Summary

- The second derivative of $y = f(x)$ can be written with any of the following notations:
  $\frac{d^2y}{dx^2}, y'', f''(x), D_x^2[f(x)]$
  The third derivative can be written in a similar way. For $n \geq 4$, the nth derivative is written as $f^{(n)}(x)$.
- The second derivative of parametric functions $x(t)$ and $y(t)$ can be found as follows:
  $$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}, \quad Put \frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)}$$
- The popular notation for the Taylor theorem of order n is:
  $$f(x_0+h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2!} + \dots + f^n(x_0)\frac{h^n}{n!} + \dots$$
- The popular notation for the Maclaurin's theorem of order n is:
  $$f(x_0+h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + \dots + f^n(0)\frac{h^n}{n!} + \dots$$
- **i.** If two lines are parallel, then their slopes are equal.
  ii. If two lines are perpendicular, then the product of their slopes equals -1.
  iii. The tangent equation at a point $P(x_0, y_0)$ is $(y - y_0) = m(x - x_0)$.
  iv. The normal equation at a point $P(x_0, y_0)$ is $(y - y_0) = \frac{-1}{m}(x - x_0)$.
- If $f(x)$ is differentiable on the open interval (a, b), then the function $f(x)$ is
  strictly increasing on (a, b) if $f'(x) > 0$ for $a < x < b$.
  strictly decreasing on (a, b) if $f'(x) < 0$ for $a < x < b$.
- If a continuous function $f(x)$ has a relative extremum at c, then c must be a critical value of $f(x)$.
- The graph of a function $f(x)$ is **concave upward** on an open interval (a, b), where $f''(x) > 0$, and it is **concave downward** where $f''(x) < 0$.
- If $y = f(x)$ is continuous on (a, b) and has an inflection point at $x = c$, then either $f''(c) = 0$ or $f''(c)$ does not exist.
  i. A point $P(c, f(c))$ on the graph of a differential function $y = f(x)$ where the concavity changes is called a **point of inflection.**
  ii. If a function has a point of inflection $P(c, f(c))$ at a partition c and it is possible to differentiate the function twice, then $f''(c) = 0$.

History

Omer Khayyam was a Persian mathematician, Astronomer and philosopher. He was born in Nishapur in north eastern Iran. He was most notable person in the history of mathematics because of his work on the classification and solution of cubic equation. Where he proved the geometric solution by the intersection of conics. He also contributed to the understanding of the parallel axiom. As an astronomer he designed the Jalali calendar, a Solor calander. He was the first person, who considered the three cases of acute, right and abtuse angle for summit angles of a Khayyam saccheri quadrilateral, three cases which are exhaustive and pairwise mutually exclusive.


Omer Khayyam (1048-1131)

Unit 5 DIFFERENTIATION OF VECTOR FUNCTIONS

By the end of this unit, the students will be able to:

5.1 Scalar and Vector Functions.
i. Define scalar and vector function.
ii. Explain domain and range of a vector function.

5.2 Limit and Continuity.
i. Define limit of a vector function and employ the usual technique for algebra of limits of scalar function to demonstrate the following properties of limits of a vector function.
- The limit of the sum (difference) of two vector functions is the sum (difference) of their limits.
- The limit of the dot product of two vector functions is the dot product of their limits.
- The limit of the cross product of two vector functions is the cross product of their limits.
- The limit of the product of a scalar function and a vector function is the product of their limits.

ii. Define continuity of a vector function and demonstrate through examples.

5.3 Derivative of Vector Function.
i. Define derivative of a vector function of a single variable and elaborate the result:
if $f(t) = f_1(t)i + f_2(t)j + f_3(t)k$, where $f_1(t), f_2(t), f_3(t)$ are differentiable functions of a scalar variable t, then $\frac{df}{dt} = \frac{df_1}{dt}i + \frac{df_2}{dt}j + \frac{df_3}{dt}k$.

5.4 Vector Differentiation.
i. Prove the following formulae of differentiation:
- $\frac{da}{dt} = 0$,
- $\frac{d}{dt}[f \pm g] = \frac{df}{dt} \pm \frac{dg}{dt}$,
- $\frac{d}{dt}[\phi f] = \phi\frac{df}{dt} + \frac{d\phi}{dt}f$.
- $\frac{d}{dt}[f \cdot g] = f \cdot \frac{dg}{dt} + \frac{df}{dt} \cdot g$,
- $\frac{d}{dt}[f \times g] = f \times \frac{dg}{dt} + \frac{df}{dt} \times g$.
- $\frac{d}{dt}\left[\frac{f}{\phi}\right] = \frac{1}{\phi^2}\left[\phi\frac{df}{dt} - \frac{d\phi}{dt}f\right]$,

where a is a constant vector function, f and g are vector functions, and ϕ is a scalar function of t.

ii. Apply vector differentiation to calculate velocity and acceleration of a position vector $r(t) = x(t)i + y(t)j + z(t)k$

Introduction

In the same way that we studied numerical calculus after we learned numerical arithmetic. We can now study vectors calculus. Since we already studied vector arithmetic in unit-3 of grade-xi Mathematics. Quite simply, we might have a vector quantity that varies with respect to another variable, either a scalar or a vector. In this unit we shall study the vector functions and the applications of the differential calculus. We shall extend the basic concepts of calculus in a simple and natural way. The study of vector calculus makes the more useful in the geometrical, physical and engineering applications.

5.1 Scalar and Vector Functions

The relationship of calculus and vector methods forms what is called **vector calculus**. The key to use vector calculus is the concept of a vector function.

i. (a) Scalar Function

A function $f(x)$ is a rule which operates on an input x (x is any scalar quantity) and produces always just a single scalar output y. This gives a proper notation of a scalar function:

$$y = f(x) \qquad (i)$$

For example.

- $C(x) = 2x + 2$ is a cost function that depends on x number of units of items. Here x is the input, y is the output and $C(x) = 2x + 2$ is the rule which operates on an input x to produce a single output quantity y. In response of $x = 2$ items (2 is scalar), the cost (cost is also scalar) is: $C(2) = 2(2) + 2 = 6$ rupees. This function is then called a **scalar (single variable) function**; because it transforms one input x to produce just one output C.
- $A(x, y) = xy$ is the area of rectangle that depends on length x and width y. Here x, and y are the two inputs and the rule $A(x, y) = xy$ which operates on two inputs x and y to produce a single output quantity A. In response of $x = 2$ and $y = 1$ (2 and 1 are scalars), the area is (is also a scalar) $A(2, 1) = (2)(1) = 2$ square units. This function is then called a **scalar (double variables) function**, because it transforms two inputs x and y to produce just one output A.

This idea can easily be extended to define a scalar multivariate function. The uniqueness of scalar function is to transform **scalar quantities in a single scalar quantity.** Is there any rule that will transform **scalar quantities in a vector quantity**? Yes, the rule is the vector functions. Vector functions are used to study curves in the plane and space.

(b) Vector Function

"A vector function $\overline{F} = (f(t), g(t), h(t))$ is a function of one variable that has only one "input value". The "output" values are in two and three dimensional vector spaces instead of simple numbers. In other words we can say $\overline{F}$ is called a vector function of 't' $\overline{F} = \overline{F}(t)$.

If $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors associated with a rectangular coordinate system (discussed in details in unit-3 of grade-xi) then a vector function $\overline{F}(t)$ is written as

i. $\overline{F}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ 2 spaces

ii. $\overline{F}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ 3 spaces

We can say that a function $\overline{F}(t)$ is defined if all its components f_1, f_2 and f_3 are defined.

