(\sqrt{1+\frac{144}{25}}\right)$

$= \frac{12}{5}x \pm 4\left(\frac{13}{5}\right) = \frac{12}{5}x \pm \frac{52}{5} = \frac{12}{5}x - \frac{52}{5}$, *choose negative sign*

History

Menaechmus was a Greek mathematician. He was teacher of Alexander the Great and a friend of Ploto. He was the first person who introduced the conic section and investigate ellipse, parabola and hyperbola. He also gave the solution to the problem of doubling the cube. He introduced parabola as $y^2 = Lx$ where 'L' is a constant called the latus rectum although he was not acute of the fact that any equation in two unknowns determines a curve. He deliberately derived these properties of conic section and other properties also. By using these information it has not possible to find a solution to the problem of the duplication of the cube by solving for the point at which two parabolas intersect. Menaechmus's work an conic section is known as primary work for conic section.



Menaechmus
(380BC)-(320BC)

Exercise 8.2

1. In each case, find the tangent and normal equations
   a. at a point (1,2) to the circle $x^2+y^2=5$.   b. at a point $(-3,-2)$ to the circle $x^2+y^2=13$.
   c. at a point (4,1) to the circle $x^2+y^2-4x+2y-3=0$.
2. In each case, find the tangent and normal equations
   a. at a point $(\cos 60°, \sin 60°)$ to the circle $36(x^2+y^2)=13$.
   b. at a point $(2\cos 45°, 2\sin 45°)$ to the circle $x^2+y^2=4$.
   c. at a point $(\cos 30°, \sin 30°)$ to the circle $x^2+y^2=1$.
3. For what value of n,
   a. the line $lx+my+n=0$ touches the circle $x^2+y^2=a^2$?
   b. the line $x+y+n=0$ touches the circle $x^2+y^2=9$?
   c. the line $2x+2y+n=0$ touches the circle $x^2+y^2=81$?
4. For what value of c
   a. the line $y=mx+c$ touches the circle $x^2+y^2=a^2$?
   b. the line $y=-x+c$ touches the circle $x^2+y^2=9$?
   c. the line $y=-x-\left(\frac{c}{2}\right)$ touches the circle $x^2+y^2=81$?
5. Find the condition at which the line $lx+my+n=0$ touches the circle $x^2+y^2+2gx+2fy+c=0$.
6. For what value of n,
   a. the line $3x+4y+n=0$ touches the circle $x^2+y^2-4x-6y-12=0$?
   b. the line $x-2y+n=0$ touches the circle $x^2+y^2+3x+6y-5=0$?
   c. the line $2x+y+n=0$ touches the circle $x^2+y^2-2x-10y+21=0$?
7. If the tangent length from the point P to the circle $x^2+y^2=a^2$ is equal to the perpendicular distance from P to the line $lx+my+n=0$, then find out the locus of P.
8. Find the locus of the point P,
   a. If the length of the tangent line from the point P to the circle $x^2+y^2=9$ is equal to the perpendicular distance form P to the line $3x+4y+3=0$.
   b. If the length of the tangent line from the point P to the circle $x^2+y^2=25$ is equal to the perpendicular distance form P to the line $4x+3y+3=0$.
9. a. The length of the tangent from (f,g) to the circle $x^2+y^2=6$ is twice the length of the tangent to the circle $x^2+y^2+3x+3y=0$. Prove that $f^2+g^2+4f+4g+2=0$.
   b. the length of the tangent from (f,g) to the circle $x^2+y^2=4$ is 4 times the length of the tangent to the circle $x^2+y^2+2x+2y=0$. Prove that $15f^2+15g^2+32f+32g+4=0$.
10. Find the equations of the tangents to the
    a. circle $x^2+y^2=4$ which are parallel to the straight line $x+2y+3=0$.
    b. circle $x^2+y^2=25$ which are parallel to the straight line $3x+4y+3=0$.
11. Prove that the lines
    a. $x=8$ and $y=7$ touch the circle $x^2+y^2-6x-4y-12=0$. Find also the contact points.
    b. $x+y-1=0$ and $x-y+1=0$ touch the circle $x^2+y^2-4x-2y+3=0$. Find also the contact points.
12. Find the equations of the tangents
    a. to the circle $x^2+y^2=2$, which make an angle of 45° with the *x-axis*.
    b. to the circle $3x^2+3y^2=1$, which make an angle of 30° with the *x-axis*.
    c. to the circle $x^2+y^2=4$, which make an angle of 60° with the *x-axis*.

