Summary

- $F(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$.
- If $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, for any real number C. It is called indefinite integral.
- If $f(x)$ and $g(x)$ are integral functions w.r.t. x, then the integral of the product of $f(x)$ and $g(x)$ w.r.t. x is:
  $\int udv = uv - \int vdu$, $v = g(x)$, $du = f'(x)dx$ and $dv = g'(x)dx$:
- If $f(x)$ is continuous on the interval $[a, b]$ and $[a, b]$ is divided into n equal subintervals whose right-hand points are $x_1, x_2, ..., x_n$, then the **definite integral** of $f(x)$ from $x = a$ to $x = b$ is:

$$\int_{x=a}^{x=b} f(x)dx = \lim_{n\to\infty} \frac{(b-a)}{n}[f(x_1) + f(x_2) + ... + f(x_n)], \quad \Delta x = \frac{b-a}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x, \quad i = 1, 2, 3, ..., n$$

- The **definite integral** of the product of two functions u and v w.r.t. x is:

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

- If $f(x)$ is continuous and $f(x) \geq 0$ on the closed interval $[a, b]$, then the area under a curve $y = f(x)$ on $[a, b]$ is given by the **definite integral** of $f(x)$ on $[a, b]$:

$$Area = \int_a^b f(x)dx = F(b) - F(a)$$

- If a function $f(x)$ is continuous on the closed interval $[a, b]$, then

$$\int_{x=a}^{x=b} f(x)dx = |F(x)|_{x=a}^{x=b} = F(b) - F(a)$$

Where $F(x)$ is any function such that $F'(x) = f(x)$ for all x in $[a, b]$.

Unit 7 PLANE ANALYTIC GEOMETRY STRAIGHT LINE

By the end of this unit, the students will be able to:

7.1 Division of a line segment
- i. Recall distance formula to calculate distance between two points given in Cartesian plane.
- ii. Find coordinates of a point that divides the line segment in given ratio (internally and externally).
- iii. Show that the medians and angle bisectors of a triangle are concurrent.

7.2 Slope of a straight line
- i. Define the slope of a line.
- ii. Derive the formula to find the slope of a line passing through two points.
- ii. Find the condition that two straight lines with given slopes may be
  - parallel to each other,
  - perpendicular to each other

7.3 Equation of a straight line parallel to Co-ordinate axes
- i. Find the equation of a straight line parallel to
  - y-axis and at a distance a from it,
  - x-axis and at a distance b from it

7.4 Standard form of equation of a straight line
- i. Define intercepts of a straight line. Derive equation of a straight line in
  - slope-intercept from,
  - point-slope form,
  - two-point form,
  - intercepts form,
  - symmetric form,
  - normal form
- ii. Show that a linear equation in two variables represents a straight line.
- iii. Reduce the general form of the equation of a straight line to the other standard forms.

7.5 Distance of a point from a line
- i. Recognize a point with respect to position of a line.
- ii. Find the perpendicular distance from a point to the given straight lines.

7.6 Angle between lines
- i. Find the angle between two coplanar intersecting straight lines.
- ii. Find the equation of family of lines passing through the point of intersection of two given lines.
- iii. Calculate angles of the triangle when the slopes of the sides are given.

7.7 Concurrency of straight lines
- i. Find the condition of concurrency of three straight lines.
- ii. Find the equation of median, altitude and right bisector of a triangle.
- iii. Show that
  - three right bisectors,
  - three medians,
  - three altitudes, of a triangle are concurrent.

7.8 Area of a triangular region
- i. Find area of a triangular region whose vertices are given.

7.9 Homogenous equation
- i. Recognize homogeneous linear and quadratic equations in two variables.
- ii. Investigate that the 2nd degree homogeneous equation in two variables x and y represents a pair of straight lines through the origin and find acute angle between them.

Introduction

We are familiar about Cartesian coordinate system, we have learnt about it in our previous classes. This Cartesian coordinate system may be helpful to know the slope formula, Pythagoras theorem and distance formula. In this lesson we will learn in details and write the equations involving arbitrary points. Most of the geometric ideas can be expressed using algebraic equations. Analytic geometry is defined as:

"The study of relationship between geometry and algebra is called analytic geometry".

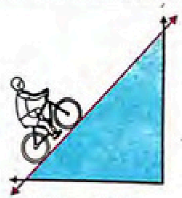


Figure 7.1

For example to calculate the slope/gradient between two given points, the numerator is the difference in the y-coordinates some times called it "Rise" and the denominator is the difference between x-coordinates, some time called it "run" e.g.

$$\text{Slop between two points} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{Rise}{Run}$$

Do You Know?

Analytic Geometry was independent and simultaneous invention of Pierre De Fermat and Rene Descartes. The fundamental idea of Analytic Geometry and the representation of curved lines by algebraic equations relating two variables say, x and y was given in seventeenth century by them.

7.1 Division of a line segment

We are familiar with the set of real numbers as well as with several of its subsets, including natural numbers and real numbers. The real numbers can easily be visualized by using a one dimensional coordinate system call real number line.

7.1.1 Calculation of distance between two given points

The study of plane analytic geometry is greatly facilitated by the use of vectors. The distance between any two given points can be calculated by using the distance formula.

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the xy–plane and θ is the angle in between the positive directions of the x and y axes, then, PQ is the directed line segment associated to initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$.

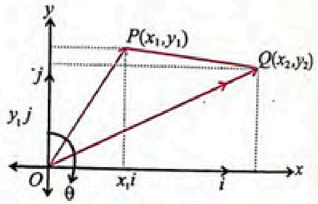


Figure 7.2

The components of the directed line segment PQ are:

$OP + PQ = OQ$

$PQ = OQ - OP$, position vectors

$= (x_2, y_2) - (x_1, y_1)$

$PQ = (x_2 - x_1, y_2 - y_1)$

$= (x_2 - x_1)i + (y_2 - y_1)j$

Pythagoras Theorem: If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the two points in the xy-plane, then the distance d between the given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is obtained by applying the theorem of Pythagoras to triangle PQR:

$(PQ)^2 = (PR)^2 + (QR)^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d, say$

Squaring both side of the directed line segment PQ to obtain

$(PQ)^2 = [(x_2 - x_1)i + (y_2 - y_1)j]^2 \quad \therefore (a+b)^2 = a^2 + b^2 + 2ab$

$= (x_2 - x_1)^2 i.i + (y_2 - y_1)^2 j.j + 2(x_2 - x_1)(y_2 - y_1)i.j$

$= (x_2 - x_1)^2 i.i + (y_2 - y_1)^2 j.j + 2(x_2 - x_1)(y_2 - y_1)|i||j|\cos\theta \quad \therefore i.i = j.j = 1,\ i.j = |i||j|\cos\theta,\ |i| = |j| = 1$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\theta$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\frac{\pi}{2} \quad \therefore \theta = \frac{\pi}{2}$

$(PQ)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \therefore \cos\frac{\pi}{2} = 0$

$|PQ|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \therefore (PQ)^2 = |PQ|^2$

$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d, say \qquad (i)$

This is the distance from point $P(x_1, y_1)$ to point $Q_2(x_2, y_2)$ in the Cartesian coordinate plane.

Note

- The distance from the origin O(0,0) to point $P(x_1, y_1)$ is obtained by inserting $x_2 = y_2 = 0$ in result (1): $d = |OP| = \sqrt{x_1^2 + y_1^2}$
- The distance from the origin O(0,0) to point $Q(x_2, y_2)$ is obtained by inserting $x_1 = y_1 = 0$ in result (1): $d = |OQ| = \sqrt{x_2^2 + y_2^2}$
- If the line segment PQ is horizontal, then the distance from the point $P(x_1, y_1)$ to point $Q(x_2, y_2)$ is obtained by inserting $y_1 = y_2$ in result (1): $d = |PQ| = \sqrt{(x_2 - x_1)^2}$
- If the line segment PQ is vertical, then the distance from point $P(x_1, y_1)$ to point $Q(x_2, y_2)$ is obtained by inserting $x_1 = x_2$ in result (1): $d = |PQ| = \sqrt{(y_2 - y_1)^2}$

Example 1 Find the distance between the two points P(3,–2) and Q(–1, –5).

Solution $P(x_1, y_1) = (3, -2)$, $Q(x_2, y_2) = (-1, -5)$ is used to obtain the distance d in between the two points P and Q:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-1-3)^2 + [(-5-(-2)]^2} = \sqrt{(-4)^2 + (-3)^2} = \sqrt{25} = 5$$

7.1.2 Co-ordinates of a point that divides the line segment in given ratio (Internally and externally)

Take $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the initial and terminal points of a line segment PQ and $R(x, y)$ is a point that divides PQ in the ratio $m_1 : m_2$. If r_1, r_2 and r are the position vectors of P, Q and R, then

$r_1 = (x_1, y_1) = x_1 i + y_1 j, \quad r_2 = (x_2, y_2) = x_2 i + y_2 j, \quad r = (x, y) = xi + yj$

If $\frac{PR}{RQ} = \frac{m_1}{m_2}$, then, $PR = \frac{m_1}{m_1 + m_2} PQ = \frac{m_1}{m_1 + m_2}(OQ - OP) = \frac{m_1}{m_1 + m_2}(r_2 - r_1) \qquad \therefore OP + PQ = OQ$

If $OP + PR = r_1 + \frac{m_1}{m_1 + m_2}(r_2 - r_1)$, $\quad OP + PR = OR$

then the position vector of OR is:

$OR = OP + PR = r_1 + \frac{m_1}{m_1 + m_2}(r_2 - r_1) = \frac{r_1 m_1 + r_1 m_2 + r_2 m_1 - r_1 m_1}{m_1 + m_2}$

$r = \frac{m_2 r_1 + m_1 r_2}{m_1 + m_2}, \qquad OR = r$

$(x, y) = \frac{m_2(x_1, y_1) + m_1(x_2, y_2)}{m_1 + m_2}$, *components form*

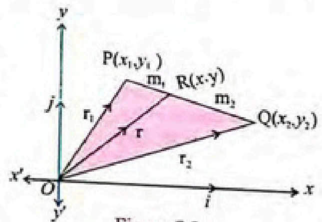


Figure 7.3

Equating x and y components to obtain the coordinates of $R(x, y)$

$$(x, y) = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right) \qquad (i)$$

that divides the line segment PQ in the ratio $m_1 : m_2$.

Remember

- If R is the midpoint of the line segment PQ, then, $m_1 = m_2$ and the coordinates of the midpoint R of the line segment PQ are: $(x,y)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$ (A)
- The coordinates of the point that divides the line segment PQ joining two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ externally in the ratio $m_1:m_2$ (m_1 or m_2 is negative) are: $(x,y)=\left(\frac{m_1x_2-m_2x_1}{m_1-m_2},\frac{m_1y_2-m_2y_1}{m_1-m_2}\right)$. (B)

Example 2 Find the coordinates of the point which divides the line segment PQ joining the two points *(a)*. P(1, 2) and Q(3, 4) in the ratio 5:7. *(b)*. P(3, 4) and Q(–6, 2) in the ratio 3: –2.

Solution

a. If R(*x*, *y*) is a point that divides the line segment PQ in the ratio 5:7, then the coordinates of R(*x*, *y*) is obtained through result (B):

$$(x,y)=\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right) \qquad \therefore m_1=5, m_2=7, P(1,2), Q(3,4)$$

$$=\left(\frac{5(3)+7(1)}{5+7},\frac{5(4)+7(2)}{5+7}\right)=\left(\frac{11}{6},\frac{17}{6}\right)$$

b. If R(*x*, *y*) is a point that divides the segment PQ in the ratio 3:–2, then the coordinates of R(*x*, *y*) is obtained through result (*B*):

$$(x,y)=\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$$

$$=\left(\frac{3(-6)+(-2)(3)}{3-2},\frac{3(2)+(-2)(4)}{3-2}\right)=(-24,-2) \qquad \therefore m_1=3,\ m_2=-2$$

7.1.3 The medians and angle bisectors of a triangle are concurrent

I. The medians of a triangle are concurrent

Proof: If $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$ are the vertices of a triangle ABC and P, Q and R are the midpoints of the sides *AB*, BC and CA, then the coordinates of the midpoint Q through mid point formula. $Q\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right)$ $\therefore m_1=m_2$

If G(*x*, *y*) is the centroid (in centre) of the triangle ABC, then, the coordinates of the point G that divides the median AQ in the ratio $m_1:m_2=2:1$ are:

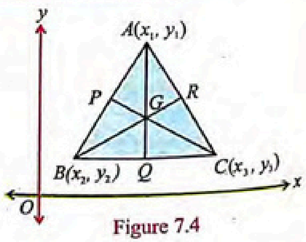


Figure 7.4

$$G(x,y)=\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+x_1}{2+1},\frac{2\left(\frac{y_2+y_3}{2}\right)+y_1}{2+1}\right)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right) \quad \text{(i)}$$

Similarly, the coordinates of the point G(*x*, *y*) that divides the medians BR and CP each in the ratio 2: 1 are respectively:

$$G(x,y)=\left(\frac{2\left(\frac{x_1+x_3}{2}\right)+x_2}{2+1},\frac{2\left(\frac{y_1+y_3}{2}\right)+y_2}{2+1}\right)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right) \quad \text{(ii)}$$

$$G(x,y)=\left(\frac{2\left(\frac{x_1+x_2}{2}\right)+x_3}{2+1},\frac{2\left(\frac{y_1+y_2}{2}\right)+y_2}{2+1}\right)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right) \quad \text{(iii)}$$

Therefore, the point G(*x*, *y*) lies on each median and consequently the medians of the triangle ABC are concurrent.