Example 1 Find $\overline{F}\left(\frac{\pi}{2}\right)$ and $\overline{F}(\pi)$ if $\overline{F}(t) = \sin(t)\hat{i} + \cos(t)\hat{j}$

Solution We have given $\overline{F}(t) = \sin(t)\hat{i} + \cos(t)\hat{j}$

As, sin (t) and cos (t) are defined for all values of t, so, $\overline{F}(t)$ is defined for all t

For $t = \frac{\pi}{2}$, $\quad F\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)\hat{i} + \cos\left(\frac{\pi}{2}\right)\hat{j} = \hat{i}$

For $t = \pi$, $\quad F(\pi) = \sin(\pi)\hat{i} + \cos(\pi)\hat{j} = -\hat{j}$

ii. Domain and range of a vector function

a. Domain

"The set of all t values used as input in $\overline{F}(t)$ is called the domain of a vector-valued function $\overline{F}(t)$".

b. Range

The set of $\overline{F}(t)$ values that the vector function $\overline{F}(t)$ takes as t varies, is called the range of a vector valued function $\overline{F}(t)$.

Example 2 Find the domain for the following vector functions:

(a) $\overline{F}(t) = 2t\hat{i} - 3t\hat{j} + t^{-1}\hat{k}$ (b) $\overline{F}(t) = \sin t\hat{i} + (1-t)^{-1}\hat{j} + \ln t\hat{k}$

Solution

a. The vector function is:

$\overline{F}(t) = (f_1(t), f_2(t), f_3(t)) = 2t\hat{i} - 3t\hat{j} + t^{-1}\hat{k}$

The function $f_1(t) = 2t$ is defined for all t; $f_2(t) = 3t$ is defined for all values of t; $f_3(t) = t^{-1}$ is defined for all values of t except $t = 0$. Thus, the domain of a function $\overline{F}(t)$ is $R - \{0\}$.

b. $\overline{F}(t) = (f_1(t), f_2(t), f_3(t)) = \sin t\hat{i} + (1-t)^{-1}\hat{j} + \ln t\hat{k}$

The function $f_1(t) = \sin t$ is defined for all t; $f_2(t) = (1-t)^{-1}$ is defined for all values of t except $t = 1$; $f_3(t) = \ln t$ is defined for $t > 0$. Thus, the domain of a function $\overline{F}(t)$ is $t > 0$, $t \neq 1$. The range in each case is of course a vector quantity.

Do You Know?

Operations with vector functions

It follows from the definition of vector operations that vector functions can be added, subtracted, multiplied by a scalar function, and multiplied together e.g.

If $\overline{F}$ and $\overline{G}$ are vector functions of the real variable t, and h(t) is any scalar function, then $\overline{F} + \overline{G}$, $\overline{F} - \overline{G}$ and $\overline{F} \times \overline{G}$ are vector functions, and $\overline{F} \cdot \overline{G}$ is a scalar function.

5.2 Limit and Continuity

For the most part, vector limits behave like scalar limits. The proper definition of the limit of a vector function is given below.

5.2.1 Limit of a vector function and properties of limits of a vectors function

"Let a vector function $\overline{F}(t)$ be defined for all values of t in some neighbourhood about a point $t = t_0$ except possibly at itself and let $\overline{L}$ be a constant vector called limit vector. The function $\overline{F}(t)$ is said to approach the limit vector $\overline{L}$ as "t approaches t_0" if for any given real number $\varepsilon > 0$ such that $\left|\overline{F}(t) - \overline{L}\right| \angle \varepsilon$ whenever $0 \angle |t - t_0| \angle \delta$ symbolically, it is written as $\lim_{t \to t_0} \overline{F}(t) = \overline{L}$

Now look at the following useful properties of vector valued functions.

i. The limit of the sum (difference of two vector functions is the sum (difference) of their limits.

If $\lim_{t \to t_0} \overline{F}(t) = \overline{L}$ and $\lim_{t \to t_0} \overline{G}(t) = \overline{M}$, where $\overline{L}$ and $\overline{M}$ are constant vector functions then:

a. $\lim_{t \to t_0}\left[\overline{F}(t) + \overline{G}(t)\right] = \lim_{t \to t_0}\overline{F}(t) + \lim_{t \to t_0}\overline{G}(t) = \overline{L} + \overline{M}$ b. $\lim_{t \to t_0}\left[\overline{F}(t) - \overline{G}(t)\right] = \lim_{t \to t_0}\overline{F}(t) - \lim_{t \to t_0}\overline{G}(t) = \overline{L} - \overline{M}$

ii. The limit of the dot product of two vector functions is the dot product of their limits.

If $\lim_{t \to t_0} \overline{F}(t) = \overline{L}$ and $\lim_{t \to t_0} \overline{G}(t) = \overline{M}$, where $\overline{L}$ and $\overline{M}$ are constant vector functions then:

$$\lim_{t \to t_0}\left[\overline{F}(t).\overline{G}(t)\right] = \lim_{t \to t_0}\overline{F}(t).\lim_{t \to t_0}\overline{G}(t) = \overline{L}.\overline{M}$$

iii. The limit of the cross product of two vector functions is the cross product of their limits.

If $\lim_{t \to t_0} \overline{F}(t) = \overline{L}$ and $\lim_{t \to t_0} \overline{G}(t) = \overline{M}$, where $\overline{L}$ and $\overline{M}$ are constant vector functions then:

$$\lim_{t \to t_0}\left[\overline{F}(t) \times \overline{G}(t)\right] = \lim_{t \to t_0}\overline{F}(t) \times \lim_{t \to t_0}\overline{G}(t) = \overline{L} \times \overline{M}$$

iv. The limit of the product of a scalar function and a vector function is the product of their limits. If $\lim_{t\to t_0}\overline{F}(t)=\overline{L}$ and $\lim_{t\to t_0}\bar{h}(t)=c$, where $\overline{L}$ is a constant vector and $\bar{c}$ is a scalar constant then:

$$\lim_{t\to t_0}\left[\bar{h}(t)\times\overline{F}(t)\right]=\lim_{t\to t_0}\bar{h}(t)\times\lim_{t\to t_0}\overline{F}(t)=c\times\overline{L}$$

Example 3 Find $\lim_{t\to 2}F(t)$, when the vector function is $\overline{F}(t)=(t^2-3)\hat{i}+e^t\hat{j}+\sin\pi t\hat{k}$.

Solution $\lim_{t\to 2}\overline{F}(t)=\lim_{t\to 2}\left[\overline{f}_1(t)\hat{i}+\overline{f}_2(t)\hat{j}+\overline{f}_3(t)\hat{k}\right]$

$$=\left[\lim_{t\to 2}(t^2-3)\right]\hat{i}+\left[\lim_{t\to 2}(e^t)\right]\hat{j}+\left[\lim_{t\to 2}\sin\pi t\right]\hat{k}$$

$$=(4-3)\hat{i}+e^2\hat{j}+\sin 2\pi\hat{k}=\hat{i}+e^2\hat{j},\quad \sin 2\pi=0$$

5.2.2 Continuity of a vector function

A vector function $F(t)$ is said to be continuous at $t=t_0$ if

i. t_0 is in the domain of a vector function $F(t)$

ii. $\lim_{t\to t_0}F(t)=F(t_0)$

Do You Know? A continuous vector value function is also continuous at every point in its domain.

Example 4 For what values of t is the vector function $F(t)=\left(\sin t,(1-t)^{-1}\right)$ continuous?