8.4 Properties of circle

There are some properties of a circle that are listed as under.

8.4.1 Perpendicular from the center of a circle on a chord bisects the chord

Let the circle be $x^2+y^2+2gx+2fy+c=0$ (i)

and PQ be any chord of a circle, whose end points are $P(x_1,y_1)$ and $Q(x_2,y_2)$ respectively.

If PQ is a chord of the circle, then P and Q are the points lying on the circle:
$$x_1^2+y_1^2+2gx_1+2fy_1+c=0$$
$$x_2^2+y_2^2+2gx_2+2fy_2+c=0$$

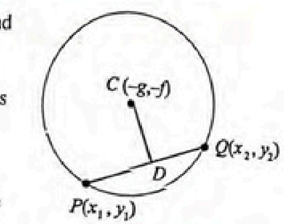


Figure 8.14

The subtraction of these two circles equations gives the slope of the chord $\overline{PQ}$:
$$(x_2^2-x_1^2)+(y_2^2-y_1^2)+2g(x_2-x_1)+2f(y_2-y_1)=0$$
$$(x_2^2-x_1^2)+2g(x_2-x_1)+(y_2^2-y_1^2)+2f(y_2-y_1)=0$$
$$(x_2-x_1)(x_1+x_2+2g)+(y_2-y_1)(y_1+y_2+2f)=0$$
$$\frac{y_2-y_1}{x_2-x_2}=-\frac{x_1+x_2+2g}{y_1+y_2+2f}=m_1,\ say \qquad \text{(ii)}$$

If the center of the circle is C(− g,− f) and the midpoint of the chord $\overline{PQ}$ is $D\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$, then the slope of the perpendicular line $\overline{CD}$ is:
$$m_2=\frac{\frac{y_1+y_2}{2}+f}{\frac{x_1+x_2}{2}+g}=\frac{y_1+y_2+2f}{x_1+x_2+2g} \qquad \text{(iii)}$$

From the Figure 8.14 the chord $\overline{PQ}$ and the line $\overline{CD}$ are perpendicular if and only if the product of their slopes equals −1:
$$m_1m_2=-\frac{x_1+x_2+2g}{y_1+y_2+2f}\cdot\frac{y_1+y_2+2f}{x_1+x_2+2g}=-1$$

Thus, $\overline{CD}$ is bisector of the chord $\overline{PQ}$.

Note

- the perpendicular bisector of any chord $\overline{PQ}$ of a circle passes through the center of the circle. This is our **second property.**
- the line joining the two points of the circle that touches the center of the circle is called the diameter of the circle. This diameter acts as the perpendicular bisector to the chord $\overline{PQ}$, if the diameter of a circle bisects the chord $\overline{PQ}$. This is our **third property**. The proof is similar to property first, but the graphical view is shown in the Figure 8.14.

Example 15 If $A(-3,4)$ and $B(1,5)$ are the end points of the chord $\overline{AB}$ of the circle $x^2+y^2+x-5y-2=0$, then show that

(a). the line from the center of the circle is perpendicular to $\overline{AB}$, also bisects the chord $\overline{AB}$.

(b). the line from the center of the circle to the midpoint of the chord $\overline{AB}$ is perpendicular to the chord $\overline{AB}$.

(c). the perpendicular bisector $\overline{CD}$ of the chord $\overline{AB}$ passes though the center of the given circle.

Solution The equation of the circle with center $C\left(\frac{-1}{2},\frac{5}{2}\right)$ is:

$$x^2+y^2+x-5y-2=0$$

If the center of the circle is $C\left(\frac{-1}{2},\frac{5}{2}\right)$ and the midpoint of the chord $\overline{AB}$ is $D\left(-1,\frac{9}{2}\right)$, then the slopes of the chord AB and the perpendicular line CD are respectively:

$$\text{slope of } \overline{AB}=m_1=\frac{5-4}{1+3}=\frac{1}{4}, \qquad \text{slope of } \overline{CD}=m_2=\frac{\frac{9}{2}-\frac{5}{2}}{-1+\frac{1}{2}}=-4$$

The chord $\overline{AB}$ and the line $\overline{CD}$ are perpendicular if and only if the product of their slopes equals –1:

$$m_1m_1=\frac{1}{4}.(-4)=-1$$

Therefore, $\overline{CD}$ is perpendicular bisector of the chord $\overline{AB}$. This result is automatically valid for parts *b* and *c*.