Example 3 Find the centroid of the triangle ABC, whose vertices are A(3,–5), B(–7, 4) and C(10, –2).

Solution Let A(3, –5), B(–7, 4) and C(10, –2) are the vertices of the triangle ABC, If G(*x*, *y*) is the centroid of the triangle ABC then, the coordinates of the point G(*x*, *y*) are:

$$G(x,y)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)=\left(\frac{3-7+10}{3},\frac{-5+4-2}{3}\right)=(2,-1)$$

II. The bisectors of a triangle are concurrent

Proof: If ABC is a triangle with vertices $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$, whose lengths are $|AB|=c, |BC|=a$ and $|CA|=b$, then, the position vectors of *A*, *B* and *C* are respectively:

$r_1=(x_1,y_1)=x_1i+y_1j,\ r_2=(x_2,y_2)=x_2i+y_2j,\ r_3=(x_3,y_3)=x_3i+y_3i$

Consider AD, BE and CF are the internal bisectors of the angles *A*, *B* and *C* that meet at centroid G. This is shown in Figure 7.5.

If AD is the internal bisector of angle A, then:

$$\frac{BD}{DC}=\frac{BA}{AC} \text{ or } \frac{BD}{DC}=\frac{c}{b} \Rightarrow BD:DC=c:b \quad \text{(i)}$$

This means that D divides BC internally in the ratio *c*:*b* and the position vector of *D* is therefore: $\frac{cr_3+br_2}{c+b}$

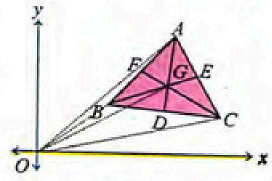


Figure 7.5

Again, $\frac{BD}{c}=\frac{DC}{b}=\frac{BD+DC}{c+b}=\frac{a}{c+b} \Rightarrow BD=\frac{ac}{c+b}$ (ii)

If *BG* is the internal bisector of the angle *B*, then, $\frac{DG}{AG}=\frac{BD}{AB}=\frac{\frac{ac}{b+c}}{c}=\frac{a}{b+c} \Rightarrow DG:GA=a:(b+c)$

The position vector of G(*x*, *y*) is: $r=\frac{ar_1+(b+c).\left(\frac{cr_3+br_2}{b+c}\right)}{a+b+c}=\frac{ar_1+br_2+cr_3}{a+b+c}=\frac{a(x_1,y_1)+b(x_2,y_2)+c(x_3,y_3)}{a+b+c}$ (iii)

The coordinates of the centroid G(x, y) is obtained from equation (iii) by equating the x and y components:

$$G(x,y)=\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right) \qquad \text{(iv)}$$

Similarly, the internal bisector of the angle C also passes through the point G(x, y). Thus, the angle bisectors of a triangle ABC are concurrent and G(x, y) is the point of concurrency.

Example 4 Muhammad Ayaan has a triangular piece of backyard where he wants to build a swimming pool. How can he find the largest circular pool that can be built there?

Solution The largest possible circular pool would have the same size as the largest circle that can be inscribed in the triangular backyard. The largest circle that can be inscribed in a triangle is incircle. This can be determined by finding the point of concurrency of the angle bisectors of each corner of the backyard and then making a circle with this point as center and the shortest distance from this point to the boundary as radius.

Example 5 Find the length JO.

Solution Here, O is the point of concurrency of the three angle bisectors of ΔLMN and therefore is the incenter. The incenter is equidistant from the sides of the triangle. That is, JO = HO = IO.

We have the measures of two sides of the right triangle ΔHOL, so it is possible to find the length of the third side.

Use the Pythagorean Theorem to find the length HO.

$=\sqrt{(LO)^2-(HL)^2}=\sqrt{13^2-12^2}=\sqrt{169-144}=\sqrt{25}=5$

Since JO = HO, the length JO also equals 5 units.

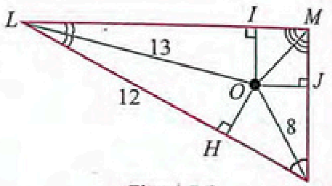


Figure 7.6

Exercise 7.1

1. The three points are A (–1,3), B (2,1) and C (5,–1). Show that $|AB|+|BC|=|AC|$.
2. In each case, find the midpoint of the line segment PQ joining the two points $P(x_1,y_1)$ and $Q(x_2,y_2)$:
   a. P(10,20), Q(–12,–8)   b. P(a,–b), Q(–a, b)   c. $P\left(\frac{1}{2},-\frac{1}{4}\right),Q\left(\frac{3}{5},\frac{4}{7}\right)$
3. In each case, find the coordinates of the point R(x,y) which divides the line segment PQ joining the two points
   a. P(1,2), Q(3,4) in the ratio 5:7.   b. P(3,4), Q(–6,2) in the ratio 3:–2.
   c. P(–6,7), Q(5,–4) in the ratio $\frac{2}{7}:1$.
4. In each case, in what ratio is the line segment PQ (joining the two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ divided by the point R(x, y):
   a. P(8,10), Q(–12,6), $R\left(-\frac{4}{7},\frac{58}{7}\right)$.   b. P(–2,4), Q(3,6), $R\left(\frac{4}{5},\frac{3}{5}\right)$.
5. Find the centroid of the triangle ABC, whose vertices are the following:
   a. A(4,–2), B(–2,4), C(5,5)   b. A(3,5), B(4,6), C(3,–1)   c. A(1,1), B(–2,–2), C(4,5)

7.2 Slope of a Straight Line

The slope of a line is a measure of the when "steepness" of the line, and whether it rises, or falls when moving from left to right. The line from A to B rises up, while the line from C to D goes down are depicted in the Figure 7.6:

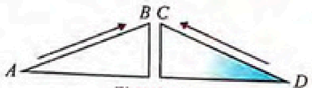


Figure 7.6

7.2.1 Slope of a line

The graph of a line can be drawn knowing only one point on the line if the "steepness" of the line is known, too.

"A number that measure the "steepness" of a line is called slope of a line."

If move off the line horizontally to the right first or move up or down (vertically) to return to the line, then the slope of the line is the "steepness" defined as the ratio of the vertical rise to the horizontal run: $slope=\frac{rise}{run}$, the run is always a movement to the right

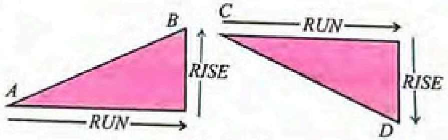


Figure 7.7

7.2.2 Formula to find the slope of a line passing through two points

Mathematically, if any two points on a line are available, then their join makes a constant angle with a fixed direction and the angle so formed is independent of the choice of the two points on the line. This is a precise way of saying that any line has a constant slope. It is customary to measure the angle θ which a line makes with the positive direction of the x-axis. The quantity tanθ is defined to be the slope of the line and is denoted by **m**. The slope of a line is also referred to **gradient** of the line.

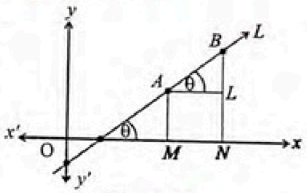


Figure 7.8

For illustration, if $A(x_1,y_1)$ and $B(x_2,y_2)$, where $x_1\neq x_2$, are any two points, then their join develops a line L that makes a constant angle θ with the x-axis. Draw AM, and BN parallel to y-axis and AL parallel to x-axis.

The slope m of a line L through the two points $A(x_1,y_1)$ and $B(x_2,y_2)$, is therefore:

$$m=\tan\theta=\frac{LB}{AL}=\frac{NB-NL}{MN}=\frac{NB-AM}{ON-OM}=\frac{y_2-y_1}{x_2-x_1} \qquad \text{(i)}$$

Example 6 Find the slope m of the line L through the points

(a). E(2,4) and F(4,6)   (b). M(3,1) and N(–1,3)

Remember
The standard equation of a line is $y=mx+c$ where m is a slope.

Solution

a. The given two points E(2,4) and F(4,6) form a line L, whose slope is: $m=\frac{y_2-y_1}{x_2-x_1}=\frac{6-4}{4-2}=\frac{2}{2}=1$

b. The given two points M(3,1) and N(–1,3) is form a line L, whose slope is:

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{3-1}{-1-3}=\frac{2}{-4}=-\frac{1}{2}$$

7.2.3 Condition for two straight lines with given slopes are

(a). **Parallel to each other** *(b).* **Perpendicular to each other**

a. **Parallel to each other**

If L_1 and L_2 are the two lines having slopes m_1 and m_2, then the lines L_1 and L_2 are **parallel** if they make the same angle with the *x-axis*, that means they have the same slope. Conversely, if two lines L_1 and L_2 have the same slope, then they will make the same angle with the *x-axis* and the lines L_1 and L_2 are therefore parallel for which: $m_1 = m_2$ (i)

It is important to note that the lines parallel to *x*-axis have zero slopes whereas the lines parallel to y-axis have the slope ∞.

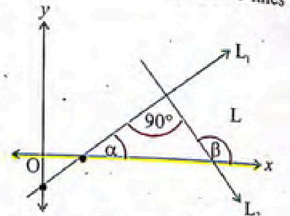


Figure 7.9

b. **Perpendicular to each other**

If L_1 and L_2 are the two perpendicular lines make the angles α and β with the *x*-axis, then the slopes of the lines L_1 and L_2 are respectively $m_1 = \tan\alpha$ and $m_2 = \tan\beta$. From the Figure 7.9, it is clear that

$$\frac{\pi}{2} = \beta - \alpha \Rightarrow \beta = \frac{\pi}{2} + \alpha$$

$$\tan\beta = \tan\left(\frac{\pi}{2} + \alpha\right), \text{ take tan of both sides}$$

$$= -\cot\alpha = -\frac{1}{\tan\alpha} \qquad \text{(ii)}$$

The given lines L_1 and L_2 are found perpendicular, since the product of their slopes equals –1:

$$m_1 m_2 = \tan\alpha \tan\beta = \tan\alpha\left(-\frac{1}{\tan\alpha}\right) = -1 \qquad \text{(iii)}$$

7.3 Equation of a Straight Line Parallel to Co-ordinate Axes

7.3.1 Equation of a straight line parallel to

- *y*-axis and at distance '*a*' from it.
- *x*-axis and at a distance '*b*' from it.

i. **y-axis and at a distance '*a*' from it**

Let PQ be a straight line parallel to *y-axis* at a distance '*a*' units from it see Figure 7.10. This is very clear, that all the points on the line PQ have the same ordinate say '*b*'. Therefore, PQ can be considered as the locus of a point at a distance '*a*' from y-axis and all points on the PQ satisfy the condition $x = a$ therefore, the equation of straight line is parallel to *y-axis* at a distance '*a*' from it. e.g $x = a$.

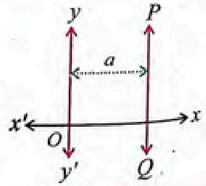


Figure 7.10

Remember

i. If $a = 0$, then the straight line coincides with the *y*-axis and its equation becomes $x = 0$.
ii. If PQ is parallel and to the left of y-axis at a distance '*a*', then its equation is $x = -b$.

Example 7 Find the equation of straight line parallel to *y*-axis at a distance 5 units on the right side of *y*-axis.

Solution Since, $x = a$ (i)

As, the distance is 5 units to right side of *y*-axis, so, equation (i) becomes $x = 5$

ii. **x-axis and at a distance '*b*' from it**

Let PQ be a straight line parallel to *x-axis* at a distance '*b*' units from it see Figure 7.11. This is every clear that all the points on the same ordinate say, '*b*'. Therefore, PQ can be considered as the locus of a point at a distance '*b*' from *x-axis* and all points on the PQ satisfy the condition $y = b$. Therefore, the equation of a straight line is parallel to *x-axis* at a distance *b* from it if e.g, $y = b$.