Solution The components of a vector function are: $f_1(t)=\sin t,\ f_2(t)=(1-t)^{-1},\quad t\in R$

The function $f_1(t)$ is continuous for all t; $f_2(t)$ is continuous where $1-t\neq 0\ (t\neq 1)$. Thus, $F(t)$ is continuous, when t is a real number other than 1.

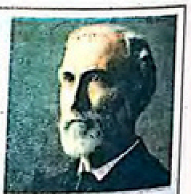
Example 5 For what values of t is $F(t)=\left(\sin t,(1-t)^{-1},\ln t\right)$ continuous?

Solution The components of a vector function are: $f_1(t)=\sin t,\quad f_2(t)=(1-t)^{-1},\quad f_3(t)=\ln t,\ t\in R$

The function $f_1(t)$ is continuous for all t; $f_2(t)$ is continuous where $1-t\neq 0$ (that is, where $t\neq 1$); $f_3(t)$ is continuous for $t>0$. Thus, F (t) is continuous function whenever t is any positive number other than 1. That is $t>0,\ t\neq 1$.

History

J. Willard Gibbs was an American scientist. He made his great contributions in the field of mathematics, physics and chemistry. He was the first American who obtained his doctorate degree in engineering after spending three years in Europe, he joined Yale university as professor of mathematical physics from 1871 to his death. He earned international reputation while working in relative isolation. A great scientist Albert Einstein praised him as "the greatest mind in American history". Together with Oliver Heaviside (Britain and American national) Gibbs developed vector analysis to express the new laws of electromagnetism.



Josiah Willard Gibbs (1839)-(1903)

Exercise 5.1

1. Find the domain for the following vector functions:
   a. $\overline{F}(t)=2t\hat{i}-3t\hat{j}+t^{-1}\hat{k}$
   b. $\overline{F}(t)=(1-t)\hat{i}+\sqrt{t}\hat{j}-(t-2)^{-1}\hat{k}$
   c. $\overline{F}(t=\sin t\hat{i}+\cos t\hat{j}+\tan t\hat{k}$
   d. $\overline{F}(t)=\cos t\hat{i}-\cot t\hat{j}+\csc t\hat{k}$
2. Perform the operations of the following expressions with $\overline{F}(t)=2t\hat{i}-5\hat{j}+t^2\hat{k},\ \overline{G}(t)=(1-t)\hat{i}+\frac{1}{t}\hat{k},\ \overline{H}(t)=\sin t\hat{i}+e^t\hat{j}$:
   a. $2\overline{F}(t)-3\overline{G}(t)$ b. $3\overline{F}(t)+4\overline{G}(t)$ c. $\overline{G}(t)\cdot\overline{H}(t)$ d. $\overline{F}(t)\times\overline{H}(t)$
3. Evaluate the limits of the following expressions:
   a. $\lim_{t\to 1}\left[3t\hat{i}+e^{2t}\hat{j}+\sin\pi t\hat{k}\right]$
   b. $\lim_{t\to 1}\left[\frac{t^3-1}{t-1}\hat{i}+\frac{t^2-3t+2}{t^2+t-2}\hat{j}+(t^2+1)e^{t-1}\hat{k}\right]$
   c. $\lim_{t\to 0}\left[\frac{\sin t}{t}\hat{i}+\frac{1-\cos t}{t}\hat{j}+e^{1-t}\hat{k}\right]$
   d. $\lim_{t\to 0}\left[\frac{\sin(2t)}{2t}\hat{i}+\ln(4+t)\hat{j}\right]$
4. Test the continuity of the following expressions for all values of t:
   a. $\overline{F}(t)=t\hat{i}+3\hat{j}-(1-t)\hat{k}$
   b. $\overline{G}(t)=t\hat{i}-t^{-1}\hat{k}$
   c. $\overline{F}(t)=e^t\left(t\hat{i}+t^{-1}\hat{j}+3\hat{k}\right)$
   d. $\overline{G}(t)=\frac{t\hat{i}+\sqrt{t}\hat{j}}{\sqrt{t^2+t}}$

5.3 Derivative of Vector Function

A vector function $\overline{F}$ determines a curve in space as the collection of terminal points of the vectors $\overline{F}(t)$. If the curve is smooth, this is natural to ask whether $\overline{F}(t)$ has a derivative. Our experience with single variable calculus in previous units prompt us to wonder what the diferentiation of the vector valued function might be and what it might tell us. For now, let's recall some important ideas from unit 3 of this book. We defined the derivative of the scalar function $f(x)$. Which is the limit as $\Delta x\to 0$ of the difference quotient $\frac{\Delta f}{\Delta x}$ e.g. Given a function $f(t)$ that measures the position of an object, moving along an axis its, derivative $f'(t)$ is defined as. $f'(t)=\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}$ (i)

and measure the instantanous rate of change of $f(t)$ with rspect to t in particular for a fixed value $t=a$, $f'(a)$ measure the velocity of the moving object as well as the slope of the tangent line to the curve $y=f(t)$ at the point $(a,f(a))$. As we are working with vector valued functions, we will use the above ideas and perspectives into the context of curves in space and output that are vectors.

Derivative of a vector function of a single variable

If $f(t)=f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}$, where $f_1(t),f_2(t),f_3(t)$ are differentiable functions of a scalar variable t, then

$$\frac{df}{dt}=\frac{df_1}{dt}\hat{i}+\frac{df_2}{dt}\hat{j}+\frac{df_3}{dt}\hat{k}$$

The derivative of a vector-valued function $\overline{F}(t)$ is defined to be. $\overline{F}'(t)=\lim_{\Delta t\to 0}\frac{\overline{F}(t+\Delta t)-\overline{F}(t)}{\Delta t}$ (ii) for these values of t at which the limit exists, we can also use the Leibniz notation $\frac{dF}{dt}$, for derivative of $F(t)$, and $\frac{d}{dt}\left[\overline{F}(t)\right]$ The following theorem establishes a convenient method for computing the derivative of a vector function.

Theorem–1: The vector function $\overline{F}(t)=\left(f_1(t),f_2(t),f_3(t)\right)=f_1(t)i+f_2(t)j+f_3(t)k$ is differentiable at a point $t=t_o$ whenever the component functions $f_1(t),f_2(t),f_3(t)$ of $\overline{F}(t)$ are all differentiable at a point $t=t_o$: i.e. $F'(t)=\left(f_1'(t),f_2'(t),f_3'(t)\right)=f_1'(t)\hat{i}+f_2'(t)\hat{j}+f_3'(t)\hat{k}$

Proof: If a vector function $\overline{F}(t)$ is differentiable, then their component functions $f_1(t), f_2(t)$ and $f_3(t)$ exist, then the scalar derivatives $f_1'(t), f_2'(t)$ and $f_3'(t)$ by first-principle rule

$$\overline{F}'(t)=\lim_{\Delta t\to 0}\frac{\overline{F}(t+\Delta t)-\overline{F}(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{\left[f_1(t+\Delta t)\hat{i}+f_2(t+\Delta t)\hat{j}+f_3(t+\Delta t)\hat{k}\right]-\left[f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}\right]}{\Delta t}$$

$$=\left[\lim_{\Delta t\to 0}\frac{f_1(t+\Delta t)-f_1(t)}{\Delta t}\right]\hat{i}+\left[\lim_{\Delta t\to 0}\frac{f_2(t+\Delta t)-f_2(t)}{\Delta t}\right]\hat{j}+\left[\lim_{\Delta t\to 0}\frac{f_3(t+\Delta t)-f_3(t)}{\Delta t}\right]\hat{k}$$

$$=f_1'(t)\hat{i}+f_2'(t)\hat{j}+f_3'(t)\hat{k}$$

In the Leibniz notation, the derivative of $\overline{F}(t)$ is denoted by: $\frac{d\overline{F}}{dt}=\frac{df_1}{dt}\hat{i}+\frac{df_2}{dt}\hat{j}+\frac{df_3}{dt}\hat{k}$ (iii)

Example 6 For what values of t is $G(t)=|t|\hat{i}+(\cos t)\hat{j}+(t-5)\hat{k}$ differentiable?

