Summary

- $F(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$.
- If $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, for any real number C. It is called indefinite integral.
- If $f(x)$ and $g(x)$ are integral functions w.r.t. x, then the integral of the product of $f(x)$ and $g(x)$ w.r.t. x is:
  $\int u dv = uv - \int v du$, $v = g(x)$, $du = f'(x)dx$ and $dv = g'(x)dx$:
- If $f(x)$ is continuous on the interval $[a, b]$ and $[a, b]$ is divided into n equal subintervals whose right-hand points are $x_1, x_2, \ldots, x_n$, then the **definite integral** of $f(x)$ from $x = a$ to $x = b$ is:

$$\int_{x=a}^{x=b} f(x)dx = \lim_{n\to\infty} \frac{(b-a)}{n}[f(x_1) + f(x_2) + \ldots + f(x_n)], \quad \Delta x = \frac{b-a}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x, \quad i = 1, 2, 3, \ldots, n$$

- The **definite integral** of the product of two functions u and v w.r.t. x is:

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$$

- If $f(x)$ is continuous and $f(x) \geq 0$ on the closed interval $[a, b]$, then the area under a curve $y = f(x)$ on $[a, b]$ is given by the **definite integral** of $f(x)$ on $[a, b]$:

$$Area = \int_a^b f(x)dx = F(b) - F(a)$$

- If a function $f(x)$ is continuous on the closed interval $[a, b]$, then

$$\int_{x=a}^{x=b} f(x)dx = |F(x)|_{x=a}^{x=b} = F(b) - F(a)$$

Where $F(x)$ is any function such that $F'(x) = f(x)$ for all x in $[a, b]$.

Unit 7 PLANE ANALYTIC GEOMETRY STRAIGHT LINE

By the end of this unit, the students will be able to:

7.1 Division of a line segment
- i. Recall distance formula to calculate distance between two points given in Cartesian plane.
- ii. Find coordinates of a point that divides the line segment in given ratio (internally and externally).
- iii. Show that the medians and angle bisectors of a triangle are concurrent.

7.2 Slope of a straight line
- i. Define the slope of a line.
- ii. Derive the formula to find the slope of a line passing through two points.
- ii. Find the condition that two straight lines with given slopes may be
  - parallel to each other,
  - perpendicular to each other

7.3 Equation of a straight line parallel to Co-ordinate axes
- i. Find the equation of a straight line parallel to
  - y-axis and at a distance a from it,
  - x-axis and at a distance b from it

7.4 Standard form of equation of a straight line
- i. Define intercepts of a straight line. Derive equation of a straight line in
  - slope-intercept from,
  - point-slope form,
  - two-point form,
  - intercepts form,
  - symmetric form,
  - normal form
- ii. Show that a linear equation in two variables represents a straight line.
- iii. Reduce the general form of the equation of a straight line to the other standard forms.

7.5 Distance of a point from a line
- i. Recognize a point with respect to position of a line.
- ii. Find the perpendicular distance from a point to the given straight lines.

7.6 Angle between lines
- i. Find the angle between two coplanar intersecting straight lines.
- ii. Find the equation of family of lines passing through the point of intersection of two given lines.
- iii. Calculate angles of the triangle when the slopes of the sides are given.

7.7 Concurrency of straight lines
- i. Find the condition of concurrency of three straight lines.
- ii. Find the equation of median, altitude and right bisector of a triangle.
- iii. Show that
  - three right bisectors,
  - three medians,
  - three altitudes, of a triangle are concurrent.

7.8 Area of a triangular region
- i. Find area of a triangular region whose vertices are given.

7.9 Homogenous equation
- i. Recognize homogeneous linear and quadratic equations in two variables.
- ii. Investigate that the 2nd degree homogeneous equation in two variables x and y represents a pair of straight lines through the origin and find acute angle between them.

Introduction

We are familiar about Cartesian coordinate system, we have learnt about it in our previous classes. This Cartesian coordinate system may be helpful to know the slope formula, Pythagoras theorem and distance formula. In this lesson we will learn in details and write the equations involving arbitrary points. Most of the geometric ideas can be expressed using algebraic equations. Analytic geometry is defined as:

"The study of relationship between geometry and algebra is called analytic geometry".

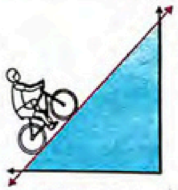


Figure 7.1

For example to calculate the slope/gradient between two given points, the numerator is the difference in the y-coordinates some times called it "Rise" and the denominator is the difference between x-coordinates, some time called it "run" e.g.

$$\text{Slop between two points} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{Rise}{Run}$$

Do You Know?

Analytic Geometry was independent and simultaneous invention of Pierre De Fermat and Rene Descartes. The fundamental idea of Analytic Geometry and the representation of curved lines by algebraic equations relating two variables say, x and y was given in seventeenth century by them.

7.1 Division of a line segment

We are familiar with the set of real numbers as well as with several of its subsets, including natural numbers and real numbers. The real numbers can easily be visualized by using a one dimensional coordinate system call real number line.

7.1.1 Calculation of distance between two given points

The study of plane analytic geometry is greatly facilitated by the use of vectors. The distance between any two given points can be calculated by using the distance formula.

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the xy-plane and θ is the angle in between the positive directions of the x and y axes, then, PQ is the directed line segment associated to initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$.

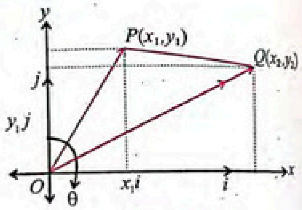


Figure 7.2

The components of the directed line segment PQ are:

$OP + PQ = OQ$

$PQ = OQ - OP$, position vectors

$= (x_2, y_2) - (x_1, y_1)$

$PQ = (x_2 - x_1, y_2 - y_1)$

$= (x_2 - x_1)i + (y_2 - y_1)j$

Pythagoras Theorem: If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the two points in the xy-plane, then the distance d between the given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is obtained by applying the theorem of Pythagoras to triangle PQR:

$(PQ)^2 = (PR)^2 + (QR)^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d, say$

Squaring both side of the directed line segment PQ to obtain

$(PQ)^2 = [(x_2 - x_1)i + (y_2 - y_1)j]^2 \quad \therefore (a+b)^2 = a^2 + b^2 + 2ab$

$= (x_2 - x_1)^2 i.i + (y_2 - y_1)^2 j.j + 2(x_2 - x_1)(y_2 - y_1)i.j$

$= (x_2 - x_1)^2 i.i + (y_2 - y_1)^2 j.j + 2(x_2 - x_1)(y_2 - y_1)|i||j|\cos\theta \quad \therefore i.i = j.j = 1,\ i.j = |i||j|\cos\theta,\ |i| = |j| = 1$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\theta$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\frac{\pi}{2} \quad \therefore \theta = \frac{\pi}{2}$

$(PQ)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \therefore \cos\frac{\pi}{2} = 0$

$|PQ|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \therefore (PQ)^2 = |PQ|^2$

$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d, say \quad$ (i)

This is the distance from point $P(x_1, y_1)$ to point $Q_2(x_2, y_2)$ in the Cartesian coordinate plane.

Note

- The distance from the origin O(0,0) to point $P(x_1, y_1)$ is obtained by inserting $x_2 = y_2 = 0$ in result (1): $d = |OP| = \sqrt{x_1^2 + y_1^2}$
- The distance from the origin O(0,0) to point $Q(x_2, y_2)$ is obtained by inserting $x_1 = y_1 = 0$ in result (1): $d = |OQ| = \sqrt{x_2^2 + y_2^2}$
- If the line segment PQ is horizontal, then the distance from the point $P(x_1, y_1)$ to point $Q(x_2, y_2)$ is obtained by inserting $y_1 = y_2$ in result (1): $d = |PQ| = \sqrt{(x_2 - x_1)^2}$
- If the line segment PQ is vertical, then the distance from point $P(x_1, y_1)$ to point $Q(x_2, y_2)$ is obtained by inserting $x_1 = x_2$ in result (1): $d = |PQ| = \sqrt{(y_2 - y_1)^2}$

Example 1 Find the distance between the two points P(3,–2) and Q(–1, –5).

Solution $P(x_1, y_1) = (3, -2)$, $Q(x_2, y_2) = (-1, -5)$ is used to obtain the distance d in between the two points P and Q:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-1-3)^2 + [(-5-(-2)]^2} = \sqrt{(-4)^2 + (-3)^2} = \sqrt{25} = 5$$

7.1.2 Co-ordinates of a point that divides the line segment in given ratio (Internally and externally)

Take $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the initial and terminal points of a line segment PQ and $R(x, y)$ is a point that divides PQ in the ratio $m_1 : m_2$. If r_1, r_2 and r are the position vectors of P, Q and R, then

$r_1 = (x_1, y_1) = x_1 i + y_1 j, \quad r_2 = (x_2, y_2) = x_2 i + y_2 j, \quad r = (x, y) = xi + yj$

If $\frac{PR}{RQ} = \frac{m_1}{m_2}$, then, $PR = \frac{m_1}{m_1 + m_2}PQ = \frac{m_1}{m_1 + m_2}(OQ - OP) = \frac{m_1}{m_1 + m_2}(r_2 - r_1) \quad \therefore OP + PQ = OQ$

If $OP + PR = r_1 + \frac{m_1}{m_1 + m_2}(r_2 - r_1)$, $\quad OP + PR = OR$

then the position vector of OR is:

$OR = OP + PR = r_1 + \frac{m_1}{m_1 + m_2}(r_2 - r_1) = \frac{r_1 m_1 + r_1 m_2 + r_2 m_1 - r_1 m_1}{m_1 + m_2}$

$r = \frac{m_2 r_1 + m_1 r_2}{m_1 + m_2}, \quad OR = r$

$(x, y) = \frac{m_2(x_1, y_1) + m_1(x_2, y_2)}{m_1 + m_2}$, *components form*

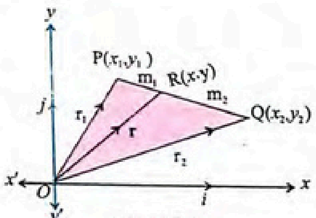


Figure 7.3

Equating x and y components to obtain the coordinates of $R(x,y)$

$$(x, y) = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right) \quad \text{(i)}$$

that divides the line segment PQ in the ratio $m_1 : m_2$.

Remember

- If R is the midpoint of the line segment PQ, then, $m_1 = m_2$ and the coordinates of the midpoint R of the line segment PQ are: $(x,y)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$ (A)
- The coordinates of the point that divides the line segment PQ joining two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ externally in the ratio $m_1:m_2$ (m_1 or m_2 is negative) are: $(x,y)=\left(\frac{m_1x_2-m_2x_1}{m_1-m_2},\frac{m_1y_2-m_2y_1}{m_1-m_2}\right)$. (B)

Example 2 Find the coordinates of the point which divides the line segment PQ joining the two points *(a).* P(1, 2) and Q(3, 4) in the ratio 5:7. *(b).* P(3, 4) and Q(–6, 2) in the ratio 3: –2.