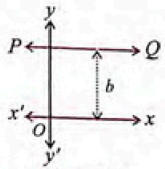


Figure 7.11

Remember

i. If $b - 0$, then the straight line coincides with the *x*-axis and its equation becomes $y = 0$.
ii. If PQ is parallel and below the *x*-axis at a distance '*b*', then its equation is $y = -b$.

7.4 Standard Form of Equation of a Straight Line

Because of their simplicity, linear equation (line) is used in many applications to describe relationships between two variables. We shall see some of these applications in this unit. First, we need to develop some standard forms that are related to linear equations.

(i) Intercepts of a straight line

"If a straight line AB intersects *x–axis* at C and *y–axis* at D, then OC is called the **x-intercept** of AB on the *x–axis* and OD is called the **y-intercept** of AB on the *y–axis*.

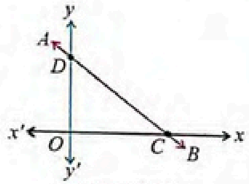


Figure 7.12

Example 8 Find the *x* and *y* intercepts of a line $2x+4y+6=0$.

Solution The *x*-intercept of a line is obtained by putting $y = 0$ in a line:

$$2x + 4y + 6 = 0$$

$$2x + 4(0) = -6 \Rightarrow 2x = -6 \Rightarrow x = -3$$

The y-intercept of a line is obtained by putting $x = 0$ in a line:

$$2x + 4y + 6 = 0$$

$$2(0) + 4y = -6 \Rightarrow 4y = -6 \Rightarrow y = -\frac{3}{2}$$

The general criteria are that a line in two dimensional space can be determined by specifying its slope and just one point.

(ii) Slope-Intercept Form

Let L be the line see Figure 7.13 develops the *y*-intercept *c* on the *y-axis*. The line L also makes an angle θ with the positive direction of the *x-axis* that develops a slope $m = \tan\theta$.

Let P(*x*,*y*) be any point on the line L. Draw PM parallel to *y-axis* and CN parallel to *x-axis* that give

$$CN = OM = x,$$

$$NP = MP - MN = MP - OC = y - c$$

In ΔPCN, the angle is $\angle PNC = 90^\circ$ and the slope of the line L is giving the slope-intercept form of the line L:

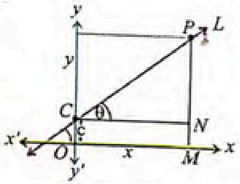


Figure 7.13

$$\frac{NP}{CN} = \tan\theta \Rightarrow \frac{y-c}{x} = \tan\theta$$

$$\Rightarrow y - c = x\tan\theta$$

$$\Rightarrow y = x\tan\theta + c = mx + c \qquad \text{(i)}$$

If the straight line L passes through the origin (0, 0), then $c = 0$ and the equation of line L becomes $y = mx$. In $y = mx + c$, m denotes the slope and c denotes the y-intercept of the line L on the axis of y.

Example 9 Determine the slopes of the following lines: *(a).* $x - y = 5$ *(b).* $2x + 3y = 6$

Solution

a. For the slope, solve the given line for y to obtain: $x - y = 5 \Rightarrow -y = -x + 5 \Rightarrow y = x - 5$

Thus, the slope of the line is the coefficient of x–term which is $m = 1$.

b. For the slope of the line, solve the given line for y to obtain:

$$2x + 3y = 6 \Rightarrow 3y = -2x + 6 \Rightarrow y = -\frac{2}{3}x + 2$$

Thus, the slope of the line is the coefficient of x–term which is $m = -\frac{2}{3}$.

Example 10 Find an equation of the line with slope 4, when the y–intercept is 6.

Solution Result (i) is used for the assumptions $m = 4$, $c = 6$ to obtain the required slope–intercept form of a line: $y = 4x + 6$

ii. Point–Slope Form

If L is a line see Figure 7.14 passing through the point $A(x_1, y_1)$ and $P(x, y)$ is any point on a line L, then the slope of the line L is giving the point-slope form of a line L:

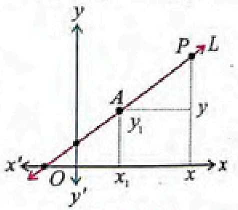


Figure 7.14

$$m = \frac{y - y_1}{x - x_1}$$

$$y - y_1 = m(x - x_1) \qquad \text{(ii)}$$

Example 11 Find an equation of a line with slope 4 and passes through the point (2,4).

Solution Result (ii) is used for the assumptions $m = 4$, $A(x_1, y_1) = A(2,4)$ to obtain the required point–slope form of a line:

$$y - 4 = 4(x - 2)$$

$$-4x + y - 4 + 8 = 0 \Rightarrow -4x + y + 4 = 0 \Rightarrow 4x - y - 4 = 0$$

iii. Two–Point Form

If L is a line see Figure 7.15 passing through the two points $A(x_1, y_1)$ and $B(x_2, y_2)$, then the slope of the line L is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(iii)}$$

If the equation of a line L through the $A(x_1, y_1)$ with slope m is

$$y - y_1 = m(x - x_1) \qquad \text{(iv)}$$

then the equation of a line L through the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the equation of the two-point form of a line L:

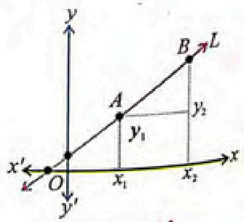


Figure 7.15

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{(v)}$$

Example 12 Find an equation of a line that passes through the two points P(–1,–2) and Q(–5,0).

Solution Result (v) is used for the assumptions $P(x_1, y_1) = P(-1,-2)$, $Q(x_2, y_2) = Q(-5,0)$ to obtain the required two–point form of a line:

$$y - (-2) = \frac{0-(-2)}{-5-(-1)}[x - (-1)] \Rightarrow y + 2 = \frac{2}{-4}(x+1)$$

$$\Rightarrow -4y - 8 = 2x + 2 \Rightarrow 2x + 4y + 10 = 0 \Rightarrow x + 2y + 5 = 0$$

iv. Double–Intercepts Form

If a line L intersects the x-*axis* and y-*axis* at points A and B, then OA = a and OB = b are the x and y-intercepts of the line L.

Let $P(x,y)$ be any point on the line L. Draw PM parallel to y-*axis* and PN parallel to x-*axis*. From the Figure 7.16, the comparison of similar triangles ΔBNP and ΔPMA is giving the equation of double-intercept form of a line L: $\frac{NB}{MP} = \frac{NP}{MA}$

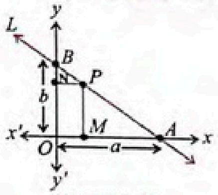


Figure 7.16

$$\frac{OB - ON}{ON} = \frac{OM}{OA - OM} \Rightarrow \frac{b-y}{y} = \frac{x}{a-x} \Rightarrow bx + ay = ab$$

$$\frac{bx}{ab} + \frac{ay}{ab} = 1 \Rightarrow \frac{x}{a} + \frac{y}{b} = 1 \qquad \text{(vi)}$$

Example 13 Find the equation of a line whose x and y intercepts are (3,0) and (0,4) respectively.

Solution Result (vi) is used for the assumptions $a = 3$, $b = 4$ to obtain the required line:

$$\frac{x}{3} + \frac{y}{4} = 1$$

$$\frac{4x + 3y}{12} = 1 \Rightarrow 4x + 3y = 12 \Rightarrow 4x + 3y - 12 = 0$$

v. Symmetric Form

Let a line L through point $A(x_1, y_1)$ makes an angle θ with the positive direction of the x-*axis*.

If $P(x, y)$ is any point on the line L, then AP = r. If we allow r to vary with any positive or negative values, then P will take any position on the line L. Conversely, if P is given to be any point on the line L, then the unique value of r can be found which in fact is the distance of P from A. Thus, it follows that r serves as a parameter of point P.

To find the coordinates of a point P in terms of the parameter r, let us draw AL and PM parallel to y-*axis* and AN parallel to x-*axis*, that with the following assumptions

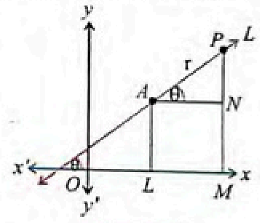


Figure 7.17

$$OM = OL + LM = OL + AN$$

$$MP = MN + NP = LA + NP \qquad \text{(vii)}$$

dvelops the parametric equations of a line L through the point $A(x_1, y_1)$ at an angle θ:

$$OM = OL + LM \qquad MP = MN + NP$$

$$= OL + AN \qquad = LA + NP \qquad \text{(viii)}$$

$$x = x_1 + r\cos\theta,\ \cos\theta = \frac{AN}{r} \qquad y = y_1 + r\sin\theta,\ \sin\theta = \frac{NP}{r}$$

The parametric equations (viii) automatically give the symmetric form of a line L after simplification:

$$\left.\begin{array}{l}\dfrac{x-x_1}{\cos\theta}=r\\[2mm]\dfrac{y-y_1}{\sin\theta}=r\end{array}\right\}\Rightarrow \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \qquad \text{(ix)}$$

Example 14 Find the equation of a straight line with inclination 45° and passing through the point $\left(2,\sqrt{2}\right)$.

Solution Here we have inclination $\alpha=45°$ and point $(x_1,y_1)=\left(2,\sqrt{2}\right)$. The equation of line in its symmetric form is: $\dfrac{x-x_1}{\cos\alpha}=\dfrac{y-y_1}{\sin\alpha}$

Substitute the above values in the formula to get the equation of a straight line.

$$\frac{x-2}{\cos 45°}=\frac{y-\sqrt{2}}{\sin 45°}$$

$$\Rightarrow \sin 45°(x-2)=\cos 45°\left(y-\sqrt{2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{2}}(x-2)=\frac{1}{\sqrt{2}}\left(y-\sqrt{2}\right)$$

$$\Rightarrow x-y-2+\sqrt{2}=0$$

vi. Normal Form

The normal form of a line is the equation of a line in terms of the length of the perpendicular on it from the origin and that perpendicular makes an angle with the *x-axis*.

If a line L intersects the *x*-axis and *y*-axis at points A and B, then OA and OB are the *x* and *y*-intercepts of the line L. Draw ON perpendicular to line L that provides the perpendicular distance p from the origin on the line L which is denoted by ON = p. If ON makes an angle θ with the positive direction of the *x-axis*, then the *x* and *y*-intercepts of the line L are respectively:

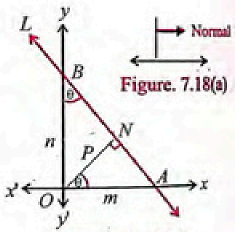


Figure. 7.18(a)
Figure 7.18(b)

$$\left.\begin{array}{l}\cos\theta=\dfrac{p}{\text{OA}}\Rightarrow \text{OA}=p\sec\theta\\[2mm]\sin\theta=\dfrac{p}{\text{OB}}\Rightarrow \text{OB}=p\,\text{cosec}\,\theta\end{array}\right\} \qquad \text{(x)}$$

If OA and OB are the *x* and *y*-intercepts of a line L, then through result (x), the equation of a normal line L in terms of perpendicular distance p and angle θ is: $\dfrac{x}{\text{OA}}+\dfrac{y}{\text{OB}}=1$

$$\frac{x}{p\sec\theta}+\frac{y}{p\,\text{cosec}\,\theta}=1 \Rightarrow \quad x\cos\theta+y\sin\theta=p \qquad \text{(xi)}$$

The normal form of a line is also referred to **perpendicular form** of a line.

Example 15 Find the corresponding equation of a line, if the length of the perpendicular distance from the origin on a line is 3 units that makes an angle of 120°.

Solution Result (xi) is used for the assumptions $p=3$, $\theta=120°$ to obtain the required equation of a line:

$$x\cos 120°+y\sin 120°=3 \Rightarrow \frac{-1}{2}x+\frac{\sqrt{3}}{2}y=3 \Rightarrow -x+\sqrt{3}y=6 \Rightarrow x-\sqrt{3}y+6=0$$

7.4.2 A linear equation in two variables is a straight line

A first degree polynomial $p_1(x)=a_1x+a_0$ is rearranged to obtain an equation of the form

$$-a_1x+p_1(x)-a_0=0 \quad \text{with } -a_1=a, p_1(x)=y, -a_0=b$$

$$ax+by+c=0$$

is then called the **general equation of the straight line**. Here a, b and c are constants while x and y are variables.

> **Remember**
> The first degree polynomial $p_1(x)$ is also called the linear algebraic equation and is denoted by $p_1(x)=f(x)$.