8.4.4 Congruent chords of a circle are equidistant from its center and its converse

If the perpendicular distances form the center of a circle to its two chords are equal, then the chords are congruent.

Let the circle equation with center $C(-g,-f)$ is:

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{(i)}$$

If $\overline{AB}$ and $\overline{DE}$ are the two chords of the circle(i), then the coordinates of the end points of the chord $\overline{AB}$ and $\overline{DE}$ are respectively:

$$A(x_1,y_1), D(x_2,y_2), B(x_3,y_3), E(x_4,y_4).$$

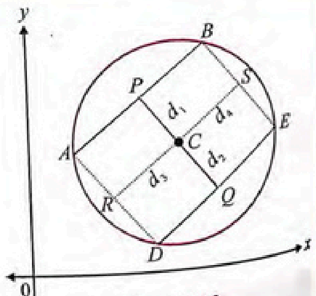


Figure 8.15

From the Figure 8.15, it is clear that the perpendicular distance $d_1=\overline{CP}$ from the center C on the chord $\overline{AB}$ equals the perpendicular distance $d_2=\overline{CQ}$ from C on the chord DE, if and only if the chords $\overline{AB}$ and $\overline{DE}$ are with equal lengths: $|\overline{AB}|=|\overline{DE}|$ (ii)

Thus, the chords AB and DE are equidistant from C on the circle (i) if and only if $d_1=d_2$

In similar manner, the chords $\overline{AB}$ (join A to D) and $\overline{BE}$ (join B to E) are congruent chords, if the perpendicular distance $d_3=\overrightarrow{CR}$ from C on the chord $\overline{AD}$ equals the perpendicular $d_4=\overline{CS}$ from C on the chord BE: $d_3=d_4$, $|\overline{AD}|=|\overline{BE}|$ (iii)

Example 16 Show that the chords $\overline{AB}$ and $\overline{DE}$ are equidistant from the center C(0,0) of the circle $x^2+y^2=4$. The coordinates of the end points of the two chords are A(0,2), B(– 2,0), D(0,– 2) and E(2,0).

Solution The circle equation with center C(0,0) is: $x^2+y^2=4$ (v)

If $\overline{AB}$ and $\overline{DE}$ are the two chords of the circle (i), whose coordinates are respectively:

$$A(0,2),\quad B(-2,0),\quad D(0,-2),\quad E(2,0)$$

From the Figure 8.16, it is clear that the chords AB and DE are with equal length:

$$\overline{AB}=(-2-0,0-2),\quad |\overline{AB}|=\sqrt{(-2)^2+(-2)^2}=2\sqrt{2}$$

$$\overline{DE}=(2-0,0+2),\quad |\overline{DE}|=\sqrt{(2)^2+(2)^2}=2\sqrt{2}$$

Thus, the two chords $\overline{AB}$ and $\overline{DE}$ are equal. For equidistant, the procedure is as under:

The equations of the chords $\overline{AB}$ and $\overline{DE}$ (through two-point form of the line) are respectively:

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-2}{x-0}=\frac{0-2}{-2-0},\quad A(x_1,y_1)=A(0,2),\ B(x_2,y_2)=B(-2,0)$$

$$\frac{y-2}{x}=1 \Rightarrow x-y+2=0$$

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y+2}{x-0}=\frac{0+2}{2-0},\quad D(x_1,y_1)=D(0,-2),\ E(x_2,y_2)=E(2,0)$$

$$\frac{y+2}{x}=1 \Rightarrow x-y-2=0$$

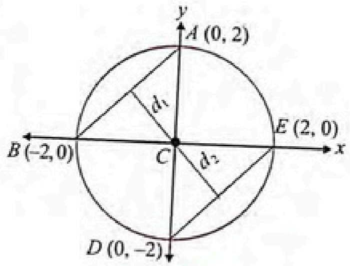


Figure 8.16

The perpendicular distance d_1 from C(0,0) on the chord AB is: $d_1=\left|\frac{0-0+2}{\sqrt{1+1}}\right|=\frac{2}{\sqrt{2}}$

The perpendicular distance d_2 from C(0,0) on the chord DE is: $d_2=\left|\frac{0-0-2}{\sqrt{1+1}}\right|=\frac{2}{\sqrt{2}}$

The perpendicular distance d_1 from C(0,0) on the chord AB is equal to the perpendicular distance d_2 form C(0,0) on the chord DE: $d_1=d_2=\frac{2}{\sqrt{2}}$

Thus, the chords AB and DE are equidistant from the center C(0,0) of the circle (i).