Solution

a. If R(*x*, *y*) is a point that divides the line segment PQ in the ratio 5:7, then the coordinates of R(*x*, *y*) is obtained through result (B):

$$(x,y)=\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right) \qquad \therefore m_1=5, m_2=7, P(1,2), Q(3,4)$$

$$=\left(\frac{5(3)+7(1)}{5+7},\frac{5(4)+7(2)}{5+7}\right)=\left(\frac{11}{6},\frac{17}{6}\right)$$

b. If R(*x*, *y*) is a point that divides the segment PQ in the ratio 3:–2, then the coordinates of R(*x*, *y*) is obtained through result *(B)*:

$$(x,y)=\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$$

$$=\left(\frac{3(-6)+(-2)(3)}{3-2},\frac{3(2)+(-2)(4)}{3-2}\right)=(-24,-2) \qquad \therefore m_1=3,\ m_2=-2$$

7.1.3 The medians and angle bisectors of a triangle are concurrent

i. The medians of a triangle are concurrent

Proof: If $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$ are the vertices of a triangle ABC and P, Q and R are the midpoints of the sides *AB*, BC and CA, then the coordinates of the midpoint Q through mid point formula. $Q\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right) \qquad \therefore m_1=m_2$

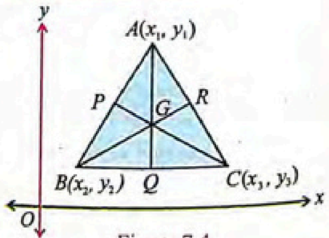


Figure 7.4

If G(*x*, *y*) is the centroid (in centre) of the triangle ABC, then, the coordinates of the point G that divides the median AQ in the ratio $m_1:m_2=2:1$ are:

$$G(x,y)=\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+x_1}{2+1},\frac{2\left(\frac{y_2+y_3}{2}\right)+y_1}{2+1}\right)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right) \quad \text{(i)}$$

Similarly, the coordinates of the point G(*x*, *y*) that divides the medians BR and CP each in the ratio 2: 1 are respectively:

$$G(x,y)=\left(\frac{2\left(\frac{x_1+x_3}{2}\right)+x_2}{2+1},\frac{2\left(\frac{y_1+y_3}{2}\right)+y_2}{2+1}\right)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right) \quad \text{(ii)}$$

$$G(x,y)=\left(\frac{2\left(\frac{x_1+x_2}{2}\right)+x_3}{2+1},\frac{2\left(\frac{y_1+y_3}{2}\right)+y_2}{2+1}\right)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right) \quad \text{(iii)}$$

Therefore, the point G(*x*, *y*) lies on each median and consequently the medians of the triangle ABC are concurrent.

Example 3 Find the centroid of the triangle ABC, whose vertices are A(3,–5), B(–7, 4) and C(10, –2).

Solution Let A(3, –5), B(–7, 4) and C(10, –2) are the vertices of the triangle ABC, If G(*x*, *y*) is the centroid of the triangle ABC then, the coordinates of the point G(*x*, *y*) are:

$$G(x,y)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)=\left(\frac{3-7+10}{3},\frac{-5+4-2}{3}\right)=(2,-1)$$

ii. The bisectors of a triangle are concurrent

Proof: If ABC is a triangle with vertices $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$, whose lengths are $|AB|=c, |BC|=a$ and $|CA|=b$, then, the position vectors of *A*, *B* and *C* are respectively:

$r_1=(x_1,y_1)=x_1i+y_1j,\ r_2=(x_2,y_2)=x_2i+y_2j,\ r_3=(x_3,y_3)=x_3i+y_3i$

Consider AD, BE and CF are the internal bisectors of the angles *A*, *B* and *C* that meet at centroid G. This is shown in Figure 7.5.

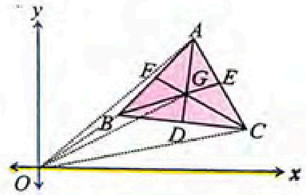


Figure 7.5

If AD is the internal bisector of angle A, then:

$$\frac{BD}{DC}=\frac{BA}{AC} \quad or \quad \frac{BD}{DC}=\frac{c}{b} \Rightarrow BD:DC=c:b \quad \text{(i)}$$

This means that D divides BC internally in the ratio *c*:*b* and the position vector of *D* is therefore: $\frac{cr_3+br_2}{c+b}$

$$\text{Again, } \frac{BD}{c}=\frac{DC}{b}=\frac{BD+DC}{c+b}=\frac{a}{c+b} \Rightarrow BD=\frac{ac}{c+b} \quad \text{(ii)}$$

If *BG* is the internal bisector of the angle *B*, then, $\frac{DG}{AG}=\frac{BD}{AB}=\frac{\frac{ac}{b+c}}{c}=\frac{a}{b+c} \Rightarrow DG:GA=a:(b+c)$

$$\text{The position vector of } G(x,y) \text{ is: } r=\frac{ar_1+(b+c).\left(\frac{cr_3+br_2}{b+c}\right)}{a+b+c}=\frac{ar_1+br_2+cr_3}{a+b+c}=\frac{a(x_1,y_1)+b(x_2,y_2)+c(x_3,y_3)}{a+b+c} \quad \text{(iii)}$$

The coordinates of the centroid G(x, y) is obtained from equation (iii) by equating the x and y components:

$$G(x,y)=\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right) \quad \text{(iv)}$$

Similarly, the internal bisector of the angle C also passes through the point G(x, y). Thus, the angle bisectors of a triangle ABC are concurrent and G(x, y) is the point of concurrency.

Example 4 Muhammad Ayaan has a triangular piece of backyard where he wants to build a swimming pool. How can he find the largest circular pool that can be built there?

Solution The largest possible circular pool would have the same size as the largest circle that can be inscribed in the triangular backyard. The largest circle that can be inscribed in a triangle is incircle. This can be determined by finding the point of concurrency of the angle bisectors of each corner of the backyard and then making a circle with this point as center and the shortest distance from this point to the boundary as radius.

Example 5 Find the length JO.

Solution Here, O is the point of concurrency of the three angle bisectors of ΔLMN and therefore is the incenter. The incenter is equidistant from the sides of the triangle. That is, JO = HO = IO.

We have the measures of two sides of the right triangle ΔHOL, so it is possible to find the length of the third side.

Use the Pythagorean Theorem to find the length HO.

$=\sqrt{(LO)^2-(HL)^2}=\sqrt{13^2-12^2}=\sqrt{169-144}=\sqrt{25}=5$

Since JO = HO, the length JO also equals 5 units.

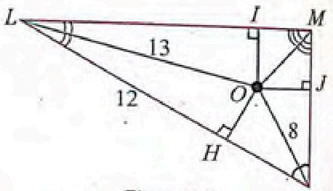


Figure 7.6

Exercise 7.1

1. The three points are A (–1,3), B (2,1) and C (5,–1). Show that $|AB|+|BC|=|AC|$.
2. In each case, find the midpoint of the line segment PQ joining the two points $P(x_1,y_1)$ and $Q(x_2,y_2)$:
   a. P(10,20), Q(–12,–8) b. P(a,–b), Q(–a, b) c. $P\left(\frac{1}{2},-\frac{1}{4}\right),Q\left(\frac{3}{5},\frac{4}{7}\right)$
3. In each case, find the coordinates of the point R(x,y) which divides the line segment PQ joining the two points
   a. P(1,2), Q(3,4) in the ratio 5:7. b. P(3,4), Q(–6,2) in the ratio 3:–2.
   c. P(–6,7), Q(5,–4) in the ratio $\frac{2}{7}:1$.
4. In each case, in what ratio is the line segment PQ (joining the two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ divided by the point R(x, y):
   a. P(8,10), Q(–12,6), $R\left(-\frac{4}{7},\frac{58}{7}\right)$. b. P(–2,4), Q(3,6), $R\left(\frac{4}{5},\frac{3}{5}\right)$.
5. Find the centroid of the triangle ABC, whose vertices are the following:
   a. A(4,–2), B(–2,4), C(5,5) b. A(3,5), B(4,6), C(3,–1) c. A(1,1), B(–2,–2), C(4,5)

7.2 Slope of a Straight Line

The slope of a line is a measure of the when **"steepness"** of the line, and whether it rises, or falls when moving from left to right. The line from A to B rises up, while the line from C to D goes down are depicted in the Figure 7.6:

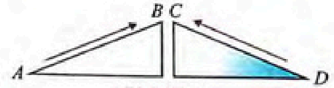


Figure 7.6

7.2.1 Slope of a line

The graph of a line can be drawn knowing only one point on the line if the "steepness" of the line is known, too.

"A number that measure the "steepness" of a line is called slope of a line."

If move off the line horizontally to the right first or move up or down (vertically) to return to the line, then the slope of the line is the "steepness" defined as the ratio of the vertical rise to the horizontal run: $slope=\frac{rise}{run}$, the run is always a movement to the right

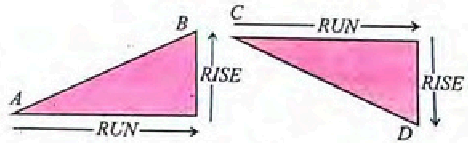


Figure 7.7

7.2.2 Formula to find the slope of a line passing through two points

Mathematically, if any two points on a line are available, then their join makes a constant angle with a fixed direction and the angle so formed is independent of the choice of the two points on the line. This is a precise way of saying that any line has a constant slope. It is customary to measure the angle θ which a line makes with the positive direction of the *x-axis*. The quantity tan θ is defined to be the slope of the line and is denoted by ***m***. The slope of a line is also referred to **gradient** of the line.

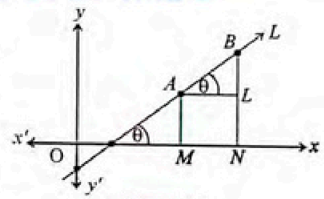


Figure 7.8

For illustration, if $A(x_1,y_1)$ and $B(x_2,y_2)$, where $x_1 \neq x_2$, are any two points, then their join develops a line L that makes a constant angle θ with the *x-axis*. Draw AM, and BN parallel to *y-axis* and AL parallel to *x*-axis.

The slope *m* of a line L through the two points $A(x_1,y_1)$ and $B(x_2,y_2)$, is therefore:

$$m=\tan\theta=\frac{LB}{AL}=\frac{NB-NL}{MN}=\frac{NB-AM}{ON-OM}=\frac{y_2-y_1}{x_2-x_1} \quad \text{(i)}$$

Example 6 Find the slope *m* of the line L through the points

(a). E(2,4) and F(4,6) (b). M(3,1) and N(–1,3)

Remember
The standard equation of a line is $y = mx + c$ where *m* is a slope.

Solution

a. The given two points E(2,4) and F(4,6) form a line L, whose slope is: $m=\frac{y_2-y_1}{x_2-x_1}=\frac{6-4}{4-2}=\frac{2}{2}=1$

b. The given two points M(3,1) and N(–1,3) is form a line L, whose slope is:

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{3-1}{-1-3}=\frac{2}{-4}=-\frac{1}{2}$$

7.2.3 Condition for two straight lines with given slopes are

(a). **Parallel to each other** (b). **Perpendicular to each other**

a. **Parallel to each other**

If L_1 and L_2 are the two lines having slopes m_1 and m_2, then the lines L_1 and L_2 are **parallel** if they make the same angle with the *x-axis*, that means they have the same slope. Conversely, if two lines L_1 and L_2 have the same slope, then they will make the same angle with the *x-axis* and the lines L_1 and L_2 are therefore parallel for which: $m_1 = m_2$ (i)

It is important to note that the lines parallel to *x*-axis have zero slopes whereas the lines parallel to y-axis have the slope ∞.

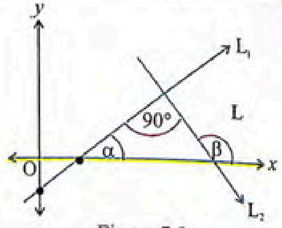


Figure 7.9

b. **Perpendicular to each other**

If L_1 and L_2 are the two perpendicular lines make the angles α and β with the *x*-axis, then the slopes of the lines L_1 and L_2 are respectively $m_1 = \tan\alpha$ and $m_2 = \tan\beta$. From the Figure 7.9, it is clear that

$$\frac{\pi}{2} = \beta - \alpha \Rightarrow \beta = \frac{\pi}{2} + \alpha$$

$$\tan\beta = \tan\left(\frac{\pi}{2} + \alpha\right), \text{ take tan of both sides}$$

$$= -\cot\alpha = -\frac{1}{\tan\alpha} \quad \text{(ii)}$$

The given lines L_1 and L_2 are found perpendicular, since the product of their slopes equals –1:

$$m_1 m_2 = \tan\alpha \tan\beta = \tan\alpha\left(-\frac{1}{\tan\alpha}\right) = -1 \quad \text{(iii)}$$

7.3 Equation of a Straight Line Parallel to Co-ordinate Axes

7.3.1 Equation of a straight line parallel to

- **y-axis and at distance '*a*' from it.**
- **x-axis and at a distance '*b*' from it.**

i. **y-axis and at a distance '*a*' from it**

Let PQ be a straight line parallel to *y-axis* at a distance '*a*' units from it see Figure 7.10. This is very clear, that all the points on the line PQ have the same ordinate say '*b*'. Therefore, PQ can be considered as the locus of a point at a distance '*a*' from y-axis and all points on the PQ satisfy the condition $x = a$ therefore, the equation of straight line is parallel to *y-axis* at a distance '*a*' from it. e.g $x = a$.