Consider, If $P(x_1,y_1)$, $Q(x_2,y_2)$ and $R(x_3,y_3)$ are the three points on the locus represented by the straight line

$$ax+by+c=0 \qquad \text{(i)}$$

then, $P(x_1,y_1)$ on the line (i) gives: $ax_1+by_1+c=0$ (ii)

$Q(x_2,y_2)$ on the line (i) gives: $ax_2+by_2+c=0$ (iii)

$R(x_3,y_3)$ on the line (i) gives: $ax_3+by_3+c=0$ (iv)

The three lines from equation (ii) to equation (iv) develops a homogeneous system of three linear equations in three unknowns a, b and c:

$$\begin{pmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow Ax=0 \qquad \text{(v)}$$

The homogeneous system of linear equations (v) defines a nontrivial solution only if the determinant of a coefficient matrix A of the system (v) is zero:

$$\det(A)=0$$

$$\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=0 \qquad \text{(vi)}$$

$x_1(y_2-y_3)-y_1(x_2-x_3)+(x_2y_3-x_3y_2)=0$ Equation (vi) is rearranged to obtain:

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0 \qquad \text{(vii)}$$

Multiply both sides of equation (vii) by $\frac{1}{2}$ to obtain the area of the triangle formed by P, Q and R that equals zero: $\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]=0$

$$\textit{Area of the triangle}\ \text{PQR}=0 \qquad \text{(viii)}$$

Since the three points P, Q and R lying on the locus (i) are collinear. Hence, the locus (i) represents a straight line.

7.4.3 General form of a straight line is reducible to other standard forms

Any standard form of a line can also be determined from the general form of a line (i).

i. To reduce the general form (i) to the slope intercept form of a line, we need to involve the following steps:

$$ax+by+c=0$$

$$by=-ax-c$$

$$y = -\frac{a}{b}x - \frac{c}{b}$$

$$= mx + c_1,\ slope = m = -\frac{a}{b},\ \ y\text{-intercept} = c_1 = -\frac{c}{b}$$

ii. To reduce the general form (i) to the double–intercept form, we need to involve the following steps:

$$ax + by + c = 0$$

$$ax + by = -c$$

$$\frac{ax}{-c} + \frac{by}{-c} = 1,\quad dividing\ by -c$$

$$\frac{x}{-\frac{c}{a}} + \frac{y}{-\frac{c}{b}} = 1$$

$$\frac{x}{a_1} + \frac{y}{b_1} = 1,\quad x\text{-}intercept = a_1 = -\frac{c}{a},\ y\text{-}intercept = b_1 = -\frac{c}{b}$$

iii. To reduce the general form (i) to the normal form, we need to involve the following steps:
From the Figure 7.18(b), the angles along the positive directions of the x and y-axis are the following:

$$\cos\theta = \frac{p}{m},\ \sin\theta = \frac{p}{n}$$

The values of $\cos\theta$ and $\sin\theta$ are used in the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ to obtain p:

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{p^2}{m^2} + \frac{p^2}{n^2} = 1$$

$$p^2\left(\frac{1}{m^2} + \frac{1}{n^2}\right) = 1 \qquad \text{(ix)}$$

$$p^2\left(\frac{m^2 + n^2}{m^2n^2}\right) = 1$$

$$p^2 = \frac{m^2n^2}{m^2 + n^2} \Rightarrow p = \pm\frac{mn}{\sqrt{m^2 + n^2}}$$

This p is the perpendicular distance from the origin to the line $\frac{x}{m} + \frac{y}{n} = 1$ (i.e. $nx + my - mn = 0$). Of course, the perpendicular distance from the origin to the line $ax + by + c = 0$ must be:

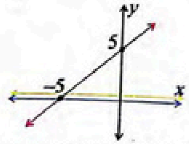
$$p = \frac{|c|}{\sqrt{a^2 + b^2}} \qquad \text{(x)}$$

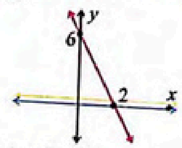
For converting the general form (i) to normal form, divide the line $ax + by + c = 0$ by $\frac{1}{\sqrt{a^2+b^2}}$ to obtain the conversion of the general form (i) in the normal form:

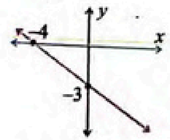
$$\frac{ax}{\sqrt{a^2+b^2}} + \frac{by}{\sqrt{a^2+b^2}} + \frac{c}{\sqrt{a^2+b^2}} = 0 \qquad \text{(xi)}$$

Exercise 7.2

1. Find the equation of lines that are represented on the coordinate planes:

a. 

b. 

c. 

2. In each case, find the slope, if it is defined:
   a. (−5, 4) and (3,6)
   b. $4x + 7y = 1$
   c. Parallel to $2y - 4x = 7$
   d. Perpendicular to $6x = y - 3$
3. What are the x– and y–intercepts for each of the following lines?
   a. $y = 2x + 6$
   b. $y = -3x + 9$
   c. $y = x + 2$
4. In each case, show that the pair of lines are parallel or perpendicular or neither:
   a. $x - 2y - 6 = 0,\ 2x + y - 5 = 0$
   b. $3x + 4y - 8 = 0,\ x + 3y - 2 = 0$
   c. $2x - y - 7 = 0,\ 4x - 2y - 5 = 0$
   d. $2x - 5y - 7 = 0,\ 6x - 15y + 5 = 0$
5. In each case, find the equation of a line that passes through the pair of points:
   a. O(0,0) and A(2,6)
   b. E(1,0) and F(2,5)
   c. I(1,1) and J(3,3)
6. In each case, find the equation of a line that passes through the point $A(x_1, y_1)$ having slope m:
   a. $A(1,2),\ m = 4$
   b. $A(-1,-2),\ m = -\frac{1}{2}$
   c. $A(-3,5),\ m = -3$
   d. $A(7,-8),\ m = 5$
7. In each case, find the equation of a line that exists the y–intercept c and slope m:
   a. $c = 2,\ m = 2$
   b. $c = 4,\ m = 8$
   c. $c = -4,\ m = \frac{1}{2}$
8. Transform the equation $7x - 10y + 13 = 0$ into:
   a. slop intercept from
   b. symmetric form
   c. normal form

Project

How can you calculate the midpoint between your home and school/college? Calculate this distance and write the procedure.

7.5 Distance of a point to a line

In Euclidean geometry, the distance from a point to a line is the shortest distance from a given point to any point on an infinite straight line. It is the perpendicular distance of the point to the line, the length of the line segment which joins the point to nearest point on the line.

7.5.1 Position of a point with respect to a line

To show that the point $P(x_1, y_1)$ is on one side or on the other side of the straight line $ax + by + c = 0$ according as the expression $ax_1 + by_1 + c < 0$ or $ax_1 + by_1 + c > 0$, the procedure developed is as under:

Let AB be the straight line $ax + by + c = 0$ and $P(x_1, y_1)$ is a point above the line AB Figure 7.18 and $P(x_1, y_1)$ is also a point below the line AB Figure 7.19. From P draw perpendicular PM on the x-axis that cuts the line AB at a point Q whose coordinates are $Q(x_1, y_2)$.

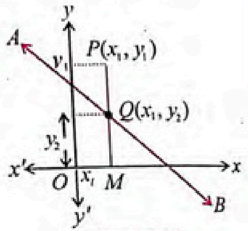


Figure 7.18

If $Q(x_1, y_2)$ lies on the line AB, then it give:

$$ax + by + c = 0$$

$$ax_1 + by_2 + c = 0, \quad Q(x_1, y_2) \text{ lies on AB}$$

$$by_2 = -(ax_1 + c)$$

$$y_2 = -\left(\frac{ax_1 + c}{b}\right) \qquad \text{(i)}$$

If P lies above AB as in Figure 7.18, then:

$$MP - MQ > 0$$

$$y_1 - y_2 > 0$$

$$y_1 + \frac{ax_1 + c}{b} > 0 \Rightarrow \frac{ax_1 + by_1 + c}{b} > 0$$

$$ax_1 + by_1 + c > 0, \quad b > 0$$

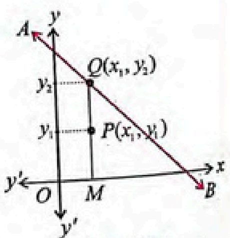


Figure 7.19

If P lies below AB as in Figure 7.19, then:

$$MP - MQ < 0$$

$$y_1 - y_2 < 0$$

$$y_1 + \frac{ax_1 + c}{b} < 0 \Rightarrow \frac{ax_1 + by_1 + c}{b} < 0$$

$$ax_1 + by_1 + c < 0, \quad b > 0$$

Hence P lies on one side or on the other side of the line $ax + by + c = 0$ according as $ax_1 + by_1 + c > 0$ or $ax_1 + by_1 + c < 0$.

Example 16 Determine whether the point P(10, –6) lies above or below the line $9x + 10y - 3 = 0$. Show that the point and the origin lie on the same or on the opposite sides of the given line.

Solution The given line $9x + 10y - 3 = 0$ is compared to the line $ax + by + c = 0$ to obtain the coefficient of y is $b = 10 > 0$:

i. The given point P(10, –6) is substituted in the given line to obtain:

$$9(10) + 10(-6) - 3 = 90 - 63 = 27 > 0$$

NOT FOR SALE

Thus, the point P(10, –6) lies above the given line $9x + 10y - 3 = 0$

ii. The given point P(10, –6) and the origin O(0,0) are substituted in the given line to obtain:

$$\left.\begin{aligned} 9(10) + 10(-6) - 3 = 90 - 63 = 27 > 0 \\ 9(0) + 10(0) - 3 = -3 < 0 \end{aligned}\right\} \text{opposite in signs}$$

Hence, the point P(10, –6) and the origin O(0,0) lie on the opposite side of the given line $9x + 10y - 3 = 0$.

7.5.2 Perpendicular distance from a point to the given straight line

If $Q(x_1, y_1)$ is any point on a line

$$ax + by + c = 0 \qquad \text{(i)}$$

and $n = (a, b)$ is a nonzero vector perpendicular to the line (i) at a point $Q(x_1, y_1)$, then the distance D is the scalar projection of a vector QP (associated to any point $(P(x_1, y_1))$ onto n:

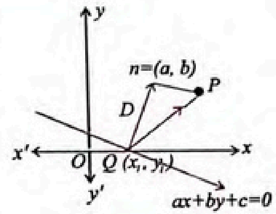


Figure 7.20

$$D = |proj_n\, QP|$$

$$= \frac{|QP.n|}{|n|}$$

$$d = \frac{|(x_0 - x_1, y_0 - y_1).(a,b)|}{\sqrt{a^2 + b^2}} \qquad n = (a,b), |n| = \sqrt{a^2 + b^2}$$

$$d = \frac{|a(x_0 - x_1) + b(y_0 - y_1))|}{\sqrt{a^2 + b^2}}$$

$$d = \frac{|ax_0 + by_0 - ax_1 - by_1)|}{\sqrt{a^2 + b^2}} = \frac{|ax_0 + by_0 + c)|}{\sqrt{a^2 + b^2}} \qquad \text{(ii)}$$

the perpendicular distance from a line $ax + by + c = 0$ to a point $P(x_1, y_1)$.

Example 17 Find the perpendicular distance from a line $7x + 3y - 9 = 0$ to a point P(2,3).

Solution Result (ii) is used for the assumptions $P(x_1, y_1) = P(2,3)$, $c = -9$, $a = 7$, $b = 3$ to obtain the perpendicular distance d from the line $7x + 3y - 9 = 0$ to the point $P(2,3)$:

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right| = \frac{7(2) + 3(3) - 9}{\sqrt{7^2 + 3^2}} = \frac{14}{\sqrt{58}}$$

7.6 Angle between Lines

If the two lines are available, then the angle between these two lines can found as follows:

7.6.1 The angle between two coplanar intersecting straight lines

The unit vectors are the vectors lie in the same directions of the given lines. The unit vectors along the line AB and CD are respectively $u = (\cos\theta_1, \sin\theta_1)$ and $v = (\cos\theta_2, \sin\theta_2)$.