8.4.5 Measure of the central angle of a minor arc is double the measure of the angle subtended by the corresponding major arc

Let the circle be $x^2+y^2=a^2$ (i)

The arc BC is the minor arc of the circle (i), whose coordinates are $B(-x_1,-y_1)$ and $C(x_1,-y_1)$, and the minor arc BC subtended the angle from the center of the circle is $\angle BOC$.

If A(0, *a*) is a point on the major arc, then join AB and AC that develops the angle of the minor arc which is two times the angle subtended by the major arc: $\angle BOC=2\angle BAC$ (ii)

From the Figure 8.17, if $\angle BAC = \theta$ and $\angle BOC = 2\theta$, then, result (ii) can be verified as follows:

If the slopes of BA and AC are

$$m_1 = \frac{a+y_1}{x_1},\ m_2 = \frac{-y_1-a}{x_1} = \frac{-(a+y_1)}{x_1}, \text{ then,}$$

the angle $\angle BAC = \theta$ from $\overline{BA}$ to $\overline{AC}$ is:

$$\tan\theta = \frac{m_1-m_2}{1+m_1m_2} = \frac{\frac{a+y_1}{x_1}+\frac{a+y_1}{x_1}}{1-\frac{a+y_1}{x_1}\cdot\frac{a+y_1}{x_1}} = \frac{2(a+y_1)}{x_1}\cdot\frac{x_1^2}{x_1^2-(a+y_1)^2}$$

$$= \frac{2x_1(a+y_1)}{x_1^2-a^2-y_1^2-2ay_1} = \frac{2x_1(a+y_1)}{-2y_1^2-2ay_1} \qquad \text{(iii)}$$

$$= \frac{2x_1(a+y_1)}{-2y_1(a+y_1)} = -\frac{x_1}{y_1}, \quad x_1^2+y_1^2=a^2$$

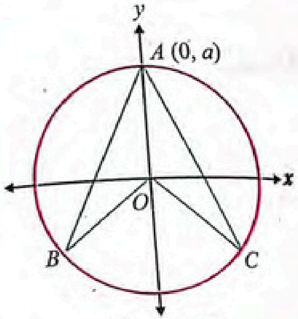


Figure 8.17

If the slopes of $\overline{BO}$ and CO are $m_3 = \frac{y_1}{x_1}$, $m_4 = \frac{y_1}{-x_1}$, then, the angle $\angle BOC = 2\theta$ from BO to OC is:

$$\tan 2\theta = \frac{m_3-m_4}{1+m_3m_4} = \frac{\frac{y_1}{x_1}+\frac{y_1}{x_1}}{1-\frac{y_1}{x_1}\cdot\frac{y_1}{x_1}} = \frac{2x_1^2y_1}{x_1(x_1^2-y_1^2)} = \frac{2x_1y_1}{x_1^2-y_1^2} \qquad \text{(iv)}$$

The trigonometric identity $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{\frac{-2x_1}{y_1}}{1-\frac{x_1^2}{y_1^2}} = \frac{-2x_1y_1}{y_1^2-x_1^2} = \frac{-2x_1y_1}{-(x_1^2-y_1^2)} = \frac{2x_1y_1}{x_1^2-y_1^2}$

is proving result (iv) with result (iii). Thus $\angle BOC = 2\angle BAC$.

Example 17 Show that the angle subtended by the minor arc BC of the circle $x^2+y^2=9$ is two times the angle subtended in the major arc. The coordinates of the minor arc are $B(2,\sqrt{5}), C(2,-\sqrt{5})$.

Solution The circle $x^2+y^2=9$, whose center is O(0,0). The arc $\overline{BC}$ is the minor arc of the given circle, whose coordinates are $B(2,\sqrt{5}), C(2,-\sqrt{5})$ and the minor arc $\overline{BC}$ subtended the angle from the center of the circle is $\angle BOC$.