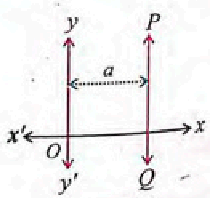


Figure 7.10

Remember

i. If a = 0, then the straight line coincides with the *y*-axis and its equation becomes $x = 0$.
ii. If PQ is parallel and to the left of y-axis at a distance '*a*', then its equation is $x = -b$.

Example 7 Find the equation of straight line parallel to *y*-axis at a distance 5 units on the right side of *y*-axis.

Solution Since, $x = a$ (i)

As, the distance is 5 units to right side of *y*-axis, so, equation (i) becomes $x = 5$

ii. **x-axis and at a distance '*b*' from it**

Let PQ be a straight line parallel to *x-axis* at a distance '*b*' units from it see Figure 7.11. This is every clear that all the points on the same ordinate say, '*b*'. Therefore, PQ can be considered as the locus of a point at a distance '*b*' from *x-axis* and all points on the PQ satisfy the condition $y = b$. Therefore, the equation of a straight line is parallel to *x-axis* at a distance *b* from it if e.g, $y = b$.

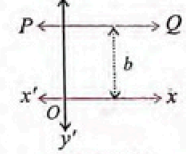


Figure 7.11

Remember

i. If $b = 0$, then the straight line coincides with the *x*-axis and its equation becomes $y = 0$.
ii. If PQ is parallel and below the *x*-axis at a distance '*b*', then its equation is $y = -b$.

7.4 Standard Form of Equation of a Straight Line

Because of their simplicity, linear equation (line) is used in many applications to describe relationships between two variables. We shall see some of these applications in this unit. First, we need to develop some standard forms that are related to linear equations.

(i) Intercepts of a straight line

"If a straight line AB intersects *x–axis* at C and *y–axis* at D, then OC is called the **x-intercept** of AB on the *x–axis* and OD is called the ***y*-intercept** of AB on the *y–axis*.

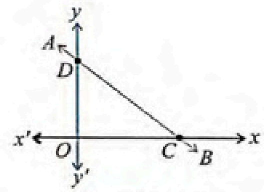


Figure 7.12

Example 8 Find the *x* and *y* intercepts of a line 2x+4y+6=0.

Solution The *x*-intercept of a line is obtained by putting $y = 0$ in a line:

$$2x + 4y + 6 = 0$$
$$2x + 4(0) = -6 \Rightarrow 2x = -6 \Rightarrow x = -3$$

The y-intercept of a line is obtained by putting $x = 0$ in a line:

$$2x + 4y + 6 = 0$$
$$2(0) + 4y = -6 \Rightarrow 4y = -6 \Rightarrow y = -\frac{3}{2}$$

The general criteria are that a line in two dimensional space can be determined by specifying its slope and just one point.

(ii) Slope-Intercept Form

Let L be the line see Figure 7.13 develops the *y*-intercept *c* on the *y-axis*. The line L also makes an angle θ with the positive direction of the *x-axis* that develops a slope $m = \tan\theta$.

Let P(*x*,*y*) be any point on the line L. Draw PM parallel to *y-axis* and CN parallel to *x-axis* that give

$$CN = OM = x,$$
$$NP = MP - MN = MP - OC = y - c$$

In ΔPCN, the angle is $\angle PNC = 90^0$ and the slope of the line L is giving the slope-intercept form of the line L:

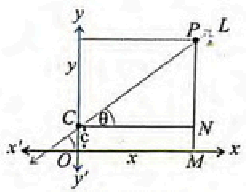


Figure 7.13

$$\frac{NP}{CN} = \tan\theta \Rightarrow \frac{y-c}{x} = \tan\theta$$

$$\Rightarrow y - c = x\tan\theta$$

$$\Rightarrow y = x\tan\theta + c = mx + c \qquad \text{(i)}$$

If the straight line L passes through the origin (0, 0), then $c = 0$ and the equation of line L becomes $y = mx$. In $y = mx + c$, m denotes the slope and c denotes the y-intercept of the line L on the axis of y.

Example 9 Determine the slopes of the following lines: *(a).* $x - y = 5$ *(b).* $2x + 3y = 6$

Solution

a. For the slope, solve the given line for y to obtain: $x - y = 5 \Rightarrow -y = -x + 5 \Rightarrow y = x - 5$

Thus, the slope of the line is the coefficient of x–term which is $m = 1$.

b. For the slope of the line, solve the given line for y to obtain:

$$2x + 3y = 6 \Rightarrow 3y = -2x + 6 \Rightarrow y = -\frac{2}{3}x + 2$$

Thus, the slope of the line is the coefficient of x–term which is $m = -\frac{2}{3}$.

Example 10 Find an equation of the line with slope 4, when the y–intercept is 6.

Solution Result (i) is used for the assumptions $m = 4$, $c = 6$ to obtain the required slope–intercept form of a line: $y = 4x + 6$

ii. Point–Slope Form

If L is a line see Figure 7.14 passing through the point $A(x_1, y_1)$ and $P(x, y)$ is any point on a line L, then the slope of the line L is giving the point-slope form of a line L:

$$m = \frac{y - y_1}{x - x_1}$$

$$y - y_1 = m(x - x_1) \qquad \text{(ii)}$$

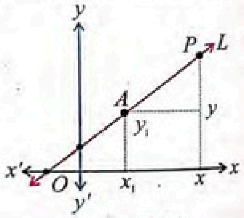


Figure 7.14

Example 11 Find an equation of a line with slope 4 and passes through the point (2,4).

Solution Result (ii) is used for the assumptions $m = 4$, $A(x_1, y_1) = A(2, 4)$ to obtain the required point–slope form of a line: $y - 4 = 4(x - 2)$

$$-4x + y - 4 + 8 = 0 \Rightarrow -4x + y + 4 = 0 \Rightarrow 4x - y - 4 = 0$$

iii. Two–Point Form

If L is a line see Figure 7.15 passing through the two points $A(x_1, y_1)$ and $B(x_2, y_2)$, then the slope of the line L is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(iii)}$$

If the equation of a line L through the $A(x_1, y_1)$ with slope m is

$$y - y_1 = m(x - x_1) \qquad \text{(iv)}$$

then the equation of a line L through the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the equation of the **two-point form** of a line L:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{(v)}$$

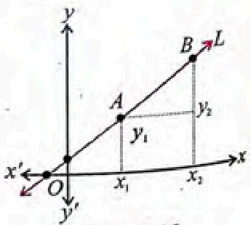


Figure 7.15

Example 12 Find an equation of a line that passes through the two points P(–1,–2) and Q(–5,0).

Solution Result (v) is used for the assumptions $P(x_1, y_1) = P(-1, -2)$, $Q(x_2, y_2) = Q(-5, 0)$ to obtain the required two–point form of a line:

$$y - (-2) = \frac{0 - (-2)}{-5 - (-1)}[x - (-1)] \Rightarrow y + 2 = \frac{2}{-4}(x + 1)$$

$$\Rightarrow -4y - 8 = 2x + 2 \Rightarrow 2x + 4y + 10 = 0 \Rightarrow x + 2y + 5 = 0$$

iv. Double–Intercepts Form

If a line L intersects the *x-axis* and *y-axis* at points A and B, then OA = a and OB = b are the x and y-intercepts of the line L.

Let P(x,y) be any point on the line L. Draw PM parallel to *y-axis* and PN parallel to *x-axis*. From the Figure 7.16, the comparison of similar triangles ΔBNP and ΔPMA is giving the equation of double-intercept form of a line L: $\frac{NB}{MP} = \frac{NP}{MA}$

$$\frac{OB - ON}{ON} = \frac{OM}{OA - OM} \Rightarrow \frac{b - y}{y} = \frac{x}{a - x} \Rightarrow bx + ay = ab$$

$$\frac{bx}{ab} + \frac{ay}{ab} = 1 \Rightarrow \frac{x}{a} + \frac{y}{b} = 1 \qquad \text{(vi)}$$

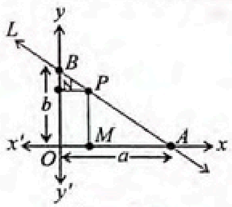


Figure 7.16

Example 13 Find the equation of a line whose x and y intercepts are (3,0) and (0,4) respectively.

Solution Result (vi) is used for the assumptions $a = 3$, $b = 4$ to obtain the required line:

$$\frac{x}{3} + \frac{y}{4} = 1$$

$$\frac{4x + 3y}{12} = 1 \Rightarrow 4x + 3y = 12 \Rightarrow 4x + 3y - 12 = 0$$

v. Symmetric Form

Let a line L through point $A(x_1, y_1)$ makes an angle θ with the positive direction of the *x-axis*.

If $P(x, y)$ is any point on the line L, then AP = r. If we allow r to vary with any positive or negative values, then P will take any position on the line L. Conversely, if P is given to be any point on the line L, then the unique value of r can be found which in fact is the distance of P from A. Thus, it follows that r serves as a parameter of point P.

To find the coordinates of a point P in terms of the parameter r, let us draw AL and PM parallel to *y-axis* and AN parallel to *x-axis*, that with the following assumptions

OM = OL + LM = OL + AN

MP = MN + NP = LA + NP (vii)

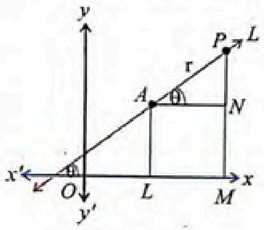


Figure 7.17

dvelops the parametric equations of a line L through the point $A(x_1, y_1)$ at an angle θ:

OM = OL + LM MP = MN + NP

= OL + AN = LA + NP (viii)

$$x = x_1 + r\cos\theta,\ \cos\theta = \frac{AN}{r} \qquad y = y_1 + r\sin\theta,\ \sin\theta = \frac{NP}{r}$$

The parametric equations automatically give the symmetric form of a line L after simplification:

$$\left.\begin{aligned}\frac{x-x_1}{\cos\theta}=r\\ \frac{y-y_1}{\sin\theta}=r\end{aligned}\right\}\Rightarrow \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \qquad \text{(ix)}$$

14 Find the equation of a straight line with inclination 45° and passing through the point $\left(2,\sqrt{2}\right)$.

Solution Here we have inclination $\alpha=45°$ and point $(x_1,y_1)=\left(2,\sqrt{2}\right)$. The equation of line in its symmetric form is: $\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}$

Substitute the above values in the formula to get the equation of a straight line.

$$\frac{x-2}{\cos 45°}=\frac{y-\sqrt{2}}{\sin 45°}$$

$$\Rightarrow \sin 45°(x-2)=\cos 45°\left(y-\sqrt{2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{2}}(x-2)=\frac{1}{\sqrt{2}}\left(y-\sqrt{2}\right)$$

$$\Rightarrow x-y-2+\sqrt{2}=0$$

vi. Normal Form

The normal form of a line is the equation of a line in terms of the length of the perpendicular on it from the origin and that perpendicular makes an angle with the *x-axis.*

Figure. 7.18(a)

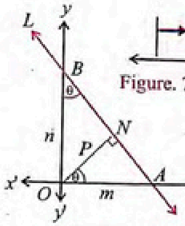


Figure 7.18(b)

If a line L intersects the *x*-axis and *y*-axis at points A and B, then OA and OB are the *x* and *y*-intercepts of the line L. Draw ON perpendicular to line L that provides the perpendicular distance *p* from the origin on the line L which is denoted by ON = *p*. If ON makes an angle θ with the positive direction of the *x-axis,* then the *x* and *y*-intercepts of the line L are respectively:

$$\left.\begin{aligned}\cos\theta=\frac{p}{\text{OA}}\Rightarrow \text{OA}=p\sec\theta\\ \sin\theta=\frac{p}{\text{OB}}\Rightarrow \text{OB}=p\,\text{cosec}\,\theta\end{aligned}\right\} \qquad \text{(x)}$$

If OA and OB are the *x* and *y*-intercepts of a line L, then through result (x), the equation of a normal line L in terms of perpendicular distance *p* and angle θ is: $\frac{x}{\text{OA}}+\frac{y}{\text{OB}}=1$

$$\frac{x}{p\sec\theta}+\frac{y}{p\,\text{cosec}\,\theta}=1 \Rightarrow x\cos\theta+y\sin\theta=p \qquad \text{(xi)}$$

The normal form of a line is also referred to **perpendicular form** of a line.