The unit vector u of a line AB is:

$$u = (\cos\theta_1, \sin\theta_1) \qquad \cos\theta_1 = \frac{1}{\sec\theta_1}$$

NOT FOR SALE

$$=\left(\frac{1}{\sec\theta_1},\frac{\tan\theta_1}{\sec\theta_1}\right),\qquad |u|=1$$

$$=\left(\frac{1}{\sqrt{1+m_1^2}},\frac{m_1}{\sqrt{1+m_1^2}}\right),\qquad \sec^2\theta_1=1+\tan^2\theta_1=1+m_1^2$$

The unit vector v of a line CD is:

$$v=(\cos\theta_2,\sin\theta_2),|v|=1$$

$$=\left(\frac{1}{\sec\theta_2},\frac{\tan\theta_2}{\sec\theta_2}\right)$$

$$=\left(\frac{1}{\sqrt{1+m_2^2}},\frac{m_1}{\sqrt{1+m_2^2}}\right),\qquad \sec^2\theta_2=1+\tan^2\theta_2=1+m_2^2$$

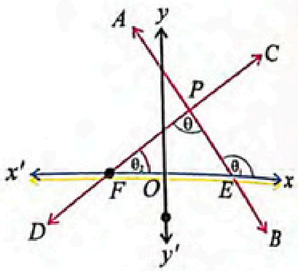


Figure 7.21

The angle of intersection between the lines AB and CD is the angle of intersection in between their unit vector u and v that can be found by taking the dot product in between the unit vectors u and v:

$$\cos\theta=\frac{u.v}{|u||v|},\quad |u|=|v|=1$$

$$=u.v$$

$$=\left(\frac{1}{\sqrt{1+m_1^2}},\frac{m_1}{\sqrt{1+m_1^2}}\right)\cdot\left(\frac{1}{\sqrt{1+m_2^2}},\frac{m_2}{\sqrt{1+m_2^2}}\right)$$

$$=\frac{1}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}+\frac{m_1m_2}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}$$

$$=\frac{1+m_1m_2}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}$$

The standard form of the angle is obtained if

$$\tan^2\theta=\sec^2\theta-1$$

$$=\frac{1}{\cos^2\theta}-1,\ \text{use value of}\cos\theta$$

$$=\frac{(1+m_1^2)(1+m_2^2)}{(1+m_1m_2)^2}-1$$

$$=\frac{(1+m_1^2)(1+m_2^2)-(1+m_1m_2)^2}{(1+m_1m_2)^2}=\frac{(m_1-m_2)^2}{(1+m_1m_2)^2}$$

$$\tan\theta=\pm\sqrt{\frac{(m_1-m_2)^2}{(1+m_1m_2)^2}}$$

$$=\pm\frac{m_1-m_2}{1+m_1m_2}\qquad \text{(i)}$$

Remember

- If $\frac{m_1-m_2}{1+m_1m_2}$ is positive, then result (i) gives the acute angle between the lines AB and CD.
- If $\frac{m_1-m_2}{1+m_1m_2}$ is negative, then result (i) gives the obtuse angle between the lines AB and CD.
- If one of the given lines is parallel to the y-axis, then the angle θ is not possible to obtain by formula: $\tan\theta=\pm\frac{m_1-m_2}{1+m_1m_2}$

  Because 90^0 is the angle made by that line with the positive x-axis and $\tan 90^0=\infty$. In such a case, the angle between the lines will be calculated by drawing the figure.
- The lines are parallel, if the cross product in between the unit vector u and v is zero:

$$u\times v=0$$

$$\left(\frac{1}{\sqrt{1+m_1^2}},\frac{m_1}{\sqrt{1+m_1^2}}\right)\times\left(\frac{1}{\sqrt{1+m_2^2}},\frac{m_2}{\sqrt{1+m_2^2}}\right)=0$$

$$\frac{m_1-m_2}{\sqrt{(1+m_1^2)(1+m_2^2)}}k=0$$

$$\Rightarrow m_1-m_1=0\Rightarrow m_1=m_2$$

where k is normal to the plane of the lines.

- The lines are perpendicular, if the dot product in between the unit vector u and v is zero:

$$u.v=0$$

$$\left(\frac{1}{\sqrt{1+m_1^2}},\frac{m_1}{\sqrt{1+m_1^2}}\right)\cdot\left(\frac{1}{\sqrt{1+m_2^2}},\frac{m_2}{\sqrt{1+m_2^2}}\right)=0$$

$$\frac{1+m_1m_2}{\sqrt{(1+m_1^2)(1+m_2^2)}}=0$$

$$\Rightarrow 1+m_1m_2=0\Rightarrow m_1m_2=-1$$

Example 18 Find the angle from the line $7x+3y-9=0$ to the line $5x-2y+2=0$.

Solution The slope of a line $7x+3y-9=0$ is $m_1=-\frac{7}{3}$

The slope of a line $5x-2y+2=0$ is $m_2=\frac{5}{2}$

If θ is the angle from first line to line second, then

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{-\frac{7}{3}-\frac{5}{2}}{1+\frac{-7}{3}\cdot\frac{5}{2}}=\frac{-29}{6\left(1-\frac{35}{6}\right)}=\frac{-29}{-29}=1\Rightarrow\theta=\tan^{-1}(1)=45^0$$

The angle $\theta=45^0$ is acute.

7.6.2 The equation of family of lines passing through the point of intersection of two given lines

Suppose, the two lines are

$L_1 : a_1x+b_1y+c_1=0$ (i)

$L_2 : a_2x+b_2y+c_2=0$ (ii)

and $P(x_1, y_1)$ is their point of intersection. The given lines L_1 and L_2 are used to obtain a first degree equation of a straight line in x and y: $(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0$, λ is constant (iii)

The coordinates of a point P will reduce each line in (iii) to zero, since, by hypothesis, P is the point of intersection, i.e., it lies on each line. Therefore P satisfies (iii) and represents the family of lines through the point of intersection of $L_1=0$ and $L_2=0$.

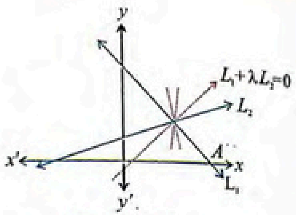


Figure 7.22

Example 19 Develop the family of lines through the point of intersection of the lines $2x-3y+4=0$ and $2x+y-1=0$. Find the line from the family of lines which is

(a). parallel to the line whose slope is $m_1=-\frac{2}{3}$. (b). perpendicular to the line $4x+3y-1=0$.

Solution Result (iii) is used for the lines $2x-3y+4=0$, $2x+y-1=0$ to obtain the family of lines:

$$\left.\begin{array}{l}(2x-3y+4)+\lambda(2x+y-1)=0\\(2+2\lambda)x+(-3+\lambda)y+(4-\lambda)=0\end{array}\right\} \quad \text{(iv)}$$

The slope of the family of lines is:

$$(2+2\lambda)x+(-3+\lambda)y+(4-\lambda)=0 \Rightarrow (-3+\lambda)y=-(2+2\lambda)x-(4-\lambda)$$

$$y=\frac{-(2+2\lambda)}{-3+\lambda}x-\frac{4-\lambda}{-3+\lambda},\ m_2=\frac{-(2+2\lambda)}{-3+\lambda}$$

a. The family of lines (iv) is parallel to the line with slope $m_1=-\frac{2}{3}$ if and only if their slopes are equal:

$$\frac{-(2+2\lambda)}{-3+\lambda}=-\frac{2}{3} \Rightarrow 6+6\lambda=-6+2\lambda \Rightarrow 4\lambda=-12 \Rightarrow \lambda=-3$$

The value of $\lambda=-3$ is used in (iv) to obtain the particular line from the family of lines (iv):

$$(2x-3y+4)-3(2x+y-1)=0 \quad \Rightarrow 2x-3y+4-6x-3y+3=0$$

$$\Rightarrow -4x-6y+7=0 \quad \Rightarrow 4x+6y-7=0$$

b. The slope of the given line $4x+3y-1=0$ is $m_3=-\frac{4}{3}$. The family of lines (iv) is perpendicular to the line $4x+3y-1=0$, if and only if the product of their slopes equals -1:

$$\left[\frac{-(2+2\lambda)}{-3+\lambda}\right]\left(-\frac{4}{3}\right)=-1 \Rightarrow \frac{8+8\lambda}{-9+3\lambda}=-1 \Rightarrow 8+8\lambda=9-3\lambda \Rightarrow 11\lambda=1 \Rightarrow \lambda=\frac{1}{11}$$

The value of $\lambda=\frac{1}{11}$ is used in (iv) to obtain the particular line from the family of lines:

$$(2x-3y+4)+\frac{1}{11}(2x+y-1)=0 \Rightarrow 22x-33y+44+2x+y-1=0 \Rightarrow 24x-32y+43=0$$

7.6.3 The angels of the triangle when the slopes of the sides are given

If $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are the vertices of a triangle ABC and the slopes of the sides AB, BC and CA of the triangle ABC are respectively:

$$m_1=\frac{y_2-y_1}{x_2-x_1},\ \textit{slope of side AB},$$

$$m_2=\frac{y_3-y_2}{x_3-x_2},\ \textit{slope of side BC}$$

$$m_3=\frac{y_1-y_3}{x_1-x_3},\ \textit{slope of side CA}$$

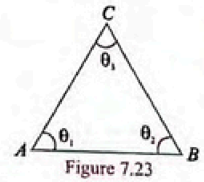


Figure 7.23

If θ_1, θ_2 and θ_3 are the angles in between their sides AB to AC, BC to BA and CB to CA respectively, then the angles can be found through results $\tan\theta=\pm\frac{m_1-m_2}{1+m_1m_2}$;

The angles from the sides AB to AC, BC to BA, and CA to CB of a triangle ABC are respectively:

$$\tan\theta_1=\frac{m_1-m_3}{1+m_1m_3},\quad \tan\theta_2=\frac{m_2-m_1}{1+m_1m_2},\quad \tan\theta_3=\frac{m_3-m_2}{1+m_2m_3} \quad \text{(v)}$$

Example 20 Find the angles of the triangle ABC, whose vertices are A(–2,–3), B(4,–1) and C(2,3).

Solution If ABC is a triangle and the slopes of their sides AB, BC and CA are respectively:

$$m_1=\frac{y_2-y_1}{x_2-x_1}=\frac{-1+3}{4+2}=\frac{2}{6}=\frac{1}{3},\ \text{slope of side AB}$$

$$m_2=\frac{y_3-y_2}{x_3-x_2}=\frac{3+1}{2-4}=-\frac{4}{2}=-2,\ \text{slope of side BC}$$

$$m_3=\frac{y_1-y_3}{x_1-x_3}=\frac{-3-3}{-2-2}=\frac{-6}{-4}=\frac{3}{2},\ \text{slope of side CA}$$

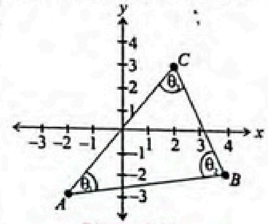


Figure 7.24

Result (v) to obtain the angle θ_1 from the sides AB to AC use:

$$\tan\theta_1=\frac{m_1-m_3}{1+m_1m_3}=\frac{\frac{1}{3}-\frac{3}{2}}{1+\left(\frac{1}{3}\right)\left(\frac{3}{2}\right)}=\frac{2-9}{6\left(1+\frac{1}{2}\right)}=-\frac{7}{6\left(\frac{3}{2}\right)}=-\frac{7}{9}\Rightarrow\theta_1=\tan^{-1}\left(\frac{-7}{9}\right)$$

Result (v) to obtain the angle θ_2 from the sides BC to BA use:

$$\tan\theta_2=\frac{m_2-m_1}{1+m_1m_2}=\frac{-2-\frac{1}{3}}{1+(-2)\left(\frac{1}{3}\right)}=\frac{-7}{3\left(\frac{1}{3}\right)}=-7\Rightarrow\theta_2=\tan^{-1}(-7)$$

Result (v) to obtain the angle θ_3 from the sides CA to CB use: $\tan\theta_3=\frac{m_3-m_2}{1+m_2m_3}$

$$=\frac{\frac{3}{2}+2}{1+(-2)\left(\frac{3}{2}\right)}=\frac{7}{2(1-3)}=\frac{7}{-4}=\frac{7}{-4}\Rightarrow\theta_3=\tan^{-1}\left(\frac{-7}{4}\right)$$

Exercise 7.3

1. Show that the point $P(x_1, y_1)$ lies above or below the line $ax + by + c = 0$. Also show that the point $P(x_1, y_1)$ and the origin lie on the same side or on the opposite side of the line $ax + by + c = 0$:
   a. $P(4,-5)$, $4x - 3y - 17 = 0$ b. $P(-3,8)$, $5x + 7y + 9 = 0$
2. In each case, show that the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are on the same side or on the opposite side of the line $ax + by + c = 0$:
   a. $P(-4,2)$, $Q(11,-3)$; $5x + 14y - 11 = 0$ b. $P(-3,5)$, $Q(1,-2)$; $2x - 3y - 10 = 0$
3. In each case, find the perpendicular distance from the line $ax + by + c = 0$ to a point $P(x_1, y_1)$:
   a. $P(3,-4)$, $4x - 3y + 6 = 0$ b. $P(5,8)$, $3x - 2y + 7 = 0$ c. $P(3,-1)$, $5x + 12y - 16 = 0$
4. In each case, find the angle θ from the line L_1 to line L_2, if the slopes of the lines L_1 and L_2 are the following:
   a. $L_1 : m_1 = \frac{1}{2}$, $L_2 : m_2 = 3$ b. $L_1 : m_1 = 2$, $L_2 : m_2 = 3$
5. In each case, find the angle θ from the line L_1 to the line L_2:
   a. L_1 : joins (1,2) and (7,–1), L_2 : joins (3,2) and (5,6).
   b. L_1 : joins (2,7) and (7,10), L_2 : joins (1,1) and (–5,3).
   Try to obtain acute angles.
6. In each case, find the angle θ from the line L_1 to the line L_2:
   a. $L_1 : x - 2y + 3 = 0$, $L_2 : 3x - y + 7 = 0$ b. $L_1 : 2x + 4y - 10 = 0$, $L_2 : 5x - 3y + 1 = 0$
7. In each case, find the angles of the triangle ABC whose vertices are the following:
   a. A(1,2), B(4,2) and C(–2,3) b. A(3,–4), B(1,5) and C(2,–4).
8. Find the equation of the straight line from the family of straight lines through the point of intersection of the lines
   a. $2x - 3y + 4 = 0$, $3x + 4y - 5 = 0$ and is perpendicular to the line $6x - 7y - 18 = 0$.
   b. $3x - 4y + 1 = 0$, $5x + y - 1 = 0$ and cuts off equal intercepts from the axes.
   c. $x - 2y = a$, $x + 3y = 2a$ and is parallel to the line $3x + 4y = 0$.
   d. $2x - y = 0$, $3x + 2y = 0$ and is perpendicular to the line $3x + y - 6 = 0$.