Solution The component functions $f_2(t)=\cos t$ and $f_3(t)=t-5$ are differentiable for all values of t, but $f_1(t)=|t|$ is not differentiable at $t=0$. Thus, the vector function $G(t)$ is differentiable for all $t\neq 0$.

Example 7 Find the derivative of the vector function $\overline{F}(t)=e^t\hat{i}+\sin t\,\hat{j}+(t^3+5t)\hat{k}$.

Solution Since, the given function is $\overline{F}(t)=e^t\hat{i}+\sin t\,\hat{j}+(t^3+5t)\hat{k}$.

Differentiate both sides w.r.t, "t"

$$\frac{d\overline{F}(t)}{dt}=\frac{d}{dt}\left[e^t\hat{i}+\sin t\,\hat{j}+(t^3+5t)\hat{k}\right]$$

$$\frac{d\overline{F}}{dt}=\frac{d}{dt}(e^t)\hat{i}+\frac{d}{dt}(\sin t)\hat{j}+\frac{d}{dt}(t^3+5t)\hat{k}=e^t\hat{i}+\cos t\,\hat{j}+(3t^2+5)\hat{k}$$

5.4 Vector Differentiation

Several rules for computing derivatives of vector functions are listed below, which can be proved by applying rules for limits of vector functions to appropriate theorems for scalar derivatives.

> **Remember**
> A vector $\overline{F}$ also written as F.

5.4.1 Formula of differentiation

i. $\frac{da}{dt}=0$ ii. $\frac{d}{dt}[f\pm g]=\frac{df}{dt}\pm\frac{dg}{dt}$ iii. $\frac{d}{dt}[\varphi f]=\varphi\frac{df}{dt}+\frac{d\varphi}{dt}f$

iv. $\frac{d}{dt}[f.g]=f.\frac{dg}{dt}+\frac{df}{dt}.g$ v. $\frac{d}{dt}[f\times g]=f\times\frac{dg}{dt}+\frac{df}{dt}\times g$ vi. $\frac{d}{dt}\left[\frac{f}{\varphi}\right]=\frac{1}{\varphi^2}\left[\varphi\frac{df}{dt}-\frac{d\varphi}{dt}f\right]$

Where, a is a constant vector function, f and g are vector functions and φ is a scalar function of t.

i. $\frac{da}{dt}=0$

Proof: i. Let a be a constant vector function then $\frac{d}{dt}(a)=\frac{d}{dt}(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})=\frac{d}{dt}a_1\hat{i}+\frac{d}{dt}a_2\hat{j}+\frac{d}{dt}a_3\hat{k}$

ii. $\frac{d}{dt}[f\pm g]=\frac{df}{dt}\pm\frac{dg}{dt}$

Proof: $\frac{d}{dt}[f]=\underset{\Delta t\to 0}{Lim}\frac{[f(t+\Delta t)\pm g(t+\Delta t)]-[f(t)\pm g(t)]}{\Delta t}=\underset{\Delta t\to 0}{Lim}\frac{f(t+\Delta t)-f(t)}{\Delta t}\pm\underset{\Delta t\to 0}{Lim}\frac{f(t+\Delta t)-g(t)}{\Delta t}$

$$=\frac{df}{dt}\pm\frac{dg}{dt}\quad or\quad \frac{d}{dt}[f+g]=\frac{df}{dt}\pm\frac{dg}{dt}$$

iii. $\frac{d}{dt}[\varphi f\pm g]=\varphi\frac{df}{dt}+\frac{d\varphi}{dt}$

Proof: $\frac{d}{dt}[\varphi f]=\underset{\Delta t\to 0}{Lim}\left[\frac{\varphi(t+\Delta t)f(t+\Delta t)-\varphi(t)f(t)}{\Delta t}\right]$

$$=\underset{\Delta t\to 0}{Lim}\left[\frac{\varphi(t+\Delta t)f(t+\Delta t)-\varphi(t+\Delta t)f(t)}{\Delta t}\right]+\underset{\Delta t\to 0}{Lim}\left[\frac{\varphi(t+\Delta t).f(t)-\varphi(t).f(t)}{\Delta t}\right]$$

$$=\underset{\Delta t\to 0}{Lim}\,\varphi(t+\Delta t)\left[\frac{f(t+\Delta t)-f(t)}{\Delta t}\right]+\underset{\Delta t\to 0}{Lim}\left[\frac{\varphi(t+\Delta t)-\varphi(t)}{\Delta t}\right]f(t)=\varphi\frac{df}{dt}+\frac{d\varphi}{dt}f(t)$$

Hence, $\frac{d}{dt}[\varphi f]=\varphi\frac{df}{dt}+\frac{d\varphi}{dt}.f$ or $\frac{d}{dt}[\varphi f]=\varphi\frac{df}{dt}+\frac{d\varphi}{dt}.f$

iv. $\frac{d}{dt}[f.g]=f.\frac{dg}{dt}+\frac{dg}{dt}.g$

Proof: $\frac{d}{dt}[f.g]=\underset{\Delta t\to 0}{Lim}\frac{[f(t+\Delta t).g(t+\Delta t)-f(t).g(t)]}{\Delta t}$

$$=\underset{\Delta t\to 0}{Lim}\frac{[f(t+\Delta t).g(t+\Delta t)-f(t+\Delta t).g(t)]}{\Delta t}+\underset{\Delta t\to 0}{Lim}\frac{[f(t+\Delta t).g(t)-f(t)g(t)]}{\Delta t}$$

$$=\underset{\Delta t\to 0}{Lim}\,f(t+\Delta t).\left[\frac{g(t+\Delta t)-g(t)}{\Delta t}\right]+\underset{\Delta t\to 0}{Lim}\left[\frac{f(t+\Delta t)-f(t)}{\Delta t}\right].g(t)=f\frac{dg}{dt}+\frac{df}{dt}g$$

Hence, $\Rightarrow=\frac{d}{dt}[f.g]=f.\frac{dg}{dt}+\frac{df}{dt}.g$

v. $\frac{d}{dt}[f\times g]=f\times\frac{dg}{dt}+\frac{df}{dt}+g$

Proof: $\frac{d}{dt}[f\times g]=\underset{\Delta t\to 0}{Lim}\frac{[f(t+\Delta t)\times g(t+\Delta t)]-[f(t)\times g(t)]}{\Delta t}$

$$=\underset{\Delta t\to 0}{Lim}\frac{[f(t+\Delta t)\times g(t+\Delta t)-f(t+\Delta t)\times g(t)]}{\Delta t}+\underset{\Delta t\to 0}{Lim}\frac{[f(t+\Delta t)\times g(t)-f(t)\times g(t)]}{\Delta t}$$

$$=\underset{\Delta t\to 0}{Lim}\,f(t+\Delta t)\times\frac{[g(t+\Delta t)-g(t)]}{\Delta t}+\underset{\Delta t\to 0}{Lim}\frac{[f(t+\Delta t)-f(t)]}{\Delta t}\times g(t)=f\times\frac{dg}{dt}+\frac{dg}{dt}\times g(t)$$