15 Find the corresponding equation of a line, if the length of the perpendicular distance from the origin on a line is 3 units that makes an angle of 120°.

Solution Result is used for the assumptions $p=3,\ \theta=120°$ to obtain the required equation of a line:

$$x\cos 120°+y\sin 120°=3 \Rightarrow \frac{-1}{2}x+\frac{\sqrt{3}}{2}y=3\Rightarrow -x+\sqrt{3}y=6\Rightarrow x-\sqrt{3}y+6=0$$

Now, use trigonometric substitutions in equation (ii)

$$u=\tan\theta,\frac{du}{d\theta}=\sec^2\theta\Rightarrow du=\sec^2\theta.d\theta$$

$$\int\frac{u}{u^2+1}du=\int\frac{\tan\theta.\sec^2\theta}{\tan^2\theta+1}.d\theta=\int\frac{\tan\theta.\sec^2\theta}{\sec^2\theta}.d\theta=\int\tan\theta\,d\theta=\int\frac{\sin\theta}{\cos\theta}.d\theta=-\ln|\cos\theta|+C \qquad \text{(iii)}$$

Return substitutions in (iii)

$$\int\frac{x-2}{x^2+4x+5}dx=-\ln\left|\left(\cos(\tan^{-1}(u)\right)\right|+C$$

$$=-\ln\left|\frac{\sqrt{1+u^2}}{1+u^2}\right|+C$$

$$=-\ln\left|\frac{\sqrt{1+(x+2)^2}}{1+(x+2)^2}\right|+C$$

$$=-\ln\left|\frac{1}{\sqrt{1+(x+2)^2}}\right|+C$$

$$=-\ln\left|\left\{1+(x+2)^2\right\}^{-\frac{1}{2}}\right|+C=\frac{1}{2}\ln\left|1+(x+2)^2\right|+C$$

Hence, $\int\frac{x-2}{x^2+4x+5}dx=\frac{1}{2}\ln\left|1+(x+2)^2\right|+C$

$$=\frac{1}{2}\ln\left|x^2+4x+5\right|+C$$

Example 9 Evaluate the integral $\int\frac{x+2}{\sqrt{x^2+4x+5}}dx$

Solution $\int\frac{x+2}{\sqrt{x^2+4x+5}}dx \qquad \text{(i)}$

By completing square in denominator

$$\int\frac{x+2}{\sqrt{x^2+4x+5}}dx=\int\frac{x+2}{\sqrt{x^2+4x+4+1}}dx=\int\frac{x+2}{\sqrt{(x+2)^2+1}}dx$$

Let $u=x+2,\frac{du}{dx}=1\Rightarrow du=dx$

$$\int\frac{x+2}{\sqrt{(x+2)^2+1}}dx=\int\frac{u}{\sqrt{u^2+1}}du \qquad \text{(ii)}$$

Now, use trigonometric substitutions in equation (ii)

$$u=\tan\theta,\frac{du}{d\theta}=\sec^2\theta\Rightarrow du=\sec^2\theta.d\theta$$

$$\int\frac{u}{\sqrt{u^2+1}}du=\int\frac{\tan\theta.\sec^2\theta}{\sqrt{\tan^2\theta+1}}d\theta=\int\frac{\tan\theta\sec^2\theta}{\sqrt{\sec^2\theta}}d\theta=\int\frac{\tan\theta.\sec^2\theta}{\sec\theta}d\theta=\int\tan\theta\cdot\sec\theta d\theta=\sec\theta+C \qquad \text{(iii)}$$

Return substitutions in (iii)

$$\int \frac{x+2}{\sqrt{x^2+4x+5}}dx = \sec\left(\tan^{-1}(u)\right)+C$$
$$= \sqrt{1+u^2}+C = \sqrt{1+(x+2)^2}+C$$

Hence, $\int \frac{x+2}{\sqrt{x^2+4x+5}}dx = \sqrt{x^2+4x+5}+C$

6.4 Integration by Parts

In previous sections, we have learnt some of the basic techniques of integration to solve problems like $\int x^2dx$ and $\int \sin x dx$. But, how do we evaluate an integral whose integrand is the product of two functions such as $\int x\sin x dx$, $\int xe^x dx$, $\int x\ln x dx$

To solve integral of the type like that, we have a technique called **integration by parts.**

6.4.1 Recognition of integration by parts

For this technique, recall the differentiation of the product of two functions $f(x)$ and $g(x)$ w.r.t x:

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}f(x) = f(x)g'(x)+g(x)f'(x)$$

$$f(x)g'(x) = \frac{d}{dx}[f(x)g(x)] - g(x)f'(x) \quad \text{(i)}$$

The integral of (i) with respect to x is giving

$$\int f(x)g'(x)dx = \int \frac{d}{dx}[f(x)g(x)]dx - \int g(x)f'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

The equation that can be transformed into more convenient form by substituting $u=f(x)$ and $v=g(x)$, $du=f'(x)dx$ and $dv=g'(x)dx$: $\int udv = uv - \int vdu$ (ii)

This is the standard form of the integration by parts formula.

Example 10 Evaluate the integral $\int xe^x dx$.

Solution The integral rule (ii) with $u=x$ and $\frac{dv}{dx}=e^x$ is used to obtain:

$$\int xe^x dx = xe^x - \int e^x(1)dx, \quad \frac{du}{dx}=1 \qquad dv=e^x dx,\ v=e^x$$
$$= xe^x - e^x + C$$

6.4.2 Applying method of integration by parts to evaluate integrals of the following types $\int\sqrt{a^2-x^2}dx$, $\int\sqrt{a^2+x^2}$, $\int\sqrt{x^2-a^2}$

i. Evaluate $\int\sqrt{a^2-x^2}dx$

Solution The given integral is

$$I = \int\sqrt{a^2-x^2}dx \quad \text{(i)}$$

In this problem, we choose $u=\sqrt{a^2-x^2}$ and $\frac{dv}{dx}=1$ to integrate the integrand of (i):

$$I = \int\sqrt{a^2-x^2}dx = \sqrt{a^2-x^2}(x) - \int\frac{(x)(-x)}{\sqrt{a^2-x^2}}dx \qquad \therefore \frac{du}{dx}=\frac{1}{2}(a^2-x^2)^{\frac{1}{2}-1}(-2x),\ dv=1dx,\ v=x$$

$$= x\sqrt{a^2-x^2} + \int\frac{x^2}{\sqrt{a^2-x^2}}dx = x\sqrt{a^2-x^2} + \int\frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}, \qquad \text{add and subtract } a^2$$

$$= x\sqrt{a^2-x^2} + \int\frac{a^2dx}{\sqrt{a^2-x^2}} - \int\sqrt{a^2-x^2}dx = x\sqrt{a^2-x^2} + a^2\int\frac{dx}{\sqrt{a^2-x^2}} - I$$

$$2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a} + C, \qquad \int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\Rightarrow I = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{C}{2}$$

ii. Evaluate $\int\sqrt{a^2+x^2}dx$

Solution $I = \int\sqrt{a^2+x^2}dx$ (i)

In this problem, we choose $u=\sqrt{a^2+x^2}$ and $\frac{dv}{dx}=1$ to integrate the integrand of (i):

$$I = \int\sqrt{a^2+x^2}dx$$

$$= \sqrt{a^2+x^2}(x) - \int\frac{(x)(x)}{\sqrt{a^2+x^2}}dx \qquad \therefore \frac{du}{dx}=\frac{1}{2}(a^2+x^2)^{\frac{1}{2}-1}(2x),\ dv=1dx,\ x=v$$

$$= x\sqrt{a^2+x^2} - \int\frac{x^2}{\sqrt{a^2+x^2}}dx = x\sqrt{a^2+x^2} - \int\frac{(a^2+x^2)-a^2}{\sqrt{a^2+x^2}}, \qquad \textit{add and subtract } a^2$$

$$= x\sqrt{a^2+x^2} + \int\frac{a^2dx}{\sqrt{a^2+x^2}} - \int\sqrt{a^2+x^2}dx = x\sqrt{a^2+x^2} + a^2\int\frac{dx}{\sqrt{a^2+x^2}} - I$$

$$2I = x\sqrt{a^2+x^2} + a^2\ln\left|x+\sqrt{a^2+x^2}\right| + C \qquad \therefore \int\frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x+\sqrt{a^2+x^2}\right|$$

$$I = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + \frac{C}{2}$$

Thus, $\int\sqrt{a^2+x^2}dx = \frac{x}{2}\sqrt{x^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + C$

iii. Evaluate $\int\sqrt{x^2-a^2}dx$

Solution $I = \int\sqrt{x^2-a^2}dx$ (i)

In this problem, we choose u $=\sqrt{x^2-a^2}$ and $\frac{dv}{dx}=1$ to integrate the integrand of (i)

$$I = \int\sqrt{x^2-a^2}dx$$

$$= \sqrt{x^2-a^2} - \int\frac{(x)(x)}{\sqrt{x^2-a^2}}dx \qquad \therefore \frac{du}{dx}=\frac{1}{2}(x^2-a^2)^{\frac{1}{2}-1}2x,\ dv=1dx \Rightarrow v=x$$

$$= x\sqrt{x^2-a^2} - \int\frac{x^2}{\sqrt{x^2-a^2}}dx = x\sqrt{x^2-a^2} - \int\frac{x^2}{\sqrt{x^2-a^2}}dx$$

$$= x\sqrt{x^2-a^2} - \int\frac{a^2+(x^2-a^2)}{\sqrt{x^2-a^2}} \qquad \text{add and subtract } a^2$$

$$= x\sqrt{x^2-a^2} - \int\frac{a^2dx}{\sqrt{x^2-a^2}} - \int\sqrt{x^2-a^2}dx = x\sqrt{x^2-a^2} - a^2\int\frac{dx}{\sqrt{x^2-a^2}} - I$$

$$I + I = x\sqrt{x^2-a^2} - a^2.\ln\left(\frac{\sqrt{x^2-a^2}+x}{|a|}\right)+C$$

$$I = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}.\ln\left(\frac{\sqrt{x^2-a^2}+x}{|a|}\right)+\frac{C}{2}$$

Thus, $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left(\frac{\left|\sqrt{x^2-a^2}+x\right|}{|a|}\right)+C$

6.4.3 Evaluation of integrals using integration by parts

Example 11 Evaluate the integral $\int x\ln x\,dx$.

Solution The integral rule $\int u\,dv = uv - \int v\,du$ with substitution $u=\ln x$ and $\frac{dv}{dx}=x$ is used to obtain:

$$\int x\ln x\,dx = \ln x\left(\frac{x^2}{2}\right) - \int\frac{x^2}{2}\left(\frac{1}{x}\right)dx \qquad \therefore \frac{du}{dx}=\frac{1}{x},\ \frac{dv}{dx}=x,\ v=\frac{x^2}{2}$$

$$=\frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx + C = \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right)+C = \frac{x^2}{2}\ln x - \frac{x^2}{4}+C$$

Example 12 Evaluate the integral $\int e^x \sin x\,dx$.