History

Pierre de Fermat was French lawyer and a mathematician. He was credited for the early development of calculus. In particular he was recognized because of his discovery of an original method of finding the greatest and smallest ordinates of curved lines which are very important for the differential calculus. He also made some contribution to number theory, but a magnificent contribution to analytical geometry, optics and probability. He became famous in the community of mathematics because of Fermat's principle for light propagation and his Fermat's last theorem in the field of number theory. Fermat's work in analytical geometry was circulated in manuscript form in 1636. He also developed a method for determining maxima, minima and tangents to various curves that was equivalent to differentiate calculus.

Pierre de Fermat (1607-1665)

7.7 Concurrency of Straight Lines

Before to touch the concurrency of straight lines, we need to develop the concept of intersection of lines. Logically, the solution of the system of lines exists only, if the lines intersect. For illustration, the two lines $x + y = 1$ and $x - y = 0$ is forming the system of two linear equations

$$x + y = 1$$
$$x - y = 0$$

that in matrix form is represented by $Ax = b$:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The augmented matrix of the system $Ax = b$ is

$$A/b = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

reduced in an echelon form through row operations

$$A/b = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \end{pmatrix} \text{ by } R_2 + R_1(-1)$$

that gives the system of equations:

$$\left.\begin{matrix} x + y = 1 \\ -2y = -1 \end{matrix}\right\}, y = \frac{1}{2}, x = \frac{1}{2}$$

The second equation is giving $y = \frac{1}{2}$ which is used in first equation to obtain $x = \frac{1}{2}$. The solution set $(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right)$ of the system of two linear equations is unique (one solution set). This unique solution is the unique point of intersection at which the given two lines intersect.

Remember

It is important to note that the system of two lines

- $x + y = 1$ and $x - y = 0$ is giving a **unique** solution set, since the lines are intersecting at just a single point.
- $x + y = 1$ and $x + y = 0$ is not giving a solution set, since the lines are not intersecting, because the line are **parallel**.
- $x + y = 1$ and $2x + 2y = 2$ is giving an **infinite** set of solutions, since the lines are intersecting more than one points, because the lines make a sense of coincident lines.

7.7.1 Condition of concurrency of three straight lines

The condition of concurrency of three straight lines is the point of intersection at which the three straight lines intersect. For illustration, if the given three lines are

$$\left.\begin{matrix} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \\ a_3x + b_3y + c_3 = 0 \end{matrix}\right\} \quad \text{(i)}$$

then, the three lines develop a homogeneous system of three linear equations

$Ax = 0$

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{(ii)}$$

In homogeneous system of three linear equations lines (ii), the homogeneous coordinates are used:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \text{(iii)}$$

Concurrency means that the three lines must intersect at a point $G(x,y)$, say, that can be found by solving the system of linear equations (i). The system (ii) has a nontrivial solution if the determinant of the coefficient matrix A of the system (ii) is zero:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \qquad \text{(iv)}$$

This is the condition of concurrency of three lines.

Remember

For required point of concurrency follow the steps given below:

- Choose any two lines from the given three system of linear equations (i).
- Develop the system of these two linear equations.
- Develop the augmented matrix $\frac{A}{b}$ of the system of two linear equations and reduce it in an echelon form to obtain the point of intersection.
- Substitute the developed point of intersection in the remaining third line. If the point of intersection satisfies the remaining third line, then that point of intersection should be taken as the point of concurrency of the given three lines.

Example 21 Show that the three lines $x+4y+3=0$, $5x-4y-5=0$ and $2x+2y+1=0$ are concurrent. If the lines are concurrent, then find out the point of concurrency.

Solution The condition of concurrency (iv) in light of the given three lines is going to be zero:

$$\begin{vmatrix} 1 & 4 & 3 \\ 5 & -4 & -5 \\ 2 & 2 & 1 \end{vmatrix} = 1(-4+10)-4(5+10)+3(10+8)=6-60+54=0$$

The given three lines are concurrent. For the point of concurrency G(*x*,*y*), choose the first two lines

$x+4y=-3,$

$5x-4y=5$

that develops the system of two linear equations, whose augmented matrix A/b is reduced in an echelon form

$$A/b=\begin{pmatrix} 1 & 4 & -3 \\ 5 & -4 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -3 \\ 0 & -24 & 20 \end{pmatrix} \quad by\ R_{21}(-5)=R_2+R_1(-5)$$

to obtain the reduced system of linear equations:

$x+4y=-3,$

$-24y=20$

The second equation is giving $y=-\frac{5}{6}$ which is used in first equation to obtain $x=\frac{1}{3}$. The third line with substitution of the point of intersection $(x,y)=\left(\frac{1}{3},-\frac{5}{6}\right)$ is going to be zero:

$$2x+2y+1=0 \Rightarrow 2\ (2\left(\frac{1}{3}\right)+2\left(-\frac{5}{6}\right)+1=\left(\frac{2}{3}\right)-\left(\frac{5}{3}\right)+1=0$$

Thus, the given three lines are concurrent at a point $G\left(\frac{1}{3},-\frac{5}{6}\right)$.

7.7.2 Equation of median, altitude and right bisector of a triangle

A. Equation of median of a triangle

A median is a line segment from an interior angle of a triangle to the mid point of the opposite side. Look at the following example, the procedure to find the equation of median of triangle is illustrated in this example.

Example 22 Find the equation of the median of a triangle having vertices are $A(8,-5)$, $B(6,5)$ and $C(-6,9)$.

Solution If C' is midd point of side $\overline{AB}$ of ΔABC

Then its co-ordinates are given as $\left(\frac{8+6}{2},\frac{-5+5}{2}\right)=(7,0)$

Since, the median CC' passes through points C and C', using the two-point form of the equation of a straight line, the equation of median CC' can be found as

$$\frac{y-0}{9-0}=\frac{x-7}{-6-7}\Rightarrow\frac{y}{9}=\frac{x-7}{-13}\Rightarrow 13y+9x-63=0$$

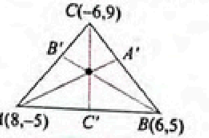


Figure 7.25

If A' is the midpoint of side BC of ΔABC then its coordinates are given as

$$\left(\frac{6-6}{2},\frac{5+9}{2}\right)=\left(\frac{0}{2},\frac{14}{2}\right)=(0,7)$$

Since, the median AA' passes through the point A and A' respectively. By using the two point form of the equation of a straight line, the equation of median AA' can be found as

$$\frac{y-7}{7-(-5)}=\frac{x-0}{0-8}\Rightarrow\frac{y-5}{12}=\frac{x}{-8}\Rightarrow -8y-12x+40=0$$

If B' is the midpoint of side AC of ΔABC then its coordinated are given as

$$\left(\frac{8-6}{2},\frac{-5+9}{2}\right)=\left(\frac{2}{2},\frac{4}{2}\right)=(1,2)$$

Since, the median BB' passes through the point B and B' respectively. By using the two point form of the equation of a straight line, the equation of median BB' can be found as

$$\frac{y-2}{2-5}=\frac{x-1}{1-6}\Rightarrow\frac{y-2}{-3}=\frac{x-1}{-5}\Rightarrow 3x-5y+7=0$$

B. Equation of altitude of a triangle

Altitude of a triangle is a perpendicular drawn from the vertex of the triangle to the opposite side. This is also known as the height of the triangle. Mostly it is used to find the area of the triangle. Look at the following example the procedure to find the equation of altitude is illustrated in example 22.

Example 23 Find the equation of altitude of triangle ABC having vertices are $A(-7, 4)$, $B(9, 6)$ and $C(7, -10)$.

Solution First we find slope of side

$$AB=m_1=\frac{6-4}{9+7}=\frac{2}{16}=\frac{1}{8}$$

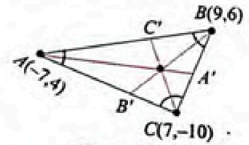


Figure 7.26

The altitude CC' is perpendicular to side AB, so, the slope of

$$CC'=-\frac{1}{m_1}=-8$$

Since the altitude CC' passes through the point $C(7,-10)$, by using point slope form of the equation of a line, the equation of CC' is

$$y-(-10)=-8(x-7)$$

$$\Rightarrow y+10=-8x+56 \Rightarrow 8x+y-46=0$$

Which is the equation of the altitude from C to AB.

The slope of side $\overline{BC}=m_2=\frac{-10-6}{7-9}=\frac{-16}{-2}=8$

The altitude AA' is perpendicular to side BC, so, the slope of $AA' = \frac{1}{m_2} = -\frac{1}{8}$

Since, the altitude passes through the point $A(-7, 4)$, by using the point slope form of the equation of a line, the equation of AA' is $y-4=-\frac{1}{8}(x+7) \Rightarrow x+8y-27=0$

Which is the equation of the altitude from A to BC

The slope of side $AC = m_3 = \frac{-10-(-7)}{7-4} = \frac{-3}{3} = -1$. The altitude BB' is perpendicular to side AC. So,

The slope of $BB' = -\frac{1}{m_3} = -\frac{1}{(-1)} = 1$, since, the altituded passes through the point $B(9,6)$. By using the pint slope form of the equation of a line, the equation of BB' is $y-6=1(x-9) \Rightarrow x-y-3=0$. Which is the equation of the altitude from B to AC.

C. Equation of bisector of a triangle

The bisector of a triangle is a line perpendicular to the side and passing through its midpoint. The three perpendicular bisectors of the sides of a triangle meet in a single point. The procedure to find the equation of a bisector is illustrate in the following example.

Example 24 Find the equation of the right bisector of a triangle having pints are $A(-7, 4)$, $B(10, 8)$ and $C(6, -12)$.