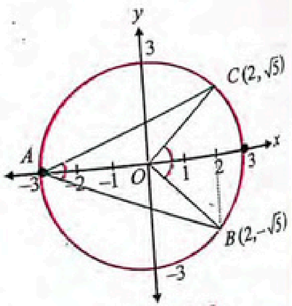


Figure 8.18

If A(–3,0) is a point on the major arc, then join $\overline{AB}$ and $\overline{AC}$ that develops the angle of the minor arc which is two times the angle subtended by the major arc: $\angle BOC = 2\angle BAC$ (i)

From the Figure 8.18, if $\angle BAC = \theta$ and $\angle BOC = 2\theta$, then, result (v) can be verified as follows:

If the slopes of $\overline{BA}$ and $\overline{AC}$ are $m_1 = \frac{0+\sqrt{5}}{-3-2} = -\frac{1}{\sqrt{5}}$, $m_2 = \frac{\sqrt{5}-0}{2+3} = \frac{1}{\sqrt{5}}$,

then, the angle $\angle BAC = \theta$ from $\overline{BA}$ to $\overline{AC}$ is $\tan\theta = \frac{m_1-m_2}{1+m_1m_2} = \frac{-\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{5}}}{1-\frac{1}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}}} = \frac{-\frac{2}{\sqrt{5}}}{1-\frac{1}{5}} = \frac{\sqrt{5}}{2}$ (ii)

If the slopes of $\overline{BO}$ and $\overline{OC}$ are $m_3 = \frac{0+\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$, $m_4 = \frac{\sqrt{5}-0}{2} = \frac{\sqrt{5}}{2}$

then, the angle $\angle BOC = 2\theta$ from OC to BO is $\tan 2\theta = \frac{m_4-m_3}{1+m_3m_4} = \frac{\frac{\sqrt{5}}{2}+\frac{\sqrt{5}}{2}}{1-\frac{\sqrt{5}}{2}\cdot\frac{\sqrt{5}}{2}} = \frac{\sqrt{5}}{1-\frac{5}{4}} = -4\sqrt{5}$ (iii)

The trigonometric identity $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{\sqrt{5}}{2}\right)}{1-\frac{5}{4}} = -4\sqrt{5}$

is proving result (ii) with result (iii). Thus $\angle BOC = 2\angle BAC$.

8.4.6 An angle in a semi-circle is a right angle

Let the circle equation be $x^2+y^2=a^2$ (i)

If $P(x_1, y_1)$ is any point on the semicircle and $\overline{BA}$ is fixed as the diameter of the circle (i) on the x-axis, whose coordinates are A (a, 0) and B (–a, 0), then the point $P(x_1, y_1)$ lies on the circle (i) that changes the circle equation to: $x_1^2+y_1^2=a^2$

Join $\overline{PA}$ and $\overline{PB}$ that develops a right angle $\angle APB$. The angle $\angle APB$ is a right angle, if $\overline{AP}$ and $\overline{PB}$ are perpendicular to each other, for which the slopes of $\overline{AP}$ and $\overline{PB}$ are respectively:

$$m_1 = \frac{y_1-0}{x_1-a} = \frac{y_1}{x_1-a}, \quad m_2 = \frac{y_1-0}{x_1+a} = \frac{y_1}{x_1+a}$$

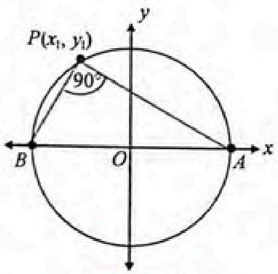


Figure 8.19

The product of the slopes of $\overline{AP}$ and $\overline{BP}$ is

$$m_1m_2 = \left(\frac{y_1}{x_1-a}\right)\cdot\left(\frac{y_1}{x_1+a}\right) = \frac{y_1^2}{x_1^2-a^2} = \frac{y_1^2}{-y_1^2} = -1, \quad x_1^2+y_1^2=a^2$$

Thus, $\overline{PA}$ and $\overline{PB}$ are perpendicular and the angle $\angle APB = 90°$ is of course a right-angle.

If $\angle APB = 90°$, then P is a point lies on the semicircle, for which the Pythagorean rule

$$\left|\overline{PA}\right|^2 + \left|\overline{PB}\right|^2 = \left|\overline{AB}\right|^2 \qquad \text{(iii)}$$

with substitution of $\overline{PA} = (a-x_1, 0-y_1)$, $\Rightarrow \left|\overline{PA}\right|^2 = \left[\sqrt{(a-x_1)^2+y_1^2}\right]^2 = (a-x_1)^2+y_1^2$

$\overline{PB} = (-a-x_1, -0-y_1)$ $\Rightarrow \left|\overline{PB}\right|^2 = \left[\sqrt{(a+x_1)^2+y_1^2}\right]^2 = (a+x_1)^2+y_1^2$