Solution The integral is

$$I = \int e^x \sin x\,dx \qquad \text{(i)}$$

The integral rule $\int u\,dv = uv - \int v\,du$ with substitution $u=e^x$ (let u be either e^x or $\sin x$) and $dv/dx = \sin x$ is used to obtain:

$$I = \int e^x \sin x\,dx = e^x(-\cos x) - \int(-\cos x)e^x dx \qquad \therefore \frac{du}{dx}=e^x,\ \frac{dv}{dx}=\sin x$$

$$= -e^x\cos x + \int e^x \cos x\,dx \qquad \text{(ii)}$$

It appears that we have not made any progress since we cannot evaluate the new integral. However, the form of the new integral prompts us to apply the technique a second time and see what happens.

Again, the integral of the integral part of equation (ii) with substitution $u=e^x$ and $\frac{dv}{dx}=\cos x$ is used in (ii) to obtain:

$$I = \int e^x \sin x\,dx = -e^x\cos x + \left[\int e^x \cos x\,dx\right]$$

$$= -e^x\cos x + \left[e^x\sin x - \int(\sin x)(e^x)dx\right]+C \qquad \therefore \frac{du}{dx}=e^x,\ \frac{dv}{dx}=\cos x$$

$$I = -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx + C$$

$$I = -e^x\cos x + e^x\sin x - I + C$$

$$\Rightarrow 2I = e^x(\sin x - \cos x) + C$$

$$\Rightarrow I = \frac{e^x(\sin x-\cos x)}{2}+\frac{C}{2}$$

Thus, $\int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2}+C$

Remember

Here's very helpful mnemonic for an order of priority for which factor the derivative must be passed to.

I	Inverse, trigonometric
L	Logarithms
A	Algebraic
T	Trigonometric
E	Exponential

Exercise 6.2

1. Evaluate the following indefinite integrals by method of substitution:

   a. $\int \sin^4 x\cos x\,dx$ b. $\int \sqrt[3]{\sin^2 x}\cos x\,dx$ c. $\int\frac{\sin x\ln(\cos x)}{\cos x}dx$

   d. $\int e^x \sin e^x dx$ e. $\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx$ f. $\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$

2. Use suitable substitutions and tables to evaluate the following indefinite integrals:

   a. $\int\frac{dx}{x^2+16}$ b. $\int\frac{\sin x}{\cos^2 x+1}dx$ c. $\int\frac{dx}{\sqrt{e^{2x}-4}}$

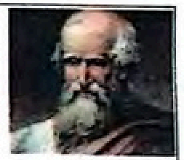
   d. $\int\frac{2x+5}{x^2+4x+5}dx$ e. $\int\frac{2+x}{\sqrt{4-2x-x^2}}dx$

3. Evaluate the following by using integration by parts.

   a. $\int x^2 e^{2x}dx$ b. $\int x\cos x\,dx$ c. $\int e^{2x}\sin(x)\,dx$

   d. $\int e^x \cos x\,dx$ e. $\int \sin^{-1}(x)\,dx$ f. $\int e^{3t}\sin(e^{3t})\,dt$

History



Archimedes
(207BC-212BC)

Archimedes was a Greek mathematician, physicist and astronomer. He was known as the leading scientist in classical antiquity. His mathematical work is to modern in technique that it is barely distinguishable from that of 17th century mathematicians. It was all done without the benefits of algebra or a convenient number system. He also developed general method for finding the areas and volumes. He used the method to find areas banded by parabolas and spirals and to find volume of cylinders, paraboloids and segments of spheres. Archimedes also gave a procedure to find approximating values of π and banded its value $\left[3\frac{10}{11}, 3\frac{1}{7}\right]$. He also invented a method to find the square roots and proposed another method based on the Greek myriad for representing numbers as large as one followed by 80 million billion zeros.

Archimedes was most proud of his discovery of a method for finding the volume of a sphere. He showed that the volume of a sphere is $\frac{2}{3}$ the volume of the cylinder. The method of mechanical theorems, which was the part of palimpsest found in the Constantinople in 1906. In that treatise Archimedes explain how he made some of his discoveries that are participating in the main idea of the integral calculus.

6.5 Integration using partial fractions

Partial fraction decomposition has great value as a tool for integration. This process may be thought of as the "reverse" of adding fractional algebraic expressions, and it allows us to break up rational expressions into simpler terms. Partial fraction decomposition is an algebraic procedure for expressing a reduced rational function as a sum of fractional parts. For example, the rational expression

$$f(x)=\frac{P(x)}{D(x)} \qquad \text{(i)}$$

can be decomposed into partial fractions only if $P(x)$ and $D(x)$ have no common factors and if the degree of $P(x)$ is less than the degree of $D(x)$. If the degree of $P(x)$ is greater than or equal to the degree of $D(x)$, then use division to obtain a polynomial plus a proper fraction. For example, the rational function after division is:

$$\frac{x^4+2x^3-4x^2+x-3}{x^2-x-2}=x^2+3x+1+\frac{8x-1}{x^2-x-2} \qquad \text{(ii)}$$

x^2+3x+1 is our polynomial term

$\frac{8x-1}{x^2-x-2}$ is our proper fraction (this is the part which requires decomposition into partial fractions).

In algebra, the theory of equations tells us that any polynomial $P(x)$ with real coefficients can be expressed as a product of linear and irreducible quadratic powers, some of which may be repeated. This fact can be used to justify the following general procedure for obtaining the partial fraction decomposition of a rational function.

Let $f(x)=\frac{P(x)}{D(x)}$, where $P(x)$ and $D(x)$ have no common factors and $D(x)\neq 0$.

The steps involved in decomposing the rational function are the following:

1. If the degree of $P(x)$ is greater than or equal to the degree of $D(x)$, use long division to express $\frac{P(x)}{D(x)}$ as the sum of a polynomial and a fraction $\frac{R(x)}{D(x)}$ in which the degree of the remainder polynomial $R(x)$ is less than the degree of the denominator polynomial $D(x)$.
2. Factorize the denominator $D(x)$ into the product of linear and irreducible quadratic powers.
3. Express $\frac{P(x)}{D(x)}$ as a cascading sum of partial fractions of the form $\frac{A_i}{(x-r)^n}$ and $\frac{A_j+B_k}{(x^2+sx+t)^m}$

Verify that the number of constants used is identical to the degree of the denominator.

6.5.1 Use of partial fraction to find $\int\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are algebraic functions, such that $g(x)\neq 0$

Example 13 Evaluate the following integrals:

(a). $\int\frac{8x-1}{x^2-x-2}dx$ (b). $\int\frac{x^2-6x+3}{(x-2)^3}dx$ (c). $\int\frac{2x^3+x^2+2x+4}{(x^2+1)^2}dx$

Solution

a. $\int\frac{8x-1}{x^2-x-2}dx$

The integrand is a proper fraction, so we start by factoring the denominator $x^2-x-2=(x-2)(x+1)$

The denominator factors are the two distinct linear factors, so we can set the rational function equal to the sum of the two partial fractions

$$\frac{8x-1}{x^2-x-2}=\frac{A_1}{x-2}+\frac{A_2}{x+1} \qquad \text{(i)}$$

To determine the constants A_1 and A_2 we multiply both sides of the equation (i) by $(x-2)(x+1)$ to obtain:

$$8x-1=A_1(x+1)+A_2(x-2) \qquad \text{(ii)}$$

Set $x-2=0 \Rightarrow x=2$ in equation (ii) to obtain: $8(2)-1=A_1(2+1)+A_2(2-2) \Rightarrow 15=3A_1 \Rightarrow A_1=5$

Set $x+1=0 \Rightarrow x=-1$ in equation (ii) to obtain :

$$8(-1)-1=A_1(-1+1)+A_2(-1-2) \Rightarrow -9=-3A_2 \Rightarrow A_2=3$$

Use these constants values in equation (i) to obtain: $\frac{8x-1}{x^2-x-2}=\frac{A_1}{x-2}+\frac{A_2}{x+1}=\frac{5}{x-2}+\frac{3}{x+1}$

Use this decomposition instead of rational expression in the given integral to obtain:

$$\int\frac{8x-1}{x^2-x-2}dx=\int\left[\frac{5}{x-2}+\frac{3}{x+1}\right]dx=\int\frac{5}{x-2}dx+\int\frac{3}{x+1}dx=5\ln(x-2)+3\ln(x+1)+\ln C$$
$$=\ln(x-2)^5+\ln(x+1)^3+\ln C=\ln C(x-2)^5(x+1)^3$$

b. $\int\frac{x^2-6x+3}{(x-2)^3}dx$

The integrand is a proper fraction, so we start by factoring the denominator

$$(x-2)^3=(x-2)(x-2)(x-2)$$

The denominator factors are the three repeated linear factors, so we can set the rational function equal to the sum of the three partial fractions

$$\frac{x^2-6x+3}{(x-2)^3}=\frac{A_1}{x-2}+\frac{A_2}{(x-2)^2}+\frac{A_3}{(x-2)^3} \qquad \text{(i)}$$

To determine the constants A_1, A_2 and A_3, we multiply both sides of the equation (i) by $(x-2)^3$ to obtain:

$$x^2-6x+3=A_1(x-2)^2+A_2(x-2)+A_3=A_1(x^2-4x+4)+A_2(x-2)+A_3 \qquad \text{(ii)}$$

Set $x-2=0 \Rightarrow x=2$ in equation (ii) to obtain:

$$(2)^2-6(2)+3=A_1(2-2)^2+A_2(2-2)+A_3 \Rightarrow -5=A_3 \Rightarrow A_3=-5$$

For constants A_1, A_2, equate the coefficients of x^2 and x on each side of equation (ii) to obtain:

$1=A_1$ x^2 terms $\qquad -6=-4A_1+A_2$ x terms

Solving this system of equations for the unknowns A_1 and A_2 to obtain $A_1=1$ and $A_2=-2$

Use these constants values in equation (ii) to obtain:

$$\frac{x^2-6x+3}{(x-2)^3}=\frac{A_1}{x-2}+\frac{A_2}{(x-2)^2}+\frac{A_3}{(x-2)^3}=\frac{1}{x-2}-\frac{2}{(x-2)^2}-\frac{5}{(x-2)^3}$$

Use this decomposition instead of rational expression in the given integral to obtain:

$$\int\frac{x^2-6x+3}{(x-2)^3}dx=\int\left[\frac{1}{x-2}-\frac{2}{(x-2)^2}-\frac{5}{(x-2)^3}\right]dx=\ln(x-2)-2\frac{(x-2)^{-2+1}}{-2+1}-\frac{5(x-2)^{-3+1}}{-3+1}+C$$
$$=\ln(x-2)+\frac{2}{(x-2)}+\frac{5}{2(x-2)^2}+C$$

c. $\int\frac{2x^3+x^2+2x+4}{(x^2-1)^2}dx$

The intergand is a proper fraction and the denominator factors are the two repeated quadratic factors, so we can set the rational function equal to the sum of the two partial fractions:

$$\frac{2x^3+x^2+2x+4}{(x^2+1)^2}=\frac{A_1x+B_1}{(x^2+1)^2}+\frac{A_2x+B_2}{(x^2+1)} \qquad \text{(i)}$$

To determine the constants values, the similar procedure is used to obtain $A_1=0, A_2=2, B_1=3, B_2=1$.

With these substitutions, the equation (i) becomes:

$$\frac{2x^3+x^2+2x+4}{(x^2+1)^2}=\frac{3}{(x^2+1)^2}+\frac{2x+1}{(x^2+1)} \qquad \text{(ii)}$$

Integrate this decomposition to obtain:

$$\int\frac{2x^3+x^2+2x+4}{(x^2+1)^2}dx=\int\frac{3}{(x^2+1)^2}dx+\int\frac{2x+1}{(x^2+1)}dx$$

$$\int\frac{2x^3+x^2+2x+4}{(x^2+1)^2}dx=\int\frac{3}{(x^2+1)^2}dx+\int\frac{2x}{(x^2+1)}dx+\int\frac{1}{(x^2+1)}dx$$

Now the readers are in position, how to find the complete solution of the question.