Solution Since, the equation of a perpendicular bisector is given as,

$$y-\frac{y_1+y_2}{2} = -\frac{x_2-x_1}{y_2-y_1}\left(x-\frac{x_1+x_2}{2}\right) \qquad \text{(i)}$$

Fore bisector of $A(-7, 4)$ and $B(10, 8)$ put the values in equation (i)

$$y-\frac{8+4}{2} = -\frac{10+7}{8-7}\left(x-\frac{10-7}{2}\right) \Rightarrow y-6=\frac{17}{4}\left(x-\frac{3}{2}\right) \Rightarrow 34x-8y-3=0$$

For bisector of $B(10, 8)$ and $C(6, -12)$, put the values in equation (i)

$$y-\frac{8-12}{2} = -\left(\frac{6-10}{-12-8}\right)\left(x-\frac{10+6}{2}\right)$$

$$\Rightarrow y+2=-\frac{1}{5}(x-8) \Rightarrow x+5y+2=0$$

For bisector of $A(-7, 4)$ and $C(6, -12)$ put the values in equation (i)

$$y-\frac{4-12}{2} = -\left(\frac{6+7}{-12+4}\right)\left(x-\frac{-7+6}{2}\right)$$

$$\Rightarrow y+4=\frac{13}{8}\left(x+\frac{1}{2}\right) \Rightarrow 26x-16y-51=0$$

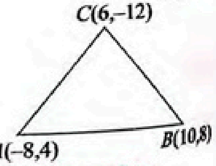


Figure 7.27

7.7.3 Show that, three right bisectors, three medians and three altitudes of a triangle are concurrent

i. Three right bisectors of a triangle are concurrent

To show the concurrency of the right bisectors of a triangle, the procedure developed is as under:

Let ABC be a triangle, whose vertices are $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$, D, E, F are the midpoints of the sides BC, CA, AB of a triangle ABC whose coordinates are respectively:

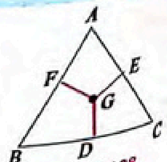


Figure 7.28

$$D\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right),\quad E\left(\frac{x_1+x_3}{2},\frac{y_1+y_3}{2}\right),\quad F\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

If the slope of the side BC and the slope of the right bisector DG of the side BC are respectively:

$$m_1=\frac{y_3-y_2}{x_3-x_2},\quad m_2=\frac{-1}{m_1}=-\frac{x_3-x_2}{y_3-y_2}$$

then, the equation of the right bisector DG of side BC is obtained by point–slope form of a line:

$$\left(y-\frac{y_2+y_3}{2}\right)=-\frac{x_3-x_2}{y_3-y_2}\left(x-\frac{x_2+x_3}{2}\right)$$

$$(y_3-y_2)\left(y-\frac{y_2+y_3}{2}\right)=(x_2-x_3)\left(x-\frac{x_2+x_3}{2}\right)$$

$$x(x_2-x_3)+y(y_2-y_3)-\frac{1}{2}(y_2^2-y_3^2)-\frac{1}{2}(x_2^2-x_3^2)=0$$

Similarly, the equations of the right bisectors EG (of side CA), FG (of side AB) is respectively:

$$x(x_3-x_1)+y(y_3-y_1)-\frac{1}{2}(y_3^2-y_1^2)-\frac{1}{2}(x_3^2-x_1^2)=0$$

$$x(x_1-x_2)+y(y_1-y_2)-\frac{1}{2}(y_1^2-y_2^2)-\frac{1}{2}(x_1^2-x_2^2)=0$$

The right bisectors DG, EG and FG is concurrent, if the determinant of the coefficient matrix A of the related system of equations of the right bisectors DG, EG and FG equals zero:

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -\frac{1}{2}(y_2^2-y_3^2)-\frac{1}{2}(x_2^2-x_3^2) \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}(y_3^2-y_1^2)-\frac{1}{2}(x_3^2-x_1^2) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}(y_1^2-y_2^2)-\frac{1}{2}(x_1^2-x_2^2) \end{vmatrix}=0 \qquad \text{(i)}$$

The operation of addition of rows $R_1+R_2+R_3$ to row R_1 is used to obtain:

$$\sim\begin{vmatrix} 0 & 0 & 0 \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}(y_3^2-y_1^2)-\frac{1}{2}(x_3^2-x_1^2) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}(y_1^2-y_2^2)-\frac{1}{2}(x_1^2-x_2^2) \end{vmatrix}=0 \qquad \text{(ii)}$$

The value of the determinant is zero. Hence, the right bisectors DG, EG and FG of a triangle ABC are concurrent.

Example 25 Let ABC be a triangle with vertices $A(0,0)$, $B(8,6)$ and $C(12,0)$. Show that the right bisectors DG, EG and FG of the triangle ABC are concurrent.

Solution The vertices $A(x_1,y_1)=A(0,0)$, $B(x_2,y_2)=B(8,6)$ and $C(x_3,y_3)=C(12,0)$ of the triangle ABC are used in the determinant (i) to obtain:

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -\frac{1}{2}(y_2^2-y_3^2)-\frac{1}{2}(x_2^2-x_3^2) \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}(y_3^2-y_1^2)-\frac{1}{2}(x_3^2-x_1^2) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}(y_1^2-y_2^2)-\frac{1}{2}(x_1^2-x_2^2) \end{vmatrix}=0 \Rightarrow \begin{vmatrix} -4 & 6 & 22 \\ 12 & 0 & -72 \\ -8 & -6 & 50 \end{vmatrix}=0$$

$$\Rightarrow -4(-432)-6(600-576)+22(-72)=0$$

$$\Rightarrow 1728-1728=0$$

The determinant (ii) equals zero. Hence, the right bisectors DG, EG and FG of the triangle ABC are concurrent.

ii. Three median of a triangle are concurrent

See topic 7.1.3 at page no 181.

iii. Three altitudes of a triangle are concurrent

Let ABC be a triangle, whose vertices are $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$. The altitudes of the triangle ABC are AD, BE and CF.

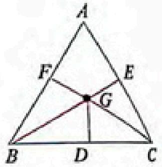


Figure 7.29

If the slope of the side BC and the slope of the altitude AD are respectively $m_1=\frac{y_3-y_2}{x_3-x_2}$, $m_2=-\frac{1}{m_1}=-\frac{x_3-x_2}{y_3-y_2}$,

then, the equation of the altitude AD is obtained by point–slope form of a line:

$$(y-y_1)=-\frac{x_3-x_2}{y_3-y_2}(x-x_1)$$

$$(y-y_1)(y_3-y_2)=(x_2-x_3)(x-x_1)x(x_2-x_3)+y(y_2-y_3)-x_1(x_2-x_3)-y_1(y_2-y_3)=0$$

Similarly, the equations of the altitudes BE and CF are respectively:

$$x(x_3-x_1)+y(y_3-y_1)-x_2(x_3-x_1)-y_2(y_3-y_1)=0$$

$$x(x_1-x_2)+y(y_1-y_2)-x_3(x_1-x_2)-y_3(y_1-y_2)=0$$

The altitudes AD, BE and CF are concurrent, if the determinant of the coefficient matrix A of the related system of equations of the altitudes AD, BE and CF equals zero:

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -x_1(x_2-x_3)-y_1(y_2-y_3) \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-y_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix}=0 \qquad \text{(iii)}$$

The operation of addition of rows $R_1+R_2+R_3$ to row R_1 is used to obtain:

$$\begin{vmatrix} 0 & 0 & 0 \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-y_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix}=0$$

The value of the determinant is zero. Therefore, the altitudes AD, BE and CF of a triangle are concurrent at a point G.

The conclusion drawn from the above results is that the three medians AD, BE and CF of a triangle ABC will also make concurrency at a point say, $G(x,y)$.

Example 26 Let ABC be a triangle with vertices $A(0,0)$, $B(8,6)$ and $C(12,0)$. Show that the altitudes AD, BE and CF of the triangle ABC are concurrent.

Solution The vertices $A(x_1,y_1)=A(0,0)$, $B(x_2,y_2)=B(8,6)$ and $C(x_3,y_3)=C(12,0)$ of the triangle ABC are used in the determinant (iii) to obtain:

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -x_1(x_2-x_3)-y_1(y_2-y_3) \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-y_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix}=0 \Rightarrow \begin{vmatrix} -4 & 6 & 0 \\ 12 & 0 & -96 \\ -8 & -6 & 96 \end{vmatrix}=0$$

$$-4(-576)-6(1152-768)=0 \Rightarrow 2304-6(384)=0 \Rightarrow 2304-2304=0$$

The determinant (iii) equals zero. Hence, the altitudes AD, BE and CF of the triangle ABC are concurrent.

7.8 Area of a Triangular Region

Let ABC be a triangle whose vertices are $P_1(x_1,y_1)$, $P_2(x_2,y_2)$ and $P_3(x_3,y_3)$. Project P_1A, P_2B and P_3C upon the x-axis that develops the trapezia P_1ACP_3, P_3CBP_2 and P_1ABP_2.

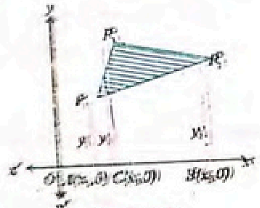


Figure 7.30

The area A of the triangular region $P_1P_2P_3$ is the sum of the areas of the trapezia P_1ACP_3, P_3CBP_2 and P_1ABP_2 minus the area of the trapezium P_1ABP_2.

$$A=\frac{1}{2}[(y_1+y_3)(x_3-x_1)]+\frac{1}{2}[(y_3+y_2)(x_2-x_3)]-\frac{1}{2}[(y_1+y_2)(x_2-x_1)]$$

$$=\frac{1}{2}[x_3y_1-x_1y_1+x_3y_3-x_1y_3+x_2y_3-x_3y_3+x_2y_2-x_3y_2-x_2y_1+x_1y_1-x_2y_2+x_1y_2] \qquad \text{(i)}$$

$$=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

$$=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad \text{(ii)}$$

It is important to note that the area A of the triangular region $P_1P_2P_3$ equals zero, when the vertices of the triangular region are collinear points.

Example 27 Find the area of the triangular region $P_1P_2P_3$ whose vertices are $P_1(4,-5)$, $P_2(5,-6)$ and $P_3(3,1)$.

Solution Result (i) is used for points $P_1(4,-5)$, $P_2(5,-6)$, $P_3(3,1)$ to obtain the area of the triangular region $P_1P_2P_3$:

$$A=\frac{1}{2}\begin{vmatrix} 4 & -5 & 1 \\ 5 & -6 & 1 \\ 3 & 1 & 1 \end{vmatrix}, \text{ first row expansion}$$

$$=\frac{1}{2}[4(-6-1)+5(5-3)+(5+18)]=\frac{1}{2}[-28+10+23]=\frac{5}{2} \text{ square units}$$

7.9 Homogeneous Equation

In general, any line equation in two variables that passes through the origin is called a homogeneous equation.

7.9.1 Homogeneous linear and quadratic equation in two variables

i. **Homogeneous linear equation in two variables**

"An equation of the form in two variables x and y ... $c \neq 0$, a, b and c are constants (i)
is called a nonhomogeneous equation of a line." For c = 0, the nonhomogeneous equation (i) gives the **homogeneous equation of the form** $ax+by=0$ (ii)
that passes through the origin definitely. This also defines a homogeneous equation of degree 1, since the indices of x and y in every term of (ii) is the same, the degree being 1. For example, the equation of line $x+y=0$ is homogeneous line, since it defines a homogeneous equation of degree 1.

ii. **Homogeneous quadratic equation in two variables**

"An equation of the form $ax^2+2hxy+by^2=0$, $a \neq 0$, where a, b, c are constants (iii)
is called a homogeneous quadratic equation of second degree in variables x and y". Since the sum of the indices of x and y in every term are the same number "2". For example,

$$3x^2-4xy+5y^2=0 \quad \text{and} \quad lx^2+mxy+ny^2=0$$

are homogeneous quadratic equations of the second degree in x and y. On the other hand, the equation of the form $3xy^2-4xy+5y^2=0$ is not a homogeneous equation, since the sum of the indices of x and y are not the same in each and every term.

7.9.2 Second degree homogeneous equations represents a pair of straight lines through the origin

i. **Standard form of second degree homogeneous equation**

If $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are the two straight lines, then the simple product of the given two nonhomogeneous lines defines a joint equation of a line:

$$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)=0 \qquad \text{(i)}$$

The joint equation of the homogeneous straight lines is obtained from (i) by putting $c_1=c_2=0$:

$$(a_1x+by_1)(ax_2+by_2)=0 \qquad \text{(ii)}$$

The product of homogeneous lines (ii) is giving the standard form of the second degree homogeneous equation:

$$(a_1x+b_1y)(a_2x+b_2y)=0$$
$$a_1a_2x^2+a_1b_2xy+b_1a_2xy+b_1b_2y^2=0$$
$$a_1a_2x^2+(a_1b_2+a_2b_1)xy+b_1b_2y^2=0 \qquad \text{(iii)}$$

If $a_1a_2=a$, $(a_1b_2+a_2b_1)=2h$, $b_1b_2=b$, then (iii) gives: $ax^2+2hxy+by^2=0$ (iv)

Any point $P(x, y)$ that satisfies first line $a_1x+b_1y=0$ or second line $a_2x+b_2y=0$ will also satisfies the joint homogeneous equation of (ii).

ii. **Representation as pair of straight lines**

The product of (iv) to constant quantity $\frac{a}{a}$ is giving the joint equation of the two first degree homogeneous equations in x and y:

Note

The lines are
- real and distant, if $h^2-ab>0$.
- real and coincident, if $h^2-ab=0$.
- imaginary, if $h^2-ab<0$.

$$\frac{a}{a}\left[ax^2+2hxy+by^2\right]=0$$
$$a^2x^2+2ahxy+aby^2=0, \text{ Add and subtract } h^2y^2$$
$$(ax+hy)^2-h^2y^2+aby^2=0 \Rightarrow (ax+hy)^2-y^2(h^2-ab)=0$$
$$(ax+hy)^2-\left(y\sqrt{(h^2-ab)}\right)^2=0 \Rightarrow \left(ax+hy+y\sqrt{h^2-ab}\right)\left(ax+hy-y\sqrt{h^2-ab}\right)=0$$
$$ax+hy+y\sqrt{h^2-ab}=ax+\left(h+\sqrt{h^2-ab}\right)y=0 \qquad \text{(v)}$$
$$ax+hy-y\sqrt{h^2-ab}=ax+\left(h-\sqrt{h^2-ab}\right)y=0 \qquad \text{(vi)}$$

The lines (v) and (vi) are therefore first degree equations in x and y.