Hence, $\frac{d}{dt}[f\times g]=f\times\frac{dg}{dt}+\frac{df}{dt}\times g$

Example 8 Let $\overline{F}(t)=\hat{i}+t\hat{j}+t^2\hat{k}$ and $\overline{G}(t)=t\hat{i}+e^t\hat{j}+3\hat{k}$ are the vector functions. Verify the derivative: $\frac{d}{dt}\left(\overline{F}\times\overline{G}\right)(t)=\frac{d\overline{F}}{dt}\times\overline{G}+\overline{F}\times\frac{d\overline{G}}{dt}$

Solution For verification, the L.H.S is:

$$L.H.S = \frac{d}{dt}(F\times G)(t) = \frac{d}{dt}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & t & t^2\\ t & e^t & 3\end{vmatrix} = \frac{d}{dt}\left[\hat{i}\begin{vmatrix}t & t^2\\ e^t & 3\end{vmatrix} - \hat{j}\begin{vmatrix}1 & t^2\\ t & 3\end{vmatrix} + \hat{k}\begin{vmatrix}1 & t\\ t & e^t\end{vmatrix}\right]$$

$$= \frac{d}{dt}[(3t - t^2e^t)\hat{i} - (3-t^3)\hat{j} + (e^t - t^2)\hat{k}] = \frac{d}{dt}(3t - t^2e^t)\hat{i} - \frac{d}{dt}(3-t^3)\hat{j} + \frac{d}{dt}(e^t - t^2)\hat{k}$$

$$= (3 - 2te^t - t^2e^t)\hat{i} + 3t^2\hat{j} + (e^t - 2t)\hat{k}$$

Now, R.H.S $= \frac{d\overline{F}}{dt}\times\overline{G} + \overline{F}\times\frac{d\overline{G}}{dt}$

The expressions $\frac{d\overline{F}}{dt} = \hat{j} + 2t\hat{k}$, $\frac{d\overline{G}}{dt} = \hat{i} + e^t\hat{j}$

$$\Rightarrow \frac{d\overline{F}}{dt}\times\overline{G} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & 2t\\ t & e^t & 3\end{vmatrix} = (3 - 2te^t)\hat{i} - (-2t^2)\hat{j} + (-t)\hat{k}$$

$$\Rightarrow \overline{F}\times\frac{d\overline{G}}{dt} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & t & t^2\\ 1 & e^t & 0\end{vmatrix} = (-t^2e^t)\hat{i} - (-t^2)\hat{j} + (e^t - t)\hat{k}$$ are used in the RHS to obtain

$$R.H.S = \frac{d\overline{F}}{dt}\times\overline{G} + \overline{F}\times\frac{d\overline{G}}{dt} = (3 - 2te^t - t^2e^t)\hat{i} + (2t^2 + t^2)\hat{j} + (-t + e^t - t)\hat{k}$$

$= (3 - 2te^t - t^2e^t)\hat{i} + (3t^2)\hat{j} + (e^t - 2t)\hat{k}$ which is identical to the L.H.S.

Thus, the L.H.S = R.H.S.

Example 9 If $\overline{F}(t) = \hat{i} + e^t\hat{j} + t^2\hat{k}$ and $\overline{G}(t) = 3t^2\hat{i} + e^{-t}\hat{j} - 2t\hat{k}$ are the two vector functions and $h(t)$ is any scalar function, then evaluate the following derivatives: (a). $\frac{d}{dt}\left(2\overline{F} + t^3\overline{G}\right)$ (b). $\frac{d}{dt}\left(\overline{F}.\overline{G}\right)$

Solution

a. $\frac{d}{dt}\left(2\overline{F} + t^3\overline{G}\right) = \frac{d}{dt}\left[2\left(\hat{i} + e^t\hat{j} + t^2\hat{k}\right) + t^3\left(3t^2\hat{i} + e^{-t}\hat{j} - 2t\hat{k}\right)\right] = \frac{d}{dt}\left[(2 + 3t^5)\hat{i} + (2e^t + t^3e^{-t})\hat{j} \ldots\right]$

$= 15t^4\hat{i} + \left(2e^t + 3t^2e^{-t} - t^3e^{-t}\right)\hat{j} + (4t - 8t^3)\hat{k} = 15t^4\hat{i} + \left(2e^t + t^2e^{-t}(3-t)\right)\hat{j} + 4t(1 - 2t^2)\hat{k}$

b. $\frac{d}{dt}\left(\overline{F}.\overline{G}\right) = \frac{d}{dt}\left[(\hat{i} + e^t\hat{j} + t^2\hat{k}).(3t^2\hat{i} + e^{-t}\hat{j} - 2t\hat{k})\right]$ $\quad\therefore \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

$= \frac{d}{dt}(3t^2 + 1 - 2t^3) = 6t + 0 - 6t^2 = -(6t^2 - 6t)$

5.4.2 Applications of vector differentiation to calculate velocity and acceleration

In the calculus of single variable the velocity is defined as the derivative of the position function. For vector calculus we use the same definition.

i. Velocity

Let $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be a differential vector valued function representing the position vector of a particle at time 't' then the velocity vector is the derivative of position vector.

$$v(t) = \bar{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k} \quad \text{or} \quad \frac{d\bar{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Example 10 Find the velocity of the particle whose position vector is $\bar{r} = \bar{r}(t) = 5t\hat{i} + 4t\hat{j} - \cos(t)\hat{k}$

Solution Since, $\bar{r} = 5t\hat{i} + 4t\hat{j} - \cos(t)\hat{k}$

$$\text{Velocity} = \frac{d}{dt}(\bar{r}) = \frac{d}{dt}(5t\hat{i} + 4t\hat{j} - \cos(t)\hat{k}) \Rightarrow \frac{d\bar{r}}{dt} = 5\hat{i} + 4\hat{j} + \sin(t)\hat{k}$$

ii. Acceleration

In the calculus of single variable, we defined the acceleration of a particle as the second derivative of the position vector. There is no change for the vector calculus.

"Let $\bar{r} = \bar{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$ be a twice differentiable vector valued function, representing the position vector of a particle at time 't'. Then the acceleration vector is the second derivative of the position vector $\bar{r}(t)$

$$\bar{a} = \bar{r}''(t) = x''(t)\hat{i} + y''(t)\hat{j} + z''(t)\hat{k} \quad \text{or} \quad \bar{a} = \frac{d^2\bar{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

Example 11 Find the acceleration of the particle whose position vector is

$\bar{r}(t) = (3t^2 + 5)\hat{i} - (4t^2 + 2t - 1)\hat{j} + \sin(t)\hat{k}$

Solution Since, $\bar{r}(t) = (3t^2 + 5)\hat{i} - (4t^2 + 2t - 1)\hat{j} + \sin(t)\hat{k}$

$$\frac{d}{dt}(\bar{r}) = \frac{d}{dt}\left[(3t^2 + 5)\hat{i} - (4t^2 + 2t - 1)\hat{j} + \sin(t)\hat{k}\right] = \frac{d}{dt}(3t^2 + 5)\hat{i} - \frac{d}{dt}(4t^2 - 2t - 1)\hat{j} + \frac{d}{dt}\sin(t)\hat{k}$$

$$\frac{d\bar{r}}{dt} = 6t\hat{i} - (8t + 2)\hat{j} + \cos(t)\hat{k}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{d\bar{r}}{dt}\right] = \frac{d}{dt}\left[6t\hat{i} - (8t + 2)\hat{j} + \cos(t)\hat{k}\right]$$

$$\Rightarrow \left[\frac{d^2\bar{r}}{dt^2}\right] = 6\hat{i} - 8\hat{j} - \sin\hat{k}$$

iii. Speed

In the calculus of single variable the speed was the absolute value of the velocity. In the vector calculus it is the magnitude of velocity vector.