Hint: $\int\frac{1}{(x^2+1)}dx=\tan^{-1}x$, $\int\frac{2x}{(x^2+1)}dx=\ln u, u=x^2+1$

$\int\frac{3}{(x^2+1)^2}dx=?$, $x=\tan\theta, dx=\sec^2\theta$

Exercise 6.3

1. Evaluate the indefinite integrals after decomposing the following rational functions into partial fractions:

a. $\int\frac{1}{x(x-3)}$ b. $\int\frac{3x^2+2x-1}{x(x+1)}$ c. $\int\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}$ d. $\int\frac{1}{x^3-1}$

e. $\int\frac{x^4-x^2+2}{x^2(x-1)}$ f. $\int\frac{dx}{x^2-1}$ g. $\int\frac{-x-3}{2x^2-x-1}dx$ h. $\int\frac{x^2-1}{x^2-2x-15}dx$

i. $\int\frac{x}{(x+1)(x^2+1)}dx$ j. $\int\frac{x^2+2}{(x^2+1)^2}dx$

2. The rate at which the body eliminates a drug (in milliliters per hour) is given by

$$\frac{R(t)}{dt}=\frac{60t}{(t+1)^2(t+2)}$$

where t is the number of hours since the drug was administered. If $R(0)=0$ is the current drug elimination, how much of the drug is eliminated during the first hour after it was administered? The fourth hour, after it was administered?

3. The rate of change of the voting population of a city with respect to time t (in years) is estimated to be $\frac{dN}{dt}=\frac{100t}{(1+t^2)^2}$

where N(t) is in thousands. If N(0) is the current voting population, then how much will this population increase during the next 3 years?

4. An oil tanker aground on a reef is losing oil and producing an oil slick that is radiating outward at a rate approximated by $\frac{dr}{dt}=\frac{100}{\sqrt{t^2+9}}, t\geq 0$

where r is the radius (in feet) of the circular slick after t minutes. Find the radius of the slick after 4 minutes if the radius is $r=0$ when $t=0$.

6.6 Definite Integrals

"A definite integral is an integral that contains start and end value, say a and b, where interval [a, b] are limits or boundaries".

Look at the following figures, Figure 6.1 is showing the indefinite integral while the Figure 6.2 is showing definite integral.

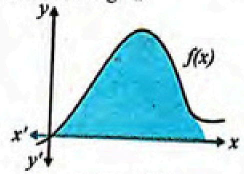


Figure 6.1

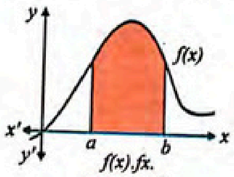


Figure 6.2

Definite integrals can be calculated in a same way as we have learnt in previous section for calculation of indefinite integral, but there is slight difference, to find definite integral simply calculate indefinite integral at point a and at point b, then subtract the result.

Example 14 Evaluate $\int_1^4 x^2dx$

Solution Here $a=1$, $b=4$ and $n=2$.

$$\int_1^4 x^2dx=\left|\frac{x^{2+1}}{2+1}\right|_1^4+C=\left|\frac{x^3}{3}\right|_1^4+C \qquad \int_a^b x^n dx=\left|\frac{x^{n+1}}{n+1}\right|_a^b+C$$

For $a=1, \frac{1}{3}+C$ (i)

For $b=4, \frac{64}{3}+C$ (ii)

Subtract equation (i) from equation (ii)

Thus, $\int_1^4 x^2dx=\left|\frac{x^3}{3}\right|_1^4=\frac{64}{3}+\cancel{C}-\frac{1}{3}-\cancel{C}=21$

Do You Know?

To calculate the approximate area of any mountain we use integration.

6.6.1 Definite integral as the limit of a sum

This limiting process is what we mean when we say the area is the **definite integral** of $f(x)=x^2$ from $x=0$ to $x=2$. It is written symbolically as

$$A=\int_{x=0}^{x=2}x^2dx=\frac{8}{3} \qquad \text{(i)}$$

We read symbol as "the area A equals the integral from $x=0$ to $x=2$ of the function $f(x)=x^2$." The number 0 is called the **lower limit of integration**, the number 2 is called the **upper limit of integration**, the function $f(x)=x^2$ is called the **integrand** and the dx tells us that we are integrating the function $f(x)=x^2$ with respect to the variable x.

If $f(x)$ is continuous on the interval $[a, b]$ and $[a, b]$ is divided into n equal subintervals whose right-hand points are $x_1, x_2, ..., x_n$, then the definite integral of $f(x)$ from $x=a$ to $x=b$ is:

$$\int_{x=a}^{x=b}f(x)dx=\lim_{n\to\infty}\frac{(b-a)}{n}[f(x_1)+f(x_2)+...+f(x_n)] \qquad \therefore \Delta x=\frac{b-a}{n}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\Delta x, \quad i=1,2,3,....,n \qquad \text{(i)}$$

Example 15 Find the actual area of the region bounded by the curve $f(x)=x^2$ and the x-axis in the interval [0, 2].

Solution For n subintervals, the width of each rectangle is $\Delta x=\frac{b-a}{n}=\frac{2-0}{n}=\frac{2}{n}$

The right end points of the subintervals are $\frac{2}{n},\frac{4}{n},\frac{6}{n},\ldots,2$.

Substitute $a=0, b=2,\ x_1=\frac{2}{n},\ x_2=\frac{4}{n},\ x_3=\frac{6}{n},\ldots,x_n=\frac{2n}{n}$ in equation (i) to obtain the actual area:

$$A=\int_0^2 x^2dx=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$=\lim_{n\to\infty}\Delta x\left[f(x_1)+f(x_2)+\ldots+f(x_n)\right]$$

$$=\lim_{n\to\infty}\frac{2}{n}\left[f\left(\frac{2}{n}\right)+f\left(\frac{4}{n}\right)+f\left(\frac{6}{n}\right)+\ldots+f(2)\right] \qquad \because f(x)=x^2$$

$$=\lim_{n\to\infty}\frac{2}{n}\left[\frac{4}{n^2}+\frac{16}{n^2}+\frac{36}{n^2}+\ldots+\frac{4n^2}{n^2}\right] \qquad \therefore 4=\frac{4n^2}{n^2}$$

$$=\lim_{n\to\infty}\frac{8}{n^3}\left[1+2^2+3^2+\ldots+n^2\right]=\lim_{n\to\infty}\frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$=\lim_{n\to\infty}\frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right]=\lim_{n\to\infty}\left[\frac{16}{6}+\frac{24}{6n}+\frac{8}{6n^2}\right]=\frac{8}{3}+0+0=\frac{8}{3}$$

6.6.2 Fundamental theorem of integral calculus

In previous section, we learned that we can determine the area of a region with a definite integral. However, with the tools available to us at this time, evaluating a definite integral using the summation process is rather tedious and time consuming. To provide us with a more efficient method of evaluating the definite integral, we now consider a very important theorem in calculus, the **"fundamental theorem of integral calculus"**. This explanation will show that the definite integral can be applied in a general manner and not only to the concept of area.

To help provide a better understanding of the meaning of the fundamental theorem of integral calculus, let us begin with area of a region using definite integral

$$Area=\int_{x=a}^{x=b} f(x)dx, \qquad \text{(i)}$$

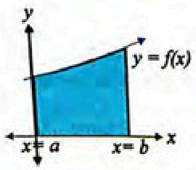


Figure 6.3

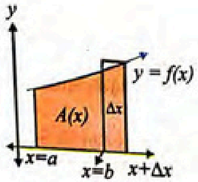


Figure 6.4

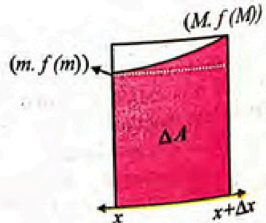


Figure 6.5

To develop the theorem, we need to introduce a new function called the area function $A(x)$. The function indicates the area of the region under the graph of the function from $x=a$ to $x=b$ in the Figure 6.3

The area function $A(x)$ is the area from a to x that must be continuous and non-negative on the interval $[a, b]$.

If we increase x by Δx, then the area $A(x)$ under the curve will increase by an amount that we call ΔA in Figure 6.4. We can see that ΔA is slightly bigger than the area of the inscribed rectangle and slightly smaller than the area of the circumscribed rectangle. In Figure 6.5, the smaller rectangle is inscribed (within the curve) and the large rectangle is circumscribed.

For the area of the inscribed rectangle, we take the minimum value of $f(x)$ within the closed interval $[x, x+\Delta x]$. We call this minimum value $f(m)$.

For the area of the circumscribed rectangle, we take the maximum value within the closed interval $[x, x+\Delta x]$. We refer to this value as $f(M)$. Hence the minimum area is $f(m)\Delta x$ and the maximum area is $f(M)\Delta x$.

Algebraically, we can write $\quad f(m)\Delta x\le \Delta A\le f(M)\Delta x$

$$f(m)\le\frac{\Delta A}{\Delta x}\le f(M),\quad \Delta x\ne 0 \qquad \text{(ii)}$$

If we take the limit as $\Delta x\to 0$, then $f(m)$ and $f(M)$ approach the same point on the curve and both approach $f(x)$

$$f(x)\le\lim_{\Delta x\to 0}\frac{\Delta A}{\Delta x}\le f(x)$$

which states that $\quad \frac{dA}{dx}=f(x),\quad \lim_{\Delta x\to 0}\frac{\Delta A}{\Delta x}=\frac{dA}{dx} \qquad$ (iii)

Integrating (iii) to obtain $\quad A(x)=F(x)+C \qquad$ (iv)

Here $F(x)$ is the antiderivative of $f(x)$. To determine a real value of $A(x)$, we must solve equation (iv) for C.

Put $x=a$ in (iv) to obtain: $\quad A(a)=F(a)+C \Rightarrow 0=F(a)+C,\ A(a)=0 \Rightarrow C=-F(a)$

Put $x=b$ in (iv) to obtain: $\quad A(b)=F(b)+C \Rightarrow A(b)=F(b)-F(a),\ C=-F(a) \qquad$ (v)

The last equation (v) tells us that if it is possible to find an antiderivative of $f(x)$, then we can evaluate the definite integral $\int_{x=a}^{x=b} f(x)dx$. This is nicely condensed in the fundamental theorem.

Statement: If a function $f(x)$ is continuous on the closed interval [a,b], then the definite integral of a function $f(x)$ in the interval $[a,b]$ is:

$$\int_{x=a}^{x=b} f(x)dx=\left|F(x)\right|_{x=a}^{x=b}=F(b)-F(a)$$

Proof: Here $F(x)$ is any function such that $F'(x)=f(x)$ for all x in $[a, b]$.

It is important to recognize that the fundamental theorem of integral calculus describes a means for evaluating a definite integral. It does not provide us with a technique for finding the antiderivative. To find the antiderivative of a definite integral, we use the same techniques we used to find the antiderivative of the indefinite integral. But what happens to the constant C? This constant C drops out as illustrated below:

$$\int_{x=a}^{x=b} f(x)dx=\left|F(x)+C\right|_{x=a}^{x=b}=\left[(F(b)+C)-(F(a)+C)\right]$$

$$=F(b)-F(a)+C-C=F(b)-F(a) \qquad \text{(vii)}$$

Basic properties of the definite integrals

In computations involving integrals, it is often helpful to use the seven basic properties related to fundamental theorem of calculus that are listed below:

i. $\int_a^a f(x)dx = 0$

Proof: By the definition of the definite integral

$$\int_a^a f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)(0)$$

$$\therefore \Delta x = \frac{a-a}{n} = 0$$

$$= \lim_{n\to 0}(0) = 0 \qquad \text{hence, } \int_a^a f(x)\,dx = 0$$

> **Challenge!** Show that $\int_a^a f(x)dx = 0$ by using fundamental theorem of integral calculus.

ii. $\int_a^b f(x)dx = \int_a^b f(y)dy$

Proof:

Let $F(x) = f(x)$, $[a, b]$ be an interval then by the fundamental theorem of integral calculus.

$$\int_a^b f(x)dx = F(b) - F(a) \qquad \text{(i)}$$

$$\text{also, } \int_a^b f(y)dy = F(b) - F(a) \qquad \text{(ii)}$$

Hence, by (i) and (ii) $\int_a^b f(x)dx = \int_a^b f(y)dy$ (proved)

> **Challenge!** Show that $\int_a^b f(x)dx = \int_a^b f(y)dy$ by using the definition of definite integral.

iii. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Proof: By using the definition of definite integrate

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)$$

and $\int_b^a f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$ where $\Delta x = \frac{a-b}{n}$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\left(\frac{a-b}{n}\right) = \lim_{n\to\infty}\left(-\sum_{i=1}^{n} f(x_i)\frac{b-a}{n}\right)$$

$$= -\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{b-a}{n} = -\int_b^a f(x)dx \quad \text{Hence, } \int_a^b f(x)dx = -\int_b^a f(x)dx$$

> **Challenge!** Use fundamental theorem of integral calculus to show $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

iv. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$

Proof: Let $F'(x) = f(x)$, $a \le x \le b$

By using the fundamental theorem of integral calculus.