$\overline{AB} = (-a-a, 0-0)$ $\Rightarrow \left|\overline{AB}\right|^2 = \left[\sqrt{(-2a)^2+0}\right]^2 = 4a^2$

gives the locus of $P(x_1, y_1)$

$$\left|\overline{PA}\right|^2 + \left|\overline{PB}\right|^2 = \left|\overline{AB}\right|^2, \quad \overline{AB} = (-a, 0)-(a, 0)$$

$$(a-x_1)^2+y_1^2+(a+x_1)^2+y_1^2 = 4a^2$$

$$a^2+x_1^2-2ax_1+y_1^2+a^2+x_1^2-2ax_1 = 4a^2$$

$$2a^2+2x_1^2+2y_1^2 = 4a^2 \Rightarrow 2x_1^2+2y_1^2 = 2a^2 \Rightarrow x_1^2+y_1^2=a^2$$

which is a circle, P may lie on the upper or the lower semicircle.

Example 18 Show that the angle in the semicircle of the circle $(x-h)^2+y^2=a^2$ is a right-angle.

Solution The circle equation with center $C(h, 0)$ is : $(x-h)^2+y^2=a^2$

If $P(x_1, y_1)$ is any point on the semicircle and $\overline{AB}$ is fixed as the diameter of the given circle on the x-axis, then $(x_1-h)^2+y_1^2=a^2$

The coordinates of A and B are respectively:

$\overline{OA}=\overline{OC}-\overline{AC}=h-a \quad \Rightarrow A(h-a, 0)$

$\overline{OB}=\overline{OA}+\overline{AB}=(h-a)+2a=h+a \quad \Rightarrow B(h+a, 0)$

Join $\overline{PA}$ and $\overline{PB}$ that develops a right angle $\angle APB$. This angle $\angle APB$ is a right angle, if AP and BP are perpendicular. They are perpendicular, if the product of their slopes equals -1:

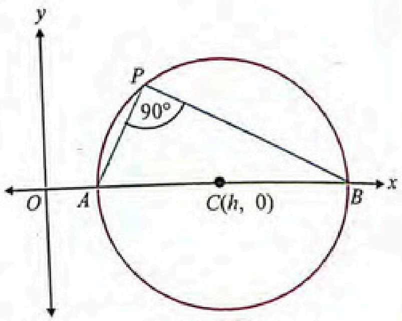


Figure 8.20

$$m_1m_2=\frac{y_1}{x_1-h+a}\cdot\frac{y_1}{x_1-h-a}=\frac{y_1^2}{(x_1-h+a)(x_1-h-a)}$$

$$=\frac{y_1^2}{(x_1-h)^2-a^2}=\frac{y_1^2}{-y_1^2}=-1, \qquad y_1^2=a^2-(x_1-h)^2=-\left[(x_1-h)^2-a^2\right]$$

when $m_1=\dfrac{y_1-0}{x_1-(h-0)}=\dfrac{y_1}{x_1-h+a}$ and $m_2=\dfrac{y_1-0}{x_1-(h+a)}=\dfrac{y_1}{x_1-h-a}$

Thus, the angle $\angle APB=90°$ is right angle.

8.4.7 The Perpendicular at the outer end of radial segment is tangent to the circle

The circle equation with center $C(-g,-f)$ is : $x^2+y^2+2gx+2fy+c=0$ (i)

If $P(x_1, y_1)$ is a point on the circle and $C(-g,-f)$ is the center of the circle (i), then $\overline{CP}$ is the radial segment of the circle.

The equation of the tangent line on the circle (i) at point P is

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0$

whose slope is $m_1=-\dfrac{x_1+g}{y_1+f}$ and the slope of $\overline{CP}$ is

$m_2=\dfrac{y_1+f}{x_1+g}$.

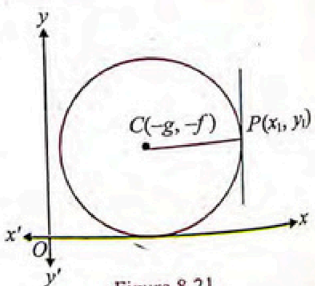


Figure 8.21

The perpendicular at the outer end P of the radial segment $\overline{CP}$ is tangent to the circle (i) if the product of the slopes of the radial segment $\overline{CP}$ and the line of the outer end of the radial segment $\overline{CP}$ is -1: $m_1m_2=-\dfrac{x_1+g}{y_1+f}\cdot\dfrac{y_1+f}{x_1+g}=-1$

Thus, the perpendicular at the outer end P of the radial segment is tangent to the circle (i).