Remember

The direction of motion can be calculate by using $\frac{\overline{V}}{|\overline{V}|}$

Let $\bar{r}(t)$ be a differentiable vector valued function representation of the position of a particle in time 't' the speed 's' of the particle is the magnitude of the velocity vector. Speed $= \overline{S}(t) = |\overline{V}(t)| = |\bar{r}'(t)|$

Example 12 Fin the speed of particle whose position vector is $\bar{r}(t) = 3t\hat{i} + 4\hat{j} + \sin(t)\hat{k}$ after 30 seconds.

Solution Since, $\bar{r}(t) = 3t\hat{i} + 4\hat{j} + \sin(t)\hat{k} \Rightarrow \overline{V}(t) = \frac{d}{dt}(\bar{r}) = \frac{d}{dt}(3t\hat{i} + 4\hat{j} + \sin(t)\hat{k})$

$$\overline{V}(t) = 3\hat{i} + \cos(t)\hat{k}) \Rightarrow \overline{V}(30) = 3\hat{i} + \cos(30)\hat{k}) \Rightarrow \overline{V} = 3\hat{i} + \frac{\sqrt{3}}{2}\hat{k}$$

$$\text{Speed} = |\overline{V}| = \sqrt{(3)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{9 + \frac{3}{4}} = \sqrt{\frac{36+3}{4}} = \frac{\sqrt{39}}{2} = 3.12m/s$$

Example 13 A particle's position at time 't' is determined by the vector $\bar{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} + t^3\hat{k}$. Find the particle's velocity, speed, direction and acceleration at a time $t = 2$. Interpret the particle's motion;

Solution: If the particle's position at a time t is , then $\vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}+t^3\hat{k}$ then, the particle's velocity and acceleration are: $\vec{V}(t)=\frac{d}{dt}(\vec{r})=\frac{d}{dt}\left[\cos(t)\hat{i}+\sin(t)\hat{j}+t^3\hat{k}\right]=-\sin t\,\hat{i}+\cos t\,\hat{j}+3t^2\hat{k}$

$$a(t)=\frac{d\vec{V}}{dt}=\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)=\frac{d}{dt}\left[-\sin(t)\hat{i}+\cos(t)\hat{j}+3t^2\hat{k}\right]=-\cos(t)\hat{i}-\sin(t)\hat{j}+6(t)\hat{k}$$

The velocity at a time $t=2$ is $\vec{V}(2)=-\sin(2)\hat{i}+\cos(2)\hat{j}+3(4)\hat{k}\approx-0.91\hat{i}-0.42\hat{j}+12\hat{k}$, *use radians*

The acceleration at a time t = 2 is $\vec{a}(2)=-\cos(2)\hat{i}-\sin(2)\hat{j}+6(2)\hat{k}\approx0.42\hat{i}-0.91\hat{j}+12\hat{k}$

The speed is $\left|\vec{V}\right|=\sqrt{(-\sin t)^2+(\cos t)^2+(3t^2)^2}=\sqrt{1+9t^4}$ .At a time $t=2$,

The speed is $\left|\vec{V}\right|=\sqrt{1+9(2)^4}=\sqrt{145}$.

The direction of motion is: $\frac{\vec{V}}{\left|\vec{V}\right|}=\frac{1}{\sqrt{145}}\left[-\sin t\,\hat{i}+\cos t\,\hat{j}+3t^2\hat{k}\right]$

At a time t=2, the direction of motion is: $\frac{\vec{V}}{\left|\vec{V}\right|}=\frac{1}{\sqrt{145}}\left[-\sin 2\,\hat{i}+\cos 2\,\hat{j}+12\hat{k}\right]\approx-0.91\hat{i}-0.42\hat{j}+12\hat{k}$

Exercise 5.2

1. Find the vector derivative of the following vector functions:
   a. $\vec{F}(t)=t\hat{i}+t^2\hat{j}+(t+t^3)\hat{k}$
   b. $\vec{F}(s)=(s\hat{i}+s^2\hat{j}+s^2\hat{k})+(2s^2\hat{i}-s\hat{j}+3\hat{k})$
   c. $\vec{F}(\theta)=\cos\theta\left[\hat{i}+\tan\theta\,\hat{j}+3\hat{k}\right]$
2. Find the second order derivatives of the following vector valued functions.
   a. $\vec{F}(t)=t^2\hat{i}+3t^3\hat{j}-8t^2\hat{k}$
   b. $\vec{F}(s)=(3+s^2)\hat{i}-(s+1)^2\hat{j}+3s^4\hat{k}$
   c. $\vec{F}(x)=\ln x\hat{i}-x^2\hat{k}$
   d. $\vec{F}(\theta)=\sin^2\theta\,\hat{i}-\cos^2\theta\,\hat{j}$
3. Differentiate the following scalar functions:
   a. $\vec{f}(x)=[x\hat{i}+(x+1)\hat{j}].[2x\hat{i}-3x^2\hat{j}]$
   b. $\vec{g}(x)=\left|\sin x\hat{i}-2x\hat{j}+\cos x\hat{k}\right|$
4. Find the particle's velocity, acceleration, speed and direction of motion for the indicated value of t, when the position vector of a particle's in space at time t is $\vec{r}(t)$:
   a. $\vec{r}(t)=t\hat{i}+t^2\hat{j}+2t\hat{k}$ *at* $t=1$
   b. $\vec{r}(t)=\cos t\hat{i}+\sin t\,\hat{j}+3t\hat{k}$ *at* $t=\frac{\pi}{4}$
   c. $\vec{r}(t)=e^t\hat{i}+e^{-t}\hat{j}+e^{2t}\hat{k}$ *at* $t=\ln 2$
5. If $F(t)$ is a differentiable vector functions of t such that $F(t)\neq0$, then show that

$$\frac{d}{dt}\frac{F(t)}{|F(t)|}=\frac{F'(t)}{|F(t)|}-\frac{[F(t).F'(t)]F(t)}{|F(t)|^3}$$

Review Exercise 5

I. Choose the correct option.

i. A quantity having magnitude and direction is called:
   (a). vector (b). scalar (c). velocity (d). derivative

ii. A quantity having magnitude only and no direction is called:
   (a). vector (b). scalar (c). velocity (d). derivative

iii. If $\vec{F}=t^2\hat{i}+t\hat{j}-\sin(t)\hat{k}$ and $\vec{G}=t\hat{i}+t^{-1}\hat{j}+5\hat{k}$ then $\vec{F}\times\vec{G}=$
   (a). $\left(5t^2+\frac{\sin(t)}{t}\right)\hat{i}-(5t^2+t\sin(t))\hat{j}-(t+t^2)\hat{k}$
   (b). $\left(5t+\frac{\sin(t)}{t}\right)\hat{i}+(5t^2+t\sin(t))\hat{j}+(t^2-t)\hat{k}$
   (c). $\left(5t+\frac{\sin(t)}{t}\right)\hat{i}-(5t^2+t\sin(t))\hat{j}+(t-t^2)\hat{k}$
   (d). $\left(5t^2+\frac{\sin(t)}{t}\right)\hat{i}-(5t^2+t\sin(t))\hat{j}+(t-t^2)\hat{k}$