$\int_a^b f(x)dx = F(b) - F(a)$ where f is continuous on $[a, b]$ then

$$\int_a^c f(x)dx = F(c) - F(a) \qquad \text{(i)}$$

$$\int_c^b f(x)dx = F(b) - F(c) \qquad \text{(ii)}$$

By adding (i) and (ii)

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \cancel{F(c)} - F(a) + F(b) - \cancel{F(c)}$$

$$= F(b) - F(a) = \int_a^b f(x)dx$$

Hence, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

> **Challenge!** Use definition of the definite integral to show that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

v. $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{when } f(-x) = f(x) \\ 0 & \text{when } f(-x) = -f(x) \end{cases}$

Proof: If $f(x)$ is integral on interval $[-a, a]$ w.r.t 'x', then for a number 0 in the interval $[-a, a]$, the definite integral of $f(x)$ from $-a$ to a is 2 times the definite integral of $f(x)$ from 0 to a:

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = \int_a^0 f(-x)d(-x) + \int_0^a f(x)dx = -\int_a^0 f(-x)dx + \int_0^a f(x)dx$$

$$= \int_0^a f(-x)dx + \int_0^a f(x)dx = \int_0^a [f(-x) + f(x)]dx = \begin{cases} 2\int_0^a f(x)dx, & \text{when } f(-x) = f(x) \\ 0, & \text{when } f(-x) = -f(x) \end{cases}$$

6.6.3 Extend techniques of integration using properties to calculate definite integral

Example 16 Evaluate the following definite integrals:

(a). $\int_1^2 (2x^2+4x+1)dx$ (b). $\int_1^2 (2x^2+4x+1)dx = \int_1^2 (2y^2+4y+1)dy$ (c). $\int_0^2 (x^2+1)dx = -\int_2^0 (x^2+1)dx$

(d). $\int_0^2 (x^2+1)dx = \int_0^1 (x^2+1)dx + \int_1^2 (x^2+1)dx$ (e). $\int_{-1}^{+1} x^2 dx = \int_{-1}^{0} x^2 dx + \int_0^1 x^2 dx = 2\int_0^1 x^2 dx$

Solution a. $\int_1^2 (2x^2+4x+1)dx = \left|\frac{2x^3}{3} + \frac{4x^2}{2} + x\right|_1^2$

$$= \left(\frac{16}{3} + \frac{16}{2} + 2\right) - \left(\frac{2}{3} + \frac{4}{2} + 1\right) = \frac{92}{6} - \frac{11}{3} = \frac{70}{6} = \frac{35}{3}$$

b. $\int_1^2 (2x^2+4x+1)\,dx = \int_1^2 (2y^2+4y+1)\,dy$,

$$\Rightarrow \left|\frac{2x^3}{3} + \frac{4x^2}{2} + x\right|_1^2 = \left|\frac{2y^3}{3} + \frac{4y^2}{2} + y\right|_1^2 \Rightarrow \frac{35}{3} = \frac{35}{3}, \text{ since } f(x) = g(y)$$

c. $\int_0^2 (x^2+1)dx = -\int_2^0 (x^2+1)dx$,

$$\left|\frac{x^3}{3}+x\right|_0^2=-\left|\frac{x^3}{3}+x\right|_2^0\Rightarrow\frac{8}{3}+2=-\left(0-\left(\frac{8}{3}+2\right)\right)\Rightarrow\frac{14}{3}=\frac{14}{3}$$

d. $\int_0^2(x^2+1)dx=\int_0^1(x^2+1)dx+\int_1^2(x^2+1)dx,$

$$\left|\frac{x^3}{3}+x\right|_0^2=\left|\frac{x^3}{3}+x\right|_0^1+\left|\frac{x^3}{3}+x\right|_1^2\Rightarrow\frac{14}{3}=\frac{4}{3}+\frac{8}{3}+2-\left(\frac{1}{3}+1\right)\Rightarrow\frac{14}{3}=\frac{14}{3}$$

e. $\int_{-1}^{+1}x^2dx=\int_{-1}^{0}x^2dx+\int_0^1x^2dx=2\int_0^1x^2dx,$

$$\left|\frac{x^3}{3}\right|_{-1}^1=\left|\frac{x^3}{3}\right|_{-1}^0+\left|\frac{x^3}{3}\right|_0^1=2\left|\frac{x^3}{3}\right|_0^1\Rightarrow\frac{1}{3}+\frac{1}{3}=0+\frac{1}{3}+\frac{1}{3}=2\left(\frac{1}{3}\right)\Rightarrow\frac{2}{3}=\frac{2}{3}=\frac{2}{3}$$

Example 17 Evaluate the following definite integrals: *(a).* $\int_0^1\frac{2e^{4x}-3}{e^{2x}}dx$ *(b).* $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}x\sin x^2dx$

Solution

a. $\int_0^1\frac{2e^{4x}-3}{e^{2x}}dx=\int_0^1\left(2e^{2x}-3e^{-2x}\right)dx$

$$=\left|2\frac{e^{2x}}{2}-3\frac{e^{-2x}}{-2}\right|_0^1=\left|e^{2x}+\frac{3}{2}e^{-2x}\right|_0^1=\left(e^2+\frac{3}{2}e^{-2}\right)-\left(e^0+\frac{3}{2}e^0\right)=e^2+\frac{3}{2}e^{-2}-1-\frac{3}{2}=5.092$$

b. We need to substitute a new variable $u(x)$:

$$x^2=u,\ \frac{d}{dx}(x^2)=\frac{du}{dx}\Rightarrow 2x=\frac{du}{dx}\Rightarrow xdx=\frac{du}{2}$$

The lower and upper limit of $x=\frac{\pi}{3}$ and $x=\frac{\pi}{2}$ are used in $x^2=u$ to obtain the lower and upper limit of u:

$$x=\frac{\pi}{3}:x^2=u\Rightarrow\left(\frac{\pi}{3}\right)^2=u\Rightarrow\frac{\pi^2}{9}=u$$

$$\Rightarrow x=\frac{\pi}{2}:x^2=u\Rightarrow\left(\frac{\pi}{2}\right)^2=u\Rightarrow\frac{\pi^2}{4}=u$$

Substitute all these in the given integral to obtain:

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}x\sin x^2dx=\frac{1}{2}\int_{\frac{\pi^2}{9}}^{\frac{\pi^2}{4}}\sin u\,du=\left|\frac{-\cos u}{2}\right|_{\frac{\pi^2}{9}}^{\frac{\pi^2}{4}}=-\frac{1}{2}\left(\cos\frac{\pi^2}{4}-\cos\frac{\pi^2}{9}\right)$$

$$=\frac{-1}{2}[\cos(2.4647)-\cos(1.0966)]=\frac{-1}{2}(-0.7812-0.4566)=\frac{-1}{2}(-1.2378)=0.6189$$

Example 18 Evaluate the following definite integrals: *(a).* $\int_0^1xe^xdx$ *(b).* $\int_0^1e^x\sin xdx$

Solution a. The technique of integration by parts with $u=x$ and $\frac{dv}{dx}=e^x$ is used to obtain:

$$\int_0^1xe^xdx=\left|xe^x\right|_0^1-\int_0^1e^x1dx$$

$\therefore u=x, du=dx, \frac{dv}{dx}=e^x, v=e^x$

$$=(e^1-0)-\left|e^x\right|_0^1=(e^1-0)-(e^1-e^0)=e^1-e^1+e^0=1$$

b. The integral is $I=\int_0^1e^x\sin xdx$

The integration by parts rule with substitution $u=e^x$ and $\frac{dv}{dx}=\sin x$ is used to obtain:

$$I=\int_0^1e^x\sin xdx$$

$=\left|e^x(-\cos x)\right|_0^1-\int_0^1(-\cos x)e^xdx,\ u=e^x$ $\qquad\therefore du=e^xdx, \frac{dv}{dx}=\sin x, v=-\cos x$

$$=-(e^1\cos 1-e^0\cos 0)+\int_0^1e^x\cos xdx$$

$=-2.718(0.540)+1+\int_0^1e^x\cos xdx=-0.468+\int_0^1e^x\cos xdx$ $\qquad$ Use radians

$=-0.468+\int_0^1e^x\cos xdx$ $\qquad$ Again integration by parts

$=-0.468+\left|e^x\sin x\right|_0^1-\int_0^1(\sin x)(e^x)dx$ $\qquad\therefore u=e^x, \frac{dv}{dx}=\cos x, v=\sin x$

$$=-0.468+(e^1\sin 1-e^0\sin 0)-\int_0^1e^x\sin xdx$$

$$I=-0.468+(2.718(0.841)-(1)(0))-I$$

$$2I=-0.468+2.287=1.819$$

$$I=\frac{1.819}{2}=0.91$$

6.6.4 Definite integral as the area under the curve

If $f(x)$ is continuous and $f(x)\geq 0$ on the closed interval $[a, b]$, then the area under a curve $y=f(x)$ on the interval $[a, b]$ is given by the definite integral of $f(x)$ on $[a, b]$:

$$Area=\int_a^bf(x)dx=F(b)-F(a) \qquad \text{(i)}$$

Area between a curve and the x-axis

The steps involved in finding the area between a curve and the x-axis are the following:

i. The definite integral $\int_a^bf(x)dx$ presents the sum of the signed areas between the graph of $y=f(x)$ and the x-axis from $x=a$ to $x=b$, where the area above the x-axis (peak) are counted positively and the areas below the x-axis (valley) are counted negatively. This is shown in the Figure 6.6.

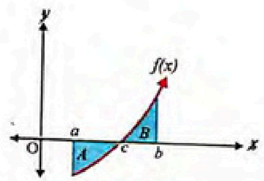


Figure 6.6

ii. If $f(x)$ is a continuous function over the interval $[a, b]$, then the area between $y = f(x)$ and the x-axis from $x = a$ to $x = b$ can be found using definite integrals as follows:

- For $f(x) \geq 0$ over $[a,b]$, the area is: $Area = \int_a^b [+f(x)]dx$
- For $f(x) \leq 0$ over $[a,b]$, the area is: $Area = \int_a^b [-f(x)]dx$

If $f(x)$ is positive for some values of x and negative for others on an interval (as in Figure 6.6), then, the area between the graph of and the x-axis can be found by (dividing the interval into subintervals over which f (x) is always positive or always negative) taking the sum of the areas of subregions over each subinterval:

$$Area = \int_a^b f(x)dx = \int_a^c [-f(x)dx] + \int_c^b [+f(x)dx] = -A + B \qquad \text{(ii)}$$

In Figure 6.6, A represents the area between $y = f(x)$ and the *x-axis* from $x = a$ to $x = c$, and B represents the area between $y = f(x)$ and the *x-axis* from $x = c$ to $x = b$. Both A and B are positive quantities. Since $f(x) \geq 0$ on the interval $[c, b]$, the area is

$\int_c^b [+f(x)]dx = B$ and $f(x) \leq 0$ on the interval $[a, c]$, the area is $\int_a^c [-f(x)dx] = -A$.