Remember

- the tangent line is perpendicular to the radial segment, if the radial segment is the segment through the point of contact of the tangent and the center of the circle.
- if a line is perpendicular to the tangent of the circle at the point of contact, then it passes through the center of the circle:

Example 19 Show that the perpendicular at the outer end point P(1,1) of the radial segment is tangent to the circle $x^2+y^2-13x-5y+16=0$.

Solution The circle equation with center $C\left(\frac{13}{2},\frac{5}{2}\right)$ is: $x^2+y^2-13x-5y+16=0$

The equation of the tangent line on the given circle at a point $P(1,1)$ is

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

$x(1)+y(1)-\frac{13}{2}(x+1)+\frac{5}{2}(y+1)+16=0, \quad P(x_1, y_1)=P(1,1)$

$11x+3y-16=0$

whose slope is $m_1=\dfrac{-11}{3}$ and the slope of the radial segment $\overline{CP}$ is $m_2=\dfrac{1-\frac{5}{2}}{1-\frac{13}{2}}=\dfrac{3}{11}$

The product of the slopes of the tangent to the circle and the radial segment $\overline{CP}$ is $m_1m_2=\dfrac{-11}{3}\cdot\dfrac{3}{11}=-1$

Thus, the perpendicular at the outer end P of the radial segment is tangent to the given circle at point P.

Exercise 8.3

1. If A(2,2) and B(3,1) are the end points of the chord $\overline{AB}$ of the circle $x^2+y^2-4x-2y+4=0$, then show that
   a. the line from the center of the circle is perpendicular to $\overline{AB}$, also bisects the chord $\overline{AB}$.
   b. the line from the center of the circle to the midpoint of the chord $\overline{AB}$ is perpendicular to the chord $\overline{AB}$.
2. If A(0,0) and B(0,3) are the end points of the chord $\overline{AB}$ of the circle $x^2+y^2+4x-5y=0$, then show that
   a. the line from the center of the circle is perpendicular to $\overline{AB}$, also bisects the chord $\overline{AB}$.
   b. the line from the center of the circle to the midpoint of the chord $\overline{AB}$ is perpendicular to the chord $\overline{AB}$.
3. Show that the chords $\overline{AB}$ and $\overline{DE}$ are equidistant from the center C(0,0) of the circle
   a. $x^2+y^2=4$. The coordinates of the end points of the two chords $\overline{AB}$ and $\overline{DE}$ are A(−2,0), B(0,2), D(0,2) and E(2,0).
   b. $x^2+y^2=16$. The coordinates of the end points of the two chords $\overline{AB}$ and $\overline{DE}$ are A(−4,0), B(0,4), D(0,4) and E(4,0).
4. Show that the angle subtended by the minor arc $\overline{AB}$ of the circle
   a. $x^2+y^2=9$ is two times the angle subtended in the major arc. The coordinates of the minor arc $\overline{AB}$ are A $(2, \sqrt{5})$, B $(2, -\sqrt{5})$.

b. $x^2+y^2=4$ is two times the angle subtended in the major arc. The coordinates of the minor arc $\overline{AB}$ are $A(1,\sqrt{3})$, $B(1,-\sqrt{3})$.

5. Show that the angle in the semicircle of the circle

a. $(x-h)^2+y^2=a^2,\ h=1,\ a=2$ is a right-angle.

b. $(x-h)^2+y^2=a^2,\ h=3,\ a=4$ is a right-angle.

Note that the diameter of the circle (in each case) is considered to be $\overline{AB}$.

6. Show that the perpendicular at the outer end point

a. P(1,5) of the radial segment is tangent to the circle $x^2+y^2+x-5y-2=0$.

b. P(5,6) of the radial segment is tangent to the circle $x^2+y^2-22x-4y+25=0$.