iv. Let $\vec{F}=t^2\hat{i}+t\hat{j}-\sin(t)\hat{k}$ and $\vec{G}=t\hat{i}+t^{-1}\hat{j}+5\hat{k}$ then $\vec{F}\cdot\vec{G}=$
   (a). $t^3+1+5\sin(t)$ (b). $t^2-1+4\sin(t)$ (c). $3t^2-1+4\sin(t)$ (d). $t^3+1-5\sin(t)$

v. If $\vec{F}(t)=(t+3)\hat{i}+2t^2\hat{j}-(1-t)\hat{k}$ then $\lim_{x\to5}\vec{F}(-2)=$ is:
   (a). $8\hat{i}+50\hat{j}+4\hat{k}$ (b). $8\hat{i}+10\hat{j}-4\hat{k}$ (c). $8\hat{i}+60\hat{j}-4\hat{k}$ (d). $8\hat{i}+50\hat{j}+4\hat{k}$

vi. If $\vec{F}=t^3\hat{i}-(3+t^2)\hat{k}$ then $\vec{F}'(-2)$ is:
   (a). $4\hat{i}-12\hat{k}$ (b). $12\hat{i}-4\hat{k}$ (c). $12\hat{i}+4\hat{k}$ (d). $12\hat{i}-8\hat{k}$

vii. If $\vec{r}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\left|\vec{r}\right|=$:
   (a). $\sqrt{29}$ (b). $\sqrt{30}$ (c). 5 (d). 6

viii. If $\vec{r}=5t^2\hat{i}+3t\hat{j}+\hat{k}$ then velocity vector $\vec{v}$ is:
   (a). $10t\hat{i}+3\hat{j}$ (b). $5t\hat{i}+3\hat{k}$ (c). $2t\hat{i}+3\hat{j}$ (d). $10\hat{t}-3\hat{k}$

ix. If velocity vector $\vec{v}=\sin(t)\hat{i}-2\cos(t)\hat{j}+4\hat{k}$ then acceleration $\vec{a}=$
   (a). $-\sin(t)\hat{i}+2\cos(t)\hat{j}-4\hat{k}$ (b). $\cos(t)\hat{i}-2\sin(t)\hat{j}+4\hat{k}$
   (c). $-\cos(t)\hat{i}+2\sin(t)\hat{j}$ (d). $\cos(t)\hat{i}-2\sin(t)\hat{j}$

x. $\lim_{t\to t_0}(C.\vec{F}(t))=$
   (a). $C\cdot\vec{F}(t)$ (b). $C\cdot F(t_0)$ (c). $C+\vec{F}(t_0)$ (d). $\frac{C}{\vec{F}(t_0)}$

Summary

- The **parametric equation** for the plane curve C generated by the set of ordered pairs in 2-space is: $(x, y) = (x(t), y(t)) = (f(t), g(t))$
- The **parametric equation** for the plane curve C generated by the set of ordered triples in 3-space is: $(x, y, z) = (x(t), y(t), z(t)) = (f(t), g(t), h(t))$
- A vector function $\vec{F}(t)$ is **continuous** at $t = t_0$ if t_0 is in the domain of $F(t)$ $\lim_{t \to t_0} F(t) = F(t_0)$
- The derivative of a vector function $F(t)$ is the vector function $F'(t)$ determined by the limit

$$F'(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{F}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t},$$

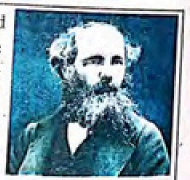
whenever this limit exists. In the Leibniz notation, the derivative of $F(t)$ is denoted by:

$$\frac{dF}{dt} = \lim_{\Delta t \to 0} \frac{\Delta F}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}$$

- If an object moves in such a way that its position at any time t is the **position vector** or displacement $R(t)$, then the
  - **Velocity** is $\vec{V} = \frac{d\vec{r}}{dt} = \vec{r}'(t)$
  - **Acceleration** is $\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{V}}{dt^2} = \vec{r}''(t)$
  - At any time t, **the speed** is $|\vec{V}| = |V_1\hat{i} + V_2\hat{j} + V_3\hat{k}| = \sqrt{V_1^2 + V_2^2 + V_3^2}$, the magnitude of the velocity and **the direction of motion** is $\frac{\vec{V}}{|\vec{V}|}$.

History

J.C. Maxwell was Scottish mathematician. He made a great contributions in the field of mathematical physics. He formulated the classical theory of electromagnetic radiation. Maxwell's equations for electromagnetism has been called the second great unification in physics. But the first unification in physics was raised by Sir Isaac Newton. In his publication "A Dynamical theory of electromagnetic field" he demonstrated electric and magnetic fields travel through space as waves moving at the speed of light. On the bases of his idea in electromagnetism Gibbs and Oliver developed vector analysis.



James Clark Maxwell (1831)-(1879)

Project

Create an art on a chart paper by hand or use any technological mean. Your creation should demonstrate a topic from this unit.

Create something using your imagination or use the mathematical concepts discussed in this unit to create your real world object

Example 20 Find the area between the x-axis and the curve $f(x) = x^2 - 2x$ from $x = -1$ to $x = 3$.

Solution First find out the x-intercepts of a curve $f(x) = x^2 - 2x$ that can be found by solving the equation of a curve:

$$x^2 - 2x = 0 \Rightarrow x = 0, 2$$

The subintervals of the interval $[-1,3]$ are therefore $[-1,0]$, $[0,2]$and $[2,3]$. The total area of the region in the required interval $[-1,3]$ is the sum of the areas of the sub regions in the subintervals $[-1,0]$, $[0,2]$and $[2,3]$:

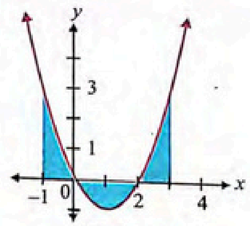


Figure 6.8

$$A = \int_{-1}^{0}[+f(x)]dx + \int_{0}^{2}[-f(x)]dx + \int_{2}^{3}[+f(x)]dx,\ f(x) \geq 0 \, in[-1,0],[2,3]$$

$$= \int_{-1}^{0}(x^2-2x)dx - \int_{0}^{2}(x^2-2x)dx + \int_{2}^{3}(x^2-2x)dx = \left|\frac{x^3}{3} - \frac{2x^2}{2}\right|_{-1}^{0} - \left|\frac{x^3}{3} - \frac{2x^2}{2}\right|_{0}^{2} + \left|\frac{x^3}{3} - \frac{2x^2}{2}\right|_{2}^{3}$$

$$= (0-0) - \left(\frac{-1}{3} - 1\right) - \left|\left(\frac{8}{3} - 4\right) - (0-0)\right| + \left(\frac{27}{3} - 9\right) - \left(\frac{8}{3} - 4\right) = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 3\left(\frac{4}{3}\right) = 4$$

The sketch of the region is shown in the Figure 6.8.

6.6.6 MAPLE command "int" to evaluate definite and indefinite integrals

The use of maple common '*int*' is illustrated in the following example.

Example 21 Use MAPLE command 'int' to solve.

(a). Indefinite integral of a function $f(x) = x^4 + x^3 + x^2 + x + 1$ w.r.t variable x.

(b). Definite integral of a function $f(x) = x^2$ w.r.t variable x.

(c). Definite integral of a function $f(x) = xe^x$ in the interval $[0,1]$.