6.6.5 Application of definite integral as the area under a curve

Example 19 Find the area between the x-axis and the curve $f(x) = x^2 - 4$ from $x = 0$ to $x = 4$.

Solution First find out the x-intercepts of a curve $f(x) = x^2 - 4$ that can be found by solving the equation of a curve:

$x^2 - 4 = 0 \Rightarrow x = 2, -2$

The subintervals of the interval [0,4] are therefore [-2,0], [0,2]and [2,4]. The total area of the region in the required interval [0,4] is the sum of the areas of the sub regions in the subintervals [0,2] and [2,4]:

$$Area = \int_0^2 [-f(x)]dx + \int_2^4 [+f(x)]dx,\ f(x) \leq 0\, in[0,2]\, and\ f(x) \geq 0\, in[2,4]$$

$$= -\int_0^2 (x^2-4)dx + \int_2^4 (x^2-4)dx = -\left|\frac{x^3}{3} - 4x\right|_0^2 + \left|\frac{x^3}{3} - 4x\right|_2^4$$

$$= -\left|\frac{8}{3} - 8 - (0-0)\right| + \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right) = \frac{16}{3} + \frac{16}{3} + \frac{16}{3} = 16 \text{ square units}$$

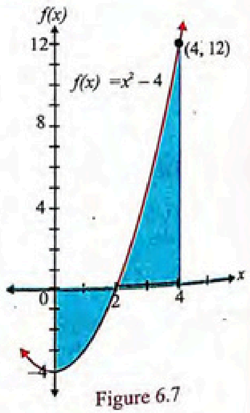


Figure 6.7

The sketch of the region is shown in the Figure 6.7.

The area over the entire interval [0,4] $\quad A = \int_0^4 (x^2-4)dx = x^3 - 4x\Big|_0^4 = \frac{16}{3}$

is not the correct area. This definite integral does not represent the area over the entire interval [0, 4], but is just a real number.

Example 20 Find the area between the x-axis and the curve $f(x) = x^2 - 2x$ from $x = -1$ to $x = 3$.

Solution First find out the x-intercepts of a curve $f(x) = x^2 - 2x$ that can be found by solving the equation of a curve:

$x^2 - 2x = 0 \Rightarrow x = 0, 2$

The subintervals of the interval [–1,3] are therefore [–1,0], [0,2]and [2,3]. The total area of the region in the required interval [–1,3] is the sum of the areas of the sub regions in the subintervals [–1,0], [0,2]and [2,3]:

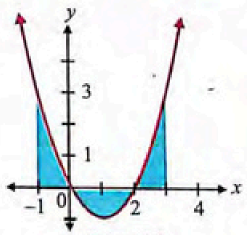


Figure 6.8

$$A = \int_{-1}^0 [+f(x)]dx + \int_0^2 [-f(x)]dx + \int_2^3 [+f(x)]dx,\ f(x) \geq 0\, in[-1,0],[2,3]$$

$$= \int_{-1}^0 (x^2-2x)dx - \int_0^2 (x^2-2x)dx + \int_2^3 (x^2-2x)dx = \left|\frac{x^3}{3} - \frac{2x^2}{2}\right|_{-1}^0 - \left|\frac{x^3}{3} - \frac{2x^2}{2}\right|_0^2 + \left|\frac{x^3}{3} - \frac{2x^2}{2}\right|_2^3$$

$$= (0-0) - \left(\frac{-1}{3} - 1\right) - \left|\left(\frac{8}{3} - 4\right) - (0-0)\right| + \left(\frac{27}{3} - 9\right) - \left(\frac{8}{3} - 4\right) = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 3\left(\frac{4}{3}\right) = 4$$

The sketch of the region is shown in the Figure 6.8.

6.6.6 MAPLE command "*int*" to evaluate definite and indefinite integrals

The use of maple common '*int*' is illustrated in the following example.

Example 21 Use MAPLE command 'int' to solve.

(a). Indefinite integral of a function $f(x) = x^4 + x^3 + x^2 + x + 1$ w.r.t variable x.

(b). Definite integral of a function $f(x) = x^2$ w.r.t variable x.

(c). Definite integral of a function $f(x) = xe^x$ in the interval [0,1].

Solution

a. **Command:**

```
> int(x^4 + x^3 + x^2 + x + 1, x);
```

$$\frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$$

Using Palettes: Use cursor button to select integral palette. Click-integral palette, insert the function required, then press "ENTER" key to obtain the integral of a given function:

$> \int x^4 + x^3 + x^2 + x + 1 dx$

$$\frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$$

b. **Command:**

```
> int(x^2, x = 0..1);
```

$$\frac{1}{3}$$

Using Palettes:

$> \int_0^1 x^2 dx$

$$\frac{1}{3}$$

c. **Command:**

```
> int(x·exp(x), x = 0..1);
```

$$1$$

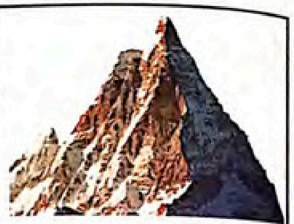
Using Plalettes:

$> \int_0^1 x \cdot \exp(x) dx$

$$1$$

Do you know a 200 year old problem?

The relationship between derivative and integrals as an inverse operation was noticed first time by Isaac barrow (1630-1677) in the 17th century. He was a teacher of Sir Isaac Newton. Newton and Leibniz are known as key inventor of calculus. They made the use of calculus as conjuctor, that is as a mathematical statement which is suspected to be true. But has not proven yet. The fundamental theorem of integral calculus was not officially proven in all its glory until Bernhard Riemann (1826-1866) demonstrated it in the 19th century. During this 200-years a lot of mathematic like real analysis had invented before Riemann could prove that derivatives and integrals are inverse.



Exercise 6.4

1. Evaluate the following definite integrals:

a. $\int_{3}^{4} 5x\,dx$ b. $\int_{12}^{20} x^3\,dx$ c. $\int_{1}^{2} (2x^{-2}-3)\,dx$ d. $\int_{1}^{4} 3\sqrt{x}\,dx$

c. $\int_{2}^{3} 12(x^2-4)^5 x\,dx$ f. $\int_{-1}^{1} \frac{e^{-x}-e^{x}}{(e^{-x}+e^{x})^2}\,dx$ g. $\int_{\frac{\pi}{2}}^{\pi} \cos\left(\frac{x}{2}+\pi\right)dx$ h. $\int_{0}^{\frac{\pi}{4}} \sec^2\theta\,d\theta$

2. Evaluate the following definite integrals:

a. $\int_{1}^{2} \frac{5t^2-3t+18}{t(9-t^2)}\,dt$ b. $\int_{1}^{2} \frac{4}{t^3+4t}\,dt$

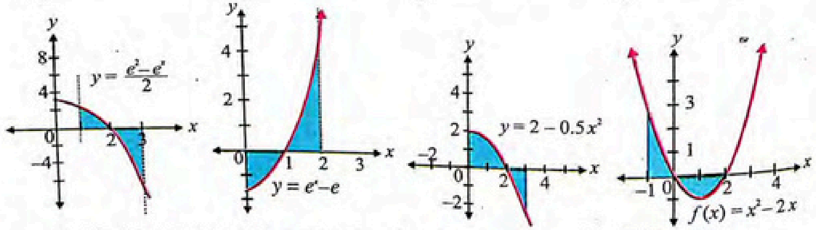
3. Use definite integral to find out the area between the curve f(x) and the x-axis over the indicated interval [a, b]:

a. $f(x)=4-x^2$, [0,3] b. $f(x)=x^2-5x+6$, [0,3]

c. $f(x)=x^2-6x+8$, [0,4] d. $f(x)=5x-x^2$, [1,3]

4. Setup definite integrals in problems a to d that represent the indicated shaded areas:

a. b. c. d.



5. An oil tanker is leaking oil at a rate given in barrels per hour by $\frac{dL}{dt}=\frac{80\ln(t+1)}{(t+1)}$

Where t is the time in hours after the tanker hits a hidden rock (when $t=0$).

a. Find the total number of barrels that the ship will leak on the first day.
b. Find the total number of barrels that the ship will leak on the second day.
c. What is happening over the long run to the amount of oil leaked per day?

6. Use MAPLE command *'int'* to evaluate

a. $f(x)=x^2+3x+1$ *w.r.t. 'x'* b. $f(x)=e^{2x}.\sin x$ *w.r.t. 'x'*

Review Exercise 6

1. **Choose the correct option.**

i. The process of finding antiderivative is called.
(a). differentiation (b). integration (c). probability (d). linear equations

ii. $\int \tan\theta\,d\theta=$ ______
(a). $\ln|\sin\theta|+C$ (b). $\ln|\cos\theta|+C$ (c). $-\ln|\cos\theta|+C$ (d). $-\ln|sin\theta|+C$

iii. $\int \frac{dx}{\sqrt{a^2-x^2}}=$ ______
(a). $\ln\left|x+\sqrt{x^2+a^2}\right|+C$ (b). $\sin^{-1}\frac{a}{x}+C$ (c). $\sin^{-1}\left(\frac{x}{a}\right)+C$ (d). $\cos^{-1}\left(\frac{x}{a}\right)+C$

iv. $\int \frac{1}{\sqrt{t^2-36}}\,dx=$ ______.
(a). $\frac{1}{2a}\ln\left|\frac{t-6}{t+6}\right|+C$ (b). $\frac{1}{2}\ln\left|\frac{t-6}{t+6}\right|+C$
(c). $-\frac{1}{2}\left(\ln\left|\frac{t}{6}+1\right|-\ln\left|\frac{t}{6}-1\right|+C\right)$ (d). $\frac{1}{6}\left(\ln\left|\frac{t}{6}+1\right|+6\right)+C$

v. $\int (x^3-4)\,dx=$ ______
(a). $\frac{x^4}{4}-4x+C$ (b). $\frac{x^3}{3}-4x+C$
(c). $-\frac{x^4}{4}+4x+C$ (d). $\frac{x^4}{4}+\frac{4x^3}{3}-\frac{4x^2}{2}-4x+C$

vi. $\int f(x)g'(x)\,dx=$ ______
(a). $f'(x)g(x)+\int g'(x)f(x)\,dx$ (b). $f(x)g(x)-\int g(x)f'(x)\,dx$
(c). $f'(x)'g(x)-\int g'(x)f'(x)\,dx$ (d). $f(x)g(x)-\int g(x)f(x)\,dx$

vii. $\int \tan^4(x)\,dx=$ ______
(a). $\frac{2}{3}\tan^2(x)+x-\tan(x)+C$ (b). $\frac{1}{3}\tan^3(x)+x-\tan(x)+C$
(c). $\frac{3}{4}\tan^2(x)-x+\tan(x)+C$ (d). $3\tan^3(x)+x+\tan(x)+C$

viii. $\int \frac{x+8}{x^2-64}\,dx=$
(a). $\ln|x-8|+C$ (b). $\frac{1}{2}\ln|x^2-64|+\frac{1}{2}\left|\frac{x}{8}+1\right|-\frac{1}{2}\ln\left|\frac{x}{2}-1\right|+C$
(c). $\frac{1}{2}\ln|x^2-64|-\frac{1}{2}\ln\left|\frac{x}{8}+1\right|+\frac{1}{2}\ln\left|\frac{x}{8}-1\right|+C$ (d). $\frac{1}{2}\ln\left|\frac{x}{8}+1\right|+\frac{1}{2}\ln\left|\frac{x}{8}-1\right|+C$

ix. $\int_{0}^{2} e^{2x}\,dx=$ ______
(a). $\frac{e^4-1}{2}$ (b). $\frac{e^3-1}{2}$ (c). $\frac{e^2-1}{4}$ (d). $\frac{e^4+1}{2}$

x. $\int_{0}^{2} x^3\,dx=$ ______
(a). 1 (b). 2 (c). 3 (d). 4