Review Exercise 8

1. **Choose the correct option.**

i. If radius of a circle is 2*cm* then its equation will be:
(a) $x^2+y^2=2$ (b) $x^2+y^2=\sqrt{2}$ (c) $x^2+y^2=8$ (d) $x^2+y^2=4$

ii. In the general equation of circle the coordinates of centre are:
(a) (x,y) (b) $(-x,-y)$ (c) (f,g) (d) $(-f,-g)$

iii. For the general equation of the circle the radius can be calculated by r =
(a) $\sqrt{x^2+y^2-c^2}$ (b) $\sqrt{(-x)^2+(-y)^2-c^2}$
(c) $\sqrt{(-g)^2+(-f)^2-c^2}$ (d) $\sqrt{(-g)^2+(-f)^2+c^2}$

iv. A point of the circle at which a tangent meets the circle is called
(a) point of contact (b) normal (c) center point (d) none of these

v. If the discriminant of the quadratic equations $m^2(x_1^2-a_2)-2mx_1y_1+(y_1^2-a_2)=0$ is greater than zero than tangent are
(a) real and different (b) imaginary
(c) real an coincident (d) equal

vi. The length of the tangent drawn from the point p(3,4) on the circle $x^2+y^2-9=0$ is
(a) 1 (b) 2 (c) 3 (d) 4

vii. The angle subtended can be calculated by using the trigonometric identity.
(a) $\tan 2\theta=\frac{m_2-m_1}{1+m_1m_2}$ (b) $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$
(c) $\tan 2\theta=\frac{2\tan^2\theta}{1+\tan^2\theta}$ (d) $\tan 2\theta=\frac{\tan^2\theta}{1-2\tan^2\theta}$

viii. What is the contact point for the lines $x=7$ touches the $x^2+y^2-4x-6y-12=0$
(a) (2,8) (b) (8.2) (c) (3,7) (d) (7,3)

ix. For the condition of tangent the line $y=mx+c$ should touch the circle $x^2+y^2=r^2$ if
(a) $c=r\sqrt{1+m^2}$ (b) $c=-r\sqrt{1+m^2}$
(c) both option a & b (d) not option (b) nor option (c)

x. A line perpendicular to the contact point to a circle is called
(a) Tangent (b) Normal (c) Chord (d) Diameter

Summary

- Standard Form of a Circle: The standard form of a circle with radius r and center C(*h*, *k*) is: $(x-h)^2+(y-k)^2=r^2$
- General Form of a Circle: The general form of a circle with radius $r=\sqrt{(-g)^2+(f)^2-c}$ and center C(−*g*, −*f*) is: $x^2+y^2+2gx+2fy+c=0$
  The coefficient of x^2 is equal to the coefficient of y^2, and there is no term containing *xy* and the square of the radius is $r^2\geq 0$.
- Nature of the circle:
  If $g^2+f^2-c>0$, then the circle is real and different from zero.
  If $g^2+f^2-c=0$, then the circle shrinks to a point (−*g*, −*f*). It is called a point circle.
  If $g^2+f^2-c<0$, then the circle is imaginary or virtual.
- Condition of Tangency:
  The condition at which the line $y=mx+c$ should touch the circle $x^2+y^2=a^2$ is: $c=\pm a\sqrt{1+m^2}$
  The equation of any tangent to the circle $x^2+y^2=a^2$ in the slope-form is: $y=mx\pm a\sqrt{1+m^2}$
  The condition at which the line $lx+my+n=0$ should touch the circle $x^2+y^2=a^2$ is: $n=\pm a\sqrt{l^2+m^2}$
  The condition at which the line $lx+my+n=0$ should touch the circle $x^2+y^2+2gx+2fy+c=0$ is: $(c-f^2)l^2+2fglm+(c-g^2)m^2-2n(gl+fm)+n^2=0$
- The tangent equation to the circle $x^2+y^2+2gx+2fy+c=0$ at a point $A(x_1,y_1)$ is: $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$
  The tangent equation to the circle $x^2+y^2=a^2$ at a point $A(x_1,y_1)$ is: $xx_1+yy_1=a^2$
- The normal equation to the circle $x^2+y^2+2gx+2fy+c=0$ at a point $A(x_1,y_1)$ is: $x(y_1+f)-y(x_1+g)+(gy_1-fx_1)=0$
  The normal equation to the circle $x^2+y^2=a^2$ at a point $A(x_1,y_1)$ is: $xy_1-yx_1=0$
- The length of the tangent drawn from the point $P(x_1,y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$ is: $|\overline{PT}|=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$
  The length of the tangent drawn from the point $P(x_1,y_1)$ to the circle $x^2+y^2=a^2$ is: $|\overline{PT}|=\sqrt{x_1^2+y_1^2-a^2}$
  The lengths of the two tangents drawn from the point $P(x_1,y_1)$ on the given circle are equal.