Solution

a. **Command:**

```
> int(x^4 + x^3 + x^2 + x + 1, x);
```

$$\frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$$

Using Palettes: Use cursor button to select integral palette. Click-integral palette, insert the function required, then press "ENTER" key to obtain the integral of a given function:

> $\int x^4 + x^3 + x^2 + x + 1 dx$

$$\frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$$

b. **Command:**

```
> int(x^2, x = 0..1);
```

$$\frac{1}{3}$$

Using Palettes:

> $\int_0^1 x^2 dx$

$$\frac{1}{3}$$

c. **Command:**

```
> int(x·exp(x), x = 0..1);
```

$$1$$

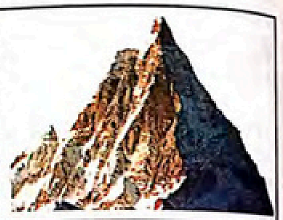
Using Plalettes:

> $\int_0^1 x \cdot \exp(x) dx$

$$1$$

Do you know a 200 year old problem ?

The relationship between derivative and integrals as an inverse operation was noticed first time by Isaac barrow (1630-1677) in the 17th century. He was a teacher of Sir Isaac Newton. Newton and Leibniz are known as key inventor of calculus. They made the use of calculus as conjuctor, that is as a mathematical statement which is suspected to be true. But has not proven yet. The fundamental theorem of integral calculus was not officially proven in all its glory until Bernhard Riemann (1826-1866) demonstrated it in the 19th century. During this 200-years a lot of mathematic like real analysis had invented before Riemann could prove that derivatives and integrals are inverse.



Exercise 6.4

1. Evaluate the following definite integrals:

a. $\int_3^4 5x\,dx$ b. $\int_{12}^{20} x^3 dx$ c. $\int_1^2 (2x^{-2}-3)dx$ d. $\int_1^4 3\sqrt{x}\,dx$

e. $\int_2^3 12(x^2-4)^5 x\,dx$ f. $\int_{-1}^{1} \frac{e^{-x}-e^x}{(e^{-x}+e^x)^2}dx$ g. $\int_{\frac{\pi}{2}}^{\pi} \cos\left(\frac{x}{2}+\pi\right)dx$ h. $\int_0^{\frac{\pi}{4}} \sec^2\theta\, d\theta$

2. Evaluate the following definite integrals:

a. $\int_1^2 \frac{5t^2-3t+18}{t(9-t^2)}dt$ b. $\int_1^2 \frac{4}{t^3+4t}dt$

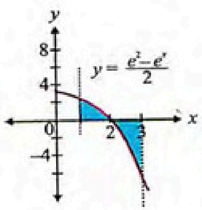
3. Use definite integral to find out the area between the curve f(x) and the x-axis over the indicated interval [a, b]:

a. $f(x)=4-x^2$, [0,3] b. $f(x)=x^2-5x+6$, [0,3]

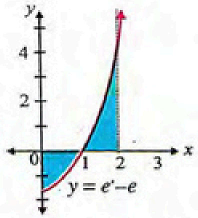
c. $f(x)=x^2-6x+8$, [0,4] d. $f(x)=5x-x^2$, [1,3]

4. Setup definite integrals in problems a to d that represent the indicated shaded areas:

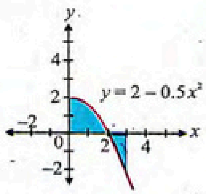
a.



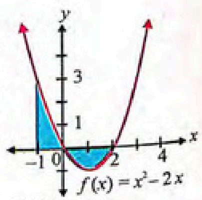
b.



c.



d.



5. An oil tanker is leaking oil at a rate given in barrels per hour by $\frac{dL}{dt}=\frac{80\ln(t+1)}{(t+1)}$

Where t is the time in hours after the tanker hits a hidden rock (when $t=0$).

a. Find the total number of barrels that the ship will leak on the first day.

b. Find the total number of barrels that the ship will leak on the second day.

c. What is happening over the long run to the amount of oil leaked per day?

6. Use MAPLE command *'int'* to evaluate

a. $f(x)=x^2+3x+1$ w.r.t. 'x' b. $f(x)=e^{2x}.\sin x$ w.r.t. 'x'

Review Exercise 6

1. **Choose the correct option.**

i. The process of finding antiderivative is called.

(a). differentiation (b). integration (c). probability (d). linear equations

ii. $\int \tan\theta\, d\theta =$ ________

(a). $\ln|\sin\theta|+C$ (b). $\ln|\cos\theta|+C$ (c). $-\ln|\cos\theta|+C$ (d). $-\ln|sin\theta|+C$

iii. $\int \frac{dx}{\sqrt{a^2-x^2}} =$ ________

(a). $\ln\left|x+\sqrt{x^2+a^2}\right|+C$ (b). $\sin^{-1}\frac{a}{x}+C$ (c). $\sin^{-1}\left(\frac{x}{a}\right)+C$ (d). $\cos^{-1}\left(\frac{x}{a}\right)+C$

iv. $\int \frac{1}{\sqrt{t^2-36}}dx =$ ________,

(a). $\frac{1}{2a}\ln\left|\frac{t-6}{t+6}\right|+C$ (b). $\frac{1}{2}\ln\left|\frac{t-6}{t+6}\right|+C$

(c). $-\frac{1}{2}\left(\ln\left|\frac{t}{6}+1\right|-\ln\left|\frac{t}{6}-1\right|+C\right)$ (d). $\frac{1}{6}\left(\ln\left|\frac{t}{6}+1\right|+6\right)+C$

v. $\int (x^3-4)dx =$ ________

(a). $\frac{x^4}{4}-4x+C$ (b). $\frac{x^3}{3}-4x+C$

(c). $-\frac{x^4}{4}+4x+C$ (d). $\frac{x^4}{4}+\frac{4x^3}{3}-\frac{4x^2}{2}-4x+C$

vi. $\int f(x)g'(x)dx =$ ________

(a). $f'(x)g(x)+\int g'(x)f(x)dx$ (b). $f(x)g(x)-\int g(x)f'(x)dx$

(c). $f'(x)'g(x)-\int g'(x)f'(x)dx$ (d). $f(x)g(x)-\int g(x)f(x)dx$

vii. $\int \tan^4(x)dx =$ ________

(a). $\frac{2}{3}\tan^2(x)+x-\tan(x)+C$ (b). $\frac{1}{3}\tan^3(x)+x-\tan(x)+C$

(c). $\frac{3}{4}\tan^2(x)-x+\tan(x)+C$ (d). $3\tan^3(x)+x+\tan(x)+C$

viii. $\int \frac{x+8}{x^2-64}dx =$

(a). $\ln|x-8|+C$ (b). $\frac{1}{2}\ln|x^2-64|+\frac{1}{2}\left|\frac{x}{8}+1\right|-\frac{1}{2}\ln\left|\frac{x}{2}-1\right|+C$

(c). $\frac{1}{2}\ln|x^2-64|-\frac{1}{2}\ln\left|\frac{x}{8}+1\right|+\frac{1}{2}\ln\left|\frac{x}{8}-1\right|+C$ (d). $\frac{1}{2}\ln\left|\frac{x}{8}+1\right|+\frac{1}{2}\ln\left|\frac{x}{8}-1\right|+C$

ix. $\int_0^2 e^{2x}dx =$ ________

(a). $\frac{e^4-1}{2}$ (b). $\frac{e^3-1}{2}$ (c). $\frac{e^2-1}{4}$ (d). $\frac{e^4+1}{2}$

x. $\int_0^2 x^3 dx =$ ________.

(a). 1 (b). 2 (c). 3 (d). 4