Example 28 Find two first degree straight lines in x and y when the second degree homogeneous equation is $5x^2+3xy-8y^2=0$.

Solution The Standard form of second degree homogenous equations (iv) is compared to the given second degree homogeneous equation $5x^2+3xy-8y^2=0$ to obtain: $a=5$, $2h=3 \Rightarrow h=\frac{3}{2}$, $b=-8$

These values are used in the standard form of two first degree homogeneous lines (v) and (vi) to obtain the required two homogeneous lines:

$$ax+\left(h+\sqrt{h^2-ab}\right)y=0$$
$$5x+\left[\frac{3}{2}+\sqrt{\frac{9}{4}-(5)(-8)}\right]y=0$$
$$5x+\left[\frac{3}{2}+\sqrt{\frac{9}{4}+40}\right]y=0$$
$$5x+\left[\frac{3}{2}+\frac{13}{2}\right]y=0$$
$$5x+8y=0$$

$$ax+\left(h-\sqrt{h^2-ab}\right)y=0$$
$$5x+\left[\frac{3}{2}-\sqrt{\frac{9}{4}-(5)(-8)}\right]y=0$$
$$5x+\left[\frac{3}{2}-\sqrt{\frac{9}{4}+40}\right]y=0$$
$$5x+\left[\frac{3}{2}-\frac{13}{2}\right]y=0$$
$$5x-5y=0$$
$$x-y=0$$

iii. **Angle between pair of straight lines**

If the standard form of the second degree homogeneous equation in two variables x and y

$$ax^2+2hxy+by^2=0 \qquad \text{(vii)}$$

is decomposed into the product of two homogeneous straight lines $y=m_1x$ and $y=m_2x$:

$$(y-m_1x)(y-m_2x)=0$$
$$\Rightarrow y^2-m_2xy-m_1xy+m_1m_2x^2=0$$
$$y^2-(m_1+m_2)xy+m_1m_2x^2=0 \qquad \text{(viii)}$$

The comparison of equations (vii) and (viii) gives:

$$b=1, -(m_1+m_2)=2h \Rightarrow (m_1+m_2)=\frac{-2h}{1}=\frac{-2h}{b},\ m_1m_2=a=\frac{a}{1}=\frac{a}{b}$$

The angle θ in between the given two straight lines is:

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{\sqrt{(m_1-m_2)^2}}{1+m_1m_2}, \qquad (m_1-m_2)^2=(m_1+m_2)^2-4m_1m_2$$

$$=\frac{\sqrt{(m_1+m_2)-4m_1m_2}}{1+m_1m_2}=\frac{\sqrt{\frac{4h^2}{b^2}-\frac{4a}{b}}}{1+\frac{a}{b}}=\frac{2\sqrt{h^2-ab}}{a+b} \qquad \text{(ix)}$$

Remember

Result (ix) developed the idea that:

- the given two straight lines are perpendicular, if the angle between them is 90^0 that makes $a+b=0$.
- the given two straight lines are coinciding if the angle between them is zero that makes $h^2=ab$.

Example 29 Find the angle in between the lines represented by the second degree homogeneous equation $x^2-xy-6y^2=0$

Solution The standard form of second degree homogeneous equation (iv) is compared to the given second degree homogeneous equation $x^2-xy-6y^2=0$ to obtain: $a=1, 2h=-1 \Rightarrow h=\frac{-1}{2}, b=-b$

These values are used in result (ix) to obtain the angle in between the two straight lines represented by $x^2-xy-6y^2=0$:

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{\frac{1}{4}-(1)(-6)}}{1+(-6)}=\frac{2\sqrt{25/4}}{-5}=\frac{\sqrt{25}}{-5}=\frac{5}{-5}=-1$$

$$\theta=\tan^{-1}(-1)$$

$$\theta=\frac{3\pi}{4}=135^0$$

[illegible] Find the first degree straight lines in x and y when the second degree homogeneous equation is $3x^2-4xy-3y^2=0$. Show that the resultant lines are coincident or perpendicular.

Solution The standard form of second degree homogeneous equations (iv) is compared to the given second degree homogeneous equation $3x^2-4xy-3y^2=0$ to obtain: $a=3, 2h=-4 \Rightarrow h=-2, b=-3$

These values are used in the standard forms (v) and (vi) of the two first degree homogeneous lines to obtain the required two homogeneous lines:

$$ax+\left(h+\sqrt{h^2-ab}\right)y=0 \qquad ax+\left(h-\sqrt{h^2-ab}\right)y=0$$

$$3x+\left[-2+\sqrt{(4+9)}\right]y=0 \qquad 3x+\left[-2-\sqrt{(4+9)}\right]y=0$$

$$3x+\left[-2+\sqrt{13}\right]y=0 \qquad 3x+\left[-2-\sqrt{13}\right]y=0$$

The two lines are perpendicular, since $a+b=3-3=0$ is zero for $a=3$ and $b=-3$.

Exercise 7.4

1. In each case, find the point of intersection $P(x, y)$ of the pair of lines:
   a. $2x+4y-10=0,\ 5x-3y+1=0$
   b. $2x+y-8=0,\ 3x+2y-2=0$
2. Show that the following lines are concurrent. If the lines are concurrent, then find out the point at which the given lines can make concurrency:
   a. $x-y-2=0, 2x-y-5=0, 11x-5y-28=0$
   b. $x+2y-3=0, 2x-y+4=0, x+4y-7=0$
   c. $3x+2y-1=0, 2x-3y+4=0, x+y-2=0$
3. If ABC is a triangle with vertices A(0,0), B(8,6) and C(12,0), then show that
   a. the right bisectors of the triangle ABC are concurrent.
   b. the altitudes of the triangle ABC are concurrent.
   c. the medians of the triangle ABC are concurrent.
4. Find the area of the triangular region whose vertices are the following:
   a. $P_1(0,0), P_2(2,4), P_3(-2,2)$.
   b. $P_1(-1,-2), P_2(2,5), P_3(5,2)$.
   c. $P_1(4,-5), P_2(5,-6), P_3(3,1)$.
5. Find the area bounded by the triangle ABC whose vertices are the following:
   a. A(−3, 6), B(3, 2) C(6, 0)
   b. A(−2, 4), B(3, −6) C(1, −2)
   c. Are the vertices in parts a and b of the triangle ABC collinear?
6. Find two first degree straight lines in x and y, when the second degree homogeneous equations are the following:
   a. $3x^2-2xy-5y^2=0$
   b. $4x^2-9xy+5y^2=0$
7. Show the two first degree straight lines in x and y are coincident, perpendicular or neither, when they are represented by the following second degree homogeneous equations:
   a. $x^2+5xy-y^2=0$
   b. $2x^2-xy-y^2=0$
8. Find a joint equation of the straight line that passes through the origin and
   a. perpendicular to the lines represented by $3x^2-7xy+2y^2=0$.
   b. perpendicular to the lines represented by $x^2-2\tan\theta xy-y^2=0$.
   c. perpendicular to the lines represented by $ax^2+2hxy+by^2=0$.

Review Exercise 7

1. **Choose the correct option.**

i. The distance between point A(7,5) and B(−5,−7) is:

(a) $2\sqrt{37}$ (b) $12\sqrt{2}$ (c) $2\sqrt{2}$ (d) 0

ii. To find the mid point of a line we use the formula:

(a) $\frac{x_1-x_2}{2}, \frac{y_1-y_2}{2}$ (b) $\frac{m_1x_2-m_2x_1}{2}, \frac{m_1y_2-m_2y_1}{2}$

(c) $\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}$ (d) $\frac{m_1x_2+m_2x_1}{2}, \frac{m_1y_2+m_2y_1}{2}$

iii. For the points (a, b) and (5,7) the slope of line is:

(a) $\frac{7-b}{5-a}$ (b) $\frac{5}{7}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

iv. y intercept of the line $2x+4y=-6$ is:

(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

v. $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$ is a

(a) slope intercept form (b) two point form

(c) point slop form (d) double intercept form

vi. The normal form of an equation is:

(a) $x\cos\theta+y\sin\theta=p$ (b) $x\cos\theta-y\sin\theta=p$

(c) $x\sin\theta+y\tan\theta=0$ (d) $x\cos\theta-y\sin\theta=0$

vii. Two given line are perpendciula if there slopes m_1 and m_2 are

(a) $m_1+m_2=1$ (b) $m_2-m_1=0$ (c) $m_1m_2=1$ (d) $m_1m_2=-1$

viii. $5x+7y=0$ is a

(a) homogeneous linear equation (b) non-homogeneous lines equation

(c) quadratic equation (d) only linear equation.

ix. The area of a triangular region is A(4,5),B(−7, 4) and C(3,1) is:

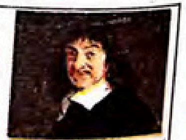
(a) 31 (b) −31 (c) 47 (d) −47

x. The angle between two pair of lines can be calculated by using the formula:

(a) $\frac{2\sqrt{h^2+ab}}{a+b}$ (b) $\frac{2-\sqrt{a^2+hb}}{a+b}$ (c) $\frac{2\sqrt{h^2+pq}}{a-b}$ (d) $\frac{2\sqrt{h^2-ab}}{a+b}$

History

Rene Descartes was a French mathematician, Philosopher and scientist. He was a creative mathematician of first order. He developed the techniques that made possible algebraic or analytic geometry. In metaphysics, he provided arguments for the existence of God, to show that the essence of matter is extension and that the essence of mind is thought. He claimed early on to possess a special method, that was variously exhibited in mathematics.



Rene Descartes (1596)-(1650)

Descartes has been called the father of analytic geometry. He laid the foundation for 17th century continental rationalism. Because of his contribution considered a well versed in mathematics. Descartes's was influence in mathematics is equally apparent the Cartesian coordinated system was named after him. He was also considered one of the key figure in the scientific revolution.

Summary

- The distance from point $P(x_1,y_1)$ to point $Q(x_2,y_2)$ in the coordinate plane is: $d=|PQ|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
- If L_1 and L_2 are the two lines having slopes m_1 and m_2, then, these two lines are
  a. **parallel** if and only if they have the same slopes: $m_1=m_2$.
  b. **perpendicular** if and only if the product of their slopes equals −1: $m_1m_2=-1$
- The equation of a straight line parallel to the *x-axis* and at a distance **a** from it, is $y=a$. The equation of the *x-axis* is $y=0$ and the vector equation of *x-axis* is $r.j=0$.
- The equation of a straight line parallel to the *y-axis* and at a distance **b** from it, is $x=b$. The equation of the *y-axis* is $x=0$ and the vector equation of *y-axis* is $r.i=0$.
- The standard forms of the line are the following:
  a. $y=mx+c$, Slope–Intercept form
  b. $y-y_1=m(x-x_1)$, Point–Slope form
  c. $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$ Two–Point form
  d. $\frac{x}{a}+\frac{y}{b}=1$ Double–Intercept form
  e. $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$ Symmetric Form
  f. $x\cos\theta+y\sin\theta=p$ Normal Form
- The standard form of a line is $ax+by+c=0$
- For $c=0$, the line is homogeneous that passes through the origin.
- For $c\neq 0$, the line is nonhomogeneous that does not pass through the origin.
- The perpendicular distance from a line $ax+by+c=0$ to a point $P(x_1,y_1)$ is: $d=\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$
- The angle between the two lines $y=m_1x+c_1$ and $y=m_2x+c_2$ is: $\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$
- The general equation of a straight line that passes through the point of intersection of the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is: $(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0$, λ constant
- The condition of concurrency of the three lines is: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$
- An equation of the form $ax^2+2hxy+by^2=0$, $a\neq 0, b, c$ *are* constants is called a homogeneous equation of second degree in x and y when the sum of the indices of x and y in every term is the same, the sum being 2.
- The angle between two homogeneous straight lines $y=m_1x$ and $y=m_2x$ is:

  $\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$

  i. The given two straight lines are perpendicular, if the angle between them is 90° that makes $a+b=0$.
  ii. The given two straight lines are coinciding if the angle between them is zero that makes $h^2=ab$.